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On the Root solution to the Skorokhod embedding problem given full marginals ^{*}

Alexandre Richard [†] Xiaolu Tan [‡] Nizar Touzi [§]

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Abstract

This paper examines the Root solution of the Skorokhod embedding problem given full marginals on some compact time interval. Our results are obtained by limiting arguments based on finitely-many marginals Root solution of Cox, Obłój, and Touzi [9]. Our main result provides a characterization of the corresponding potential function by means of a convenient parabolic PDE.

1 Introduction

The Skorokhod embedding problem, initially suggested by Skorokhod [30], consists in finding a stopping time τ together with a Brownian motion B such that $B_\tau \sim \mu$ for a given marginal distribution μ on \mathbb{R} . The existing literature contains various solutions suggested in different contexts. Some of them satisfy an optimality property among all possible solutions, e.g. the Root solution [28], the Rost solution [29], the Azéma-Yor solution [1], the Vallois solution [31], the Perkins solution [27], etc. This problem has been extensively revived in the recent literature due to the important connexion with the problem of robust hedging in financial mathematics. We refer to Obłój [25] and Hobson [19] for a survey on different solutions and the applications in finance.

Our interest in this paper is on the Root solution of the Skorokhod embedding problem, which is characterized as a hitting time of the Brownian motion B of some time-space domain \mathcal{R} unlimited to the right, that is, $\tau_R := \inf\{t \geq 0 : (t, B_t) \in \mathcal{R}\}$. This solution was shown by Rost [29] to have the minimal variance among all solutions to the embedding problem. As an application in finance, it can be used to deduce robust no-arbitrage price bounds for a class of variance options (see e.g. Hobson [19]). To find the barrier \mathcal{R} in the description of the Root solution, Cox and Wang [8] provided a construction by solving a variational inequality. This

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approach is then explored in Gassiat, Oberhauser, and dos Reis [13] and Gassiat, Mijatović, and Oberhauser [12] to construct \mathcal{R} under more general conditions. We also refer to the remarkable work of Beiglböck, Cox, and Huesmann [4] which derives the Root embedding, among other solutions, as a natural consequence of the monotonicity principle in optimal transport.

It is also natural to extend the Skorohod embedding problem to the multiple marginals case. Let $(\mu_k)_{0 \leq k \leq n}$ be a family of marginal distributions, nondecreasing in the convex order, i.e. $\mu_{k-1}(\phi) \leq \mu_k(\phi)$, $k = 1, \dots, n$, for all convex functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$. The multiple-marginals Skorohod embedding problem is to find a Brownian motion B , together with an increasing sequence of stopping times $(\tau_k)_{1 \leq k \leq n}$, such that $B_{\tau_k} \sim \mu_k$ for each $k = 1, \dots, n$. Madan and Yor [23] provided a sufficient condition on the marginals, under which the Azéma-Yor embedding stopping times corresponding to each marginal are automatically ordered, so that the iteration of Azéma-Yor solutions provides a solution to the multiple marginals Skorohod embedding problem. In general, the Azéma-Yor embedding stopping times may not be ordered. An extension of the Azéma-Yor embedding was obtained by Brown, Hobson, and Rogers [6] in the two-marginals case, and later by Oblój and Spoida [26] for an arbitrary finite number of marginals. Moreover, the corresponding embeddings enjoys the similar optimality property as in the one marginal case. In Claisse, Guo, and Henry-Labordère [7], an extension of the Vallois solution to the two-marginals case is obtained for a specific class of marginals. We also refer to Beiglböck, Cox, and Huesmann [5] for a geometric representation of the optimal Skorohod embedding solutions given multiple marginals.

The Root solution of the Skorohod embedding problem was recently extended by Cox, Oblój, and Touzi [9] to the multiple marginals case. Our objective in this paper is to characterize the limit case with a family of full marginals $\mu = (\mu_t)_{t \in [0,1]}$. Let us assume that each μ_t has finite first moment and $t \mapsto \mu_t$ is right continuous and increasing in convex order. Such a family is called a peacock (or PCOC “Processus Croissant pour l’Ordre Convexe” in French) by Hirsch, Profeta, Roynette, and Yor [18]. Then Kellerer’s Theorem [22] ensures the existence of a right-continuous martingale $M = (M_t)_{0 \leq t \leq 1}$ such that $M_t \sim \mu_t$ for each $t \in [0, 1]$. Further, by Monroe’s result [24], one can find an increasing sequence of stopping times $(\tau_t)_{0 \leq t \leq 1}$ together with a Brownian motion $B = (B_s)_{s \geq 0}$ such that $B_{\tau_t} \sim \mu_t$ for each $t \in [0, 1]$. This consists in an embedding for the full marginals μ . We refer to [18] for different explicit constructions of the martingales or embeddings fitting the peacock marginals. Among all martingales or μ -embeddings, it is interesting to find solutions enjoying some optimality properties. In the context of Madan and Yor [23], the Azéma-Yor embedding τ_t^{AY} of the one marginal problem with μ_t is ordered w.r.t. t , and thus $(\tau_t^{AY})_{0 \leq t \leq 1}$ is the embedding maximizing the expected maximum among all embedding solutions. This optimality is further extended by Källblad, Tan, and Touzi [21] allowing for non-ordered barriers. Hobson [20] gave a construction of a martingale with minimal expected total variation among all martingales fitting the marginals. Henry-Labordère, Tan, and Touzi [17] provided a local Lévy martingale, as limit of the left-monotone martingales introduced by Beiglböck and Juillet [3] (see also Henry-Labordère and Touzi [16]), which inherits its optimality property. For general existence of the optimal solution and the associated duality result, one needs a tightness argument, which is studied in Guo, Tan, and Touzi [15] by using the S-topology on the Skorohod space, and in Källblad, Tan, and Touzi [21] by using the Skorohod embedding approach.

The aim of this paper is to study the full marginals limit of the multiple-marginals Root embedding as derived in [9]. This leads to a natural extension of the Root solution for the

embedding problem given full marginals. Using the tightness result in [21], we can easily obtain the existence of such limit as well as its optimality. We then provide some characterization of the limit Root solution as well as that of the associated optimal stopping problem, which is used in the finitely many marginals case to describe the barriers.

In the rest of the paper, we will first formulate our main results in Section 2. Then in Section 3, we recall some details on the Root solution given finitely many marginals in [9] and the limit argument of [21], which induces the existence of the limit Root solution for the embedding problem given full marginals. We then provide the proofs of our main results on some characterization of the limit Root solution in Section 4. Some further discussions are finally provided in Section 5.

2 Problem formulation and main results

We are given a family of probability measures $\mu = (\mu_s)_{s \in [0,1]}$ on \mathbb{R} , such that μ_s is centred with finite first moment for all $s \in [0, 1]$, $s \mapsto \mu_s$ is càdlàg under the weak convergence topology, and the family μ is non-decreasing in convex order, i.e. for any convex function $\phi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}} \phi(x) \mu_s(dx) \leq \int_{\mathbb{R}} \phi(x) \mu_t(dx) \quad \text{for all } s \leq t.$$

Definition 2.1. (i) A stopping rule is a term

$$\alpha = (\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha, \mathbb{P}^\alpha, B^\alpha, (\tau_s^\alpha)_{s \in [0,1]}),$$

such that $(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha, \mathbb{P}^\alpha)$ is a filtered probability space equipped with a standard Brownian motion B^α and a family of stopping times $(\tau_s^\alpha)_{s \in [0,1]}$ such that $s \mapsto \tau_s^\alpha$ is càdlàg and non-decreasing. We denote

$$\mathcal{A} := \{\text{All stopping rules}\}, \quad \text{and} \quad \mathcal{A}_t := \{\alpha \in \mathcal{A} : \tau_1^\alpha \leq t\}, \quad \text{for all } t \geq 0.$$

(ii) A stopping rule $\alpha \in \mathcal{A}$ is called a μ -embedding if $(B_{t \wedge \tau_1^\alpha}^\alpha)_{t \geq 0}$ is uniformly integrable and $B_{\tau_s^\alpha}^\alpha \sim \mu_s$ for all $s \in [0, 1]$. We denote by $\mathcal{A}(\mu)$ the collection of all μ -embeddings.

