



**HAL**  
open science

# Weyl's law for singular Riemannian manifolds

Yacine Chitour, Dario Prandi, Luca Rizzi

► **To cite this version:**

Yacine Chitour, Dario Prandi, Luca Rizzi. Weyl's law for singular Riemannian manifolds. 2018. hal-01902740v1

**HAL Id: hal-01902740**

**<https://hal.science/hal-01902740v1>**

Preprint submitted on 23 Oct 2018 (v1), last revised 18 Aug 2023 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Weyl's law for singular Riemannian manifolds

Y. Chitour\*, D. Prandi†, L. Rizzi‡

October 23, 2018

## Abstract

We study the asymptotic growth of the eigenvalues of the Laplace-Beltrami operator on singular Riemannian manifolds, where all geometrical invariants appearing in classical spectral asymptotics are unbounded, and the total volume can be infinite. Under suitable assumptions, we prove that the leading term of the Weyl's asymptotics contains information on the singularity, i.e. its Minkowski dimension and its regularized measure. We apply our results to a particular class of almost-Riemannian structures. A key tool in the proof is a universal estimate for the remainder of the heat trace on Riemannian manifolds, which is of independent interest.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>7</b>
<b>3</b>	<b>Heat kernel estimates with remainder</b>	<b>10</b>
<b>4</b>	<b>Weyl's asymptotics for singular manifolds</b>	<b>18</b>
<b>5</b>	<b>Strongly regular ARS</b>	<b>26</b>
<b>A</b>	<b>Auxiliary geometric estimates</b>	<b>31</b>
<b>B</b>	<b>Hardy inequality and discreteness of the spectrum</b>	<b>34</b>

## 1 Introduction

In [35], Hermann Weyl studied the distribution of eigenvalues for the Laplace operator on bounded domains of  $\mathbb{R}^n$ , and proved the following asymptotic formula:

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \text{vol}(\Omega) \lambda^{n/2} \quad \text{as } \lambda \rightarrow +\infty. \quad (1)$$

---

\*Laboratoire des Signaux et Systèmes, Université Paris-Sud, CNRS, CentraleSupélec, Gif-sur-Yvette

†CNRS, Laboratoire des Signaux et Systèmes, CentraleSupélec, Gif-sur-Yvette

‡Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France

Here,  $N(\lambda)$  is the number of eigenvalues smaller than  $\lambda$  for the Dirichlet Laplacian on a bounded domain  $\Omega \subset \mathbb{R}^n$  (also called the eigenvalue counting or Weyl's function), and  $\text{vol}(\Omega)$  is its Lebesgue measure. The classical proof of this result employs the variational method known as Dirichlet-Neumann bracketing, and the explicit estimate with remainder of the eigenvalue counting function on cubes (see e.g. [13]).

Weyl's asymptotic law (1) has been proved to hold for the Laplace-Beltrami operator on compact Riemannian manifolds, where the Lebesgue measure is replaced by the Riemannian one. In this case (1) is a consequence of the celebrated Minakshisundaram-Pleijel heat kernel asymptotics and the Karamata Tauberian theorem, see e.g. [7] and [2, 37] for recent generalizations to the case of metric measure spaces satisfying synthetic curvature conditions.

It is interesting to notice that the finiteness of the volume of  $M$  is not a necessary condition for the discreteness of the spectrum of the Laplace-Beltrami operator. Moreover, if such volume is infinite, the r.h.s. of (1) is infinite as well and one might wonder what the leading order of  $N(\lambda)$  is, as  $\lambda \rightarrow +\infty$ , and which spectral invariant it encodes.

A well understood example is that of quasi-bounded domains of  $\mathbb{R}^2$ , as the horn-shaped domains  $\{(x, y) \mid |x|^\alpha |y| \leq 1\}$ . It turns out that the parameter  $\alpha$  is a spectral invariant encoding the leading order of  $N(\lambda)$ , which is of the form  $\lambda \log \lambda$  for  $\alpha = 1$ , and  $\lambda^{(1+\alpha)/2}$  otherwise, see [31, 34] and references therein for details. In these cases the difficulty in the study of the Weyl function arises from the unboundedness of the domain, since the geometry of the problem is trivial.

## 1.1 The Grushin sphere model

In this paper we are concerned with non-complete Riemannian structures where the metric is singular at the boundary of the metric completion of the manifold, and in particular all geometric invariants, including the total measure, can be unbounded. We first discuss a simple example representing the class of singularities under investigation, which is an almost-Riemannian structure in the sense of [1, 6].

Consider the two dimensional sphere  $\mathbb{S}^2 \subset \mathbb{R}^3$ . Let  $X$  and  $Y$  be the standard generators of rotations around the  $x$  and  $y$  axis, respectively. These vector fields are collinear on the equator  $\mathcal{Z} = \{(x, y, z) \in \mathbb{S}^2 \mid z = 0\}$ , and linearly independent elsewhere. By declaring  $X, Y$  to be orthonormal, we define a Riemannian structure on the sphere, which is singular on  $\mathcal{Z}$  (the coefficients of the metric explode). Nevertheless, the associated Laplace-Beltrami operator, with domain  $C_c^\infty(N \setminus \mathcal{Z})$  is essentially self-adjoint on  $L^2(N \setminus \mathcal{Z}, d\mu_g)$  and has discrete spectrum [6]. Due to the high symmetry of the problem the spectrum can be explicitly computed, and in [5] the following non-classical asymptotics has been obtained:

$$N(\lambda) \sim \frac{1}{4} \lambda \log \lambda, \quad \lambda \rightarrow +\infty. \quad (2)$$

It is easy to check that the total Riemannian volume of  $N \setminus \mathcal{Z}$  is infinite, and the curvature is unbounded<sup>1</sup>. This is a general feature of ARS, and more particularly

---

<sup>1</sup>More precisely, it explodes to  $-\infty$  at the equator, and this is the case for all ARS in dimension  $n = 2$ . However, let us point out that for  $n > 2$  one can build similar structures where the curvature is unbounded both above and below, see Example 5.1.

of singular Riemannian structures under investigation. In addition, all heat kernel and heat trace estimates are bound to blow up close to the singular region, and it is not clear how to apply Tauberian techniques to deduce the asymptotic behaviour of  $N(\lambda)$ .

## 1.2 Main results

We formalize next our class of singular structures. Let  $\mathcal{M}$  and  $\widehat{\mathcal{M}}$  be a non-complete Riemannian manifold of dimension  $n$  and its metric completion, respectively. The *singular set* is the (possibly empty) subset of the metric boundary  $\partial\mathcal{M} = \widehat{\mathcal{M}} \setminus \mathcal{M}$  approaching whom geometric quantities such as the curvature, the measure of small balls, et cetera, blow up. We also consider the Friedrichs extension  $\Delta$  of the Laplace-Beltrami operator with domain  $C_c^\infty(\mathcal{M})$ , that is the unique self-adjoint operator in  $L^2(\mathcal{M}, d\mu_g)$  associated with the quadratic form

$$Q(u) = \int_{\mathcal{M}} |\nabla u|^2 d\mu_g, \quad \forall u \in C_c^\infty(\mathcal{M}). \quad (3)$$

It is crucial to identify several geometrical quantities that control the spectral properties of the problem as the distance from the metric boundary  $\hat{d}_\partial \rightarrow 0$ . To this purpose, let  $\mathcal{M}_a^b$ , for  $0 \leq a < b \leq +\infty$ , denotes the level sets  $\mathcal{M}_a^b = \{a \leq \hat{d}_\partial \leq b\} \cap \mathcal{M}$ . We consider the following regularity assumption of the metric boundary.

**Definition 1.1.** We say that  $\mathcal{M}$  has *regular metric boundary* if there exists  $\varepsilon_0 > 0$  such that  $\hat{d}_\partial$  is smooth on  $\mathcal{M}_0^{\varepsilon_0}$ .

Our main result reads as follows.

**Theorem 1.2.** *Let  $\mathcal{M}$  be a non-complete Riemannian manifold, with compact metric completion and regular metric boundary. Assume that there exist  $C, \varepsilon_0 > 0$  such that, for all  $\varepsilon \leq \varepsilon_0$ , we have*

$$\text{Ric}(\mathcal{M}_\varepsilon^\infty) \geq -\frac{C}{\varepsilon^2}, \quad \text{Sec}(\mathcal{M}_\varepsilon^\infty) \leq \frac{C}{\varepsilon^2}. \quad (4)$$

*Assume also that for all  $\varepsilon \leq \varepsilon_0$ , the boundary of  $\mathcal{M}_\varepsilon^\infty$  is convex, and that*

$$m(\partial\mathcal{M}_\varepsilon^\infty) \geq -\frac{C}{\varepsilon}, \quad \text{inj}_{\mathcal{M}_\varepsilon^\infty} \geq \frac{\varepsilon}{C}. \quad (5)$$

*Then there exist  $C_\pm > 0$  and  $\Lambda > 0$  such that*

$$C_- \leq \frac{N(\lambda)}{\lambda^{n/2} \text{vol}(\mathcal{M}_{1/\sqrt{\lambda}}^\infty)} \leq C_+, \quad \forall \lambda \geq \Lambda. \quad (6)$$

Several remarks are in order. Firstly, almost all quantitative assumptions in Theorem 1.2 are concerned with the local geometry of  $\mathcal{M}$  close to the metric boundary. The only non-local assumption is that on the injectivity radius, which might be difficult to check in concrete examples. However this assumption is automatically verified provided that the boundaries are *strictly* convex, see Remark 4.1.

Secondly, let us stress the importance of the convexity assumption. It is used to prove the compactness of the resolvent of  $\Delta$  and, in turn, the discreteness of its spectrum. Furthermore it plays a fundamental role in several comparison estimates of the remainder of the Weyl asymptotics for the truncations.

Lastly, in this general setting we are not able to deduce the existence of a limit in (6) as  $\lambda \rightarrow +\infty$ , but only the leading order. Under more precise assumptions on the Minkowski dimension of the singularity we are able to improve this result. For the definition of Minkowski dimension, see Section 4.2.

**Corollary 1.3.** *Let  $\mathcal{M}$  be an  $n$ -dimensional non-complete Riemannian manifold satisfying the assumptions of Theorem 1.2. Assume that  $\partial\mathcal{M}$  has Minkowski dimension  $d > 0$ . Then*

$$N(\lambda) \asymp \begin{cases} \lambda^{n/2} & d < n, \\ \lambda^{n/2} \log \lambda & d = n, \\ \lambda^{d/2} & d > n, \end{cases} \quad \lambda \rightarrow +\infty, \quad (7)$$

where the notation  $f \asymp g$  means that  $f/g$  has finite and positive  $\limsup$  and  $\liminf$ .

In particular, the Minkowski dimension of the singularity is a spectral invariant, which determines the leading term of the asymptotics of  $N(\lambda)$ . Notice that the asymptotics is non-classical, as soon as  $d \geq n$ . Finally, in the case  $d = n$  we are able to prove a more precise result.

**Theorem 1.4.** *Let  $\mathcal{M}$  be a non-complete Riemannian manifold satisfying the assumptions of Theorem 1.2. Assume that  $\partial\mathcal{M}$  has Minkowski dimension  $d = n$  with regularized volume  $|\partial\mathcal{M}|_{reg}$ . Then,*

$$\lim_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \log \lambda} = \frac{\omega_n}{(2\pi)^n} \frac{|\partial\mathcal{M}|_{reg}}{2}. \quad (8)$$

The assumptions of Theorem 1.4 are verified for generic 2-dimensional ARS without tangency points, and more precisely, for strongly regular  $n$ -dimensional ARS which are defined in Section 5. This class includes the Grushin example discussed at the beginning of this introduction. For a more precise statement concerning ARS, see Theorem 5.3.

### 1.3 Structure of the proof

We give here a sketch of the proof of Theorem 1.2, which is carried out in detail in Section 4, and present a key technical tool occurring in the proof. The argument relies on a combination of variational and Tauberian techniques. In its simplest instance, we split  $\mathcal{M} = \mathcal{M}_0^\varepsilon \cup \mathcal{M}_\varepsilon^\infty$ . By Dirichlet-Neumann bracketing, we bound  $N(\lambda)$  with the sum of the counting functions for the Laplace-Beltrami operator on the two domains, with Neumann (+) or Dirichlet (-) boundary conditions, respectively:

$$N_{[0,\varepsilon]}^-(\lambda) + N_{[\varepsilon,\infty]}^-(\lambda) \leq N(\lambda) \leq N_{[0,\varepsilon]}^+(\lambda) + N_{[\varepsilon,\infty]}^+(\lambda). \quad (9)$$

Formula (9), as  $\varepsilon \rightarrow 0$ , should determine the asymptotics of  $N(\lambda)$  as  $\lambda \rightarrow +\infty$ .

The implementation of this simple strategy is quite delicate. Thanks to the convexity assumption,  $\mathcal{M}_0^\varepsilon$  supports a Hardy-type inequality. As a consequence,  $N_{[0,\varepsilon(\lambda)]}^\pm(\lambda) = 0$ , provided that  $\varepsilon \rightarrow 0$  sufficiently fast (in a quantitative way) as  $\lambda \rightarrow +\infty$ . In this regime the asymptotics of  $N(\lambda)$  is controlled by the Weyl function of the truncation  $\mathcal{M}_{\varepsilon(\lambda)}^\infty$ . The latter is a Riemannian manifold with boundary and finite measure, which satisfies indeed the classical Weyl law given by

$$N_{[\varepsilon(\lambda),\infty]}(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \text{vol}(\mathcal{M}_{\varepsilon(\lambda)}^\infty) \lambda^{n/2}. \quad (10)$$

The remainder in (10), which depends on the parameter of the truncation  $\varepsilon(\lambda)$ , must be carefully controlled as  $\lambda \rightarrow +\infty$ . The key tool here is the following heat-trace asymptotic formula with universal remainder<sup>2</sup>, proved in Section 3.

**Theorem 1.5** (Heat trace asymptotics with universal remainder). *Let  $(M, g)$  be a smooth compact  $n$ -dimensional Riemannian manifold with convex boundary  $\partial M$ . Let  $K \geq 0$  such that  $\text{Sec} \leq K$ ,  $\text{Ric} \geq -(n-1)K$  and let  $H \geq 0$  such that*

$$-H \leq \text{Hess}(d_\partial) \leq 0, \quad \text{for } d_\partial < \text{inj}_{\partial M}. \quad (11)$$

Moreover, set

$$\mathbf{i} = \min \left\{ \text{inj}_M, \frac{\text{inj}_{\partial M}}{2} \right\}. \quad (12)$$

Then, there exist positive constants  $c_1, c_2, c_3, c_4 > 0$ , depending only on  $n$ , such that the following estimate for the Dirichlet and Neumann heat kernels holds:

$$\left| \frac{(4\pi t)^{n/2}}{\text{vol}(M)} \int_M p^\pm(t, x, x) d\mu_g(x) - 1 \right| \leq c_1 K t + c_2 \frac{\text{vol}(\partial M)}{\text{vol}(M)} \sqrt{t} + c_3 e^{-c_4 \frac{\mathbf{i}^2}{4t}}, \quad (13)$$

for all values of  $t \in \mathbb{R}_+$  such that

$$\sqrt{t} \leq \sqrt{t_0} = \min \left\{ \text{inj}_M, \frac{\text{inj}_{\partial M}}{2}, \frac{\pi}{\sqrt{K}}, \frac{1}{H} \right\}. \quad (14)$$

As a consequence of Theorem 1.5, and a suitable Karamata-type theorem with remainder (due to Ingham [19]), we obtain an asymptotic formula with universal remainder for the eigenvalue counting function of the Laplace-Beltrami operator on a compact Riemannian manifold with boundary as  $\lambda \rightarrow +\infty$ , which is given in Theorem 3.5.

When applied to the truncations  $M = \mathcal{M}_{\varepsilon(\lambda)}^\infty$ , this result singles out the quantities whose explosion must be controlled as  $\lambda \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ , concluding the proof of Theorem 1.2.

## 1.4 Comparison with other singular structures

**Conical singularity.** There is a sharp difference between our class of singular structures and conical singularities [8]. In this case, our techniques do not apply since

---

<sup>2</sup>By “universal”, we mean that the remainder only depends on a handful of geometrical invariant quantities, and fixed dimensional constants.

the boundaries of the truncations  $\mathcal{M}_\varepsilon^\infty$  are concave (hence non-convex) as  $\varepsilon \rightarrow 0$ . However, the spectrum of the Laplace-Beltrami is still discrete, the total volume is finite, and the classical Weyl law (1) holds. In this sense, conical singularities are more gentle, and do not modify the leading order of the Weyl function as in Theorem 1.2 and Corollary 1.3. Indeed, for the conical case, the presence of the singularity is detected only at much higher order, see [32, 33].

**Structures with locally bounded geometry** In [23] the author considers non-complete Riemannian structures  $(\mathcal{M}, g)$ , equipped with a weighted measure  $\sigma^2 d\mu_g$ , and the unique self-adjoint operator in  $L^2(\mathcal{M}, \sigma^2 d\mu_g)$ , associated with the Friedrichs extension of the quadratic form

$$Q(u) = \int_{\mathcal{M}} |\nabla u|^2 \sigma^2 d\mu_g, \quad u \in C_c^\infty(\mathcal{M}). \quad (15)$$

The Riemannian measure  $d\mu_g$  and the weight  $\sigma$  might be singular at the metric boundary, and no regularity of the metric boundary is assumed. The author derives in this setting Weyl's asymptotics similar to the ones of Theorem 1.2 and Corollary 1.3. Despite the similarities, the setting and methods of [23] are rather different with respect to ours. The assumptions in [23] imply that  $\mathcal{M}$  is locally uniformly bi-Lipschitz equivalent to an Euclidean ball. If the metric completion is compact, this implies that the Riemannian measure of  $\mathcal{M}$  is finite. In particular [23] cannot be applied to the simplest model of our class of singularities, that is the Grushin sphere.

