Bridging the Hybrid High-Order and Virtual Element methods

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Simon Lemaire*[1]

[1]Inria Lille - Nord Europe (RAPSODI project-team), 40 avenue Halley, 59 650 Villeneuve d'Ascq, France

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Abstract

We build a bridge between the Hybrid High-Order and the Virtual Element methods on general polytopal meshes in dimension 2 or 3, in the setting of a model Poisson problem. To do so, we reformulate the conforming-Virtual Element method as a (newborn) *conforming-Hybrid High-Order* method. This allows us to perform a unified study of conforming/nonconforming-Virtual Element/Hybrid High-Order methods, which happens to shed new light on the analysis of Virtual Element methods.

1 Introduction

The design of arbitrary-order Galerkin methods that support meshes with general polygonal/polyhedral cells, has been attracting the attention of the community for more than 40 years now. In practice, the use of general meshes, when not an inherent constraint like e.g. in subsurface modeling, can bring major advantages. In particular, it increases the flexibility in meshing complex geometries, and simplifies the refinement/coarsening procedures in adaptive simulations. Classical arbitrary-order polytopal discretization approaches encompass the (polytopal) Finite Element (FE) method [36, 34], and the (polytopal) discontinuous Galerkin (dG) method [1, 25, 3, 13]. The construction of FE shape functions on arbitrarily-shaped cells that both (i) satisfy the desired conformity prescriptions, and (ii) for which closed-form expressions can be obtained (and numerically integrated), is highly challenging. However, when the shape functions are available, one can fully benefit from the fact that FE are a skeletal method, i.e. cell degrees of freedom can be locally eliminated in terms of the skeletal degrees of freedom, thus reducing the number of globally coupled unknowns. The dG method, for which stability is enforced by penalization of the discrete bilinear form, can handle completely nonconforming discrete spaces. One hence has the opportunity to consider simple polynomial local approximation spaces. The price to pay for such a flexibility is, naturally, an increased number of globally coupled degrees of freedom, which makes of dG a computationally more expensive method than FE on standard meshes (and this is all the more true that the order of approximation increases). When considering meshes featuring cells with an important number of faces, things are not that clear anymore, and dG may definitely become a competitive computational approach. Yet, in that case, (polynomial-based) dG suffers from an important limitation. On general cells, it is not clear at all whether the polynomial local approximation spaces are sufficiently rich to robustly approximate tricky operators like the divergence (think, e.g., of a linear elasticity model in the quasi-incompressible limit) or curl operators.

More recently, a new paradigm has emerged. The idea is to define a finite element whose construction is generic with respect to the shape of the element. The underlying local approximation space (i) is spanned by functions that are implicitly defined (usually as the solutions to some PDEs posed in the cell), (ii) is built so that the desired conformity properties can be obtained at the global level, and (iii) is constructed so as to enjoy sufficient approximation properties (for instance, so as to contain the polynomial functions up to a given degree). The fact that one cannot obtain a closed-form expression for the shape functions is the reason why they are called virtual in that context. In practice, the numerical method is defined using computable (in terms of the degrees of freedom) projections of the virtual functions, and is stabilized through a subtle penalization, that shall also be computable. The most salient example of such an approach is the (polynomial-based) Virtual Element (VE) method [5, 6], which has first been introduced under its conforming version (denoted c-VE). Another example is the Hybrid High-Order (HHO) method [28], first introduced/analyzed for linear elasticity [26], and then for the Poisson problem [27]. Both the VE and the

*simon.lemaire@inria.fr

HHO methods are skeletal methods. In [19, Section 2.4], the HHO method has been proved equivalent (up to identical stabilization and right-hand side) to the nonconforming version of the VE method (denoted nc-VE) introduced in [32], and posteriorly analyzed in [2]. In [19], the HHO method has also been bridged to the so-called Hybridizable Discontinuous Galerkin (HDG) method [21, 18], in the sense that it is possible to recast the HHO method as a HDG method, with distinctive numerical flux trace. This work has shed light on the fact that the quite subtle choice of stabilization advocated in HHO, results in HDG formulation in a numerical flux trace that ensures superconvergence on general polytopal meshes. Note that efforts towards superconvergence for standard HDG methods are still being undertaken [20].

Let Ω be a bounded and connected open subset of $\mathbb{R}^d$, $d \in \{2,3\}$, whose boundary is assumed to be composed of a finite union of portions of affine hyperplanes. We focus on the following model Poisson problem: find $u \in H^1_0(\Omega)$ solution to

$$a(u,v) := \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = \int_\Omega f\, v =: l(v) \qquad \text{for all } v \in H^1_0(\Omega), \tag{1}$$

with source term $f \in L^2(\Omega)$. In this work, by reformulating the c-VE method as a (newborn) *conforming-HHO* method, we complete the construction of the bridge between HHO and VE undertaken in [19, Section 2.4]. Building upon existing contributions, especially [24] on the analysis of schemes in fully discrete formulation, [14] on the unified analysis of c/nc-VE methods, and [33, 10, 15] (see also [8, 12] for the treatment of faces of arbitrarily small measure) on the analysis of c-VE, we introduce a unifying framework, taking inspiration from the HHO framework, in which we perform a common study of c/nc-VE/HHO methods. Doing so, we shed new light on the similarities/differences between conforming and nonconforming VE/HHO methods, and we introduce substantial simplifications in the analysis of VE methods. For simplicity of exposition, we consider throughout this paper standard VE spaces (i.e., neither enhanced VE spaces [14], nor Serendipity VE spaces [7]), with cell degrees of freedom even in the lowest-order case, which simplifies the treatment of the right-hand side without compromising the computational efficiency.

A crucial observation that is made clear in this article is the following. The main difference between the conforming and nonconforming VE/HHO methods is that, in the nonconforming case, the (local) interpolation operator (say, $\mathcal{I}_T$) is the elliptic projector on the local virtual space. It is not true in the conforming case. This has two important consequences.

- Letting Π_T denote the elliptic projector on a polynomial subspace $\mathbb{P}(T)$ of the local virtual space, $\Pi_T \circ \mathcal{I}_T$ is actually equal to Π_T in the nonconforming case, whereas this is not true in the conforming case. Whereas the HHO error in broken H^1-seminorm naturally splits along

  $$\|\boldsymbol{\nabla}_h(u - \Pi_h u_h)\| \leqslant \|\boldsymbol{\nabla}_h(u - \Pi_h u)\| + \|\boldsymbol{\nabla}_h \Pi_h(\mathcal{I}_h u - u_h)\|,$$

  where the first addend in the right-hand side is the approximation error (directly available from the properties of Π_h), and the second is bounded by the consistency error of the scheme, the VE error is usually split in the literature along

  $$\begin{aligned}\|\boldsymbol{\nabla}_h(u - \Pi_h u_h)\| &\leqslant \|\boldsymbol{\nabla}_h(u - \Pi_h u)\| + \|\boldsymbol{\nabla}_h \Pi_h(u - \mathcal{I}_h u)\| + \|\boldsymbol{\nabla}_h \Pi_h(\mathcal{I}_h u - u_h)\| \\ &\leqslant \|\boldsymbol{\nabla}_h(u - \Pi_h u)\| + \|\boldsymbol{\nabla}_h(u - \mathcal{I}_h u)\| + \|\boldsymbol{\nabla}_h \Pi_h(\mathcal{I}_h u - u_h)\|,\end{aligned}$$

  where the approximation properties of the local virtual spaces frankly invite themselves into the picture. In this article, we split the error along

  $$\|\boldsymbol{\nabla}_h(u - \Pi_h u_h)\| \leqslant \|\boldsymbol{\nabla}_h(u - (\Pi_h \circ \mathcal{I}_h)u)\| + \|\boldsymbol{\nabla}_h \Pi_h(\mathcal{I}_h u - u_h)\|,$$

  and we directly investigate the approximation properties of the polynomial projector $\Pi_h \circ \mathcal{I}_h$. Since $\Pi_T \circ \mathcal{I}_T$ preserves the polynomials in $\mathbb{P}(T)$, and since the operator Π_T is stable, all we have to prove are stability properties for $\mathcal{I}_T$. We then follow the ideas of [11, Chapter 4] to derive the desired approximation properties. This is the object of Sections 4.1 and 4.2 (cf. in particular Theorem 4.17).

- In the conforming case, in order to prove stability properties for $\mathcal{I}_T$, one must prove the existence, on any cell T, of a (linear) lifting operator $\mathcal{L}_T$ such that, for any function v on ∂T that is equal to the trace of a virtual function, one has the following scaled estimate:

  $$h_T^{-1}\|\mathcal{L}_T v\|_T + \|\boldsymbol{\nabla}\mathcal{L}_T v\|_T \leqslant c\left(h_T^{-1/2}\|v\|_{\partial T} + h_T^{1/2}\|\boldsymbol{\nabla}_t v\|_{\partial T}\right), \tag{2}$$

  where $\boldsymbol{\nabla}_t$ denotes the tangential derivative, and $c > 0$ is a constant independent of h_T. When T is star-shaped with respect to a ball of radius comparable to h_T (in standard analyses of c-VE

methods, the latter assumption is supplemented by the fact that all the faces of T are assumed to have diameter comparable to h_T), the result above is true for any function $v \in H^1(\partial T)$ (cf., e.g., [12, Section 2.7]). When T is not star-shaped, it is less clear whether optimal scaling can be obtained. In the nonconforming case, owing to the fact that the local interpolation operator is the elliptic projector on the local virtual space, a similar estimate is not needed to prove stability for $\mathcal{I}_T$, and regularity assumptions on mesh sequences can be relaxed. All one needs to assume in that case is that the cell T can be remeshed by a (matching, shape-regular, and with meshsize comparable to h_T) simplicial submesh (cf. Definition 2.1). These assumptions are sufficient to derive all the necessary analysis tools. We believe that, following [10, Lemma 5.3], it should be possible to prove a result such as (2) under the sole assumptions of Definition 2.1. Yet, this is beyond the scope of this article. We will hence do not pursue in that direction, and will make the further assumption, in the conforming case, that each cell is star-shaped with respect to a ball (with radius comparable to its diameter).

The material is organized as follows. In Section 2, we introduce the notation, we detail our admissibility assumptions on mesh sequences, and we introduce a number of analysis tools that will be useful in the sequel. In Section 3, we undertake a general description of skeletal methods, such as the VE or HHO methods. Finally, in Section 4, we provide the unified formulation/analysis of c/nc-VE/HHO methods, as well as a description of the general workflow of the methods.

2 Notation, mesh assumptions, and basic analysis tools

We collect in this section all the conventions, mathematical tools, and results that will be useful in the sequel.

2.1 Notation

2.1.1 Geometry

For $l \in \{1, \ldots, d\}$, we let $|\cdot|_l$ denote the l-dimensional Hausdorff measure. In what follows, the term polytope refers to polygons if $d = 2$, and to polyhedra if $d = 3$. The discretization of the domain Ω is described in the following manner.

- $\mathcal{T}_h$ denotes a mesh of the domain Ω, i.e. a collection of disjoint open polytopes T (the cells) such that $\bigcup_{T\in\mathcal{T}_h} \overline{T} = \overline{\Omega}$. The parameter h is the meshsize, defined as $h := \max_{T\in\mathcal{T}_h} h_T$, where h_T stands for the diameter of the cell T.

- $\mathcal{F}_h$ denotes the collection of faces of the mesh $\mathcal{T}_h$. Since the cells of $\mathcal{T}_h$ are polytopes, their boundary is composed of a finite union of closed portions of affine hyperplanes, called facets. A closed subset F of $\overline{\Omega}$ with $|F|_{d-1} \neq 0$ is a face as soon as (i) F is equal to the intersection, for T_1, T_2 two cells of $\mathcal{T}_h$, of a facet of T_1 and a facet of T_2, or (ii) F is equal to the intersection, for T cell of $\mathcal{T}_h$, of a facet of T and a facet of Ω. In the first case, F is termed an interface, whereas in the second, F is termed a boundary face. Interfaces are collected in the set $\mathcal{F}_h^{\mathrm{i}}$, boundary faces in the set $\mathcal{F}_h^{\mathrm{b}}$, in such a way that $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \bigcup \mathcal{F}_h^{\mathrm{b}}$. For a cell $T \in \mathcal{T}_h$, we let $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ be the collection of faces composing its boundary, and $\boldsymbol{n}_T$ be the unit normal vector to ∂T pointing outward T (that is defined almost everywhere on ∂T). For $F \in \mathcal{F}_T$, we also let $\boldsymbol{n}_{T,F} := \boldsymbol{n}_{T|F}$; remark that $\boldsymbol{n}_{T,F}$ is a constant vector on F since F is planar. Finally, for any face $F \in \mathcal{F}_h$, we denote h_F its diameter (and, for $T \in \mathcal{T}_h$, we let $h_{\partial T}$ be such that $h_{\partial T|F} := h_F$ for all $F \in \mathcal{F}_T$), and we let $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$ be the collection of cells sharing F (two cells for an interface, one for a boundary face).

- $\partial\mathcal{T}_h$ denotes the $(d-1)$-dimensional skeleton of the mesh $\mathcal{T}_h$, that is $\partial\mathcal{T}_h = \bigcup_{F\in\mathcal{F}_h} F$.

- When $d = 3$, $\mathcal{E}_h$ denotes the collection of edges e (with $|e|_1 \neq 0$) of the mesh $\mathcal{T}_h$, defined from the collection $\mathcal{F}_h$ of faces. For a cell $T \in \mathcal{T}_h$ (respectively a face $F \in \mathcal{F}_h$), we let $\mathcal{E}_T := \{e \in \mathcal{E}_h \mid e \subset \partial T\}$ (respectively $\mathcal{E}_F := \{e \in \mathcal{E}_h \mid e \subset \partial F\}$, that will also be denoted $\mathcal{F}_F$ with a slight abuse in notation) be the collection of edges composing its boundary.

- $\mathcal{V}_h$ denotes the collection of vertices ν of the mesh $\mathcal{T}_h$. For a cell $T \in \mathcal{T}_h$ (respectively a face $F \in \mathcal{F}_h$), we let $\mathcal{V}_T := \{\nu \in \mathcal{V}_h \mid \nu \in \partial T\}$ (respectively $\mathcal{V}_F := \{\nu \in \mathcal{V}_h \mid \nu \in \partial F\}$) be the collection of its vertices. The position of any vertex $\nu \in \mathcal{V}_h$ is denoted $\boldsymbol{x}_\nu \in \Omega$.

When $d = 2$, faces are sometimes called edges in the literature. Note that we will not use this vocable in this article. The term edge will always refer to a 1-manifold in dimension $d = 3$. We finally introduce, for $\nu \in \mathcal{V}_h$, the notation $\mathcal{F}_\nu := \{F \in \mathcal{F}_h \mid \nu \in \mathcal{V}_F\}$ and, when $d = 3$, for $e \in \mathcal{E}_h$, the notation $\mathcal{F}_e := \{F \in \mathcal{F}_h \mid e \in \mathcal{E}_F\}$.

2.1.2 Functions spaces

For $X \subset \overline{\Omega}$, and $m \geqslant 0$, we let $|\cdot|_{m,X}$ and $\|\cdot\|_{m,X}$ respectively denote the seminorm and norm on the Sobolev space $H^m(X)$ (or $H^m(X)^d$), with the convention that $H^0(X) \equiv L^2(X)$ (hence, $|\cdot|_{0,X} \equiv \|\cdot\|_{0,X}$). We also define $\|\cdot\|_{\infty,X}$ as the norm on $L^\infty(X)$. We finally let $\langle\cdot,\cdot\rangle_{-m,X}$ be the duality pairing between $H^{-m}(X)$ and its topological dual.

For $q \in \mathbb{N}$ and $l \in \{1,\ldots,d\}$, we let $\mathbb{P}_l^q$ be the vector space of l-variate polynomial functions of total degree less than or equal to q. We also let

$$\mathrm{N}_l^q := \dim(\mathbb{P}_l^q) = \binom{q+l}{q},$$

and we adopt the conventions $\mathbb{P}_l^{-1} := \{0\}$ and $\mathrm{N}_l^{-1} := 0$. For $T \in \mathcal{T}_h$, we define $\mathbb{P}_d^q(T)$ as the restriction of $\mathbb{P}_d^q$ to T. For $F \in \mathcal{F}_h$, we define $\mathbb{P}_{d-1}^q(F)$ as the restriction of $\mathbb{P}_d^q$ to F. When $d = 3$, for $e \in \mathcal{E}_h$, we define $\mathbb{P}_1^q(e)$ as the restriction of $\mathbb{P}_3^q$ to e. For $T \in \mathcal{T}_h$, we also define the broken space

$$\mathbb{P}_{d-1}^q(\mathcal{F}_T) := \left\{v \in L^2(\partial T) \mid v_{|F} \in \mathbb{P}_{d-1}^q(F)\ \forall F \in \mathcal{F}_T\right\},$$

and when $d = 3$, for $F \in \mathcal{F}_h$, we let $\mathbb{P}_1^q(\mathcal{E}_F) := \{v \in L^2(\partial F) \mid v_{|e} \in \mathbb{P}_1^q(e)\ \forall e \in \mathcal{E}_F\}$. We finally introduce, for any $T \in \mathcal{T}_h$, a set of basis functions for $\mathbb{P}_d^q(T)$, that we denote $\{\psi_{T,i}^q\}_{i\in\{1,\ldots,\mathrm{N}_d^q\}}$, and for any $F \in \mathcal{F}_h$, a set of basis functions for $\mathbb{P}_{d-1}^q(F)$, that we denote $\{\psi_{F,j}^q\}_{j\in\{1,\ldots,\mathrm{N}_{d-1}^q\}}$. When $d = 3$, we further introduce, for any $e \in \mathcal{E}_h$, a set of basis functions for $\mathbb{P}_1^q(e)$, denoted $\{\psi_{e,m}^q\}_{m\in\{1,\ldots,\mathrm{N}_1^q\}}$.

Given a mesh $\mathcal{T}_h$ of Ω, we introduce the following notation for broken functions spaces on $\mathcal{T}_h$:

$$X(\mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_{h|T} \in X(T)\ \forall T \in \mathcal{T}_h\}.$$

We introduce on $H^1(\mathcal{T}_h)$ the so-called broken gradient operator $\boldsymbol{\nabla}_h : H^1(\mathcal{T}_h) \to L^2(\Omega)$ such that, for any $v_h \in H^1(\mathcal{T}_h)$ and $T \in \mathcal{T}_h$, $\boldsymbol{\nabla}_h v_{h|T} := \boldsymbol{\nabla}(v_{h|T})$. For any interface $F \in \mathcal{F}_h^{\mathrm{i}}$ with $\mathcal{T}_F = \{T_1, T_2\}$, we define the jump $[\![v_h]\!]_F$ along F of $v_h \in H^s(\mathcal{T}_h)$, $s > 1/2$, by $[\![v_h]\!]_F := (v_{h|T_1})_{|F} - (v_{h|T_2})_{|F}$, and we let $\boldsymbol{n}_F := \boldsymbol{n}_{T_1,F}$. For any $F \in \mathcal{F}_h^{\mathrm{b}}$ with $\mathcal{T}_F = \{T\}$, we let $[\![v_h]\!]_F := (v_{h|T})_{|F}$, and $\boldsymbol{n}_F := \boldsymbol{n}_{T,F}$. We finally introduce the operator $[\![\cdot]\!] : H^s(\mathcal{T}_h) \to L^2(\partial\mathcal{T}_h)$ such that, for any $v_h \in H^s(\mathcal{T}_h)$, $[\![v_h]\!]_{|F} := [\![v_h]\!]_F$ for all $F \in \mathcal{F}_h$. Note that the quantity $[\![v_h]\!] \in L^2(\partial\mathcal{T}_h)$ may be multi-valued (whenever this has a sense) at vertices when $d = 2$, and on edges/vertices when $d = 3$. Assume, for simplicity, that $v_h \in H^s(\mathcal{T}_h)$ is such that $v_{h|\partial T} \in C^0(\partial T)$ for all $T \in \mathcal{T}_h$. Considering a vertex $\nu \in \mathcal{V}_h$, there are $\mathrm{card}(\mathcal{F}_\nu)$ potentially different values for $[\![v_h]\!]$ at $\boldsymbol{x}_\nu$, that are the $([\![v_h]\!]_F(\boldsymbol{x}_\nu))_{F\in\mathcal{F}_\nu}$. When $d = 3$, considering an edge $e \in \mathcal{E}_h$, there are, identically, $\mathrm{card}(\mathcal{F}_e)$ potentially different functions $[\![v_h]\!]$ on e, that are the $\left([\![v_h]\!]_{F|e}\right)_{F\in\mathcal{F}_e}$.

2.2 Mesh assumptions

We define the notion of admissible mesh family.

Definition 2.1. *The mesh family $(\mathcal{T}_h)_h$ is admissible if, for all h, $\mathcal{T}_h$ admits a matching simplicial submesh, denoted $\mathcal{S}_h$, and there exists $\gamma > 0$, called mesh regularity parameter, so that, for all h,*

(i) for all $S \in \mathcal{S}_h$ of diameter h_S and inradius r_S, $\gamma h_S \leqslant r_S$ (in other words, $\mathcal{S}_h$ is shape-regular);

(ii) for all $T \in \mathcal{T}_h$, and all $S \in \mathcal{S}_T := \{S \in \mathcal{S}_h \mid S \subseteq T\}$, $\gamma h_T \leqslant h_S$.

By matching simplicial submesh, we mean that $\mathcal{S}_h$ is a (conforming, i.e. free of hanging node) simplicial mesh, and that, for all $S \in \mathcal{S}_h$, there exists a unique $T \in \mathcal{T}_h$ such that $S \subseteq T$, and for all $Z \in \mathcal{Z}_h$, where $\mathcal{Z}_h$ collects the faces of $\mathcal{S}_h$, there exists at most one $F \in \mathcal{F}_h$ such that $Z \subseteq F$ (cf. [25, Definition 1.37]). Henceforth, we will use the symbol $\lesssim$ to notify that an estimate is valid up to a multiplicative constant $c > 0$, with c only depending on the dimension d, the mesh regularity parameter γ, and, if need be, the underlying polynomial degree; in particular, the bound is uniform with respect to the meshsize.

Let us mention three important consequences of Definition 2.1: for all h, and all $T \in \mathcal{T}_h$,

(a) for all $S \in \mathcal{S}_T$, $\gamma h_T \leqslant h_S \leqslant h_T$, and $\mathrm{card}(\mathcal{S}_T) \lesssim 1$ (cf. [25, Lemma 1.40]);

(b) for all $F \in \mathcal{F}_T$, $\mathrm{card}(\mathcal{Z}_F) \lesssim 1$, where $\mathcal{Z}_F := \{Z \in \mathcal{Z}_h \mid Z \subseteq F\}$ (cf. [25, Lemma 1.41]);

(c) for all $F \in \mathcal{F}_T$, $\gamma^2 h_T \leqslant h_F \leqslant h_T$, and $\mathrm{card}(\mathcal{F}_T) \lesssim 1$ (cf. [25, Lemmas 1.42 and 1.41]).

