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# Bridging the Hybrid High-Order and Virtual Element methods

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## Abstract

We build a bridge between the Hybrid High-Order and the Virtual Element methods on general polytopal meshes in dimension 2 or 3, in the setting of a model Poisson problem. To do so, we reformulate the conforming-Virtual Element method as a (newborn) *conforming-Hybrid High-Order* method, and we derive  $H^s$ -approximation properties for the associated local polynomial projector (on which hinges the local discrete bilinear form). This allows us to perform a unified study of conforming/nonconforming-Virtual Element/Hybrid High-Order methods, which simplifies the classical error analysis of Virtual Element methods in broken  $H^1$ -seminorm, and sheds new light on the differences between the conforming and nonconforming cases (in particular in terms of mesh assumptions).

## 1 Introduction

The design of arbitrary-order Galerkin methods that support meshes with general polygonal/polyhedral cells has been attracting the attention of the community for more than 40 years now. In practice, the use of general meshes, when not an inherent constraint like e.g. in subsurface modelling, can bring major advantages. In particular, it increases the flexibility in meshing complex geometries, and simplifies the refinement/coarsening procedures in adaptive simulations. Classical arbitrary-order polytopal discretisation approaches encompass the (polytopal) Finite Element (FE) method [40, 38], and the (polytopal) discontinuous Galerkin (dG) method [1, 28, 3, 13]. The construction of FE shape functions on arbitrarily-shaped cells that both (i) satisfy the desired conformity prescriptions, and (ii) for which closed-form expressions can be obtained (and numerically integrated), is highly challenging. However, when the shape functions are available, one can fully benefit from the fact that FE are a skeletal method, i.e. cell degrees of freedom can be locally eliminated in terms of the skeletal degrees of freedom, thus reducing the number of globally coupled unknowns. The dG method, for which stability is enforced by penalization of the discrete bilinear form, can handle completely nonconforming discrete spaces. One hence has the opportunity to consider simple polynomial local approximation spaces. The price to pay for such a flexibility is, naturally, an increased number of globally coupled degrees of freedom, which makes of dG a computationally more expensive method than FE on standard meshes (and this is all the more true that the order of approximation increases). When considering meshes featuring cells with an important number of faces, things are not that clear anymore, and dG may definitely become a competitive computational approach. Yet, in that case, (polynomial-based) dG suffers from an important limitation. On general cells, it is not clear at all whether the polynomial local approximation spaces are sufficiently rich to robustly approximate tricky operators like the divergence (think, e.g., of a linear elasticity model in the quasi-incompressible limit) or curl operators.

More recently, a new paradigm has emerged. The idea is to define a finite element whose construction is generic with respect to the shape of the element. The underlying local approximation space (i) is spanned by functions that are (at least for some of them) implicitly defined (usually as the solutions to some PDEs posed in the cell), (ii) is built so that the desired conformity properties can be obtained at the global level, and (iii) is constructed so as to enjoy sufficient approximation properties (for instance, so as to contain the polynomial functions up to a given degree). The fact that one cannot obtain a closed-form expression for all shape functions is the reason why they are called virtual in that context. In practice, the numerical method is defined using computable (in terms of the degrees of freedom) projections of the virtual functions, and is stabilized through a subtle penalization, that shall also be computable. The most salient example of such an approach is the (polynomial-based) Virtual Element (VE) method [5, 6], which has first been

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introduced under its conforming version (denoted c-VE). Another example is the Hybrid High-Order (HHO) method [31], first introduced/analysed for linear elasticity [29], and then for the Poisson problem [30]. Both the VE and the HHO methods are skeletal methods. In [21, Section 2.4], the HHO method has been proved equivalent (up to identical stabilization and right-hand side) to the nonconforming version of the VE method (denoted nc-VE) introduced in [36], and posteriorly analysed in [2]. In [21], the HHO method has also been bridged to the so-called Hybridizable Discontinuous Galerkin (HDG) method [23, 20], in the sense that it is possible to recast the HHO method as a HDG method, with distinctive numerical flux trace. This work has shed light on the fact that the quite subtle choice of stabilization advocated in HHO results in HDG formulation in a numerical flux trace that ensures superconvergence on general polytopal meshes. Note that efforts towards superconvergence for standard HDG methods are still being undertaken [22]. Finally, in [27], the nc-VE/HHO methods are proven to be Gradient Discretisation methods [32].

Let  $\Omega$  be a bounded and connected open subset of  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ , whose boundary is assumed to be composed of a finite union of portions of affine hyperplanes. We focus on the following model Poisson problem: find  $u \in H_0^1(\Omega)$  solution to

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v =: l(v) \quad \text{for all } v \in H_0^1(\Omega), \quad (1)$$

with source term  $f \in L^2(\Omega)$ . In this work, we complete the construction of the bridge between HHO and VE undertaken in [21, Section 2.4]. To do so, we reformulate the c-VE method as a (newborn) *conforming-HHO* method, and we prove  $H^s$ -approximation properties for the associated local polynomial projector  $\mathcal{P}_T$  (on which hinges the local discrete bilinear form). This enables us to perform a unified study of c/nc-VE/HHO methods, which introduces substantial simplifications in the error analysis of VE methods in broken  $H^1$ -seminorm, and sheds new light on the differences between the conforming and nonconforming cases (in terms of mesh assumptions in particular). We build upon existing contributions, especially [26] on the analysis of schemes in fully discrete formulation, [14] on the unified analysis of c/nc-VE methods, and [37, 10, 16] (see also [8, 12, 15] for the treatment of faces with arbitrarily small measure) on the analysis of c-VE. For simplicity of exposition, we consider throughout this paper standard VE spaces (i.e., neither enhanced VE spaces [14], nor Serendipity VE spaces [7]), with cell degrees of freedom even in the lowest-order case, which simplifies the treatment of the right-hand side without compromising the computational efficiency.

A crucial observation that is made clear in this paper is the following. The main difference between the conforming and nonconforming VE/HHO methods is that, in the nonconforming case, the local canonical interpolation operator (say,  $\mathcal{I}_T$ ) is the elliptic projector on the local virtual space (cf. Eq. (27)). It is not true in the conforming case. This has two important consequences.

- Letting  $\Pi_T$  denote the elliptic projector on a polynomial subspace  $\mathbb{P}(T)$  of the local virtual space, the local polynomial projector  $\mathcal{P}_T := \Pi_T \circ \mathcal{I}_T$ , on which hinges the local discrete bilinear form, is actually equal to  $\Pi_T$  in the nonconforming case (this has already been pointed out in [26, Remark 25]), which is not true in the conforming case. Hence, whereas the HHO error in broken  $H^1$ -seminorm naturally splits along (recall that the only computable/computed quantity is  $\mathcal{P}_h u_h = \Pi_h u_h$ )

$$\begin{aligned} \|\nabla_h(u - \Pi_h u_h)\| &\leq \|\nabla_h(u - \mathcal{P}_h u)\| + \|\nabla_h \Pi_h(\mathcal{I}_h u - u_h)\| \\ &= \|\nabla_h(u - \Pi_h u)\| + \|\nabla_h \Pi_h(\mathcal{I}_h u - u_h)\| \end{aligned}$$

where the first term in the right-hand side is the approximation error (directly available from the properties of  $\Pi_h$ ), and the second is bounded by the consistency error of the scheme (energy-norm error), the VE error is usually split in the literature (including [26, Theorem 19]) along

$$\begin{aligned} \|\nabla_h(u - \Pi_h u_h)\| &\leq \|\nabla_h(u - \Pi_h u)\| + \|\nabla_h \Pi_h(u - \mathcal{I}_h u)\| + \|\nabla_h \Pi_h(\mathcal{I}_h u - u_h)\| \\ &\leq \|\nabla_h(u - \Pi_h u)\| + \|\nabla_h(u - \mathcal{I}_h u)\| + \|\nabla_h \Pi_h(\mathcal{I}_h u - u_h)\| \end{aligned}$$

where the approximation properties of the local virtual spaces frankly invite themselves into the picture. In [10], approximation properties for  $\mathcal{I}_T$  in the  $L^2$ -norm and  $H^1$ -seminorm have been proven by resorting to an inverse inequality for virtual functions. This reasoning cannot be extended to prove general  $H^s$ -approximation properties since derivatives of virtual functions may not be virtual functions. In this article, we directly investigate the approximation properties of the *polynomial* projector  $\mathcal{P}_T$ , for which  $H^s$ -inverse inequalities hold. Since  $\mathcal{P}_T = \Pi_T \circ \mathcal{I}_T$  preserves the polynomials in  $\mathbb{P}(T)$ , and since the operator  $\Pi_T$  is stable, all we have to prove are stability properties for  $\mathcal{I}_T$ ; we then follow the ideas of [11, Chapter 4] to derive the desired  $H^s$ -approximation properties. This is the object of Sections 4.1

(stability properties for  $\mathcal{I}_T$ , where the proofs are revisited with respect to [10] to highlight the difference between the conforming and nonconforming cases), and 4.2 (approximation properties for  $\mathcal{P}_T$ ; cf. in particular Theorem 4.17). One can then split the error in broken  $H^1$ -seminorm, irrespectively of the conformity of the method, along

$$\|\nabla_h(u - \Pi_h u_h)\| \leq \|\nabla_h(u - \mathcal{P}_h u)\| + \|\nabla_h \Pi_h(\mathcal{I}_h u - u_h)\|,$$

thereby allowing for a completely unified analysis inspired from the one of HHO methods (cf. Theorem 4.26 and Corollary 4.28).

- In the conforming case, in order to prove stability properties for  $\mathcal{I}_T$  (which actually is, as we explained above, the cornerstone of the analysis), one has to estimate a dual norm of the boundary flux for virtual functions (cf. Lemmas 4.6 and 4.11). For that, one must prove the existence, on any cell  $T$ , of a (linear) lifting operator  $\mathcal{L}_T$  such that, for any function  $v$  on  $\partial T$  that is equal to the trace of a virtual function, one has the following scaled estimate:

$$h_T^{-1} \|\mathcal{L}_T v\|_T + \|\nabla \mathcal{L}_T v\|_T \leq c \left( h_T^{-1/2} \|v\|_{\partial T} + h_T^{1/2} \|\nabla_t v\|_{\partial T} \right), \quad (2)$$

where  $\nabla_t$  denotes the tangential derivative, and  $c > 0$  is a constant independent of  $h_T$ . When  $T$  is star-shaped with respect to a ball of radius comparable to  $h_T$  (in standard analyses of c-VE methods, the latter assumption is supplemented by the fact that all the faces of  $T$  are assumed to have diameter comparable to  $h_T$ ), the result (2) is true (cf. [10, Eq. (2.48) and Lemma 5.3]). When  $T$  is not star-shaped, it is less clear whether optimal scaling can be obtained. In [16], the star-shapedness assumption is weakened by resorting in the error analysis to a well-chosen (noncanonical) interpolant, but the argument seems restricted to the 2D case. In [10, Section 4], the star-shapedness assumption is also weakened, but the details on how to construct the lifting (especially in the tricky 3D case) are not given. In the nonconforming case, owing to the fact that the local canonical interpolation operator is the elliptic projector on the local virtual space, a similar estimate is not needed to prove stability for  $\mathcal{I}_T$  (cf. Lemma 4.15 and Remark 4.16), and regularity assumptions on mesh sequences can be relaxed. All one needs to assume in that case is that the cell  $T$  can be remeshed by a (matching, shape-regular, and with meshsize comparable to  $h_T$ ) simplicial submesh (cf. Definition 2.1). These assumptions are sufficient to derive all the necessary analysis tools. We believe that, following [10, Lemma 5.3], it should be possible to prove a result such as (2) under the sole assumptions of Definition 2.1. Yet, this is beyond the scope of this work. We will hence do not pursue in that direction, and make the further assumption, in the conforming case, that each cell is star-shaped with respect to a ball (with radius comparable to its diameter).

The material is organized as follows. In Section 2, we introduce the notation, we detail our admissibility assumptions on mesh sequences, and we introduce a number of analysis tools that will be useful in the sequel. In Section 3, we undertake a general description of skeletal methods, such as the VE or HHO methods. Finally, in Section 4, we provide our unified formulation/study of c/nc-VE/HHO methods, as well as a description of the general workflow of the methods.

## 2 Notation, mesh assumptions, and basic analysis tools

We collect in this section all the conventions, tools, and results that will be useful in the sequel.

### 2.1 Notation

#### 2.1.1 Geometry

For  $l \in \{1, \dots, d\}$ , we let  $|\cdot|_l$  denote the  $l$ -dimensional Hausdorff measure. In what follows, the term polytope refers to polygons if  $d = 2$ , and to polyhedra if  $d = 3$ . The discretisation of the domain  $\Omega$  is described in the following manner.

- $\mathcal{T}_h$  denotes a mesh of the domain  $\Omega$ , i.e. a collection of disjoint open polytopes  $T$  (the cells) such that  $\bigcup_{T \in \mathcal{T}_h} \bar{T} = \bar{\Omega}$ . The parameter  $h$  is the meshsize, defined as  $h := \max_{T \in \mathcal{T}_h} h_T$ , where  $h_T$  stands for the diameter of the cell  $T$ .

- $\mathcal{F}_h$  denotes the collection of faces of the mesh  $\mathcal{T}_h$ . Since the cells of  $\mathcal{T}_h$  are polytopes, their boundary is composed of a finite union of closed portions of affine hyperplanes, called facets. A closed subset  $F$  of  $\bar{\Omega}$  with  $|F|_{d-1} \neq 0$  is a face as soon as (i)  $F$  is equal to the intersection, for  $T_1, T_2$  two cells of  $\mathcal{T}_h$ , of a facet of  $T_1$  and a facet of  $T_2$ , or (ii)  $F$  is equal to the intersection, for  $T$  cell of  $\mathcal{T}_h$ , of a facet of  $T$  and a facet of  $\Omega$ . In the first case,  $F$  is termed an interface, whereas in the second,  $F$  is termed a boundary face. Interfaces are collected in the set  $\mathcal{F}_h^i$ , boundary faces in the set  $\mathcal{F}_h^b$ , in such a way that  $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$ . For a cell  $T \in \mathcal{T}_h$ , we let  $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$  be the collection of faces composing its boundary, and  $\mathbf{n}_T$  be the unit normal vector to  $\partial T$  pointing outward  $T$  (that is defined almost everywhere on  $\partial T$ ). For  $F \in \mathcal{F}_T$ , we also let  $\mathbf{n}_{T,F} := \mathbf{n}_{T|F}$ ; remark that  $\mathbf{n}_{T,F}$  is a constant vector on  $F$  since  $F$  is planar. Finally, for any face  $F \in \mathcal{F}_h$ , we denote  $h_F$  its diameter (and, for  $T \in \mathcal{T}_h$ , we let  $h_{\partial T}$  be such that  $h_{\partial T|F} := h_F$  for all  $F \in \mathcal{F}_T$ ), and we let  $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$  be the collection of cells sharing  $F$  (two cells for an interface, one for a boundary face).
- $\partial\mathcal{T}_h$  denotes the  $(d-1)$ -dimensional skeleton of the mesh  $\mathcal{T}_h$ , that is  $\partial\mathcal{T}_h = \bigcup_{F \in \mathcal{F}_h} F$ .
- When  $d = 3$ ,  $\mathcal{E}_h$  denotes the collection of edges  $e$  (with  $|e|_1 \neq 0$ ) of the mesh  $\mathcal{T}_h$ , defined from the collection  $\mathcal{F}_h$  of faces. For a cell  $T \in \mathcal{T}_h$  (respectively a face  $F \in \mathcal{F}_h$ ), we let  $\mathcal{E}_T := \{e \in \mathcal{E}_h \mid e \subset \partial T\}$  (respectively  $\mathcal{E}_F := \{e \in \mathcal{E}_h \mid e \subset \partial F\}$ , that will also be denoted  $\mathcal{F}_F$  with a slight abuse in notation) be the collection of edges composing its boundary.
- $\mathcal{V}_h$  denotes the collection of vertices  $\nu$  of the mesh  $\mathcal{T}_h$ . For a cell  $T \in \mathcal{T}_h$  (respectively a face  $F \in \mathcal{F}_h$ ), we let  $\mathcal{V}_T := \{\nu \in \mathcal{V}_h \mid \nu \in \partial T\}$  (respectively  $\mathcal{V}_F := \{\nu \in \mathcal{V}_h \mid \nu \in \partial F\}$ ) be the collection of its vertices. The position of any vertex  $\nu \in \mathcal{V}_h$  is denoted  $\mathbf{x}_\nu \in \Omega$ .

When  $d = 2$ , faces are sometimes called edges in the literature. We will not use this vocable in this article. The term edge will always refer to a 1-manifold in dimension  $d = 3$ . We finally introduce, for  $\nu \in \mathcal{V}_h$ , the set  $\mathcal{F}_\nu := \{F \in \mathcal{F}_h \mid \nu \in \mathcal{V}_F\}$  and, when  $d = 3$ , for  $e \in \mathcal{E}_h$ , the set  $\mathcal{F}_e := \{F \in \mathcal{F}_h \mid e \in \mathcal{E}_F\}$ .

### 2.1.2 Functions spaces

For  $X \subset \bar{\Omega}$ , and  $m \geq 0$ , we let  $|\cdot|_{m,X}$  and  $\|\cdot\|_{m,X}$  respectively denote the seminorm and norm on the Sobolev space  $H^m(X; \mathbb{R}^l)$  ( $l \in \{1; d\}$ ), with the convention that  $H^0(X; \mathbb{R}^l) \equiv L^2(X; \mathbb{R}^l)$  (hence,  $|\cdot|_{0,X} \equiv \|\cdot\|_{0,X}$ ). We also define  $\|\cdot\|_{\infty,X}$  as the norm on  $L^\infty(X)$ . We finally let  $\langle \cdot, \cdot \rangle_{-m,X}$  be the duality pairing between  $H^{-m}(X)$  and its topological dual.

For  $q \in \mathbb{N}$  and  $l \in \{1, \dots, d\}$ , we let  $\mathbb{P}_l^q$  be the vector space of  $l$ -variate polynomial functions of total degree less than or equal to  $q$ . We also let

$$\mathbb{N}_l^q := \dim(\mathbb{P}_l^q) = \binom{q+l}{l},$$

and we adopt the conventions  $\mathbb{P}_l^{-1} := \{0\}$  and  $\mathbb{N}_l^{-1} := 0$ . For  $T \in \mathcal{T}_h$ , we define  $\mathbb{P}_d^q(T)$  as the restriction of  $\mathbb{P}_d^q$  to  $T$ . For  $F \in \mathcal{F}_h$ , we define  $\mathbb{P}_{d-1}^q(F)$  as the restriction of  $\mathbb{P}_d^q$  to  $F$ . When  $d = 3$ , for  $e \in \mathcal{E}_h$ , we define  $\mathbb{P}_1^q(e)$  as the restriction of  $\mathbb{P}_3^q$  to  $e$ . For  $T \in \mathcal{T}_h$ , we also define the broken space

$$\mathbb{P}_{d-1}^q(\mathcal{F}_T) := \{v \in L^2(\partial T) \mid v|_F \in \mathbb{P}_{d-1}^q(F) \forall F \in \mathcal{F}_T\},$$

and when  $d = 3$ , for  $F \in \mathcal{F}_h$ , we let  $\mathbb{P}_1^q(\mathcal{E}_F) := \{v \in L^2(\partial F) \mid v|_e \in \mathbb{P}_1^q(e) \forall e \in \mathcal{E}_F\}$ . We finally introduce, for any  $T \in \mathcal{T}_h$ , a set of basis functions for  $\mathbb{P}_d^q(T)$ , that we denote  $\{\psi_{T,i}^q\}_{i \in \{1, \dots, \mathbb{N}_d^q\}}$ , and for any  $F \in \mathcal{F}_h$ , a set of basis functions for  $\mathbb{P}_{d-1}^q(F)$ , that we denote  $\{\psi_{F,j}^q\}_{j \in \{1, \dots, \mathbb{N}_{d-1}^q\}}$ . When  $d = 3$ , we further introduce, for any  $e \in \mathcal{E}_h$ , a set of basis functions for  $\mathbb{P}_1^q(e)$ , denoted  $\{\psi_{e,m}^q\}_{m \in \{1, \dots, \mathbb{N}_1^q\}}$ .