**ARS with smooth measures.** An analogue to Theorem 1.4 for 2-dimensional ARS was announced in [11], as a consequence of a more general local Weyl law for sub-Laplacians [12, 10]. There, the authors are concerned with the Friedrichs extension associated with the quadratic form

$$Q(u) = \int_N |\nabla u|^2 d\omega, \quad u \in C^\infty(N), \quad (16)$$

where  $N$  is a smooth compact manifold carrying a smooth almost-Riemannian structure (the reader not familiar with AR geometry can think at the example of the Grushin sphere discussed above, where  $N = \mathbb{S}^2$ ). The norm of the gradient is well defined and smooth everywhere, including at the singular region  $\mathcal{Z} \subset N$  and the measure  $\omega$  is positive and smooth on  $N$ . Even if the Weyl's asymptotics is known to be invariant when replacing  $\omega$  by  $e^\phi \omega$ , for  $\phi \in C^\infty(N)$ , it is not clear how the spectral asymptotics are related when  $\phi$  explodes at  $\mathcal{Z}$ . It is actually surprising that, for generic 2-ARS, we obtain the same Weyl law in our setting, where  $d\omega = d\mu_g$  is singular on  $\mathcal{Z}$ , and the domain of the form (16) is  $C_c^\infty(N \setminus \mathcal{Z})$ .

**Magnetic bottles.** It would be interesting to extend our results to the magnetic Laplacian, that is the self-adjoint operator  $-\Delta_A$  associated with the quadratic form

$$Q_A(u) = \int_{\mathcal{M}} |du - iuA|^2 d\mu_g, \quad u \in C_c^\infty(\mathcal{M}), \quad (17)$$

where  $\mathcal{M}$  is a Riemannian manifold,  $A$  is a one-form representing the magnetic potential, and  $|\cdot|$  here is the dual Riemannian norm on the complexified cotangent bundle. The two-form  $B = dA$  is the magnetic field. When  $-\Delta_A$  has compact resolvent and is essentially self-adjoint on  $C_c^\infty(\mathcal{M})$ , one talks about *magnetic bottles*. The Weyl law for magnetic bottles on  $\mathbb{R}^n$  has been studied in [9], for the Poincaré half-plane in [25], and more generally for geometrically finite hyperbolic surfaces in [26]. In all these cases, the results are proved through the variational method and localization on suitable small cubes. To our knowledge, the problem on manifolds with non-constant and possibly exploding curvature has not been yet addressed.

## Acknowledgments

This work was supported by the Grant ANR-15-CE40-0018 of the ANR, and by a public grant as part of the Investissement d’avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH, in a joint call with the “FMJH Program Gaspard Monge in optimization and operation research”. This work has been supported by the ANR project ANR-15-IDEX-02 and by a public grant overseen by the French National Research Agency (ANR) as part of the Investissement d’Avenir program, through the iCODE project funded by the IDEX Paris-Saclay, ANR-11-IDEX-0003-02.

We wish to thank R. Neel, S. Gallot and Y. Colin de Verdière for stimulating discussions.

## 2 Preliminaries

In this section we collect some basic definitions which will be necessary in the rest of the paper. In particular, the notions presented in Section 2.1 will be used for the heat kernel and trace estimates of Section 3, while those presented in Section 2.2 will be used in Sections 4 and 5.

### 2.1 Riemannian manifolds with boundary

Let  $(M, g)$  be a smooth  $n$ -dimensional Riemannian manifold, with smooth boundary  $\partial M$ . We let  $d_\partial : M \rightarrow [0, +\infty)$  be the Riemannian distance from  $\partial M$ , that is,

$$d_\partial(x) = \inf_{z \in \partial M} d(x, z). \quad (18)$$

A geodesic  $\gamma : [0, t] \rightarrow M$ ,  $\gamma(0) \in \partial M$  is optimal w.r.t. the boundary if for all  $0 \leq s < t$  it holds  $d_{\partial M}(\gamma(s)) = s$  and  $\gamma(0)$  is the only point of  $\partial M$  realising  $d_{\partial M}(\gamma(s))$ . In particular, it necessarily holds that  $\dot{\gamma}(0) \perp T_z \partial M$ .

**Definition 2.1.** The *injectivity radius at  $x \in M$* , denoted by  $\text{inj}_M(x)$  is the supremum of times  $t > 0$  such that every optimal geodesic emanating from  $x$  is either optimal up to time  $t$  or crosses the boundary at some time  $t' < t$ . The *injectivity radius of  $M$*  is then equal to

$$\text{inj}_M = \inf_{x \in M} \text{inj}_M(x). \quad (19)$$



The *injectivity radius from  $\partial M$  at  $z \in \partial M$* , denoted by  $\text{inj}_{\partial M}(z)$ , is defined similarly, by replacing optimal geodesics with optimal geodesics w.r.t. the boundary. We then let the *injectivity radius of  $\partial M$*  to be

$$\text{inj}_{\partial M} = \inf_{z \in \partial M} \text{inj}_{\partial M}(z). \quad (20)$$

Observe that, with the above definition, that  $\text{inj}_M > 0$  for any compact manifold. Moreover, if  $M$  is a bounded smooth open subset of  $\mathbb{R}^d$ , then  $\text{inj}_M = +\infty$ . In general, it could happen that  $\text{inj}_{\partial M} = 0$  (e.g., for the Euclidean square in  $\mathbb{R}^n$ ).

For a smooth function  $f : M \rightarrow \mathbb{R}$ , we denote by  $\text{Hess}(f)$  the Riemannian Hessian of  $f$ , that is,

$$\text{Hess}(f)(X, Y) = g(\nabla_X \nabla f, Y), \quad X, Y \in \Gamma(M). \quad (21)$$

Here, we denoted by  $\nabla_X$  the covariant derivative in the direction  $X$ . In the following, the notation  $\text{Hess}(f) \geq c$  (resp.  $\leq c$ ) for some constant  $c \in \mathbb{R}$  is to be understood in the sense of symmetric forms and w.r.t. the metric  $g$ .

Let  $0 < \varepsilon < \text{inj}_{\partial M}$ . Notice that  $d_{\partial}$  is smooth on  $\{d_{\partial} < \text{inj}_{\partial M}\}$ , and thus  $Z_{\varepsilon} = \{d_{\partial} = \varepsilon\}$  is a smooth embedded hypersurface. The second fundamental form and the mean curvature of  $Z_{\varepsilon}$  are equal to  $\text{Hess}(d_{\partial})|_{TZ_{\varepsilon}}$  and its trace, respectively.

**Definition 2.2.** The boundary  $\partial M$  is convex (resp. strictly convex) if its second fundamental form  $\text{Hess}(d_{\partial})|_{\partial M}$  is non-positive (resp. negative). Moreover, it is mean convex if  $\text{Tr Hess}(d_{\partial})|_{\partial M} \leq 0$ .

**Laplace-Beltrami operator.** We use  $L^2(\Omega)$  to denote the space of square integrable functions on  $\Omega$  w.r.t. the Riemannian volume  $\mu_g$ . For any  $u \in L^2(\Omega)$ , we let  $\nabla u$  be the distributional Riemannian gradient of  $u$ , and define its  $H^1$  norm as

$$\|u\|_{H^1(\Omega)}^2 = \int_M |u|^2 d\mu + \int_M |\nabla u|^2 d\mu. \quad (22)$$

Then,  $H_0^1(\Omega)$  and  $H^1(\Omega)$  are, respectively, the closure of  $C_c^\infty(\Omega)$  and of  $\{u \in C^\infty(\Omega) \mid \|u\|_{H^1(\Omega)} < +\infty\}$  w.r.t. this norm. We remark that  $H^1(\Omega) \subset W^1(\Omega)$ , where the latter is the space of measurable functions such that  $\|u\|_{H^1(\Omega)} < +\infty$  in the distributional sense.

The Laplace-Beltrami operator is the differential operator defined by

$$\Delta = \text{div}_{\mu_g} \circ \nabla. \quad (23)$$

We will consider two realizations of the Laplace-Beltrami operator as an operator on  $L^2(\Omega)$ : the Friedrichs (or Dirichlet) and the Neumann Laplace-Beltrami operators, denoted, by  $\Delta_{\Omega}^-$  and  $\Delta_{\Omega}^+$ , respectively. When  $\Omega = M$ , we will often drop the subscript, i.e.,  $\Delta^\pm = \Delta_M^\pm$ . These are the operators associated with the non-negative symmetric form

$$\mathcal{E}(u, v) = \int_M g(\nabla u, \nabla v) d\mu, \quad (24)$$

with domains  $H_0^1(\Omega)$  and  $H^1(\Omega)$ , respectively.

*Remark 2.1.* Let  $\Omega = M$ . When  $M$  is complete, both the Friedrichs and Neumann Laplace-Beltrami operators coincide since the Laplace-Beltrami operator with domain  $C_c^\infty(M)$  is essentially self-adjoint [16]. On the other hand, if  $\Omega \subset M$  is a smooth domain, these corresponds to consider Dirichlet and Neumann boundary conditions, respectively.

**Heat kernel.** The Dirichlet (resp. Neumann) heat kernel of  $M$ , denoted by  $p^-$  (resp.  $p^+$ ) is the minimal fundamental solution of the heat equation associated with the Dirichlet (resp. Neumann) Laplace-Beltrami operator, i.e.,  $p^\pm : \mathbb{R}_+ \times \Omega \times \Omega \rightarrow \mathbb{R}$  is a smooth function such that

$$\partial_t p^\pm(t, x, \cdot) = \Delta_\Omega^\pm p^\pm(t, x, \cdot), \quad \lim_{t \rightarrow 0} p^\pm(t, x, \cdot) = \delta_x, \quad \forall x \in M. \quad (25)$$

Here,  $\delta_x$  denotes the Dirac delta mass at  $x \in M$ . In particular, since  $M$  has smooth boundary, the definition of  $\Delta_\Omega^\pm$  implies that  $p^-$  satisfies Dirichlet boundary conditions at  $\partial M$ , while  $p^+$  satisfies Neumann ones.

**Spectrum.** If  $\Omega$  is a relatively compact set in  $M$ , the spectrum of  $-\Delta_\Omega^\pm$  is a discrete subset of the positive real axis, i.e.,  $\sigma(-\Delta_\Omega^\pm) \subset [0, +\infty)$ , accumulating at infinity. The Weyl counting function is then defined as

$$N_\Omega^\pm(\lambda) = \#(\sigma(-\Delta_\Omega^\pm) \cap [0, \lambda]). \quad (26)$$

When  $\Omega = M$ , we will always consider the Friedrichs Laplace-Beltrami operator, and denote its Weyl counting function by  $N(\lambda)$ .

## 2.2 Non-complete Riemannian manifolds

Let  $\mathcal{M}$  be a non-complete Riemannian manifold (without boundary). We let  $\widehat{\mathcal{M}}$  be the metric completion of  $\mathcal{M}$  w.r.t. the Riemannian distance and  $\partial\mathcal{M} = \widehat{\mathcal{M}} \setminus \mathcal{M}$  be the *metric boundary*. The distance from the metric boundary is denoted by  $\hat{d}_\partial$ . We recall that  $\hat{d}_\partial$  is Lipschitz and satisfies the eikonal equation, i.e.,  $|\nabla \hat{d}_\partial| \equiv 1$ . For any  $0 \leq a < b \leq +\infty$ , we let  $\mathcal{M}_a^b = \{a \leq \hat{d}_\partial \leq b\} \cap \mathcal{M}$  and  $\mathcal{Z}_a = \{\hat{d}_\partial = a\}$ .

We also assume that  $\mathcal{M}$  has regular metric boundary (in the sense of Definition 1.1), i.e., there exists  $\varepsilon_0 > 0$  such that  $\hat{d}_\partial$  is smooth on  $\mathcal{M}_0^{\varepsilon_0}$ . Under this assumption the level sets  $\mathcal{Z}_\varepsilon$  are smooth embedded hypersurfaces for all  $\varepsilon \in (0, \varepsilon_0]$ , and there exists a uniform tubular neighborhood of the metric boundary  $\mathcal{M}_0^{\varepsilon_0} \simeq (0, \varepsilon_0] \times \mathcal{Z}_{\varepsilon_0}$  where the metric reads

$$g = dx^2 + h_x(z), \quad (x, z) \in (0, \varepsilon_0] \times \mathcal{Z}_{\varepsilon_0}. \quad (27)$$

Here,  $h_x$  is a one-parameter family of smooth metrics on  $\mathcal{Z}_{\varepsilon_0}$ . Within this identification,  $\hat{d}_\partial(x, z) = x$ . Furthermore for all  $\varepsilon \leq \varepsilon_0$  the truncation  $\mathcal{M}_\varepsilon^\infty$  admits an inward tubular neighborhood of its boundary, given by  $\mathcal{M}_\varepsilon^{\varepsilon_0}$ , on which the metric is the restriction of (27) to  $(\varepsilon, \varepsilon_0] \times \mathcal{Z}_{\varepsilon_0}$ . Hence, the distance from its boundary  $\partial\mathcal{M}_\varepsilon^\infty = \mathcal{Z}_\varepsilon \simeq \mathcal{Z}_{\varepsilon_0}$ , satisfies

$$d_\partial(x, z) = \hat{d}_\partial(x, z) - \varepsilon = x - \varepsilon. \quad (28)$$

We reformulate this last statement in an equivalent way, as follows.

**Lemma 2.3** (Boundary injectivity radius estimate). *Let  $\mathcal{M}$  be a non-complete Riemannian manifold with regular metric boundary. Then, up to halving  $\varepsilon_0$ , it holds*

$$\text{inj}_{\partial\mathcal{M}_\varepsilon^\infty} \geq \varepsilon_0, \quad \forall \varepsilon \leq \varepsilon_0. \quad (29)$$

From now on, we assume  $\mathcal{M}$  to have compact metric completion. On  $\mathcal{M}$  we are only interested in the Friedrichs Laplace-Beltrami operator, as defined in the previous section, that we denote simply by  $\Delta$ . In particular, we will consider only the Sobolev space  $H_0^1(\mathcal{M})$ .

For this reason, given  $\Omega \subset \mathcal{M}$ , we define the Sobolev space  $H_0^1(\Omega)$ , as in the case of manifolds with boundary, to be the closure of  $C_c^\infty(\Omega)$  w.r.t. the  $H^1(\Omega)$  norm, and  $H^1(\Omega)$ , to be the closure w.r.t. the same norm of the set<sup>3</sup>  $C^\infty(\Omega) \cap C_c^\infty(\mathcal{M})$ . With this definition, functions of  $H^1(\Omega)$  are restrictions of functions of  $H_0^1(\mathcal{M})$ . Observe that, if  $\Omega \subset \mathcal{M}_\varepsilon^\infty$  for some  $\varepsilon > 0$  and  $\partial\Omega$  is smooth, by [3, Theorem 2.9] we recover the definition of  $H^1(\Omega)$  given in the case of manifolds with boundary.

We need the following result, that we prove in Appendix B.

**Proposition 2.4** (Hardy inequality). *Assume that there exists  $\varepsilon_0 > 0$  such that the boundary of  $\mathcal{M}_\varepsilon^\infty$  is mean convex for any  $0 < \varepsilon \leq \varepsilon_0$ . Then, for any  $0 < \varepsilon \leq \varepsilon_0$ ,*

$$\int_{\mathcal{M}_\varepsilon^\infty} |\nabla u|^2 d\mu_g \geq \frac{1}{4} \int_{\mathcal{M}_\varepsilon^\infty} \frac{|u|^2}{\widehat{d}_\partial^2} d\mu_g, \quad \forall u \in C_c^\infty(\mathcal{M}_\varepsilon^\infty). \quad (30)$$

Assume from now on that there exists  $\varepsilon_0 > 0$  such that the boundary of  $\mathcal{M}_\varepsilon^\infty$  is mean convex for any  $0 < \varepsilon \leq \varepsilon_0$ . Then, since  $\mathcal{M}$  has compact metric completion, the spectrum of  $-\Delta$  is a discrete subset of the positive real axis, i.e.,  $\sigma(-\Delta^\pm) \subset [0, +\infty)$ , accumulating at infinity. (See Proposition B.2.) We denote the Weyl counting function associated with  $\Delta$  by  $N(\lambda)$  and, for  $0 \leq a < b \leq \infty$ , the Weyl counting function of  $\Delta_{\mathcal{M}_b^\infty}^\pm$  by  $N_{[a,b]}^\pm(\lambda)$ .

A classical tool to study  $N(\lambda)$  is the following, see [13, p. 407].

**Proposition 2.5** (Dirichlet-Neumann bracketing). *Let  $\mathcal{M}$  be a non-complete Riemannian manifold with compact metric completion and regular metric boundary. Assume, moreover, that there exists  $\varepsilon_0 > 0$  such that the boundary of  $\mathcal{M}_\varepsilon^\infty$  is mean convex for any  $0 < \varepsilon \leq \varepsilon_0$ . Then, for any  $0 = a_0 < a_1 < \dots < a_{n+1} = +\infty$ , we have*

$$\sum_{i=0}^n N_{[a_i, a_{i+1}]}^-(\lambda) \leq N(\lambda) \leq \sum_{i=0}^n N_{[a_i, a_{i+1}]}^+(\lambda), \quad \forall \lambda \geq 0. \quad (31)$$

### 3 Heat kernel estimates with remainder

In this section we prove the following on-diagonal estimates for the heat kernel, with an explicit control on the remainder term.

**Theorem 3.1.** *Let  $(M, g)$  be a smooth compact  $n$ -dimensional Riemannian manifold with convex boundary  $\partial M$ . Let  $K \geq 0$  be such that*

$$\text{Sec}(M) \leq K, \quad \text{and} \quad \text{Ric}(M) \geq -(n-1)K. \quad (32)$$

---

<sup>3</sup>By  $C^\infty(\Omega) \cap C_c^\infty(\mathcal{M})$  we mean functions on  $\Omega$  that are restrictions of functions of  $C_c^\infty(\mathcal{M})$ .