From (a) and (b), one can figure out what is the general outline to prove inverse and trace inequalities on arbitrarily-shaped (admissible) cells. One first considers the case of a simplex satisfying (i) of Definition 2.1, for which these inequalities are classical. Then, the passage to arbitrary geometries follows from the fact that any admissible cell is composed of an uniformly bounded number of simplices satisfying (i) (and, an admissible face is composed of an uniformly bounded number of subfaces belonging to simplices satisfying (i)), and whose diameters are comparable to the diameter of the cell under consideration (cf. Section 2.3.1). Note that the notion of admissible mesh we consider here allows for cells that are not necessarily star-shaped. When this latter additional assumption (i.e., that each cell T is star-shaped with respect to a ball of radius $r_T \geqslant \gamma h_T$) will be needed in the analysis, we will specify it. The assumptions of Definition 2.1 are also sufficient to prove polynomial approximation properties for the L^2-orthogonal and elliptic projectors (cf. Sections 2.3.3 and 2.3.4). As far as point (c) is concerned, it is instrumental in the analysis of numerical methods based on $\mathcal{T}_h$. It expresses the fact that the number of faces of an admissible cell is uniformly bounded, and that the diameter of a face is comparable to the diameter of the cell(s) to which it belongs. When $d = 3$, it is an easy matter to show that, under the assumptions of Definition 2.1, one also has, for any $T \in \mathcal{T}_h$, $F \in \mathcal{F}_T$, and $e \in \mathcal{E}_F$, $\gamma^3 h_F \leqslant h_e \leqslant h_F$, and $\mathrm{card}(\mathcal{E}_F) \lesssim 1$.

2.3 Basic analysis tools

Henceforth, $\mathcal{T}_h$ denotes a member of an admissible mesh family in the sense of Definition 2.1.

2.3.1 Useful inequalities

On any $T \in \mathcal{T}_h$, the following inequalities hold:

- *inverse inequality:*
$$\forall v \in \mathbb{P}_d^q(T), \qquad |v|_{1,T} \lesssim h_T^{-1} \|v\|_{0,T}; \tag{3}$$
- *continuous trace inequality:* for any $F \in \mathcal{F}_T$,
$$\forall v \in H^1(T), \qquad \|v_{|F}\|_{0,F} \lesssim h_T^{-1/2} \|v\|_{0,T} + h_T^{1/2} |v|_{1,T}; \tag{4}$$
- *discrete trace inequality:* for any $F \in \mathcal{F}_T$,
$$\forall v \in \mathbb{P}_d^q(T), \qquad \|v_{|F}\|_{0,F} \lesssim h_T^{-1/2} \|v\|_{0,T}. \tag{5}$$

For the proofs of these different results, we refer to [25, Section 1.4.3]. Note that the discrete trace inequality (5) can be seen as a consequence of the continuous trace inequality (4) and of the inverse inequality (3). We also state the classical Poincaré inequality:

$$\forall v \in H^1(T) \text{ such that } \int_T v = 0, \qquad \|v\|_{0,T} \leqslant c_{\mathrm{P}} h_T |v|_{1,T}. \tag{6}$$

If T is convex, $c_{\mathrm{P}} = \pi^{-1}$, independently of the ambient dimension; cf. [4]. For some insight on the value of c_{P} on more general element shapes, we refer to [35]. In the forthcoming analysis of conforming methods, we will also need (i) the following alternative inverse inequality, whose proof can be found, e.g., in [17, Lemma 4.4 (take $\mathbb{A}_\varepsilon = \mathbb{I}_d$)]: for all $v \in H^1(T)$ such that $\triangle v \in \mathbb{P}_d^q(T)$ for some $q \in \mathbb{N}$, there holds

$$\|\triangle v\|_{0,T} \lesssim h_T^{-1} |v|_{1,T}; \tag{7}$$

(ii) the following version of Sobolev's inequality (cf., e.g., [11, Lemma 4.3.4]):

$$\forall v \in H^2(T), \qquad \|v\|_{\infty,T} \lesssim h_T^{-\frac{d}{2}} \|v\|_{0,T} + h_T^{\frac{(2-d)}{2}} |v|_{1,T} + h_T^{\frac{(4-d)}{2}} |v|_{2,T}; \tag{8}$$

and (iii) the following flux estimate, whose proof is postponed until Appendix A.1, that is valid for any cell T star-shaped with respect to a ball whose radius is comparable to h_T, and any function $v \in H^1(T)$ such that $\triangle v \in L^2(T)$:

$$\sup_{z \in H^1(\partial T)} \frac{\langle \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T, z \rangle_{-\frac{1}{2},\partial T}}{h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}} \lesssim |v|_{1,T} + h_T \|\triangle v\|_{0,T}. \tag{9}$$

2.3.2 Finite element in the sense of Ciarlet

The following definition is directly inspired from [16, p. 94]. Let $l \in \{1, \dots, d\}$.

Definition 2.2. *A finite element consists in a triple $(X, V(X), \Sigma_X)$ where*

- *X is a bounded and connected Lipschitz subset of $\mathbb{R}^d$ such that $|X|_l \neq 0$;*
- *$V(X)$ is a finite-dimensional vector space of functions $v : X \to \mathbb{R}$;*
- *$\Sigma_X := \{\sigma_X^1, \dots, \sigma_X^{n_X}\}$, $n_X \in \mathbb{N}^\star$, is a collection of linear forms σ_X^i on $V(X)$ such that the mapping*

$$\underline{\Sigma}_X : V(X) \ni v \mapsto (\sigma_X^1(v), \dots, \sigma_X^{n_X}(v)) \in \mathbb{R}^{n_X}$$

is bijective (we then have $\dim(V(X)) = n_X$).

The operator $\underline{\Sigma}_X$ is the so-called (local) reduction operator, and $\underline{\Sigma}_X(v)$ is the so-called vector of (local) degrees of freedom (DoFs). The bijectivity of $\underline{\Sigma}_X$ is in general referred to as unisolvence in the literature. To prove unisolvence, one usually aims at proving that

$$\begin{cases} \dim(V(X)) \geqslant n_X, \\ \text{for } v \in V(X),\ (\sigma_X^i(v) = 0\ \forall i \in \{1, \dots, n_X\}) \Rightarrow (v \equiv 0). \end{cases}$$

The following proposition is a direct consequence of the unisolvence property.

Proposition 2.3. *Let $(X, V(X), \Sigma_X)$ be a finite element. There exists a basis $\{\varphi_{X,1}, \dots, \varphi_{X,n_X}\}$ (referred to as canonical) of $V(X)$ such that $\sigma_X^i(\varphi_{X,j}) = \delta_{ij}$ for all $i, j \in \{1, \dots, n_X\}$. The $(\varphi_{X,i})_{i \in \{1,\dots,n_X\}}$ are the so-called (local) shape functions.*

Let $R_X : \mathbb{R}^{n_X} \to V(X)$ be the operator such that, for any $\underline{\mathrm{v}}_X := (\mathrm{v}_X^i)_{i \in \{1,\dots,n_X\}} \in \mathbb{R}^{n_X}$, $R_X \underline{\mathrm{v}}_X := \sum_{i=1}^{n_X} \mathrm{v}_X^i \varphi_{X,i}$. One can easily remark that, for all $\underline{\mathrm{v}}_X \in \mathbb{R}^{n_X}$, $\underline{\Sigma}_X(R_X \underline{\mathrm{v}}_X) = \underline{\mathrm{v}}_X$, hence $R_X = \underline{\Sigma}_X^{-1}$ and there holds, for any $v \in V(X)$,

$$v = R_X(\underline{\Sigma}_X(v)) = \sum_{i=1}^{n_X} \sigma_X^i(v)\, \varphi_{X,i}.$$

R_X is the so-called (local, canonical) reconstruction operator.

Assume that there exists a normed vector space $W(X)$ of functions $w : X \to \mathbb{R}$ such that (i) $V(X) \subset W(X)$, and such that (ii) every linear form σ_X^i of Σ_X can be extended as a linear form $\overline{\sigma}_X^i$ on $W(X)$. We denote by $\overline{\Sigma}_X$ this new collection of extended linear forms, and by $\underline{\overline{\Sigma}}_X$ the corresponding (local) reduction operator. We can then introduce the operator $\mathcal{I}_X : W(X) \to V(X)$ such that, for any $w \in W(X)$, $\mathcal{I}_X w := R_X(\underline{\overline{\Sigma}}_X(w)) = \sum_{i=1}^{n_X} \overline{\sigma}_X^i(w) \varphi_{X,i}$. With such a definition, there holds $\underline{\Sigma}_X(\mathcal{I}_X w) = \underline{\overline{\Sigma}}_X(w)$. The operator $\mathcal{I}_X$ is the so-called (local) interpolation operator (in a broad sense). Of course, for all $w \in V(X)$, $\mathcal{I}_X w = w$, since $\underline{\overline{\Sigma}}_{X|V(X)} = \underline{\Sigma}_X = R_X^{-1}$.

2.3.3 L^2-orthogonal polynomial projectors

For $T \in \mathcal{T}_h$, we define $\pi_T^q : L^2(T) \to \mathbb{P}_d^q(T)$ so that, for all $v \in L^2(T)$,

$$\int_T \pi_T^q v\, w = \int_T v\, w \qquad \forall w \in \mathbb{P}_d^q(T).$$

For $l \in \{1, \dots, \mathrm{N}_d^q\}$, we let $\pi_T^{q,l}$ denote the linear form on $L^2(T)$ such that, for any $v \in L^2(T)$, $\pi_T^{q,l}(v) \in \mathbb{R}$ is the l^{th} coordinate of $\pi_T^q v$ on the basis $\{\psi_{T,i}^q\}_{i \in \{1,\dots,\mathrm{N}_d^q\}}$ of $\mathbb{P}_d^q(T)$.

Proposition 2.4 (Properties of π_T^q)**.** *Let $T \in \mathcal{T}_h$, and $v \in L^2(T)$.*

- preservation of polynomials: *if $v \in \mathbb{P}_d^q(T)$, $\pi_T^q v = v$;*
- stability: $\|\pi_T^q v\|_{0,T} \leqslant \|v\|_{0,T}$;
- optimality: $\|v - \pi_T^q v\|_{0,T} = \min_{z \in \mathbb{P}_d^q(T)} \|v - z\|_{0,T}$;

- approximation: *for* $v \in H^s(T)$, $s \in \{1, \ldots, q+1\}$, *there holds*

$$|v - \pi_T^q v|_{m,T} \lesssim h_T^{s-m} |v|_{s,T} \qquad \text{for } m \in \{0, \ldots, s\}, \tag{10}$$

and, for any $F \in \mathcal{F}_T$, *and any* $(\zeta_1, \ldots, \zeta_d) \in \mathbb{N}^d$ *such that* $\sum_{i=1}^d \zeta_i = m$,

$$\left| [\partial_{x_1}^{\zeta_1} \ldots \partial_{x_d}^{\zeta_d} (v - \pi_T^q v)]_{|F} \right|_{0,F} \lesssim h_T^{s-m-1/2} |v|_{s,T} \qquad \text{for } m \in \{0, \ldots, s-1\}. \tag{11}$$

In Proposition 2.4, the only nontrivial result to prove is the last one. Its proof relies on (i) the ideas of [11, Chapter 4] and [29, Section 7] (cf. also [10, Section 4]), and on (ii) two important features of π_T^q, that are the preservation of polynomials, and its stability in the $L^2(T)$-norm. For a detailed proof of the result, and its extension to more general Sobolev seminorms, we refer to [22]. We also introduce $\pi_h^q : L^2(\Omega) \to \mathbb{P}_d^q(\mathcal{T}_h)$ such that, for any $v \in L^2(\Omega)$, $\pi_h^q v_{|T} := \pi_T^q(v_{|T})$ for all $T \in \mathcal{T}_h$.

For $F \in \mathcal{F}_h$, we define $\pi_F^q : L^2(F) \to \mathbb{P}_{d-1}^q(F)$ so that, for all $v \in L^2(F)$,

$$\int_F \pi_F^q v \, w = \int_F v \, w \qquad \forall w \in \mathbb{P}_{d-1}^q(F).$$

For $l \in \{1, \ldots, \mathrm{N}_{d-1}^q\}$, we let $\pi_F^{q,l}$ denote the linear form on $L^2(F)$ such that, for any $v \in L^2(F)$, $\pi_F^{q,l}(v) \in \mathbb{R}$ is the l^{th} coordinate of $\pi_F^q v$ on the basis $\{\psi_{F,j}^q\}_{j \in \{1,\ldots,\mathrm{N}_{d-1}^q\}}$ of $\mathbb{P}_{d-1}^q(F)$. We also define, for $T \in \mathcal{T}_h$, $\pi_{\partial T}^q : L^2(\partial T) \to \mathbb{P}_{d-1}^q(\mathcal{F}_T)$ such that, for any $v \in L^2(\partial T)$, $\pi_{\partial T}^q v_{|F} := \pi_F^q(v_{|F})$ for all $F \in \mathcal{F}_T$.

When $d = 3$, for $e \in \mathcal{E}_h$, we define $\pi_e^q : L^2(e) \to \mathbb{P}_1^q(e)$ so that, for all $v \in L^2(e)$,

$$\int_e \pi_e^q v \, w = \int_e v \, w \qquad \forall w \in \mathbb{P}_1^q(e).$$

For $l \in \{1, \ldots, \mathrm{N}_1^q\}$, we let $\pi_e^{q,l}$ denote the linear form on $L^2(e)$ such that, for any $v \in L^2(e)$, $\pi_e^{q,l}(v) \in \mathbb{R}$ is the l^{th} coordinate of $\pi_e^q v$ on the basis $\{\psi_{e,m}^q\}_{m \in \{1,\ldots,\mathrm{N}_1^q\}}$ of $\mathbb{P}_1^q(e)$.

Remark 2.5 (Computation of $\pi_X^q v$)**.** *It can be easily seen that, for* $X \in \{T, F, e\}$ *and* $v \in L^2(X)$, $\pi_X^0 v = \frac{1}{|X|_l} \int_X v$ *with* $l \in \{d, d-1, 1\}$ *respectively. To compute* $\pi_X^q v$ *for* $q \geqslant 1$, *it suffices to solve a symmetric positive-definite (SPD) system of size* $\mathrm{N}_l^q \times \mathrm{N}_l^q$. *This can be done effectively via, e.g., a Cholesky factorization.*

2.3.4 Elliptic polynomial projector

For $T \in \mathcal{T}_h$, we define $\Pi_T^{q+1} : H^1(T) \to \mathbb{P}_d^{q+1}(T)$ so that, for all $v \in H^1(T)$,

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla} \Pi_T^{q+1} v \cdot \boldsymbol{\nabla} w = \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} w \qquad \forall w \in \mathbb{P}_d^{q+1}(T), \\ \displaystyle\int_T \Pi_T^{q+1} v = \int_T v. \end{cases} \tag{12}$$

Remark that $\Pi_T^0 = \pi_{T|H^1(T)}^0$, which is the reason why we only consider here Π_T^{q+1}, $q \in \mathbb{N}$.

Proposition 2.6 (Properties of Π_T^{q+1})**.** *Let* $T \in \mathcal{T}_h$, *and* $v \in H^1(T)$.

- preservation of polynomials: *if* $v \in \mathbb{P}_d^{q+1}(T)$, $\Pi_T^{q+1} v = v$;
- stability: $\left\| \Pi_T^{q+1} v \right\|_{0,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T}$;
- optimality: $\left| v - \Pi_T^{q+1} v \right|_{1,T} = \min_{z \in \mathbb{P}_d^{q+1}(T)} |v - z|_{1,T}$;
- approximation: *for* $v \in H^s(T)$, $s \in \{1, \ldots, q+2\}$, *there holds*

$$\left| v - \Pi_T^{q+1} v \right|_{m,T} \lesssim h_T^{s-m} |v|_{s,T} \qquad \text{for } m \in \{0, \ldots, s\}, \tag{13}$$

and, for any $F \in \mathcal{F}_T$, *and any* $(\zeta_1, \ldots, \zeta_d) \in \mathbb{N}^d$ *such that* $\sum_{i=1}^d \zeta_i = m$,

$$\left| \left[\partial_{x_1}^{\zeta_1} \ldots \partial_{x_d}^{\zeta_d} (v - \Pi_T^{q+1} v) \right]_{|F} \right|_{0,F} \lesssim h_T^{s-m-1/2} |v|_{s,T} \qquad \text{for } m \in \{0, \ldots, s-1\}. \tag{14}$$

As for Proposition 2.4, the only nontrivial result to prove in Proposition 2.6 is the last one. Its proof also relies on (i) the ideas of [11, Chapter 4] and [29, Section 7] (cf. also [10, Section 4]), and on (ii) two important features of Π_T^{q+1}, that are the preservation of polynomials, and its stability in the $L^2(T)$-norm (here, by-product of the stability in the $H^1(T)$-seminorm, and of the fact that Π_T^{q+1} is mean-value preserving). For a detailed proof of the result, and its extension to more general Sobolev seminorms, we refer to [23]. We also introduce $\Pi_h^{q+1} : H^1(\mathcal{T}_h) \to \mathbb{P}_d^{q+1}(\mathcal{T}_h)$ such that, for any $v \in H^1(\mathcal{T}_h)$, $\Pi_h^{q+1} v_{|T} := \Pi_T^{q+1}(v_{|T})$ for all $T \in \mathcal{T}_h$.

Remark 2.7 (Computation of $\Pi_T^{q+1} v$)**.** *One possibility (another one is to solve a problem posed on the quotient space $\mathbb{P}_d^{q+1}(T)/\mathbb{P}_d^0(T)$) to compute $\Pi_T^{q+1} v$ is to consider the following coercive problem: find $z \in \mathbb{P}_d^{q+1}(T)$ such that, for all $w \in \mathbb{P}_d^{q+1}(T)$,*

$$\int_T \boldsymbol{\nabla} z \cdot \boldsymbol{\nabla} w + \int_T \pi_T^0 z \, \pi_T^0 w = -\int_T \pi_T^{q-1} v \, \triangle w + \int_{\partial T} \pi_{\partial T}^q (v_{|\partial T}) \, \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T.$$

The elliptic projection is obtained through $\Pi_T^{q+1} v = z + \pi_T^0 v$. According to the expression above, to compute $\Pi_T^{q+1} v$, it suffices to know selected moments of $v \in H^1(T)$, namely $\pi_T^\mu v$ for $\mu = \max(q-1, 0)$, and $\pi_{\partial T}^q(v_{|\partial T})$. The computation of $\Pi_T^{q+1} v$ then requires to solve a SPD system of size $\mathrm{N}_d^{q+1} \times \mathrm{N}_d^{q+1}$, which can be done effectively via, e.g., a Cholesky factorization. The rewriting of the right-hand side of (12) advocated above is based on an integration by parts formula, and on the fact that $\triangle w \in \mathbb{P}_d^{q-1}(T)$ and $\boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T \in \mathbb{P}_{d-1}^q(\mathcal{F}_T)$ for any $w \in \mathbb{P}_d^{q+1}(T)$.

3 Skeletal methods

Galerkin methods on $\mathcal{T}_h$, among which c/nc-FE, c/nc-VE, or dG methods, seek for an approximation u_h of the solution $u \in H_0^1(\Omega)$ to Problem (1) in a broken space $V_{h,0} \subseteq V(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$, with $V(\mathcal{T}_h)$ such that, for all $T \in \mathcal{T}_h$, $V(T)$ is a finite-dimensional vector space of functions on which exists a collection Σ_T of linear forms so that $(T, V(T), \Sigma_T)$ is a finite element in the sense of Ciarlet. The linear forms collected in Σ_T are the local DoFs of the method. For dG methods, $V_{h,0} = V(\mathcal{T}_h)$, whereas for c/nc-FE/VE, $V_{h,0}$ is a strict subspace of $V(\mathcal{T}_h)$ that embeds more or less stringent conformity prescriptions (with respect to $H_0^1(\Omega)$); for c-FE/VE for instance, $V_{h,0}$ is a subspace of $H_0^1(\Omega)$.

For those methods (like c/nc-FE/VE) based on a space $V_{h,0}$ embedding conformity prescriptions, locally to any $T \in \mathcal{T}_h$, the collection Σ_T of DoFs splits into (i) linear forms on $V(\partial T) := \{v_{|\partial T},\, v \in V(T)\}$ (collected in the set Σ_T^∂) and, (ii) if need be, those linear forms on $V(T)$ that cannot be written as linear forms on $V(\partial T)$ (collected in the set Σ_T°). The linear forms in Σ_T^∂ are skeletal DoFs, whereas those in Σ_T° are cell DoFs. We let $n_T^\partial := \mathrm{card}(\Sigma_T^\partial)$, $n_T^\circ := \mathrm{card}(\Sigma_T^\circ)$, and $n_T := n_T^\partial + n_T^\circ = \mathrm{card}(\Sigma_T) = \dim(V(T))$. From a global viewpoint, the collection Σ_h of global DoFs of the method splits into (i) linear forms on

$$V(\mathcal{F}_h) := \left\{ v_h \in L^2(\partial\mathcal{T}_h) \mid v_{h|F} \in \dotplus_{T \in \mathcal{T}_F} V_T(F) \, \forall F \in \mathcal{F}_h \right\}$$

with $V_T(F) := \{v_{|F},\, v \in V(T)\}$ (collected in the set Σ_h^∂) and, (ii) if need be, linear forms on $V(\mathcal{T}_h)$ that cannot be written as linear forms on $V(\mathcal{F}_h)$ (collected in the set Σ_h°). Whereas $\Sigma_h^\circ := \bigcup_{T \in \mathcal{T}_h} \check{\Sigma}_T^\circ$, where, for any $T \in \mathcal{T}_h$, $\check{\Sigma}_T^\circ$ is the collection of the n_T° linear forms $\check{\sigma}_T^\circ$ on $V(\mathcal{T}_h)$ so that $\check{\sigma}_T^\circ(\cdot) := \sigma_T^\circ(\cdot_{|T})$ for $\sigma_T^\circ \in \Sigma_T^\circ$, the global skeletal DoFs in Σ_h^∂ (that are linear forms on $V(\mathcal{F}_h)$) are intrinsically defined, and their local counterparts are obtained by localization. More precisely, for any $T \in \mathcal{T}_h$, letting $\hat{\Sigma}_T^\partial$ be the collection of the n_T^∂ linear forms $\hat{\sigma}_T^\partial$ on

$$V_T(\mathcal{F}_h) := \left\{ v_h \in V(\mathcal{F}_h) \mid v_{h|\partial T} \in V(\partial T),\, v_{h|F} \equiv 0 \, \forall F \in \mathcal{F}_h \backslash \mathcal{F}_T \right\}$$

so that $\hat{\sigma}_T^\partial(\cdot) := \sigma^\partial(\cdot)$ for $\sigma^\partial \in \Sigma_h^\partial$ such that $\sigma^\partial_{|V_T(\mathcal{F}_h)} \not\equiv 0$, and $\mathcal{Z}_h(v)$ be the zero extension of $v \in V(\partial T)$ to $V_T(\mathcal{F}_h)$, the linear forms σ_T^∂ in Σ_T^∂ are obtained by

$$\left[\forall v \in V(\partial T), \sigma_T^\partial(v) := c \, (\hat{\sigma}_T^\partial \circ \mathcal{Z}_h)(v) \right] \text{ for } \hat{\sigma}_T^\partial \in \hat{\Sigma}_T^\partial, \tag{15}$$

where c is a constant depending on the skeletal DoF under consideration, and on T, whose default value is 1 (cf. Remark 3.1). Whereas $n_h^\circ := \mathrm{card}(\Sigma_h^\circ) = \sum_{T \in \mathcal{T}_h} n_T^\circ$, $n_h^\partial := \mathrm{card}(\Sigma_h^\partial) < \sum_{T \in \mathcal{T}_h} n_T^\partial$, and $n_h := n_h^\partial + n_h^\circ = \mathrm{card}(\Sigma_h) < \dim(V(\mathcal{T}_h)) =: n_{\mathcal{T}_h}$.