Given a mesh  $\mathcal{T}_h$  of  $\Omega$ , we introduce the following notation for broken functions spaces on  $\mathcal{T}_h$ :

$$X(\mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_h|_T \in X(T) \forall T \in \mathcal{T}_h\}.$$

We introduce on  $H^1(\mathcal{T}_h)$  the so-called broken gradient operator  $\nabla_h : H^1(\mathcal{T}_h) \rightarrow L^2(\Omega)$  such that, for any  $v_h \in H^1(\mathcal{T}_h)$  and  $T \in \mathcal{T}_h$ ,  $\nabla_h v_h|_T := \nabla(v_h|_T)$ . For any interface  $F \in \mathcal{F}_h^i$  with  $\mathcal{T}_F = \{T_1, T_2\}$ , we define the jump  $[[v_h]]_F$  along  $F$  of  $v_h \in H^s(\mathcal{T}_h)$ ,  $s > 1/2$ , by  $[[v_h]]_F := (v_h|_{T_1})|_F - (v_h|_{T_2})|_F$ , and we let  $\mathbf{n}_F := \mathbf{n}_{T_1,F}$ . For any  $F \in \mathcal{F}_h^b$  with  $\mathcal{T}_F = \{T\}$ , we let  $[[v_h]]_F := (v_h|_T)|_F$ , and  $\mathbf{n}_F := \mathbf{n}_{T,F}$ . We finally introduce the operator  $[[\cdot]] : H^s(\mathcal{T}_h) \rightarrow L^2(\partial\mathcal{T}_h)$  such that, for any  $v_h \in H^s(\mathcal{T}_h)$ ,  $[[v_h]]|_F := [[v_h]]_F$  for all  $F \in \mathcal{F}_h$ . Note that the quantity  $[[v_h]] \in L^2(\partial\mathcal{T}_h)$  may be multi-valued (whenever this has a sense) at vertices when  $d = 2$ , and on edges/vertices when  $d = 3$ . Assume, for simplicity, that  $v_h \in H^s(\mathcal{T}_h)$  is such that  $v_h|_{\partial T} \in C^0(\partial T)$  for all

$T \in \mathcal{T}_h$ . Considering a vertex  $\nu \in \mathcal{V}_h$ , there are  $\text{card}(\mathcal{F}_\nu)$  potentially different values for  $\llbracket v_h \rrbracket$  at  $\mathbf{x}_\nu$ , that are the  $(\llbracket v_h \rrbracket_F(\mathbf{x}_\nu))_{F \in \mathcal{F}_\nu}$ . When  $d = 3$ , considering an edge  $e \in \mathcal{E}_h$ , there are, identically,  $\text{card}(\mathcal{F}_e)$  potentially different functions  $\llbracket v_h \rrbracket$  on  $e$ , that are the  $(\llbracket v_h \rrbracket_{F|e})_{F \in \mathcal{F}_e}$ .

## 2.2 Mesh assumptions

We define the notion of admissible mesh family.

**Definition 2.1.** *The mesh family  $(\mathcal{T}_h)_h$  is admissible if, for all  $h$ ,  $\mathcal{T}_h$  admits a matching simplicial submesh, denoted  $\mathcal{S}_h$ , and there exists  $\gamma > 0$ , called mesh regularity parameter, so that, for all  $h$ ,*

- (i) *for all  $S \in \mathcal{S}_h$  of diameter  $h_S$  and inradius  $r_S$ ,  $\gamma h_S \leq r_S$  (in other words,  $\mathcal{S}_h$  is shape-regular);*
- (ii) *for all  $T \in \mathcal{T}_h$ , and all  $S \in \mathcal{S}_T := \{S \in \mathcal{S}_h \mid S \subseteq T\}$ ,  $\gamma h_T \leq h_S$ .*

By matching simplicial submesh, we mean that  $\mathcal{S}_h$  is a (conforming, i.e. free of hanging node) simplicial mesh, and that, for all  $S \in \mathcal{S}_h$ , there exists a unique  $T \in \mathcal{T}_h$  such that  $S \subseteq T$ , and for all  $Z \in \mathcal{Z}_h$ , where  $\mathcal{Z}_h$  collects the faces of  $\mathcal{S}_h$ , there exists at most one  $F \in \mathcal{F}_h$  such that  $Z \subseteq F$  (cf. [28, Definition 1.37]). Henceforth, we will use the symbol  $\lesssim$  to notify that an estimate is valid up to a multiplicative constant  $c > 0$ , with  $c$  only depending on the dimension  $d$ , the mesh regularity parameter  $\gamma$ , and, if need be, the underlying polynomial degree; in particular, the bound is uniform with respect to the meshsize.

Let us mention three important consequences of Definition 2.1: for all  $h$ , and all  $T \in \mathcal{T}_h$ ,

- (a) for all  $S \in \mathcal{S}_T$ ,  $\gamma h_T \leq h_S \leq h_T$ , and  $\text{card}(\mathcal{S}_T) \lesssim 1$  (cf. [28, Lemma 1.40]);
- (b) for all  $F \in \mathcal{F}_T$ ,  $\text{card}(\mathcal{Z}_F) \lesssim 1$ , where  $\mathcal{Z}_F := \{Z \in \mathcal{Z}_h \mid Z \subseteq F\}$  (cf. [28, Lemma 1.41]);
- (c) for all  $F \in \mathcal{F}_T$ ,  $\gamma^2 h_T \leq h_F \leq h_T$ , and  $\text{card}(\mathcal{F}_T) \lesssim 1$  (cf. [28, Lemmas 1.42 and 1.41]).

From (a) and (b), one can picture the general outline to prove inverse and trace inequalities on arbitrarily-shaped (admissible) cells. One first considers the case of a simplex satisfying (i) of Definition 2.1, for which these inequalities are standard. Then, the passage to arbitrary geometries follows from the fact that any admissible cell is composed of an uniformly bounded number of simplices satisfying (i) (and, an admissible face is composed of an uniformly bounded number of subfaces belonging to simplices satisfying (i)), and whose diameters are comparable to the diameter of the cell under consideration (cf. Section 2.3.1). Note that the notion of admissible mesh we consider here allows for cells that are not necessarily star-shaped. When this latter additional assumption (i.e., that each cell  $T$  is star-shaped with respect to a ball of radius  $\geq \gamma h_T$ ) will be needed in the analysis, we will specify it. The assumptions of Definition 2.1 are also sufficient to prove  $H^s$ -approximation properties for standard polynomial projectors (cf. Section 2.3.3). As for point (c), it is instrumental in the analysis of numerical methods based on  $\mathcal{T}_h$ . When  $d = 3$ , it is an easy matter to show that, under the assumptions of Definition 2.1, one also has, for any  $T \in \mathcal{T}_h$ ,  $F \in \mathcal{F}_T$ , and  $e \in \mathcal{E}_F$ ,  $\gamma^3 h_F \leq h_e \leq h_F$ , and  $\text{card}(\mathcal{E}_F) \lesssim 1$ .

## 2.3 Basic analysis tools

Henceforth,  $\mathcal{T}_h$  denotes a member of an admissible mesh family in the sense of Definition 2.1.

### 2.3.1 Useful inequalities

On any  $T \in \mathcal{T}_h$ , the following inequalities hold:

- *inverse inequality:*

$$\forall v \in \mathbb{P}_d^q(T), \quad |v|_{1,T} \lesssim h_T^{-1} \|v\|_{0,T}; \quad (3)$$

- *continuous trace inequality:* for any  $F \in \mathcal{F}_T$ ,

$$\forall v \in H^1(T), \quad \|v|_F\|_{0,F} \lesssim h_T^{-1/2} \|v\|_{0,T} + h_T^{1/2} |v|_{1,T}; \quad (4)$$

- *discrete trace inequality:* for any  $F \in \mathcal{F}_T$ ,

$$\forall v \in \mathbb{P}_d^q(T), \quad \|v|_F\|_{0,F} \lesssim h_T^{-1/2} \|v\|_{0,T}. \quad (5)$$

For the proofs of these different results, we refer to [28, Section 1.4.3] (note that therein, faces may even be nonplanar). We also state the classical Poincaré inequality:

$$\forall v \in H^1(T) \text{ such that } \int_T v = 0, \quad \|v\|_{0,T} \leq c_P h_T |v|_{1,T}. \quad (6)$$

If  $T$  is convex,  $c_P = \pi^{-1}$  is optimal, independently of the ambient dimension; cf. [4]. For some insight on the value of  $c_P$  on more general element shapes, we refer to [39]. In the forthcoming analysis of conforming methods, we will also need (i) the following alternative inverse inequality, whose proof can be found, e.g., in [19, Lemma 4.4 (take  $\mathbb{A}_\varepsilon = \mathbb{I}_d$ )]: for all  $v \in H^1(T)$  such that  $\Delta v \in \mathbb{P}_d^q(T)$  for some  $q \in \mathbb{N}$ , there holds

$$\|\Delta v\|_{0,T} \lesssim h_T^{-1} |v|_{1,T}; \quad (7)$$

(ii) the following version of Sobolev's inequality (cf., e.g., [11, Lemma 4.3.4]):

$$\forall v \in H^2(T), \quad \|v\|_{\infty,T} \lesssim h_T^{-\frac{d}{2}} \|v\|_{0,T} + h_T^{\frac{(2-d)}{2}} |v|_{1,T} + h_T^{\frac{(4-d)}{2}} |v|_{2,T}; \quad (8)$$

and (iii) the following estimate on a dual norm of the boundary flux, whose proof is postponed until Appendix A.1, that is valid for any cell  $T$  star-shaped with respect to a ball whose radius is comparable to  $h_T$ , and any function  $v \in H^1(T)$  such that  $\Delta v \in L^2(T)$ :

$$\sup_{z \in H^1(\partial T)} \frac{\langle \nabla v|_{\partial T} \cdot \mathbf{n}_T, z \rangle_{-\frac{1}{2}, \partial T}}{h_T^{-1/2} \|z\|_{0, \partial T} + h_T^{1/2} |z|_{1, \partial T}} \lesssim |v|_{1,T} + h_T \|\Delta v\|_{0,T}. \quad (9)$$

### 2.3.2 Finite element in the sense of Ciarlet

The following definition is directly inspired from [17, p. 94]. Let  $l \in \{1, \dots, d\}$ .

**Definition 2.2.** *A finite element consists in a triple  $(X, V(X), \Sigma_X)$  where*

- $X$  is a bounded and connected Lipschitz subset of  $\mathbb{R}^d$  such that  $|X|_l \neq 0$ ;
- $V(X)$  is a finite-dimensional vector space of functions  $v : X \rightarrow \mathbb{R}$ ;
- $\Sigma_X := \{\sigma_X^1, \dots, \sigma_X^{n_X}\}$ ,  $n_X \in \mathbb{N}^*$ , is a collection of linear forms on  $V(X)$  such that the mapping

$$\underline{\Sigma}_X : V(X) \ni v \mapsto (\sigma_X^1(v), \dots, \sigma_X^{n_X}(v)) \in \mathbb{R}^{n_X}$$

is bijective (we then have  $\dim(V(X)) = n_X$ ).

The operator  $\underline{\Sigma}_X$  is the so-called (local) reduction operator, and  $\underline{\Sigma}_X(v)$  is the so-called vector of (local) degrees of freedom (DoFs). The bijectivity of  $\underline{\Sigma}_X$  is in general referred to as unisolvence in the literature. The following proposition is a direct consequence of the unisolvence property.

**Proposition 2.3.** *Let  $(X, V(X), \Sigma_X)$  be a finite element. There exists a basis  $\{\varphi_{X,1}, \dots, \varphi_{X,n_X}\}$  (referred to as canonical) of  $V(X)$  such that  $\sigma_X^i(\varphi_{X,j}) = \delta_{ij}$  for all  $i, j \in \{1, \dots, n_X\}$ . The  $(\varphi_{X,i})_{i \in \{1, \dots, n_X\}}$  are the so-called (local) shape functions.*

Let  $r_X : \mathbb{R}^{n_X} \rightarrow V(X)$  be the operator such that, for any  $\underline{v}_X := (v_X^i)_{i \in \{1, \dots, n_X\}} \in \mathbb{R}^{n_X}$ ,  $r_X \underline{v}_X := \sum_{i=1}^{n_X} v_X^i \varphi_{X,i}$ . One can easily remark that, for all  $\underline{v}_X \in \mathbb{R}^{n_X}$ ,  $\underline{\Sigma}_X(r_X \underline{v}_X) = \underline{v}_X$ , hence  $r_X = \underline{\Sigma}_X^{-1}$  and, for any  $v \in V(X)$ ,

$$v = r_X(\underline{\Sigma}_X(v)) = \sum_{i=1}^{n_X} \sigma_X^i(v) \varphi_{X,i}.$$

$r_X$  is the so-called (local, canonical) reconstruction operator.

Assume that there exists a normed vector space  $W(X)$  of functions  $w : X \rightarrow \mathbb{R}$  such that (i)  $V(X) \subset W(X)$ , and such that (ii) every linear form  $\sigma_X^i$  of  $\Sigma_X$  can be extended as a linear form  $\bar{\sigma}_X^i$  on  $W(X)$ . We denote by  $\bar{\Sigma}_X$  this new collection of extended linear forms, and by  $\bar{\underline{\Sigma}}_X$  the corresponding (local) reduction operator. We can then introduce the operator  $\mathcal{I}_X : W(X) \rightarrow V(X)$  such that, for any  $w \in W(X)$ ,  $\mathcal{I}_X w := r_X(\bar{\underline{\Sigma}}_X(w)) = \sum_{i=1}^{n_X} \bar{\sigma}_X^i(w) \varphi_{X,i}$ . With such a definition, there holds  $\underline{\Sigma}_X(\mathcal{I}_X w) = \bar{\underline{\Sigma}}_X(w)$ . The operator  $\mathcal{I}_X$  is the so-called (local, canonical) interpolation operator (in a broad sense). Of course, for all  $w \in V(X)$ ,  $\mathcal{I}_X w = w$  since  $\bar{\underline{\Sigma}}_X|_{V(X)} = \underline{\Sigma}_X = r_X^{-1}$ .

### 2.3.3 Standard polynomial projectors

For  $T \in \mathcal{T}_h$ , we define the  $L^2$ -orthogonal  $\pi_T^q : L^2(T) \rightarrow \mathbb{P}_d^q(T)$  and elliptic  $\Pi_T^q : H^1(T) \rightarrow \mathbb{P}_d^q(T)$  projectors so that

$$\begin{aligned} & \text{for all } v \in L^2(T), \quad \int_T \pi_T^q v w = \int_T v w \quad \forall w \in \mathbb{P}_d^q(T); \\ & \text{for all } v \in H^1(T), \quad \begin{cases} \int_T \nabla \Pi_T^q v \cdot \nabla w = \int_T \nabla v \cdot \nabla w & \forall w \in \mathbb{P}_d^q(T), \\ \int_T \Pi_T^q v = \int_T v. \end{cases} \end{aligned} \quad (10)$$

Remark that  $\Pi_T^0 = \pi_T^0|_{H^1(T)}$ . For  $l \in \{1, \dots, N_d^q\}$ , we let  $\pi_T^{q,l}$  denote the linear form on  $L^2(T)$  such that, for any  $v \in L^2(T)$ ,  $\pi_T^{q,l}(v) \in \mathbb{R}$  is the  $l^{\text{th}}$  coordinate of  $\pi_T^q v$  on the basis  $\{\psi_{T,i}^q\}_{i \in \{1, \dots, N_d^q\}}$  of  $\mathbb{P}_d^q(T)$ .

**Proposition 2.4** (Properties of  $\pi_T^q$  and  $\Pi_T^q$ ). *Let  $T \in \mathcal{T}_h$  and  $v \in L^2(T)$ .*

- preservation of polynomials: if  $v \in \mathbb{P}_d^q(T)$ ,  $\pi_T^q v = \Pi_T^q v = v$ ;
- stability:  $\|\pi_T^q v\|_{0,T} \leq \|v\|_{0,T}$  and, if  $v \in H^1(T)$ ,  $\|\Pi_T^q v\|_{0,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T}$ ;
- optimality:  $\|v - \pi_T^q v\|_{0,T} = \min_{z \in \mathbb{P}_d^q(T)} \|v - z\|_{0,T}$  and, if  $v \in H^1(T)$ ,  $|v - \Pi_T^q v|_{1,T} = \min_{z \in \mathbb{P}_d^q(T)} |v - z|_{1,T}$ ;
- $H^s$ -approximation: for  $v \in H^s(T)$ ,  $s \in \{1, \dots, q+1\}$ , there holds, with  $\varpi_T^q \in \{\pi_T^q; \Pi_T^q\}$ ,

$$|v - \varpi_T^q v|_{m,T} \lesssim h_T^{s-m} |v|_{s,T} \quad \text{for } m \in \{0, \dots, s\}, \quad (11)$$

and, for any  $F \in \mathcal{F}_T$ , and any  $(\zeta_1, \dots, \zeta_d) \in \mathbb{N}^d$  such that  $\sum_{i=1}^d \zeta_i = m$ ,

$$|[\partial_{x_1}^{\zeta_1} \dots \partial_{x_d}^{\zeta_d} (v - \varpi_T^q v)]|_{F}|_{0,F} \lesssim h_T^{s-m-1/2} |v|_{s,T} \quad \text{for } m \in \{0, \dots, s-1\}. \quad (12)$$

In Proposition 2.4, the only nontrivial result to prove is the last one. Its proof relies on (i) the ideas of [11, Chapter 4] and [33, Section 7] (cf. also [10, Section 4]), and on (ii) two important features of  $\varpi_T^q$ , that are the preservation of polynomials and its stability in the  $L^2(T)$ -norm. For a detailed proof of the result, and its extension to more general Sobolev seminorms, we refer to [24] ( $L^2$ -orthogonal projector) and [25] (elliptic projector). We also introduce  $\pi_h^q : L^2(\Omega) \rightarrow \mathbb{P}_d^q(\mathcal{T}_h)$  and  $\Pi_h^q : H^1(\mathcal{T}_h) \rightarrow \mathbb{P}_d^q(\mathcal{T}_h)$  such that, for any  $v \in L^2(\Omega)$  (resp.,  $v \in H^1(\mathcal{T}_h)$ ),  $\pi_h^q v|_T := \pi_T^q(v|_T)$  (resp.,  $\Pi_h^q v|_T := \Pi_T^q(v|_T)$ ) for all  $T \in \mathcal{T}_h$ .

For  $F \in \mathcal{F}_h$ , we define  $\pi_F^q : L^2(F) \rightarrow \mathbb{P}_{d-1}^q(F)$  so that, for all  $v \in L^2(F)$ ,

$$\int_F \pi_F^q v w = \int_F v w \quad \forall w \in \mathbb{P}_{d-1}^q(F).$$

For  $l \in \{1, \dots, N_{d-1}^q\}$ , we let  $\pi_F^{q,l}$  denote the linear form on  $L^2(F)$  such that, for any  $v \in L^2(F)$ ,  $\pi_F^{q,l}(v) \in \mathbb{R}$  is the  $l^{\text{th}}$  coordinate of  $\pi_F^q v$  on the basis  $\{\psi_{F,j}^q\}_{j \in \{1, \dots, N_{d-1}^q\}}$  of  $\mathbb{P}_{d-1}^q(F)$ . We also define, for  $T \in \mathcal{T}_h$ ,  $\pi_{\partial T}^q : L^2(\partial T) \rightarrow \mathbb{P}_{d-1}^q(\mathcal{F}_T)$  such that, for any  $v \in L^2(\partial T)$ ,  $\pi_{\partial T}^q v|_F := \pi_F^q(v|_F)$  for all  $F \in \mathcal{F}_T$ . When  $d = 3$ , for  $e \in \mathcal{E}_h$ , we define  $\pi_e^q : L^2(e) \rightarrow \mathbb{P}_1^q(e)$  so that, for all  $v \in L^2(e)$ ,

$$\int_e \pi_e^q v w = \int_e v w \quad \forall w \in \mathbb{P}_1^q(e).$$

For  $l \in \{1, \dots, N_1^q\}$ , we let  $\pi_e^{q,l}$  denote the linear form on  $L^2(e)$  such that, for any  $v \in L^2(e)$ ,  $\pi_e^{q,l}(v) \in \mathbb{R}$  is the  $l^{\text{th}}$  coordinate of  $\pi_e^q v$  on the basis  $\{\psi_{e,m}^q\}_{m \in \{1, \dots, N_1^q\}}$  of  $\mathbb{P}_1^q(e)$ .

**Remark 2.5** (Computation of  $\pi_X^q v$ ). *It can be easily seen that, for  $X \in \{T, F\}$  and  $v \in L^2(X)$ ,  $\pi_X^0 v = \frac{1}{|X|} \int_X v$  with  $l \in \{d, d-1\}$  respectively. To compute  $\pi_X^q v$  for  $q \geq 1$ , it suffices to solve a symmetric positive-definite (SPD) system of size  $N_1^q$ .*

**Remark 2.6** (Computation of  $\Pi_T^q v$ ,  $q \geq 1$ ). *One possibility (another one is to solve a problem posed on the quotient space  $\mathbb{P}_d^q(T)/\mathbb{P}_d^0(T)$ ) to compute  $\Pi_T^q v$  is to consider the following coercive problem: find  $z \in \mathbb{P}_d^q(T)$  such that, for all  $w \in \mathbb{P}_d^q(T)$ ,*

$$\int_T \nabla z \cdot \nabla w + \int_T \pi_T^0 z \pi_T^0 w = - \int_T \pi_T^{q-2} v \Delta w + \int_{\partial T} \pi_{\partial T}^{q-1}(v|_{\partial T}) \nabla w|_{\partial T} \cdot \mathbf{n}_T.$$



The elliptic projection is obtained through  $\Pi_T^q v = z + \pi_T^0 v$ . According to the expression above, to compute  $\Pi_T^q v$ , it suffices to know selected moments of  $v \in H^1(T)$ , namely  $\pi_T^\mu v$  for  $\mu = \max(q-2, 0)$  and  $\pi_{\partial T}^{q-1}(v|_{\partial T})$ . The computation of  $\Pi_T^q v$  then requires to solve a SPD system of size  $N_d^q$ , which can be done effectively via, e.g., a Cholesky factorization.