Moreover, let

$$\rho(x) = \min \left\{ \frac{d_{\partial}(x)}{2}, \text{inj}_M \right\}, \quad \forall x \in M. \quad (33)$$

Then there exist positive constants  $c_1, c_2, c_3 > 0$ , depending only on  $n$ , such that

$$\left| (4\pi t)^{n/2} p^{\pm}(t, x, x) - 1 \right| \leq c_1 K t + c_2 e^{-c_3 \frac{\rho(x)^2}{4t}}, \quad \forall \sqrt{t} \leq \min \left\{ \rho(x), \frac{\pi}{\sqrt{K}} \right\}. \quad (34)$$

*Proof.* Consider the double  $\bar{M} = M \cup_{\partial M} M$  of  $M$ , which is a compact smooth manifold without boundary, endowed with the Lipschitz metric  $\bar{g}$  inherited from  $g$ . Let  $\bar{d}$  and  $\bar{\mu}$  denote the corresponding metric and measure on  $\bar{M}$ . Clearly,  $\bar{d}$  and  $\bar{\mu}$  coincide with  $d$  and  $\mu$ , when restricted to either isometric copy  $M \subset \bar{M}$ . Following [24], although the coefficients of the Laplace-Beltrami operator are discontinuous there is a well-defined heat kernel  $\bar{p}$ , which satisfies

$$p^{\pm}(t, x, y) = \bar{p}(t, x, y) \mp \bar{p}(t, x, y^*), \quad (35)$$

where  $y^* \in \bar{M}$  denotes the reflection of  $y$  w.r.t. the boundary  $\partial M \in \bar{M}$ . We organize the proof in several steps.

**Step 1. Gromov-Hausdorff approximation.** For  $\tau > 0$ , there exists a sequence  $\bar{g}_{\tau}$  of smooth Riemannian metrics on  $\bar{M}$  such that

- $(M, \bar{d}_{\tau}, \bar{\mu}_{\tau}) \rightarrow (\bar{M}, \bar{d}, \bar{\mu})$  in the measured Gromov-Hausdorff sense, as  $\tau \rightarrow 0$ ;
- $\text{Ric}(\bar{g}_{\tau}) \geq -K(n-1)$ , for all values of  $\tau$ ;
- for any compact set  $K$  such that  $K \cap \partial M = \emptyset$  and for sufficiently small  $\tau$ , we have  $\bar{g}_{\tau}|_K = \bar{g}|_K$ ;
- the distance to  $\partial M$  in  $\bar{M}$  w.r.t.  $\bar{g}_{\tau}$  coincides with  $d_{\partial}$ , seen as a function on  $\bar{M}$ .

The construction of this family is sketched in [27, Sec. 4] for positive Ricci curvature and strictly convex boundary. It is not hard to check that this extends to the case of convex boundary, see [36, Thm. 1.8] and references therein for details. The measured Gromov-Hausdorff convergence in the sense of Fukaya [15] follows from the fact that, in this construction,  $\bar{g}_{\tau} \rightarrow \bar{g}$  uniformly in coordinates.

As a consequence of the measured Gromov-Hausdorff convergence and the Ricci bound, we have the convergence of the corresponding heat kernels  $\bar{p}^{\tau} : \mathbb{R}_+ \times \bar{M} \times \bar{M}$

$$\lim_{\tau \downarrow 0} \bar{p}^{\tau}(t, x, y) = \bar{p}(t, x, y), \quad \forall (t, x, y) \in \mathbb{R}_+ \times \bar{M} \times \bar{M}, \quad (36)$$

uniformly on  $\bar{M} \times \bar{M}$ , for any fixed  $t$ . See [14, Theorem 2.6].

The argument consists in obtaining lower and upper bounds for  $\bar{p}^{\tau}$  that are uniform w.r.t.  $\tau$ . Passing to the limit and using (35) will then yield the statement.

**Step 2. Lower bound.** The lower bound on  $\bar{p}^\tau$  is a consequence of classical comparison theorems for the heat kernel with Ricci lower bounds on complete manifolds without boundary. Indeed, applying [7, Thm. 7, p. 196] we have  $p_{-K}(t, d(x, y)) \leq \bar{p}^\tau(t, x, y)$  for all  $(t, x, y) \in \mathbb{R}_+ \times \bar{M} \times \bar{M}$ , for all  $n \in \mathbb{N}$ , and where  $p_{-K} : \mathbb{R}_+ \times [0, +\infty) \rightarrow \mathbb{R}$  is the heat kernel space form of constant curvature  $-K$ . In particular, taking the limit, we have

$$p_{-K}(t, x, y) \leq \bar{p}(t, x, y), \quad (t, x, y) \in \mathbb{R}_+ \times \bar{M} \times \bar{M}. \quad (37)$$

**Step 3. Upper bound.** Observe that, for the upper bound, the comparison theorem cannot hold globally. We claim that there exists positive constants  $C_1, C_2, C_3 > 0$ , such that for any  $x \in M$  and  $t \leq \rho^2(x)$ , where  $\rho(x)$  is defined in (33), it holds

$$\bar{p}(t, x, x) \leq p_K(t, 0) + \frac{C_1}{t^{n/2}} e^{-C_3 \frac{\rho^2(x)}{4t}} \quad \text{and} \quad \bar{p}(t, x, x^*) \leq \frac{C_1}{t^{n/2}} e^{-C_3 \frac{\rho^2(x)}{4t}}. \quad (38)$$

For  $x \in \bar{M}$  and  $r > 0$ , we use  $\bar{B}_x(r)$  (resp.  $\bar{B}_x^\tau(r)$ ) to denote the metric ball with center  $x$  and radius  $r$  in  $\bar{M}$  with respect to the metric  $\bar{d}$  (resp.  $\bar{d}_\tau$ ). When the ball is contained in  $M \subset \bar{M}$ , we drop the bar since no confusion arises.

Fix  $x \in M$ , let  $\rho := \rho(x)$ , and consider  $\tau > 0$ . Denoting by  $p_{\bar{B}_x^\tau(\rho)}^\tau$  the heat kernel w.r.t.  $g_\tau$  on  $\bar{B}_x^\tau(\rho)$  with Dirichlet boundary condition, the Markov property of the heat kernel implies that

$$\bar{p}^\tau(t, x, y) \leq \bar{p}_{\bar{B}_x^\tau(\rho)}^\tau(t, x, y) + \sup_{\substack{0 < s \leq t \\ z \in \partial \bar{B}_x^\tau(\rho)}} \bar{p}^\tau(s, z, y). \quad (39)$$

This follows, e.g., by applying [18, Lemma 3.1] and upper-bounding the hitting probability appearing there by 1. The claim follows by considering in the above  $y \in \{x, x^*\}$  and by estimating separately the two terms appearing on the r.h.s., which we will refer to as the local and the global term, respectively.

Let us start by considering the local term. Since  $\bar{p}_{\bar{B}_x^\tau(\rho)}^\tau(t, x, y) = 0$  whenever  $y \notin B_x(\rho)$ , and  $d(x, x^*) \geq 2d_\partial(x) > \rho$ , we have

$$\bar{p}_{\bar{B}_x^\tau(\rho)}^\tau(t, x, x^*) = 0. \quad (40)$$

On the other hand, by definition of  $\rho$ , the closure of  $\bar{B}_x^\tau(\rho)$  is contained in one of the two copies of  $M \subset \bar{M}$ , and does not intersect  $\partial M$ . Hence, up to assuming  $\tau$  sufficiently small, we have  $\bar{g}_\tau|_\Omega = g|_\Omega$ . In particular,  $\bar{B}_x^\tau(\rho) = \bar{B}_x(\rho) = B_x(\rho)$ . By construction, the latter lies within the injectivity radius from  $x$  and the sectional curvature is bounded from above by  $K$  on it. Therefore, we can apply [7, Theorem 6, p. 194] and domain monotonicity of the Dirichlet heat kernel to obtain

$$\bar{p}_{\bar{B}_x^\tau(\rho)}^\tau(t, x, x) \leq p_K(t, 0). \quad (41)$$

The global term is more delicate. Observe that Li-Yau inequality (see Lemma A.2) requires only a lower bound on the Ricci curvature, and hence can be applied to the compact Riemannian manifold  $(\bar{M}, \bar{g}_\tau)$ , for which  $\text{Ric}(\bar{g}_\tau) \geq -(n-1)K$ , for all

$\tau$ . As a consequence, there exist constants  $C_1, C_2, C_3 > 0$ , depending only on the dimension  $n$  of  $\bar{M}$ , such that

$$\bar{p}^\tau(s, z, y) \leq \frac{C_1}{\sqrt{\text{vol}_\tau(B_y^\tau(\sqrt{s}))\text{vol}_\tau(B_z^\tau(\sqrt{s}))}} e^{C_2 K s - C_3 \frac{\bar{d}_\tau^2(z, y)}{4s}} \quad (42)$$

for all  $(s, z, y) \in \mathbb{R}_+ \times \bar{M} \times \bar{M}$ .

Recall that  $z \in \partial B_x(\rho)$ , and  $\rho(x) \leq d_\partial(x)/2$ . This implies that  $\bar{d}_\tau(z, x) = \rho$ , and

$$\bar{d}_\tau(z, x^*) \geq \bar{d}_\tau(x, x^*) - \bar{d}_\tau(z, x) = 2d_\partial(x) - \rho \geq 3\rho \geq \rho. \quad (43)$$

Hence, (42) yields

$$\bar{p}^\tau(s, z, y) \leq \frac{C_1}{\sqrt{\text{vol}_\tau(B_y^\tau(\sqrt{s}))\text{vol}_\tau(B_z^\tau(\sqrt{s}))}} e^{C_2 K s - C_3 \frac{\rho(x)^2}{4s}}, \quad y \in \{x, x^*\}. \quad (44)$$

Observe that, if  $\sqrt{t} \leq \rho = \min\{\frac{d_\partial(x)}{2}, \text{inj}_M\}$ , there exists a constant  $C > 0$ , depending only on  $n$ , such that

$$\text{vol}_\tau \bar{B}_y(\sqrt{s}) \geq C s^{n/2}, \quad \forall s \leq t, \quad y \in \{x, x^*\}, \quad (45)$$

and the same holds true for  $z$ . Indeed, since  $\sqrt{t} < d_\partial(x)/2$ , the closure of  $B_y^\tau(\sqrt{s})$  does not intersect  $\partial M$ . As a consequence, for fixed  $t$ , we can further restrict  $\tau$  in such a way that these balls lie inside the region of  $\bar{M}$  where the metric is unperturbed, yielding

$$\text{vol}_\tau \bar{B}_y(\sqrt{s}) = \text{vol}(B_y(\sqrt{s})), \quad \forall s \leq t. \quad (46)$$

Furthermore, by  $\sqrt{t} \leq \text{inj}_M$ , and thanks to the upper bound on the sectional curvature of  $(M, g)$ , we have

$$\text{vol}(B_y(\sqrt{s})) \geq \text{vol}(B_K(\sqrt{s})), \quad \forall s \leq t. \quad (47)$$

Finally, since  $\sqrt{t} \leq \frac{\pi}{\sqrt{K}}$ , we can easily deduce (see Lemma A.4) the existence of a constant  $C > 0$  depending only on  $n$  such that

$$B_K(\sqrt{s}) \geq C s^{n/2}, \quad \forall s \leq t. \quad (48)$$

The same argument can be applied replacing  $y \in \{x, x^*\}$  by  $z \in \partial B_x(\rho)$ .

By plugging (45) in (44), and up to redefining the constants, we deduce that

$$\bar{p}^\tau(s, z, y) \leq \frac{C_1}{s^{n/2}} e^{-C_3 \frac{\rho(x)^2}{4s}}, \quad y \in \{x, x^*\}. \quad (49)$$

Elementary arguments show that, up to replacing the constant  $C_1$  with a larger constant (still depending only on the dimension), one has

$$\sup_{\substack{0 < s < t \\ z \in \partial\Omega}} \bar{p}^\tau(s, z, y) \leq \frac{C_1}{t^{n/2}} e^{-C_3 \frac{\rho^2(x)}{4t}}, \quad \forall t \leq \rho^2, \quad y \in \{x, x^*\}, \quad (50)$$

which is the the final estimate for the global part of (39).

By (40) (resp. (41)) and (50), passing to the limit as  $\tau \rightarrow 0$  in (39), completes the proof of the upper bounds (38).

**Step 4. Conclusion.** By (35), the lower bound (37) and the upper bound (38) for the heat kernel of the double yield the following on-diagonal estimates for the Dirichlet and Neumann heat kernels of the original manifold with boundary:

$$p_{-K}(t, 0) - \frac{C_1}{t^{n/2}} e^{-C_3 \frac{\rho^2(x)}{4t}} \leq p^\pm(t, x, x) \leq p_K(t, 0) + \frac{2C_1}{t^{n/2}} e^{-C_3 \frac{\rho^2(x)}{4t}}. \quad (51)$$

In particular, by the uniform estimates of  $p_{\pm K}(t, 0)$  given in Lemma A.1 (which we apply with  $T = \pi^2$ ), we have

$$\left| (4\pi t)^{n/2} p^\pm(t, x, x) - 1 \right| \leq c_1 K t + c_2 e^{-C_3 \frac{\rho^2(x)}{4t}}, \quad (52)$$

valid for all  $0 < \sqrt{t} \leq \min\{\rho(x), \frac{\pi}{\sqrt{K}}\}$ .  $\square$

### 3.1 Heat trace bound

In this section we apply Theorem 3.1 to estimate the heat trace on  $M$ .

**Theorem 3.2** (Heat trace asymptotics with quantitative remainder). *Let  $(M, g)$  be a smooth compact  $n$ -dimensional Riemannian manifold with convex boundary  $\partial M$ . Let  $K, H \geq 0$  such that*

$$\text{Sec}(M) \leq K, \quad \text{and} \quad \text{Ric}(M) \geq -(n-1)K, \quad (53)$$

$$-H \leq \text{Hess}(d_\partial) \leq 0, \quad \text{for} \quad d_\partial < \text{inj}_{\partial M}. \quad (54)$$

Moreover, we let

$$\mathbf{i} = \min \left\{ \text{inj}_M, \frac{\text{inj}_{\partial M}}{2} \right\}. \quad (55)$$

Then, there exist constants  $c_1, c_2, c_3, c_4 > 0$ , depending only on  $n$ , such that the following estimate for the Dirichlet and Neumann heat kernels holds:

$$\left| \frac{(4\pi t)^{n/2}}{\text{vol}(M)} \int_M p^\pm(t, x, x) d\mu_g(x) - 1 \right| \leq c_1 K t + c_2 \frac{\text{vol}(\partial M)}{\text{vol}(M)} \sqrt{t} + c_3 e^{-c_4 \frac{\mathbf{i}^2}{4t}}, \quad (56)$$

for all values of  $t \in \mathbb{R}_+$  such that

$$\sqrt{t} \leq \sqrt{t_0} = \min \left\{ \text{inj}_M, \frac{\text{inj}_{\partial M}}{2}, \frac{\pi}{\sqrt{K}}, \frac{1}{H} \right\}. \quad (57)$$

*Proof.* Fix  $t$  as in our assumptions. We split the manifold into 3 disjoint components (see Figure 1):

$$\Omega_1 = \left\{ \frac{d_\partial}{2} < \sqrt{t} \right\}, \quad \Omega_2 = \left\{ \sqrt{t} \leq \frac{d_\partial}{2} < \mathbf{i} \right\}, \quad \Omega_3 = \left\{ \mathbf{i} \leq \frac{d_\partial}{2} \right\}. \quad (58)$$

We estimate the heat trace on these three sets separately.

**Estimate on  $\Omega_1$ .** By definition, and thanks to our assumption on  $t$ , we have

$$\frac{d_\partial}{2} < \sqrt{t} \leq \min \left\{ \text{inj}_M, \frac{\text{inj}_{\partial M}}{2}, \frac{\pi}{\sqrt{K}}, \frac{1}{H} \right\}. \quad (59)$$

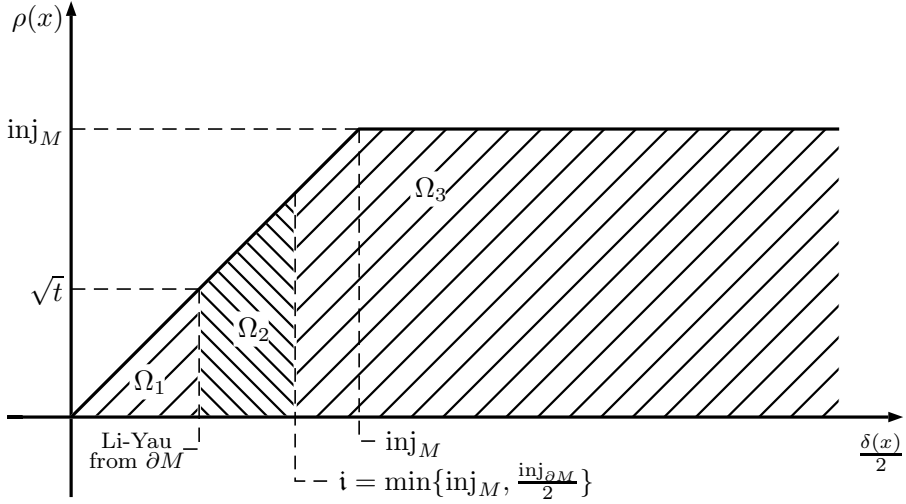


Figure 1: The regions  $\Omega_1, \Omega_2, \Omega_3$  and their relations with all parameters. The condition  $\sqrt{t} \leq \text{inj}_M$  ensures the existence of  $\Omega_2, \Omega_3$  where we apply Theorem 3.1. The condition  $\sqrt{t} \leq \min \left\{ \text{inj}_M, \frac{\pi}{\sqrt{K}}, \frac{\text{inj}_{\partial M}}{2}, \frac{1}{H} \right\}$  allows one to apply the Li-Yau estimate on  $\Omega_1$ .