Remark 3.1 (Role of the constant c in (15)). *Functions in $V(\mathcal{F}_h)$ may be multi-valued (whenever this has a sense) at vertices when $d=2$, and on edges/vertices when $d=3$. Yet, in the conforming case, one has to define DoFs at vertices when $d=2$, and on edges/vertices when $d=3$, to prescribe conformity. Let us then describe how multi-valuedness can be dealt with. Assume, for simplicity, that $V(\partial T)\subset C^0(\partial T)$ for all $T\in\mathcal{T}_h$. For a vertex $\nu\in\mathcal{V}_h$, the (global) skeletal DoF $\sigma^{\partial,\nu}\in\Sigma_h^\partial$ associated to a pointwise evaluation at $\boldsymbol{x}_\nu$, is defined, for any $v_h\in V(\mathcal{F}_h)$, by $\sigma^{\partial,\nu}(v_h):=\frac{1}{\mathrm{card}(\mathcal{F}_\nu)}\sum_{F\in\mathcal{F}_\nu}v_{h|F}(\boldsymbol{x}_\nu)$. With such a definition, it degenerates towards $v_h(\boldsymbol{x}_\nu)$ whenever v_h is single-valued at $\boldsymbol{x}_\nu$. Besides, letting $c_T^\nu=\frac{\mathrm{card}(\mathcal{F}_\nu)}{\mathrm{card}(\mathcal{F}_\nu\bigcap\mathcal{F}_T)}$, one obtains, as expected, that the restriction of $\sigma^{\partial,\nu}$ to a cell $T\in\mathcal{T}_h$ such that $\nu\in\mathcal{V}_T$, is given, for any $v\in V(\partial T)$, by $\sigma_T^{\partial,\nu}(v)=v(\boldsymbol{x}_\nu)$. Edge DoFs when $d=3$ can be handled in a similar way. For face DoFs, one can always take $c=1$.*

For any $F\in\mathcal{F}_h$, and $v_h\in V(\mathcal{F}_h)$, we let $\mathcal{R}_F(v_h)\in V(\mathcal{F}_h)$ be such that $\mathcal{R}_F(v_h)_{|F}=v_{h|F}$, and $\mathcal{R}_F(v_h)_{|F'}\equiv 0$ for all $F'\in\mathcal{F}_h\backslash F$. The approximation space, that also takes into account the boundary conditions, writes

$$V_{h,0}:=\left\{v_h\in V(\mathcal{T}_h)\mid(\sigma^\partial\circ\mathcal{R}_F)(\llbracket v_h\rrbracket)=0\,\forall F\in\mathcal{F}_h,\,\forall\sigma^\partial\in\Sigma_h^\partial\right\},\tag{16}$$

with dimension $n_{h,0}:=\dim(V_{h,0})<n_h$. The conditions on the jumps of discrete functions enforce the conformity prescriptions. The discrete problem then reads as follows: find $u_h\in V_{h,0}$ such that

$$a_h(u_h,v_h):=\sum_{T\in\mathcal{T}_h}a_T(u_{h|T},v_{h|T})=\sum_{T\in\mathcal{T}_h}l_T(v_{h|T})=:l_h(v_h)\qquad\text{for all } v_h\in V_{h,0},\tag{17}$$

where the (bi)linear forms $a_h:V(\mathcal{T}_h)\times V(\mathcal{T}_h)\to\mathbb{R}$ and $l_h:V(\mathcal{T}_h)\to\mathbb{R}$ write as the sums of local contributions expressed by the local forms $a_T:V(T)\times V(T)\to\mathbb{R}$ and $l_T:V(T)\to\mathbb{R}$. This special structure of the discrete problem ensures that the potential cell DoFs of the method are not coupled between adjacent cells and can be eliminated locally in each cell $T\in\mathcal{T}_h$ in terms of the local skeletal DoFs. Algebraically, this consists in computing the Schur complement of the cell-cell block of the global system matrix, which is quite inexpensive as this block is itself block-diagonal. After elimination, the global system to solve is expressed in terms of the global skeletal DoFs only. This explains why methods that are based on a discrete space like (16) and on a variational formulation like (17) are referred to as skeletal methods.

Since, for all $T\in\mathcal{T}_h$, $(T,V(T),\Sigma_T)$ is a finite element in the sense of Ciarlet, any function $v_h\in V(\mathcal{T}_h)$ can be equivalently written as a vector $\mathbb{R}^{n_{\mathcal{T}_h}}\ni\underline{v}_h:=(\underline{v}_T)_{T\in\mathcal{T}_h}$, with $\mathbb{R}^{n_T}\ni\underline{v}_T:=\left(\underline{v}_T^\partial\in\mathbb{R}^{n_T^\partial},\underline{v}_T^\circ\in\mathbb{R}^{n_T^\circ}\right)$. For $T\in\mathcal{T}_h$ and $F\in\mathcal{F}_T$, we let $\underline{v}_{T,F}^\partial\in\mathbb{R}^{n_{T,F}^\partial}$ be the restriction of $\underline{v}_T^\partial$ to the face F (with possible overlaps between faces, i.e. $\sum_{F\in\mathcal{F}_T}n_{T,F}^\partial\geqslant n_T^\partial$). Since we are going to enforce conformity, we assume that, for all $F\in\mathcal{F}_h^{\mathrm{i}}$ such that $\mathcal{T}_F=\{T_1,T_2\}$, $n_{T_1,F}^\partial=n_{T_2,F}^\partial$, and we let $[\underline{v}_h]_F:=\underline{v}_{T_1,F}^\partial-\underline{v}_{T_2,F}^\partial$; for $F\in\mathcal{F}_h^{\mathrm{b}}$ such that $\mathcal{T}_F=\{T\}$, we let $[\underline{v}_h]_F:=\underline{v}_{T,F}^\partial$. We introduce

$$\mathbb{V}_{h,0}:=\left\{\underline{v}_h\in\mathbb{R}^{n_{\mathcal{T}_h}}\mid[\underline{v}_h]_F=\underline{0}\,\forall F\in\mathcal{F}_h\right\}.$$

There holds $\dim(\mathbb{V}_{h,0})=n_{h,0}$. For vectors $\underline{v}_h\in\mathbb{V}_{h,0}$, for every $F\in\mathcal{F}_h^{\mathrm{i}}$, we have $\underline{v}_{T_1,F}^\partial=\underline{v}_{T_2,F}^\partial$ and, for every $F\in\mathcal{F}_h^{\mathrm{b}}$, we have $\underline{v}_{T,F}^\partial=\underline{0}$. Letting $n_F^\partial:=n_{T_1,F}^\partial=n_{T_2,F}^\partial$ when $F\in\mathcal{F}_h^{\mathrm{i}}$, or $n_F^\partial:=n_{T,F}^\partial$ when $F\in\mathcal{F}_h^{\mathrm{b}}$, there holds $\sum_{F\in\mathcal{F}_h}n_F^\partial\geqslant n_h^\partial$. Problem (17) can be equivalently rewritten: find $\underline{u}_h\in\mathbb{V}_{h,0}$ such that

$$\mathrm{a}_h(\underline{u}_h,\underline{v}_h):=\sum_{T\in\mathcal{T}_h}\mathrm{a}_T(\underline{u}_T,\underline{v}_T)=\sum_{T\in\mathcal{T}_h}\mathrm{l}_T(\underline{v}_T)=:\mathrm{l}_h(\underline{v}_h)\qquad\text{for all }\underline{v}_h\in\mathbb{V}_{h,0},\tag{18}$$

where the (bi)linear forms $\mathrm{a}_h:\mathbb{R}^{n_{\mathcal{T}_h}}\times\mathbb{R}^{n_{\mathcal{T}_h}}\to\mathbb{R}$ and $\mathrm{l}_h:\mathbb{R}^{n_{\mathcal{T}_h}}\to\mathbb{R}$ are expressed in terms of the local forms $\mathrm{a}_T:\mathbb{R}^{n_T}\times\mathbb{R}^{n_T}\to\mathbb{R}$ and $\mathrm{l}_T:\mathbb{R}^{n_T}\to\mathbb{R}$ such that $\mathrm{a}_T:=a_T(R_T\cdot,R_T\cdot)$, and $\mathrm{l}_T:=l_T(R_T\cdot)$, where R_T is the (local) resconstruction operator. For all $T\in\mathcal{T}_h$, there holds $\underline{u}_T=\underline{\Sigma}_T(u_{h|T})$, where $\underline{u}_T\in\mathbb{R}^{n_T}$ is the restriction to T of $\underline{u}_h\in\mathbb{V}_{h,0}$ solution to Problem (18), and $u_h\in V_{h,0}$ is the solution to Problem (17). In the rest of this section, we will stick to the notation used in Problem (17), but in Section 4 we will adopt the notation used for Problem (18).

The most famous examples of skeletal methods are surely given by the c/nc-FE methods. In that case, one considers, for all $T\in\mathcal{T}_h$,

$$a_T(w,v):=\int_T\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}v\text{ and }l_T(v):=\int_T f_{|T}\,v\qquad\text{for all }w,v\in V(T),$$

in such a way that the discrete problem writes: find $u_h\in V_{h,0}$ such that

$$a_h(u_h,v_h)=\int_\Omega\boldsymbol{\nabla}_hu_h\cdot\boldsymbol{\nabla}_hv_h=\int_\Omega f\,v_h=l_h(v_h)\qquad\text{for all }v_h\in V_{h,0}.\tag{19}$$

Without describing too much the space $V_{h,0}$, one can prove interesting properties for Problem (19). Let us denote, for any $F \in \mathcal{F}_h$, by $\check{\pi}_F^0$ the linear form on $V(\mathcal{F}_h)$ so that $\check{\pi}_F^0(\cdot) := \pi_F^0(\cdot_{|F})$, with the convention (henceforth adopted) that $\mathbb{P}_l^0$ is identified to $\mathbb{R}$. Let us begin with the following lemma, whose proof is postponed until Appendix A.2.

Lemma 3.2. *If $\{\check{\pi}_F^0\}_{F\in\mathcal{F}_h} \subseteq \Sigma_h^\partial$, then $\|\boldsymbol{\nabla}_h \cdot\|_{0,\Omega}$ defines a norm on $V_{h,0}$, and the following discrete Poincaré inequalities hold:*

$$\forall v_h \in V_{h,0}, \qquad \sum_{T\in\mathcal{T}_h} h_T^{-2} \|v_{h|T}\|_{0,T}^2 \lesssim \|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}^2, \tag{20}$$

and

$$\forall v_h \in V_{h,0}, \qquad \|v_h\|_{0,\Omega} \lesssim \|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}. \tag{21}$$

Proposition 3.3. *If $\{\check{\pi}_F^0\}_{F\in\mathcal{F}_h} \subseteq \Sigma_h^\partial$, Problem (19) is well-posed in the sense of Hadamard.*

Proof. The proof relies on the results of Lemma 3.2. Since $\|\boldsymbol{\nabla}_h \cdot\|_{0,\Omega}^2 = a_h(\cdot,\cdot)$ defines a norm on $V_{h,0}$, Problem (19) admits a unique solution for all $f \in L^2(\Omega)$. Besides, testing (19) with $v_h = u_h \in V_{h,0}$, and combining Cauchy–Schwarz inequality with the discrete Poincaré inequality (21), yields the a priori estimate

$$\|\boldsymbol{\nabla}_h u_h\|_{0,\Omega} \lesssim \|f\|_{0,\Omega}. \tag{22}$$

□

The conformity assumption $\{\check{\pi}_F^0\}_{F\in\mathcal{F}_h} \subseteq \Sigma_h^\partial$ hence guarantees the well-posedness of Problem (19). Even better, as soon as the space $V_{h,0}$ guarantees as well some approximability in $H_0^1(\Omega)$ (or in a dense subspace thereof), the conformity assumption ensures that the unique solution to Problem (19) strongly converges in $H_0^1(\Omega)$ when h goes to zero towards the solution u to Problem (1). Indeed, by discrete compactness, the conformity assumption, through the resulting a priori estimate (22), first yields that, up to extraction, the family $(u_h)_h$ strongly converges in $L^2(\Omega)$ towards some $w \in L^2(\Omega)$, and the family $(\boldsymbol{\nabla}_h u_h)_h$ weakly converges in $L^2(\Omega)^d$ towards some $\boldsymbol{w} \in L^2(\Omega)^d$. Then, by conformity assumption again, one can prove that, actually, $\boldsymbol{w} = \boldsymbol{\nabla} w$ with $w \in H_0^1(\Omega)$. In a third time, the approximation properties of $V_{h,0}$ in $H_0^1(\Omega)$ (or in a dense subspace thereof) enable to infer that $w = u$, and the uniqueness of the solution to Problem (1) ensures that the whole families $(u_h)_h$ and $(\boldsymbol{\nabla}_h u_h)_h$ actually converge. Finally, an energetic argument yields the strong convergence of $(u_h)_h$ in $H_0^1(\Omega)$ (cf., e.g., [25, Sections 5.1-5.2]).

Remark 3.4. *Note that the condition $\{\check{\pi}_F^0\}_{F\in\mathcal{F}_h} \subseteq \Sigma_h^\partial$ in Lemma 3.2 and Proposition 3.3, that is reminiscent of (lowest-order) nc-FE, is sufficient to prove the well-posedness of Problem (19), but is not necessary to prove so. Note as well that, as expected, the results of Lemma 3.2 and Proposition 3.3 remain valid under more stringent conformity prescriptions (like it is for instance the case for c-FE).*

Remark 3.5. *The results of Lemma 3.2 and Proposition 3.3, that do not assume that discrete functions in $V_{h,0}$ are piecewise polynomial, apply to the mixed-order multiscale HHO method of [17, Section 5.1], for which the local (oscillatory) basis functions are explicitly known on each coarse cell (in practice, they are in fact approximated using a fine submesh of the coarse cell).*

Other, more recent, examples of skeletal methods include the c/nc-VE methods, as well as the HHO method. The specificity of VE methods is to consider virtual local spaces $V(T)$, i.e. local spaces which are spanned by functions that are (i) in general, not fully computable, and (ii) by definition, never computed. The local virtual functions are usually implicitly defined as the solutions to some PDEs posed in the cell. The VE methods are defined using computable (in terms of the DoFs) projections of the virtual functions, and are stabilized through computable penalizations. Polynomial-based VE methods hinge on local virtual spaces (i) spanned by the solutions to PDEs that feature polynomial data, and (ii) that contain the space $\mathbb{P}_d^k(T)$ for some $k \geqslant 1$. They are the VE methods that are classically encountered in the literature. As such, we will henceforth refer to them simply as VE methods, and we will exclusively focus on them in the sequel. An example of (nonconforming) method that hinges on a different kind of virtual spaces is given by the equal-order multiscale HHO method of [17, Section 5.2], for which local virtual functions do solve (oscillatory) PDEs with polynomial data, but the local virtual spaces do not contain polynomials in general. Whereas the nc-VE method has been bridged in [19, Section 2.4] to the (nc-)HHO method, the fact that the c-VE method can be recast into a conforming-HHO method has not been demonstrated yet in the literature. This is one object of the next section, that provides a unified formulation/analysis of c/nc-VE/HHO methods.

4 Bridging Hybrid High-Order and Virtual Element methods

In this section, we reformulate and analyze, within a unified framework inspired from HHO methods, the conforming and nonconforming versions of the (polynomial-based) VE method. By recasting the c-VE method into a conforming-HHO method, we end up bridging, by the same, the two families of methods. We let $k \geqslant 1$ be a given integer, that will stand for the order of the method. Doing so, we adopt the classical VE notation. HHO methods of the same order are classically defined using $k' = k - 1 \geqslant 0$.

4.1 Local virtual spaces

4.1.1 Conforming case

Let $T \in \mathcal{T}_h$ be star-shaped with respect to a ball whose radius is comparable to h_T. For any $\nu \in \mathcal{V}_T$ (respectively $\nu \in \mathcal{V}_F$, for $F \in \mathcal{F}_T$), we let ι_T^ν (respectively ι_F^ν) denote the linear form on $C^0(\partial T)$ (respectively $C^0(\partial F)$) so that, for all $v \in C^0(\partial T)$ (respectively $v \in C^0(\partial F)$), $\iota_T^\nu(v) = v(\boldsymbol{x}_\nu)$ (respectively $\iota_F^\nu(v) = v(\boldsymbol{x}_\nu)$).

Preliminary results. Let $d = 2$, and consider a face $F \in \mathcal{F}_T$ such that $F := [\boldsymbol{x}_{\nu_1}, \boldsymbol{x}_{\nu_2}]$. Then, we have the following result.

Proposition 4.1. *The triple $(F, \mathbb{P}_1^k(F), \Sigma_{1,F}^k)$, where $\Sigma_{1,F}^k := \Sigma_{1,F}^{k,\partial} \bigcup \Sigma_{1,F}^{k,\mathrm{o}}$ is the collection of $\Sigma_{1,F}^{k,\partial} := \{\iota_F^\nu\}_{\nu\in\mathcal{V}_F}$ and $\Sigma_{1,F}^{k,\mathrm{o}} := \{\pi_{F|\mathbb{P}_1^k(F)}^{k-2,j}\}_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}$, is a finite element in the sense of Ciarlet.*

Proof. Clearly, $\dim(\mathbb{P}_1^k(F)) = n_{1,F}^k (= \mathrm{card}(\Sigma_{1,F}^k))$. Now, let $v \in \mathbb{P}_1^k(F)$ be such that $\underline{\Sigma}_{1,F}^k(v) = \underline{0}$. An integration by parts yields

$$\int_F (v')^2 = -\int_F v\, v'' + [v(\boldsymbol{x}_{\nu_2})v'(\boldsymbol{x}_{\nu_2}) - v(\boldsymbol{x}_{\nu_1})v'(\boldsymbol{x}_{\nu_1})] = -\int_F \pi_F^{k-2} v\, v'' = 0,$$

where we have used that $v'' \in \mathbb{P}_1^{k-2}(F)$, and that $v(\boldsymbol{x}_\nu) = 0$ for both $\nu \in \mathcal{V}_F$ and $\pi_F^{k-2} v \equiv 0$. Hence, v is constant on F, and since it is zero on ∂F, it is identically zero. This concludes the proof. □

Let $r_{1,F}^k : \mathbb{R}^{n_{1,F}^k} \to \mathbb{P}_1^k(F)$ be the operator such that, for any

$$\underline{\mathrm{v}}_F := \big((\mathrm{v}_F^{\partial,\nu})_{\nu\in\mathcal{V}_F}, (\mathrm{v}_F^{\mathrm{o},j})_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}\big) \in \mathbb{R}^{n_{1,F}^k},$$

$r_{1,F}^k \underline{\mathrm{v}}_F \in \mathbb{P}_1^k(F)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_F (r_{1,F}^k \underline{\mathrm{v}}_F)' w' = -\int_F \mathrm{v}_F^{\mathrm{o}}\, w'' + [\mathrm{v}_F^{\partial,\nu_2} w'(\boldsymbol{x}_{\nu_2}) - \mathrm{v}_F^{\partial,\nu_1} w'(\boldsymbol{x}_{\nu_1})] \qquad \forall w \in \mathbb{P}_1^k(F), \\ r_{1,F}^k \underline{\mathrm{v}}_F(\boldsymbol{x}_{\nu_1}) = \mathrm{v}_F^{\partial,\nu_1}, \end{cases} \tag{23}$$

where we have introduced the notation $\mathrm{v}_F^{\mathrm{o}} := \sum_{j=1}^{\mathrm{N}_1^{k-2}} \mathrm{v}_F^{\mathrm{o},j} \psi_{F,j}^{k-2} \in \mathbb{P}_1^{k-2}(F)$.

Proposition 4.2. *The operator $r_{1,F}^k$ defined by (23) coincides with the (canonical) reconstruction operator $R_{1,F}^k = (\underline{\Sigma}_{1,F}^k)^{-1}$.*

Proof. Let $\mathbb{R}^{n_{1,F}^k} \ni \underline{\mathrm{v}}_F := \underline{\Sigma}_{1,F}^k(v)$ for some $v \in \mathbb{P}_1^k(F)$. Plugging $\underline{\mathrm{v}}_F$ into (23), we obtain by integration by parts

$$\int_F (r_{1,F}^k \underline{\mathrm{v}}_F)' w' = \int_F v'\, w' \qquad \forall w \in \mathbb{P}_1^k(F),$$

which, combined to the condition $r_{1,F}^k \underline{\mathrm{v}}_F(\boldsymbol{x}_{\nu_1}) = \mathrm{v}_F^{\partial,\nu_1} = v(\boldsymbol{x}_{\nu_1})$, finally yields that $r_{1,F}^k \underline{\mathrm{v}}_F = r_{1,F}^k(\underline{\Sigma}_{1,F}^k(v)) = v$. This is true for any $v \in \mathbb{P}_1^k(F)$, hence $r_{1,F}^k = (\underline{\Sigma}_{1,F}^k)^{-1} = R_{1,F}^k$. □

Since the operators $r_{1,F}^k$ and $R_{1,F}^k$ coincide, we will henceforth drop the notation $R_{1,F}^k$ and stick to the notation $r_{1,F}^k$.

It is an easy matter to see that the collection $\Sigma_{1,F}^k$ of linear forms on $\mathbb{P}_1^k(F)$ can be extended to a collection $\overline{\Sigma}_{1,F}^k$ of linear forms on $C^0(F)$. We can hence define the interpolation operator $\mathcal{I}_{1,F}^k : C^0(F) \to \mathbb{P}_1^k(F)$ such that $\mathcal{I}_{1,F}^k := r_{1,F}^k \circ \overline{\Sigma}_{1,F}^k$.

Lemma 4.3 (Stability of $\mathcal{I}_{1,F}^k$). *For all $v \in C^0(F)$, there holds*

$$\left\|\mathcal{I}_{1,F}^k v\right\|_{\infty,F} \lesssim \|v\|_{\infty,F}.$$

Proof. To prove the result, we rewrite $\mathcal{I}_{1,F}^k v$ as

$$\mathcal{I}_{1,F}^k v = v(\boldsymbol{x}_{\nu_1})\varphi_{F,\nu_1}^{\partial} + v(\boldsymbol{x}_{\nu_2})\varphi_{F,\nu_2}^{\partial} + \sum_{j=1}^{\mathrm{N}_1^{k-2}} \pi_F^{k-2,j}(v)\varphi_{F,j}^{\mathrm{o}},$$

where we recall that the shape functions $\varphi_{F,\nu_1}^{\partial}, \varphi_{F,\nu_2}^{\partial}, \varphi_{F,j}^{\mathrm{o}}$ belong to $\mathbb{P}_1^k(F)$. By stability of the L^2-orthogonal projector π_F^{k-2} in the $L^\infty(F)$-norm (cf., e.g., [22, Lemma 3.2 with $N = 1$]), we infer the desired estimate. □

The case $d = 2$. If $d = 2$, the conforming local virtual space on T is defined as

$$V_2^k(T) := \left\{ v \in H^1(T) \mid \triangle v \in \mathbb{P}_2^{k-1}(T),\ v_{|\partial T} \in \mathbb{P}_1^{k,c}(\mathcal{F}_T) \right\}, \tag{24}$$

where $\mathbb{P}_1^{k,c}(\mathcal{F}_T) := \mathbb{P}_1^k(\mathcal{F}_T) \bigcap C^0(\partial T)$. One can show (cf., e.g., [10, Remark 2.3]) that functions in $V_2^k(T)$ belong to $C^0(\overline{T})$. Besides, $\mathbb{P}_2^k(T) \subset V_2^k(T)$.