(iii) Let $\pi_n = \{0 = s_0 < s_1 < \dots < s_n = 1\}$ be a partition of $[0, 1]$. A stopping rule $\alpha \in \mathcal{A}$ is called a (μ, π_n) -embedding if $(B_{t \wedge \tau_1^\alpha}^\alpha)_{t \geq 0}$ is uniformly integrable and $B_{\tau_{s_k}^\alpha}^\alpha \sim \mu_{s_k}$ for all $k = 1, \dots, n$. We denote by $\mathcal{A}(\mu, \pi_n)$ the collection of all (μ, π_n) -embeddings.

Our aim is to study the Root solution of the Skorohod embedding problem (SEP, hereafter) given full marginals $(\mu_s)_{s \in [0,1]}$. To this end, we first recall the Root solution of the SEP given finitely many marginals, constructed in [9]. Let $(\pi_n)_{n \geq 1}$ be a sequence of partitions of $[0, 1]$, where $\pi_n = \{0 = s_0^n < s_1^n < \dots < s_n^n = 1\}$ and $|\pi_n| := \max_{k=1}^n |s_k^n - s_{k-1}^n| \rightarrow 0$ as $n \rightarrow \infty$. Then for every fixed n , one obtains n marginal distributions $(\mu_{s_k^n})_{1 \leq k \leq n}$ and has the following Root solution to the corresponding SEP.

Theorem (Cox, Obłój and Touzi, 2018). *For any $n \geq 1$, there exists a (μ, π_n) -embedding α_n^* called Root embedding, where $\sigma_k^n := \tau_{s_k^n}^{\alpha_n^*}$ is defined by*

$$\sigma_0^n := 0 \quad \text{and} \quad \sigma_k^n := \inf\{t \geq \sigma_{k-1}^n : (t, B_t^{\alpha_n^*}) \in \mathcal{R}_k^n\},$$

for some family of barriers $(\mathcal{R}_k^n)_{1 \leq k \leq n}$ in $\mathbb{R}_+ \times \mathbb{R}$. Moreover, for any non-decreasing and non-negative function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, one has

$$\mathbb{E}^{\mathbb{P}^{\alpha_n^*}} \left[\int_0^{\tau_1^{\alpha_n^*}} f(t) dt \right] = \inf_{\alpha \in \mathcal{A}(\mu, \pi_n)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^{\tau_1^\alpha} f(t) dt \right].$$

The barriers $(\mathcal{R}_k^n)_{1 \leq k \leq n}$ are given explicitly in [9] by solving an optimal stopping problem, see Section 3.1 below.

Let us denote by $\mathbb{A}([0, 1], \mathbb{R}_+)$ the space of all càdlàg non-decreasing functions $a : [0, 1] \rightarrow \mathbb{R}_+$, which is a Polish space under the Lévy metric. Notice also that the Lévy metric metrizes the weak convergence topology on $\mathbb{A}([0, 1], \mathbb{R}_+)$ seen as a space of finite measures. Denote also by $C(\mathbb{R}_+, \mathbb{R})$ the space of all continuous paths $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $\omega_0 = 0$, which is a Polish space under the compact convergence topology. Then for a given embedding α , one can see (B^α, τ^α) as a random element taking values in $C(\mathbb{R}_+, \mathbb{R}) \times \mathbb{A}([0, 1], \mathbb{R}_+)$, which allows to define their weak convergence. Our first main result ensures that the (μ, π_n) -Root embedding has a limit in sense of the weak convergence, which enjoys the same optimality property, and thus can be considered as the full marginals Root solution of the SEP. Our proof requires the following technical condition.

Assumption 2.2. *Let $U : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be the potential function of μ defined by*

$$U(s, x) := - \int_{\mathbb{R}} |x - y| \mu_s(dy). \quad (2.1)$$

Assume that U is C^1 in s , with partial derivative $\partial_s U$ such that $x \mapsto \sup_{s \in [0, 1]} \partial_s U(s, \cdot)$ has polynomial growth.

Theorem 1. (i) *Let $(\pi_n)_{n \geq 1}$ be a sequence of partitions of $[0, 1]$ such that $|\pi_n| \rightarrow 0$ as $n \rightarrow \infty$. Denote by α_n^* the corresponding (μ, π_n) -Root embedding solution. Then there exists $\alpha^* \in \mathcal{A}(\mu)$ such that the sequence $(B^{\alpha_n^*}, \tau^{\alpha_n^*})_{n \geq 1}$ weakly converges to $(B^{\alpha^*}, \tau^{\alpha^*})$. Moreover, for all non-decreasing and non-negative functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, one has*

$$\mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[\int_0^{\tau_1^{\alpha^*}} f(t) dt \right] = \inf_{\alpha \in \mathcal{A}(\mu)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^{\tau_1^\alpha} f(t) dt \right].$$

(ii) *Under Assumption 2.2, for all fixed $(s, t) \in [0, 1] \times \mathbb{R}_+$, the law of $B_{\tau_s^{\alpha^*} \wedge t}^{\alpha^*}$ is independent of the sequence of partitions $(\pi_n)_{n \geq 1}$ and of the limit α^* .*

We next provide some characterization of the full marginals Root solution of the SEP α^* given in Theorem 1. Let

$$u(s, t, x) := -\mathbb{E}^{\mathbb{P}^{\alpha^*}} [|B_{t \wedge \tau_s^{\alpha^*}}^{\alpha^*} - x|], \quad (s, t, x) \in \mathbf{Z} := [0, 1] \times \mathbb{R}_+ \times \mathbb{R}. \quad (2.2)$$

Our next main result, Theorem 2 below, provides a unique characterization of u which is independent of the nature of the limit α^* , thus justifying Claim (ii) of Theorem 1. Moreover, it follows by direct computation that one has

$$-|x| - \sqrt{t} \mathbb{E}|B_1| \leq U_{\mathbf{N}(0, t)}(x) \leq u(1, t, x) \leq u(s, t, x) \leq U(0, x),$$

for all $(s, t, x) \in [0, 1] \times \mathbb{R}_+ \times \mathbb{R}$, where we denoted by $U_{\mathbf{N}(0,t)}$ the potential function of the $\mathbf{N}(0, t)$ distribution (see (2.1) for the definition of the potential function).