It follows that  $\rho(x) = d_{\partial}(x)/2$  for any  $x \in \Omega_1$ , where  $\rho$  is defined in (33). Furthermore,  $d_{\partial}(x) < \text{inj}_{\partial M}$ , that is,  $\Omega_1$  is contained inside the injectivity radius from the boundary. Observe that, by construction,  $\sqrt{t} \geq \rho(x)$ , and one cannot apply the bound of Theorem 3.1. However, the assumption on  $t$  allows one to apply Li-Yau estimate (194) of Lemma A.2. This yields,

$$(4\pi t)^{n/2} \int_{\Omega_1} p^{\pm}(t, x, x) d\mu_g(x) \leq C_4 \text{vol}(\Omega_1). \quad (60)$$

In order to complete the estimate on  $\Omega_1$ , one has to bound its volume. We have,

$$\text{vol}(\Omega_1) = \int_0^{2\sqrt{t}} \text{vol}(Z_{\xi}) d\xi, \quad (61)$$

where  $\text{vol}(Z_{\xi})$  denotes the Riemannian volume of the level set  $Z_{\xi} = \{d_{\partial} = \xi\}$  (a smooth  $(n-1)$ -dimensional hypersurface for  $\xi > 0$ , with  $Z_0 = \partial M$ ). Thanks to the convexity assumption, for all  $0 < \xi \leq \text{inj}_{\partial M}$ , it holds

$$\frac{d}{d\xi} \text{vol}(Z_{\xi}) = \int_{Z_{\xi}} \text{Tr}(\text{Hess}(d_{\partial})) d\sigma_{\xi} \leq 0, \quad \Rightarrow \quad \text{vol}(Z_{\xi}) \leq \text{vol}(\partial M). \quad (62)$$

This and (60) conclude the estimate on  $\Omega_1$ , giving

$$(4\pi t)^{n/2} \int_{\Omega_1} p^{\pm}(t, x, x) d\mu_g(x) \leq 2C_4 \text{vol}(\partial M) \sqrt{t}. \quad (63)$$

**Estimate on  $\Omega_2$ .** By construction,  $\Omega_2$  still lies in the region within the injectivity radius from  $\partial M$ . Furthermore, it still holds  $\rho(x) = d_{\partial}(x)/2$  for  $x \in \Omega_2$ . Here, however,  $\sqrt{t} \leq \rho(x)$ , and hence we can apply the result of Theorem 3.1. In particular,

$$\int_{\Omega_2} \left| (4\pi t)^{n/2} p^{\pm}(t, x, x) - 1 \right| d\mu_g(x) \leq \int_{\Omega_2} \left( c_1 K t + c_2 e^{-c_3 \frac{\rho^2(x)}{4t}} \right) d\mu_g(x) \quad (64)$$



$$= c_1 K \text{vol}(\Omega_2)t + c_2 \int_{\Omega_2} e^{-c_3 \frac{d_\partial^2(x)}{16t}} d\mu_g(x) \quad (65)$$

$$= c_1 K \text{vol}(\Omega_2)t + c_2 \int_{2\sqrt{t}}^i e^{-c_3 \frac{\xi^2}{16t}} \text{vol}(Z_\xi) d\xi \quad (66)$$

$$\leq c_1 K \text{vol}(\Omega_2)t + c_2 \text{vol}(\partial M) \int_0^\infty e^{-c_3 \frac{\xi^2}{16t}} d\xi \quad (67)$$

$$\leq c_1 K \text{vol}(\Omega_2)t + c_4 \text{vol}(\partial M) \sqrt{t}. \quad (68)$$

Here, we used the same argument we employed for  $\Omega_1$  to disintegrate the measure  $d\mu$ , and to estimate  $\text{vol}(Z_\xi)$ .

**Estimate on  $\Omega_3$ .** For  $x \in \Omega_3$ , it does not necessarily hold  $\rho(x) = d_\partial(x)/2$ , neither  $x$  is forcibly within the injectivity radius from  $\partial M$ . However, it holds  $\rho(x) \geq i$ . Since we have  $\sqrt{t} \leq i$ , this implies that  $\sqrt{t} \leq \rho(x)$ , and we can apply Theorem 3.1 again. Hence, we obtain

$$\int_{\Omega_3} \left| (4\pi t)^{n/2} p^\pm(t, x, x) - 1 \right| d\mu_g(x) \leq \int_{\Omega_3} \left( c_1 K t + c_2 e^{-c_3 \frac{\rho^2(x)}{4t}} \right) d\mu_g(x) \quad (69)$$

$$\leq \left( c_1 K t + c_2 e^{-c_3 \frac{i^2}{4t}} \right) \text{vol}(\Omega_3). \quad (70)$$

Since  $\text{vol}(\Omega_i)/\text{vol}(M) \leq 1$ , splitting the l.h.s. of (56) in the subsets  $\Omega_1, \Omega_2, \Omega_3$ , using (63), (68), (70), and renaming the constants, yield the result.  $\square$

The remainder estimate of Theorem 3.2 can be expressed in many equivalent ways. For example, one can control the ratio  $\text{vol}(\partial M)/\text{vol}(M)$  in terms of the lower bound on the mean curvature of the boundary and the lower bound on the injectivity radius from the boundary (and hence  $i$ ). We prefer (56) since it is closer to the well known small time heat kernel asymptotics for Riemannian manifolds with boundary. In the next corollary, we present a weaker but simpler statement, in global form.

**Corollary 3.3** (Heat trace asymptotics with quantitative remainder, global form). *In the setting of Theorem 3.2, there exists a constant  $c > 0$ , depending only on the dimension  $n$ , such that*

$$\left| \frac{(4\pi t)^{n/2}}{\text{vol}(M)} \int_M p^\pm(t, x, x) d\mu_g(x) - 1 \right| \leq c \begin{cases} (t/t_0)^{1/2} & t \leq t_0, \\ (t/t_0)^{n/2} & t \geq t_0, \end{cases} \quad (71)$$

where  $t_0$  is defined in (57).

*Proof.* Recall the remainder term of (56) in its original form

$$R(t) = c_1 K t + c_2 \frac{\text{vol}(\partial M)}{\text{vol}(M)} \sqrt{t} + c_3 e^{-c_4 \frac{i^2}{4t}}, \quad \forall t \leq t_0. \quad (72)$$

The inequality  $e^{-1/x} \leq x/e$  for  $x > 0$  yields

$$R(t) \leq c \left( K t + \frac{\text{vol}(\partial M)}{\text{vol}(M)} \sqrt{t} + \frac{t}{i^2} \right). \quad (73)$$

Recall that  $Z_x = \{d_\partial = x\}$ . The lower bound on the Hessian yields

$$\frac{d}{dx} \text{vol}(Z_x) = \int_{Z_x} \text{Tr}(\text{Hess}(d_\partial)) d\sigma_x \geq -H(n-1) \text{vol}(Z_x), \quad \forall x \leq \text{inj}_{\partial M}. \quad (74)$$

Integrating the above, we have

$$\text{vol}(Z_x) \geq \text{vol}(\partial M) e^{-H(n-1)x}, \quad \forall x \leq \text{inj}_{\partial M}. \quad (75)$$

Therefore, since  $t_0 \leq \mathfrak{i}^2$  and  $H\sqrt{t_0} \leq 1$ , we have

$$\frac{\text{vol}(M)}{\text{vol}(\partial M)} \geq \int_0^{\sqrt{t_0}/(n-1)} \frac{\text{vol}(Z_x)}{\text{vol}(\partial M)} dx \geq \frac{1 - e^{-H\sqrt{t_0}}}{H(n-1)} \geq \frac{\sqrt{t_0}(1 - e^{-1})}{n-1}. \quad (76)$$

Thus, up to modifying  $c$  with another constant, depending only on  $n$ , we obtain

$$R(t) \leq c \left( \frac{t}{t_0} \right)^{1/2}, \quad \forall t \leq t_0, \quad (77)$$

where we used the definition of  $t_0$ , and elementary manipulations.

Assume now that  $t \geq t_0$ , and let here  $W(t) = \frac{(4\pi)^{n/2}}{\text{vol}(M)} \int_M p^\pm(t, x, x) d\mu_g(x)$ . Since  $W(t)$  is decreasing and positive, we have

$$|W(t) - t^{-n/2}| \leq W(t_0) + t_0^{-n/2} \quad (78)$$

$$\leq |W(t_0) - t_0^{-n/2}| + 2t_0^{-n/2} \quad (79)$$

$$\leq t_0^{-n/2} R(t_0) + 2t_0^{-n/2} \quad (80)$$

$$\leq t_0^{-n/2} (c + 2). \quad (81)$$

Therefore, up to modifying again  $c$ , we obtain

$$|t^{n/2} W(t) - 1| \leq c \left( \frac{t}{t_0} \right)^{n/2}, \quad \forall t \geq t_0, \quad (82)$$

concluding the proof.  $\square$

## 3.2 Weyl's law with remainder

It is well known that heat trace asymptotics imply Weyl law asymptotics, by means of Tauberian theorems in the form of Karamata [20]. We need here a Karamata-type of result with explicit remainder, due to Ingham [19, Theorem B]. For our purposes, we need to know the explicit dependence of the constants with respect to all parameters and functions at play. For this reason, the statement below is slightly more precise than the one in [19]. Since the proof is unchanged, we omit it.

**Theorem 3.4** (Karamata theorem with remainder [19]). *Suppose that*

$$F(t) = t \int_0^\infty A(\lambda) e^{-t\lambda} d\lambda, \quad s > 0, \quad (83)$$

where  $A(\lambda)$  is positive and non decreasing for  $\lambda > 0$ . Suppose  $\alpha > -1$ . Let  $\chi(\lambda)$  be a function such that

$$\chi(\lambda) > 0, \quad \chi(\lambda) \nearrow \infty, \quad \lambda^{-\alpha-1}\chi(\lambda) \searrow, \quad \forall \lambda > 0 \quad (84)$$

Suppose that there exists  $c_1 > 0$  such that

$$\left| \frac{F(1/\lambda)}{\lambda^\alpha} - 1 \right| \leq \frac{c_1}{\chi(\lambda)}, \quad \forall \lambda > 0. \quad (85)$$

Then there exists another constant  $c_2 = c_2(c_1, \alpha) > 0$  such that

$$\left| \frac{\Gamma(\alpha + 1)A(\lambda)}{\lambda^\alpha} - 1 \right| \leq \frac{c_2}{\log(\chi(\lambda) + 1)}, \quad \forall \lambda > 0. \quad (86)$$

We use Corollary 3.3 to derive the Weyl law with remainder for  $M$ . In order to do that, we define the function  $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}$  by

$$\frac{1}{\chi(\lambda)} := \begin{cases} (\lambda_0/\lambda)^{1/2} & \lambda \geq \lambda_0, \\ (\lambda_0/\lambda)^{n/2} & \lambda \leq \lambda_0. \end{cases} \quad (87)$$

Observe that  $\chi$  is obtained by replacing  $t$  with  $1/\lambda$  in the remainder given by Corollary 3.3. Here  $\lambda_0 = 1/t_0$  which, using (57), is

$$\sqrt{\lambda_0} = \frac{1}{\min \left\{ \text{inj}_M, \frac{\text{inj}_{\partial M}}{2}, \frac{\pi}{\sqrt{K}}, \frac{1}{H} \right\}}. \quad (88)$$

**Theorem 3.5** (Weyl law with quantitative remainder). *In the setting of Theorem 3.2, there exists a constant  $c > 0$  depending on  $n$ , such that the following estimate holds for the Weyl counting function for the Dirichlet or Neumann eigenvalues:*

$$\left| \frac{N(\lambda)}{\frac{\omega_n}{(2\pi)^n} \text{vol}(M) \lambda^{n/2}} - 1 \right| \leq \frac{c}{\log(\chi(\lambda) + 1)}, \quad \forall \lambda > 0. \quad (89)$$

*Proof.* The proof is an application of Theorem 3.4. Indeed, observe that

$$\int_M p^\pm(t, x, x) d\mu_g(x) = \sum_{i=1}^{\infty} e^{-t\lambda_i} = t \int_0^{\infty} N(\lambda) e^{-t\lambda} d\lambda. \quad (90)$$

Moreover, by Corollary 3.3, the assumptions in Theorem 3.4 are satisfied with  $\alpha = n/2$ ,  $\chi$  defined in (87), and letting

$$A(\lambda) = \frac{(4\pi)^{n/2}}{\text{vol}(M)} N(\lambda), \quad F(t) = t \int_0^{\infty} A(\lambda) e^{-t\lambda} d\lambda. \quad (91)$$

Recalling that  $\Gamma(n/2 + 1) = \pi^{n/2}/\omega_n$  yields the result.  $\square$

## 4 Weyl's asymptotics for singular manifolds

Let  $\mathcal{M}$  be a non-complete Riemannian manifold. For the notations and some preliminary facts used in this section, we refer to Section 2.2.

## 4.1 Proof of the Weyl law

Here we prove the Weyl law for the counting function  $N(\lambda)$  of the Friedrichs extension of the Laplace-Beltrami operator of  $\mathcal{M}$ . The strategy of proof (based on Dirichlet-Neumann bracketing, see Proposition 2.5) consists in the following steps:

1. Consider the exhaustion  $\mathcal{M}_\varepsilon^\infty$ , for  $\varepsilon \leq \varepsilon_0$ . Use then Corollary 3.5 to obtain a Weyl law  $N_{[\varepsilon, +\infty]}(\lambda)$  of  $\mathcal{M}_\varepsilon^\infty$ , with an explicit remainder term.
2. Relate  $\varepsilon$  to  $\lambda$  so that  $\varepsilon \rightarrow 0$  as  $\lambda \rightarrow +\infty$  so as to control the remainder term in  $N_{[\varepsilon(\lambda), +\infty]}(\lambda)$  as  $\lambda \rightarrow \infty$ .

A first essential fact is the following, which allows to discard the contributions of the regions near the boundary to  $N(\lambda)$  for quantitatively small values of  $\lambda$ .

**Lemma 4.1** (Spectral estimates at the metric boundary). *Let  $\mathcal{M}$  be a non-complete Riemannian manifold with compact metric completion and regular metric boundary. Assume that, for all  $\varepsilon \leq \varepsilon_0$ , the boundary of  $\mathcal{M}_\varepsilon^\infty$  is mean convex. Then for all  $\varepsilon \leq \varepsilon_0/2$  it holds*

$$N_{[0, \varepsilon]}^\pm(\lambda) = 0, \quad \forall \lambda < \frac{1}{8\varepsilon^2}. \quad (92)$$

*Proof.* A minmax argument combined with Proposition 2.4 yields a spectral lower bound for the Dirichlet spectrum, and hence  $N_{[0, \varepsilon]}^-(\lambda) = 0$  if  $\lambda < 1/4\varepsilon^2$ .

To prove the analogous statement for the Neumann spectrum, we claim that the Hardy inequality of Proposition 2.4 holds also for the restrictions to  $\mathcal{M}_0^\varepsilon$  of functions  $C_c^\infty(\mathcal{M})$ . To prove the claim, thanks to the regularity of the metric boundary, we consider coordinates  $(x, z) \in (0, \varepsilon] \times \mathcal{Z}_{\varepsilon_0} \simeq \mathcal{M}_0^\varepsilon$ , where the metric is as in (27). In these coordinates, the Riemannian measure is  $d\mu = e^{2\theta(x, z)} dx dz$ , for some smooth function  $\theta : (0, \varepsilon] \times \mathcal{Z}_{\varepsilon_0} \rightarrow \mathbb{R}$ . Furthermore,  $\partial_x \theta = \frac{1}{2} \text{Tr Hess } d_\partial \leq 0$ . Let  $u \in C_c^\infty(\mathcal{M})$  and  $\tilde{u}$  be the function on  $\mathcal{M}_0^{2\varepsilon}$  defined in the above coordinates by

$$\tilde{u}(x, z) = u(\min\{x, 2\varepsilon - x\}, z). \quad (93)$$

By construction,  $\tilde{u}$  is continuous, smooth on  $\mathcal{M}_0^\varepsilon$  and  $\mathcal{M}_\varepsilon^{2\varepsilon}$ , compactly supported in  $\mathcal{M}_0^{2\varepsilon}$ , and its restriction to  $\mathcal{M}_0^\varepsilon$  coincides with  $u$ . Furthermore, it holds

$$\int_{\mathcal{M}_\varepsilon^{2\varepsilon}} |\nabla \tilde{u}|^2 d\mu = \int_\varepsilon^{2\varepsilon} dx \int_{\mathcal{Z}_{\varepsilon_0}} |\nabla u(2\varepsilon - x, z)|^2 e^{2\theta(x, z)} dx dz \quad (94)$$

$$\leq \int_\varepsilon^{2\varepsilon} dx \int_{\mathcal{Z}_{\varepsilon_0}} |\nabla u(2\varepsilon - x, z)|^2 e^{2\theta(2\varepsilon - x, z)} dx dz \quad (95)$$

$$= \int_0^\varepsilon dx \int_{\mathcal{Z}_{\varepsilon_0}} |\nabla u(x, z)|^2 e^{2\theta(x, z)} dx dz \quad (96)$$

$$= \int_{\mathcal{M}_0^\varepsilon} |\nabla u|^2 d\mu, \quad (97)$$

where in the second line we used the fact that  $x \mapsto e^{2\theta(x, z)}$  is non-increasing by mean convexity. A straightforward approximation argument shows that the Hardy

inequality of Proposition 2.4 holds for  $\tilde{u}$  on  $\mathcal{M}_0^{2\varepsilon}$ , even if  $\tilde{u}$  is not smooth (this is why we required  $2\varepsilon \leq \varepsilon_0$ ). Hence, we obtain

$$\int_{\mathcal{M}_0^\varepsilon} |\nabla u|^2 d\mu_g = \frac{1}{2} \int_{\mathcal{M}_0^\varepsilon} |\nabla u|^2 d\mu_g + \frac{1}{2} \int_{\mathcal{M}_0^\varepsilon} |\nabla u|^2 d\mu_g \quad (98)$$

$$\geq \frac{1}{2} \int_{\mathcal{M}_0^{2\varepsilon}} |\nabla \tilde{u}|^2 d\mu_g \quad (99)$$

$$\geq \frac{1}{8\varepsilon^2} \int_{\mathcal{M}_0^{2\varepsilon}} |\tilde{u}|^2 d\mu_g \quad (100)$$

$$\geq \frac{1}{8\varepsilon^2} \int_{\mathcal{M}_0^\varepsilon} |u|^2 d\mu_g, \quad \forall u \in C_c^\infty(\mathcal{M}), \quad (101)$$

concluding the proof of the claim.  $\square$

Crucial in the proof is the following relative volume estimate.