### 3 Skeletal methods

#### 3.1 Generic structure

Galerkin methods on  $\mathcal{T}_h$ , among which c/nc-FE, c/nc-VE, or dG methods, seek for an approximation  $u_h$  of the solution  $u \in H_0^1(\Omega)$  to Problem (1) in a broken space  $V_{h,0} \subseteq V(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$ , with  $V(\mathcal{T}_h)$  such that, for all  $T \in \mathcal{T}_h$ ,  $V(T)$  is a finite-dimensional vector space of functions on which exists a collection  $\Sigma_T$  of linear forms so that  $(T, V(T), \Sigma_T)$  is a finite element in the sense of Ciarlet. The linear forms collected in  $\Sigma_T$  are the local DoFs of the method. For dG methods,  $V_{h,0} = V(\mathcal{T}_h)$ , whereas for c/nc-FE/VE,  $V_{h,0}$  is a strict subspace of  $V(\mathcal{T}_h)$  that embeds more or less stringent conformity prescriptions (with respect to  $H_0^1(\Omega)$ ); for c-FE/VE for instance,  $V_{h,0}$  is a subspace of  $H_0^1(\Omega)$ .

For those methods (like c/nc-FE/VE) based on a space  $V_{h,0}$  embedding conformity prescriptions, locally to any  $T \in \mathcal{T}_h$ , the collection  $\Sigma_T$  of DoFs can be split into (i) linear forms on  $V(\partial T) := \{v|_{\partial T}, v \in V(T)\}$  (collected in the set  $\Sigma_T^\partial$ ) and, (ii) if need be, those linear forms on  $V(T)$  that cannot be written as linear forms on  $V(\partial T)$  (collected in the set  $\Sigma_T^\circ$ ). The linear forms in  $\Sigma_T^\partial$  are skeletal DoFs, whereas those in  $\Sigma_T^\circ$  are cell DoFs. We let  $n_T^\partial := \text{card}(\Sigma_T^\partial)$ ,  $n_T^\circ := \text{card}(\Sigma_T^\circ)$ , and  $n_T := n_T^\partial + n_T^\circ = \dim(V(T))$ . From a global viewpoint, the collection of global DoFs of the method splits into (i) linear forms on

$$V(\mathcal{F}_h) := \left\{ v_h \in L^2(\partial\mathcal{T}_h) \mid v_h|_F \in \bigoplus_{T \in \mathcal{T}_F} V_T(F) \forall F \in \mathcal{F}_h \right\}$$

with  $V_T(F) := \{v|_F, v \in V(T)\}$  (collected in the set  $\Sigma_h^\partial$ ) and, (ii) if need be, linear forms on  $V(\mathcal{T}_h)$  that cannot be written as linear forms on  $V(\mathcal{F}_h)$  (collected in the set  $\Sigma_h^\circ$ ). Whereas  $\Sigma_h^\circ := \bigcup_{T \in \mathcal{T}_h} \check{\Sigma}_T^\circ$  where, for any  $T \in \mathcal{T}_h$ ,  $\check{\Sigma}_T^\circ$  is the collection of linear forms on  $V(\mathcal{T}_h)$  given by  $\check{\Sigma}_T^\circ := \{\sigma_T^\circ(\cdot|_T), \sigma_T^\circ \in \Sigma_T^\circ\}$ , the global skeletal DoFs in  $\Sigma_h^\partial$  (that are linear forms on  $V(\mathcal{F}_h)$ ) are intrinsically defined, and their local counterparts are obtained by localization. More precisely, for any  $T \in \mathcal{T}_h$ , letting

$$V_T(\mathcal{F}_h) := \left\{ v_h \in V(\mathcal{F}_h) \mid v_h|_{\partial T} \in V(\partial T), v_h|_F \equiv 0 \forall F \in \mathcal{F}_h \setminus \mathcal{F}_T \right\} \text{ and } \hat{\Sigma}_T^\partial := \left\{ \sigma_{V_T(\mathcal{F}_h)}^\partial \neq 0, \sigma^\partial \in \Sigma_h^\partial \right\},$$

one has

$$\Sigma_T^\partial := \left\{ c(\hat{\sigma}_T^\partial \circ \mathcal{Z}_h), \hat{\sigma}_T^\partial \in \hat{\Sigma}_T^\partial \right\}, \quad (13)$$

where  $\mathcal{Z}_h$  is the zero extension operator from  $V(\partial T)$  to  $V_T(\mathcal{F}_h)$ , and  $c$  is a constant depending on the skeletal DoF under consideration and on  $T$ , whose default value is 1 (cf. Remark 3.1). Note that whereas  $\text{card}(\Sigma_h^\circ) = \sum_{T \in \mathcal{T}_h} n_T^\circ$ ,  $\text{card}(\Sigma_h^\partial) < \sum_{T \in \mathcal{T}_h} n_T^\partial$ .

**Remark 3.1** (Role of the constant  $c$  in (13)). *Functions in  $V(\mathcal{F}_h)$  may be multi-valued (whenever this has a sense) at vertices when  $d = 2$ , and on edges/vertices when  $d = 3$ . Yet, in the conforming case, one has to define DoFs at vertices when  $d = 2$ , and on edges/vertices when  $d = 3$ , to prescribe conformity. Let us then describe how multi-valuedness can be dealt with. Assume, for simplicity, that  $V(\partial T) \subset C^0(\partial T)$  for all  $T \in \mathcal{T}_h$ . For a vertex  $\nu \in \mathcal{V}_h$ , the (global) skeletal DoF  $\sigma^{\partial, \nu} \in \Sigma_h^\partial$  associated to a pointwise evaluation at  $\mathbf{x}_\nu$  is defined, for any  $v_h \in V(\mathcal{F}_h)$ , by  $\sigma^{\partial, \nu}(v_h) := \frac{1}{\text{card}(\mathcal{F}_\nu)} \sum_{F \in \mathcal{F}_\nu} v_h|_F(\mathbf{x}_\nu)$ . With such a definition, it degenerates towards  $v_h(\mathbf{x}_\nu)$  whenever  $v_h$  is single-valued at  $\mathbf{x}_\nu$ . Besides, letting  $c_T^\nu = \frac{\text{card}(\mathcal{F}_\nu)}{\text{card}(\mathcal{F}_\nu \cap \mathcal{F}_T)}$ , one obtains as expected that the restriction of  $\sigma^{\partial, \nu}$  to a cell  $T \in \mathcal{T}_h$  such that  $\nu \in \mathcal{V}_T$  is given, for any  $v \in V(\partial T)$ , by  $\sigma_T^{\partial, \nu}(v) = v(\mathbf{x}_\nu)$ . Edge DoFs when  $d = 3$  can be handled in a similar way. For face DoFs, one can always take  $c = 1$ .*

For any  $F \in \mathcal{F}_h$  and  $v_h \in V(\mathcal{F}_h)$ , we let  $\mathcal{R}_F(v_h) \in V(\mathcal{F}_h)$  be such that  $\mathcal{R}_F(v_h)|_F = v_h|_F$  and  $\mathcal{R}_F(v_h)|_{F'} \equiv 0$  for all  $F' \in \mathcal{F}_h \setminus F$ . The approximation space, that also takes into account the boundary conditions, writes

$$V_{h,0} := \left\{ v_h \in V(\mathcal{T}_h) \mid (\sigma^\partial \circ \mathcal{R}_F)(\llbracket v_h \rrbracket) = 0 \forall F \in \mathcal{F}_h, \forall \sigma^\partial \in \Sigma_h^\partial \right\}, \quad (14)$$

with dimension  $n_{h,0} := \dim(V_{h,0}) < \dim(V(\mathcal{T}_h)) := n_h$ . The conditions on the jumps of discrete functions enforce the conformity prescriptions. The discrete problem then reads as follows: find  $u_h \in V_{h,0}$  such that

$$a_h(u_h, v_h) := \sum_{T \in \mathcal{T}_h} a_T(u_h|_T, v_h|_T) = \sum_{T \in \mathcal{T}_h} l_T(v_h|_T) =: l_h(v_h) \quad \text{for all } v_h \in V_{h,0}, \quad (15)$$

where the (bi)linear forms  $a_h : V(\mathcal{T}_h) \times V(\mathcal{T}_h) \rightarrow \mathbb{R}$  and  $l_h : V(\mathcal{T}_h) \rightarrow \mathbb{R}$  write as the sums of local contributions expressed by the local forms  $a_T : V(T) \times V(T) \rightarrow \mathbb{R}$  and  $l_T : V(T) \rightarrow \mathbb{R}$ . This special structure of the discrete problem ensures that the potential cell DoFs of the method are not coupled between adjacent cells and can be eliminated locally in each cell  $T \in \mathcal{T}_h$  in terms of the local skeletal DoFs. Algebraically, the elimination consists in computing the Schur complement of the cell-cell block of the global system matrix, which is quite inexpensive as this block is itself block-diagonal. After elimination, the global system to solve is expressed in terms of the global skeletal DoFs only. This explains why methods that are based on a discrete space like (14) and on a variational formulation like (15) are referred to as skeletal methods.

Without describing too much the space  $V_{h,0}$ , one can prove interesting structural properties on it. Let us denote, for any  $F \in \mathcal{F}_h$ , by  $\tilde{\pi}_F^0$  the linear form on  $V(\mathcal{F}_h)$  so that  $\tilde{\pi}_F^0 := \pi_F^0(\cdot|_F)$ , with the convention (henceforth adopted) that  $\mathbb{P}_l^0$  is identified to  $\mathbb{R}$ .

**Lemma 3.2.** *If  $\{\tilde{\pi}_F^0\}_{F \in \mathcal{F}_h} \subseteq \Sigma_h^\partial$ , then  $\|\nabla_h \cdot\|_{0,\Omega}$  defines a norm on  $V_{h,0}$ , and the following discrete Poincaré inequality holds:*

$$\forall v_h \in V_{h,0}, \quad \|v_h\|_{0,\Omega} \lesssim \|\nabla_h v_h\|_{0,\Omega}. \quad (16)$$

Lemma 3.2, whose proof (which is quite classical but recalled for completeness) is postponed until Appendix A.2, gives sufficient conformity conditions for  $V_{h,0}$  (that are reminiscent of lowest-order nc-FE) so as to ensure that a discrete Poincaré inequality holds on it.

**Remark 3.3.** *Obviously, the result of Lemma 3.2 remains valid under more stringent conformity prescriptions (like it is for instance the case for c-FE).*

### 3.2 Equivalent algebraic viewpoint

Since, for all  $T \in \mathcal{T}_h$ ,  $(T, V(T), \Sigma_T)$  is a finite element in the sense of Ciarlet, any function  $v_h \in V(\mathcal{T}_h)$  can be equivalently written as a vector  $\mathbb{R}^{n_h} \ni \underline{v}_h := (\underline{v}_T)_{T \in \mathcal{T}_h}$  with  $\mathbb{R}^{n_T} \ni \underline{v}_T := (\underline{v}_T^\partial \in \mathbb{R}^{n_T^\partial}, \underline{v}_T^o \in \mathbb{R}^{n_T^o})$ . For  $T \in \mathcal{T}_h$  and  $F \in \mathcal{F}_T$ , we let  $\underline{v}_{T,F}^\partial \in \mathbb{R}^{n_{T,F}^\partial}$  be the restriction of  $\underline{v}_T^\partial$  to the face  $F$  (with possible overlaps between faces, i.e.  $\sum_{F \in \mathcal{F}_T} n_{T,F}^\partial \geq n_T^\partial$ ). Since the skeletal DoFs are intrinsically defined at the global level, we necessarily have that, for all  $F \in \mathcal{F}_h^i$  such that  $\mathcal{T}_F = \{T_1, T_2\}$ ,  $n_{T_1,F}^\partial = n_{T_2,F}^\partial$ . We hence can let  $[\underline{v}_h]_F := \underline{v}_{T_1,F}^\partial - \underline{v}_{T_2,F}^\partial$ ; for  $F \in \mathcal{F}_h^b$  such that  $\mathcal{T}_F = \{T\}$ , we let  $[\underline{v}_h]_F := \underline{v}_{T,F}^\partial$ . We introduce

$$\mathbb{V}_{h,0} := \{\underline{v}_h \in \mathbb{R}^{n_h} \mid [\underline{v}_h]_F = \underline{0} \ \forall F \in \mathcal{F}_h\}. \quad (17)$$

There holds  $\dim(\mathbb{V}_{h,0}) = n_{h,0}$ . For vectors  $\underline{v}_h \in \mathbb{V}_{h,0}$ , for every  $F \in \mathcal{F}_h^i$  we have  $\underline{v}_{T_1,F}^\partial = \underline{v}_{T_2,F}^\partial$ , and for every  $F \in \mathcal{F}_h^b$  we have  $\underline{v}_{T,F}^\partial = \underline{0}$ . Problem (15) can be equivalently rewritten: find  $\underline{u}_h \in \mathbb{V}_{h,0}$  such that

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_h} l_T(\underline{v}_T) =: l_h(\underline{v}_h) \quad \text{for all } \underline{v}_h \in \mathbb{V}_{h,0}, \quad (18)$$

where the (bi)linear forms  $a_h : \mathbb{R}^{n_h} \times \mathbb{R}^{n_h} \rightarrow \mathbb{R}$  and  $l_h : \mathbb{R}^{n_h} \rightarrow \mathbb{R}$  are expressed in terms of the local forms  $a_T : \mathbb{R}^{n_T} \times \mathbb{R}^{n_T} \rightarrow \mathbb{R}$  and  $l_T : \mathbb{R}^{n_T} \rightarrow \mathbb{R}$  such that  $a_T := a_T(r_T \cdot, r_T \cdot)$  and  $l_T := l_T(r_T \cdot)$ , where  $r_T$  is the (local, canonical) reconstruction operator. For all  $T \in \mathcal{T}_h$ , there holds  $\underline{u}_T = \underline{\Sigma}_T(u_h|_T)$ , where  $\underline{u}_T \in \mathbb{R}^{n_T}$  is the restriction to  $T$  of  $\underline{u}_h \in \mathbb{V}_{h,0}$  solution to Problem (18), and  $u_h \in V_{h,0}$  is the solution to Problem (15). In the rest of this section, we will stick to the functional notation used in Problem (15), but in Section 4 we will adopt the algebraic notation used for Problem (18).

### 3.3 Examples of skeletal methods

The most famous examples of skeletal methods are surely given by the c/nc-FE methods. In that case, one considers, for all  $T \in \mathcal{T}_h$ ,

$$a_T(w, v) := \int_T \nabla w \cdot \nabla v \quad \text{and} \quad l_T(v) := \int_T f|_T v \quad \text{for all } w, v \in V(T),$$

in such a way that the discrete problem writes: find  $u_h \in V_{h,0}$  such that

$$a_h(u_h, v_h) = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h = l_h(v_h) \quad \text{for all } v_h \in V_{h,0}. \quad (19)$$

For FE methods, it is assumed that closed-form expressions are available for the local shape functions. In that case, the assumptions of Lemma 3.2 are sufficient to ensure well-posedness of the discrete problem in the

sense of Hadamard (from (16) directly results the stability estimate  $\|\nabla_h u_h\|_{0,\Omega} \lesssim \|f\|_{0,\Omega}$  on the solution  $u_h$  to Problem (19)). As soon as the space  $V_{h,0}$  guarantees as well some approximability in a dense subspace of  $H_0^1(\Omega)$ , one can classically also infer strong convergence in  $H_0^1(\Omega)$  of  $u_h$  to the solution  $u$  to Problem (1) [32]. The result of Lemma 3.2, that does not assume that discrete functions in  $V_{h,0}$  are piecewise polynomial, also applies to the mixed-order multiscale HHO method of [19, Section 5.1], for which the local (oscillatory) basis functions are considered to be all explicitly known on each coarse cell (in practice, they are in fact approximated using a fine submesh of the coarse cell).

Other, more recent examples of skeletal methods are the c/nc-VE methods, as well as the HHO method. The specificity of VE methods with respect to FE methods is to consider virtual local spaces  $V(T)$ , i.e. local spaces which are spanned by functions that are (i) in general, not all computable, and (ii) by definition, never all computed. The local virtual functions are usually implicitly defined as the solutions to some PDEs posed in the cell. The VE methods are defined using computable (in terms of the DoFs) projections of the virtual functions, and are stabilized through computable penalizations. Polynomial-based VE methods hinge on local virtual spaces (i) spanned by the solutions to PDEs featuring polynomial data, and (ii) that contain the space  $\mathbb{P}_d^k$  for some  $k \geq 1$ . They are the VE methods that are classically encountered in the literature. As such, we will henceforth refer to them simply as VE methods, and we will exclusively focus on them in the sequel. An example of (nonconforming) method that hinges on a different kind of virtual space is given by the equal-order multiscale HHO method of [19, Section 5.2], for which local virtual functions do solve (oscillatory) PDEs with polynomial data, but the local virtual space does not contain polynomials in general.

## 4 Bridging the Hybrid High-Order and Virtual Element methods

In this section, we reformulate and analyse, within a unified framework inspired from HHO methods, the conforming and nonconforming versions of the (polynomial-based) VE method. By recasting the c-VE method into a conforming-HHO method, we end up bridging, by the same, the two families of methods. We let  $k \geq 1$  be a given integer, that will stand for the order of the method. Doing so, we adopt the classical VE notation. HHO methods of the same order are classically defined using  $k' = k - 1 \geq 0$ .

### 4.1 Local virtual space

Let  $T \in \mathcal{T}_h$  be a given cell.

#### 4.1.1 Conforming case

In that case, we further assume that  $T$  is star-shaped with respect to a ball whose radius is comparable to  $h_T$ . For any  $\nu \in \mathcal{V}_T$  (respectively  $\nu \in \mathcal{V}_F$ , for  $F \in \mathcal{F}_T$ ), we let  $\iota_T^\nu$  (respectively  $\iota_F^\nu$ ) denote the linear form on  $C^0(\partial T)$  (respectively  $C^0(\partial F)$ ) so that, for all  $v \in C^0(\partial T)$  (respectively  $v \in C^0(\partial F)$ ),  $\iota_T^\nu(v) = v(\mathbf{x}_\nu)$  (respectively  $\iota_F^\nu(v) = v(\mathbf{x}_\nu)$ ).

**Preliminary results.** Let  $d = 2$ , and consider a face  $F \in \mathcal{F}_T$  such that  $F := [\mathbf{x}_{\nu_1}, \mathbf{x}_{\nu_2}]$ . Then, we have the following classical result.

**Proposition 4.1.** *The triple  $(F, \mathbb{P}_1^k(F), \Sigma_{1,F}^k)$ , where the collection  $\Sigma_{1,F}^k$  splits into  $\Sigma_{1,F}^{k,\partial} := \{\iota_F^\nu\}_{\nu \in \mathcal{V}_F}$  and  $\Sigma_{1,F}^{k,o} := \{\pi_F^{k-2,j}\}_{j \in \{1, \dots, N_1^{k-2}\}}$ , is a finite element in the sense of Ciarlet.*

To alleviate the notation, we have kept in the collection  $\Sigma_{1,F}^{k,o}$  the symbol  $\pi_F^{k-2,j}$  to actually denote  $\pi_{F|\mathbb{P}_1^k(F)}^{k-2,j}$ . This slight abuse in notation will henceforth be adopted. Letting  $n_{1,F}^k := \dim(\mathbb{P}_1^k(F))$ , we introduce the operator  $r_{1,F}^k : \mathbb{R}^{n_{1,F}^k} \rightarrow \mathbb{P}_1^k(F)$  such that, for any

$$\underline{v}_F := ((v_F^{\partial,\nu})_{\nu \in \mathcal{V}_F}, (v_F^{o,j})_{j \in \{1, \dots, N_1^{k-2}\}}) \in \mathbb{R}^{n_{1,F}^k},$$

$r_{1,F}^k \underline{v}_F \in \mathbb{P}_1^k(F)$  solves the well-posed problem

$$\begin{cases} \int_F (r_{1,F}^k \underline{v}_F)' w' = - \int_F v_F^o w'' + [v_F^{\partial,\nu_2} w'(\mathbf{x}_{\nu_2}) - v_F^{\partial,\nu_1} w'(\mathbf{x}_{\nu_1})] & \forall w \in \mathbb{P}_1^k(F), \\ r_{1,F}^k \underline{v}_F(\mathbf{x}_{\nu_1}) = v_F^{\partial,\nu_1}, \end{cases} \quad (20)$$

where we have introduced the notation  $v_F^o := \sum_{j=1}^{N_1^{k-2}} v_F^{o,j} \psi_{F,j}^{k-2} \in \mathbb{P}_1^{k-2}(F)$ .