In the finitely many marginals case in [9], the function u is obtained from an optimal stopping problem and is then used to define the barriers in the construction of the Root solution. Similar to equations (2.10) and (3.1) in [9], we can characterize u as value function of an optimal stopping problem, and then as unique viscosity solution of the variational inequality:

$$\begin{cases} \min \left\{ \partial_t u - \frac{1}{2} \partial_{xx}^2 u, \partial_s(u - U) \right\} = 0, & \text{on } \text{int}(\mathbf{Z}), \\ u|_{t=0} = U \quad \text{and} \quad u|_{s=0} = U(0, \cdot). \end{cases} \quad (2.3)$$

Let Du and D^2u denote the gradient and Hessian of u w.r.t. $z = (s, t, x)$, and set:

$$F(Du, D^2u) := \min \left\{ \partial_t u - \frac{1}{2} \partial_{xx}^2 u, \partial_s(u - U) \right\}. \quad (2.4)$$

Definition 2.3. (i) *An upper semicontinuous function $v : \mathbf{Z} \rightarrow \mathbb{R}$ is a viscosity subsolution of (2.3) if $v|_{s=0} \leq U(0, \cdot)$, $v|_{t=0} \leq U$ and $F(D\varphi, D^2\varphi)(z_0) \leq 0$ for all $(z_0, \varphi) \in \text{int}(\mathbf{Z}) \times C^2(\mathbf{Z})$ satisfying $(v - \varphi)(z_0) = \max_{z \in \mathbf{Z}} (v - \varphi)(z)$.*

(ii) *A lower semicontinuous function $w : [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a viscosity supersolution of (2.3) if $w|_{s=0} \geq U(0, \cdot)$, $w|_{t=0} \geq U$ and $F(D\varphi, D^2\varphi)(z_0) \geq 0$ for all $(z_0, \varphi) \in \text{int}(\mathbf{Z}) \times C^2(\mathbf{Z})$ satisfying $(w - \varphi)(z_0) = \min_{z \in \mathbf{Z}} (w - \varphi)(z)$.*

(iii) *A continuous function v is a viscosity solution of (2.3) if it is both viscosity subsolution and supersolution.*

Theorem 2. *Let Assumption 2.2 hold true.*

(i) *The function u can be expressed as value function of an optimal stopping problem,*

$$u(s, t, x) = \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}^{\mathbb{P}^\alpha} \left[U(0, x + B_{\tau_\alpha}^\alpha) + \int_0^s \partial_s U(s - k, x + B_{\tau_k}^\alpha) \mathbf{1}_{\{\tau_k^\alpha < t\}} dk \right]. \quad (2.5)$$

(ii) *The function $u(s, t, x)$ is decreasing and locally Lipschitz in s , uniformly Lipschitz in x and uniformly $\frac{1}{2}$ -Hölder in t . Moreover, u is a viscosity solution of equation (2.3).*

(iii) *Moreover, u is the unique viscosity solution of (2.3) satisfying*

$$|u(s, t, x)| \leq C(1 + t + |x|), \quad (s, t, x) \in \mathbf{Z}, \quad \text{for some constant } C > 0.$$

3 Multiple marginals Root solution of the SEP and its limit

The main objective of this section is to recall the construction of the Root solution to the SEP given multiple marginals from [9]. As an extension to the one marginal Root solution studied in [8] and [12], the solution to the multiple marginals' case enjoys some optimality property among all embeddings. We then also recall the limit argument in [21] to show how the optimality property is preserved in the limit case.

3.1 The Root solution of the SEP given multiple marginals

Let $n \in \mathbb{N}$ and π_n be a partition of $[0, 1]$, with $\pi_n = \{0 = s_0^n < s_1^n < \dots < s_n^n = 1\}$, we then obtain n marginal distributions $\mu^n := \{\mu_{s_j^n}\}_{j=1, \dots, n}$ and recall the Root solution to the corresponding embedding problem.

Let $\Omega = C(\mathbb{R}_+, \mathbb{R})$ denote the canonical space of all continuous paths $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $\omega_0 = 0$, B be the canonical process, $B^x := x + B$, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the canonical filtration, $\mathcal{F} := \mathcal{F}_\infty$, and \mathbb{P}_0 the Wiener measure under which B is a standard Brownian motion. For each $t \geq 0$, let $\mathcal{T}_{0,t}$ denote the collection of all \mathbb{F} -stopping times taking values in $[0, t]$. Denote

$$\delta^n U(s_j^n, x) := U(s_j^n, x) - U(s_{j-1}^n, x), \quad x \in \mathbb{R},$$

which is non-positive since $\{\mu_s\}_{s \in [0,1]}$ is non-decreasing in convex ordering. We then define the function $u^n(\cdot)$ by a sequence of optimal stopping problems:

$$u^n|_{s=0} := U(s_0^n, \cdot), \text{ and } u^n(s_j^n, t, x) := \sup_{\theta \in \mathcal{T}_{0,t}} \mathbb{E} [u^n(s_{j-1}^n, t - \theta, B_\theta^x) + \delta^n U(s_j^n, B_\theta^x) \mathbf{1}_{\{\theta < t\}}]. \quad (3.1)$$

Denoting similarly $\delta^n u(s_j^n, t, x) = u^n(s_j^n, t, x) - u^n(s_{j-1}^n, t, x)$, we define the corresponding stopping regions

$$\mathcal{R}_j^n := \{(t, x) \in [0, \infty] \times [-\infty, \infty] : \delta^n u(s_j^n, t, x) = \delta^n U(s_j^n, x)\}, \quad j = 1, \dots, n. \quad (3.2)$$

Given the above, the Root solution on the Brownian motion B in the space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, is given by the family $\sigma^n = (\sigma_1^n, \dots, \sigma_n^n)$ of stopping times

$$\sigma_0^n := 0, \text{ and } \sigma_j^n := \inf \{t \geq \sigma_{j-1}^n : (t, B_t) \in \mathcal{R}_j^n\}, \quad \forall j \in \{1, \dots, n\}. \quad (3.3)$$

The stopping times σ^n induce a stopping rule α_n^* in the sense of Definition 2.1:

$$\alpha_n^* = (\Omega^{\alpha_n^*}, \mathcal{F}^{\alpha_n^*}, \mathbb{F}^{\alpha_n^*}, \mathbb{P}^{\alpha_n^*}, B^{\alpha_n^*}, \tau^{\alpha_n^*}) := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_0, B, \tau^{\alpha_n^*}), \quad (3.4)$$

with $\tau_s^{\alpha_n^*} := \sigma_j^n$ for $s \in [s_j^n, s_{j+1}^n)$.

Theorem (Cox, Obłój, and Touzi [9]). *The stopping rule α_n^* is a (μ, π_n) -embedding, with*

$$u^n(s_j^n, t, x) = -\mathbb{E}[B_{t \wedge \sigma_j^n} - x]. \quad (3.5)$$

Moreover, for all non-decreasing and non-negative $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we have

$$\mathbb{E}^{\mathbb{P}_0} \left[\int_0^{\sigma_n^n} f(t) dt \right] = \mathbb{E}^{\mathbb{P}_0} \left[\int_0^{\tau_1^{\alpha_n^*}} f(t) dt \right] = \inf_{\alpha \in \mathcal{A}(\mu, \pi_n)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^{\tau_1^\alpha} f(t) dt \right].$$

Using a dynamic programming argument, one can also reformulate the definition of u^n in (3.1) by induction as a global multiple optimal stopping problem. Let us denote by $\mathcal{T}_{0,t}^n$ the collection of all terms (τ_1, \dots, τ_n) , where each τ_j , $j = 1, \dots, n$, is a \mathbb{F} -stopping time on $(\Omega, \mathcal{F}, \mathbb{P}_0)$ satisfying $0 \leq \tau_1 \leq \dots \leq \tau_n \leq t$.