Proposition 4.4. *The triple $(T, V_2^k(T), \Sigma_{2,T}^k)$, where $V_2^k(T)$ is given by (24), and $\Sigma_{2,T}^k := \Sigma_{2,T}^{k,\partial} \bigcup \Sigma_{2,T}^{k,\mathrm{o}}$ is the collection of*

- $\Sigma_{2,T}^{k,\partial} := \left\{\iota^{\nu}_{T|\mathbb{P}_1^{k,c}(\mathcal{F}_T)}\right\}_{\nu\in\mathcal{V}_T} \bigcup \left\{\pi^{k-2,j}_{F|\mathbb{P}_1^{k,c}(\mathcal{F}_T)}(\cdot_{|F})\right\}_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}^{F\in\mathcal{F}_T}$,
- $\Sigma_{2,T}^{k,\mathrm{o}} := \left\{\pi^{k-1,i}_{T|V_2^k(T)}\right\}_{i\in\{1,\ldots,\mathrm{N}_2^{k-1}\}}$,

is a finite element in the sense of Ciarlet.

Proof. For any

$$\underline{\mathrm{v}}_T := \left((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}^{F\in\mathcal{F}_T}, (\mathrm{v}_T^{\mathrm{o},i})_{i\in\{1,\ldots,\mathrm{N}_2^{k-1}\}}\right) \in \mathbb{R}^{n_{2,T}^k},$$

we have $\underline{\mathrm{v}}_T^{\partial} = \left((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}^{F\in\mathcal{F}_T}\right) \in \mathbb{R}^{n_{2,T}^{k,\partial}}$,

$$\underline{\mathrm{v}}_{T,F}^{\partial} = \left((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_F}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}\right) \in \mathbb{R}^{n_{1,F}^k} \text{ for any } F \in \mathcal{F}_T,$$

and $\underline{\mathrm{v}}_T^{\mathrm{o}} = (\mathrm{v}_T^{\mathrm{o},i})_{i\in\{1,\ldots,\mathrm{N}_2^{k-1}\}} \in \mathbb{R}^{n_{2,T}^{k,\mathrm{o}}}$. Let us define the operator $\Phi_{2,T}^k : \mathbb{R}^{n_{2,T}^k} \to V_2^k(T)$ such that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{2,T}^k}$,

$$-\triangle\Phi_{2,T}^k\underline{\mathrm{v}}_T = \mathrm{v}_T^{\mathrm{o}} \text{ in } T, \qquad \Phi_{2,T}^k\underline{\mathrm{v}}_{T|F} = r_{1,F}^k\underline{\mathrm{v}}_{T,F}^{\partial} \text{ on all } F \in \mathcal{F}_T,$$

where we have introduced the notation $\mathrm{v}_T^{\mathrm{o}} := \sum_{i=1}^{\mathrm{N}_2^{k-1}} \mathrm{v}_T^{\mathrm{o},i}\psi_{T,i}^{k-1} \in \mathbb{P}_2^{k-1}(T)$. Since this Dirichlet problem is well-posed, if $\Phi_{2,T}^k\underline{\mathrm{v}}_T \equiv 0$, then necessarily $\mathrm{v}_T^{\mathrm{o}} \equiv 0$ and $r_{1,F}^k\underline{\mathrm{v}}_{T,F}^{\partial} \equiv 0$ for all $F \in \mathcal{F}_T$. Owing to Proposition 4.1, we thus have $\underline{\mathrm{v}}_T = \underline{0}$. This means that $\Phi_{2,T}^k$ is injective, and hence that $\dim(V_2^k(T)) \geqslant n_{2,T}^k$. Now, let $v \in V_2^k(T)$ be such that $\underline{\Sigma}_{2,T}^k(v) = \underline{0}$. Owing to Proposition 4.2, for all $F \in \mathcal{F}_T$, we have $v_{|F} = r_{1,F}^k\big(\underline{\Sigma}_{1,F}^k(v_{|F})\big) = r_{1,F}^k\big(\underline{\Sigma}_{2,T,F}^{k,\partial}(v)\big) \equiv 0$, which yields by integration by parts

$$\int_T |\boldsymbol{\nabla} v|^2 = -\int_T v\,\triangle v = -\int_T \pi_T^{k-1} v\,\triangle v = 0.$$

Hence, $v \equiv 0$, and $\underline{\Sigma}_{2,T}^k$ is injective. Combined to the fact that $\dim(V_2^k(T)) \geqslant n_{2,T}^k$, we finally prove that $\underline{\Sigma}_{2,T}^k$ is bijective. □

Let us introduce the operator $r_{1,\partial T}^k : \mathbb{R}^{n_{2,T}^{k,\partial}} \to \mathbb{P}_1^{k,c}(\mathcal{F}_T)$ so that, for any $\underline{\mathrm{v}}_T^{\partial} \in \mathbb{R}^{n_{2,T}^{k,\partial}}$, $r_{1,\partial T}^k\underline{\mathrm{v}}_{T|F}^{\partial} := r_{1,F}^k\underline{\mathrm{v}}_{T,F}^{\partial}$ for all $F \in \mathcal{F}_T$. We are now in position to introduce the operator $r_{2,T}^k : \mathbb{R}^{n_{2,T}^k} \to V_2^k(T)$ so that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{2,T}^k}$, $r_{2,T}^k\underline{\mathrm{v}}_T \in V_2^k(T)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla} r_{2,T}^k\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^{\mathrm{o}}\,\triangle w + \left\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, r_{1,\partial T}^k\underline{\mathrm{v}}_T^{\partial}\right\rangle_{-\frac{1}{2},\partial T} & \forall w \in V_2^k(T), \\ \displaystyle\int_T r_{2,T}^k\underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^{\mathrm{o}}. \end{cases} \tag{25}$$

Proposition 4.5. *The operator $r^k_{2,T}$ defined by (25) coincides with the (canonical) reconstruction operator $R^k_{2,T} = (\underline{\Sigma}^k_{2,T})^{-1}$.*

Proof. Let $\mathbb{R}^{n^k_{2,T}} \ni \underline{\mathrm{v}}_T := \underline{\Sigma}^k_{2,T}(v)$ for some $v \in V^k_2(T)$. Plugging $\underline{\mathrm{v}}_T$ into (25), we infer by integration by parts

$$\int_T \boldsymbol{\nabla} r^k_{2,T}\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla} w = -\int_T \pi^{k-1}_T v\,\triangle w + \Big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, r^k_{1,\partial T}\big(\underline{\Sigma}^{k,\partial}_{2,T}(v)\big)\Big\rangle_{-\frac12,\partial T}$$
$$= \int_T \boldsymbol{\nabla} v\cdot\boldsymbol{\nabla} w \qquad \forall w\in V^k_2(T),$$

where we have used that $\triangle w \in \mathbb{P}^{k-1}_2(T)$ and that, for all $F\in\mathcal{F}_T$, $r^k_{1,\partial T}\big(\underline{\Sigma}^{k,\partial}_{2,T}(v)\big)_{|F} = r^k_{1,F}\big(\underline{\Sigma}^{k,\partial}_{2,T,F}(v)\big) = r^k_{1,F}\big(\underline{\Sigma}^k_{1,F}(v_{|F})\big) = v_{|F}$ (cf. Proposition 4.2). Since $\int_T r^k_{2,T}\underline{\mathrm{v}}_T = \int_T \pi^{k-1}_T v = \int_T v$, we finally infer that $r^k_{2,T}\underline{\mathrm{v}}_T = r^k_{2,T}\big(\underline{\Sigma}^k_{2,T}(v)\big) = v$. This is true for any $v\in V^k_2(T)$, hence $r^k_{2,T} = (\underline{\Sigma}^k_{2,T})^{-1} = R^k_{2,T}$. □

As above, since the operators $r^k_{2,T}$ and $R^k_{2,T}$ coincide, we henceforth drop the notation $R^k_{2,T}$ and stick to $r^k_{2,T}$.

Let us introduce the operator $\mathcal{I}^k_{1,\partial T} : C^0(\partial T)\to \mathbb{P}^{k,c}_1(\mathcal{F}_T)$ so that, for any $v\in C^0(\partial T)$, $\mathcal{I}^k_{1,\partial T}v_{|F} := \mathcal{I}^k_{1,F}(v_{|F})$ for all $F\in\mathcal{F}_T$. It is clear that the collection $\Sigma^k_{2,T}$ of linear forms on $V^k_2(T)$ can be extended to a collection $\overline{\Sigma}^k_{2,T}$ of linear forms on

$$H^{1,c}(T) := \{v\in H^1(T)\mid v_{|\partial T}\in C^0(\partial T)\}.$$

We can hence define the interpolation operator $\mathcal{I}^k_{2,T} : H^{1,c}(T)\to V^k_2(T)$ such that $\mathcal{I}^k_{2,T} := r^k_{2,T}\circ\overline{\Sigma}^k_{2,T}$. Besides, we remark that, for all $v\in H^{1,c}(T)$, $r^k_{1,\partial T}\big(\overline{\Sigma}^{k,\partial}_{2,T}(v)\big) = \mathcal{I}^k_{1,\partial T}(v_{|\partial T})$. We have the following stability result for $\mathcal{I}^k_{2,T}$.

Lemma 4.6 (Stability of $\mathcal{I}^k_{2,T}$)**.** *For all $v\in H^2(T)$, there holds*

- $\big|\mathcal{I}^k_{2,T}v\big|_{1,T} \lesssim |v|_{1,T} + h_T|v|_{2,T}$;
- $\big\|\mathcal{I}^k_{2,T}v\big\|_{0,T} \lesssim \|v\|_{0,T} + h_T|v|_{1,T} + h^2_T|v|_{2,T}$.

Proof. First, remark that if $v\in H^2(T)$, $v\in C^0(\overline{T})$, and hence $v\in H^{1,c}(T)$; consequently, $\mathcal{I}^k_{2,T}v$ has a sense. Let us test (25) with $\underline{\mathrm{v}}_T := \overline{\Sigma}^k_{2,T}(v)$. There holds

$$\int_T \boldsymbol{\nabla}\mathcal{I}^k_{2,T}v\cdot\boldsymbol{\nabla} w = -\int_T v\,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, \mathcal{I}^k_{1,\partial T}(v_{|\partial T})\big\rangle_{-\frac12,\partial T} \qquad \forall w\in V^k_2(T),$$

where we have used that $\triangle w\in\mathbb{P}^{k-1}_2(T)$ and that $r^k_{1,\partial T}\big(\overline{\Sigma}^{k,\partial}_{2,T}(v)\big) = \mathcal{I}^k_{1,\partial T}(v_{|\partial T})$. We equivalently rewrite the equality above as

$$\int_T \boldsymbol{\nabla}\mathcal{I}^k_{2,T}v\cdot\boldsymbol{\nabla} w = -\int_T (v-\pi^0_T v)\,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, \mathcal{I}^k_{1,\partial T}(v_{|\partial T}-\pi^0_T v)\big\rangle_{-\frac12,\partial T}$$
$$\forall w\in V^k_2(T),$$

and, by Cauchy–Schwarz inequality, we infer, for any $w\in V^k_2(T)$,

$$\left|\int_T \boldsymbol{\nabla}\mathcal{I}^k_{2,T}v\cdot\boldsymbol{\nabla} w\right| \lesssim \|v-\pi^0_T v\|_{0,T}\|\triangle w\|_{0,T} + \Big(h_T^{-\frac12}\big\|\mathcal{I}^k_{1,\partial T}(v_{|\partial T}-\pi^0_T v)\big\|_{0,\partial T}$$
$$+h_T^{\frac12}\big|\mathcal{I}^k_{1,\partial T}(v_{|\partial T}-\pi^0_T v)\big|_{1,\partial T}\Big)\sup_{z\in H^1(\partial T)}\frac{\big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, z\big\rangle_{-\frac12,\partial T}}{h_T^{-1/2}\|z\|_{0,\partial T} + h_T^{1/2}|z|_{1,\partial T}} =: \mathfrak{T}_1+\mathfrak{T}_2.$$

To estimate the first term $\mathfrak{T}_1$, we apply (7) to $w\in V^k_2(T)$, and the Poincaré inequality (6) to $(v-\pi^0_T v)$. We get

$$\mathfrak{T}_1 \lesssim |v|_{1,T}|w|_{1,T}.$$

To estimate the second term $\mathfrak{T}_2$, we make use of the estimate (9). This yields, using again (7),

$$\sup_{z\in H^1(\partial T)} \frac{\langle \boldsymbol{\nabla} w_{|\partial T}\cdot \boldsymbol{n}_T, z\rangle_{-\frac12,\partial T}}{h_T^{-1/2}\|z\|_{0,\partial T}+h_T^{1/2}|z|_{1,\partial T}} \lesssim |w|_{1,T}.$$

To estimate the second factor, we first use an inverse inequality (cf., e.g., [30, Lemma 1.138]) for $\mathcal{I}_{1,F}^k\big(v_{|F}-\pi_T^0 v\big)\in\mathbb{P}_1^k(F)$ on the 1-simplex F for all $F\in\mathcal{F}_T$, to infer

$$\begin{aligned} h_T^{-\frac12}\big\|\mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\big\|_{0,\partial T}+h_T^{\frac12}\big|\mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\big|_{1,\partial T}&\\ \lesssim h_T^{-\frac12}\big\|\mathcal{I}_{1,\partial T}^k\big(v_{|\partial T}-\pi_T^0 v\big)\big\|_{0,\partial T}\lesssim \big\|\mathcal{I}_{1,\partial T}^k\big(v_{|\partial T}-\pi_T^0 v\big)\big\|_{\infty,\partial T}.&\end{aligned}$$

Then, applying Lemma 4.3, and since $v\in C^0(\overline{T})$, there holds

$$\begin{aligned} h_T^{-\frac12}\big\|\mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\big\|_{0,\partial T}+h_T^{\frac12}\big|\mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\big|_{1,\partial T}&\lesssim \big\|v_{|\partial T}-\pi_T^0 v\big\|_{\infty,\partial T}\\ &\lesssim \big\|v-\pi_T^0 v\big\|_{\infty,T},\end{aligned}$$

which, by application of the Sobolev's inequality (8), and of the Poincaré inequality (6), yields

$$h_T^{-\frac12}\big\|\mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\big\|_{0,\partial T}+h_T^{\frac12}\big|\mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\big|_{1,\partial T}\lesssim |v|_{1,T}+h_T|v|_{2,T}.$$

Finally, we get

$$\mathfrak{T}_2\lesssim \big(|v|_{1,T}+h_T|v|_{2,T}\big)|w|_{1,T}.$$

In conclusion, there holds, for any $w\in V_2^k(T)$,

$$\left|\int_T \boldsymbol{\nabla}\mathcal{I}_{2,T}^k v\cdot\boldsymbol{\nabla} w\right|\lesssim \big(|v|_{1,T}+h_T|v|_{2,T}\big)|w|_{1,T}.$$

Taking $w=\mathcal{I}_{2,T}^k v\in V_2^k(T)$ provides the expected estimate in the $H^1(T)$-seminorm. To obtain the estimate in the $L^2(T)$-norm, it suffices to remark that $\int_T \mathcal{I}_{2,T}^k v=\int_T \pi_T^{k-1}v=\int_T v$. Hence, by the triangle inequality and the Poincaré inequality (6), we infer

$$\big\|\mathcal{I}_{2,T}^k v\big\|_{0,T}\leqslant \|v\|_{0,T}+\big\|v-\mathcal{I}_{2,T}^k v\big\|_{0,T}\lesssim \|v\|_{0,T}+h_T\big(|v|_{1,T}+\big|\mathcal{I}_{2,T}^k v\big|_{1,T}\big).$$

The conclusion then follows from the estimate in the $H^1(T)$-seminorm. □

Remark 4.7. *We note that the only moment in the proof of Lemma 4.6 (the same observation applies to the case $d=3$; see Lemma 4.11) where the assumption that T is star-shaped with respect to a ball is needed is when it comes to use (9). Hidden behind this, is the result (67) that states the existence of a lifting operator with optimal scaling for functions in $H^1(\partial T)$. Besides, it is worth noticing that, in the proofs of Lemmas 4.6 and 4.11, the liftings actually need only be constructed, respectively, for functions in $\mathbb{P}_1^{k,c}(\mathcal{F}_T)$ and $V_2^{k,c}(\mathcal{F}_T)$, both subspaces of $H^1(\partial T)$. One can hence use, instead of (67) and (9), the results of [10, Eq. (2.48) and Lemma 5.3].*

In view of the case $d=3$, we state a sharper stability estimate for the interpolation operator.

Lemma 4.8 (Sharper stability estimate for $\mathcal{I}_{2,T}^k$)**.** *For all $v\in H^{3/2}(T)$, there holds*

$$\big\|\mathcal{I}_{2,T}^k v\big\|_{0,T}+h_T\big|\mathcal{I}_{2,T}^k v\big|_{1,T}\lesssim \|v\|_{0,T}+h_T|v|_{1,T}+h_T^{3/2}|v|_{\frac32,T}.$$

Proof. The proof exactly follows the one of Lemma 4.6, with a slight variation when it comes to apply the Sobolev's inequality (8). We make use of the following sharper estimate (cf., e.g., [12, Eq. (2.4)]): for all $z\in H^{3/2}(T)$,

$$\|z\|_{\infty,T}\lesssim h_T^{-1}\|z\|_{0,T}+|z|_{1,T}+h_T^{1/2}|z|_{\frac32,T}.$$

We thus obtain

$$\mathfrak{T}_2\lesssim \big(|v|_{1,T}+h_T^{1/2}|v|_{\frac32,T}\big)|w|_{1,T},$$

which finally yields the desired estimate. □

The case $d = 3$. If $d = 3$, the conforming local virtual space on T is defined as

$$V_3^k(T) := \left\{ v \in H^1(T) \mid \triangle v \in \mathbb{P}_3^{k-1}(T),\, v_{|\partial T} \in V_2^{k,c}(\mathcal{F}_T) \right\}, \tag{26}$$

where $V_2^{k,c}(\mathcal{F}_T) := V_2^k(\mathcal{F}_T) \bigcap C^0(\partial T)$ with broken space

$$V_2^k(\mathcal{F}_T) := \{ v \in L^2(\partial T) \mid v_{|F} \in V_2^k(F)\, \forall F \in \mathcal{F}_T \},$$

where $V_2^k(F)$ is given by (24) with $T \leftarrow F$. Since $V_2^k(F) \subset C^0(F)$ for all $F \in \mathcal{F}_T$, one has $V_3^k(T) \subset H^{1,c}(T)$. Besides, $\mathbb{P}_3^k(T) \subset V_3^k(T)$.

Proposition 4.9. *The triple $(T, V_3^k(T), \Sigma_{3,T}^k)$, where $V_3^k(T)$ is given by (26), and $\Sigma_{3,T}^k := \Sigma_{3,T}^{k,\partial} \bigcup \Sigma_{3,T}^{k,\mathrm{o}}$ is the collection of*

- $\Sigma_{3,T}^{k,\partial} := \left\{ \iota^{\nu}_{T|V_2^{k,c}(\mathcal{F}_T)} \right\}_{\nu \in \mathcal{V}_T} \bigcup \left\{ \pi^{k-2,m}_{e|V_2^{k,c}(\mathcal{F}_T)}(\cdot_{|e}) \right\}_{m \in \{1,\ldots,\mathrm{N}_1^{k-2}\}}^{e \in \mathcal{E}_T} \bigcup \left\{ \pi^{k-1,j}_{F|V_2^{k,c}(\mathcal{F}_T)}(\cdot_{|F}) \right\}_{j \in \{1,\ldots,\mathrm{N}_2^{k-1}\}}^{F \in \mathcal{F}_T}$,
- $\Sigma_{3,T}^{k,\mathrm{o}} := \left\{ \pi^{k-1,i}_{T|V_3^k(T)} \right\}_{i \in \{1,\ldots,\mathrm{N}_3^{k-1}\}}$,

is a finite element in the sense of Ciarlet.