**Proposition 4.2.** *The operator  $r_{1,F}^k$  defined by (20) coincides with the (canonical) reconstruction operator  $(\underline{\Sigma}_{1,F}^k)^{-1}$ .*

*Proof.* Let  $\mathbb{R}^{n_{1,F}^k} \ni \underline{v}_F := \underline{\Sigma}_{1,F}^k(v)$  for some  $v \in \mathbb{P}_1^k(F)$ . Plugging  $\underline{v}_F$  into (20), we obtain by integration by parts

$$\int_F (r_{1,F}^k \underline{v}_F)' w' = \int_F v' w' \quad \forall w \in \mathbb{P}_1^k(F),$$

which, combined to the condition  $r_{1,F}^k \underline{v}_F(\mathbf{x}_{\nu_1}) = v_F^{\partial, \nu_1} = v(\mathbf{x}_{\nu_1})$ , yields that  $r_{1,F}^k \underline{v}_F = r_{1,F}^k(\underline{\Sigma}_{1,F}^k(v)) = v$ . This is true for any  $v \in \mathbb{P}_1^k(F)$ , hence  $r_{1,F}^k = (\underline{\Sigma}_{1,F}^k)^{-1}$ .  $\square$

The collection  $\Sigma_{1,F}^k$  of linear forms on  $\mathbb{P}_1^k(F)$  can patently be extended to a collection  $\overline{\Sigma}_{1,F}^k$  of linear forms on  $C^0(F)$ . We can hence define the interpolation operator  $\mathcal{I}_{1,F}^k : C^0(F) \rightarrow \mathbb{P}_1^k(F)$  such that  $\mathcal{I}_{1,F}^k := r_{1,F}^k \circ \overline{\Sigma}_{1,F}^k$ .

**Lemma 4.3** (Stability of  $\mathcal{I}_{1,F}^k$ ). *For all  $v \in C^0(F)$ , there holds  $\|\mathcal{I}_{1,F}^k v\|_{\infty, F} \lesssim \|v\|_{\infty, F}$ .*

*Proof.* To prove the result, we rewrite  $\mathcal{I}_{1,F}^k v$  as

$$\mathcal{I}_{1,F}^k v = v(\mathbf{x}_{\nu_1}) \varphi_{F, \nu_1}^{\partial} + v(\mathbf{x}_{\nu_2}) \varphi_{F, \nu_2}^{\partial} + \sum_{j=1}^{N_1^{k-2}} \pi_F^{k-2, j}(v) \varphi_{F, j}^{\circ},$$

where we recall that the shape functions  $\varphi_{F, \nu_1}^{\partial}, \varphi_{F, \nu_2}^{\partial}, \varphi_{F, j}^{\circ}$  belong to  $\mathbb{P}_1^k(F)$ . By stability of the  $L^2$ -orthogonal projector  $\pi_F^{k-2}$  in the  $L^\infty(F)$ -norm (cf., e.g., [24, Lemma 3.2 with  $N = 1$ ]), we infer the desired estimate.  $\square$

**The case  $d = 2$ .** If  $d = 2$ , the conforming local virtual space on  $T$  is defined as

$$V_2^k(T) := \left\{ v \in H^1(T) \mid \Delta v \in \mathbb{P}_2^{k-1}(T), v|_{\partial T} \in \mathbb{P}_1^{k,c}(\mathcal{F}_T) \right\}, \quad (21)$$

where  $\mathbb{P}_1^{k,c}(\mathcal{F}_T) := \mathbb{P}_1^k(\mathcal{F}_T) \cap C^0(\partial T)$ . One can show (cf., e.g., [10, Remark 2.3]) that functions in  $V_2^k(T)$  belong to  $C^0(\overline{T})$ . Besides,  $\mathbb{P}_2^k(T) \subset V_2^k(T)$ . The following result is standard.

**Proposition 4.4.** *The triple  $(T, V_2^k(T), \Sigma_{2,T}^k)$ , where  $V_2^k(T)$  is given by (21) and the collection  $\Sigma_{2,T}^k$  splits into*

$$\Sigma_{2,T}^{k, \partial} := \{ \nu_T^{\nu} \}_{\nu \in \mathcal{V}_T} \cup \{ \pi_F^{k-2, j}(\cdot|_F) \}_{j \in \{1, \dots, N_1^{k-2}\}}^{F \in \mathcal{F}_T} \quad \text{and} \quad \Sigma_{2,T}^{k, \circ} := \{ \pi_T^{k-1, i} \}_{i \in \{1, \dots, N_2^{k-1}\}},$$

*is a finite element in the sense of Ciarlet.*

Letting  $n_{2,T}^k := \dim(V_2^k(T))$ , for any

$$\underline{v}_T := \left( (v_T^{\partial, \nu})_{\nu \in \mathcal{V}_T}, (v_T^{\partial, j})_{j \in \{1, \dots, N_1^{k-2}\}}^{F \in \mathcal{F}_T}, (v_T^{\circ, i})_{i \in \{1, \dots, N_2^{k-1}\}} \right) \in \mathbb{R}^{n_{2,T}^k},$$

we have  $\underline{v}_T^{\partial} = \left( (v_T^{\partial, \nu})_{\nu \in \mathcal{V}_T}, (v_T^{\partial, j})_{j \in \{1, \dots, N_1^{k-2}\}}^{F \in \mathcal{F}_T} \right) \in \mathbb{R}^{n_{2,T}^{k, \partial}}$ ,  $\underline{v}_T^{\circ} = \left( (v_T^{\partial, \nu})_{\nu \in \mathcal{V}_T}, (v_T^{\partial, j})_{j \in \{1, \dots, N_1^{k-2}\}} \right) \in \mathbb{R}^{n_{1,F}^k}$  for any  $F \in \mathcal{F}_T$ , and we let  $v_T^{\circ} := \sum_{i=1}^{N_2^{k-1}} v_T^{\circ, i} \psi_{T, i}^{k-1} \in \mathbb{P}_2^{k-1}(T)$ . Defining  $r_{1, \partial T}^k : \mathbb{R}^{n_{2,T}^{k, \partial}} \rightarrow \mathbb{P}_1^{k,c}(\mathcal{F}_T)$  so that, for any  $\underline{v}_T^{\partial} \in \mathbb{R}^{n_{2,T}^{k, \partial}}$ ,  $r_{1, \partial T}^k \underline{v}_T^{\partial}|_F := r_{1,F}^k \underline{v}_T^{\partial}|_F$  for all  $F \in \mathcal{F}_T$ , we are now in position to introduce the operator  $r_{2,T}^k : \mathbb{R}^{n_{2,T}^k} \rightarrow V_2^k(T)$  such that, for any  $\underline{v}_T \in \mathbb{R}^{n_{2,T}^k}$ ,  $r_{2,T}^k \underline{v}_T \in V_2^k(T)$  solves the well-posed problem

$$\begin{cases} \int_T \nabla r_{2,T}^k \underline{v}_T \cdot \nabla w = - \int_T v_T^{\circ} \Delta w + \langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, r_{1, \partial T}^k \underline{v}_T^{\partial} \rangle_{-\frac{1}{2}, \partial T} & \forall w \in V_2^k(T), \\ \int_T r_{2,T}^k \underline{v}_T = \int_T v_T^{\circ}. \end{cases} \quad (22)$$

**Proposition 4.5.** *The operator  $r_{2,T}^k$  defined by (22) coincides with the (canonical) reconstruction operator  $(\underline{\Sigma}_{2,T}^k)^{-1}$ .*

*Proof.* Let  $\mathbb{R}^{n_{2,T}^k} \ni \underline{v}_T := \underline{\Sigma}_{2,T}^k(v)$  for some  $v \in V_2^k(T)$ . Plugging  $\underline{v}_T$  into (22), we infer by integration by parts

$$\int_T \nabla r_{2,T}^k \underline{v}_T \cdot \nabla w = - \int_T \pi_T^{k-1} v \Delta w + \left\langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, r_{1,\partial T}^k(\underline{\Sigma}_{2,T}^k(v)) \right\rangle_{-\frac{1}{2},\partial T} = \int_T \nabla v \cdot \nabla w \quad \forall w \in V_2^k(T),$$

where we have used that  $\Delta w \in \mathbb{P}_2^{k-1}(T)$  and that, for all  $F \in \mathcal{F}_T$ ,  $r_{1,\partial T}^k(\underline{\Sigma}_{2,T}^k(v))|_F = r_{1,F}^k(\underline{\Sigma}_{2,T,F}^k(v)) = r_{1,F}^k(\underline{\Sigma}_{1,F}^k(v|_F)) = v|_F$  (cf. Proposition 4.2). Since  $\int_T r_{2,T}^k \underline{v}_T = \int_T \pi_T^{k-1} v = \int_T v$ , we finally infer that  $r_{2,T}^k \underline{v}_T = r_{2,T}^k(\underline{\Sigma}_{2,T}^k(v)) = v$ . This is true for any  $v \in V_2^k(T)$ , hence  $r_{2,T}^k = (\underline{\Sigma}_{2,T}^k)^{-1}$ .  $\square$

Let us introduce the operator  $\mathcal{I}_{1,\partial T}^k : C^0(\partial T) \rightarrow \mathbb{P}_1^{k,c}(\mathcal{F}_T)$  so that, for any  $v \in C^0(\partial T)$ ,  $\mathcal{I}_{1,\partial T}^k v|_F := \mathcal{I}_{1,F}^k(v|_F)$  for all  $F \in \mathcal{F}_T$ . It is clear that the collection  $\Sigma_{2,T}^k$  of linear forms on  $V_2^k(T)$  can be extended to a collection  $\overline{\Sigma}_{2,T}^k$  of linear forms on

$$H^{1,c}(T) := \{v \in H^1(T) \mid v|_{\partial T} \in C^0(\partial T)\}.$$

We can hence define the interpolation operator  $\mathcal{I}_{2,T}^k : H^{1,c}(T) \rightarrow V_2^k(T)$  such that  $\mathcal{I}_{2,T}^k := r_{2,T}^k \circ \overline{\Sigma}_{2,T}^k$ . Besides, we remark that, for all  $v \in H^{1,c}(T)$ ,  $r_{1,\partial T}^k(\overline{\Sigma}_{2,T}^k(v)) = \mathcal{I}_{1,\partial T}^k(v|_{\partial T})$ . We have the following stability result for  $\mathcal{I}_{2,T}^k$ .

**Lemma 4.6** (Stability of  $\mathcal{I}_{2,T}^k$ ). *For all  $v \in H^2(T)$ , there holds*

- $|\mathcal{I}_{2,T}^k v|_{1,T} \lesssim |v|_{1,T} + h_T |v|_{2,T}$ ;
- $\|\mathcal{I}_{2,T}^k v\|_{0,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T} + h_T^2 |v|_{2,T}$ .

*Proof.* First, remark that if  $v \in H^2(T)$ ,  $v \in C^0(\overline{T})$  and hence  $v \in H^{1,c}(T)$ ; consequently,  $\mathcal{I}_{2,T}^k v$  has a sense. Let us test (22) with  $\underline{v}_T := \overline{\Sigma}_{2,T}^k(v)$ . There holds

$$\int_T \nabla \mathcal{I}_{2,T}^k v \cdot \nabla w = - \int_T v \Delta w + \left\langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, \mathcal{I}_{1,\partial T}^k(v|_{\partial T}) \right\rangle_{-\frac{1}{2},\partial T} \quad \forall w \in V_2^k(T),$$

where we have used that  $\Delta w \in \mathbb{P}_2^{k-1}(T)$  and that  $r_{1,\partial T}^k(\overline{\Sigma}_{2,T}^k(v)) = \mathcal{I}_{1,\partial T}^k(v|_{\partial T})$ . We equivalently rewrite the equality above as

$$\int_T \nabla \mathcal{I}_{2,T}^k v \cdot \nabla w = - \int_T (v - \pi_T^0 v) \Delta w + \left\langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, \mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v) \right\rangle_{-\frac{1}{2},\partial T} \quad \forall w \in V_2^k(T)$$

and, by Cauchy–Schwarz inequality we infer, for any  $w \in V_2^k(T)$ ,

$$\left| \int_T \nabla \mathcal{I}_{2,T}^k v \cdot \nabla w \right| \lesssim \|v - \pi_T^0 v\|_{0,T} \|\Delta w\|_{0,T} + \left( h_T^{-\frac{1}{2}} \|\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{0,\partial T} + h_T^{\frac{1}{2}} |\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)|_{1,\partial T} \right) \sup_{z \in H^1(\partial T)} \frac{\langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, z \rangle_{-\frac{1}{2},\partial T}}{h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}} =: \mathfrak{I}_1 + \mathfrak{I}_2.$$

To estimate  $\mathfrak{I}_1$ , we apply (7) to  $w \in V_2^k(T)$ , and the Poincaré inequality (6) to  $(v - \pi_T^0 v)$ . We get

$$\mathfrak{I}_1 \lesssim |v|_{1,T} |w|_{1,T}.$$

To estimate the second factor in  $\mathfrak{I}_2$ , we make use of the estimate (9). This yields, using again (7),

$$\sup_{z \in H^1(\partial T)} \frac{\langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, z \rangle_{-\frac{1}{2},\partial T}}{h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}} \lesssim |w|_{1,T}.$$

To estimate the first factor in  $\mathfrak{I}_2$ , we first use an inverse inequality (cf., e.g., [34, Lemma 1.138]) for  $\mathcal{I}_{1,F}^k(v|_F - \pi_T^0 v) \in \mathbb{P}_1^k(F)$  on the 1-simplex  $F$  for all  $F \in \mathcal{F}_T$ , to infer

$$\begin{aligned} h_T^{-\frac{1}{2}} \|\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{0,\partial T} + h_T^{\frac{1}{2}} |\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)|_{1,\partial T} \\ \lesssim h_T^{-\frac{1}{2}} \|\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{0,\partial T} \lesssim \|\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{\infty,\partial T}. \end{aligned}$$

Then, applying Lemma 4.3, and since  $v \in C^0(\bar{T})$ , there holds

$$h_T^{-\frac{1}{2}} \|\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{0,\partial T} + h_T^{\frac{1}{2}} |\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)|_{1,\partial T} \lesssim \|v|_{\partial T} - \pi_T^0 v\|_{\infty,\partial T} \lesssim \|v - \pi_T^0 v\|_{\infty,T},$$

which, by application of the Sobolev's inequality (8), and of the Poincaré inequality (6), yields

$$h_T^{-\frac{1}{2}} \|\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{0,\partial T} + h_T^{\frac{1}{2}} |\mathcal{I}_{1,\partial T}^k(v|_{\partial T} - \pi_T^0 v)|_{1,\partial T} \lesssim |v|_{1,T} + h_T |v|_{2,T}.$$

Finally, we get

$$\mathfrak{A}_2 \lesssim (|v|_{1,T} + h_T |v|_{2,T}) |w|_{1,T}.$$

In conclusion, there holds, for any  $w \in V_2^k(T)$ ,

$$\left| \int_T \nabla \mathcal{I}_{2,T}^k v \cdot \nabla w \right| \lesssim (|v|_{1,T} + h_T |v|_{2,T}) |w|_{1,T}.$$

Taking  $w = \mathcal{I}_{2,T}^k v \in V_2^k(T)$  provides the expected estimate in the  $H^1(T)$ -seminorm. To obtain the estimate in the  $L^2(T)$ -norm, it suffices to remark that  $\int_T \mathcal{I}_{2,T}^k v = \int_T \pi_T^{k-1} v = \int_T v$ . Hence, by the triangle inequality and the Poincaré inequality (6), we infer

$$\|\mathcal{I}_{2,T}^k v\|_{0,T} \leq \|v\|_{0,T} + \|v - \mathcal{I}_{2,T}^k v\|_{0,T} \lesssim \|v\|_{0,T} + h_T (|v|_{1,T} + |\mathcal{I}_{2,T}^k v|_{1,T}).$$

The conclusion then follows from the estimate in the  $H^1(T)$ -seminorm.  $\square$

**Remark 4.7.** We note that the only moment in the proof of Lemma 4.6 (the same observation applies to the case  $d = 3$ ; see Lemma 4.11) where the assumption that  $T$  is star-shaped with respect to a ball is needed is when it comes to use (9). Hidden behind this is the result (61) that states the existence of a lifting operator with optimal scaling for functions in  $H^1(\partial T)$ . Besides, it is worth noticing that, in the proofs of Lemmas 4.6 and 4.11, the liftings actually need only be constructed, respectively, for functions in  $\mathbb{P}_1^{k,c}(\mathcal{F}_T)$  and  $V_2^{k,c}(\mathcal{F}_T)$  (both strict subspaces of  $H^1(\partial T)$ ). One could hence use, instead of (61) and (9), the results of [10, Eq. (2.48) and Lemma 5.3].

In view of the case  $d = 3$ , we state a sharper stability estimate for the interpolation operator.

**Lemma 4.8** (Sharper stability estimate for  $\mathcal{I}_{2,T}^k$ ). *For all  $v \in H^{3/2}(T)$ , there holds*

$$\|\mathcal{I}_{2,T}^k v\|_{0,T} + h_T |\mathcal{I}_{2,T}^k v|_{1,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T} + h_T^{3/2} |v|_{\frac{3}{2},T}.$$

*Proof.* The proof exactly follows the one of Lemma 4.6, with a slight variation when it comes to apply the Sobolev's inequality (8). We make use of the following sharper estimate (cf., e.g., [12, Eq. (2.4)]): for all  $z \in H^{3/2}(T)$ ,

$$\|z\|_{\infty,T} \lesssim h_T^{-1} \|z\|_{0,T} + |z|_{1,T} + h_T^{1/2} |z|_{\frac{3}{2},T}.$$

We thus obtain

$$\mathfrak{A}_2 \lesssim (|v|_{1,T} + h_T^{1/2} |v|_{\frac{3}{2},T}) |w|_{1,T},$$

which finally yields the desired estimate.  $\square$

**The case  $d = 3$ .** If  $d = 3$ , the conforming local virtual space on  $T$  is defined as

$$V_3^k(T) := \left\{ v \in H^1(T) \mid \Delta v \in \mathbb{P}_3^{k-1}(T), v|_{\partial T} \in V_2^{k,c}(\mathcal{F}_T) \right\}, \quad (23)$$

where  $V_2^{k,c}(\mathcal{F}_T) := V_2^k(\mathcal{F}_T) \cap C^0(\partial T)$  with broken space

$$V_2^k(\mathcal{F}_T) := \{v \in L^2(\partial T) \mid v|_F \in V_2^k(F) \forall F \in \mathcal{F}_T\},$$

where  $V_2^k(F)$  is given by (21) with  $T \leftarrow F$ . Since  $V_2^k(F) \subset C^0(F)$  for all  $F \in \mathcal{F}_T$ , one has  $V_3^k(T) \subset H^{1,c}(T)$ . Besides,  $\mathbb{P}_3^k(T) \subset V_3^k(T)$ . The following result is standard.

**Proposition 4.9.** *The triple  $(T, V_3^k(T), \Sigma_{3,T}^k)$ , where  $V_3^k(T)$  is given by (23) and the collection  $\Sigma_{3,T}^k$  splits into*

- $\Sigma_{3,T}^{k,\partial} := \{v_T^\nu\}_{\nu \in \mathcal{V}_T} \cup \{\pi_e^{k-2,m}(\cdot|_e)\}_{m \in \{1, \dots, N_1^{k-2}\}} \cup \{\pi_F^{k-1,j}(\cdot|_F)\}_{j \in \{1, \dots, N_2^{k-1}\}}^{F \in \mathcal{F}_T}$  and
- $\Sigma_{3,T}^{k,o} := \{\pi_T^{k-1,i}\}_{i \in \{1, \dots, N_3^{k-1}\}},$

is a finite element in the sense of Ciarlet.