Proposition 3.1. *For all $j = 1, \dots, n$, we have*

$$u^n(s_j^n, t, x) = \sup_{(\tau_1, \dots, \tau_n) \in \mathcal{T}_{0,t}^n} \mathbb{E} \left[U(0, x + B_{\tau_j}) + \sum_{k=1}^j \delta^n U(s_k^n, x + B_{\tau_j - k + 1}) \mathbf{1}_{\{\tau_j - k + 1 < t\}} \right]. \quad (3.6)$$

Proof. We will use a backward induction argument. First, let us denote by $\mathcal{T}_{r,t}$ the collection of all \mathbb{F} -stopping times taking values in $[r, t]$, and by $B_s^{r,x} := x + B_s - B_r$ for all $s \geq r$. Then it follows from the expression (3.1) that

$$u^n(s_{j-1}^n, t - r, x) = \sup_{\tau \in \mathcal{T}_{r,t}} \mathbb{E} \left[u^n(s_{j-2}^n, t - \tau, B_\tau^{r,x}) + \delta^n U(s_{j-1}^n, B_\tau^{r,x}) \mathbf{1}_{\{\tau < t\}} \right].$$

Using the dynamic programming principle, one has

$$\begin{aligned} u^n(s_j^n, t, x) &= \sup_{\tau_1 \in \mathcal{T}_{0,t}} \mathbb{E} \left[u^n(s_{j-1}^n, t - \tau_1, B_{\tau_1}^{0,x}) + \delta^n U(s_j^n, B_{\tau_1}^{0,x}) \mathbf{1}_{\{\tau_1 < t\}} \right] \\ &= \sup_{\tau_1 \in \mathcal{T}_{0,t}} \mathbb{E} \left[\text{ess sup}_{\tau_2 \in \mathcal{T}_{\tau_1,t}} \mathbb{E} \left[u^n(s_{j-2}^n, t - \tau_2, B_{\tau_2}^{\tau_1, B_{\tau_1}^{0,x}}) + \delta^n U(s_{j-1}^n, B_{\tau_2}^{\tau_1, B_{\tau_1}^{0,x}}) \mathbf{1}_{\{\tau_1 < t\}} \middle| \mathcal{F}_{\tau_1} \right] \right. \\ &\quad \left. + \delta^n U(s_j^n, B_{\tau_1}^{0,x}) \mathbf{1}_{\{\tau_1 < t\}} \right] \\ &= \sup_{(\tau_1, \tau_2) \in \mathcal{T}_{0,t}^2} \mathbb{E} \left[u^n(s_{j-2}^n, t - \tau_2, B_{\tau_2}^{0,x}) + \sum_{k=j-1}^j \delta^n U(s_k^n, B_{\tau_{j-k+1}}^{0,x}) \mathbf{1}_{\{\tau_{j-k+1} < t\}} \right]. \end{aligned}$$

To conclude, it is enough to apply the same argument to iterate and to use the fact that $u^n(0, t, x) = U(0, x)$ for any t and x . \square

Remark 3.2. For later uses, we also observe that it is not necessary to restrict the stopping times w.r.t. the Brownian filtration, in the optimal stopping problem (3.6). In fact, one can consider a larger filtration with respect to which B is still a Brownian motion.

More precisely, let \mathcal{A}_t^n denote the collection of all stopping rules

$$\alpha = (\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha, \mathbb{P}^\alpha, B^\alpha, \tau_j^\alpha, j = 1, \dots, n)$$

such that $(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{F}^\alpha, \mathbb{P}^\alpha)$ is a filtered probability space equipped with a standard Brownian motion B^α and $(\tau_j^\alpha)_{j=1, \dots, n}$ is a sequence of stopping times satisfying $0 \leq \tau_1^\alpha \leq \dots \leq \tau_n^\alpha \leq t$. Then one has

$$u^n(s_j^n, t, x) = \sup_{\alpha \in \mathcal{A}_t^n} \mathbb{E}^{\mathbb{P}^\alpha} \left[U(0, x + B_{\tau_j^\alpha}^\alpha) + \sum_{k=1}^j \delta^n U(s_k^n, x + B_{\tau_{j-k+1}^\alpha}^\alpha) \mathbf{1}_{\{\tau_{j-k+1}^\alpha < t\}} \right].$$

This equivalence is standard and very well known in case $n = 1$, see also Lemma 4.9 of [14] for the multiple stopping problem where $n \geq 1$.

3.2 The Root solution given full marginals (Theorem 1.(i))

In Källblad, Tan, and Touzi [21], it is shown that the sequence of Root stopping times $(\sigma_1^n, \dots, \sigma_n^n)_{n \geq 1}$ is tight in some sense and any limit provides an embedding solution given full marginals.

More precisely, let $(\sigma_k^n)_{k=1, \dots, n}$ be the Root embedding given n -marginals $(\mu_{s_k^n})_{k=1, \dots, n}$ defined in (3.3), we define α_n^* by (3.4) as a (μ, π_n) -embedding in sense of Definition 2.1. Notice that $\overline{\mathbb{P}}^n := \mathbb{P}^{\alpha_n^*} \circ (B^{\alpha_n^*}, \tau^{\alpha_n^*})^{-1}$ is a probability measure on $C(\mathbb{R}_+, \mathbb{R}) \times \mathbb{A}([0, 1], \mathbb{R}_+)$, which is a Polish space if $C(\mathbb{R}_+, \mathbb{R})$ is equipped with the compact convergence topology and $\mathbb{A}([0, 1], \mathbb{R}_+)$

is equipped with the Lévy metric. This allows us to consider the weak convergence of the sequence $(\alpha_n^*)_{n \geq 1}$. Theorem 1 is then a consequence of the following convergence theorem, which can be gathered from several results in [21]. Recall also that $\mathcal{A}(\mu)$ and $\mathcal{A}(\mu, \pi_n)$ are defined in Definition 2.1.

Proposition 3.3. *Let $(\pi_n)_{n \geq 1}$ be a sequence of partitions of $[0, 1]$ with mesh $|\pi_n| \rightarrow 0$, and let α_n^* be the corresponding multiple-marginals Root embedding (3.4). Then, the sequence $(B_{\tau_s^{\alpha_n^*}}^{\alpha_n^*}, \tau_s^{\alpha_n^*})$ is tight, and any limit α^* is a full marginals embedding, i.e. $\alpha^* \in \mathcal{A}(\mu)$, with*

$$\mathbb{P}^{\alpha_{n_k}^*} \circ (B_{\tau_s^{\alpha_{n_k}^*}}^{\alpha_{n_k}^*})^{-1} \longrightarrow \bar{\mathbb{P}}^{\alpha^*} \circ (B_{\tau_s^{\alpha^*}}^{\alpha^*})^{-1}, \quad \text{for all } s \in [0, 1] \setminus \mathbb{T},$$

for some countable set $\mathbb{T} \subset [0, 1]$, and some subsequence $(n_k)_{k \geq 1}$, and

$$\mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[\int_0^{\tau_1^{\alpha^*}} f(t) dt \right] = \inf_{\alpha \in \mathcal{A}(\mu)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\int_0^{\tau_1^\alpha} f(t) dt \right],$$

for any non-decreasing and non-negative function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Proof. (i) The first item is a direct consequence of Lemma 4.5 of [21].

(ii) For the second item, we notice that $\Phi(\omega, \theta) := -\int_0^{\theta} f(t) dt$ is a continuous function defined on $C(\mathbb{R}_+, \mathbb{R}) \times \mathbb{A}([0, 1], \mathbb{R}_+)$ and bounded from above. Then it is enough to apply Theorem 2.3 and Proposition 3.6 of [21] to obtain the optimality of α^* . \square

4 Proof of Theorems 2 and 1.(ii)

Recall that $u(s, t, \cdot)$ is defined in (2.2) as the potential function of $B_{t \wedge \tau_s^{\alpha^*}}^{\alpha^*}$ for an arbitrary Root solution α^* given full marginals. We provide an optimal stopping problem characterization as well as a PDE characterization for the function u under Assumption 2.2, which consists in a proof of Theorem 2. Further, the uniqueness of the solution to the PDE induces the uniqueness result in part (ii) of Theorem 1.