Proof. For any

$$\underline{\mathrm{v}}_T := \left((\mathrm{v}_T^{\partial,\nu})_{\nu \in \mathcal{V}_T}, (\mathrm{v}_{T,e}^{\partial,m})_{m \in \{1,\ldots,\mathrm{N}_1^{k-2}\}}^{e \in \mathcal{E}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j \in \{1,\ldots,\mathrm{N}_2^{k-1}\}}^{F \in \mathcal{F}_T}, (\mathrm{v}_T^{\mathrm{o},i})_{i \in \{1,\ldots,\mathrm{N}_3^{k-1}\}} \right) \in \mathbb{R}^{n_{3,T}^k},$$

we have $\underline{\mathrm{v}}_T^{\partial} = \left((\mathrm{v}_T^{\partial,\nu})_{\nu \in \mathcal{V}_T}, (\mathrm{v}_{T,e}^{\partial,m})_{m \in \{1,\ldots,\mathrm{N}_1^{k-2}\}}^{e \in \mathcal{E}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j \in \{1,\ldots,\mathrm{N}_2^{k-1}\}}^{F \in \mathcal{F}_T} \right) \in \mathbb{R}^{n_{3,T}^{k,\partial}}$,

$$\underline{\mathrm{v}}_{T,F}^{\partial} = \left((\mathrm{v}_T^{\partial,\nu})_{\nu \in \mathcal{V}_F}, (\mathrm{v}_{T,e}^{\partial,m})_{m \in \{1,\ldots,\mathrm{N}_1^{k-2}\}}^{e \in \mathcal{E}_F}, (\mathrm{v}_{T,F}^{\partial,j})_{j \in \{1,\ldots,\mathrm{N}_2^{k-1}\}} \right) \in \mathbb{R}^{n_{2,F}^k} \text{ for any } F \in \mathcal{F}_T,$$

and $\underline{\mathrm{v}}_T^{\mathrm{o}} = (\mathrm{v}_T^{\mathrm{o},i})_{i \in \{1,\ldots,\mathrm{N}_3^{k-1}\}} \in \mathbb{R}^{n_{3,T}^{k,\mathrm{o}}}$. We introduce the operator $\Phi_{3,T}^k : \mathbb{R}^{n_{3,T}^k} \to V_3^k(T)$ such that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{3,T}^k}$,

$$-\triangle \Phi_{3,T}^k \underline{\mathrm{v}}_T = \mathrm{v}_T^{\mathrm{o}} \text{ in } T, \qquad \Phi_{3,T}^k \underline{\mathrm{v}}_{T|F} = r_{2,F}^k \underline{\mathrm{v}}_{T,F}^{\partial} \text{ on all } F \in \mathcal{F}_T,$$

where $r_{2,F}^k$ is defined by (25) with $T \leftarrow F$, and where we make use of the notation $\mathrm{v}_T^{\mathrm{o}} := \sum_{i=1}^{\mathrm{N}_3^{k-1}} \mathrm{v}_T^{\mathrm{o},i} \psi_{T,i}^{k-1} \in \mathbb{P}_3^{k-1}(T)$. Since this Dirichlet problem is well-posed, if $\Phi_{3,T}^k \underline{\mathrm{v}}_T \equiv 0$, then necessarily $\mathrm{v}_T^{\mathrm{o}} \equiv 0$ and $r_{2,F}^k \underline{\mathrm{v}}_{T,F}^{\partial} \equiv 0$ for all $F \in \mathcal{F}_T$. Proposition 4.4 then implies that $\underline{\mathrm{v}}_T = \underline{0}$, which means that $\Phi_{3,T}^k$ is injective, and that $\dim(V_3^k(T)) \geqslant n_{3,T}^k$. Let now $v \in V_3^k(T)$ be such that $\underline{\Sigma}_{3,T}^k(v) = \underline{0}$. Owing to Proposition 4.5, for all $F \in \mathcal{F}_T$, we have $v_{|F} = r_{2,F}^k\big(\underline{\Sigma}_{2,F}^k(v_{|F})\big) = r_{2,F}^k\big(\underline{\Sigma}_{3,T,F}^{k,\partial}(v)\big) \equiv 0$, which yields by integration by parts

$$\int_T |\boldsymbol{\nabla} v|^2 = -\int_T v \triangle v = -\int_T \pi_T^{k-1} v \triangle v = 0.$$

Hence, $v \equiv 0$, and $\underline{\Sigma}_{3,T}^k$ is injective. Combined to the fact that $\dim(V_3^k(T)) \geqslant n_{3,T}^k$, we conclude that $\underline{\Sigma}_{3,T}^k$ is bijective. □

Let us define the operator $r_{2,\partial T}^k : \mathbb{R}^{n_{3,T}^{k,\partial}} \to V_2^{k,c}(\mathcal{F}_T)$ so that, for any $\underline{\mathrm{v}}_T^{\partial} \in \mathbb{R}^{n_{3,T}^{k,\partial}}$, $r_{2,\partial T}^k \underline{\mathrm{v}}_{T|F}^{\partial} := r_{2,F}^k \underline{\mathrm{v}}_{T,F}^{\partial}$ for all $F \in \mathcal{F}_T$. We can now introduce the operator $r_{3,T}^k : \mathbb{R}^{n_{3,T}^k} \to V_3^k(T)$ so that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{3,T}^k}$, $r_{3,T}^k \underline{\mathrm{v}}_T \in V_3^k(T)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla} r_{3,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^{\mathrm{o}} \triangle w + \left\langle \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T, r_{2,\partial T}^k \underline{\mathrm{v}}_T^{\partial} \right\rangle_{-\frac{1}{2},\partial T} & \forall w \in V_3^k(T), \\ \displaystyle\int_T r_{3,T}^k \underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^{\mathrm{o}}. \end{cases} \tag{27}$$

Proposition 4.10. *The operator $r_{3,T}^k$ defined by (27) coincides with the (canonical) reconstruction operator $R_{3,T}^k = (\underline{\Sigma}_{3,T}^k)^{-1}$.*

Proof. Let $\mathbb{R}^{n^k_{3,T}} \ni \underline{\mathrm{v}}_T := \underline{\Sigma}^k_{3,T}(v)$ for some $v \in V^k_3(T)$. Plugging $\underline{\mathrm{v}}_T$ into (27), we infer by integration by parts

$$\int_T \boldsymbol{\nabla} r^k_{3,T}\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla} w = -\int_T \pi^{k-1}_T v\,\triangle w + \Big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, r^k_{2,\partial T}\big(\underline{\Sigma}^{k,\partial}_{3,T}(v)\big)\Big\rangle_{-\frac12,\partial T}$$
$$= \int_T \boldsymbol{\nabla} v\cdot\boldsymbol{\nabla} w \qquad \forall w \in V^k_3(T),$$

where we have used that $\triangle w \in \mathbb{P}^{k-1}_3(T)$ and that, for all $F \in \mathcal{F}_T$, $r^k_{2,\partial T}\big(\underline{\Sigma}^{k,\partial}_{3,T}(v)\big)_{|F} = r^k_{2,F}\big(\underline{\Sigma}^{k,\partial}_{3,T,F}(v)\big) = r^k_{2,F}\big(\underline{\Sigma}^k_{2,F}(v_{|F})\big) = v_{|F}$ (cf. Proposition 4.5). Since $\int_T r^k_{3,T}\underline{\mathrm{v}}_T = \int_T \pi^{k-1}_T v = \int_T v$, we finally infer that $r^k_{3,T}\underline{\mathrm{v}}_T = r^k_{3,T}\big(\underline{\Sigma}^k_{3,T}(v)\big) = v$. This is valid for any $v \in V^k_3(T)$, hence $r^k_{3,T} = (\underline{\Sigma}^k_{3,T})^{-1} = R^k_{3,T}$. □

Since the operators $r^k_{3,T}$ and $R^k_{3,T}$ coincide, we henceforth drop the notation $R^k_{3,T}$ and we stick to $r^k_{3,T}$.

We introduce the operator $\mathcal{I}^k_{2,\partial T} : C^0(\partial T) \to V^{k,c}_2(\mathcal{F}_T)$ so that, for any $v \in C^0(\partial T)$, $\mathcal{I}^k_{2,\partial T}v_{|F} := \mathcal{I}^k_{2,F}(v_{|F})$ for all $F \in \mathcal{F}_T$. The collection $\Sigma^k_{3,T}$ of linear forms on $V^k_3(T)$ can clearly be extended to a collection $\overline{\Sigma}^k_{3,T}$ of linear forms on $H^{1,c}(T)$. Thus, we can introduce the interpolation operator $\mathcal{I}^k_{3,T} : H^{1,c}(T) \to V^k_3(T)$ such that $\mathcal{I}^k_{3,T} := r^k_{3,T}\circ\overline{\Sigma}^k_{3,T}$. We remark that, for all $v \in H^{1,c}(T)$, $r^k_{2,\partial T}\big(\overline{\Sigma}^{k,\partial}_{3,T}(v)\big) = \mathcal{I}^k_{2,\partial T}(v_{|\partial T})$. We have the following stability result for $\mathcal{I}^k_{3,T}$.

Lemma 4.11 (Stability of $\mathcal{I}^k_{3,T}$)**.** *For all $v \in H^2(T)$, there holds*

- $\big|\mathcal{I}^k_{3,T}v\big|_{1,T} \lesssim |v|_{1,T} + h_T|v|_{2,T}$;
- $\big\|\mathcal{I}^k_{3,T}v\big\|_{0,T} \lesssim \|v\|_{0,T} + h_T|v|_{1,T} + h^2_T|v|_{2,T}$.

Proof. First, remark that if $v \in H^2(T)$, $v \in C^0(\overline{T})$, and hence $v \in H^{1,c}(T)$; consequently, $\mathcal{I}^k_{3,T}v$ has a sense. Let us test (27) with $\underline{\mathrm{v}}_T := \overline{\Sigma}^k_{3,T}(v)$. There holds

$$\int_T \boldsymbol{\nabla}\mathcal{I}^k_{3,T}v\cdot\boldsymbol{\nabla} w = -\int_T v\,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, \mathcal{I}^k_{2,\partial T}(v_{|\partial T})\big\rangle_{-\frac12,\partial T} \qquad \forall w \in V^k_3(T),$$

where we have used that $\triangle w \in \mathbb{P}^{k-1}_3(T)$ and that $r^k_{2,\partial T}\big(\overline{\Sigma}^{k,\partial}_{3,T}(v)\big) = \mathcal{I}^k_{2,\partial T}(v_{|\partial T})$. We equivalently rewrite the equality above as

$$\int_T \boldsymbol{\nabla}\mathcal{I}^k_{3,T}v\cdot\boldsymbol{\nabla} w = -\int_T (v-\pi^0_T v)\,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, \mathcal{I}^k_{2,\partial T}(v_{|\partial T}-\pi^0_T v)\big\rangle_{-\frac12,\partial T}$$
$$\forall w \in V^k_3(T),$$

and, by Cauchy–Schwarz inequality, we infer, for any $w \in V^k_3(T)$,

$$\left|\int_T \boldsymbol{\nabla}\mathcal{I}^k_{3,T}v\cdot\boldsymbol{\nabla} w\right| \lesssim \|v-\pi^0_T v\|_{0,T}\|\triangle w\|_{0,T} + \Big(h_T^{-\frac12}\big\|\mathcal{I}^k_{2,\partial T}(v_{|\partial T}-\pi^0_T v)\big\|_{0,\partial T}$$
$$+h_T^{\frac12}\big|\mathcal{I}^k_{2,\partial T}(v_{|\partial T}-\pi^0_T v)\big|_{1,\partial T}\Big)\sup_{z\in H^1(\partial T)}\frac{\big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, z\big\rangle_{-\frac12,\partial T}}{h_T^{-1/2}\|z\|_{0,\partial T} + h_T^{1/2}|z|_{1,\partial T}} =: \mathfrak{T}_1 + \mathfrak{T}_2.$$

To estimate the first term $\mathfrak{T}_1$, we apply (7) to $w \in V^k_3(T)$, and the Poincaré inequality (6) to $(v-\pi^0_T v)$. We get

$$\mathfrak{T}_1 \lesssim |v|_{1,T}|w|_{1,T}.$$

To estimate the second term $\mathfrak{T}_2$, we first use (9) combined to (7) to infer

$$\sup_{z\in H^1(\partial T)}\frac{\big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, z\big\rangle_{-\frac12,\partial T}}{h_T^{-1/2}\|z\|_{0,\partial T} + h_T^{1/2}|z|_{1,\partial T}} \lesssim |w|_{1,T}.$$

Then, to estimate the second factor, we remark that, for all $F \in \mathcal{F}_T$, $\big(v_{|F}-\pi^0_T v\big) \in H^{3/2}(F)$. We can hence apply Lemma 4.8 with $T \leftarrow F$, to infer

$$h_F^{-1/2}\big\|\mathcal{I}^k_{2,F}(v_{|F}-\pi^0_T v)\big\|_{0,F} + h_F^{1/2}\big|\mathcal{I}^k_{2,F}(v_{|F}-\pi^0_T v)\big|_{1,F}$$
$$\lesssim h_F^{-1/2}\|v_{|F}-\pi^0_T v\|_{0,F} + h_F^{1/2}|v_{|F}|_{1,F} + h_F|v_{|F}|_{\frac32,F}.$$

By (a sharper version of) the continuous trace inequality (4), and the Poincaré inequality (6), we finally infer, summing over $F \in \mathcal{F}_T$,

$$h_T^{-\frac{1}{2}} \left\| \mathcal{I}_{2,\partial T}^k (v_{|\partial T} - \pi_T^0 v) \right\|_{0,\partial T} + h_T^{1/2} \left| \mathcal{I}_{2,\partial T}^k (v_{|\partial T} - \pi_T^0 v) \right|_{1,\partial T} \lesssim |v|_{1,T} + h_T |v|_{2,T}.$$

Hence, we get

$$\mathfrak{T}_2 \lesssim \big(|v|_{1,T} + h_T |v|_{2,T} \big) |w|_{1,T},$$

and there holds, for any $w \in V_3^k(T)$,

$$\left| \int_T \boldsymbol{\nabla} \mathcal{I}_{3,T}^k v \cdot \boldsymbol{\nabla} w \right| \lesssim \big(|v|_{1,T} + h_T |v|_{2,T} \big) |w|_{1,T}.$$

Taking $w = \mathcal{I}_{3,T}^k v \in V_3^k(T)$ provides the expected estimate in the $H^1(T)$-seminorm. To obtain the estimate in the $L^2(T)$-norm, we remark that $\int_T \mathcal{I}_{3,T}^k v = \int_T \pi_T^{k-1} v = \int_T v$. Hence, by the triangle inequality and the Poincaré inequality (6), we infer

$$\left\| \mathcal{I}_{3,T}^k v \right\|_{0,T} \leqslant \|v\|_{0,T} + \left\| v - \mathcal{I}_{3,T}^k v \right\|_{0,T} \lesssim \|v\|_{0,T} + h_T \big(|v|_{1,T} + \left| \mathcal{I}_{3,T}^k v \right|_{1,T} \big).$$

The conclusion then follows easily from the estimate in the $H^1(T)$-seminorm. □

Remark 4.12. *In view of Lemmas 4.6 and 4.11, the stability result on $\mathcal{I}_{d,T}^k$ in [24, Eq. (35)], that is taken as an assumption for the subsequent analysis, seems a bit too optimistic.*

4.1.2 Nonconforming case

The nonconforming local virtual space on $T \in \mathcal{T}_h$ in dimension d is defined as

$$V_d^k(T) := \left\{ v \in H^1(T) \mid \triangle v \in \mathbb{P}_d^{k-1}(T),\ \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T) \right\}. \tag{28}$$

As opposed to the conforming case, the construction of the nonconforming local virtual space does not depend on the ambient dimension. We have $\mathbb{P}_d^k(T) \subset V_d^k(T)$.

Proposition 4.13. *The triple $(T, V_d^k(T), \Sigma_{d,T}^k)$, where $V_d^k(T)$ is given by (28), and $\Sigma_{d,T}^k := \Sigma_{d,T}^{k,\partial} \bigcup \Sigma_{d,T}^{k,\mathrm{o}}$ is the collection of*

- $\Sigma_{d,T}^{k,\partial} := \left\{ \pi_{F|V_d^k(\partial T)}^{k-1,j}(\cdot_{|F}) \right\}_{j \in \{1,\dots,\mathrm{N}_{d-1}^{k-1}\}}^{F \in \mathcal{F}_T}$,
- $\Sigma_{d,T}^{k,\mathrm{o}} := \left\{ \pi_{T|V_d^k(T)}^{k-1,i} \right\}_{i \in \{1,\dots,\mathrm{N}_d^{k-1}\}}$,

is a finite element in the sense of Ciarlet.

Proof. For any

$$\underline{\mathrm{v}}_T := \big((\mathrm{v}_{T,F}^{\partial,j})_{j \in \{1,\dots,\mathrm{N}_{d-1}^{k-1}\}}^{F \in \mathcal{F}_T}, (\mathrm{v}_T^{\mathrm{o},i})_{i \in \{1,\dots,\mathrm{N}_d^{k-1}\}} \big) \in \mathbb{R}^{n_{d,T}^k},$$

we have $\underline{\mathrm{v}}_T^\partial = (\mathrm{v}_{T,F}^{\partial,j})_{j \in \{1,\dots,\mathrm{N}_{d-1}^{k-1}\}}^{F \in \mathcal{F}_T} \in \mathbb{R}^{n_{d,T}^{k,\partial}}$ and $\underline{\mathrm{v}}_T^\mathrm{o} = (\mathrm{v}_T^{\mathrm{o},i})_{i \in \{1,\dots,\mathrm{N}_d^{k-1}\}} \in \mathbb{R}^{n_{d,T}^{k,\mathrm{o}}}$. We introduce the operator $\Phi_{d,T}^k : \mathbb{R}^{n_{d,T}^k} \to V_d^k(T)$ such that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{d,T}^k}$,

$$\begin{cases} -\triangle \Phi_{d,T}^k \underline{\mathrm{v}}_T = \mathrm{v}_T^\mathrm{o} - |T|_d^{-1} \left(\displaystyle\int_T \mathrm{v}_T^\mathrm{o} + \int_{\partial T} \mathrm{v}_T^\partial \right) & \text{in } T, \\ \boldsymbol{\nabla} \Phi_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} \cdot \boldsymbol{n}_T = \mathrm{v}_T^\partial & \text{on } \partial T, \\ \displaystyle\int_T \Phi_{d,T}^k \underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^\mathrm{o}, \end{cases}$$

where we have introduced the notation $\mathrm{v}_T^\mathrm{o} := \sum_{i=1}^{\mathrm{N}_d^{k-1}} \mathrm{v}_T^{\mathrm{o},i} \psi_{T,i}^{k-1} \in \mathbb{P}_d^{k-1}(T)$ and $\mathrm{v}_T^\partial \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$ is such that $\mathrm{v}_{T|F}^\partial := \sum_{j=1}^{\mathrm{N}_{d-1}^{k-1}} \mathrm{v}_{T,F}^{\partial,j} \psi_{F,j}^{k-1} \in \mathbb{P}_{d-1}^{k-1}(F)$ for all $F \in \mathcal{F}_T$. Since this Neumann problem is well-posed, if $\Phi_{d,T}^k \underline{\mathrm{v}}_T \equiv 0$, then necessarily $\mathrm{v}_T^\partial \equiv 0$ and $\mathrm{v}_T^\mathrm{o} \equiv 0$, i.e. $\underline{\mathrm{v}}_T = \underline{0}$. Thus, $\Phi_{d,T}^k$ is injective, and $\dim(V_d^k(T)) \geqslant n_{d,T}^k$. Let now $v \in V_d^k(T)$ be such that $\underline{\Sigma}_{d,T}^k(v) = \underline{0}$. An integration by parts formula yields

$$\int_T |\boldsymbol{\nabla} v|^2 = -\int_T \pi_T^{k-1} v \triangle v + \int_{\partial T} \pi_{\partial T}^{k-1}(v_{|\partial T})\, \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T = 0.$$

Hence, $v \equiv 0$ (since $\int_T v = 0$), and $\underline{\Sigma}_{d,T}^k$ is injective. Combined to the fact that $\dim(V_d^k(T)) \geqslant n_{d,T}^k$, we have proved that $\underline{\Sigma}_{d,T}^k$ is bijective. □

Let us define the operator $r_{d,T}^k : \mathbb{R}^{n_{d,T}^k} \to V_d^k(T)$ so that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{d,T}^k}$, $r_{d,T}^k \underline{\mathrm{v}}_T \in V_d^k(T)$ solves the well-posed problem

$$\left\{ \begin{aligned} &\int_T \boldsymbol{\nabla} r_{d,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^{\mathrm{o}} \triangle w + \int_{\partial T} \mathrm{v}_T^{\partial} \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T \qquad \forall w \in V_d^k(T), \\ &\int_T r_{d,T}^k \underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^{\mathrm{o}}. \end{aligned} \right. \tag{29}$$

Proposition 4.14. *The operator $r_{d,T}^k$ defined by (29) coincides with the (canonical) reconstruction operator $R_{d,T}^k = (\underline{\Sigma}_{d,T}^k)^{-1}$.*

Proof. Let $\mathbb{R}^{n_{d,T}^k} \ni \underline{\mathrm{v}}_T := \underline{\Sigma}_{d,T}^k(v)$ for some $v \in V_d^k(T)$. Plugging $\underline{\mathrm{v}}_T$ into (29), we infer by integration by parts

$$\begin{aligned} \int_T \boldsymbol{\nabla} r_{d,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w &= -\int_T \pi_T^{k-1} v \triangle w + \int_{\partial T} \pi_{\partial T}^{k-1}(v_{|\partial T}) \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T \\ &= \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} w \qquad \forall w \in V_d^k(T), \end{aligned}$$

where we have used that $\triangle w \in \mathbb{P}_d^{k-1}(T)$ and that, for all $F \in \mathcal{F}_T$, $\boldsymbol{\nabla} w_{|F} \cdot \boldsymbol{n}_{T,F} \in \mathbb{P}_{d-1}^{k-1}(F)$. Since $\int_T r_{d,T}^k \underline{\mathrm{v}}_T = \int_T \pi_T^{k-1} v = \int_T v$, we finally deduce that $r_{d,T}^k \underline{\mathrm{v}}_T = r_{d,T}^k\big(\underline{\Sigma}_{d,T}^k(v)\big) = v$. This is valid for any $v \in V_d^k(T)$, hence $r_{d,T}^k = (\underline{\Sigma}_{d,T}^k)^{-1} = R_{d,T}^k$. □

Since the operators $r_{d,T}^k$ and $R_{d,T}^k$ coincide, we henceforth drop the notation $R_{d,T}^k$ to stick to the notation $r_{d,T}^k$.

The collection $\Sigma_{d,T}^k$ of linear forms on $V_d^k(T)$ can be extended to a collection $\overline{\Sigma}_{d,T}^k$ of linear forms on $H^1(T)$. Thus, we can introduce the interpolation operator $\mathcal{I}_{d,T}^k : H^1(T) \to V_d^k(T)$ such that $\mathcal{I}_{d,T}^k := r_{d,T}^k \circ \overline{\Sigma}_{d,T}^k$. We have the following stability result for $\mathcal{I}_{d,T}^k$.

Lemma 4.15 (Stability of $\mathcal{I}_{d,T}^k$)**.** *For all $v \in H^1(T)$, there holds*

- $\left|\mathcal{I}_{d,T}^k v\right|_{1,T} \leqslant |v|_{1,T}$;
- $\left\|\mathcal{I}_{d,T}^k v\right\|_{0,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T}$.

Proof. We test (29) with $\underline{\mathrm{v}}_T := \overline{\Sigma}_{d,T}^k(v)$, for some $v \in H^1(T)$. By integration by parts, there holds

$$\int_T \boldsymbol{\nabla} \mathcal{I}_{d,T}^k v \cdot \boldsymbol{\nabla} w = \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} w \qquad \text{for all } w \in V_d^k(T). \tag{30}$$

Testing with $w = \mathcal{I}_{d,T}^k v \in V_d^k(T)$, we immediately obtain the estimate in the $H^1(T)$-seminorm. Then, since $\int_T \mathcal{I}_{d,T}^k v = \int_T \pi_T^{k-1} v = \int_T v$, by the triangle and Poincaré (6) inequalities, there holds

$$\left\|\mathcal{I}_{d,T}^k v\right\|_{0,T} \leqslant \|v\|_{0,T} + \left\|v - \mathcal{I}_{d,T}^k v\right\|_{0,T} \lesssim \|v\|_{0,T} + h_T\big(|v|_{1,T} + \left|\mathcal{I}_{d,T}^k v\right|_{1,T}\big),$$

which enables to conclude. □

Remark 4.16. *In the proof of Lemma 4.15, and as opposed to the proofs of Lemmas 4.6 and 4.11, no flux estimate is needed. As a consequence, there is no need to construct a lifting (with optimal scaling) for traces of virtual functions, and $T \in \mathcal{T}_h$ needs not be assumed star-shaped to lead the analysis. The difference is related to the fact that, in the nonconforming case, the interpolation operator $\mathcal{I}_{d,T}^k$ is the elliptic projector on the virtual space $V_d^k(T)$ (see (30)).*

4.2 Local approximation properties

Henceforth, in the conforming case (and in the conforming case only), we will assume that any cell $T \in \mathcal{T}_h$ is star-shaped with respect to a ball whose radius is comparable to h_T.