Letting  $n_{3,T}^k := \dim(V_3^k(T))$ , for any

$$\underline{v}_T := ((v_T^{\partial,\nu})_{\nu \in \mathcal{V}_T}, (v_{T,e}^{\partial,m})_{m \in \{1, \dots, N_1^{k-2}\}}, (v_{T,F}^{\partial,j})_{j \in \{1, \dots, N_2^{k-1}\}}^{F \in \mathcal{F}_T}, (v_T^{o,i})_{i \in \{1, \dots, N_3^{k-1}\}}) \in \mathbb{R}^{n_{3,T}^k},$$

we have  $\underline{v}_T^\partial = ((v_T^{\partial,\nu})_{\nu \in \mathcal{V}_T}, (v_{T,e}^{\partial,m})_{m \in \{1, \dots, N_1^{k-2}\}}, (v_{T,F}^{\partial,j})_{j \in \{1, \dots, N_2^{k-1}\}}^{F \in \mathcal{F}_T}) \in \mathbb{R}^{n_{3,T}^{k,\partial}},$

$$\underline{v}_{T,F}^\partial = ((v_T^{\partial,\nu})_{\nu \in \mathcal{V}_F}, (v_{T,e}^{\partial,m})_{m \in \{1, \dots, N_1^{k-2}\}}, (v_{T,F}^{\partial,j})_{j \in \{1, \dots, N_2^{k-1}\}}) \in \mathbb{R}^{n_{2,F}^k} \text{ for any } F \in \mathcal{F}_T,$$

and we let  $v_T^o := \sum_{i=1}^{N_3^{k-1}} v_T^{o,i} \psi_{T,i}^{k-1} \in \mathbb{P}_3^{k-1}(T)$ . Defining  $r_{2,\partial T}^k : \mathbb{R}^{n_{3,T}^{k,\partial}} \rightarrow V_2^{k,c}(\mathcal{F}_T)$  so that, for any  $\underline{v}_T^\partial \in \mathbb{R}^{n_{3,T}^{k,\partial}}$ ,  $r_{2,\partial T}^k \underline{v}_T^\partial|_F := r_{2,F}^k \underline{v}_{T,F}^\partial$  for all  $F \in \mathcal{F}_T$ , we can now introduce the operator  $r_{3,T}^k : \mathbb{R}^{n_{3,T}^k} \rightarrow V_3^k(T)$  such that, for any  $\underline{v}_T \in \mathbb{R}^{n_{3,T}^k}$ ,  $r_{3,T}^k \underline{v}_T \in V_3^k(T)$  solves the well-posed problem

$$\begin{cases} \int_T \nabla r_{3,T}^k \underline{v}_T \cdot \nabla w = - \int_T v_T^o \Delta w + \langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, r_{2,\partial T}^k(\underline{v}_T^\partial) \rangle_{-\frac{1}{2}, \partial T} & \forall w \in V_3^k(T), \\ \int_T r_{3,T}^k \underline{v}_T = \int_T v_T^o. \end{cases} \quad (24)$$

**Proposition 4.10.** *The operator  $r_{3,T}^k$  defined by (24) coincides with the (canonical) reconstruction operator  $(\underline{\Sigma}_{3,T}^k)^{-1}$ .*

*Proof.* Let  $\mathbb{R}^{n_{3,T}^k} \ni \underline{v}_T := \underline{\Sigma}_{3,T}^k(v)$  for some  $v \in V_3^k(T)$ . Plugging  $\underline{v}_T$  into (24), we infer by integration by parts

$$\int_T \nabla r_{3,T}^k \underline{v}_T \cdot \nabla w = - \int_T \pi_T^{k-1} v \Delta w + \langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, r_{2,\partial T}^k(\underline{\Sigma}_{3,T}^k(v)) \rangle_{-\frac{1}{2}, \partial T} = \int_T \nabla v \cdot \nabla w \quad \forall w \in V_3^k(T),$$

where we have used that  $\Delta w \in \mathbb{P}_3^{k-1}(T)$  and that, for all  $F \in \mathcal{F}_T$ ,  $r_{2,\partial T}^k(\underline{\Sigma}_{3,T}^k(v))|_F = r_{2,F}^k(\underline{\Sigma}_{3,T,F}^k(v)) = r_{2,F}^k(\underline{\Sigma}_{2,F}^k(v|_F)) = v|_F$  (cf. Proposition 4.5). Since  $\int_T r_{3,T}^k \underline{v}_T = \int_T \pi_T^{k-1} v = \int_T v$ , we finally infer that  $r_{3,T}^k \underline{v}_T = r_{3,T}^k(\underline{\Sigma}_{3,T}^k(v)) = v$ . This is valid for any  $v \in V_3^k(T)$ , hence  $r_{3,T}^k = (\underline{\Sigma}_{3,T}^k)^{-1}$ .  $\square$

Let us introduce the operator  $\mathcal{I}_{2,\partial T}^k : C^0(\partial T) \rightarrow V_2^{k,c}(\mathcal{F}_T)$  so that, for any  $v \in C^0(\partial T)$ ,  $\mathcal{I}_{2,\partial T}^k v|_F := \mathcal{I}_{2,F}^k(v|_F)$  for all  $F \in \mathcal{F}_T$ . The collection  $\Sigma_{3,T}^k$  of linear forms on  $V_3^k(T)$  can clearly be extended to a collection  $\overline{\Sigma}_{3,T}^k$  of linear forms on  $H^{1,c}(T)$ . Thus, we can define the interpolation operator  $\mathcal{I}_{3,T}^k : H^{1,c}(T) \rightarrow V_3^k(T)$  such that  $\mathcal{I}_{3,T}^k := r_{3,T}^k \circ \overline{\Sigma}_{3,T}^k$ . We remark that, for all  $v \in H^{1,c}(T)$ ,  $r_{2,\partial T}^k(\overline{\Sigma}_{3,T}^k(v)) = \mathcal{I}_{2,\partial T}^k(v|_{\partial T})$ . We have the following stability result for  $\mathcal{I}_{3,T}^k$ .

**Lemma 4.11** (Stability of  $\mathcal{I}_{3,T}^k$ ). *For all  $v \in H^2(T)$ , there holds*

- $|\mathcal{I}_{3,T}^k v|_{1,T} \lesssim |v|_{1,T} + h_T |v|_{2,T};$
- $\|\mathcal{I}_{3,T}^k v\|_{0,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T} + h_T^2 |v|_{2,T}.$

*Proof.* First, remark that if  $v \in H^2(T)$ ,  $v \in C^0(\overline{T})$  also in that case and hence  $v \in H^{1,c}(T)$ ; consequently,  $\mathcal{I}_{3,T}^k v$  has a sense. Let us test (24) with  $\underline{v}_T := \overline{\Sigma}_{3,T}^k(v)$ . There holds

$$\int_T \nabla \mathcal{I}_{3,T}^k v \cdot \nabla w = - \int_T v \Delta w + \langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, \mathcal{I}_{2,\partial T}^k(v|_{\partial T}) \rangle_{-\frac{1}{2}, \partial T} \quad \forall w \in V_3^k(T).$$

We equivalently rewrite the equality above as

$$\int_T \nabla \mathcal{I}_{3,T}^k v \cdot \nabla w = - \int_T (v - \pi_T^0 v) \Delta w + \langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, \mathcal{I}_{2,\partial T}^k(v|_{\partial T} - \pi_T^0 v) \rangle_{-\frac{1}{2}, \partial T} \quad \forall w \in V_3^k(T),$$

and, by Cauchy–Schwarz inequality, we infer, for any  $w \in V_3^k(T)$ ,

$$\begin{aligned} \left| \int_T \nabla \mathcal{I}_{3,T}^k v \cdot \nabla w \right| &\lesssim \|v - \pi_T^0 v\|_{0,T} \|\Delta w\|_{0,T} + \left( h_T^{-\frac{1}{2}} \|\mathcal{I}_{2,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{0,\partial T} \right. \\ &\quad \left. + h_T^{\frac{1}{2}} |\mathcal{I}_{2,\partial T}^k(v|_{\partial T} - \pi_T^0 v)|_{1,\partial T} \right) \sup_{z \in H^1(\partial T)} \frac{\langle \nabla w|_{\partial T} \cdot \mathbf{n}_T, z \rangle_{-\frac{1}{2},\partial T}}{h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}} =: \mathfrak{I}_1 + \mathfrak{I}_2. \end{aligned}$$

The term  $\mathfrak{I}_1$  and the second factor in  $\mathfrak{I}_2$  can be handled as in the proof of Lemma 4.6. To estimate the first factor in  $\mathfrak{I}_2$ , we remark that for all  $F \in \mathcal{F}_T$ ,  $(v|_F - \pi_T^0 v) \in H^{3/2}(F)$ . We can hence apply Lemma 4.8 with  $T \leftarrow F$  to infer

$$\begin{aligned} h_F^{-1/2} \|\mathcal{I}_{2,F}^k(v|_F - \pi_T^0 v)\|_{0,F} + h_F^{1/2} |\mathcal{I}_{2,F}^k(v|_F - \pi_T^0 v)|_{1,F} \\ \lesssim h_F^{-1/2} \|v|_F - \pi_T^0 v\|_{0,F} + h_F^{1/2} |v|_F|_{1,F} + h_F |v|_F|_{\frac{3}{2},F}. \end{aligned}$$

By (a sharper version of) the continuous trace inequality (4), and the Poincaré inequality (6), we obtain, summing over  $F \in \mathcal{F}_T$ ,

$$h_T^{-\frac{1}{2}} \|\mathcal{I}_{2,\partial T}^k(v|_{\partial T} - \pi_T^0 v)\|_{0,\partial T} + h_T^{1/2} |\mathcal{I}_{2,\partial T}^k(v|_{\partial T} - \pi_T^0 v)|_{1,\partial T} \lesssim |v|_{1,T} + h_T |v|_{2,T}.$$

Hence, there holds, for any  $w \in V_3^k(T)$ ,

$$\left| \int_T \nabla \mathcal{I}_{3,T}^k v \cdot \nabla w \right| \lesssim (|v|_{1,T} + h_T |v|_{2,T}) |w|_{1,T}.$$

Taking  $w = \mathcal{I}_{3,T}^k v \in V_3^k(T)$  provides the expected estimate in the  $H^1(T)$ -seminorm. To obtain the estimate in the  $L^2(T)$ -norm, we follow the same reasoning as in the proof of Lemma 4.6.  $\square$

**Remark 4.12.** *In view of Lemmas 4.6 and 4.11, the stability result on  $\mathcal{I}_{d,T}^k$  in [26, Eq. (35)], that is taken as an assumption for the subsequent analysis, seems a bit optimistic.*

#### 4.1.2 Nonconforming case

The nonconforming local virtual space on  $T \in \mathcal{T}_h$  in dimension  $d$  is defined as

$$V_d^k(T) := \{v \in H^1(T) \mid \Delta v \in \mathbb{P}_d^{k-1}(T), \nabla v|_{\partial T} \cdot \mathbf{n}_T \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)\}. \quad (25)$$

As opposed to the conforming case, the construction of the nonconforming local virtual space does not depend on the ambient dimension. We have  $\mathbb{P}_d^k(T) \subset V_d^k(T)$ . The following result is standard.

**Proposition 4.13.** *The triple  $(T, V_d^k(T), \Sigma_{d,T}^k)$ , where  $V_d^k(T)$  is given by (25) and the collection  $\Sigma_{d,T}^k$  splits into*

$$\Sigma_{d,T}^{k,\partial} := \{\pi_F^{k-1,j}(\cdot|_F)\}_{j \in \{1, \dots, N_d^{k-1}\}}^{F \in \mathcal{F}_T} \quad \text{and} \quad \Sigma_{d,T}^{k,o} := \{\pi_T^{k-1,i}\}_{i \in \{1, \dots, N_d^{k-1}\}},$$

*is a finite element in the sense of Ciarlet.*

Letting  $n_{d,T}^k := \dim(V_d^k(T))$ , for any

$$\underline{v}_T := \left( (v_{T,F}^{\partial,j})_{j \in \{1, \dots, N_d^{k-1}\}}^{F \in \mathcal{F}_T}, (v_T^{o,i})_{i \in \{1, \dots, N_d^{k-1}\}} \right) \in \mathbb{R}^{n_{d,T}^k},$$

we let  $v_T^o := \sum_{i=1}^{N_d^{k-1}} v_T^{o,i} \psi_{T,i}^{k-1} \in \mathbb{P}_d^{k-1}(T)$  and  $v_T^\partial \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$  so that  $v_T^\partial|_F := \sum_{j=1}^{N_d^{k-1}} v_{T,F}^{\partial,j} \psi_{F,j}^{k-1} \in \mathbb{P}_{d-1}^{k-1}(F)$  for all  $F \in \mathcal{F}_T$ . We can now introduce the operator  $r_{d,T}^k : \mathbb{R}^{n_{d,T}^k} \rightarrow V_d^k(T)$  such that, for any  $\underline{v}_T \in \mathbb{R}^{n_{d,T}^k}$ ,  $r_{d,T}^k \underline{v}_T \in V_d^k(T)$  solves the well-posed problem

$$\begin{cases} \int_T \nabla r_{d,T}^k \underline{v}_T \cdot \nabla w = - \int_T v_T^o \Delta w + \int_{\partial T} v_T^\partial \nabla w|_{\partial T} \cdot \mathbf{n}_T & \forall w \in V_d^k(T), \\ \int_T r_{d,T}^k \underline{v}_T = \int_T v_T^o. \end{cases} \quad (26)$$



**Proposition 4.14.** *The operator  $r_{d,T}^k$  defined by (26) coincides with the (canonical) reconstruction operator  $(\underline{\Sigma}_{d,T}^k)^{-1}$ .*

*Proof.* Let  $\mathbb{R}^{n_{d,T}} \ni \underline{v}_T := \underline{\Sigma}_{d,T}^k(v)$  for some  $v \in V_d^k(T)$ . Plugging  $\underline{v}_T$  into (26), we infer by integration by parts

$$\int_T \nabla r_{d,T}^k \underline{v}_T \cdot \nabla w = - \int_T \pi_T^{k-1} v \Delta w + \int_{\partial T} \pi_{\partial T}^{k-1}(v|_{\partial T}) \nabla w|_{\partial T} \cdot \mathbf{n}_T = \int_T \nabla v \cdot \nabla w \quad \forall w \in V_d^k(T),$$

where we have used that  $\Delta w \in \mathbb{P}_d^{k-1}(T)$  and that, for all  $F \in \mathcal{F}_T$ ,  $\nabla w|_F \cdot \mathbf{n}_{T,F} \in \mathbb{P}_{d-1}^{k-1}(F)$ . Since  $\int_T r_{d,T}^k \underline{v}_T = \int_T \pi_T^{k-1} v = \int_T v$ , we finally deduce that  $r_{d,T}^k \underline{v}_T = r_{d,T}^k(\underline{\Sigma}_{d,T}^k(v)) = v$ . This is valid for any  $v \in V_d^k(T)$ , hence  $r_{d,T}^k = (\underline{\Sigma}_{d,T}^k)^{-1}$ .  $\square$

The collection  $\Sigma_{d,T}^k$  of linear forms on  $V_d^k(T)$  can be patently extended to a collection  $\overline{\Sigma}_{d,T}^k$  of linear forms on  $H^1(T)$ . Thus, we can define the interpolation operator  $\mathcal{I}_{d,T}^k : H^1(T) \rightarrow V_d^k(T)$  such that  $\mathcal{I}_{d,T}^k := r_{d,T}^k \circ \overline{\Sigma}_{d,T}^k$ . We have the following stability result for  $\mathcal{I}_{d,T}^k$ .

**Lemma 4.15** (Stability of  $\mathcal{I}_{d,T}^k$ ). *For all  $v \in H^1(T)$ , there holds*

- $|\mathcal{I}_{d,T}^k v|_{1,T} \leq |v|_{1,T}$ ;
- $\|\mathcal{I}_{d,T}^k v\|_{0,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T}$ .

*Proof.* We test (26) with  $\underline{v}_T := \overline{\Sigma}_{d,T}^k(v)$ , for some  $v \in H^1(T)$ . By integration by parts, there holds

$$\int_T \nabla \mathcal{I}_{d,T}^k v \cdot \nabla w = \int_T \nabla v \cdot \nabla w \quad \text{for all } w \in V_d^k(T). \quad (27)$$

Testing with  $w = \mathcal{I}_{d,T}^k v \in V_d^k(T)$ , we immediately obtain the estimate in the  $H^1(T)$ -seminorm. Then, since  $\int_T \mathcal{I}_{d,T}^k v = \int_T \pi_T^{k-1} v = \int_T v$ , by the triangle and Poincaré (6) inequalities, there holds

$$\|\mathcal{I}_{d,T}^k v\|_{0,T} \leq \|v\|_{0,T} + \|v - \mathcal{I}_{d,T}^k v\|_{0,T} \lesssim \|v\|_{0,T} + h_T (|v|_{1,T} + |\mathcal{I}_{d,T}^k v|_{1,T}),$$

which enables to conclude.  $\square$

**Remark 4.16.** *In the proof of Lemma 4.15, and as opposed to the proofs of Lemmas 4.6 and 4.11, no estimate on a dual norm of the boundary flux is needed for virtual functions. As a consequence, there is no need to construct a lifting (with optimal scaling) for traces of virtual functions, and  $T \in \mathcal{T}_h$  needs not be assumed star-shaped to lead the analysis. The difference is related to the fact that, in the nonconforming case, the canonical interpolation operator  $\mathcal{I}_{d,T}^k$  is the elliptic projector on the virtual space  $V_d^k(T)$  (see (27)). In the nonconforming case, one can actually control the  $L^2(\partial T)$ -norm of the boundary flux by the  $H^1(T)$ -seminorm of the virtual function (cf. [19, Lemma 4.4 (take  $\mathbb{A}_\varepsilon = \mathbb{I}_d$ )]).*

## 4.2 Local polynomial projector

In this section, we prove  $H^s$ -approximation properties for the local polynomial projector on which the local discrete bilinear form of the c/nc-VE/HHO methods hinges. Henceforth, in the conforming case (and in the conforming case only), we assume that any cell  $T \in \mathcal{T}_h$  is star-shaped with respect to a ball whose radius is comparable to  $h_T$ .

Let  $\underline{s}$  be an integer such that  $\underline{s} = 2$  in the conforming case and  $\underline{s} = 1$  in the nonconforming case. We let  $W_d^k(T) := H^{1,c}(T)$  in the conforming case and  $W_d^k(T) := H^1(T)$  in the nonconforming case. In view of Lemmas 4.6, 4.11, and 4.15, the interpolation operator  $\mathcal{I}_{d,T}^k : W_d^k(T) \rightarrow V_d^k(T)$  satisfies the following stability estimates:

$$\forall v \in H^{\underline{s}}(T), \quad |\mathcal{I}_{d,T}^k v|_{1,T} \lesssim \sum_{\alpha=1}^{\underline{s}} h_T^{\alpha-1} |v|_{\alpha,T}, \quad \|\mathcal{I}_{d,T}^k v\|_{0,T} \lesssim \sum_{\alpha=0}^{\underline{s}} h_T^\alpha |v|_{\alpha,T}. \quad (28)$$

For any  $\underline{v}_T^\partial \in \mathbb{R}^{n_{d,T}^\partial}$ , we define  $\mathfrak{t}_{\partial T}^k \underline{v}_T^\partial \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)$  so that  $\mathfrak{t}_{\partial T}^k \underline{v}_T^\partial := r_{1,\partial T}^k \underline{v}_T^\partial \in \mathbb{P}_1^{k,c}(\mathcal{F}_T)$  in the conforming case with  $d = 2$ ,  $\mathfrak{t}_{\partial T}^k \underline{v}_T^\partial := \pi_{\partial T}^{k-1}(r_{2,\partial T}^k \underline{v}_T^\partial) \in \mathbb{P}_2^{k-1}(\mathcal{F}_T)$  in the conforming case with  $d = 3$ , and  $\mathfrak{t}_{\partial T}^k \underline{v}_T^\partial := \underline{v}_T^\partial \in$

$\mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$  in the nonconforming case. We introduce the operator  $p_{d,T}^k : \mathbb{R}^{n_{d,T}^k} \rightarrow \mathbb{P}_d^k(T)$  such that, for any  $\underline{v}_T \in \mathbb{R}^{n_{d,T}^k}$ ,  $p_{d,T}^k \underline{v}_T \in \mathbb{P}_d^k(T)$  solves the well-posed problem

$$\begin{cases} \int_T \nabla p_{d,T}^k \underline{v}_T \cdot \nabla w = - \int_T v_T^\circ \Delta w + \int_{\partial T} t_{\partial T}^k \underline{v}_T^\circ \cdot \nabla w|_{\partial T} \cdot \mathbf{n}_T & \forall w \in \mathbb{P}_d^k(T), \\ \int_T p_{d,T}^k \underline{v}_T = \int_T v_T^\circ, \end{cases} \quad (29)$$

and the associated operator  $\mathcal{P}_{d,T}^k : W_d^k(T) \rightarrow \mathbb{P}_d^k(T)$  such that  $\mathcal{P}_{d,T}^k := p_{d,T}^k \circ \overline{\Sigma}_{d,T}^k$ . Doing so, we mimick the standard formalism of HHO methods, where the polynomial operator  $p_{d,T}^k$  is meant to define the principal part of the local discrete bilinear form. Since  $\mathbb{P}_d^k(T) \subset V_d^k(T)$ , by comparing (29) to (22), (24), and (26), we easily infer using the definition (10) that  $p_{d,T}^k = \Pi_T^k \circ r_{d,T}^k$ , and  $\mathcal{P}_{d,T}^k = \Pi_T^k \circ \mathcal{I}_{d,T}^k$ . Hence, the polynomial operator  $\mathcal{P}_{d,T}^k$  is a projector, that satisfies:

- $\mathcal{P}_{d,T}^k v = \Pi_T^k v$  for all  $v \in V_d^k(T)$  (and, in particular,  $\mathcal{P}_{d,T}^k v = v$  for all  $v \in \mathbb{P}_d^k(T)$ );
- in the nonconforming case, owing to (27),  $\mathcal{P}_{d,T}^k = \Pi_T^k$ .

Combining the stability result for  $\Pi_T^k$  of Proposition 2.4 with (28) directly yields stability for the local polynomial projector  $\mathcal{P}_{d,T}^k$ :

$$\forall v \in H^s(T), \quad \|\mathcal{P}_{d,T}^k v\|_{0,T} \lesssim \sum_{\alpha=0}^s h_T^\alpha |v|_{\alpha,T}. \quad (30)$$

We can then prove  $H^s$ -approximation properties for  $\mathcal{P}_{d,T}^k$ .