**Lemma 4.2** (Relative volume estimates). *Let  $\mathcal{M}$  be a non-complete Riemannian manifold with regular metric boundary. Assume that there exists  $C > 0$  such that*

$$m(\partial\mathcal{M}_\varepsilon^\infty) \geq -\frac{C}{\varepsilon}, \quad \forall \varepsilon \leq \varepsilon_0. \quad (102)$$

Then there exists  $C' > 0$  such that, for any  $0 < a \leq b \leq \varepsilon_0$  we have

$$\frac{\text{vol}(\mathcal{M}_b^\infty)}{\text{vol}(\mathcal{M}_a^\infty)} \geq \left(\frac{a}{b}\right)^{1/C'}. \quad (103)$$

*Proof.* Thanks to the regularity of the metric boundary and in particular (27), we have

$$\text{vol}(\mathcal{M}_\varepsilon^\infty) = (\mathcal{M}_{\varepsilon_0}^\infty) + \int_\varepsilon^{\varepsilon_0} \text{vol}(\mathcal{Z}_x) dx. \quad (104)$$

The lower bound on the mean curvature yields

$$\frac{d}{dx} \text{vol}(\mathcal{Z}_x) = \int_{\mathcal{Z}_x} m(\partial\mathcal{M}_x^\infty) d\sigma_x \geq -\frac{C}{x} \text{vol}(\mathcal{Z}_x), \quad \forall x \leq \varepsilon_0. \quad (105)$$

By Gronwall's Lemma, this implies

$$\frac{\text{vol}(\mathcal{Z}_x)}{\text{vol}(\mathcal{Z}_\varepsilon)} \geq \left(\frac{\varepsilon}{x}\right)^C, \quad \forall \varepsilon \leq x \leq \varepsilon_0. \quad (106)$$

Combining (104) with (106) we obtain

$$\frac{\text{vol}(\mathcal{M}_\varepsilon^\infty)}{\text{vol}(\mathcal{Z}_\varepsilon)} \geq \int_\varepsilon^{\varepsilon_0} \left(\frac{\varepsilon}{x}\right)^C dx = \varepsilon \int_1^{\varepsilon_0/\varepsilon} \left(\frac{1}{x}\right)^C dx. \quad (107)$$

The r.h.s. of the above inequality is larger than or equal to  $C'\varepsilon$  provided that, e.g.,  $\varepsilon \leq \varepsilon_0/2$ . By continuity, it follows that

$$\frac{\text{vol}(\mathcal{M}_\varepsilon^\infty)}{\text{vol}(\mathcal{Z}_\varepsilon)} \geq C'\varepsilon, \quad \forall \varepsilon \leq \varepsilon_0. \quad (108)$$

Note that (108) is equivalent to

$$-\frac{d}{d\varepsilon} \log \text{vol}(\mathcal{M}_\varepsilon^\infty) \leq \frac{1}{C'\varepsilon}, \quad \forall \varepsilon \leq \varepsilon_0, \quad (109)$$

which yields (103) upon integration.  $\square$

We are now ready to state and prove the main result of this section.

**Theorem 4.3.** *Let  $\mathcal{M}$  be a non-complete Riemannian manifold with compact metric completion and regular metric boundary. Assume that there exist two constants  $C, \varepsilon_0 > 0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$ , we have*

$$\operatorname{Ric}(\mathcal{M}_\varepsilon^\infty) \geq -\frac{(n-1)C}{\varepsilon^2}, \quad \operatorname{Sec}(\mathcal{M}_\varepsilon^\infty) \leq \frac{C}{\varepsilon^2}. \quad (110)$$

Assume also that for all  $0 < \varepsilon \leq \varepsilon_0$  the boundary of  $\mathcal{M}_\varepsilon^\infty$  is convex, and that

$$m(\partial\mathcal{M}_\varepsilon^\infty) \geq -\frac{C}{\varepsilon}, \quad \operatorname{inj}_{\mathcal{M}_\varepsilon^\infty} \geq \frac{\varepsilon}{C}. \quad (111)$$

Then there exist  $C_\pm > 0$  and  $\Lambda > 0$  such that

$$C_- \leq \frac{N(\lambda)}{\lambda^{n/2} \operatorname{vol}(\mathcal{M}_{1/\sqrt{\lambda}}^\infty)} \leq C_+, \quad \forall \lambda \geq \Lambda. \quad (112)$$

*Remark 4.1.* Thanks to Proposition 4.8, the assumption on the injectivity radius in Theorem 4.3 is automatically satisfied if the boundary is *strictly* convex.

*Remark 4.2.* We recall that a function  $v : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is slowly varying at 0 in the sense of Karamata [4] if

$$\lim_{x \rightarrow 0} \frac{v(ax)}{v(x)} = 1, \quad \forall a > 0. \quad (113)$$

Then, if the function  $x \mapsto \operatorname{vol}(\mathcal{M}_x^\infty)$  is slowly varying at 0, we can asymptotically improve the lower bound in (112):

$$\frac{\omega_n}{(2\pi)^{n/2}} \leq \liminf_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \operatorname{vol}(\mathcal{M}_{1/\sqrt{\lambda}}^\infty)}. \quad (114)$$

This is the case, in particular, if  $\operatorname{vol}(\mathcal{M})$  is finite.

*Proof.* Let  $0 < \varepsilon \leq \varepsilon_0$ . We split  $\mathcal{M}$  into two parts  $\mathcal{M} = \mathcal{M}_0^\varepsilon \cup \mathcal{M}_\varepsilon^\infty$ . Observe that, and thanks to all our assumptions, by Theorem 3.5, we have

$$\left| \frac{(2\pi)^n}{\omega_n \lambda^{n/2} \operatorname{vol}(\mathcal{M}_\varepsilon^\infty)} N_{[\varepsilon, \infty]}^\pm(\lambda) - 1 \right| \leq \frac{c}{\log(\chi(\lambda) + 1)}, \quad \forall \lambda > 0, \quad (115)$$

where  $\chi(\lambda)$  is given in (87). By our assumptions, there exists a constant  $b > 0$ , depending only on the dimension, such that  $\lambda_0 \leq b/\varepsilon^2$  (without loss of generality, we assume this constant to be equal to 1). In particular, we have

$$\chi(\lambda) = \min \left\{ (\varepsilon^2 \lambda)^{1/2}, (\varepsilon^2 \lambda)^{d/2} \right\}. \quad (116)$$

We would like to send  $\varepsilon \rightarrow 0$  as fast as possible, with  $\lambda \rightarrow +\infty$ , while keeping the remainder term under control. During this process, we remark that  $\chi$  is bounded if

and only if  $\varepsilon^2\lambda$  is bounded. Hence, we relate  $\varepsilon$  and  $\lambda$  in such a way that  $\varepsilon^2\lambda$  is a given constant parameter. More precisely, we set  $\varepsilon = \varepsilon_a(\lambda)$  where

$$\varepsilon_a(\lambda) = \sqrt{\frac{1}{a\lambda}}, \quad a > 0. \quad (117)$$

In this case the remainder in (115) is bounded by a constant, depending only on the dimension and on  $a$ , which can be made arbitrarily small as  $a \rightarrow 0$ .

In the remainder of the proof, by considering the cases in which  $a$  is very large and very small, we obtain the upper and lower bound for  $N(\lambda)$ , respectively.

**Upper bound.** Choose  $a_+ > 8$  and  $\varepsilon = \varepsilon_{a_+}(\lambda)$  as described above. In this case, Lemma 4.1 yields  $N_{[0,\varepsilon]}^\pm(\lambda) = 0$ . Hence, by Neumann bracketing (e.g., the r.h.s. of the statement of Proposition 2.5) we obtain that there exists  $C_+ > 0$  independent of  $\lambda$  and such that

$$N(\lambda) \leq N_{[0,\varepsilon]}^+(\lambda) + N_{[\varepsilon,\infty]}^+(\lambda) \leq N_{[\varepsilon,\infty]}^+(\lambda) \leq C_+ \lambda^{n/2} \text{vol} \left( \mathcal{M}_{1/\sqrt{a_+\lambda}}^\infty \right), \quad (118)$$

provided that  $\lambda \geq \Lambda_+ = 1/\varepsilon_0^2 a_+$ .

**Lower bound.** In this case we can always neglect the boundary contribution, since  $N_{[0,\varepsilon]}^-(\lambda) \geq 0$ . By Dirichlet bracketing (e.g., the l.h.s. of the statement of Proposition 2.5), we have

$$N(\lambda) \geq N_{[0,\varepsilon]}^-(\lambda) + N_{[\varepsilon,\infty]}^-(\lambda) \geq N_{[\varepsilon,\infty]}^-(\lambda), \quad \forall \varepsilon \leq \varepsilon_0. \quad (119)$$

Choose  $a = a_-$  sufficiently small, and  $\varepsilon = \varepsilon_a(\lambda)$ , in such a way that the remainder term in (115) is smaller than  $< 1$ . We deduce that there exists a constant  $C_- > 0$  such that

$$N(\lambda) \geq N_{[\varepsilon,\infty]}^-(\lambda) \geq C_- \lambda^{n/2} \text{vol} \left( \mathcal{M}_{1/\sqrt{a_-\lambda}}^\infty \right), \quad (120)$$

provided that  $\lambda \geq \Lambda_- = 1/\varepsilon_0^2 a_-$ .

We conclude the proof of (112) thanks to Lemma 4.2, and setting  $\Lambda = \Lambda_- \geq \Lambda_+$ .

Assume now that the function  $v(x) = \text{vol}(\mathcal{M}_x^\infty)$  is slowly varying, that is,

$$\lim_{x \rightarrow 0} \frac{v(ax)}{v(x)} = 1, \quad \forall a > 0. \quad (121)$$

In this case, in the proof of the lower bound, we can let  $a_- \rightarrow 0$ , in which case  $\Lambda \rightarrow +\infty$ ,  $C_- \rightarrow \frac{\omega_n}{(2\pi)^n}$ , and  $\text{vol}(\mathcal{M}_{1/\sqrt{a_-\lambda}}^\infty) \sim \text{vol}(\mathcal{M}_{1/\sqrt{\lambda}})$ . This proves (114). Clearly we cannot improve the upper bound with the same argument, sending  $a_+ \rightarrow 0$ , since in order to neglect the boundary term one must keep  $a_+ > 8$ .  $\square$

*Remark 4.3.* The proof of Theorem 4.3 shows that the estimate of Theorem 3.2 (and in turn Theorem 4.3 itself) cannot be improved. To see that, suppose that we are able to deduce a better remainder, in such a way that by setting

$$\varepsilon = \frac{1}{\sqrt{a\lambda}}, \quad (122)$$

the remainder of Theorem 3.2 is negligible, and not simply bounded, as  $\lambda \rightarrow +\infty$ . For the upper bound, arguing as above, we obtain the estimate

$$\limsup_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \text{vol}(\mathcal{M}_{1/\sqrt{\lambda}}^\infty)} \leq \frac{\omega_n}{(2\pi)^n} a_+^{1/2C'}, \quad (123)$$

where we used Lemma 4.2 to compute  $\text{vol}(\mathcal{M}_{1/\sqrt{a\lambda}}^\infty)/\text{vol}(\mathcal{M}_{1/\sqrt{\lambda}}^\infty) = a^{1/2C'}$  for some constant  $C' > 0$ . Note that, to get rid of the boundary terms by Lemma 4.1, we need to choose  $a_+ > 8$ . Hence, the best upper bound in (123) is obtained for  $a_+ = 8$ , and cannot be improved further. For the lower bound, we obtain

$$\liminf_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \text{vol}(\mathcal{M}_{1/\sqrt{\lambda}}^\infty)} \geq \frac{\omega_n}{(2\pi)^n} a_-^{1/2C'}. \quad (124)$$

In this case there is no constraint on  $a_-$ , which can be chosen as large as we want, obtaining a contradiction with (123).

## 4.2 Generalized Minkowski dimension

In this section we refine the Weyl asymptotics for non-complete manifolds, under stronger conditions on the volume explosion at infinity.

**Definition 4.4** (Minkowski dimension). Let  $\mathcal{M}$  be a non-complete Riemannian manifold of dimension  $n$ . We say that the metric boundary  $\partial\mathcal{M}$  has *generalized Minkowski dimension*  $d$  if  $\varepsilon \mapsto \frac{\text{vol}(\mathcal{M}_\varepsilon^{2\varepsilon})}{\varepsilon^{n-d}}$  admits positive and finite lim inf and lim sup as  $\varepsilon \rightarrow 0$ . If these two limits agree, we define the *regularized volume of  $\partial\mathcal{M}$*  as

$$|\partial\mathcal{M}|_{\text{reg}} = \frac{1}{C_d} \lim_{\varepsilon \rightarrow 0} \frac{\text{vol}(\mathcal{M}_\varepsilon^{2\varepsilon})}{\varepsilon^{n-d}}, \quad \text{where } C_d = \begin{cases} \log 2 & \text{if } d = n, \\ \frac{2^{n-d}-1}{n-d} & \text{otherwise.} \end{cases} \quad (125)$$

*Remark 4.4.* It is easy to check that  $\partial\mathcal{M}$  has Minkowski dimension  $d \geq n$  if and only if there exists positive and finite lim inf and lim sup as  $\varepsilon \rightarrow 0$  of the quantity:

$$\frac{\text{vol}(\mathcal{M}_\varepsilon^\infty)}{\log \varepsilon^{-1}}, \quad \text{if } d = n, \quad \text{or} \quad \frac{\text{vol}(\mathcal{M}_\varepsilon^\infty)}{\varepsilon^{n-d}}, \quad \text{if } d > n. \quad (126)$$

Moreover,  $\text{vol}(\mathcal{M})$  is finite if and only if  $d < n$ .

The following is a direct consequence of Theorem 4.3, and Remark 4.4 and 4.2.

**Corollary 4.5.** *Let  $\mathcal{M}$  be a non-complete Riemannian manifold satisfying the assumptions of Theorem 4.3. Assume that  $\partial\mathcal{M}$  has Minkowski dimension  $d$ . Then*

$$N(\lambda) \asymp \begin{cases} \lambda^{n/2} & d < n, \\ \lambda^{n/2} \log \lambda & d = n, \\ \lambda^{d/2} & d > n, \end{cases} \quad \lambda \rightarrow +\infty, \quad (127)$$

where the notation  $f \asymp g$  means that  $f/g$  has finite and positive lim sup and lim inf.



The main result of this section is an improvement on the above result in the case where the Minkowski dimension of  $\partial\mathcal{M}$  coincides with the dimension of  $\mathcal{M}$ .

**Theorem 4.6.** *Let  $\mathcal{M}$  be a non-complete Riemannian manifold satisfying the assumptions of Theorem 4.3. Assume that  $\partial\mathcal{M}$  has Minkowski dimension  $d = n$  with regularized volume  $|\partial\mathcal{M}|_{\text{reg}}$ . Then,*

$$\lim_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \log \lambda} = \frac{\omega_n}{(2\pi)^n} \frac{|\partial\mathcal{M}|_{\text{reg}}}{2}. \quad (128)$$

In order to prove Theorem 4.6, we need the following technical result.

**Lemma 4.7.** *Let  $\mathcal{M}$  be an  $n$ -dimensional non-complete Riemannian manifold such that  $\partial\mathcal{M}$  has Minkowski dimension  $d = n$  and regularized volume  $|\partial\mathcal{M}|_{\text{reg}}$ . Then, for any  $0 < \varepsilon_1 < \varepsilon_2 < 1$ , we have*

$$\text{vol}(\mathcal{M}_{\varepsilon_1}^{\varepsilon_2}) = \left(1 + R(\varepsilon_1, \varepsilon_2)\right) |\partial\mathcal{M}|_{\text{reg}} \log \frac{\varepsilon_2}{\varepsilon_1}, \quad (129)$$

where  $R(\varepsilon_1, \varepsilon_2) \rightarrow 0$  as  $\varepsilon_1 \rightarrow 0$  and  $\frac{\varepsilon_2}{\varepsilon_1} \rightarrow +\infty$ .

*Proof.* By assumption one has that

$$\text{vol}(\mathcal{M}_{\varepsilon}^{2\varepsilon}) = |\partial\mathcal{M}|_{\text{reg}} \log 2 + o(1), \quad \text{as } \varepsilon \rightarrow 0. \quad (130)$$

Let  $0 < \varepsilon_1 < \varepsilon_2 < 1$  be such that  $\varepsilon_1$  is sufficiently small and  $\varepsilon_2/\varepsilon_1$  is sufficiently large (w.r.t. 1). Let  $k \in \mathbb{N}$  be defined by

$$k = \left\lfloor \frac{\log \frac{\varepsilon_2}{\varepsilon_1}}{\log 2} \right\rfloor. \quad (131)$$

We apply (130) with  $2\varepsilon = 2^{j+1}\varepsilon_1$  and  $\varepsilon = 2^j\varepsilon_1$ ,  $j \in \mathbb{N}$ . Summing up for  $0 \leq j \leq k$ , and letting  $\varepsilon_1 \rightarrow 0$  one obtains that

$$\text{vol}(\mathcal{M}_{\varepsilon_1}^{\varepsilon_2}) = \left(1 + o(1)\right) k |\partial\mathcal{M}|_{\text{reg}} \log 2. \quad (132)$$

Assuming further that  $\varepsilon_2/\varepsilon_1$  tends to  $+\infty$  we get the result.  $\square$

*Proof of Theorem 4.6.* We prove the following inequalities:

$$\limsup_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \log \lambda} \leq \frac{\omega_n}{(2\pi)^n} \frac{|\partial\mathcal{M}|_{\text{reg}}}{2} \leq \liminf_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \log \lambda}. \quad (133)$$

The proof of the right inequality in (133) starts as in the proof of Theorem 4.3, i.e., by splitting  $\mathcal{M} = \mathcal{M}_{\varepsilon}^{\infty} \cup \mathcal{M}_0^{\varepsilon}$ . Thanks to Dirichlet bracketing (e.g., the l.h.s. of the statement of Proposition 2.5), we can neglect the contribution of  $\mathcal{M}_0^{\varepsilon}$ , hence obtaining

$$N(\lambda) \geq N_{[0, \varepsilon]}^-(\lambda) + N_{[\varepsilon, \infty]}^-(\lambda) \geq N_{[\varepsilon, \infty]}^-(\lambda), \quad \forall \varepsilon > 0. \quad (134)$$

Note that  $\text{vol}(\mathcal{M}_1^{\infty}) < +\infty$ . Then, by Lemma 4.7 we have

$$\text{vol}(\mathcal{M}_{\varepsilon}^{\infty}) = \left(1 + o(1)\right) |\partial\mathcal{M}|_{\text{reg}} \log \frac{1}{\varepsilon}, \quad \text{where } \lim_{\varepsilon \rightarrow 0} o(1) = 0. \quad (135)$$

Given  $\alpha \in (0, 1)$ , we let  $\varepsilon(\lambda) := \frac{1}{\sqrt{\lambda}^{(1-\alpha)}}$ . In this case, applying Theorem 3.5, the remainder tends to zero as  $\lambda \rightarrow +\infty$ . We therefore obtain

$$\liminf_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \log \lambda} \geq \frac{\omega_n}{(2\pi)^n} \frac{|\partial\mathcal{M}|_{\text{reg}}}{2} (1 - \alpha), \quad \forall \alpha \in (0, 1). \quad (136)$$

Letting  $\alpha \rightarrow 0$  in the above estimate we obtain the desired inequality.