Let $\underline{s}$ be an integer such that $\underline{s} = 2$ in the conforming case, and $\underline{s} = 1$ in the nonconforming case. We let $W_d^k(T) := H^{1,c}(T)$ in the conforming case, and $W_d^k(T) := H^1(T)$ in the nonconforming case. In view of Lemmas 4.6, 4.11, and 4.15, the interpolation operator $\mathcal{I}_{d,T}^k : W_d^k(T) \to V_d^k(T)$ satisfies, for all $v \in H^{\underline{s}}(T)$,

$$\left|\mathcal{I}_{d,T}^k v\right|_{1,T} \lesssim \sum_{\alpha=1}^{\underline{s}} h_T^{\alpha-1} |v|_{\alpha,T}, \qquad \left\|\mathcal{I}_{d,T}^k v\right\|_{0,T} \lesssim \sum_{\alpha=0}^{\underline{s}} h_T^{\alpha} |v|_{\alpha,T}. \tag{31}$$

For any $\underline{\mathrm{v}}_T^\partial \in \mathbb{R}^{n_{d,T}^{k,\partial}}$, we define $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^\partial \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)$ so that $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^\partial := r_{1,\partial T}^k \underline{\mathrm{v}}_T^\partial \in \mathbb{P}_1^{k,c}(\mathcal{F}_T)$ in the conforming case with $d = 2$, $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^\partial := \pi_{\partial T}^{k-1}\big(r_{2,\partial T}^k \underline{\mathrm{v}}_T^\partial\big) \in \mathbb{P}_2^{k-1}(\mathcal{F}_T)$ in the conforming case with $d = 3$, and $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^\partial := \mathrm{v}_T^\partial \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$ in the nonconforming case. We define the operator $p_{d,T}^k : \mathbb{R}^{n_{d,T}^k} \to \mathbb{P}_d^k(T)$ so that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{d,T}^k}$, $p_{d,T}^k \underline{\mathrm{v}}_T \in \mathbb{P}_d^k(T)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^{\mathrm{o}} \triangle w + \int_{\partial T} \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^\partial \, \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T & \forall w \in \mathbb{P}_d^k(T), \\ \displaystyle\int_T p_{d,T}^k \underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^{\mathrm{o}}. \end{cases} \tag{32}$$

One can introduce the interpolation operator $\mathcal{P}_{d,T}^k : W_d^k(T) \to \mathbb{P}_d^k(T)$ such that $\mathcal{P}_{d,T}^k := p_{d,T}^k \circ \overline{\Sigma}_{d,T}^k$. Since $\mathbb{P}_d^k(T) \subset V_d^k(T)$, and since $\boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$, by comparing (32) to (25), (27), and (29), we infer, using the definition (12), that $p_{d,T}^k = \Pi_T^k \circ r_{d,T}^k$. As a consequence, $\mathcal{P}_{d,T}^k = \Pi_T^k \circ \mathcal{I}_{d,T}^k$. In particular,

- $\mathcal{P}_{d,T}^k v = \Pi_T^k v$ for all $v \in V_d^k(T)$;
- owing to Proposition 2.6, $\mathcal{P}_{d,T}^k v = v$ for all $v \in \mathbb{P}_d^k(T)$;
- in the nonconforming case, owing to (30), $\mathcal{P}_{d,T}^k = \Pi_T^k$.

Combining the stability result of Proposition 2.6 with (31) yields, for all $v \in H^{\underline{s}}(T)$,

$$\left\|\mathcal{P}_{d,T}^k v\right\|_{0,T} \lesssim \sum_{\alpha=0}^{\underline{s}} h_T^{\alpha} |v|_{\alpha,T}. \tag{33}$$

Theorem 4.17 (Approximation properties of $\mathcal{P}_{d,T}^k$)**.** *Let $v \in H^s(T)$, for $s \in \{\underline{s}, \dots, k+1\}$. Then, the following holds:*

$$\left|v - \mathcal{P}_{d,T}^k v\right|_{m,T} \lesssim h_T^{s-m} |v|_{s,T} \qquad \text{for } m \in \{0, \dots, s\}, \tag{34}$$

and, for any $F \in \mathcal{F}_T$, and any $(\zeta_1, \dots, \zeta_d) \in \mathbb{N}^d$ such that $\sum_{i=1}^d \zeta_i = m$,

$$\left|\left[\partial_{x_1}^{\zeta_1} \dots \partial_{x_d}^{\zeta_d} (v - \mathcal{P}_{d,T}^k v)\right]_{|F}\right|_{0,F} \lesssim h_T^{s-m-1/2} |v|_{s,T} \qquad \text{for } m \in \{0, \dots, s-1\}. \tag{35}$$

Proof. In the nonconforming case, there is nothing to prove. Since $\mathcal{P}_{d,T}^k = \Pi_T^k$, the result follows from (13) and (14) of Proposition 2.6. For the conforming case, we follow the ideas of [11, Chapter 4]. We proceed by density of $C^\infty(T) \bigcap H^s(T)$ in $H^s(T)$. Since in that case T is star-shaped with respect to every point of a ball of radius γh_T, $v \in C^\infty(T) \bigcap H^s(T)$ admits the following Sobolev representation:

$$v = Q_T^s v + R_T^s v,$$

where $Q_T^s v \in \mathbb{P}_d^{s-1}(T) \subseteq \mathbb{P}_d^k(T)$ is the averaged Taylor polynomial, and the remainder $R_T^s v$ satisfies, for $r \in \{0, \dots, s\}$, the Bramble–Hilbert lemma (cf., e.g., [11, Lemma 4.3.8]):

$$|R_T^s v|_{r,T} \lesssim h_T^{s-r} |v|_{s,T}. \tag{36}$$

One can easily see that, since $Q_T^s v \in \mathbb{P}_d^k(T)$ and $\mathcal{P}_{d,T}^k v = v$ for all $v \in \mathbb{P}_d^k(T)$, there holds $v - \mathcal{P}_{d,T}^k v = R_T^s v - \mathcal{P}_{d,T}^k(R_T^s v)$. Thus,

$$\left|v - \mathcal{P}_{d,T}^k v\right|_{m,T} \leqslant |R_T^s v|_{m,T} + \left|\mathcal{P}_{d,T}^k(R_T^s v)\right|_{m,T}.$$

Applying m times the inverse inequality (3) to $\mathcal{P}_{d,T}^k(R_T^s v) \in \mathbb{P}_d^k(T)$, and then the stability result (33) in the $L^2(T)$-norm on $\mathcal{P}_{d,T}^k$, we infer

$$\left|v - \mathcal{P}_{d,T}^k v\right|_{m,T} \lesssim |R_T^s v|_{m,T} + h_T^{-m}\left\|\mathcal{P}_{d,T}^k(R_T^s v)\right\|_{0,T} \lesssim |R_T^s v|_{m,T} + \sum_{\alpha=0}^{s} h_T^{-m+\alpha}|R_T^s v|_{\alpha,T}.$$

Applying the Bramble–Hilbert lemma (36) for $r = m$ and $r = \alpha \in \{0, \ldots, \underline{s}\}$, we finally obtain (34). For $F \in \mathcal{F}_T$ now, the continuous trace inequality (4) yields

$$\left|\left[\partial_{x_1}^{\zeta_1} \ldots \partial_{x_d}^{\zeta_d}\left(v - \mathcal{P}_{d,T}^k v\right)\right]_{|F}\right|_{0,F} \lesssim h_T^{-1/2}\left|v - \mathcal{P}_{d,T}^k v\right|_{m,T} + h_T^{1/2}\left|v - \mathcal{P}_{d,T}^k v\right|_{m+1,T}.$$

The conclusion then follows from (34) (remark that, for $m \in \{0, \ldots, s-1\}$, $m+1 \in \{1, \ldots, s\}$). □

Remark 4.18. *For any $v \in W_d^k(T)$, there holds*

$$\mathcal{P}_{d,T}^k v - \mathcal{I}_{d,T}^k v = \mathcal{I}_{d,T}^k\left(\mathcal{P}_{d,T}^k v - v\right). \tag{37}$$

Remark 4.19. *The definition (32) of the operator $p_{d,T}^k$ is entirely based on some polynomial-valued linear combinations of the DoFs, that are $\mathrm{v}_T^{\mathrm{o}} \in \mathbb{P}_d^{k-1}(T)$, and $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial} \in \mathbb{P}_{d-1}^k(\mathcal{F}_T)$. In the conforming case with $d = 2$, $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}$ only depends on $\left(\mathrm{v}_T^{\partial,\nu} \in \mathbb{R}\right)_{\nu \in \mathcal{V}_T}$, and $\left(\mathrm{v}_F^{\mathrm{o}} \in \mathbb{P}_1^{k-2}(F)\right)_{F \in \mathcal{F}_T}$ through (23). In the conforming case with $d = 3$, $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}$ is such that, for all $F \in \mathcal{F}_T$, $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_{T|F}^{\partial} = \mathrm{v}_{T,F}^{\partial} := \sum_{j=1}^{\mathrm{N}_2^{k-1}} \mathrm{v}_{T,F}^{\partial,j} \psi_{F,j}^{k-1} \in \mathbb{P}_2^{k-1}(F)$. In the nonconforming case, $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial} = \mathrm{v}_T^{\partial} \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$.*

4.3 Discrete problem

Let us define, for any $T \in \mathcal{T}_h$, the following seminorm on $\mathbb{R}^{n_{d,T}^k}$: for all $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{d,T}^k}$,

- $|\underline{\mathrm{v}}_T|_T^2 := \|\boldsymbol{\nabla} \mathrm{v}_T^{\mathrm{o}}\|_{0,T}^2 + \left\|h_{\partial T}^{-1/2}\left(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\right)\right\|_{0,\partial T}^2$ in the conforming case with $d = 2$, and in the nonconforming case,
- $|\underline{\mathrm{v}}_T|_T^2 := \|\boldsymbol{\nabla} \mathrm{v}_T^{\mathrm{o}}\|_{0,T}^2 + \left\|h_{\partial T}^{-1/2}\left(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \pi_{\partial T}^{k-1}(r_{2,\partial T}^k \underline{\mathrm{v}}_T^{\partial})\right)\right\|_{0,\partial T}^2$
  $+ \sum_{F \in \mathcal{F}_T} \left\|\mathrm{v}_{T|\partial F}^{\mathrm{o}} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F}^2$ in the conforming case with $d = 3$.

Defining $|\underline{\mathrm{v}}_h|_h^2 := \sum_{T \in \mathcal{T}_h} |\underline{\mathrm{v}}_T|_T^2$ for any $\underline{\mathrm{v}}_h \in \mathbb{R}^{n_{d,\mathcal{T}_h}^k}$, one can easily see that $|\cdot|_h$ defines a norm on $\mathbb{V}_{d,h,0}^k$.

Remark 4.20. *In view of Lemma 3.2 and Remark 3.4, the quantity*

$$\left(\sum_{T \in \mathcal{T}_h} \left\|\boldsymbol{\nabla} r_{d,T}^k \underline{\mathrm{v}}_T\right\|_{0,T}^2\right)^{1/2},$$

which can be equivalently written $\|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}$ for $v_h \in V_d^k(\mathcal{T}_h)$ such that $v_{h|T} := r_{d,T}^k \underline{\mathrm{v}}_T$ for all $T \in \mathcal{T}_h$, defines a norm on $\mathbb{V}_{d,h,0}^k$. Furthermore, one can easily show (we omit the proof for brevity) that, for any $\underline{\mathrm{v}}_h \in \mathbb{R}^{n_{d,\mathcal{T}_h}^k}$,

$$\left(\sum_{T \in \mathcal{T}_h} \left\|\boldsymbol{\nabla} r_{d,T}^k \underline{\mathrm{v}}_T\right\|_{0,T}^2\right)^{1/2} \lesssim |\underline{\mathrm{v}}_h|_h.$$

We henceforth consider Problem (18), with local linear form

$$\mathrm{l}_T(\underline{\mathrm{v}}_T) := \int_T f_{|T} \mathrm{v}_T^{\mathrm{o}},$$

and (symmetric) local bilinear form

$$\mathrm{a}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) := \int_T \boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{w}}_T \cdot \boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T + \mathrm{s}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T),$$

with stabilization, letting $l = k$ in the conforming case with $d = 2$, and $l = k-1$ in the other cases,

- $\mathrm{s}_T(\underline{\mathrm{w}}_T,\underline{\mathrm{v}}_T) := \int_{\partial T} h_{\partial T}^{-1}\pi_{\partial T}^{l}\big(\delta_{d,T}^{k}\underline{\mathrm{w}}_{T|\partial T} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{w}}_T^{\partial}\big)\pi_{\partial T}^{l}\big(\delta_{d,T}^{k}\underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big)$ in the conforming case with $d=2$, and in the nonconforming case,
- $\mathrm{s}_T(\underline{\mathrm{w}}_T,\underline{\mathrm{v}}_T) := \int_{\partial T} h_{\partial T}^{-1}\pi_{\partial T}^{k-1}\big(\delta_{3,T}^{k}\underline{\mathrm{w}}_{T|\partial T} - r_{2,\partial T}^{k}\underline{\mathrm{w}}_T^{\partial}\big)\pi_{\partial T}^{k-1}\big(\delta_{3,T}^{k}\underline{\mathrm{v}}_{T|\partial T} - r_{2,\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big) + \sum_{F\in\mathcal{F}_T}\int_{\partial F}\big(\delta_{3,T}^{k}\underline{\mathrm{w}}_{T|\partial F} - r_{1,\partial F}^{k}\underline{\mathrm{w}}_{T,F}^{\partial,\partial}\big)\big(\delta_{3,T}^{k}\underline{\mathrm{v}}_{T|\partial F} - r_{1,\partial F}^{k}\underline{\mathrm{v}}_{T,F}^{\partial,\partial}\big)$ in the conforming case with $d=3$,

where $\delta_{d,T}^{k}\underline{\mathrm{v}}_T := p_{d,T}^{k}\underline{\mathrm{v}}_T + \pi_T^{k-1}\big(r_{d,T}^{k}\underline{\mathrm{v}}_T - p_{d,T}^{k}\underline{\mathrm{v}}_T\big) \in \mathbb{P}_d^k(T)$. Other choices of stabilization are possible (cf., e.g., [8]). Here, we build upon the standard HHO choice of stabilization (see [26, 27]).

Remark 4.21. *Letting $w,v\in V_d^k(T)$ such that $w := r_{d,T}^{k}\underline{\mathrm{w}}_T$ and $v := r_{d,T}^{k}\underline{\mathrm{v}}_T$, and since $\mathcal{P}_{d,T}^{k}z = \Pi_T^k z$ for all $z\in V_d^k(T)$, one can equivalently consider Problem (17), with local (bi)linear forms*

$$l_T(v) = \int_T f_{|T}\pi_T^{k-1}v, \qquad a_T(w,v) = \int_T \boldsymbol{\nabla}\Pi_T^k w\cdot\boldsymbol{\nabla}\Pi_T^k v + s_T(w,v),$$

with stabilization

$$s_T(w,v) = \int_{\partial T} h_{\partial T}^{-1}\pi_{\partial T}^{l}\big([\Delta_{d,T}^{k}w - w]_{|\partial T}\big)\pi_{\partial T}^{l}\big([\Delta_{d,T}^{k}v - v]_{|\partial T}\big)$$

in the conforming case with $d=2$, and in the nonconforming case, and stabilization

$$s_T(w,v) = \int_{\partial T} h_{\partial T}^{-1}\pi_{\partial T}^{k-1}\big([\Delta_{3,T}^{k}w - w]_{|\partial T}\big)\pi_{\partial T}^{k-1}\big([\Delta_{3,T}^{k}v - v]_{|\partial T}\big) + \sum_{F\in\mathcal{F}_T}\int_{\partial F}\big(\Delta_{3,T}^{k}w - w\big)_{|\partial F}\big(\Delta_{3,T}^{k}v - v\big)_{|\partial F}$$

in the conforming case with $d=3$, where $\Delta_{d,T}^{k}v := \Pi_T^k v + \pi_T^{k-1}\big(v - \Pi_T^k v\big) \in \mathbb{P}_d^k(T)$.

We prove well-posedness for Problem (18).

Lemma 4.22 (Local coercivity and boundedness)**.** *For all $T\in\mathcal{T}_h$, and all $\underline{\mathrm{v}}_T\in\mathbb{R}^{n_{d,T}^{k}}$, the following holds:*

$$|\underline{\mathrm{v}}_T|_T^2 \lesssim a_T(\underline{\mathrm{v}}_T,\underline{\mathrm{v}}_T) \lesssim |\underline{\mathrm{v}}_T|_T^2.$$

Proof. We begin by showing local coercivity. Testing (32) with $w = \mathrm{v}_T^{\mathrm{o}}\in\mathbb{P}_d^{k-1}(T)\subset\mathbb{P}_d^k(T)$, we infer

$$-\int_T \mathrm{v}_T^{\mathrm{o}}\triangle\mathrm{v}_T^{\mathrm{o}} = \int_T \boldsymbol{\nabla}p_{d,T}^{k}\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}} - \int_{\partial T}\mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\,\boldsymbol{\nabla}\mathrm{v}_{T|\partial T}^{\mathrm{o}}\cdot\boldsymbol{n}_T.$$

Integrating by parts the left-hand side, there holds

$$\|\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}}\|_{0,T}^2 = \int_T \boldsymbol{\nabla}p_{d,T}^{k}\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}} + \int_{\partial T}\pi_{\partial T}^{l}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big)\boldsymbol{\nabla}\mathrm{v}_{T|\partial T}^{\mathrm{o}}\cdot\boldsymbol{n}_T,$$

which yields, by Cauchy–Schwarz inequality, and the discrete trace inequality (5),

$$\|\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}}\|_{0,T} \lesssim \|\boldsymbol{\nabla}p_{d,T}^{k}\underline{\mathrm{v}}_T\|_{0,T} + \big\|h_{\partial T}^{-1/2}\pi_{\partial T}^{l}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big)\big\|_{0,\partial T}. \tag{38}$$

Now, adding/subtracting $\delta_{d,T}^{k}\underline{\mathrm{v}}_{T|\partial T}$, using the fact that $\pi_T^{k-1}\big(r_{d,T}^{k}\underline{\mathrm{v}}_T\big) = \mathrm{v}_T^{\mathrm{o}}$, and by stability of π_F^l in the $L^2(F)$-norm for all $F\in\mathcal{F}_T$, we infer

$$\big\|h_{\partial T}^{-1/2}\pi_{\partial T}^{l}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big)\big\|_{0,\partial T} \leqslant \big\|h_{\partial T}^{-1/2}\big(p_{d,T}^{k}\underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{d,T}^{k}\underline{\mathrm{v}}_T)\big)_{|\partial T}\big\|_{0,\partial T} + \big\|h_{\partial T}^{-1/2}\pi_{\partial T}^{l}\big(\delta_{d,T}^{k}\underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big)\big\|_{0,\partial T}.$$

By the discrete trace inequality (5), and an application of (10) with $s=1$ and $m=0$, we infer

$$\big\|h_{\partial T}^{-1/2}\pi_{\partial T}^{l}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big)\big\|_{0,\partial T} \lesssim \|\boldsymbol{\nabla}p_{d,T}^{k}\underline{\mathrm{v}}_T\|_{0,T} + \big\|h_{\partial T}^{-1/2}\pi_{\partial T}^{l}\big(\delta_{d,T}^{k}\underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T}^{k}\underline{\mathrm{v}}_T^{\partial}\big)\big\|_{0,\partial T}. \tag{39}$$

With $l=k$ in the conforming case with $d=2$, and $l=k-1$ in the nonconforming case, we infer local coercivity in those two cases as a direct consequence of (38) and (39). In the conforming case with $d=3$,

one has to estimate two terms in the stabilization part of the seminorm. The first one is handled as above with $l = k-1$. It then remains to estimate the term $\left\|\mathrm{v}_{T|\partial F}^{\mathrm{o}} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F}$ for all $F \in \mathcal{F}_T$. Adding/subtracting $\delta_{3,T}^k \underline{\mathrm{v}}_{T|\partial F}$, and since $\pi_T^{k-1}\big(r_{3,T}^k \underline{\mathrm{v}}_T\big) = \mathrm{v}_T^{\mathrm{o}}$, we have, for any $F \in \mathcal{F}_T$,

$$\left\|\mathrm{v}_{T|\partial F}^{\mathrm{o}} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F} \leqslant \left\|\big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial F}\right\|_{0,\partial F} + \left\|\delta_{3,T}^k \underline{\mathrm{v}}_{T|\partial F} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F}.$$

Since $\big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|F} \in \mathbb{P}_2^k(F)$, we can apply a first time the discrete trace inequality (5) (with $T \leftarrow F$) to obtain

$$\left\|\big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial F}\right\|_{0,\partial F} \lesssim h_F^{-1/2}\left\|\big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|F}\right\|_{0,F},$$

and a second time (as it is) to obtain

$$\left\|\big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial F}\right\|_{0,\partial F} \lesssim h_T^{-1}\left\|p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\right\|_{0,T}.$$

An application of (10) with $s = 1$ and $m = 0$ finally yields

$$\left\|\mathrm{v}_{T|\partial F}^{\mathrm{o}} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F} \lesssim \left\|\boldsymbol{\nabla} p_{3,T}^k \underline{\mathrm{v}}_T\right\|_{0,T} + \left\|\delta_{3,T}^k \underline{\mathrm{v}}_{T|\partial F} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F},$$

which concludes the proof of local coercivity in the conforming case with $d = 3$.

Let us now show local boundedness. Testing (32) with $w = p_{d,T}^k \underline{\mathrm{v}}_T \in \mathbb{P}_d^k(T)$, and integrating by parts the right-hand side, we infer

$$\left\|\boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T\right\|_{0,T}^2 = \int_T \boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}} \cdot \boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T - \int_{\partial T} \pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} \cdot \boldsymbol{n}_T,$$

which yields, by Cauchy–Schwarz inequality, and the discrete trace inequality (5),

$$\left\|\boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T\right\|_{0,T} \lesssim \left\|\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}}\right\|_{0,T} + \left\|h_{\partial T}^{-1/2}\pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T}. \tag{40}$$

Then, writing, by the triangle inequality, and the stability of π_F^l in the $L^2(F)$-norm for all $F \in \mathcal{F}_T$,

$$\left\|h_{\partial T}^{-1/2}\pi_{\partial T}^l\big(\delta_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T} \leqslant \left\|h_{\partial T}^{-1/2}\big(\delta_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} - \mathrm{v}_{T|\partial T}^{\mathrm{o}}\big)\right\|_{0,\partial T} + \left\|h_{\partial T}^{-1/2}\pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T},$$

we infer, since $\pi_T^{k-1}\big(r_{d,T}^k \underline{\mathrm{v}}_T\big) = \mathrm{v}_T^{\mathrm{o}}$,

$$\left\|h_{\partial T}^{-1/2}\pi_{\partial T}^l\big(\delta_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T} \leqslant \left\|h_{\partial T}^{-1/2}\big(p_{d,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{d,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial T}\right\|_{0,\partial T} + \left\|h_{\partial T}^{-1/2}\pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T}.$$

By the discrete trace inequality (5), and (10) with $s = 1$ and $m = 0$, we finally obtain

$$\left\|h_{\partial T}^{-1/2}\pi_{\partial T}^l\big(\delta_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T} \lesssim \left\|\boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T\right\|_{0,T} + \left\|h_{\partial T}^{-1/2}\pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T}, \tag{41}$$

which, combined to (40), proves local boundedness in the conforming case with $d = 2$ ($l = k$), and in the nonconforming case ($l = k-1$). In the conforming case with $d = 3$, one has to estimate two terms in the stabilization. The first one is handled as above with $l = k-1$. We do not detail the treatment of the second, which relies on the exact same kind of arguments. □

Well-posedness follows as an immediate consequence of Lemma 4.22, and of the fact that $|\cdot|_h$ defines a norm on $\mathbb{V}_{d,h,0}^k$.