**Theorem 4.17** ( $H^s$ -approximation properties for  $\mathcal{P}_{d,T}^k$ ). *Let  $v \in H^s(T)$ , for  $s \in \{\underline{s}, \dots, k+1\}$ . Then, the following holds:*

$$|v - \mathcal{P}_{d,T}^k v|_{m,T} \lesssim h_T^{s-m} |v|_{s,T} \quad \text{for } m \in \{0, \dots, s\}, \quad (31)$$

and, for any  $F \in \mathcal{F}_T$ , and any  $(\zeta_1, \dots, \zeta_d) \in \mathbb{N}^d$  such that  $\sum_{i=1}^d \zeta_i = m$ ,

$$\left| [\partial_{x_1}^{\zeta_1} \dots \partial_{x_d}^{\zeta_d} (v - \mathcal{P}_{d,T}^k v)]|_F \right|_{0,F} \lesssim h_T^{s-m-1/2} |v|_{s,T} \quad \text{for } m \in \{0, \dots, s-1\}. \quad (32)$$

*Proof.* In the nonconforming case, there is nothing to prove. Since  $\mathcal{P}_{d,T}^k = \Pi_T^k$ , the result follows from (11) and (12) of Proposition 2.4. For the conforming case, we follow the ideas of [11, Chapter 4]. We proceed by density of  $C^\infty(\overline{T})$  in  $H^s(T)$ . Since in that case  $T$  is star-shaped with respect to every point of a ball of radius  $\gamma h_T$ ,  $v \in C^\infty(\overline{T})$  admits the following Sobolev representation:

$$v = Q_T^s v + R_T^s v,$$

where  $Q_T^s v \in \mathbb{P}_d^{s-1}(T) \subseteq \mathbb{P}_d^k(T)$  is the averaged Taylor polynomial, and the remainder  $R_T^s v$  satisfies, for  $r \in \{0, \dots, s\}$ , the Bramble–Hilbert lemma (cf., e.g., [11, Lemma 4.3.8]):

$$|R_T^s v|_{r,T} \lesssim h_T^{s-r} |v|_{s,T}. \quad (33)$$

One can easily see that, since  $Q_T^s v \in \mathbb{P}_d^k(T)$  and  $\mathcal{P}_{d,T}^k v = v$  for all  $v \in \mathbb{P}_d^k(T)$ , there holds  $v - \mathcal{P}_{d,T}^k v = R_T^s v - \mathcal{P}_{d,T}^k (R_T^s v)$ . Thus,

$$|v - \mathcal{P}_{d,T}^k v|_{m,T} \leq |R_T^s v|_{m,T} + |\mathcal{P}_{d,T}^k (R_T^s v)|_{m,T}.$$

Applying  $m$  times the inverse inequality (3) to  $\mathcal{P}_{d,T}^k (R_T^s v) \in \mathbb{P}_d^k(T)$ , and then the stability result (30) for  $\mathcal{P}_{d,T}^k$  in the  $L^2(T)$ -norm, we infer

$$|v - \mathcal{P}_{d,T}^k v|_{m,T} \lesssim |R_T^s v|_{m,T} + h_T^{-m} \|\mathcal{P}_{d,T}^k (R_T^s v)\|_{0,T} \lesssim |R_T^s v|_{m,T} + \sum_{\alpha=0}^s h_T^{-m+\alpha} |R_T^s v|_{\alpha,T}.$$

Applying the Bramble–Hilbert lemma (33) for  $r = m$  and  $r = \alpha \in \{0, \dots, \underline{s}\}$ , we finally obtain (31). For  $F \in \mathcal{F}_T$  now, the continuous trace inequality (4) yields

$$\left| [\partial_{x_1}^{\zeta_1} \dots \partial_{x_d}^{\zeta_d} (v - \mathcal{P}_{d,T}^k v)]|_F \right|_{0,F} \lesssim h_T^{-1/2} |v - \mathcal{P}_{d,T}^k v|_{m,T} + h_T^{1/2} |v - \mathcal{P}_{d,T}^k v|_{m+1,T}.$$

The conclusion then follows from (31) (remark that, for  $m \in \{0, \dots, s-1\}$ ,  $m+1 \in \{1, \dots, s\}$ ).  $\square$

Note that, in the conforming case, general  $H^s$ -approximation properties for  $\mathcal{I}_{d,T}^k$  cannot be obtained easily, because of the fact that one needs  $H^s$ -inverse inequalities on the local virtual space (as opposed to polynomials, derivatives of virtual functions may not be virtual functions).

**Remark 4.18.** For any  $v \in W_d^k(T)$ , there holds

$$\mathcal{P}_{d,T}^k v - \mathcal{I}_{d,T}^k v = \mathcal{I}_{d,T}^k (\mathcal{P}_{d,T}^k v - v). \quad (34)$$

**Remark 4.19.** The definition (29) of the operator  $p_{d,T}^k$  is entirely based on some polynomial-valued linear combinations of the DoFs, that are  $v_T^o \in \mathbb{P}_d^{k-1}(T)$  and  $t_{\partial T}^k v_T^{\hat{o}} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)$ . In the conforming case with  $d = 2$ ,  $t_{\partial T}^k v_T^{\hat{o}}$  only depends on  $(v_T^{\hat{o},\nu} \in \mathbb{R})_{\nu \in \mathcal{V}_T}$  and  $\left( \sum_{j=1}^{N_1^{k-2}} v_{T,F}^{\hat{o},j} \psi_{F,j}^{k-2} \in \mathbb{P}_1^{k-2}(F) \right)_{F \in \mathcal{F}_T}$  through (20). In the conforming case with  $d = 3$ ,  $t_{\partial T}^k v_T^{\hat{o}}$  is such that  $t_{\partial T}^k v_T^{\hat{o}}|_F = \sum_{j=1}^{N_2^{k-1}} v_{T,F}^{\hat{o},j} \psi_{F,j}^{k-1} \in \mathbb{P}_2^{k-1}(F)$  for any  $F \in \mathcal{F}_T$ . In the nonconforming case,  $t_{\partial T}^k v_T^{\hat{o}} = v_T^o \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$ .

### 4.3 Discrete problem

Let us define, for any  $T \in \mathcal{T}_h$ , the following seminorm on  $\mathbb{R}^{n_{d,T}^k}$ : for all  $\underline{v}_T \in \mathbb{R}^{n_{d,T}^k}$ ,

- $|\underline{v}_T|_T^2 := \|\nabla v_T^o\|_{0,T}^2 + \left\| h_{\partial T}^{-1/2} (v_T^o|_{\partial T} - t_{\partial T}^k v_T^{\hat{o}}) \right\|_{0,\partial T}^2$  in the conforming case with  $d = 2$ , and in the nonconforming case,
- $|\underline{v}_T|_T^2 := \|\nabla v_T^o\|_{0,T}^2 + \left\| h_{\partial T}^{-1/2} (v_T^o|_{\partial T} - t_{\partial T}^k v_T^{\hat{o}}) \right\|_{0,\partial T}^2 + \sum_{F \in \mathcal{F}_T} \left\| v_T^o|_{\partial F} - r_{1,\partial F}^k v_{T,F}^{\hat{o},\partial} \right\|_{0,\partial F}^2$  in the conforming case with  $d = 3$ .

Letting  $|\underline{v}_h|_h^2 := \sum_{T \in \mathcal{T}_h} |\underline{v}_T|_T^2$  for any  $\underline{v}_h \in \mathbb{R}^{n_{d,h}^k}$ , one can easily see that  $|\cdot|_h$  defines a norm on  $\mathbb{V}_{d,h,0}^k$  (defined in (17)).

**Remark 4.20.** In view of Lemma 3.2 and Remark 3.3, the quantity  $\left( \sum_{T \in \mathcal{T}_h} \left\| \nabla r_{d,T}^k \underline{v}_T \right\|_{0,T}^2 \right)^{1/2}$ , which can be equivalently written  $\|\nabla_h v_h\|_{0,\Omega}$  for  $v_h \in V_d^k(\mathcal{T}_h)$  such that  $v_h|_T := r_{d,T}^k \underline{v}_T$  for all  $T \in \mathcal{T}_h$ , defines a norm on  $\mathbb{V}_{d,h,0}^k$ . Furthermore, there holds (we omit the proof for brevity), for any  $\underline{v}_h \in \mathbb{R}^{n_{d,h}^k}$ ,

$$\left( \sum_{T \in \mathcal{T}_h} \left\| \nabla r_{d,T}^k \underline{v}_T \right\|_{0,T}^2 \right)^{1/2} \lesssim |\underline{v}_h|_h.$$

Let us write down the discrete problem. We consider Problem (18), where the local linear form is given by

$$l_T(\underline{v}_T) := \int_T f|_T v_T^o,$$

and the (symmetric) local bilinear form, based on the local polynomial projector studied in Section 4.2, by

$$a_T(\underline{w}_T, \underline{v}_T) := \int_T \nabla p_{d,T}^k \underline{w}_T \cdot \nabla p_{d,T}^k \underline{v}_T + s_T(\underline{w}_T, \underline{v}_T),$$

with stabilization (letting  $l = k$  in the conforming case with  $d = 2$ , and  $l = k-1$  in the other cases)

- $s_T(\underline{w}_T, \underline{v}_T) := \int_{\partial T} h_{\partial T}^{-1} \pi_{\partial T}^l (\delta_{d,T}^k \underline{w}_T|_{\partial T} - t_{\partial T}^k \underline{w}_T^{\hat{o}}) \pi_{\partial T}^l (\delta_{d,T}^k \underline{v}_T|_{\partial T} - t_{\partial T}^k \underline{v}_T^{\hat{o}})$  in the conforming case with  $d = 2$ , and in the nonconforming case,
- $s_T(\underline{w}_T, \underline{v}_T) := \int_{\partial T} h_{\partial T}^{-1} \pi_{\partial T}^l (\delta_{d,T}^k \underline{w}_T|_{\partial T} - t_{\partial T}^k \underline{w}_T^{\hat{o}}) \pi_{\partial T}^l (\delta_{d,T}^k \underline{v}_T|_{\partial T} - t_{\partial T}^k \underline{v}_T^{\hat{o}}) + \sum_{F \in \mathcal{F}_T} \int_{\partial F} (\delta_{3,T}^k \underline{w}_T|_{\partial F} - r_{1,\partial F}^k \underline{w}_{T,F}^{\hat{o},\partial}) (\delta_{3,T}^k \underline{v}_T|_{\partial F} - r_{1,\partial F}^k \underline{v}_{T,F}^{\hat{o},\partial})$  in the conforming case with  $d = 3$ ,

where  $\delta_{d,T}^k \underline{v}_T := p_{d,T}^k \underline{v}_T + \pi_T^{k-1} (r_{d,T}^k \underline{v}_T - p_{d,T}^k \underline{v}_T) \in \mathbb{P}_d^k(T)$ . Other choices of stabilization are possible (cf., e.g., [8]). Here, we build upon the standard HHO choice of stabilization (see [29, 30]).

**Remark 4.21** (Equivalent functional viewpoint). *Letting  $w, v \in V_d^k(T)$  such that  $w := r_{d,T}^k \underline{w}_T$  and  $v := r_{d,T}^k \underline{v}_T$ , and since  $\mathcal{P}_{d,T}^k z = \Pi_T^k z$  for all  $z \in V_d^k(T)$ , one can equivalently consider Problem (15), with local (bi)linear forms*

$$l_T(v) = \int_T f|_T \pi_T^{k-1} v, \quad a_T(w, v) = \int_T \nabla \Pi_T^k w \cdot \nabla \Pi_T^k v + s_T(w, v),$$

with stabilization

$$s_T(w, v) = \int_{\partial T} h_{\partial T}^{-1} \pi_{\partial T}^l([\Delta_{d,T}^k w - w]_{|\partial T}) \pi_{\partial T}^l([\Delta_{d,T}^k v - v]_{|\partial T})$$

in the conforming case with  $d = 2$  and in the nonconforming case, and stabilization supplemented with the term

$$\sum_{F \in \mathcal{F}_T} \int_{\partial F} (\Delta_{3,T}^k w - w)_{|\partial F} (\Delta_{3,T}^k v - v)_{|\partial F}$$

in the conforming case with  $d = 3$ , where  $\Delta_{d,T}^k v := \Pi_T^k v + \pi_T^{k-1}(v - \Pi_T^k v)$ .

We prove well-posedness for Problem (18).

**Lemma 4.22** (Local coercivity and boundedness). *For all  $T \in \mathcal{T}_h$ , and all  $\underline{v}_T \in \mathbb{R}^{n_{d,T}^k}$ , the following holds:*

$$|\underline{v}_T|_T^2 \lesssim \mathfrak{a}_T(\underline{v}_T, \underline{v}_T) \lesssim |\underline{v}_T|_T^2.$$

*Proof.* Let us show local coercivity. Testing (29) with  $w = \underline{v}_T^\circ \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$ , we infer

$$- \int_T \underline{v}_T^\circ \Delta \underline{v}_T^\circ = \int_T \nabla p_{d,T}^k \underline{v}_T \cdot \nabla \underline{v}_T^\circ - \int_{\partial T} \mathfrak{t}_{\partial T}^k \underline{v}_T^\circ \nabla \underline{v}_T^\circ|_{\partial T} \cdot \mathbf{n}_T.$$

Integrating by parts the left-hand side, there holds

$$\|\nabla \underline{v}_T^\circ\|_{0,T}^2 = \int_T \nabla p_{d,T}^k \underline{v}_T \cdot \nabla \underline{v}_T^\circ + \int_{\partial T} \pi_{\partial T}^l(\underline{v}_T^\circ|_{\partial T} - \mathfrak{t}_{\partial T}^k \underline{v}_T^\circ) \nabla \underline{v}_T^\circ|_{\partial T} \cdot \mathbf{n}_T,$$

which yields, by Cauchy–Schwarz inequality, and the discrete trace inequality (5),

$$\|\nabla \underline{v}_T^\circ\|_{0,T} \lesssim \|\nabla p_{d,T}^k \underline{v}_T\|_{0,T} + \left\| h_{\partial T}^{-1/2} \pi_{\partial T}^l(\underline{v}_T^\circ|_{\partial T} - \mathfrak{t}_{\partial T}^k \underline{v}_T^\circ) \right\|_{0,\partial T}. \quad (35)$$

Now, adding/subtracting  $\pi_{\partial T}^l(\delta_{d,T}^k \underline{v}_T|_{\partial T})$ , using the fact that  $\pi_T^{k-1}(r_{d,T}^k \underline{v}_T) = \underline{v}_T^\circ$ , and by stability of  $\pi_F^l$  in the  $L^2(F)$ -norm for all  $F \in \mathcal{F}_T$ , we infer

$$\begin{aligned} \left\| h_{\partial T}^{-1/2} \pi_{\partial T}^l(\underline{v}_T^\circ|_{\partial T} - \mathfrak{t}_{\partial T}^k \underline{v}_T^\circ) \right\|_{0,\partial T} &\leq \left\| h_{\partial T}^{-1/2} (p_{d,T}^k \underline{v}_T - \pi_T^{k-1}(p_{d,T}^k \underline{v}_T)) \right\|_{0,\partial T} \\ &\quad + \left\| h_{\partial T}^{-1/2} \pi_{\partial T}^l(\delta_{d,T}^k \underline{v}_T|_{\partial T} - \mathfrak{t}_{\partial T}^k \underline{v}_T^\circ) \right\|_{0,\partial T}. \end{aligned}$$

By the discrete trace inequality (5), and an application of (11) with  $s = 1$  and  $m = 0$ , we infer

$$\left\| h_{\partial T}^{-1/2} \pi_{\partial T}^l(\underline{v}_T^\circ|_{\partial T} - \mathfrak{t}_{\partial T}^k \underline{v}_T^\circ) \right\|_{0,\partial T} \lesssim \|\nabla p_{d,T}^k \underline{v}_T\|_{0,T} + \left\| h_{\partial T}^{-1/2} \pi_{\partial T}^l(\delta_{d,T}^k \underline{v}_T|_{\partial T} - \mathfrak{t}_{\partial T}^k \underline{v}_T^\circ) \right\|_{0,\partial T}. \quad (36)$$

With  $l = k$  in the conforming case with  $d = 2$ , and  $l = k-1$  in the nonconforming case, we infer local coercivity in those two cases as a direct consequence of (35) and (36). In the conforming case with  $d = 3$ , one has to estimate two terms in the stabilization part of the seminorm. The first one is handled as above with  $l = k-1$ . It then remains to estimate the term  $\left\| \underline{v}_T^\circ|_{\partial F} - r_{1,\partial F}^k \underline{v}_{T,F}^{\partial,\partial} \right\|_{0,\partial F}$  for all  $F \in \mathcal{F}_T$ . Adding/subtracting  $\delta_{3,T}^k \underline{v}_T|_{\partial F}$ , and since  $\pi_T^{k-1}(r_{3,T}^k \underline{v}_T) = \underline{v}_T^\circ$ , we have for any  $F \in \mathcal{F}_T$ ,

$$\left\| \underline{v}_T^\circ|_{\partial F} - r_{1,\partial F}^k \underline{v}_{T,F}^{\partial,\partial} \right\|_{0,\partial F} \leq \left\| (p_{3,T}^k \underline{v}_T - \pi_T^{k-1}(p_{3,T}^k \underline{v}_T)) \right\|_{0,\partial F} + \left\| \delta_{3,T}^k \underline{v}_T|_{\partial F} - r_{1,\partial F}^k \underline{v}_{T,F}^{\partial,\partial} \right\|_{0,\partial F}.$$

Since  $(p_{3,T}^k \underline{v}_T - \pi_T^{k-1}(p_{3,T}^k \underline{v}_T))|_F \in \mathbb{P}_2^k(F)$ , we can apply a first time the discrete trace inequality (5) (with  $T \leftarrow F$ ) to obtain

$$\left\| (p_{3,T}^k \underline{v}_T - \pi_T^{k-1}(p_{3,T}^k \underline{v}_T)) \right\|_{0,\partial F} \lesssim h_F^{-1/2} \left\| (p_{3,T}^k \underline{v}_T - \pi_T^{k-1}(p_{3,T}^k \underline{v}_T)) \right\|_{0,F},$$

and a second time (as it is) to obtain

$$\left\| (p_{3,T}^k \underline{v}_T - \pi_T^{k-1}(p_{3,T}^k \underline{v}_T)) \Big|_{\partial F} \right\|_{0,\partial F} \lesssim h_T^{-1} \|p_{3,T}^k \underline{v}_T - \pi_T^{k-1}(p_{3,T}^k \underline{v}_T)\|_{0,T}.$$

An application of (11) with  $s = 1$  and  $m = 0$  finally yields

$$\left\| v_T^{\circ} \Big|_{\partial F} - r_{1,\partial F}^k \underline{v}_{T,F}^{\partial,\partial} \right\|_{0,\partial F} \lesssim \|\nabla p_{3,T}^k \underline{v}_T\|_{0,T} + \left\| \delta_{3,T}^k \underline{v}_T \Big|_{\partial F} - r_{1,\partial F}^k \underline{v}_{T,F}^{\partial,\partial} \right\|_{0,\partial F},$$

which concludes the proof of local coercivity in the conforming case with  $d = 3$ . We omit the proof of local boundedness, which relies on the same kind of arguments.  $\square$

Well-posedness follows as an immediate consequence of Lemma 4.22, and of the fact that  $|\cdot|_h$  defines a norm on  $\mathbb{V}_{d,h,0}^k$ .

**Corollary 4.23** (Well-posedness). *For all  $\underline{v}_h \in \mathbb{R}^{n_{d,h}^k}$ , there holds*

$$|\underline{v}_h|_h^2 \lesssim a_h(\underline{v}_h, \underline{v}_h).$$

As a consequence, Problem (18) is well-posed.

Before proceeding with the convergence analysis, let us investigate the approximation capacity of the stabilization.