We next turn to the proof of the left inequality in (133), which is more delicate. In this case, we split  $\mathcal{M}$  into three parts. Let  $0 < \varepsilon_1 < \varepsilon_2$  sufficiently small, and let

$$\mathcal{M} = \mathcal{M}_0^{\varepsilon_1} \cup \mathcal{M}_{\varepsilon_1}^{\varepsilon_2} \cup \mathcal{M}_{\varepsilon_2}^\infty. \quad (137)$$

Given  $\alpha \in (0, 1)$ , we choose

$$\varepsilon_1(\lambda) := \frac{C_H}{\sqrt{\lambda}}, \quad \varepsilon_2(\lambda) := \frac{C_H}{\sqrt{\lambda}^{(1-\alpha)}}, \quad (138)$$

where  $C_H = 1/8$  is the constant appearing in Lemma 4.1. The latter implies that  $N_{[0, \varepsilon_1(\lambda)]}^+(\lambda) = 0$ . By Neumann bracketing (e.g., the r.h.s. of the statement of Proposition 2.5) we obtain

$$N(\lambda) \leq N_{[\varepsilon_1(\lambda), \varepsilon_2(\lambda)]}^+(\lambda) + N_{[\varepsilon_2(\lambda), \infty]}^+(\lambda). \quad (139)$$

Observe that, by Lemma 4.7, we have

$$\text{vol}\left(\mathcal{M}_{\varepsilon_1(\lambda)}^{\varepsilon_2(\lambda)}\right) = \left(1 + o(1)\right) |\partial\mathcal{M}|_{\text{reg}} \frac{\alpha}{2} \log \lambda, \quad \text{where } \lim_{\lambda \rightarrow +\infty} o(1) = 0. \quad (140)$$

Applying Theorem 3.5 to  $\mathcal{M}_{\varepsilon_1(\lambda)}^{\varepsilon_2(\lambda)}$ , this implies that as  $\lambda \rightarrow +\infty$ ,

$$N_{[\varepsilon_1(\lambda), \varepsilon_2(\lambda)]}^+(\lambda) = \frac{\omega_n}{(2\pi)^n} |\partial\mathcal{M}|_{\text{reg}} \frac{\alpha}{2} \lambda^{n/2} \log \lambda (1 + O(1)). \quad (141)$$

In the above, the remainder term  $O(1)$  may not tend to zero. However, the quantity  $1/\chi(\lambda)$ , which controls the error, is uniformly bounded in terms of geometric quantities computed at the smallest distance from the boundary, that is,  $\varepsilon_1(\lambda)$ , which does not depend on  $\alpha$ . Therefore the  $O(1)$  term is uniformly bounded for  $\alpha \in (0, 1)$ . Hence, there exist a constant  $R > 0$  such that, for all  $\alpha \in (0, 1)$ , it holds

$$\limsup_{\lambda \rightarrow +\infty} \frac{N_{[\varepsilon_1(\lambda), \varepsilon_2(\lambda)]}^+(\lambda)}{\lambda^{n/2} \log \lambda} \leq \frac{\omega_n}{(2\pi)^n} |\partial\mathcal{M}|_{\text{reg}} \frac{\alpha}{2} (1 + R). \quad (142)$$

On the other hand, by (135) one has

$$\text{vol}(M_{\varepsilon_2(\lambda)}^\infty) = \left(1 + o(1)\right) |\partial\mathcal{M}|_{\text{reg}} \frac{1 - \alpha}{2} \log \lambda, \quad \text{where } \lim_{\lambda \rightarrow +\infty} o(1) = 0. \quad (143)$$

Applying Theorem 3.5 to  $\mathcal{M}_{\varepsilon_2(\lambda)}^\infty$ , we obtain,

$$N_{[\varepsilon_2(\lambda), \infty]}^+(\lambda) = \frac{\omega_n}{(2\pi)^n} \text{vol}(M_{\varepsilon_2(\lambda)}^\infty) \lambda^{n/2} \left(1 + O\left(\frac{1}{\lambda^\alpha}\right)\right) \quad (144)$$

$$= \left(1 + o(1)\right) \frac{\omega_n}{(2\pi)^n} |\partial\mathcal{M}|_{\text{reg}} \frac{1 - \alpha}{2} \lambda^{n/2} \log \lambda. \quad (145)$$

By (139), summing (142) and (145), one gets that

$$\limsup_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda^{n/2} \log \lambda} \leq \frac{\omega_n}{(2\pi)^n} |\partial\mathcal{M}|_{\text{reg}} (1 + \alpha R). \quad (146)$$

As before, one concludes by letting  $\alpha \rightarrow 0$ .  $\square$

### 4.3 On the assumptions of Theorem 4.3

Here, we show that the assumption on the injectivity radius in Theorem 4.3 is automatically satisfied if the boundary is *strictly* convex.

**Proposition 4.8** (Injectivity radius estimate). *Let  $\mathcal{M}$  be a non-complete Riemannian manifold with regular metric boundary. If for all  $\varepsilon \leq \varepsilon_0$  the boundary of  $\mathcal{M}_\varepsilon^\infty$  is strictly convex and  $\text{Sec}(\mathcal{M}_\varepsilon^\infty) \leq \frac{C}{\varepsilon^2}$  for some  $C > 0$ , then there exists  $C' > 0$  such that*

$$\text{inj}_{\mathcal{M}_\varepsilon^\infty} \geq C'\varepsilon, \quad \forall \varepsilon \leq \varepsilon_0. \quad (147)$$

*Proof.* A Klingenberg-type lemma for manifolds with convex boundary<sup>4</sup> yields that

$$\text{inj}_{\mathcal{M}_\varepsilon^\infty} \geq \min \left\{ \frac{\pi}{\sqrt{K_\varepsilon}}, \frac{\ell_\varepsilon}{2} \right\}, \quad (148)$$

where  $K_\varepsilon = C/\varepsilon^2$  is the upper bound on the sectional curvature of  $\mathcal{M}_\varepsilon^\infty$ , and  $\ell_\varepsilon$  is the shortest closed geodesic in  $\mathcal{M}_\varepsilon^\infty$ . We claim that no such closed geodesic can intersect  $\mathcal{M}_\varepsilon^{\varepsilon_0}$ . In fact, let  $\gamma$  be a shortest geodesic loop, and assume by contradiction that the closest point of  $\gamma$  to  $\partial\mathcal{M}_\varepsilon^\infty$ , say  $\gamma(t_0)$ , lies in  $\mathcal{M}_\varepsilon^{\varepsilon_0}$ . In particular,  $t \mapsto d_\partial(\gamma(t))$  has a minimum at  $t_0$ . But the strict convexity assumption yields

$$\left. \frac{d^2}{dt^2} \right|_{t=t_0} d_\partial(\gamma(t)) = \text{Hess}(d_\partial)(\dot{\gamma}(t_0), \dot{\gamma}(t_0)) < 0, \quad (149)$$

yielding a contradiction. Hence, for all  $\varepsilon < \varepsilon_0$ , the shortest closed geodesic in  $\mathcal{M}_\varepsilon^\infty$  is contained in  $\mathcal{M}_{\varepsilon_0}^\infty$ , and its length does not depend on  $\varepsilon$ . We conclude by (148).  $\square$

## 5 Strongly regular ARS

We now apply our results to a class of structures where the metric boundary can be locally described as the singular region of a Riemannian metric. This class is modeled on almost-Riemannian structures, introduced in [1]. We provide here a direct and local definition, which is sufficient for our purposes. We refer the reader to [28, Sec. 7] for a self-contained presentation closer to our approach.

Let  $N$  be a smooth, connected  $n$ -dimensional manifold, and let  $\mathcal{Z} \subset N$  be a smooth embedded hypersurface  $\mathcal{Z}$ . We assume to be given a metric  $g$ , which is Riemannian on  $N \setminus \mathcal{Z}$  and such that for all  $p \in \mathcal{Z}$  there exist local coordinates  $(x, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and smooth vector fields

$$X_0 = \partial_x, \quad X_i = \sum_{j=1}^{n-1} a_{ij}(x, z) \partial_{z_j}, \quad (150)$$

which are orthonormal for  $g$  outside of  $\mathcal{Z}$ , and  $\det a_{ij}(x, z) = 0$  if and only if  $x = 0$ .

Furthermore, we ask that there exists an  $m \in \mathbb{N}$  such that

$$a_{ij}(x, z) = x^m \hat{a}_{ij}(x, z), \quad \det \hat{a}_{ij} \neq 0. \quad (151)$$

---

<sup>4</sup>The convexity of the boundary implies geodesic convexity of the interior, and hence the proof of Klingenberg lemma is unchanged w.r.t. the case with  $\partial M = \emptyset$ . See e.g. [21, Sec. 6].

In this case, on each local chart, we have

$$X_0 = \partial_x, \quad X_i = x^m \hat{X}_i = x^m \sum_{j=1}^{n-1} \hat{a}_{ij} \partial_{z_j}, \quad (152)$$

where  $X_0, \hat{X}_1, \dots, \hat{X}_n$  have maximal rank also on the singular region. In particular we can introduce the regularized Riemannian structure in a neighborhood of  $\mathcal{Z}$  such that  $\hat{X}_0 = X_0, \hat{X}_1, \dots, \hat{X}_n$  are a smooth orthonormal frame. We denote with a hat all the quantities relative to this structure. In particular, the *regularized measure*  $\hat{\sigma}(\mathcal{Z})$  of  $\mathcal{Z}$  is well defined.

**Definition 5.1.** A singular Riemannian structure on an  $n$ -dimensional manifold  $N$  satisfying the above condition is called an  $m$ -strongly regular ARS, where  $m$  is the integer defined in (151).

The non-complete Riemannian manifold  $\mathcal{M} = N \setminus \mathcal{Z}$  has metric boundary given by at most two copies of  $\mathcal{Z}$  (depending whether the latter is one or two-sided). Finally, in the above local coordinates, and close to  $\mathcal{Z}$ , we have  $\hat{d}_\partial(x, z) = |x|$ . It follows that any strongly regular ARS has regular metric boundary.

*Remark 5.1.* As a consequence of the theory developed in [28], the Laplace-Beltrami of a strongly regular ARS is essentially self-adjoint in  $L^2(N \setminus \mathcal{Z})$ . The same result holds more generally for *regular structures*, introduced in [28], that is, when the condition (151) is replaced by the weaker one

$$\det a_{ij}(x, z) = x^k \phi(x, z), \quad \phi(x, z) \neq 0. \quad (153)$$

Indeed, any strongly regular structure is regular, for  $k = m(n - 1)$ .

## 5.1 Curvature bounds

We now discuss the relevant curvature bounds for strongly regular ARS.

**Proposition 5.2.** *For any  $m$ -strongly regular ARS on a compact  $n$ -dimensional manifold, there exists  $\varepsilon_0, C > 0$  such that, for all planes  $\sigma$ , one has*

$$|\text{Sec}(\sigma)| \leq \frac{C}{\hat{d}_\partial^2}, \quad -\frac{C}{\hat{d}_\partial} \leq \text{Hess}(\hat{d}_\partial) < 0, \quad (154)$$

whenever  $\hat{d}_\partial \leq \varepsilon_0$ . The Minkowski dimension of  $\partial\mathcal{M}$  is equal to  $d = (n-1)(m+1) \geq n$ . The regularized measure of  $\partial\mathcal{M}$  and the regularized volume of  $\mathcal{Z}$  are related by

$$|\partial\mathcal{M}|_{\text{reg}} = 2\hat{\sigma}(\mathcal{Z}). \quad (155)$$

*Proof.* To compute curvature-like quantities, we adopt the following modified Einstein convention. Latin indices run from  $1, \dots, n - 1$ , and repeated indices are summed on that range. The index 0 is reserved for the variable  $x$ , e.g.  $\partial_0 = \partial_x$ . The non vanishing structural functions are given by

$$[X_0, X_i] = c_{0i}^\ell X_\ell, \quad [X_i, X_j] = c_{ij}^\ell X_\ell. \quad (156)$$

Koszul formula for the Levi-Civita connection in terms of orthonormal frames yields

$$\nabla_i X_j = \Gamma_{ij}^\ell X_\ell + \gamma_{ij} X_0, \quad \nabla_0 X_i = \delta_{i\ell} X_\ell, \quad \nabla_i X_0 = -\gamma_{i\ell} X_\ell, \quad (157)$$

$$\delta_{i\ell} = \frac{1}{2}(c_{0i}^\ell - c_{0\ell}^i), \quad \gamma_{i\ell} = \frac{1}{2}(c_{0\ell}^i + c_{0i}^\ell), \quad \Gamma_{ij}^\ell = \frac{1}{2}(c_{ij}^\ell + c_{\ell j}^i + c_{\ell i}^j). \quad (158)$$

Notice that  $\delta = -\delta^*$ ,  $\gamma = \gamma^*$ , while  $\Gamma_{ij}^\ell = -\Gamma_{i\ell}^j$ . From the definition of curvature

$$R(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W), \quad (159)$$

we deduce the following formulas for the Riemann tensor

$$R(X_i, X_j, X_k, X_\ell) = \partial_i \Gamma_{jk}^\ell + \Gamma_{jk}^s \Gamma_{is}^\ell - \gamma_{jk} \gamma_{i\ell} - \partial_j \Gamma_{ik}^\ell - \Gamma_{ij}^s \Gamma_{js}^\ell \quad (160)$$

$$+ \gamma_{ik} \gamma_{j\ell} - c_{ij}^s \Gamma_{sk}^\ell, \quad (161)$$

$$R(X_i, X_j, X_k, X_0) = \Gamma_{jk}^\ell \gamma_{i\ell} + \partial_i \gamma_{jk} - \Gamma_{ik}^\ell \gamma_{j\ell} - \partial_j \gamma_{ik} - c_{ij}^s \gamma_{sk}, \quad (162)$$

$$R(X_0, X_i, X_j, X_0) = \partial_0 \gamma_{ij} + \gamma_{i\ell} \delta_{\ell j} + \gamma_{j\ell} \delta_{\ell i} - \gamma_{i\ell} \gamma_{\ell j}, \quad (163)$$

In particular  $\text{Sec}(X \wedge Y) = R(X, Y, Y, X)$  for any pair of unit orthogonal vectors  $X, Y$ . Furthermore, since  $d_\partial = |x|$ , we have

$$\text{Hess}(\hat{d}_\partial)(X_i, X_j) = -\gamma_{ij}. \quad (164)$$

In terms of the matrix  $a$ , the structural functions read

$$c_{0i}^\ell = a_{\ell s}^{-1} \partial_0 a_{si}, \quad c_{ij}^\ell = (a_{ri} \partial_r a_{sj} - a_{jr} \partial_r a_{si}) a_{s\ell}^{-1}. \quad (165)$$

Using (165) one obtains

$$c_{0i}^\ell = \frac{m}{x} \mathbb{1}_{i\ell} + \hat{c}_{0i}^\ell, \quad c_{ij}^\ell = x^m \hat{c}_{ij}^\ell, \quad (166)$$

which implies

$$\delta_{ij} = \hat{\delta}_{ij}, \quad \gamma_{ij} = \frac{m}{x} \mathbb{1}_{ij} + \hat{\gamma}_{ij}, \quad \Gamma_{ij}^\ell = x^m \hat{\Gamma}_{ij}^\ell. \quad (167)$$

From (161)-(163) we obtain

$$R(X_i, X_j, X_k, X_\ell) = -\frac{m^2}{x^2} (\mathbb{1}_{jk} \mathbb{1}_{i\ell} - \mathbb{1}_{ik} \mathbb{1}_{j\ell}) + O\left(\frac{1}{x}\right), \quad (168)$$

$$R(X_i, X_j, X_k, X_0) = O(1), \quad (169)$$

$$R(X_0, X_i, X_j, X_0) = -\frac{m(m+1)}{x^2} \mathbb{1}_{ij} + O\left(\frac{1}{x}\right). \quad (170)$$

and

$$\text{Hess}(d_\partial)(X_i, X_j) = -\frac{m}{x} \mathbb{1}_{ij} + O(1). \quad (171)$$

A computation of  $|\mathcal{M}_\varepsilon^{2\varepsilon}|$  in local coordinates given by the definition of strongly regular structures yields the Minkowski dimension and the regularized measure as

stated in the proposition. Assuming without loss of generality that  $\mathcal{Z}$  is contained in a single chart, we have, for sufficiently small  $\varepsilon$ ,

$$|\mathcal{M}_\varepsilon^{2\varepsilon}| = \int_{\varepsilon \leq |x| \leq 2\varepsilon} dx \int_{\mathcal{Z}} \sqrt{\det(aa^*)}(x, z) dz \quad (172)$$

$$= \int_{\varepsilon \leq |x| \leq 2\varepsilon} \frac{1}{x^{m(n-1)}} dx \int_{\mathcal{Z}} \sqrt{\det(\hat{a}\hat{a}^*(x, z))} dz \quad (173)$$

$$= \int_{\varepsilon \leq |x| \leq 2\varepsilon} \frac{1}{x^{m(n-1)}} dx \left( \int_{\mathcal{Z}} \sqrt{\det(\hat{a}\hat{a}^*(0, z))} dz \right) (1 + O(\varepsilon)) \quad (174)$$

$$= 2(1 + O(\varepsilon)) \hat{\sigma}(\mathcal{Z}) \int_\varepsilon^{2\varepsilon} \frac{1}{x^{m(n-1)}} dx \quad (175)$$

The factor 2 is due to the fact that the coordinate  $x$  can assume positive and negative values. The result now follows from the definition of Minkowski dimension and regularized measure.  $\square$

The next example shows that the results of Proposition 5.2 do not hold, in general, if one weakens the strongly regular condition (151). In particular, one can have ARS where all geometric quantities have an arbitrarily fast polynomial explosion to  $\pm\infty$ .