4.1 Characterization of u by an optimal stopping problem (Theorem 2.(i))

By a slight abuse of notation, we can extend the definition of u^n given in (3.1) to $[0, 1] \times \mathbb{R}_+ \times \mathbb{R}$ by setting

$$u^n(s, t, x) := u^n(s_j^n, t, x) \quad \text{whenever } s \in (s_{j-1}^n, s_j^n].$$

With \mathcal{A}_t in Definition 2.1, we also define \tilde{u} as a mapping from $[0, 1] \times \mathbb{R}_+ \times \mathbb{R}$ to \mathbb{R} by

$$\tilde{u}(s, t, x) := \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}^{\mathbb{P}^\alpha} \left[U(0, x + B_{\tau_s^\alpha}^\alpha) + \int_0^s \partial_s U(s - k, x + B_{\tau_k^\alpha}^\alpha) \mathbf{1}_{\{\tau_k^\alpha < t\}} dk \right]. \quad (4.1)$$

The main objective of this section is to provide some characterisation of this limit law as well as the limit problem of u^n (3.1) used in the construction of the Root solution.

Proposition 4.1. *For all (s, t, x) , one has $u^n(s, t, x) \rightarrow \tilde{u}(s, t, x)$ as $n \rightarrow \infty$.*

Proof. We start by rewriting the representation formula of $u^n(k, t, x)$ in Remark 3.2 as

$$u^n(s_j^n, t, x) = \sup_{\alpha \in \mathcal{A}_t^n} \mathbb{E}^{\mathbb{P}^\alpha} \left[U(0, x + B_{\tau_j^\alpha}^\alpha) + \sum_{k=1}^j \delta^n U(s_{j-k+1}^n, x + B_{\tau_k^\alpha}^\alpha) \mathbf{1}_{\{\tau_k^\alpha < t\}} \right].$$

(i) First, for a fixed $n \in \mathbb{N}$, one can see \mathcal{A}_t^n as a subset of \mathcal{A}_t in the following sense. Given $\alpha \in \mathcal{A}_t^n$, and assume that $s \in (s_{j-1}^n, s_j^n]$. Let us set $\hat{\tau}_k^\alpha := \tau_i^\alpha$ whenever $k \in [s - s_{j-i+1}^n, s - s_{j-i}^n)$. Notice that $U(s_j^n, x + B_{\tau_1^\alpha}^\alpha) - U(s, x + B_{\tau_1^\alpha}^\alpha) \leq 0$, then it follows by direct computation that

$$\begin{aligned} & \sum_{k=1}^j \delta^n U(s_{j-k+1}^n, x + B_{\tau_k^\alpha}^\alpha) \mathbf{1}_{\{\tau_k^\alpha < t\}} \\ & \leq U(s, x + B_{\tau_1^\alpha}^\alpha) - U(s_{j-1}^n, x + B_{\tau_1^\alpha}^\alpha) + \sum_{k=2}^j \delta^n U(s_{j-k+1}^n, x + B_{\tau_k^\alpha}^\alpha) \mathbf{1}_{\{\tau_k^\alpha < t\}} \\ & = \int_0^s \partial_s U(s - k, x + B_{\tau_k^\alpha}^\alpha) \mathbf{1}_{\{\hat{\tau}_k^\alpha < t\}} dk \end{aligned}$$

and it follows by (4.1) that $u^n(s, t, x) = u^n(s_j^n, t, x) \leq \tilde{u}(s, t, x)$.

(ii) Let $\alpha \in \mathcal{A}_t$, and define $\alpha_n \in \mathcal{A}_t^n$ by

$$\tau_j^{\alpha_n} := \tau_{s_j^n}^\alpha, \quad \text{for } j = 1, \dots, n.$$

Let j_n be such that $s_{j_n}^n$ converges to s as $n \rightarrow \infty$, then it follows that

$$\begin{aligned} X_n & := U(0, x + B_{\tau_{j_n}^{\alpha_n}}^\alpha) + \sum_{k=1}^{j_n} \delta^n U(s_{j_n-k+1}^n, x + B_{\tau_k^{\alpha_n}}^\alpha) \mathbf{1}_{\{\tau_k^{\alpha_n} < t\}} \\ & \xrightarrow{n \rightarrow \infty} U(0, x + B_{\tau_s^\alpha}^\alpha) + \int_0^s \partial_s U(s - k, x + B_{\tau_k^\alpha}^\alpha) \mathbf{1}_{\{\tau_k^\alpha < t\}} dk, \quad \text{a.s.} \end{aligned}$$

Recall that by Assumption 2.2, there exists some $C > 0$ and $p > 0$ such that $|U(0, x)| \leq C + |x|$ and $\sup_{s \in [0, 1]} |\partial_s U(s, x)| \leq C(1 + |x|^p)$ for any $x \in \mathbb{R}$, then $(X_n)_{n \geq 1}$ is in fact uniformly integrable. Hence for $\varepsilon > 0$ such that α is ε -optimal in (4.1), it follows from the previous remark and from Fatou's lemma that $\liminf_{n \rightarrow \infty} \mathbb{E}[X_n] \geq \tilde{u}(s, t, x) - \varepsilon$. Thus, $\lim_{n \rightarrow \infty} u^n(s, t, x) \geq \tilde{u}(s, t, x)$. Hence this proves that the following convergence holds: $\lim_{n \rightarrow \infty} u^n(s, t, x) = \tilde{u}(s, t, x)$. \square

Lemma 4.2. *The function $\tilde{u}(s, t, x)$ is non-increasing and locally Lipschitz in s , and is uniformly Lipschitz in x and uniformly $\frac{1}{2}$ -Hölder in t .*

Proof. First, using representation formula of u^n in (3.5) and noticing that $y \mapsto |y - x|$ is convex, we see that $s \mapsto u^n(s, t, x)$ is non-increasing. Further, using (3.1), it follows immediately that

$$u^n(s_j^n, t, x) - u^n(s_{j-1}^n, t, x) \geq U(s_j^n, x) - U(s_{j-1}^n, x).$$

Then under Assumption 2.2, one has $0 \geq \partial_s u^n(s, t, x) \geq -C(1 + |x|^p)$ for some constant $C > 0$ and $p > 0$ independent of n . By the limit result $u^n \rightarrow \tilde{u}$, it follows that $\tilde{u}(s, t, x)$ is non-increasing and locally Lipschitz in s .

Finally, using again the representation formula of u^n in (3.5), it is easy to deduce that $u^n(k, t, x)$ is uniformly Lipschitz in x and $1/2$ -Hölder in t , uniformly in n . As limit of u^n , it follows that \tilde{u} is also uniformly Lipschitz in x and $1/2$ -Hölder in t . \square

We next show that the function u defined by (2.2) is also the limit of u^n , which leads to the equivalence of u and \tilde{u} , and then Theorem 2.(i) readily follows.