Corollary 4.23 (Well-posedness). *For all* $\underline{\mathrm{v}}_h \in \mathbb{R}^{n^k_{d,\mathcal{T}_h}}$, *there holds*

$$|\underline{\mathrm{v}}_h|_h^2 \lesssim \mathrm{a}_h(\underline{\mathrm{v}}_h, \underline{\mathrm{v}}_h).$$

As a consequence, Problem (18) *is well-posed.*

Let us now investigate the approximation capacity of the stabilization term.

Lemma 4.24 (Approximation properties of the stabilization). *Let* $v \in H^s(T)$, *for* $s \in \{\underline{s}, \dots, k+1\}$. *Then, letting* $\underline{\mathrm{v}}_T = \overline{\Sigma}^k_{d,T}(v)$, *the following holds:*

$$\mathrm{s}_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T)^{1/2} \lesssim h_T^{s-1}|v|_{s,T}. \tag{42}$$

Proof. In the conforming case with $d = 3$, and in the nonconforming case, we have to estimate the quantity $\left\|h_{\partial T}^{-1/2}\pi^{k-1}_{\partial T}\big(\delta^k_{d,T}\underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}^k_{\partial T}\underline{\mathrm{v}}^\partial_T\big)\right\|_{0,\partial T}$. In both cases, we have $\pi^{k-1}_{\partial T}\big(\mathrm{t}^k_{\partial T}\underline{\mathrm{v}}^\partial_T\big) = \pi^{k-1}_{\partial T}\big(v_{|\partial T}\big)$, and $\pi^{k-1}_T\big(r^k_{d,T}\underline{\mathrm{v}}_T\big) = \pi^{k-1}_T\big(\mathcal{I}^k_{d,T}v\big) = \pi^{k-1}_T v$. Hence, by stability of $\pi^{k-1}_{\partial T}$, there holds

$$\left\|h_{\partial T}^{-1/2}\pi^{k-1}_{\partial T}\big(\delta^k_{d,T}\underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}^k_{\partial T}\underline{\mathrm{v}}^\partial_T\big)\right\|_{0,\partial T} \leqslant \left\|h_{\partial T}^{-1/2}\big(\mathcal{P}^k_{d,T}v - v\big)_{|\partial T}\right\|_{0,\partial T} + \left\|h_{\partial T}^{-1/2}\pi^{k-1}_T\big(v - \mathcal{P}^k_{d,T}v\big)_{|\partial T}\right\|_{0,\partial T}.$$

To estimate the first term in the right-hand side, we directly apply the result (35) for $m = 0$. To estimate the second term, we successively use the discrete trace inequality (5), the stability of π^{k-1}_T in the $L^2(T)$-norm, and the result (34) for $m = 0$. The conclusion easily follows. In the conforming case with $d = 2$, we have to estimate the quantity

$$\left\|h_{\partial T}^{-1/2}\pi^k_{\partial T}\big(\delta^k_{2,T}\underline{\mathrm{v}}_{T|\partial T} - r^k_{1,\partial T}\underline{\mathrm{v}}^\partial_T\big)\right\|_{0,\partial T} = \left\|h_{\partial T}^{-1/2}\big(\delta^k_{2,T}\underline{\mathrm{v}}_{T|\partial T} - r^k_{1,\partial T}\underline{\mathrm{v}}^\partial_T\big)\right\|_{0,\partial T}.$$

Since $r^k_{1,\partial T}\underline{\mathrm{v}}^\partial_T = r^k_{2,T}\underline{\mathrm{v}}_{T|\partial T}$, and $\pi^{k-1}_T\big(r^k_{2,T}\underline{\mathrm{v}}_T\big) = \pi^{k-1}_T\big(\mathcal{I}^k_{2,T}v\big) = \pi^{k-1}_T v$, there holds

$$\left\|h_{\partial T}^{-1/2}\pi^k_{\partial T}\big(\delta^k_{2,T}\underline{\mathrm{v}}_{T|\partial T} - r^k_{1,\partial T}\underline{\mathrm{v}}^\partial_T\big)\right\|_{0,\partial T} \leqslant \left\|h_{\partial T}^{-1/2}\big(\mathcal{P}^k_{2,T}v - \mathcal{I}^k_{2,T}v\big)_{|\partial T}\right\|_{0,\partial T} + \left\|h_{\partial T}^{-1/2}\pi^{k-1}_T\big(v - \mathcal{P}^k_{2,T}v\big)_{|\partial T}\right\|_{0,\partial T}.$$

The second term in the right-hand side can be estimated as previously. The first one is handled recalling the remark (37). We indeed have

$$\left\|h_{\partial T}^{-1/2}\big(\mathcal{P}^k_{2,T}v - \mathcal{I}^k_{2,T}v\big)_{|\partial T}\right\|_{0,\partial T} = \left\|h_{\partial T}^{-1/2}\mathcal{I}^k_{2,T}\big(v - \mathcal{P}^k_{2,T}v\big)_{|\partial T}\right\|_{0,\partial T}.$$

Applying Lemma 4.3 (recall that $\mathcal{I}^k_{2,T}\big(v - \mathcal{P}^k_{2,T}v\big)_{|\partial T} = \mathcal{I}^k_{1,\partial T}\big([v - \mathcal{P}^k_{2,T}v]_{|\partial T}\big)$), there holds

$$\left\|h_{\partial T}^{-1/2}\big(\mathcal{P}^k_{2,T}v - \mathcal{I}^k_{2,T}v\big)_{|\partial T}\right\|_{0,\partial T} \lesssim \big\|(v - \mathcal{P}^k_{2,T}v)_{|\partial T}\big\|_{\infty,\partial T}.$$

Since $v \in H^2(T) \subset C^0(\overline{T})$ in that case, one has

$$\big\|(v - \mathcal{P}^k_{2,T}v)_{|\partial T}\big\|_{\infty,\partial T} \leqslant \big\|v - \mathcal{P}^k_{2,T}v\big\|_{\infty,T}.$$

By (8), combined to (34) with $m = \alpha \in \{0, \dots, 2\}$, we finally infer

$$\left\|h_{\partial T}^{-1/2}\big(\mathcal{P}^k_{2,T}v - \mathcal{I}^k_{2,T}v\big)_{|\partial T}\right\|_{0,\partial T} \lesssim \sum_{\alpha=0}^{2} h_T^{\alpha-1}\big|v - \mathcal{P}^k_{2,T}v\big|_{\alpha,T} \lesssim h_T^{s-1}|v|_{s,T}.$$

We are left with estimating, in the conforming case with $d = 3$, and for all $F \in \mathcal{F}_T$, $\left\|\delta^k_{3,T}\underline{\mathrm{v}}_{T|\partial F} - r^k_{1,\partial F}\underline{\mathrm{v}}^{\partial,\partial}_{T,F}\right\|_{0,\partial F}$. We follow the same path as previously, and end up having to estimate $\left\|\mathcal{I}^k_{3,T}\big(v - \mathcal{P}^k_{3,T}v\big)_{|\partial F}\right\|_{0,\partial F}$. Since

$$\mathcal{I}^k_{3,T}\big(v - \mathcal{P}^k_{3,T}v\big)_{|\partial F} = \mathcal{I}^k_{1,\partial F}\big([v - \mathcal{P}^k_{3,T}v]_{|\partial F}\big),$$

by Lemma 4.3, the fact that $v \in H^2(T) \subset C^0(\overline{T})$ also in that case, and (8), we infer

$$\left\|\mathcal{I}_{3,T}^k\big(v-\mathcal{P}_{3,T}^k v\big)_{|\partial F}\right\|_{0,\partial F} \lesssim h_F^{1/2}\left(h_T^{-\frac{3}{2}}\left\|v-\mathcal{P}_{3,T}^k v\right\|_{0,T}+h_T^{-\frac{1}{2}}\left|v-\mathcal{P}_{3,T}^k v\right|_{1,T}\right.$$
$$\left.+h_T^{\frac{1}{2}}\left|v-\mathcal{P}_{3,T}^k v\right|_{2,T}\right).$$

The conclusion then follows from (34) with $m \in \{0,\dots,2\}$. □

4.4 Convergence analysis

Let us begin this section with the following remark.

Remark 4.25 (Regularity of the solution to (1))**.** *Since the boundary of the domain Ω is assumed to be composed of a finite union of portions of affine hyperplanes, one can prove (see, e.g., [31, Theorem 4.4.3.7]) the following elliptic regularity result: there exists $\varepsilon \in (0, 1/2]$ so that $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)$, and*

$$\|u\|_{\frac{3}{2}+\varepsilon,\Omega} \lesssim \|f\|_{0,\Omega}.$$

If Ω is in addition convex, one can actually prove full elliptic regularity, in the sense that $u \in H^2(\Omega)$, and

$$\|u\|_{2,\Omega} \lesssim \|f\|_{0,\Omega}. \tag{43}$$

In any case, and since $\mathrm{div}(\boldsymbol{\nabla} u) = \triangle u = -f \in L^2(\Omega)$, there holds: for all $F \in \mathcal{F}_h^{\mathrm{i}}$, $[\![\boldsymbol{\nabla} u]\!]_F \cdot \boldsymbol{n}_F = 0$ almost everywhere on F.

Let $|\cdot|_{\mathrm{e},h}^2 := \mathrm{a}_h(\cdot,\cdot)$. According to Corollary 4.23, $|\cdot|_{\mathrm{e},h}$ defines a norm on $\mathbb{V}_{d,h,0}^k$. For $\boldsymbol{\beta} := (\beta_T)_{T\in\mathcal{T}_h} \in \{2,\dots,k+1\}^{\mathrm{card}(\mathcal{T}_h)}$, we define

$$H^{\boldsymbol{\beta}}(\mathcal{T}_h) := \left\{v_h \in L^2(\Omega) \mid v_{h|T} \in H^{\beta_T}(T)\ \forall T \in \mathcal{T}_h\right\}.$$

Theorem 4.26 (Energy-norm error estimate)**.** *Assume that the solution $u \in H_0^1(\Omega)$ to Problem (1) further belongs to $H^{\boldsymbol{\beta}}(\mathcal{T}_h)$. Then, the following estimate holds:*

$$\left|\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h\right|_{\mathrm{e},h} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\right)^{1/2}, \tag{44}$$

where $\mathbb{R}^{n_{d,\mathcal{T}_h}^k} \ni \overline{\Sigma}_{d,h}^k(u) := \big(\overline{\Sigma}_{d,T}^k(u_{|T})\big)_{T\in\mathcal{T}_h}$, and $\underline{\mathrm{u}}_h \in \mathbb{V}_{d,h,0}^k$ is the unique solution to Problem (18).

Proof. Since $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)\bigcap H_0^1(\Omega)$ (cf. Remark 4.25), $u \in C^0(\overline{\Omega})$ and $u_{|\partial\Omega} \equiv 0$, hence $\overline{\Sigma}_{d,h}^k(u) \in \mathbb{V}_{d,h,0}^k$, and so does the difference $\big(\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h\big)$. We can then write

$$\left|\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h\right|_{\mathrm{e},h} = \max_{\underline{\mathrm{v}}_h\in\mathbb{V}_{d,h,0}^k,\,|\underline{\mathrm{v}}_h|_{\mathrm{e},h}=1} \mathrm{a}_h\big(\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h,\underline{\mathrm{v}}_h\big). \tag{45}$$

Since $\underline{\mathrm{u}}_h$ solves Problem (18), we have to estimate, for any $\underline{\mathrm{v}}_h \in \mathbb{V}_{d,h,0}^k$ such that $|\underline{\mathrm{v}}_h|_{\mathrm{e},h} = 1$,

$$\mathcal{C}_h(\underline{\mathrm{v}}_h) := \sum_{T\in\mathcal{T}_h} \mathcal{C}_T(\underline{\mathrm{v}}_T), \quad \text{with} \quad \mathcal{C}_T(\underline{\mathrm{v}}_T) := \mathrm{a}_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}),\underline{\mathrm{v}}_T\big) - \int_T f_{|T}\mathrm{v}_T^{\mathrm{o}}.$$

Using the strong form of Problem (1), and integrating by parts, we infer, for any $T \in \mathcal{T}_h$,

$$\mathcal{C}_T(\underline{\mathrm{v}}_T) = \int_T \boldsymbol{\nabla}\mathcal{P}_{d,T}^k(u_{|T})\cdot\boldsymbol{\nabla} p_{d,T}^k\underline{\mathrm{v}}_T - \int_T \boldsymbol{\nabla} u\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}}$$
$$+\int_{\partial T}\mathrm{v}_{T|\partial T}^{\mathrm{o}}\boldsymbol{\nabla} u_{|\partial T}\cdot\boldsymbol{n}_T + s_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}),\underline{\mathrm{v}}_T\big).$$

Using the definition (32), and since $\mathcal{P}_{d,T}^k(u_{|T}) \in \mathbb{P}_d^k(T)$, we then have

$$\mathcal{C}_T(\underline{\mathrm{v}}_T) = \int_T \boldsymbol{\nabla}\big(\mathcal{P}_{d,T}^k(u_{|T})-u\big)\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}} - \int_{\partial T}\pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}}-\mathrm{t}_{\partial T}^k\underline{\mathrm{v}}_T^{\partial}\big)\boldsymbol{\nabla}\mathcal{P}_{d,T}^k(u_{|T})_{|\partial T}\cdot\boldsymbol{n}_T$$
$$+\int_{\partial T}\pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}}\big)\boldsymbol{\nabla} u_{|\partial T}\cdot\boldsymbol{n}_T + s_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}),\underline{\mathrm{v}}_T\big).$$

Summing over $T \in \mathcal{T}_h$, and invoking the continuity of the flux of the exact solution along interfaces (cf. Remark 4.25), combined to the fact that $\mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_{T|F}^{\partial} \equiv 0$ for all $F \in \mathcal{F}_T \bigcap \mathcal{F}_h^{\mathrm{b}}$ since $\underline{\mathrm{v}}_h \in \mathbb{V}_{d,h,0}^k$, we infer

$$\mathcal{C}_h(\underline{\mathrm{v}}_h) = \sum_{T\in\mathcal{T}_h} \Bigg(\int_T \boldsymbol{\nabla}\big(\mathcal{P}_{d,T}^k(u_{|T}) - u\big)\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}} + \int_{\partial T} \pi_{\partial T}^l\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}^k \underline{\mathrm{v}}_T^{\partial}\big)\boldsymbol{\nabla}\big(u - \mathcal{P}_{d,T}^k(u_{|T})\big)_{|\partial T}\cdot\boldsymbol{n}_T + s_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}), \underline{\mathrm{v}}_T\big)\Bigg). \quad (46)$$

Applying Cauchy–Schwarz inequality, and the approximation results (34)–(35) with $s \in \{\beta_T\}_{T\in\mathcal{T}_h}$ and $m = 1$, we obtain

$$\mathcal{C}_h(\underline{\mathrm{v}}_h) \lesssim \left(\sum_{T\in\mathcal{T}_h}\Big(h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2 + \mathrm{s}_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}), \overline{\Sigma}_{d,T}^k(u_{|T})\big)\Big)\right)^{1/2}\Big(|\underline{\mathrm{v}}_h|_h + |\underline{\mathrm{v}}_h|_{\mathrm{e},h}\Big).$$

The conclusion then follows from Lemma 4.24 with $s \in \{\beta_T\}_{T\in\mathcal{T}_h}$, Corollary 4.23, and the fact that $|\underline{\mathrm{v}}_h|_{\mathrm{e},h} = 1$ by assumption. □

Remark 4.27. *In the nonconforming case, to prove that $\overline{\Sigma}_{d,h}^k(u)$ belongs to $\mathbb{V}_{d,h,0}^k$, it is sufficient to use that $u \in H_0^1(\Omega)$. Furthermore, the first term in the right-hand side of (46) is identically zero. Indeed, in that case, for all $T \in \mathcal{T}_h$, $\mathcal{P}_{d,T}^k = \Pi_T^k$ (recall that $\mathrm{v}_T^{\mathrm{o}} \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$). Such a property is not true in the conforming case.*

Corollary 4.28 ($H^1(\mathcal{T}_h)$-seminorm error estimate)**.** *Under the regularity assumption of Theorem 4.26, the following estimate holds:*

$$\big\|\boldsymbol{\nabla}_h\big(u - p_{d,h}^k\underline{\mathrm{u}}_h\big)\big\|_{0,\Omega} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\right)^{1/2}, \quad (47)$$

where $p_{d,h}^k : \mathbb{R}^{n_{d,\mathcal{T}_h}^k} \to \mathbb{P}_d^k(\mathcal{T}_h)$ is such that, for all $T \in \mathcal{T}_h$, $p_{d,h}\underline{\mathrm{v}}_{h|T} := p_{d,T}^k\underline{\mathrm{v}}_T$.

Proof. For any $T \in \mathcal{T}_h$, by a simple triangle inequality, we infer

$$\big\|\boldsymbol{\nabla}\big(u - p_{d,T}^k\underline{\mathrm{u}}_T\big)\big\|_{0,T} \leqslant \big\|\boldsymbol{\nabla}\big(u - \mathcal{P}_{d,T}^k(u_{|T})\big)\big\|_{0,T} + \Big\|\boldsymbol{\nabla}p_{d,T}^k\big(\overline{\Sigma}_{d,T}^k(u_{|T}) - \underline{\mathrm{u}}_T\big)\Big\|_{0,T}.$$

Squaring, summing over $T \in \mathcal{T}_h$, and using the definition of $|\cdot|_{\mathrm{e},h}$, yields

$$\big\|\boldsymbol{\nabla}_h\big(u - p_{d,h}^k\underline{\mathrm{u}}_h\big)\big\|_{0,\Omega}^2 \lesssim \sum_{T\in\mathcal{T}_h}\big\|\boldsymbol{\nabla}\big(u - \mathcal{P}_{d,T}^k(u_{|T})\big)\big\|_{0,T}^2 + \Big|\overline{\Sigma}_{d,h}^k(u) - \underline{\mathrm{u}}_h\Big|_{\mathrm{e},h}^2. \quad (48)$$

The conclusion then follows from the approximation result (34) with $s \in \{\beta_T\}_{T\in\mathcal{T}_h}$ and $m = 1$, and (44). □

In view of (48) and (45), the error in $H^1(\mathcal{T}_h)$-seminorm between the exact solution u and the computable/computed solution $p_{d,h}^k\underline{\mathrm{u}}_h$ simply splits into an approximation error (directly available from the properties of $\mathcal{P}_{d,T}^k$, $T \in \mathcal{T}_h$), and the consistency error of the scheme, that can be estimated within a few lines (cf. the proof of Theorem 4.26).

Remark 4.29. *Letting $u_h \in V_{d,h,0}^k$ (respectively, $\mathcal{I}_{d,h}^k u \in V_{d,h,0}^k$) such that $u_{h|T} := r_{d,T}^k\underline{\mathrm{u}}_T$ (respectively, $\mathcal{I}_{d,h}^k u_{|T} := \mathcal{I}_{d,T}^k(u_{|T}) = r_{d,T}^k\big(\overline{\Sigma}_{d,T}^k(u_{|T})\big)$) for all $T \in \mathcal{T}_h$, we know, by Remark 4.20 and Corollary 4.23, that*

$$\big\|\boldsymbol{\nabla}_h\big(\mathcal{I}_{d,h}^k u - u_h\big)\big\|_{0,\Omega} \lesssim \Big|\overline{\Sigma}_{d,h}^k(u) - \underline{\mathrm{u}}_h\Big|_{\mathrm{e},h}. \quad (49)$$

Since, by the triangle inequality, and letting $\mathcal{P}_{d,h}^k := \Pi_h^k \circ \mathcal{I}_{d,h}^k$,

$$\big\|\boldsymbol{\nabla}_h\big(u - u_h\big)\big\|_{0,\Omega} \leqslant \big\|\boldsymbol{\nabla}_h\big(u - \mathcal{P}_{d,h}^k u\big)\big\|_{0,\Omega} + \big\|\boldsymbol{\nabla}_h\big(\mathcal{P}_{d,h}^k u - \mathcal{I}_{d,h}^k u\big)\big\|_{0,\Omega} + \big\|\boldsymbol{\nabla}_h\big(\mathcal{I}_{d,h}^k u - u_h\big)\big\|_{0,\Omega},$$

we can prove, using (34) *with* $s \in \{\beta_T\}_{T\in\mathcal{T}_h}$ *and* $m = 1$ *for the first term in the right-hand side, the remark* (37) *combined to* (31) *and* (34) *for the second, and* (49) *combined to* (44) *for the third, that, under the same regularity assumption as in Theorem 4.26, there holds*

$$\left\|\boldsymbol{\nabla}_h\big(u-u_h\big)\right\|_{0,\Omega} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|^2_{\beta_T,T}\right)^{1/2}. \tag{50}$$

Let us now derive an estimate on the error between $p^k_{d,h}\underline{\mathrm{u}}_h$ and u in the $L^2(\Omega)$-norm.