**Lemma 4.24** (Approximation properties of the stabilization). *Let  $v \in H^s(T)$ , for  $s \in \{\underline{s}, \dots, k+1\}$ . Then, letting  $\underline{v}_T = \overline{\Sigma}_{d,T}^k(v)$ , the following holds:*

$$s_T(\underline{v}_T, \underline{v}_T)^{1/2} \lesssim h_T^{s-1} |v|_{s,T}. \quad (37)$$

*Proof.* In the conforming case with  $d = 3$  and in the nonconforming case, we have to estimate the quantity  $\left\| h_{\partial T}^{-1/2} \pi_{\partial T}^{k-1} (\delta_{d,T}^k \underline{v}_T \Big|_{\partial T} - t_{\partial T}^k \underline{v}_T^{\partial}) \right\|_{0,\partial T}$ . In both cases, we have  $\pi_{\partial T}^{k-1}(t_{\partial T}^k \underline{v}_T^{\partial}) = \pi_{\partial T}^{k-1}(v \Big|_{\partial T})$ , and  $\pi_T^{k-1}(r_{d,T}^k \underline{v}_T) = \pi_T^{k-1}(\mathcal{I}_{d,T}^k v) = \pi_T^{k-1}v$ . Hence, by stability of  $\pi_{\partial T}^{k-1}$ , there holds

$$\begin{aligned} \left\| h_{\partial T}^{-1/2} \pi_{\partial T}^{k-1} (\delta_{d,T}^k \underline{v}_T \Big|_{\partial T} - t_{\partial T}^k \underline{v}_T^{\partial}) \right\|_{0,\partial T} &\leq \left\| h_{\partial T}^{-1/2} (\mathcal{P}_{d,T}^k v - v) \Big|_{\partial T} \right\|_{0,\partial T} \\ &\quad + \left\| h_{\partial T}^{-1/2} \pi_T^{k-1} (v - \mathcal{P}_{d,T}^k v) \Big|_{\partial T} \right\|_{0,\partial T}. \end{aligned}$$

To estimate the first term in the right-hand side, we directly apply the result (32) for  $m = 0$ . To estimate the second term, we successively use the discrete trace inequality (5), the stability of  $\pi_T^{k-1}$  in the  $L^2(T)$ -norm, and the result (31) for  $m = 0$ . The conclusion easily follows. In the conforming case with  $d = 2$ , we have to estimate the quantity

$$\left\| h_{\partial T}^{-1/2} \pi_{\partial T}^k (\delta_{2,T}^k \underline{v}_T \Big|_{\partial T} - r_{1,\partial T}^k \underline{v}_T^{\partial}) \right\|_{0,\partial T} = \left\| h_{\partial T}^{-1/2} (\delta_{2,T}^k \underline{v}_T \Big|_{\partial T} - r_{1,\partial T}^k \underline{v}_T^{\partial}) \right\|_{0,\partial T}.$$

Since  $r_{1,\partial T}^k \underline{v}_T^{\partial} = r_{2,T}^k \underline{v}_T \Big|_{\partial T}$ , and  $\pi_T^{k-1}(r_{2,T}^k \underline{v}_T) = \pi_T^{k-1}(\mathcal{I}_{2,T}^k v) = \pi_T^{k-1}v$ , there holds

$$\begin{aligned} \left\| h_{\partial T}^{-1/2} \pi_{\partial T}^k (\delta_{2,T}^k \underline{v}_T \Big|_{\partial T} - r_{1,\partial T}^k \underline{v}_T^{\partial}) \right\|_{0,\partial T} &\leq \left\| h_{\partial T}^{-1/2} (\mathcal{P}_{2,T}^k v - \mathcal{I}_{2,T}^k v) \Big|_{\partial T} \right\|_{0,\partial T} \\ &\quad + \left\| h_{\partial T}^{-1/2} \pi_T^{k-1} (v - \mathcal{P}_{2,T}^k v) \Big|_{\partial T} \right\|_{0,\partial T}. \end{aligned}$$

The second term in the right-hand side can be estimated as previously. The first one is handled recalling the remark (34). We indeed have

$$\left\| h_{\partial T}^{-1/2} (\mathcal{P}_{2,T}^k v - \mathcal{I}_{2,T}^k v) \Big|_{\partial T} \right\|_{0,\partial T} = \left\| h_{\partial T}^{-1/2} \mathcal{I}_{2,T}^k (v - \mathcal{P}_{2,T}^k v) \Big|_{\partial T} \right\|_{0,\partial T}.$$

Applying Lemma 4.3 (recall that  $\mathcal{I}_{2,T}^k (v - \mathcal{P}_{2,T}^k v) \Big|_{\partial T} = \mathcal{I}_{1,\partial T}^k ([v - \mathcal{P}_{2,T}^k v] \Big|_{\partial T})$ ), there holds

$$\left\| h_{\partial T}^{-1/2} (\mathcal{P}_{2,T}^k v - \mathcal{I}_{2,T}^k v) \Big|_{\partial T} \right\|_{0,\partial T} \lesssim \|(v - \mathcal{P}_{2,T}^k v) \Big|_{\partial T}\|_{\infty,\partial T}.$$

Since  $v \in H^2(T) \subset C^0(\bar{T})$  in that case, one has

$$\|(v - \mathcal{P}_{2,T}^k v)|_{\partial T}\|_{\infty, \partial T} \leq \|v - \mathcal{P}_{2,T}^k v\|_{\infty, T}.$$

By (8), combined to (31) with  $m = \alpha \in \{0, \dots, 2\}$ , we finally infer

$$\|h_{\partial T}^{-1/2} (\mathcal{P}_{2,T}^k v - \mathcal{I}_{2,T}^k v)|_{\partial T}\|_{0, \partial T} \lesssim \sum_{\alpha=0}^2 h_T^{\alpha-1} |v - \mathcal{P}_{2,T}^k v|_{\alpha, T} \lesssim h_T^{s-1} |v|_{s, T}.$$

We are left with estimating, in the conforming case with  $d = 3$ ,  $\|\delta_{3,T}^k \mathbb{V}_T|_{\partial F} - r_{1, \partial F}^k \mathbb{V}_{T,F}^{\partial, \partial}\|_{0, \partial F}$  for all  $F \in \mathcal{F}_T$ .

We follow the same path as previously, and end up having to estimate  $\|\mathcal{I}_{3,T}^k (v - \mathcal{P}_{3,T}^k v)|_{\partial F}\|_{0, \partial F}$ . Since

$$\mathcal{I}_{3,T}^k (v - \mathcal{P}_{3,T}^k v)|_{\partial F} = \mathcal{I}_{1, \partial F}^k ([v - \mathcal{P}_{3,T}^k v]|_{\partial F}),$$

by Lemma 4.3, the fact that  $v \in H^2(T) \subset C^0(\bar{T})$  also in that case, and (8), we infer

$$\|\mathcal{I}_{3,T}^k (v - \mathcal{P}_{3,T}^k v)|_{\partial F}\|_{0, \partial F} \lesssim h_F^{1/2} \left( h_T^{-3/2} \|v - \mathcal{P}_{3,T}^k v\|_{0, T} + h_T^{-1/2} |v - \mathcal{P}_{3,T}^k v|_{1, T} + h_T^{1/2} |v - \mathcal{P}_{3,T}^k v|_{2, T} \right).$$

The conclusion then follows from (31) with  $m \in \{0, \dots, 2\}$ .  $\square$

## 4.4 Convergence analysis

Let us begin this section with the following remark.

**Remark 4.25** (Regularity of the solution to (1)). *Since the boundary of the domain  $\Omega$  is assumed to be composed of a finite union of portions of affine hyperplanes, one can prove (see, e.g., [35, Theorem 4.4.3.7]) the following elliptic regularity result: there exists  $\varepsilon \in (0, 1/2]$  so that  $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)$  and*

$$\|u\|_{\frac{3}{2}+\varepsilon, \Omega} \lesssim \|f\|_{0, \Omega}.$$

If  $\Omega$  is in addition convex, one can actually prove full elliptic regularity, in the sense that  $u \in H^2(\Omega)$  and

$$\|u\|_{2, \Omega} \lesssim \|f\|_{0, \Omega}. \quad (38)$$

In any case, and since  $\operatorname{div}(\nabla u) = \Delta u = -f \in L^2(\Omega)$ , there holds: for all  $F \in \mathcal{F}_h^i$ ,  $[[\nabla u]]_F \cdot \mathbf{n}_F = 0$  almost everywhere on  $F$ .

Let  $|\cdot|_{e, h}^2 := \mathfrak{a}_h(\cdot, \cdot)$ . According to Corollary 4.23,  $|\cdot|_{e, h}$  defines a norm on  $\mathbb{V}_{d, h, 0}^k$ . For  $\beta := (\beta_T)_{T \in \mathcal{T}_h} \in \{2, \dots, k+1\}^{\operatorname{card}(\mathcal{T}_h)}$ , we define

$$H^\beta(\mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_h|_T \in H^{\beta_T}(T) \forall T \in \mathcal{T}_h\}.$$

**Theorem 4.26** (Energy-norm error estimate). *Assume that the solution  $u \in H_0^1(\Omega)$  to Problem (1) further belongs to  $H^\beta(\mathcal{T}_h)$ . Then, the following estimate holds:*

$$\left| \overline{\Sigma}_{d, h}^k(u) - \underline{u}_h \right|_{e, h} \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T, T}^2 \right)^{1/2}, \quad (39)$$

where  $\mathbb{R}^{n_{d, h}^k} \ni \overline{\Sigma}_{d, h}^k(u) := (\overline{\Sigma}_{d, T}^k(u|_T))_{T \in \mathcal{T}_h}$ , and  $\underline{u}_h \in \mathbb{V}_{d, h, 0}^k$  is the unique solution to Problem (18).

*Proof.* Since  $u \in H^{\frac{3}{2}+\varepsilon}(\Omega) \cap H_0^1(\Omega)$  (cf. Remark 4.25),  $u \in C^0(\bar{\Omega})$  and  $u|_{\partial\Omega} \equiv 0$ , hence  $\overline{\Sigma}_{d, h}^k(u) \in \mathbb{V}_{d, h, 0}^k$ , and so does the difference  $(\overline{\Sigma}_{d, h}^k(u) - \underline{u}_h)$ . We can then write

$$\left| \overline{\Sigma}_{d, h}^k(u) - \underline{u}_h \right|_{e, h} = \max_{\underline{v}_h \in \mathbb{V}_{d, h, 0}^k, |\underline{v}_h|_{e, h} = 1} \mathfrak{a}_h(\overline{\Sigma}_{d, h}^k(u) - \underline{u}_h, \underline{v}_h). \quad (40)$$

Since  $\underline{u}_h$  solves Problem (18), we have to estimate, for any  $\underline{v}_h \in \mathbb{V}_{d, h, 0}^k$  such that  $|\underline{v}_h|_{e, h} = 1$ ,

$$\mathcal{C}_h(\underline{v}_h) := \sum_{T \in \mathcal{T}_h} \mathcal{C}_T(\underline{v}_T), \quad \text{with } \mathcal{C}_T(\underline{v}_T) := \mathfrak{a}_T(\overline{\Sigma}_{d, T}^k(u|_T), \underline{v}_T) - \int_T f|_T \mathbb{V}_T^0.$$

Using the strong form of Problem (1), and integrating by parts, we infer, for any  $T \in \mathcal{T}_h$ ,

$$\mathcal{C}_T(\underline{v}_T) = \int_T \nabla \mathcal{P}_{d,T}^k(u|_T) \cdot \nabla p_{d,T}^k \underline{v}_T - \int_T \nabla u \cdot \nabla v_T^o + \int_{\partial T} v_T^o|_{\partial T} \nabla u|_{\partial T} \cdot \mathbf{n}_T + s_T(\overline{\Sigma}_{d,T}^k(u|_T), \underline{v}_T).$$

Using the definition (29), and since  $\mathcal{P}_{d,T}^k(u|_T) \in \mathbb{P}_d^k(T)$ , we then have

$$\begin{aligned} \mathcal{C}_T(\underline{v}_T) &= \int_T \nabla (\mathcal{P}_{d,T}^k(u|_T) - u) \cdot \nabla v_T^o - \int_{\partial T} \pi_{\partial T}^l (v_T^o|_{\partial T} - t_{\partial T}^k \underline{v}_T^{\hat{c}}) \nabla \mathcal{P}_{d,T}^k(u|_T)|_{\partial T} \cdot \mathbf{n}_T \\ &\quad + \int_{\partial T} \pi_{\partial T}^l (v_T^o|_{\partial T}) \nabla u|_{\partial T} \cdot \mathbf{n}_T + s_T(\overline{\Sigma}_{d,T}^k(u|_T), \underline{v}_T). \end{aligned}$$

Summing over  $T \in \mathcal{T}_h$ , and invoking the continuity of the flux of the exact solution along interfaces (cf. Remark 4.25), combined to the fact that  $\underline{v}_h \in \mathbb{V}_{d,h,0}^k$ , we infer

$$\begin{aligned} \mathcal{C}_h(\underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \left( \int_T \nabla (\mathcal{P}_{d,T}^k(u|_T) - u) \cdot \nabla v_T^o \right. \\ &\quad \left. + \int_{\partial T} \pi_{\partial T}^l (v_T^o|_{\partial T} - t_{\partial T}^k \underline{v}_T^{\hat{c}}) \nabla (u - \mathcal{P}_{d,T}^k(u|_T))|_{\partial T} \cdot \mathbf{n}_T + s_T(\overline{\Sigma}_{d,T}^k(u|_T), \underline{v}_T) \right). \quad (41) \end{aligned}$$

Applying Cauchy–Schwarz inequality, and the approximation results (31)–(32) with  $s \in \{\beta_T\}_{T \in \mathcal{T}_h}$  and  $m = 1$ , we obtain

$$\mathcal{C}_h(\underline{v}_h) \lesssim \left( \sum_{T \in \mathcal{T}_h} \left( h_T^{2(\beta_T-1)} |u|_{\beta_T, T}^2 + s_T(\overline{\Sigma}_{d,T}^k(u|_T), \overline{\Sigma}_{d,T}^k(u|_T)) \right) \right)^{1/2} \left( |\underline{v}_h|_h + |\underline{v}_h|_{e,h} \right).$$

The conclusion then follows from Lemma 4.24 with  $s \in \{\beta_T\}_{T \in \mathcal{T}_h}$ , Corollary 4.23, and the fact that  $|\underline{v}_h|_{e,h} = 1$  by assumption.  $\square$

**Remark 4.27.** *In the nonconforming case, to prove that  $\overline{\Sigma}_{d,h}^k(u)$  belongs to  $\mathbb{V}_{d,h,0}^k$ , it is sufficient to use that  $u \in H_0^1(\Omega)$ . Furthermore, the first term in the right-hand side of (41) is identically zero. Indeed, in that case, for all  $T \in \mathcal{T}_h$ ,  $\mathcal{P}_{d,T}^k = \Pi_T^k$  (recall that  $v_T^o \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$ ). Such a property is not true in the conforming case.*

**Corollary 4.28** ( $H^1(\mathcal{T}_h)$ -seminorm error estimate). *Under the regularity assumption of Theorem 4.26, the following estimate holds:*

$$\|\nabla_h(u - p_{d,h}^k \underline{u}_h)\|_{0,\Omega} \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T, T}^2 \right)^{1/2}, \quad (42)$$

where  $p_{d,h}^k : \mathbb{R}^{n_{d,h}} \rightarrow \mathbb{P}_d^k(\mathcal{T}_h)$  is such that, for all  $T \in \mathcal{T}_h$ ,  $p_{d,h}^k \underline{v}_h|_T := p_{d,T}^k \underline{v}_T$ .

*Proof.* For any  $T \in \mathcal{T}_h$ , by a simple triangle inequality, we infer

$$\|\nabla(u - p_{d,T}^k \underline{u}_T)\|_{0,T} \leq \|\nabla(u - \mathcal{P}_{d,T}^k(u|_T))\|_{0,T} + \|\nabla p_{d,T}^k(\overline{\Sigma}_{d,T}^k(u|_T) - \underline{u}_T)\|_{0,T}.$$

Squaring, summing over  $T \in \mathcal{T}_h$ , and using the definition of  $|\cdot|_{e,h}$ , yields

$$\|\nabla_h(u - p_{d,h}^k \underline{u}_h)\|_{0,\Omega}^2 \leq 2 \left( \sum_{T \in \mathcal{T}_h} \|\nabla(u - \mathcal{P}_{d,T}^k(u|_T))\|_{0,T}^2 + \left| \overline{\Sigma}_{d,h}^k(u) - \underline{u}_h \right|_{e,h}^2 \right). \quad (43)$$

The conclusion follows from the approximation result (31) with  $s \in \{\beta_T\}_{T \in \mathcal{T}_h}$  and  $m = 1$ , and from the energy-norm error estimate (39).  $\square$

The proof of Theorem 4.26–Corollary 4.28 is inspired from the one for HHO methods, but is here valid whatever the conformity of the underlying global virtual space. It simplifies the classical analysis of VE methods in broken  $H^1$ -seminorm, by a direct splitting of the error (see (43)) into (i) an approximation error (only depending on the properties of the local polynomial projector  $\mathcal{P}_{d,T}^k$ ,  $T \in \mathcal{T}_h$ ), and (ii) the consistency error of the scheme (see (40)).

**Remark 4.29.** Letting  $u_h \in V_{d,h,0}^k$  (respectively,  $\mathcal{I}_{d,h}^k u \in V_{d,h,0}^k$ ) such that  $u_h|_T := r_{d,T}^k \underline{u}_T$  (respectively,  $\mathcal{I}_{d,h}^k u|_T := \mathcal{I}_{d,T}^k(u|_T) = r_{d,T}^k(\overline{\Sigma}_{d,T}^k(u|_T))$ ) for all  $T \in \mathcal{T}_h$ , we have, by Remark 4.20 and Corollary 4.23,

$$\|\nabla_h(\mathcal{I}_{d,h}^k u - u_h)\|_{0,\Omega} \lesssim \left| \overline{\Sigma}_{d,h}^k(u) - \underline{u}_h \right|_{e,h}. \quad (44)$$

Since, by the triangle inequality, and letting  $\mathcal{P}_{d,h}^k := \Pi_h^k \circ \mathcal{I}_{d,h}^k$ ,

$$\|\nabla_h(u - u_h)\|_{0,\Omega} \leq \|\nabla_h(u - \mathcal{P}_{d,h}^k u)\|_{0,\Omega} + \|\nabla_h(\mathcal{P}_{d,h}^k u - \mathcal{I}_{d,h}^k u)\|_{0,\Omega} + \|\nabla_h(\mathcal{I}_{d,h}^k u - u_h)\|_{0,\Omega},$$

we can prove, using (31) with  $s \in \{\beta_T\}_{T \in \mathcal{T}_h}$  and  $m = 1$  for the first term in the right-hand side, the remark (34) combined to (28) and (31) for the second, and (44) combined to (39) for the third, that, under the same regularity assumption as in Theorem 4.26, there holds

$$\|\nabla_h(u - u_h)\|_{0,\Omega} \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T,T}^2 \right)^{1/2}. \quad (45)$$

Obviously, the discrete solution  $u_h$  is not computable/computed in practice, only the polynomial projection  $p_{d,h}^k \underline{u}_h = \Pi_h^k u_h$  is. Hence, the estimate (45) only has a limited interest.

Let us now derive an estimate on the error between  $p_{d,h}^k \underline{u}_h$  and  $u$  in the  $L^2(\Omega)$ -norm. In this weaker norm, the analysis has to distinguish between the conforming and nonconforming cases. The reason for that is because the orthogonality property of the local polynomial projector plays a crucial role in the analysis. Yet, we propose a factorized analysis, that enables to show supercloseness of cell unknowns. For a different proposition of analysis, we refer to [26, Theorem 22].