Proposition 4.3. *For all $(s, t, x) \in [0, 1] \times \mathbb{R}_+ \times \mathbb{R}$, one has $u(s, t, x) = \tilde{u}(s, t, x)$.*

Proof. By Theorem 3.1 of [9] together with our extended definition in (3.4), $u^n(s, t, x) = -\mathbb{E}^{\mathbb{P}^{\alpha_n^*}} \left[\left| B_{t \wedge \tau_s^{\alpha_n^*}}^{\alpha_n^*} - x \right| \right]$. Moreover, by Proposition 3.3, there exists a countable set $\mathbb{T} \subset [0, 1]$ and a subsequence $(n_k)_{k \geq 1}$ such that $\mathbb{P}^{\alpha_{n_k}^*} \circ (B_{\tau_s^{\alpha_{n_k}^*}}^{\alpha_{n_k}^*})^{-1} \rightarrow \mathbb{P}^{\alpha^*} \circ (B_{\tau_s^{\alpha^*}}^{\alpha^*})^{-1}$. As $\mathbb{E}[\max_{0 \leq r \leq t} |B_r - x|] < \infty$ for a Brownian motion, it follows that $\mathbb{E}^{\mathbb{P}^{\alpha_{n_k}^*}} \left[\left| B_{t \wedge \tau_s^{\alpha_{n_k}^*}}^{\alpha_{n_k}^*} - x \right| \right] \rightarrow \mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[\left| B_{t \wedge \tau_s^{\alpha^*}}^{\alpha^*} - x \right| \right]$, for all $s \in [0, 1] \setminus \mathbb{T}$. Hence,

$$u^n(s, t, x) \rightarrow u(s, t, x), \quad \text{for all } s \in [0, 1] \setminus \mathbb{T}.$$

Further, by the right-continuity of $s \mapsto \tau_s^{\alpha^*}$, it is easy to deduce that $s \mapsto u(s, t, x) := -\mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[\left| B_{t \wedge \tau_s^{\alpha^*}}^{\alpha^*} - x \right| \right]$ is right-continuous.

On the other hand, we know from Proposition 4.1 that $u^n(s, t, x) \rightarrow \tilde{u}(s, t, x)$, and from Lemma 4.2 that \tilde{u} is a continuous function in all arguments, it follows that $u(s, t, x) = \tilde{u}(s, t, x)$ holds for all $(s, t, x) \in [0, 1] \times \mathbb{R}_+ \times \mathbb{R}$. \square

Remark 4.4. *Formally, we can understand the above result in the following equivalent way. The n -marginals Root solution (σ^n, B) converges weakly to a full marginal Root solution (σ, B) which then satisfies:*

$$u(s, t, x) = -\mathbb{E} \left[\left| B_{t \wedge \sigma_s} - x \right| \right].$$

4.2 PDE characterization of u (Theorem 2.(ii))

Proof of Theorem 2.(ii). *Step 1.* We first notice that the continuity of $u(s, t, x)$ in (s, t, x) follows directly by Lemma 4.2 and Proposition 4.3.

Step 2. In a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ equipped with a Brownian motion W , we denote by \mathcal{U}_t the collection of all \mathbb{F} -predictable processes $\gamma = (\gamma_r)_{r \geq 0}$ such that $\int_0^1 \gamma_r^2 dr \leq t$. Given a control process γ , we define two controlled processes X^γ and Y^γ by

$$X_s^\gamma := x + \int_0^s \gamma_r dW_r, \quad Y_s^\gamma := \int_0^s \gamma_r^2 dr.$$

By a time change argument, one can show that

$$u(s, t, x) = \sup_{\gamma \in \mathcal{U}_t} \mathbb{E} \left[U(0, X_s^\gamma) + \int_0^s \partial_s U(s - k, X_k^\gamma) \mathbf{1}_{\{Y_k^\gamma < t\}} dk \right]. \quad (4.2)$$

Indeed, given $\gamma \in \mathcal{U}_t$, one obtains a square integrable martingale X^γ which has the representation $X_s^\gamma = W_{Y_s^\gamma}$, where W is a Brownian motion and Y_s^γ are all stopping times, and it induces a stopping rule in \mathcal{A}_t . By (4.1) and Proposition 4.3, it follows that in (4.2), the left-hand side is larger than the right-hand side. On the other hand, given an increasing sequence of stopping times $(\tau_1, \dots, \tau_n) \in \mathcal{T}_{0,t}^n$, we define $\Gamma_s := \tau_j \vee \frac{s - s_j^n}{s_{j+1}^n - s_j^n} \wedge \tau_{j+1}$ for all $s \in [s_j^n, s_{j+1}^n)$. Notice that

$s \mapsto \Gamma_s$ is absolutely continuous and $\Gamma_{s_j^n} = \tau_j$ for all $j = 1, \dots, n$. Then one can construct a predictable process γ such that

$$\mathbb{P}_0 \circ \left(\int_0^\cdot \gamma_r dB_r, \int_0^\cdot \gamma_r^2 dt \right)^{-1} = \mathbb{P}_0 \circ (B_\Gamma, \Gamma)^{-1}.$$

Using the definition of u^n in (3.6) and its convergence in Proposition 4.1, one obtains that in (4.2), the right-hand side is larger than the left-hand side.

The above optimal control problem satisfies the dynamic programming principle (see e.g. [11]): for a family of stopping times $(\tau^\gamma)_{\gamma \in \mathcal{U}_t}$ dominated by s , one has

$$u(s, t, x) = \sup_{\gamma \in \mathcal{U}_t} \mathbb{E} \left[u(s - \tau^\gamma, t - Y_{\tau^\gamma}^\gamma, X_{\tau^\gamma}^\gamma) + \int_0^{\tau^\gamma} \partial_s U(s - k, X_k^\gamma) \mathbf{1}_{\{Y_k^\gamma < t\}} dk \right]. \quad (4.3)$$

Step 3 (supersolution). Let $z = (s, t, x) \in \text{int}(\mathbf{Z})$ be fixed, and $\varphi \in C^2(\mathbf{Z})$ be such that $0 = (u - \varphi)(z) = \min_{z' \in \mathbf{Z}} (u - \varphi)$. By a slight abuse of notation, denote by $\partial_s U(z')$ the quantity $\partial_s U(s', t', x')$ for any $z' = (s', t', x')$. Then by (4.3), for any family of stopping times $(\tau^\gamma)_{\gamma \in \mathcal{U}_t}$ dominated by s , one has,

$$\sup_{\gamma \in \mathcal{U}_t} \mathbb{E} \left[\int_0^{\tau^\gamma} (-\partial_s \varphi(Z_k) + \partial_s U(Z_k) \mathbf{1}_{\{Y_k^\gamma < t\}}) dk + \int_0^{\tau^\gamma} \gamma_k^2 \left(-\partial_t \varphi + \frac{1}{2} \partial_{xx}^2 \varphi \right) (Z_k) dk \right] \leq 0,$$

where $Z_k := (s - k, t - Y_k^\gamma, X_k^\gamma)$ and $X_0^\gamma = x$. Choosing $\gamma \equiv 0$ and $\tau^\gamma \equiv h$ for h small enough, we get

$$(-\partial_s \varphi + \partial_s U)(z) \leq 0.$$

On the other hand, choosing $\gamma \equiv \gamma_0$ for some constant γ_0 and $\tau^\gamma := \inf\{k \geq 0 : |X_k^\gamma - x| + |Y_k^\gamma| \geq h\}$, then by letting γ_0 be large enough and h be small enough, one can deduce that

$$\left(-\partial_t \varphi + \frac{1}{2} \partial_{xx}^2 \varphi \right) (s, t, x) \leq 0.$$

Step 4 (subsolution). Assume that u is not a viscosity sub-solution, then there exists $z = (s, t, x) \in \text{int}(\mathbf{Z})$ and $\varphi \in C^2(\mathbf{Z})$, such that $0 = (u - \varphi)(z) = \max_{z' \in \mathbf{Z}} (u - \varphi)(z')$, and

$$\min \left\{ \partial_t \varphi - \frac{1}{2} \partial_{xx}^2 \varphi, \partial_s(\varphi - U) \right\} (s, t, x) > 0.$$