Theorem 4.30 (Potential-norm error estimate)**.** *Assume that the solution* $u \in H^1_0(\Omega)$ *to Problem* (1) *further belongs to* $H^{\boldsymbol{\beta}}(\mathcal{T}_h)$ *and that, when* $k = 1$*, one also has* $f \in H^1(\mathcal{T}_h)$*. Assume full elliptic regularity for Problem* (1) *(cf. Remark 4.25). Then, there holds*

$$\|u-u_h\|_{0,\Omega} \lesssim h\left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}\big(|u|^2_{\beta_T,T}+\delta_{1k}|f|^2_{1,T}\big)\right)^{1/2}. \tag{51}$$

Proof. Setting $g := (u-u_h) \in L^2(\Omega)$, we let $z \in H^1_0(\Omega)$ be the unique solution in $H^1(\Omega)$ to $-\triangle z = g$ in Ω, with $z = 0$ on $\partial\Omega$. By (full) elliptic regularity, $z \in H^2(\Omega)$ and, in view of (43), there holds

$$\|z\|_{2,\Omega} \lesssim \|u-u_h\|_{0,\Omega}. \tag{52}$$

Writing

$$\|u-u_h\|^2_{0,\Omega} = \int_\Omega (u-u_h)g = -\int_\Omega (u-u_h)\triangle z, \tag{53}$$

we distinguish between the conforming, and the nonconforming cases. In the conforming case, using the fact that $(u-u_h) \in H^1_0(\Omega)$, we infer by integration by parts:

$$\|u-u_h\|^2_{0,\Omega} = \int_\Omega \boldsymbol{\nabla}(u-u_h)\cdot\boldsymbol{\nabla} z.$$

Then, we add/subtract $\mathcal{P}^k_{d,h}z$, to obtain

$$\|u-u_h\|^2_{0,\Omega} = \int_\Omega \boldsymbol{\nabla}(u-u_h)\cdot\boldsymbol{\nabla}_h(z-\mathcal{P}^k_{d,h}z) + \int_\Omega \big(\boldsymbol{\nabla}_h\Pi^k_h u\cdot\boldsymbol{\nabla}\mathcal{I}^k_{d,h}z - \boldsymbol{\nabla}_h\Pi^k_h u_h\cdot\boldsymbol{\nabla}_h\mathcal{P}^k_{d,h}z\big),$$

which, in turn, rewrites

$$\begin{aligned}\|u-u_h\|^2_{0,\Omega} = \int_\Omega \boldsymbol{\nabla}(u-u_h)\cdot\boldsymbol{\nabla}_h(z-\mathcal{P}^k_{d,h}z) + \left(\int_\Omega \boldsymbol{\nabla}_h\Pi^k_h u\cdot\boldsymbol{\nabla}\mathcal{I}^k_{d,h}z - \mathrm{a}_h\big(\underline{\mathrm{u}}_h,\overline{\Sigma}^k_{d,h}(z)\big)\right)\\ + \sum_{T\in\mathcal{T}_h}\mathrm{s}_T\big(\underline{\mathrm{u}}_T,\overline{\Sigma}^k_{d,T}(z_{|T})\big) =: \mathfrak{T}^c_1+\mathfrak{T}^c_2+\mathfrak{T}^c_3.\end{aligned} \tag{54}$$

In the nonconforming case now, we add/subtract $\pi^{k-1}_h(u-u_h)$ in (53), and we use that $\pi^{k-1}_T(u_{h|T}) = \mathrm{u}^{\mathrm{o}}_T$ for any $T \in \mathcal{T}_h$, to infer

$$\|u-u_h\|^2_{0,\Omega} = -\int_\Omega \big((u-u_h)-\pi^{k-1}_h(u-u_h)\big)\triangle z - \sum_{T\in\mathcal{T}_h}\int_T \big(\pi^{k-1}_T(u_{|T})-\mathrm{u}^{\mathrm{o}}_T\big)\triangle z. \tag{55}$$

By an integration by parts, and using the fact that $[\![\boldsymbol{\nabla} z]\!]_F\cdot\boldsymbol{n}_F = 0$ almost everywhere on interfaces $F \in \mathcal{F}^{\mathrm{i}}_h$ (cf. Remark 4.25), as well as the fact that $\pi^{k-1}_F(u_{|F}) = \mathrm{u}^{\partial}_{T|F} \equiv 0$ for all $F \in \mathcal{F}_T\bigcap\mathcal{F}^{\mathrm{b}}_h$, there holds

$$\begin{aligned}-\sum_{T\in\mathcal{T}_h}\int_T \big(\pi^{k-1}_T(u_{|T})-\mathrm{u}^{\mathrm{o}}_T\big)\triangle z = \sum_{T\in\mathcal{T}_h}\Bigg(\int_T \boldsymbol{\nabla}\big(\pi^{k-1}_T(u_{|T})-\mathrm{u}^{\mathrm{o}}_T\big)\cdot\boldsymbol{\nabla} z\\ -\int_{\partial T}\big[(\pi^{k-1}_T(u_{|T})-\mathrm{u}^{\mathrm{o}}_T)_{|\partial T}-(\pi^{k-1}_{\partial T}(u_{|\partial T})-\mathrm{u}^{\partial}_T)\big]\boldsymbol{\nabla} z_{|\partial T}\cdot\boldsymbol{n}_T\Bigg).\end{aligned}$$

Adding/subtracting the term $\displaystyle\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\nabla}p^k_{d,T}\big(\overline{\Sigma}^k_{d,T}(u_{|T})-\underline{\mathrm{u}}_T\big)\cdot\boldsymbol{\nabla}\mathcal{P}^k_{d,T}(z_{|T})$ to (55), using the definition (32), and letting

$$\mathfrak{T}^{nc}_{1,1} := -\int_\Omega \big((u-u_h)-\pi^{k-1}_h(u-u_h)\big)\triangle z,$$

as well as

$$\mathfrak{T}_{1,2}^{nc} := \sum_{T\in\mathcal{T}_h}\Big(\int_T \boldsymbol{\nabla}\big(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}\big)\cdot\boldsymbol{\nabla}\big(z - \mathcal{P}_{d,T}^k(z_{|T})\big) - \int_{\partial T}\big[(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}})_{|\partial T} - (\pi_{\partial T}^{k-1}(u_{|\partial T}) - \mathrm{u}_T^{\partial})\big]\boldsymbol{\nabla}\big(z - \mathcal{P}_{d,T}^k(z_{|T})\big)_{|\partial T}\cdot\boldsymbol{n}_T\Big),$$

we hence infer that

$$\|u-u_h\|_{0,\Omega}^2 = \mathfrak{T}_{1,1}^{nc} + \mathfrak{T}_{1,2}^{nc} + \Big(\int_\Omega \boldsymbol{\nabla}_h\mathcal{P}_{d,h}^k u\cdot\boldsymbol{\nabla}_h\mathcal{P}_{d,h}^k z - \mathrm{a}_h\big(\underline{\mathrm{u}}_h, \overline{\Sigma}_{d,h}^k(z)\big)\Big) + \sum_{T\in\mathcal{T}_h}\mathrm{s}_T\big(\underline{\mathrm{u}}_T, \overline{\Sigma}_{d,T}^k(z_{|T})\big) =: \mathfrak{T}_{1,1}^{nc} + \mathfrak{T}_{1,2}^{nc} + \mathfrak{T}_2^{nc} + \mathfrak{T}_3^{nc}. \quad (56)$$

We now have to estimate the different terms in the right-hand sides of (54), and of (56). To estimate $\mathfrak{T}_1^c$, we apply, successively, Cauchy–Schwarz inequality, (50), and (34) with $s=2$ and $m=1$, combined to (52). We obtain

$$\mathfrak{T}_1^c \lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\Big)^{1/2} h\,\|u-u_h\|_{0,\Omega}. \quad (57)$$

To estimate $\mathfrak{T}_{1,1}^{nc}$, we apply, successively, Cauchy–Schwarz inequality, (10) with $s=1$ and $m=0$, (50), the fact that $\|\triangle z\|_{0,\Omega}\leqslant|z|_{2,\Omega}$, and (52). We get

$$\mathfrak{T}_{1,1}^{nc} \lesssim h\Big(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\Big)^{1/2}\|u-u_h\|_{0,\Omega}. \quad (58)$$

To estimate $\mathfrak{T}_{1,2}^{nc}$, we apply, successively, the Cauchy–Schwarz inequality, (34)–(35) with $s=2$ and $m=1$, and Corollary 4.23. We get

$$\mathfrak{T}_{1,2}^{nc} \lesssim \Big|\overline{\Sigma}_{d,h}^k(u) - \underline{\mathrm{u}}_h\Big|_{\mathrm{e},h} h\,|z|_{2,\Omega},$$

which, in turn, by (44), and (52), gives

$$\mathfrak{T}_{1,2}^{nc} \lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\Big)^{1/2} h\,\|u-u_h\|_{0,\Omega}. \quad (59)$$

To estimate $\mathfrak{T}_3^c$ and $\mathfrak{T}_3^{nc}$, we use the fact that

$$\sum_{T\in\mathcal{T}_h}\mathrm{s}_T\big(\underline{\mathrm{u}}_T, \overline{\Sigma}_{d,T}^k(z_{|T})\big) = \sum_{T\in\mathcal{T}_h}\mathrm{s}_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}), \overline{\Sigma}_{d,T}^k(z_{|T})\big) - \sum_{T\in\mathcal{T}_h}\mathrm{s}_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}) - \underline{\mathrm{u}}_T, \overline{\Sigma}_{d,T}^k(z_{|T})\big),$$

to infer, using successively the Cauchy–Schwarz inequality, (42) with $s\in\{\beta_T\}_{T\in\mathcal{T}_h}$ for $u_{|T}$, (44), (42) with $s=2$ for $z_{|T}$, and (52),

$$\mathfrak{T}_3^c, \mathfrak{T}_3^{nc} \lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\Big)^{1/2} h\,\|u-u_h\|_{0,\Omega}. \quad (60)$$

We are left with estimating $\mathfrak{T}_2^c$, and $\mathfrak{T}_2^{nc}$. Since $\underline{\mathrm{u}}_h\in\mathbb{V}_{d,h,0}^k$ solves Problem (18), adding/subtracting $\int_\Omega \boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}\mathcal{I}_{d,h}^k z = \int_\Omega f\,\mathcal{I}_{d,h}^k z$ in the conforming case (remark that $\mathcal{I}_{d,h}^k z\in H_0^1(\Omega)$ in that case), and $\int_\Omega \boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}z = \int_\Omega f\,z$ in the nonconforming case, there holds

$$\mathfrak{T}_2^c = \int_\Omega \boldsymbol{\nabla}_h\big(\Pi_h^k u - u\big)\cdot\boldsymbol{\nabla}_h\big(\mathcal{I}_{d,h}^k z - \mathcal{P}_{d,h}^k z\big) + \sum_{T\in\mathcal{T}_h}\int_T f_{|T}\left(\mathcal{I}_{d,T}^k(z_{|T}) - \pi_T^{k-1}(z_{|T})\right), \quad (61)$$

and (recall that $\mathcal{P}_{d,T}^k = \Pi_T^k$ in the nonconforming case)

$$\mathfrak{T}_2^{nc} = \int_\Omega \boldsymbol{\nabla}_h\big(\Pi_h^k u - u\big)\cdot\boldsymbol{\nabla}_h\big(z - \Pi_h^k z\big) + \sum_{T\in\mathcal{T}_h}\int_T f_{|T}\left(z - \pi_T^{k-1}(z_{|T})\right). \tag{62}$$

The first term in the right-hand side of (61) is estimated by means of (i) Cauchy–Schwarz inequality, (ii) (13) with $s\in\{\beta_T\}_{T\in\mathcal{T}_h}$ and $m=1$, (iii) the remark (37) combined to (31), and to (34) with $s=2$ and $m\in\{1,2\}$, and (iv) (52). The first term in the right-hand side of (62) is estimated using (i) the Cauchy–Schwarz inequality, (ii) (13) with $m=1$, and (iii) (52). It remains to estimate the second term in the right-hand side of (61) (equivalent arguments apply to (62)). When $k=1$, there holds

$$\sum_{T\in\mathcal{T}_h}\int_T f_{|T}\left(\mathcal{I}_{d,T}^1(z_{|T}) - \pi_T^0(z_{|T})\right) = \sum_{T\in\mathcal{T}_h}\int_T \left(f_{|T} - \pi_T^0(f_{|T})\right)\mathcal{I}_{d,T}^1\big(z_{|T} - \pi_T^0(z_{|T})\big).$$

Then, by Cauchy–Schwarz inequality, (31), and (10), we infer

$$\sum_{T\in\mathcal{T}_h}\int_T f_{|T}\left(\mathcal{I}_{d,T}^1(z_{|T}) - \pi_T^0(z_{|T})\right) \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^2|f|_{1,T}^2\right)^{1/2} h\left(|z|_{1,\Omega} + h|z|_{2,\Omega}\right). \tag{63}$$

When $k\geqslant 2$ now, there holds

$$\sum_{T\in\mathcal{T}_h}\int_T f_{|T}\left(\mathcal{I}_{d,T}^k(z_{|T}) - \pi_T^{k-1}(z_{|T})\right) = \sum_{T\in\mathcal{T}_h}\int_T\Big(f_{|T} - \pi_T^{\beta_T-3}(f_{|T})\Big)\,\mathcal{I}_{d,T}^k\big(z_{|T} - \pi_T^{k-1}(z_{|T})\big).$$

By the same kind of arguments as in the case $k=1$, we infer

$$\sum_{T\in\mathcal{T}_h}\int_T f_{|T}\left(\mathcal{I}_{d,T}^k(z_{|T}) - \pi_T^{k-1}(z_{|T})\right) \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\right)^{1/2} h\,|z|_{2,\Omega}. \tag{64}$$

Collecting (61), (63)–(64), and using (52), we obtain

$$\mathfrak{T}_2^c, \mathfrak{T}_2^{nc} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}\big(|u|_{\beta_T,T}^2 + \delta_{1k}|f|_{1,T}^2\big)\right)^{1/2} h\,\|u-u_h\|_{0,\Omega}. \tag{65}$$

Collecting (54), (56), (57), (58), (59), (65), and (60), we conclude the proof. □

Corollary 4.31 ($L^2(\Omega)$-norm error estimate)**.** *Under the assumptions of Theorem 4.30, the following estimate holds:*

$$\big\|u - p_{d,h}^k\underline{\mathrm{u}}_h\big\|_{0,\Omega} \lesssim h\left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}\big(|u|_{\beta_T,T}^2 + \delta_{1k}|f|_{1,T}^2\big)\right)^{1/2}. \tag{66}$$

Proof. By the triangle inequality, and since $p_{d,h}^k\underline{\mathrm{u}}_h = \Pi_h^k u_h$, we infer

$$\big\|u - p_{d,h}^k\underline{\mathrm{u}}_h\big\|_{0,\Omega} \leqslant \big\|u - \Pi_h^k u\big\|_{0,\Omega} + \big\|\Pi_h^k(u-u_h)\big\|_{0,\Omega}.$$

By the stability property of Π_T^k for any $T\in\mathcal{T}_h$ (cf. Proposition 2.6), there holds

$$\big\|u - p_{d,h}^k\underline{\mathrm{u}}_h\big\|_{0,\Omega} \leqslant \big\|u - \Pi_h^k u\big\|_{0,\Omega} + \|u-u_h\|_{0,\Omega} + h\|\boldsymbol{\nabla}_h(u-u_h)\|_{0,\Omega}.$$

The conclusion then follows from (13) with $s\in\{\beta_T\}_{T\in\mathcal{T}_h}$ and $m=0$, (51), and (50). □

Remark 4.32 (Supercloseness of cell unknowns)**.** *Noticing that $\pi_T^{k-1}\big([u-u_h]_{|T}\big) = \pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}$ for any $T\in\mathcal{T}_h$, there holds, by stability of π_T^{k-1} in the $L^2(T)$-norm, by (51), and under the same assumptions as in Theorem 4.30,*

$$\big\|\pi_h^{k-1}u - \mathrm{u}_h^{\mathrm{o}}\big\|_{0,\Omega} \lesssim h\left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}\big(|u|_{\beta_T,T}^2 + \delta_{1k}|f|_{1,T}^2\big)\right)^{1/2},$$

where $\mathrm{u}_h^{\mathrm{o}}\in\mathbb{P}_d^{k-1}(\mathcal{T}_h)$ is such that $\mathrm{u}_{h|T}^{\mathrm{o}} := \mathrm{u}_T^{\mathrm{o}}$ for all $T\in\mathcal{T}_h$. Hence, the cell unknowns are superclose to the projections of degree $k-1$ of u.

4.5 General workflow

As for any Galerkin method, the general workflow for solving Problem (18) with the c/nc-VE/HHO methods splits into (i) an offline stage, that is independent of the source term, and which aims at performing the assembly of the problem matrix, and (ii) an online stage, that consists, for a given source term, in solving the resulting global system. A change in the data only affects the online stage (cf. Remark 4.33). The precomputations that are performed in the offline stage are all local; hence, the offline stage can naturally benefit from parallel architectures.

Let us describe, in details, these two stages, beginning with the offline stage.

1. In the conforming case with $d = 2$ (respectively, $d = 3$), one first computes the operator $r^k_{1,F}$ defined by (23) for all $F \in \mathcal{F}_h$ (respectively, $r^k_{1,e}$ for all $e \in \mathcal{E}_h$). This requires to solve a SPD system of size N^k_1 (cf. Remark 2.7), with $\mathrm{N}^{k-2}_1 + 2$ right-hand sides. In both the conforming and nonconforming cases, locally to any $T \in \mathcal{T}_h$, one computes the operator $p^k_{d,T}$ defined by (32). Its computation requires to solve a SPD system of size N^k_d (cf. also Remark 2.7), for a number of right-hand sides that is (i) $\mathrm{N}^{k-1}_2 + \mathrm{card}(\mathcal{F}_T) \times \mathrm{N}^{k-2}_1 + \mathrm{card}(\mathcal{V}_T)$ in the conforming case with $d = 2$, (ii) $\mathrm{N}^{k-1}_3 + \mathrm{card}(\mathcal{F}_T) \times \mathrm{N}^{k-1}_2$ in the conforming case with $d = 3$, and (iii) $\mathrm{N}^{k-1}_d + \mathrm{card}(\mathcal{F}_T) \times \mathrm{N}^{k-1}_{d-1}$ in the nonconforming case (cf. Remark 4.19). Once the operator $p^k_{d,T}$ has been computed, one computes, still locally to any $T \in \mathcal{T}_h$, the stabilization form(s), that express(es) in terms of the DoFs and of the different already computed quantities.

2. As common to any skeletal method, cell DoFs are locally eliminated by static condensation in terms of the local skeletal DoFs. Locally to each $T \in \mathcal{T}_h$, one has to invert a SPD matrix of size N^{k-1}_d.

Let us now describe the online stage, for a given source term $f \in L^2(\Omega)$.

3. One computes the right-hand side. This requires to integrate, locally to any $T \in \mathcal{T}_h$, $f_{|T}$ against polynomials in $\mathbb{P}^{k-1}_d(T)$.

4. One solves the resulting SPD global system, that is of size

$$\mathrm{card}(\mathcal{F}_h) \times \mathrm{N}^{k-2}_1 + \mathrm{card}(\mathcal{V}_h)$$

in the conforming case with $d = 2$,

$$\mathrm{card}(\mathcal{F}_h) \times \mathrm{N}^{k-1}_2 + \mathrm{card}(\mathcal{E}_h) \times \mathrm{N}^{k-2}_1 + \mathrm{card}(\mathcal{V}_h)$$

in the conforming case with $d = 3$ (cf. Remark 4.34), and

$$\mathrm{card}(\mathcal{F}_h) \times \mathrm{N}^{k-1}_{d-1}$$

in the nonconforming case.

Remark 4.33. *Except point* 3*, the description above of the general workflow of the method applies verbatim to the case of a Problem* (1) *featuring a source term $f \in H^{-1}(\Omega)$ (assuming that a decomposition $f = f_0 + \mathrm{div}\boldsymbol{f}$ with $f_0 \in L^2(\Omega)$ and $\boldsymbol{f} \in L^2(\Omega)^d$ is known), or nonhomogeneous mixed Dirichlet–Neumann boundary conditions. Indeed, assuming that the lines/colums of the problem matrix corresponding to Dirichlet skeletal DoFs are diagonalized, such changes only affect the right-hand side of the discrete variational formulation.*

Remark 4.34. *In the conforming case with $d = 3$, one can reduce the size of the global system by using enhanced virtual spaces [14], or Serendipity spaces [7], on the faces $F \in \mathcal{F}_T$ of the cells $T \in \mathcal{T}_h$, instead of $V^k_2(F)$.*

A Proofs of (9) and Lemma 3.2

A.1 Proof of (9)

To prove (9), we first recall (cf., e.g., [12, Section 2.7]) that, since T is star-shaped with respect to a ball whose radius is comparable to h_T, for any $z \in H^1(\partial T)$, there exists $\overline{z} \in H^1(T)$ such that $\overline{z}_{|\partial T} = z$, and

$$h_T^{-1}\|\overline{z}\|_{0,T} + |\overline{z}|_{1,T} \lesssim h_T^{-1/2}\|z\|_{0,\partial T} + h_T^{1/2}|z|_{1,\partial T}. \tag{67}$$

Now, by integration by parts, there holds, for any $z \in H^1(\partial T) \subset H^{1/2}(\partial T)$,

$$\langle \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T, z \rangle_{-\frac{1}{2},\partial T} = \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} \overline{z} + \int_T \triangle v \, \overline{z},$$

which yields, by Cauchy–Schwarz inequality,

$$\langle \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T, z \rangle_{-\frac{1}{2},\partial T} \leqslant |v|_{1,T} |\overline{z}|_{1,T} + h_T \|\triangle v\|_{0,T} h_T^{-1} \|\overline{z}\|_{0,T}.$$

The estimate (67) enables to conclude.

A.2 Proof of Lemma 3.2

Assume that, for $v_h \in V_{h,0}$, $\|\boldsymbol{\nabla}_h v_h\|_{0,\Omega} = 0$. Then, for all $T \in \mathcal{T}_h$, $\boldsymbol{\nabla}(v_{h|T}) = \boldsymbol{0}$ and there is $c_T \in \mathbb{R}$ such that $v_{h|T} = c_T$. Since $\{\check{\pi}_F^0\}_{F \in \mathcal{F}_h} \subseteq \Sigma_h^\partial$, then for all $F \in \mathcal{F}_h^{\mathrm{i}}$, $\int_F [\![v_h]\!]_F = 0$ and there exists $c \in \mathbb{R}$ such that $c_T = c$ for all $T \in \mathcal{T}_h$. The fact that $\int_F v_{h|F} = 0$ for some (here, all) $F \in \mathcal{F}_h^{\mathrm{b}}$ finally yields that $v_h \equiv 0$ on Ω. To prove (20), we start from the following discrete Poincaré inequality on $H^1(\mathcal{T}_h)$ (cf., e.g., [9]):

$$\forall v_h \in H^1(\mathcal{T}_h), \qquad \sum_{T \in \mathcal{T}_h} h_T^{-2} \|v_{h|T}\|_{0,T}^2 \lesssim \|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}^2 + \sum_{F \in \mathcal{F}_h} h_F^{-1} \|[\![v_h]\!]_F\|_{0,F}^2,$$

and we show that $\sum_{F \in \mathcal{F}_h} h_F^{-1} \|[\![v_h]\!]_F\|_{0,F}^2 \lesssim \|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}^2$ for all $v_h \in V_{h,0}$. To prove so, since for $v_h \in V_{h,0}$, $\int_F [\![v_h]\!]_F = 0$ for all $F \in \mathcal{F}_h$, there holds

$$\|[\![v_h]\!]_F\|_{0,F}^2 = \int_F [\![v_h - \pi_h^0 v_h]\!]_F [\![v_h]\!]_F \leqslant \|[\![v_h - \pi_h^0 v_h]\!]_F\|_{0,F} \|[\![v_h]\!]_F\|_{0,F},$$

and we can use the continuous trace inequality (4) and the Poincaré inequality (6) to infer

$$\begin{aligned} h_F^{-\frac{1}{2}} \|[\![v_h]\!]_F\|_{0,F} &\lesssim h_F^{-\frac{1}{2}} \sum_{T \in \mathcal{T}_F} \left(h_T^{-\frac{1}{2}} \|v_{h|T} - \pi_T^0(v_{h|T})\|_{0,T} + h_T^{\frac{1}{2}} \|\boldsymbol{\nabla}(v_{h|T})\|_{0,T} \right) \\ &\lesssim h_F^{-\frac{1}{2}} \sum_{T \in \mathcal{T}_F} h_T^{\frac{1}{2}} \|\boldsymbol{\nabla}(v_{h|T})\|_{0,T}. \end{aligned}$$

Finally, since h_F is comparable to h_T for $T \in \mathcal{T}_F$, and $\mathrm{card}(\mathcal{F}_T) \lesssim 1$ for all $T \in \mathcal{T}_h$ (cf. Section 2.2), the conclusion follows. To obtain (21), it suffices to remark that, for all $T \in \mathcal{T}_h$, $h_T \leqslant h \leqslant \mathrm{diam}(\Omega)$.

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