*Example 5.1* (Worst case curvature explosion). The example is in dimension  $n = 3$ , then it is sufficient to take the metric product with an Euclidean space of the appropriate dimension to obtain the analogue example in dimension  $n$ . We stress that the forthcoming example respects the  $k$ -regular condition (153).

Let  $k \geq 1$ , and consider the structure defined locally in coordinates  $(x, z) \in \mathbb{R} \times \mathbb{R}^2$  by the vector fields

$$X_0 = \partial_x, \quad X_1 = \partial_{z_1} + x\partial_{z_2}, \quad X_2 = x^k \partial_{z_2}. \quad (176)$$

In other words,  $X_i = a_{ji} \partial_{z_i}$ , for  $i = 1, 2$ , with

$$a = \begin{pmatrix} 1 & 0 \\ x & x^k \end{pmatrix}. \quad (177)$$

The most singular terms in the curvature arise all from the first equation of (165). In particular letting  $C_{i\ell} = c_{0i}^\ell$ , we have

$$C = (a^{-1} \partial_0 a)^* = \begin{pmatrix} 0 & \frac{1}{x^k} \\ 0 & \frac{k}{x} \end{pmatrix}. \quad (178)$$

It follows that

$$\delta = \frac{1}{2}(C - C^*) = \begin{pmatrix} 0 & \frac{1}{2x^k} \\ -\frac{1}{2x^k} & 0 \end{pmatrix}, \quad \gamma = \frac{1}{2}(C + C^*) = \begin{pmatrix} 0 & \frac{1}{2x^k} \\ \frac{1}{2x^k} & \frac{k}{x} \end{pmatrix}. \quad (179)$$

Notice that, rewriting (163) in this notation, we obtain, for  $i = 1, 2$ ,

$$\text{Sec}(X_0 \wedge X_i) = (\partial_0 \gamma + 2\gamma\delta - \gamma^2)_{ii} = \text{diag} \left( -\frac{3}{4x^{2k}}, \frac{1}{4x^{2k}} - \frac{k(k+1)}{x^2} \right), \quad (180)$$

and, for  $i, j = 1, 2$ ,

$$\text{Hess}(d_\partial)(X_i, X_j) = -\frac{1}{2} \begin{pmatrix} 0 & \frac{1}{2x^k} \\ \frac{1}{2x^k} & \frac{k}{x} \end{pmatrix}, \quad (181)$$

whose eigenvalues obey  $h_\pm = \pm \frac{1}{2x^k} (1 + o(1))$  as  $x \rightarrow 0$ .

## 5.2 Weyl asymptotics for ARS

We now apply the theory developed in the previous sections to ARS. We recall that the notation  $f \asymp g$  means that  $f/g$  has finite and positive lim sup and lim inf.

**Theorem 5.3** (Weyl asymptotics for ARS). *Consider the Laplace-Beltrami operator of an  $m$ -strongly regular ARS on an  $n$ -dimensional compact manifold. Then, with the exception of the case  $n = 2$  and  $m = 1$ , we have*

$$N(\lambda) \asymp \lambda^{d/2}, \quad \lambda \rightarrow +\infty, \quad (182)$$

where  $d = (n - 1)(m + 1) > n$  is the Minkowski dimension of the singular region.

In the special case  $n = 2$  and  $m = 1$ , in which  $d = n$ , we have

$$\lim_{\lambda \rightarrow +\infty} \frac{N(\lambda)}{\lambda \log \lambda} = \frac{|\hat{\sigma}(\mathcal{Z})|}{4\pi}. \quad (183)$$

*Proof.* As we discussed at the beginning of this section, a strongly regular ARS has regular metric boundary. Furthermore, by Proposition 5.2, it respects all assumptions of Theorem 4.3. Notice that, since the boundaries of the truncations  $\mathcal{M}_\varepsilon^\infty$  are *strictly* convex, we applied Proposition 4.8, to avoid a direct estimate of the injectivity radius (see Remark 4.1). In particular, we can apply Corollary 4.5, using the fact that the Minkowski dimension of the singular region (equivalently, of the metric boundary) is  $d = (n - 1)(m + 1)$ . The stronger estimate given by Theorem 4.6 holds when  $d = n$ , that is  $n = 2$  and  $m = 1$ .  $\square$

## 5.3 A non-regular ARS

In this section, we present an example of non-regular structure that still satisfies all curvature bounds of Theorem 4.3.

Let  $\mathcal{M}$  be a 2D smooth manifold endowed with a non-complete metric  $g$ , such that  $\widehat{\mathcal{M}}$  is compact, and  $\partial\mathcal{M} = \mathbb{S}^1$ . Assume, moreover, that in a collar neighborhood of  $\partial\mathcal{M}$  with coordinates  $(x, \theta) \in (0, \varepsilon_0) \times \mathbb{S}^1$ , an orthonormal basis for  $g$  is given by

$$X_0 = \partial_x, \quad X_1 = x \left( x^2 + \sin^2(\theta/2) \right) \partial_\theta. \quad (184)$$

This structure is non-regular, since (151) is not satisfied uniformly for  $\theta \in \mathbb{S}^1$ . Nevertheless, we have the following.

**Proposition 5.4.** *Up to reducing  $\varepsilon_0$ , for  $\hat{d}_\partial \leq \varepsilon_0$  it holds*

$$-\frac{3}{\hat{d}_\partial^2} \leq \text{Sec}(X_0, X_1) \leq 0, \quad -\frac{3}{\hat{d}_\partial} \leq \text{Hess}(\hat{d}_\partial) < 0. \quad (185)$$

Moreover, the Minkowski dimension of  $\partial\mathcal{M}$  is 3. As a consequence, we have

$$N(\lambda) \asymp \lambda^{3/2}, \quad \lambda \rightarrow +\infty. \quad (186)$$

*Proof.* By (163) and (164) we have

$$\text{Sec}(X_0 \wedge X_1) = -\frac{3}{x^2} + \frac{4 \sin^2(\theta/2) (6x^2 + 1 - \cos \theta)}{x^2 (2x^2 + 1 - \cos \theta)^2}, \quad (187)$$

$$\text{Hess}(\hat{d}_\theta) = -\frac{3}{x} + \frac{2 \sin^2(\theta/2)}{x^3 + x \sin^2(\theta/2)}. \quad (188)$$

The first part of the statement follows easily.

To conclude the proof, by Proposition 4.8 and Corollary 4.5, it suffices to directly integrate the measure  $d\mu = \frac{dx d\theta}{x(x^2 + \sin^2(\theta/2))}$ . Indeed, for any  $\varepsilon > 0$ , we have

$$\int_{[\varepsilon, 2\varepsilon] \times \mathbb{S}^1} d\mu = \int_\varepsilon^{2\varepsilon} \frac{2\pi}{x^2 \sqrt{x^2 + 1}} dx = \frac{\pi}{\varepsilon} + O(\varepsilon). \quad \square$$

## A Auxiliary geometric estimates

On the  $n$ -dimensional Riemannian space form  $M_K$  of curvature  $K \in \mathbb{R}$ , the heat kernel depends only on  $t$  and on the distance  $d(x, y)$ , hence with a slight abuse of notation we denote it with  $p_K(t, r)$ . Here and below  $r \in [0, \text{diam}(M_K)]$ , where  $\text{diam}(M_K) = +\infty$  if  $K \leq 0$ , and  $\text{diam}(M_K) = \pi/\sqrt{K}$  if  $K > 0$ .

**Lemma A.1** (Uniform heat kernel for model spaces). *Let  $M_K$  be the  $n$ -dimensional space form of curvature  $K \in \mathbb{R}$ . Then, for all  $T > 0$  there exists a constant  $C > 0$  such that*

$$|(4\pi t)^{n/2} p_K(t, 0) - 1| \leq C|K|t, \quad \forall t \leq T/|K|; \quad (189)$$

The constant  $C$  depends only on  $n$  and  $T$ .

*Proof.* If  $K = 0$ , the estimate is trivially verified. Let us consider  $K \neq 0$ . For a general Riemannian manifold  $(M, g)$ , define  $g_\alpha := \alpha^2 g$ . Then,  $\text{Sec}(g_\alpha) = \alpha^{-2} \text{Sec}(g)$ , and  $p_{(M, g_\alpha)}(t, x, y) = \alpha^{-d} p_{(M, g)}(t/\alpha^2, x, y)$ . This immediately implies

$$(4\pi t)^{n/2} p_K(t, r) = (4\pi t|K|)^{n/2} p_{\pm 1}(t|K|, r\sqrt{|K|}), \quad (190)$$

where  $\pm 1$  is the sign of  $K$ . Moreover, by the Minakshisundaram-Pleijel asymptotics<sup>5</sup>, we deduce that for all  $T > 0$  there exist a constant  $C > 0$  such that

$$|(4\pi t)^{n/2} p_{\pm 1}(t, 0) - 1| \leq Ct, \quad \forall t \leq T. \quad (191)$$

where  $C$  depends only on  $n$  and  $T$ . Together with (190), this proves the statement.  $\square$

---

<sup>5</sup>We refer to the following simplified statement, valid for any complete  $n$ -dimensional Riemannian manifold: for all  $T > 0$  and  $x \in M$  there exists  $C > 0$  such that  $|(4\pi t)^{n/2} p(t, x, x) - 1| \leq Ct$ , for all  $t \in (0, T]$ . For a proof in the compact case see e.g. [29, Prop. 3.23]. The extension to the non compact case is done via a localization argument (as in (39)) and Varadhan's formula.



**Lemma A.2** (Li-Yau inequality). *Let  $(M, g)$  be a complete  $n$ -dimensional Riemannian manifold with convex boundary, and  $\text{Ric}(g) \geq -K(n-1)$ , for some  $K \geq 0$ . Then there exist constants  $C_1, C_2, C_3 > 0$ , depending only on  $n$ , such that*

$$p(t, x, y) \leq \frac{C_1}{\sqrt{\text{vol}(B_x(\sqrt{t}))\text{vol}(B_y(\sqrt{t}))}} e^{C_2 K t - C_3 \frac{d^2(x, y)}{4t}}, \quad \forall (t, x, y) \in \mathbb{R}_+ \times M \times M. \quad (192)$$

Furthermore, let  $H \geq 0$  be such that  $-H \leq \text{Hess}(d_\partial)$  for  $d_\partial < \text{inj}_{\partial M}$ , let  $K \geq 0$  such that  $\text{Sec}(M) \leq K$ , and let

$$\sqrt{t_0} = \min \left\{ \frac{\text{inj}_{\partial M}}{2}, \text{inj}_M, \frac{\pi}{\sqrt{K}} \frac{1}{H} \right\}. \quad (193)$$

Then, there exists a constant  $C_4 > 0$ , depending only on  $n$ , such that

$$(4\pi t)^{n/2} p^\pm(t, x, y) \leq C_4 e^{-C_3 \frac{d^2(x, y)}{4t}}, \quad \forall (t, x, y) \in (0, t_0) \times M \times M. \quad (194)$$

*Proof.* The first inequality is [22, Corollary 3.1], where the parameters  $\varepsilon$  and  $\alpha$  are fixed in the allowed ranges. For the case with boundary, see [30, Thm. 4.7].

To prove the second part of the theorem, it suffices to uniformly bound from below the volumes of the balls appearing at the denominator of (192). Since  $\sqrt{t_0}$  is smaller than the  $r_0$  appearing in Lemma A.3, the latter implies the existence of a constant  $C_1 > 0$ , depending only on  $n$ , such that  $\text{vol}(B_z(\sqrt{t})) \geq C_1 \text{vol}(B_K(\sqrt{t}))$  for all  $z \in M$ . Then, to complete the proof, it suffices to observe that the bound  $\sqrt{t_0} \leq \frac{\pi}{\sqrt{K}}$  implies that  $\text{vol}(B_K(\sqrt{t})) \geq C_2 t^{n/2}$  for some dimensional constant  $C_2 > 0$ . This follows by elementary computations, as done in the proof of Lemma A.4.  $\square$

In the next Lemma we show that for any ball  $B_x(r)$  there always exists a spherical sector which points away from the boundary and whose size does not depend on the point. This yields a uniform lower bound to the measure of sufficiently small balls.

**Lemma A.3** (Uniform volume lower bound for manifolds with boundary). *Let  $(M, g)$  be a smooth compact  $n$ -dimensional Riemannian manifold with boundary. Let  $H \geq 0$  be such that  $-H \leq \text{Hess}(d_\partial)$  for  $d_\partial < \text{inj}_{\partial M}$ . Let*

$$r_0 = \min \left\{ \text{inj}_M, \frac{\text{inj}_{\partial M}}{2}, \frac{1}{H} \right\}. \quad (195)$$

Then for any  $x \in M$  and  $r \leq r_0$  there exists an open set  $S_x(r) \subset B_x(r)$  such that

- if  $B_x(r)$  does not intersect  $\partial M$ , then  $S_x(r) = B_x(r)$ ;
- if  $B_x(r)$  intersects  $\partial M$ , then the closest point of  $S_x(r)$  to  $\partial M$  is  $x$ .

Let, moreover,  $K \geq 0$  be such that  $\text{Sec}(g) \leq K$  on  $S_x(r)$ . Then, there exists a constant  $C \in (0, 1/2)$ , depending only on  $n$ , such that

$$\text{vol}(B_x(r)) \geq \text{vol}(S_x(r)) \geq C \text{vol}(B_K(r)), \quad \forall r \leq r_0. \quad (196)$$

*Proof.* If  $d_{\partial}(x) > r$ , the ball does not intersect the boundary, and we set  $S_x(r) = B_x(r)$ . By the upper bound on Sec and since the balls lie within the injectivity radius from their center, we have that their volumes are bounded from below by the volume of the metric ball with the same radius on the space form with constant curvature equal to  $K$ :

$$\text{vol}(B_x(r)) \geq \text{vol}(B_K(r)). \quad (197)$$

If  $d_{\partial}(x) \leq r$ , the ball hits the boundary of  $\partial M$ . The condition  $r < \text{inj}_{\partial M}/2$  implies that  $B_x(r)$  lies in the region where  $d_{\partial}$  is smooth and  $-H \leq \text{Hess}(d_{\partial})$ . Consider a length parametrized geodesic  $\gamma$  emanating from  $x$ , and directed towards the direction where  $d_{\partial}$  increases, that is with  $\theta \in [-\pi/2, \pi/2]$ , where  $\cos \theta = g(\dot{\gamma}, \nabla d_{\partial})$ . Hence

$$d_{\partial}(\gamma(t)) \geq -\frac{H}{2} \sin^2(\theta)t^2 + \cos(\theta)t + d_{\partial}(x), \quad \forall t \leq r. \quad (198)$$

Therefore, minimizing geodesics emanating from  $x$  and with length smaller than  $r$  do not cross the boundary  $\partial M$  provided that, e.g.,

$$\cos \theta \geq \frac{Hr}{2}. \quad (199)$$

Thanks to the assumption  $r < 1/H$  the above inequality holds if  $|\theta| < \pi/3$ . Let  $S_x(r) \subset B_x(r)$  be the corresponding spherical sector of radius  $r$ . By construction,  $x$  is the closest point to  $\partial M$ . Since  $r < r_0 \leq \text{inj}(x)$ , we can fix normal polar coordinates  $(s, \Omega) \in [0, r_0] \times \mathbb{S}^{n-1}$  at  $x$ . Therefore

$$\text{vol}(B_x(r)) \geq \text{vol}(S_x(r)) = \int_{\bar{S}_x(r)} s^{n-1} A(s, \Omega) ds d\Omega, \quad (200)$$

where  $\bar{S}(r)$  is the Euclidean spherical sector corresponding to  $S_x(r)$  in these coordinates, and  $r^{n-1} A(\rho, \Omega)$  is the Jacobian determinant of the exponential map. By standard comparison arguments, the assumption  $\text{Sec} \leq K$  yields  $A(s, \Omega) \geq A_K(s)$ , where the latter is the corresponding object on the  $n$ -dimensional space form with constant curvature equal to  $K$ . Hence

$$\text{vol}(S_x(r)) \geq \int_{\bar{S}(r)} s^{n-1} A_K(s) ds d\Omega. \quad (201)$$

Without loss of generality we can fix coordinates  $(\theta, \varphi) \in (-\pi/2, \pi/2) \times \mathbb{S}^{d-2}$  such that  $\bar{S}(r) = \{|\theta| < \pi/3, s < r\}$ . In these coordinates  $d\Omega = \sin(\theta)^{d-2} d\theta d\varphi$ , where  $d\varphi$  is the standard measure on  $\mathbb{S}^{d-2}$ . Therefore

$$\text{vol}(S_x(r)) \geq \int_0^r s^{n-1} A_K(s) ds \int_0^{\pi/3} d\theta \sin(\theta)^{d-2} \text{vol}(\mathbb{S}^{d-2}) = C \text{vol}(B_K(r)). \quad (202)$$

Simple symmetry considerations imply that  $C \in (0, 1/2)$ .  $\square$

**Lemma A.4.** *Let  $K \geq 0$ , and let  $B_K(r)$  be the ball of radius  $r \leq \pi/\sqrt{K}$  on the space form with constant curvature equal to  $K$  and dimension  $n$ . Then, there exists a constant  $C > 0$ , depending only on the dimension, such that*

$$\text{vol}(B_K(r)) \geq Cr^n, \quad \forall r \leq \pi/\sqrt{K}. \quad (203)$$

*Proof.* By Bishop-Gromov, and since  $K \geq 0$ , the function

$$r \mapsto \text{vol}(B_K(r))/\text{vol}(B_0(r)) \quad (204)$$

is non-increasing. Hence, by the same rescaling argument used to prove Lemma A.1,

$$\frac{\text{vol}(B_K(r))}{\text{vol}(B_0(r))} = \frac{\text{vol}(B_1(r\sqrt{K}))}{\sqrt{K}^n \text{vol}(B_0(r))} = \frac{\text{vol}(B_1(r\sqrt{K}))}{\text{vol}(B_0(r\sqrt{K}))} \geq \frac{\text{vol}(B_1(\pi))}{\text{vol}(B_0(\pi))}. \quad (205)$$

To conclude the proof it suffices to observe that  $\text{vol}(B_0(r)) = r^n(\text{vol}(B_0(1)))$ . In particular,  $C = \text{vol}(B_1(\pi))/\pi^n$ .  $\square$

## B Hardy inequality and discreteness of the spectrum

In this section we prove in a slightly more general version of the Hardy inequality of Proposition 2.4 and present a (mostly) self-contained argument for the compactness of the resolvent of the operators considered in Section 4.