By continuity of u and φ , we may find $R > 0$ such that

$$\min \left\{ \partial_t \varphi - \frac{1}{2} \partial_{xx}^2 \varphi, \partial_s(\varphi - U) \right\} \geq 0, \quad \text{on } B_R(z), \quad (4.4)$$

where $B_R(z)$ is the open ball with radius R and center z . Let $\tau^\gamma := \inf\{k : Z_k^\gamma \notin B_R(z) \text{ or } Y_k^\gamma \geq t\}$, and notice that $\max_{\partial B_R(s, t, x)} (u - \varphi) = -\eta < 0$, by the strict maximality property. Then it follows from (4.3) that

$$\begin{aligned} 0 &= \sup_{\gamma} \mathbb{E} \left[u(s - \tau^\gamma, t - Y_{\tau^\gamma}^\gamma, X_{\tau^\gamma}^\gamma) - u(s, t, x) + \int_0^{\tau^\gamma} \partial_s U(s - k, X_k^\gamma) \mathbf{1}_{\{Y_k^\gamma < t\}} dk \right] \\ &\leq -\eta + \sup_{\gamma} \mathbb{E} \left[\int_0^{\tau^\gamma} \left(-\partial_s(\varphi - U) \mathbf{1}_{\{Y_k^\gamma < t\}} - \left(\partial_t \varphi - \frac{1}{2} \partial_{xx}^2 \varphi \right) \gamma_k^2 \right) (Z_k) dk \right] \leq -\eta, \end{aligned}$$

where the last inequality follows by (4.4). This is the required contradiction. \square

4.3 The comparison principle of the PDE (Theorems 2.(iii) and 1.(ii))

Recall that the operator F is defined in (2.4) and we will study the PDE (2.3). For any $\eta \geq 0$, a lower semicontinuous function $w : \mathbf{Z} \rightarrow \mathbb{R}$ is called an η -strict viscosity supersolution of (2.3) if $w|_{s=0} \geq \eta + U(0, \cdot)$, $w|_{t=0} \geq \eta + U$ and $F(D\varphi, D^2\varphi)(z_0) \geq \eta$ for all $(z_0, \varphi) \in \text{int}(\mathbf{Z}) \times C^2(\mathbf{Z})$ satisfying $(w - \varphi)(z_0) = \min_{z \in \mathbf{Z}}(w - \varphi)(z)$.

Proposition 4.5 (Comparison). *Let v (resp. w) be an upper (resp. lower) semicontinuous viscosity subsolution (resp. supersolution) of the equation (2.3) satisfying*

$$v(z) \leq C(1 + t + |x|) \quad \text{and} \quad w(z) \geq -C(1 + t + |x|), \quad z \in \mathbf{Z}, \quad \text{for some constant } C > 0.$$

Then $v \leq w$ on \mathbf{Z} .

Proof. We proceed in three steps.

(i) In this step, we prove the result under the assumption that the comparison result holds true if the supersolution is η -strict for some $\eta > 0$. First, direct verification reveals that the function:

$$w^1(s, t, x) := U(s, x) + \eta(1 + s + t), \quad (s, t, x) \in \mathbf{Z},$$

is an η -strict supersolution. For all $\mu \in (0, 1)$, we claim that the function $w^\mu := (1 - \mu)w + \mu w^1$ is a $\mu\eta$ -strict viscosity supersolution. Indeed, this follows from the proof of Lemma A.3 (p.52) of Barles and Jakobsen [2], which shows that w^μ is a viscosity supersolution of both linear equations:

$$\partial_t w^\mu - \frac{1}{2} \partial_{xx}^2 w^\mu \geq \mu\eta \quad \text{and} \quad \partial_s w^\mu - \partial_s U \geq \mu\eta.$$

Assume that the comparison principle holds true if the supersolution is strict, then it follows that $v \leq w^\mu$ on \mathbf{Z} . Let $\mu \searrow 0$, we obtain $v \leq w$ on \mathbf{Z} .

(ii) In view of the previous step, we may assume without loss of generality that w is an η -strict supersolution. In order to prove the comparison result in this setting, we assume to the contrary that

$$\delta := (v - w)(\hat{z}) > 0, \quad \text{for some } \hat{z} \in \mathbf{Z}, \tag{4.5}$$

and we work toward a contradiction. Following the standard doubling variables technique, we introduce for arbitrary $\alpha, \varepsilon > 0$:

$$\Phi^{\alpha, \varepsilon}(z, z') := \frac{\alpha}{2} |z - z'|^2 + \varepsilon(\varphi(z) + \varphi(z')), \quad \text{with } \varphi(z) := \ln(1 - s) + \frac{1}{2}(t^2 + x^2), \quad z, z' \in \mathbf{Z},$$

and the corresponding maximum

$$M^{\alpha, \varepsilon} := \sup_{(z, z') \in \mathbf{Z} \times \mathbf{Z}} \{v(z) - w(z') - \Phi^{\alpha, \varepsilon}(z, z')\} \geq \delta - 2\varepsilon\varphi(\hat{z}) > 0, \tag{4.6}$$

by (4.5), for sufficiently small $\varepsilon > 0$. Also, recalling that both potential functions U and $U_{\mathbf{N}}(0, 1)$ have linear growth in x , it follows from the bounds on v and w that the above

supremum may be confined to a compact subset of $\mathbf{Z} \times \mathbf{Z}$. Then the upper semicontinuity of the objective function implies the existence of a minimizer $(z^{\alpha,\varepsilon}, z'^{\alpha,\varepsilon}) \in \mathbf{Z} \times \mathbf{Z}$, i.e.

$$M^{\alpha,\varepsilon} = v(z^{\alpha,\varepsilon}) - w(z'^{\alpha,\varepsilon}) - \frac{\alpha}{2} |z^{\alpha,\varepsilon} - z'^{\alpha,\varepsilon}|^2 - \varepsilon(\varphi(z^{\alpha,\varepsilon}) + \varphi(z'^{\alpha,\varepsilon})),$$

and there exists a converging subsequence $(z_n^\varepsilon, z_n'^\varepsilon) := (z_{\alpha_n}^\varepsilon, z'_{\alpha_n}^\varepsilon) \longrightarrow (z^\varepsilon, z'^\varepsilon) \in \mathbf{Z} \times \mathbf{Z}$, for some $(\alpha_n)_n$ converging to ∞ . Moreover, denoting by z^* any minimizer of $v - w - 2\varepsilon\varphi$, we obtain from the inequality $(v - w - 2\varepsilon\varphi)(z^*) \leq M^{\alpha_n,\varepsilon}$ that

$$\begin{aligned} \ell := \limsup_{n \rightarrow \infty} \frac{\alpha}{2} |z_n^\varepsilon - z_n'^\varepsilon|^2 &\leq \limsup_{n \rightarrow \infty} v(z_n^\varepsilon) - w(z_n'^\varepsilon) - \varepsilon(\varphi(z_n^\varepsilon) - \varphi(z_n'^\varepsilon)) - (v - w - 2\varepsilon\varphi)(z^*) \\ &\leq v(z^\varepsilon) - w(z'^\varepsilon) - \varepsilon(\varphi(z^\varepsilon) - \varphi(z'^\varepsilon)) - (v - w - 2\varepsilon\varphi)(z^*) < \infty. \end{aligned}$$

Then $z^\varepsilon = z'^\varepsilon$, and $0 \leq \ell \leq (v - w - 2\varepsilon\varphi)(z^\varepsilon) - (v - w - 2\varepsilon\varphi)(z^*) \leq 0$ by the definition of z^* . Consequently:

$$z^\varepsilon = z'^\varepsilon, \quad \alpha_n |z_n^\varepsilon - z_n'^\varepsilon|^2 \longrightarrow 0, \quad \text{and} \quad M_{\alpha_n} \longrightarrow \sup_{\mathbf{Z}} (u - v) - 2\varepsilon\varphi, \quad \text{as } n \rightarrow \infty. \quad (4.7)$$

Finally, our definition of φ implies that $s^\varepsilon < 1$. Moreover, as v is a subsolution and w a supersolution, we see that if \hat{z} lies in the remaining part of $\partial\mathbf{Z}$, we would have $\limsup_{n \rightarrow \infty} M_{\alpha_n} \leq -2\varepsilon\varphi(z^\varepsilon) \leq 0$, which is in contradiction with the positive lower bound in (4.6). Consequently z^ε is an interior point of \mathbf{Z} , and therefore both z_n^ε and $z_n'^\varepsilon$ are interior points of \mathbf{Z} for sufficiently large n .