**Theorem 4.30** (Potential-norm error estimate). *Assume that the solution  $u \in H_0^1(\Omega)$  to Problem (1) further belongs to  $H^\beta(\mathcal{T}_h)$  and that, when  $k = 1$ , one also has  $f \in H^1(\mathcal{T}_h)$ . Assume full elliptic regularity for Problem (1) (cf. Remark 4.25). Then, there holds*

$$\|u - u_h\|_{0,\Omega} \lesssim h \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} (|u|_{\beta_T,T}^2 + \delta_{1k} |f|_{1,T}^2) \right)^{1/2}. \quad (46)$$

*Proof.* We follow the standard Aubin–Nitsche argument. Setting  $g := (u - u_h) \in L^2(\Omega)$ , we let  $z \in H_0^1(\Omega)$  be the unique solution in  $H^1(\Omega)$  to  $-\Delta z = g$  in  $\Omega$ , with  $z = 0$  on  $\partial\Omega$ . By (full) elliptic regularity,  $z \in H^2(\Omega)$  and, in view of (38), there holds

$$\|z\|_{2,\Omega} \lesssim \|u - u_h\|_{0,\Omega}. \quad (47)$$

Writing

$$\|u - u_h\|_{0,\Omega}^2 = \int_{\Omega} (u - u_h)g = - \int_{\Omega} (u - u_h)\Delta z, \quad (48)$$

we distinguish between the conforming and nonconforming cases. In the conforming case, using the fact that  $(u - u_h) \in H_0^1(\Omega)$ , we infer by integration by parts:

$$\|u - u_h\|_{0,\Omega}^2 = \int_{\Omega} \nabla(u - u_h) \cdot \nabla z.$$

Then, we add/subtract  $\mathcal{P}_{d,h}^k z$  to  $z$  to obtain

$$\|u - u_h\|_{0,\Omega}^2 = \int_{\Omega} \nabla(u - u_h) \cdot \nabla(z - \mathcal{P}_{d,h}^k z) + \int_{\Omega} (\nabla_h \Pi_h^k u \cdot \nabla \mathcal{I}_{d,h}^k z - \nabla_h \Pi_h^k u_h \cdot \nabla \mathcal{P}_{d,h}^k z),$$

which, in turn, rewrites

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 = \int_{\Omega} \nabla(u - u_h) \cdot \nabla(z - \mathcal{P}_{d,h}^k z) &+ \left( \int_{\Omega} \nabla_h \Pi_h^k u \cdot \nabla \mathcal{I}_{d,h}^k z - \mathfrak{a}_h(\underline{u}_h, \overline{\Sigma}_{d,h}^k(z)) \right) \\ &+ \sum_{T \in \mathcal{T}_h} s_T(\underline{u}_T, \overline{\Sigma}_{d,T}^k(z|_T)) =: \mathfrak{I}_1^c + \mathfrak{I}_2^c + \mathfrak{I}_3^c. \end{aligned} \quad (49)$$



In the nonconforming case now, we add/subtract  $\pi_h^{k-1}(u - u_h)$  to  $(u - u_h)$  in (48) and we use that  $\pi_T^{k-1}(u_h|_T) = u_T^o$  for any  $T \in \mathcal{T}_h$ , to infer

$$\|u - u_h\|_{0,\Omega}^2 = - \int_{\Omega} ((u - u_h) - \pi_h^{k-1}(u - u_h)) \Delta z - \sum_{T \in \mathcal{T}_h} \int_T (\pi_T^{k-1}(u|_T) - u_T^o) \Delta z. \quad (50)$$

By integration by parts, and since  $[\![\nabla z]\!]_F \cdot \mathbf{n}_F = 0$  almost everywhere on interfaces  $F \in \mathcal{F}_h^i$  (cf. Remark 4.25), and  $u \in H_0^1(\Omega)$  (hence  $[\![u]\!]_F = 0$  a.e. on all  $F \in \mathcal{F}_h$ ) as well as  $\underline{u}_h \in \mathbb{V}_{d,h,0}^k$ , there holds

$$\begin{aligned} - \sum_{T \in \mathcal{T}_h} \int_T (\pi_T^{k-1}(u|_T) - u_T^o) \Delta z &= \sum_{T \in \mathcal{T}_h} \left( \int_T \nabla (\pi_T^{k-1}(u|_T) - u_T^o) \cdot \nabla z \right. \\ &\quad \left. - \int_{\partial T} [(\pi_T^{k-1}(u|_T) - u_T^o)|_{\partial T} - (\pi_{\partial T}^{k-1}(u|_{\partial T}) - u_{\partial T}^o)] \nabla z|_{\partial T} \cdot \mathbf{n}_T \right). \end{aligned}$$

Adding/subtracting the term  $\sum_{T \in \mathcal{T}_h} \int_T \nabla \mathcal{P}_{d,T}^k(\bar{\Sigma}_{d,T}^k(u|_T) - \underline{u}_T) \cdot \nabla \mathcal{P}_{d,T}^k(z|_T)$  to (50), using the definition (29), and letting

$$\mathfrak{I}_{1,1}^{nc} := - \int_{\Omega} ((u - u_h) - \pi_h^{k-1}(u - u_h)) \Delta z,$$

as well as

$$\begin{aligned} \mathfrak{I}_{1,2}^{nc} &:= \sum_{T \in \mathcal{T}_h} \left( \int_T \nabla (\pi_T^{k-1}(u|_T) - u_T^o) \cdot \nabla (z - \mathcal{P}_{d,T}^k(z|_T)) \right. \\ &\quad \left. - \int_{\partial T} [(\pi_T^{k-1}(u|_T) - u_T^o)|_{\partial T} - (\pi_{\partial T}^{k-1}(u|_{\partial T}) - u_{\partial T}^o)] \nabla (z - \mathcal{P}_{d,T}^k(z|_T))|_{\partial T} \cdot \mathbf{n}_T \right), \end{aligned}$$

we hence infer that

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= \mathfrak{I}_{1,1}^{nc} + \mathfrak{I}_{1,2}^{nc} + \left( \int_{\Omega} \nabla_h \mathcal{P}_{d,h}^k u \cdot \nabla_h \mathcal{P}_{d,h}^k z - \mathfrak{a}_h(\underline{u}_h, \bar{\Sigma}_{d,h}^k(z)) \right) \\ &\quad + \sum_{T \in \mathcal{T}_h} s_T(\underline{u}_T, \bar{\Sigma}_{d,T}^k(z|_T)) =: \mathfrak{I}_{1,1}^{nc} + \mathfrak{I}_{1,2}^{nc} + \mathfrak{I}_2^{nc} + \mathfrak{I}_3^{nc}. \quad (51) \end{aligned}$$

We now have to estimate the different terms in the right-hand sides of (49), and of (51). To estimate  $\mathfrak{I}_1^c$ , we apply successively Cauchy–Schwarz inequality, (45), and (31) with  $s = 2$  and  $m = 1$ , combined to (47). To estimate  $\mathfrak{I}_{1,1}^{nc}$ , we apply successively Cauchy–Schwarz inequality, (11) with  $s = 1$  and  $m = 0$ , (45), the fact that  $\|\Delta z\|_{0,\Omega} \leq |z|_{2,\Omega}$ , and (47). In both cases, we obtain

$$\mathfrak{I}_1^c, \mathfrak{I}_{1,1}^{nc} \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T,T}^2 \right)^{1/2} h \|u - u_h\|_{0,\Omega}. \quad (52)$$

To estimate  $\mathfrak{I}_{1,2}^{nc}$ , we apply successively the Cauchy–Schwarz inequality, (31)–(32) with  $s = 2$  and  $m = 1$ , and Corollary 4.23. We get

$$\mathfrak{I}_{1,2}^{nc} \lesssim \left| \bar{\Sigma}_{d,h}^k(u) - \underline{u}_h \right|_{e,h} h |z|_{2,\Omega},$$

which in turn, by (39) and (47), gives

$$\mathfrak{I}_{1,2}^{nc} \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T,T}^2 \right)^{1/2} h \|u - u_h\|_{0,\Omega}. \quad (53)$$

To estimate  $\mathfrak{I}_3^c$  and  $\mathfrak{I}_3^{nc}$ , we use the fact that

$$\sum_{T \in \mathcal{T}_h} s_T(\underline{u}_T, \bar{\Sigma}_{d,T}^k(z|_T)) = \sum_{T \in \mathcal{T}_h} s_T(\bar{\Sigma}_{d,T}^k(u|_T), \bar{\Sigma}_{d,T}^k(z|_T)) - \sum_{T \in \mathcal{T}_h} s_T(\bar{\Sigma}_{d,T}^k(u|_T) - \underline{u}_T, \bar{\Sigma}_{d,T}^k(z|_T)),$$

to infer, using successively the Cauchy–Schwarz inequality, (37) with  $s \in \{\beta_T\}_{T \in \mathcal{T}_h}$  for  $u|_T$ , (39), (37) with  $s = 2$  for  $z|_T$ , and (47),

$$\mathfrak{I}_3^c, \mathfrak{I}_3^{nc} \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T, T}^2 \right)^{1/2} h \|u - u_h\|_{0, \Omega}. \quad (54)$$

We are left with estimating  $\mathfrak{I}_2^c$  and  $\mathfrak{I}_2^{nc}$ . Since  $\underline{u}_h \in \mathbb{V}_{d,h,0}^k$  solves Problem (18), adding/subtracting  $\int_{\Omega} \nabla u \cdot \nabla \mathcal{I}_{d,h}^k z = \int_{\Omega} f \mathcal{I}_{d,h}^k z$  in the conforming case (remark that  $\mathcal{I}_{d,h}^k z \in H_0^1(\Omega)$  in that case), and  $\int_{\Omega} \nabla u \cdot \nabla z = \int_{\Omega} f z$  in the nonconforming case, there holds

$$\mathfrak{I}_2^c = \int_{\Omega} \nabla_h(\Pi_h^k u - u) \cdot \nabla_h(\mathcal{I}_{d,h}^k z - \mathcal{P}_{d,h}^k z) + \sum_{T \in \mathcal{T}_h} \int_T f|_T (\mathcal{I}_{d,T}^k(z|_T) - \pi_T^{k-1}(z|_T)), \quad (55)$$

and (recall that  $\mathcal{P}_{d,T}^k = \Pi_T^k$  in the nonconforming case)

$$\mathfrak{I}_2^{nc} = \int_{\Omega} \nabla_h(\Pi_h^k u - u) \cdot \nabla_h(z - \Pi_h^k z) + \sum_{T \in \mathcal{T}_h} \int_T f|_T (z - \pi_T^{k-1}(z|_T)). \quad (56)$$

The first term in the right-hand side of (55) is estimated by means of (i) Cauchy–Schwarz inequality, (ii) (11) with  $s \in \{\beta_T\}_{T \in \mathcal{T}_h}$  and  $m = 1$ , (iii) the remark (34) combined to (28) and to (31) with  $s = 2$  and  $m \in \{1, 2\}$ , and (iv) (47). The first term in the right-hand side of (56) is estimated using (i) the Cauchy–Schwarz inequality, (ii) (11) with  $m = 1$ , and (iii) (47). It remains to estimate the second term in the right-hand side of (55) (simpler arguments apply to (56)). When  $k = 1$ , there holds

$$\sum_{T \in \mathcal{T}_h} \int_T f|_T (\mathcal{I}_{d,T}^1(z|_T) - \pi_T^0(z|_T)) = \sum_{T \in \mathcal{T}_h} \int_T (f|_T - \pi_T^0(f|_T)) \mathcal{I}_{d,T}^1(z|_T - \pi_T^0(z|_T)).$$

Then, by Cauchy–Schwarz inequality, (28), and (11), we infer

$$\sum_{T \in \mathcal{T}_h} \int_T f|_T (\mathcal{I}_{d,T}^1(z|_T) - \pi_T^0(z|_T)) \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^2 |f|_{1,T}^2 \right)^{1/2} h (|z|_{1,\Omega} + h|z|_{2,\Omega}). \quad (57)$$

When  $k \geq 2$  now, there holds

$$\sum_{T \in \mathcal{T}_h} \int_T f|_T (\mathcal{I}_{d,T}^k(z|_T) - \pi_T^{k-1}(z|_T)) = \sum_{T \in \mathcal{T}_h} \int_T (f|_T - \pi_T^{\beta_T-3}(f|_T)) \mathcal{I}_{d,T}^k(z|_T - \pi_T^{k-1}(z|_T)).$$

By the same kind of arguments as in the case  $k = 1$ , we infer

$$\sum_{T \in \mathcal{T}_h} \int_T f|_T (\mathcal{I}_{d,T}^k(z|_T) - \pi_T^{k-1}(z|_T)) \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T, T}^2 \right)^{1/2} h |z|_{2,\Omega}. \quad (58)$$

Collecting (55), (57)–(58), and using (47), we obtain

$$\mathfrak{I}_2^c, \mathfrak{I}_2^{nc} \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} (|u|_{\beta_T, T}^2 + \delta_{1k} |f|_{1,T}^2) \right)^{1/2} h \|u - u_h\|_{0, \Omega}. \quad (59)$$

Collecting (49), (51), (52), (53), (59), and (54), we conclude the proof.  $\square$

**Corollary 4.31** ( $L^2(\Omega)$ -norm error estimate). *Under the assumptions of Theorem 4.30, the following estimate holds:*

$$\|u - p_{d,h}^k \underline{u}_h\|_{0, \Omega} \lesssim h \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} (|u|_{\beta_T, T}^2 + \delta_{1k} |f|_{1,T}^2) \right)^{1/2}. \quad (60)$$

*Proof.* By the triangle inequality, and since  $p_{d,h}^k \mathfrak{u}_h = \Pi_h^k u_h$ , we infer

$$\|u - p_{d,h}^k \mathfrak{u}_h\|_{0,\Omega} \leq \|u - \Pi_h^k u\|_{0,\Omega} + \|\Pi_h^k(u - u_h)\|_{0,\Omega}.$$

By the stability property of  $\Pi_T^k$  for any  $T \in \mathcal{T}_h$  (cf. Proposition 2.4), there holds

$$\|u - p_{d,h}^k \mathfrak{u}_h\|_{0,\Omega} \leq \|u - \Pi_h^k u\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} + h \|\nabla_h(u - u_h)\|_{0,\Omega}.$$

The conclusion then follows from (11) with  $s \in \{\beta_T\}_{T \in \mathcal{T}_h}$  and  $m = 0$ , (46), and (45).  $\square$

**Remark 4.32** (Supercloseness of cell unknowns). *Noticing that  $\pi_T^{k-1}([u - u_h]_T) = \pi_T^{k-1}(u|_T) - u_T^\circ$  for any  $T \in \mathcal{T}_h$ , there holds, by stability of  $\pi_T^{k-1}$  in the  $L^2(T)$ -norm, by (46), and under the same assumptions as in Theorem 4.30,*

$$\|\pi_h^{k-1} u - u_h^\circ\|_{0,\Omega} \lesssim h \left( \sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T-1)} (|u|_{\beta_T, T}^2 + \delta_{1k} |f|_{1,T}^2) \right)^{1/2},$$

where  $u_h^\circ \in \mathbb{P}_d^{k-1}(\mathcal{T}_h)$  is such that  $u_h^\circ|_T := u_T^\circ$  for all  $T \in \mathcal{T}_h$ . Hence, the cell unknowns are superclose to the projections of degree  $k-1$  of  $u$ .

## 4.5 General workflow

As for any Galerkin method, the general workflow for solving Problem (18) with the c/nc-VE/HHO methods splits into (i) an offline stage, that is independent of the source term and of the boundary conditions, and which aims at performing the assembly of the general problem matrix, and (ii) an online stage, that consists, for given source term and boundary conditions, in solving the resulting global system. A change in the data only affects the online stage (cf. Remark 4.33). The precomputations that are performed in the offline stage are all local; hence, this stage can naturally benefit from parallel architectures.

Let us describe, in details, these two stages for Problem (18), beginning with the offline stage.

1. In the conforming case with  $d = 2$  (respectively,  $d = 3$ ), one first computes the operator  $r_{1,F}^k$  defined by (20) for all  $F \in \mathcal{F}_h$  (respectively,  $r_{1,e}^k$  for all  $e \in \mathcal{E}_h$ ). This requires to solve a SPD system of size  $N_1^k$  (cf. Remark 2.6), with  $N_1^{k-2} + 2$  right-hand sides. In both the conforming and nonconforming cases now, locally to any  $T \in \mathcal{T}_h$ , one computes the operator  $p_{d,T}^k$  defined by (29). Its computation requires to solve a SPD system of size  $N_d^k$  (cf. also Remark 2.6), for a number of right-hand sides that is (i)  $N_2^{k-1} + \text{card}(\mathcal{F}_T) \times N_1^{k-2} + \text{card}(\mathcal{V}_T)$  in the conforming case with  $d = 2$ , (ii)  $N_3^{k-1} + \text{card}(\mathcal{F}_T) \times N_2^{k-1}$  in the conforming case with  $d = 3$ , and (iii)  $N_d^{k-1} + \text{card}(\mathcal{F}_T) \times N_{d-1}^{k-1}$  in the nonconforming case (cf. Remark 4.19). Once the operator  $p_{d,T}^k$  has been computed, one computes, still locally to any  $T \in \mathcal{T}_h$ , the stabilization form(s), that write(s) in terms of the DoFs and of the different already computed quantities.
2. As common to any skeletal method, cell DoFs are locally eliminated by static condensation in terms of the local skeletal DoFs. Locally to each  $T \in \mathcal{T}_h$ , one has to invert a SPD matrix of size  $N_d^{k-1}$ .

Let us now describe the online stage, for a given source term  $f \in L^2(\Omega)$ .

3. One diagonalizes the lines/columns of the general problem matrix corresponding to the boundary (skeletal) DoFs, and computes the right-hand side (with strong imposition of the zero Dirichlet boundary condition), which requires to integrate, locally to any  $T \in \mathcal{T}_h$ ,  $f|_T$  against polynomials in  $\mathbb{P}_d^{k-1}(T)$ .
4. One solves the resulting SPD global system, that is of size

$$\text{card}(\mathcal{F}_h) \times N_1^{k-2} + \text{card}(\mathcal{V}_h)$$

in the conforming case with  $d = 2$ ,

$$\text{card}(\mathcal{F}_h) \times N_2^{k-1} + \text{card}(\mathcal{E}_h) \times N_1^{k-2} + \text{card}(\mathcal{V}_h)$$

in the conforming case with  $d = 3$  (cf. Remark 4.34), and

$$\text{card}(\mathcal{F}_h) \times N_{d-1}^{k-1}$$

in the nonconforming case.

For an example of implementation of the method using generic programming, we refer to [18].

**Remark 4.33.** *Except point 3, the description above of the general workflow of the method applies verbatim to the case of a Problem (1) featuring a source term  $f \in H^{-1}(\Omega)$  (assuming that a decomposition  $f = f_0 + \operatorname{div} \mathbf{f}$  with  $f_0 \in L^2(\Omega)$  and  $\mathbf{f} \in L^2(\Omega)^d$  is known), or nonhomogeneous mixed Dirichlet–Neumann boundary conditions.*

**Remark 4.34.** *In the conforming case with  $d = 3$ , one can reduce the size of the global system by using enhanced virtual spaces [14] or Serendipity spaces [7] on the faces  $F \in \mathcal{F}_T$  of the cells  $T \in \mathcal{T}_h$ , instead of  $V_2^k(F)$ .*

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## A Proofs of (9) and Lemma 3.2

### A.1 Proof of (9)

To prove (9), we first recall (cf., e.g., [12, Section 2.7]) that, since  $T$  is star-shaped with respect to a ball whose radius is comparable to  $h_T$ , for any  $z \in H^1(\partial T)$ , there exists  $\bar{z} \in H^1(T)$  such that  $\bar{z}|_{\partial T} = z$ , and

$$h_T^{-1} \|\bar{z}\|_{0,T} + |\bar{z}|_{1,T} \lesssim h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}. \quad (61)$$

Now, by the divergence formula, there holds, for any  $z \in H^1(\partial T) \subset H^{1/2}(\partial T)$ ,

$$\langle \nabla v|_{\partial T} \cdot \mathbf{n}_T, z \rangle_{-\frac{1}{2},\partial T} = \int_T \nabla v \cdot \nabla \bar{z} + \int_T \Delta v \bar{z},$$

which yields, by Cauchy–Schwarz inequality,

$$\langle \nabla v|_{\partial T} \cdot \mathbf{n}_T, z \rangle_{-\frac{1}{2},\partial T} \leq |v|_{1,T} |\bar{z}|_{1,T} + h_T \|\Delta v\|_{0,T} h_T^{-1} \|\bar{z}\|_{0,T}.$$

The estimate (61) enables to conclude.

### A.2 Proof of Lemma 3.2

Assume that, for  $v_h \in V_{h,0}$ ,  $\|\nabla_h v_h\|_{0,\Omega} = 0$ . Then, for all  $T \in \mathcal{T}_h$ ,  $\nabla(v_h|_T) = \mathbf{0}$  and there is  $c_T \in \mathbb{R}$  such that  $v_h|_T = c_T$ . Since  $\{\tilde{\pi}_F^0\}_{F \in \mathcal{F}_h} \subseteq \Sigma_h^0$ , then for all  $F \in \mathcal{F}_h^i$ ,  $\int_F \llbracket v_h \rrbracket_F = 0$  and there exists  $c \in \mathbb{R}$  such that  $c_T = c$  for all  $T \in \mathcal{T}_h$ . The fact that  $\int_F v_h|_F = 0$  for some (here, all)  $F \in \mathcal{F}_h^b$  finally yields that  $v_h \equiv 0$  on  $\Omega$ . To prove (16), we start from the following discrete Poincaré inequality on  $H^1(\mathcal{T}_h)$  (cf., e.g., [9]):

$$\forall v_h \in H^1(\mathcal{T}_h), \quad \|v_h\|_{0,\Omega}^2 \lesssim \|\nabla_h v_h\|_{0,\Omega}^2 + \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\llbracket v_h \rrbracket_F\|_{0,F}^2,$$

and we show that  $\sum_{F \in \mathcal{F}_h} h_F^{-1} \|\llbracket v_h \rrbracket_F\|_{0,F}^2 \lesssim \|\nabla_h v_h\|_{0,\Omega}^2$  for all  $v_h \in V_{h,0}$ . To prove so, since for  $v_h \in V_{h,0}$ ,  $\int_F \llbracket v_h \rrbracket_F = 0$  for all  $F \in \mathcal{F}_h$ , there holds

$$\|\llbracket v_h \rrbracket_F\|_{0,F}^2 = \int_F \llbracket v_h - \pi_h^0 v_h \rrbracket_F \llbracket v_h \rrbracket_F \leq \|\llbracket v_h - \pi_h^0 v_h \rrbracket_F\|_{0,F} \|\llbracket v_h \rrbracket_F\|_{0,F},$$

and we can use the continuous trace inequality (4) and the Poincaré inequality (6) to infer

$$\begin{aligned} h_F^{-\frac{1}{2}} \|\llbracket v_h \rrbracket_F\|_{0,F} &\lesssim h_F^{-\frac{1}{2}} \sum_{T \in \mathcal{T}_F} (h_T^{-\frac{1}{2}} \|v_h|_T - \pi_T^0(v_h|_T)\|_{0,T} + h_T^{\frac{1}{2}} \|\nabla(v_h|_T)\|_{0,T}) \\ &\lesssim h_F^{-\frac{1}{2}} \sum_{T \in \mathcal{T}_F} h_T^{\frac{1}{2}} \|\nabla(v_h|_T)\|_{0,T}. \end{aligned}$$

Finally, since  $h_F$  is comparable to  $h_T$  for  $T \in \mathcal{T}_F$ , and  $\operatorname{card}(\mathcal{F}_T) \lesssim 1$  for all  $T \in \mathcal{T}_h$  (cf. Section 2.2), the conclusion follows.

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