(iii) We now use the viscosity properties of v and w at the interior points z_n^ε and $z_n'^\varepsilon$, for large n . By the Crandall-Ishii Lemma, see e.g. Crandall, Ishii, and Lions [10], we may find for each such n two pairs $(p_n^\varepsilon, A_n^\varepsilon)$ and $(q_n^\varepsilon, B_n^\varepsilon)$ in $\mathbb{R}^3 \times \mathbb{S}_3$, such that

$$\begin{aligned} (p_n^\varepsilon + D\Phi^{\alpha,\varepsilon}(z_n^\varepsilon), A_n^\varepsilon + D^2\Phi^{\alpha,\varepsilon}(z_n^\varepsilon)) &\in \bar{J}w(z_n^\varepsilon), \\ (q_n^\varepsilon - D\Phi^{\alpha,\varepsilon}(z_n'^\varepsilon), B_n^\varepsilon - D^2\Phi^{\alpha,\varepsilon}(z_n'^\varepsilon)) &\in \underline{J}v(z_n'^\varepsilon), \\ p_n^\varepsilon = q_n^\varepsilon = \alpha_n(z_n^\varepsilon - z_n'^\varepsilon) \quad \text{and} \quad A_n^\varepsilon &\leq B_n^\varepsilon, \end{aligned}$$

where \bar{J} and \underline{J} denote the second order super and subjets, see [10]. Then, it follows from the subsolution property of v and the η -strict supersolution of w that

$$\begin{aligned} &\min \left\{ \alpha_n(t_n^\varepsilon - t_n'^\varepsilon) + \varepsilon t_n^\varepsilon - \frac{1}{2}(A_{3,3,n}^\varepsilon + \varepsilon), \alpha_n(s_n^\varepsilon - s_n'^\varepsilon) + \frac{\varepsilon}{1 - s_n^\varepsilon} - \partial_s U(s_n^\varepsilon, x_n^\varepsilon) \right\} \\ &\leq 0 \leq -\eta + \min \left\{ \alpha_n(t_n^\varepsilon - t_n'^\varepsilon) - \varepsilon t_n'^\varepsilon - \frac{1}{2}(B_{3,3,n}^\varepsilon - \varepsilon), \alpha_n(s_n^\varepsilon - s_n'^\varepsilon) - \frac{\varepsilon}{1 - s_n'^\varepsilon} - \partial_s U(s_n'^\varepsilon, x_n'^\varepsilon) \right\} \\ &\leq -\eta + \min \left\{ \alpha_n(t_n^\varepsilon - t_n'^\varepsilon) - \varepsilon t_n'^\varepsilon - \frac{1}{2}(A_{3,3,n}^\varepsilon - \varepsilon), \alpha_n(s_n^\varepsilon - s_n'^\varepsilon) - \frac{\varepsilon}{1 - s_n'^\varepsilon} - \partial_s U(s_n'^\varepsilon, x_n'^\varepsilon) \right\}, \end{aligned}$$

by the inequality $A_n^\varepsilon \leq B_n^\varepsilon$. This implies that

$$\begin{aligned} 0 &\leq -\eta - \varepsilon(t_n^\varepsilon + t_n'^\varepsilon) + 2\varepsilon - \frac{\varepsilon}{1 - s_n^\varepsilon} - \frac{\varepsilon}{1 - s_n'^\varepsilon} + |\partial_s U(s_n'^\varepsilon, x_n'^\varepsilon) - \partial_s U(s_n^\varepsilon, x_n^\varepsilon)| \\ &\leq -\eta + 2\varepsilon + |\partial_s U(s_n^\varepsilon, x_n^\varepsilon) - \partial_s U(s_n'^\varepsilon, x_n'^\varepsilon)| \longrightarrow -\eta + 2\varepsilon, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which provides the required contradiction for sufficiently small $\varepsilon > 0$. \square

Remark 4.6. *To conclude the proofs of Theorem 2 (iii) as well as Theorem 1 (ii), we notice that the comparison result in Proposition 4.5 induces immediately the uniqueness of the PDE (2.3) in part (iii) of Theorem 2. Further, this implies also the uniqueness of the potential functions of $B_{\tau_s^{\alpha^*} \wedge t}$ for all $s \in [0, 1]$ and $t \geq 0$, and hence the uniqueness of law of $B_{\tau_s^{\alpha^*} \wedge t}$ in part (ii) of Theorem 1.*

5 More discussions

Recall that in the case with finitely many marginals $(\mu_{s_j^n})_{j=1, \dots, n}$, the Root stopping times $\{\sigma_j^n\}_{j=1..n}$ are defined successively as hitting times of barriers, that is

$$\sigma_j^n := \inf \{t \geq \sigma_{j-1}^n : (t, B_t) \in \mathcal{R}_j^n\},$$

with barriers \mathcal{R}_j^n defined by

$$\mathcal{R}_j^n = \{(t, x) : \delta^n u_n(s_j^n, t, x) = \delta^n U(s_j^n, x)\}.$$

In consequence, one has for any $n \geq 1$,

$$\delta^n u_n(s_j^n, \sigma_j^n, B_{\sigma_j^n}) = \delta^n U(s_j^n, B_{\sigma_j^n}), \quad \forall j = 1, \dots, n.$$

Denote for simplicity a Root solution of the SEP given full marginals by $(\sigma_s^\infty)_{s \in [0, 1]}$. Assume that the partial derivative $\partial_s u(s, t, x)$ exists and is continuous, then one may naturally expect to have

$$\int_0^t \partial_s u(s, \sigma_s^\infty, B_{\sigma_s^\infty}) ds = \int_0^t \partial_s U(s, B_{\sigma_s^\infty}) ds, \quad \text{for all } t \in [0, 1].$$

Nevertheless, it is not easy to formulate a sufficient condition on U to ensure that $\partial_s u(s, t, x)$ is well-defined, as u is only the value function of an optimal stopping problem.

We could also expect to define the limit Root solution σ_s^∞ as a hitting time such that

$$\sigma_s^\infty = \inf \{t \geq \sigma_{s-} : (t, W_t) \in \mathcal{R}_s\}, \quad (5.1)$$

for barriers $\mathcal{R} = \{\mathcal{R}_s\}_{s \in [0, 1]}$ defined by

$$\mathcal{R}_s := \{(t, x) : \partial_s u(s, t, x) = \partial_s U(s, x)\}.$$

But again here, the definition of the partial derivative $\partial_s u(s, t, x)$ is not clear. Moreover, as the number of marginals is not countable in the full marginals case, the equation (5.1) cannot provide a definition for an uncountable family of stopping times.

Finally, an optimal solution to the dual problem of the optimal SEP has been provided in [9]. It is also interesting to look at the limit of the dual solutions. Nevertheless, as the dual solution are only defined in an inductive way using the barriers $(\mathcal{R}_k^n)_{k=1, \dots, n}$, it is not clear how to figure out the limit barriers and the limit dual solutions.

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