**Proposition B.1** (Hardy inequality). *Let  $M$  be a Riemannian manifold, and  $\varrho : M \rightarrow (0, +\infty)$  be a Lipschitz function such that  $|\nabla\varrho| \equiv 1$  and  $-\Delta\varrho \geq 0$  in the sense of distributions. Then,*

$$\int_M |\nabla u|^2 d\mu_g \geq \frac{1}{4} \int_M \frac{|u|^2}{\varrho^2} d\mu_g, \quad \forall u \in C_c^\infty(M). \quad (206)$$

*Remark B.1.* Proposition 2.4 follows from the above by choosing  $M = \mathcal{M}_0^\varepsilon$  and  $\varrho = \hat{d}_\partial$ .

*Proof.* Let  $u \in C_c^\infty(\mathcal{M}_0^\varepsilon)$ . Since the function  $\varrho$  is Lipschitz and strictly positive, it can be approximated on  $\text{supp } u$  by a sequence  $(\varrho_n)_{n \in \mathbb{N}}$  of positive  $C_c^\infty(\mathcal{M}_0^\varepsilon)$  functions. Since  $-\Delta\varrho \geq 0$  in the sense of distributions, we have

$$\begin{aligned} 0 &\leq - \int_{\mathcal{M}_0^\varepsilon} |u|^2 \varrho_n^{-1} \Delta\varrho d\mu_g \\ &= \int_{\mathcal{M}_0^\varepsilon} g(\nabla(|u|^2 \varrho_n^{-1}), \nabla\varrho) d\mu_g \\ &= - \int_{\mathcal{M}_0^\varepsilon} \frac{|u|^2}{\varrho_n^2} g(\nabla\varrho_n, \nabla\varrho) d\mu_g + 2 \int_{\mathcal{M}_0^\varepsilon} \frac{u}{\varrho_n} g(\nabla u, \nabla\varrho) d\mu_g. \end{aligned} \quad (207)$$

Passing to the limit as  $n \rightarrow +\infty$ , recalling that  $|\nabla\varrho| \equiv 1$ , and applying the Cauchy-Schwarz inequality we obtain

$$\int_{\mathcal{M}_0^\varepsilon} \frac{|u|^2}{\varrho^2} d\mu_g \leq 2 \left( \int_{\mathcal{M}_0^\varepsilon} \frac{|u|^2}{\varrho^2} d\mu_g \right)^{1/2} \left( \int_{\mathcal{M}_0^\varepsilon} |\nabla u|^2 d\mu_g \right)^{1/2}, \quad (208)$$

which implies the statement.  $\square$

The proof of the following result essentially follows the arguments in [28, Prop. 3.7].

**Proposition B.2** (Discreteness of the spectrum). *Let  $\mathcal{M}$  be a non-complete Riemannian manifold satisfying  $(\star)$ , with compact metric completion. Assume that the boundaries  $\partial\mathcal{M}_\varepsilon^\infty$  are mean convex, for  $0 < \varepsilon \leq \varepsilon_0$ . Then for any open  $\Omega \subset \mathcal{M}$  such that  $\partial\Omega \cap \mathcal{M}$  is smooth, the self-adjoint operator  $-\Delta_\Omega^\pm$  has compact resolvent. Therefore, its spectrum is discrete and consists of eigenvalues with finite multiplicity accumulating at infinity.*

*Proof.* All the operators  $-\Delta_\Omega^\pm$  are positive. Furthermore, since  $C_c^\infty(\Omega) \subset C_c^\infty(\mathcal{M})$  and  $C^\infty(\Omega) \cap C_c^\infty(\mathcal{M}) \subset C_c^\infty(\mathcal{M})$ , we have that  $\Delta = \Delta_{\mathcal{M}}^-$  is an extension of  $\Delta_\Omega^\pm$  for any  $\Omega$ . It follows that for any  $z > 0$  the resolvent  $(\Delta - z)^{-1}$  is a bounded operator on  $L^2(\mathcal{M}, d\mu)$  which extends the corresponding resolvent of  $\Delta_\Omega^\pm$ , defined on  $L^2(\Omega, d\omega)$ . Hence, it is sufficient to prove the compactness of the resolvent of  $\Delta$ .

We also observe that, in order to prove the statement, it suffices to show that there exists  $z \in \mathbb{R}$  such that the resolvent  $(\Delta - z)^{-1}$  on  $L^2(\mathcal{M})$  is compact. Indeed, this implies the compactness of  $(\Delta - w)^{-1}$  for all  $w$  in the resolvent set by the first resolvent formula and the fact that compact operators form an ideal in the bounded ones.

By Proposition 2.4, thanks to the mean convexity assumption we have the following Hardy inequality for any  $\varepsilon \in (0, \varepsilon_0)$ :

$$\int_{\mathcal{M}_\varepsilon^\infty} |\nabla u|^2 d\mu_g \geq \frac{1}{4} \int_{\mathcal{M}_\varepsilon^\infty} \frac{|u|^2}{\hat{d}_\partial^2} d\mu_g, \quad \forall u \in C_c^\infty(\mathcal{M}_\varepsilon^\infty). \quad (209)$$

With a localization argument based on the IMS formula, (see, e.g., the proof of [28, Proposition 3.3]), the above yields the global weak Hardy inequality:

$$\int_{\mathcal{M}_\varepsilon^\infty} |\nabla u|^2 d\mu_g \geq \frac{1}{4} \int_{\mathcal{M}_\varepsilon^\infty} \frac{|u|^2}{\hat{d}_\partial^2} d\mu_g + c \int_{\mathcal{M}} |u|^2 d\mu_g, \quad \forall u \in C_c^\infty(\mathcal{M}), \quad (210)$$

with  $c \in \mathbb{R}$  independent of  $\varepsilon \in (0, \varepsilon_0)$ . By density, the above holds for  $u \in H_0^1(\mathcal{M})$ .

Let  $z < c$ . By (210) and the fact that  $\Delta$  is self-adjoint, the resolvent  $(\Delta - z)^{-1}$  is well defined on  $L^2(\mathcal{M})$  and satisfies  $\|(\Delta - z)^{-1}\| \leq 1/(c - z)$ . Consider then  $(\psi_n)_{n \in \mathbb{N}}$  to be a bounded sequence in  $L^2(\mathcal{M})$ , and let  $u_n = (\Delta - z)^{-1} \psi_n \in D(\Delta)$ . Without loss of generality we can assume  $\|\psi_n\| \leq c - z$ , which yields  $\|u_n\| \leq 1$ . One has to prove that  $(u_n)_{n \in \mathbb{N}} \subset L^2(\mathcal{M})$  admits a convergent subsequence. To this end, we consider separately the behavior close and far away from the metric boundary. Let  $\varepsilon \in (0, \varepsilon_0)$  and consider the piecewise linear function  $\chi_1 : [0, +\infty) \rightarrow [0, 1]$  such that  $\chi_1 \equiv 0$  on  $[0, \varepsilon/2]$ , and  $\chi_1 \equiv 1$  on  $[\varepsilon, +\infty)$ . Define  $\chi_2 : [0, +\infty) \rightarrow [0, 1]$  as  $1 - \chi_1$ . Then,  $\chi_2 \equiv 1$  on  $[0, \varepsilon/2]$ , and  $\chi_2 \equiv 0$  on  $[\varepsilon, +\infty)$ ,  $\chi_1$  and  $\chi_2$  are Lipschitz functions such that  $\chi_1 + \chi_2 \equiv 1$ . Then, let  $u_n = u_{n,1} + u_{n,2}$ , where  $u_{n,i} = \phi_i u$  for  $\phi_i = \chi_i \circ \hat{d}_\partial$ ,  $i = 1, 2$ . Observe that, since  $u_n \in D(\Delta) \subset H_0^1(\mathcal{M})$ , we have  $u_{n,1}, u_{n,2} \in H_0^1(\mathcal{M})$ .

We start by considering  $u_{n,2}$ . Since the metric completion of  $\mathcal{M}$  is compact, the support of  $\phi_2$  is compact in  $\mathcal{M}$ , say  $\text{supp } \phi_2 \subset U$  with  $U \Subset \mathcal{M}$ . A straightforward application of Leibniz rule (see, e.g., [28, Lemma 3.5]) yields

$$\int_U |\nabla(\phi_2 u)|^2 d\mu_g = \text{Re} \left( \int_U g(\nabla u, \nabla(\phi_2^2 u)) d\mu_g \right) + \int_U |u|^2 |\nabla \phi_2|^2 d\mu_g, \quad \forall u \in C_c^\infty(\mathcal{M}). \quad (211)$$

By density, the above is true for any  $u \in H_0^1(\mathcal{M})$ . We then obtain,

$$\int_U |\nabla u_{n,2}|^2 d\mu_g = \operatorname{Re} \left( \int_U g(\nabla u_n, \nabla(\phi_2^2 u_n)) d\mu_g \right) + \int_U |u_n|^2 |\nabla \phi_2|^2 d\mu_g. \quad (212)$$

Since  $u_n \in D(\Delta)$  and  $\phi_2^2 u_n \in H^1(U)$ , integrating by parts in the above, we get

$$\begin{aligned} \int_U |\nabla u_{n,2}|^2 d\mu_g &= \operatorname{Re} \left( \int_U \phi_2^2 u_n \overline{\Delta u_n} d\mu_g \right) + \int_U |u_n|^2 |\nabla \phi_2|^2 d\mu_g \\ &= z \|\phi_2 u_n\|^2 + \operatorname{Re} \left( \int_U \phi_2^2 u_n \overline{\psi_n} d\mu_g \right) + \int_U |u_n|^2 |\nabla \phi_2|^2 d\mu_g \\ &\leq z \|u_n\|^2 + \|u_n\| \|\psi_n\| + \frac{4}{\varepsilon^2} \|u_n\|^2 \\ &\leq c + \frac{4}{\varepsilon^2}. \end{aligned} \quad (213)$$

This shows that  $(u_{n,2})_{n \in \mathbb{N}}$  is a bounded sequence in  $H_0^1(U)$ . By compact embedding of  $H_0^1(U)$  in  $L^2(U)$  [17, Cor. 10.21] we finally have that  $(u_{n,2})_{n \in \mathbb{N}}$  converges up to subsequences in  $L^2(U)$ , and thus in  $L^2(\mathcal{M})$ .

We now turn to  $u_{n,1}$ . Since  $u_n \in D(\Delta) \subset H_0^1(\mathcal{M})$  and  $\phi_1 \leq 1$ , by density and integration by parts we obtain

$$\begin{aligned} \int_{\mathcal{M}_0^\varepsilon} |u_{n,1}|^2 d\mu_g &\leq \int_{\mathcal{M}_0^\varepsilon} |u_n|^2 d\mu_g \\ &\leq 4\varepsilon^2 \left( \int_{\mathcal{M}_0^\varepsilon} |\nabla u_n|^2 d\mu_g - c \int_{\mathcal{M}} |u_n|^2 d\mu_g \right) \\ &= 4\varepsilon^2 \left( \int_{\mathcal{M}_0^\varepsilon} u_n \overline{\psi_n} d\mu_g + (z - c) \int_{\mathcal{M}} |u_n|^2 d\mu_g \right) \\ &\leq 4\varepsilon^2 (c - z). \end{aligned} \quad (214)$$

Here, in the last inequality we used the Cauchy-Schwarz inequality.

Summing up, since  $\varepsilon$  can be arbitrary small, by considering  $\varepsilon_k = (2k\sqrt{c-z})^{-1}$  we obtain that for any  $k \in \mathbb{N}$  there exists a subsequence  $n \mapsto \gamma_k(n)$  such that  $u_{\gamma_k(n)} = u_{\gamma_k(n),1} + u_{\gamma_k(n),2}$  with  $\|u_{\gamma_k(n),1}\| \leq 1/k$  and  $(u_{\gamma_k(n),2})_{n \in \mathbb{N}}$  convergent in  $L^2(\mathcal{M})$ . By the same arguments used at the end of the proof of [28, Prop. 3.7], this implies that  $(u_n)_{n \in \mathbb{N}}$  admits a subsequence that converges in  $L^2(\mathcal{M})$ , concluding the proof.  $\square$

## References

- [1] Andrei Agrachev, Ugo Boscain, and Mario Sigalotti. A Gauss-Bonnet-like formula on two-dimensional almost-Riemannian manifolds. *Discrete Contin. Dyn. Syst.*, 20(4):801–822, 2008.
- [2] Luigi Ambrosio, Shouhei Honda, and David Tewodrose. Short-time behavior of the heat kernel and Weyl’s law on  $\operatorname{RCD}^*(K, N)$  spaces. *Ann. Global Anal. Geom.*, 53(1):97–119, 2018.
- [3] Thierry Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.

- [4] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.
- [5] U. Boscain, D. Prandi, and M. Seri. Spectral analysis and the Aharonov-Bohm effect on certain almost-Riemannian manifolds. *Comm. Partial Differential Equations*, 41(1):32–50, 2016.
- [6] Ugo Boscain and Camille Laurent. The Laplace-Beltrami operator in almost-Riemannian geometry. *Ann. Inst. Fourier (Grenoble)*, 63(5):1739–1770, 2013.
- [7] Isaac Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
- [8] Jeff Cheeger. Spectral geometry of singular Riemannian spaces. *J. Differential Geom.*, 18(4):575–657 (1984), 1983.
- [9] Yves Colin de Verdière. L’asymptotique de Weyl pour les bouteilles magnétiques. *Comm. Math. Phys.*, 105(2):327–335, 1986.
- [10] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Spectral asymptotics for sub-Riemannian Laplacians. II: microlocal Weyl measures. *Work in progress*.
- [11] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Quantum ergodicity and quantum limits for sub-Riemannian Laplacians. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2014–2015*, pages Exp. No. XX, 17. Ed. Éc. Polytech., Palaiseau, 2016.
- [12] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Spectral asymptotics for sub-Riemannian Laplacians, I: Quantum ergodicity and quantum limits in the 3-dimensional contact case. *Duke Math. J.*, 167(1):109–174, 2018.
- [13] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. I*. Interscience Publishers, Inc., New York, N.Y., 1953.
- [14] Yu Ding. Heat kernels and Green’s functions on limit spaces. *Comm. Anal. Geom.*, 10(3):475–514, 2002.
- [15] Kenji Fukaya. Collapsing of Riemannian manifolds and eigenvalues of Laplace operator. *Invent. Math.*, 87(3):517–547, 1987.
- [16] Matthew P. Gaffney. A special Stokes’s theorem for complete Riemannian manifolds. *Ann. of Math. (2)*, 60:140–145, 1954.
- [17] Alexander Grigor’yan. *Heat kernel and analysis on manifolds*, volume 47 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.
- [18] Alexander Grigoryan and L Saloff-Coste. Heat kernel on manifolds with ends. *Annales de l’institut Fourier*, 59(5):1917–1997, 2009.
- [19] A. E. Ingham. On Tauberian theorems. *Proc. London Math. Soc. (3)*, 14a:157–173, 1965.

- [20] J. Karamata. Neuer Beweis und Verallgemeinerung einiger Tauberian-Sätze. *Math. Z.*, 33(1):294–299, 1931.
- [21] Shigeru Kodani. Convergence theorem for Riemannian manifolds with boundary. *Compositio Math.*, 75(2):171–192, 1990.
- [22] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.
- [23] Maria Lianantonakis. On the eigenvalue counting function for weighted Laplace-Beltrami operators. *J. Geom. Anal.*, 10(2):299–322, 2000.
- [24] H. P. McKean, Jr. and I. M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.
- [25] Abderemane Morame and Françoise Truc. Magnetic bottles on the Poincaré half-plane: spectral asymptotics. *J. Math. Kyoto Univ.*, 48(3):597–616, 2008.
- [26] Abderemane Morame and Françoise Truc. Magnetic bottles on geometrically finite hyperbolic surfaces. *J. Geom. Phys.*, 59(7):1079–1085, 2009.
- [27] G. Perelman. Construction of manifolds of positive Ricci curvature with big volume and large Betti numbers. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 157–163. Cambridge Univ. Press, Cambridge, 1997.
- [28] D. Prandi, L. Rizzi, and M. Seri. Quantum confinement on non-complete Riemannian manifolds. *Journal of Spectral Theory (in press)*, September 2016.
- [29] Steven Rosenberg. *The Laplacian on a Riemannian manifold*, volume 31 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. An introduction to analysis on manifolds.
- [30] R. Schoen and S.-T. Yau. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.
- [31] Barry Simon. Nonclassical eigenvalue asymptotics. *J. Funct. Anal.*, 53(1):84–98, 1983.
- [32] A. Suleymanova. Heat trace expansion on manifolds with conic singularities. *ArXiv e-prints*, January 2017.
- [33] A. Suleymanova. On the spectral geometry of manifolds with conic singularities. *ArXiv e-prints*, October 2017.
- [34] M. van den Berg. Dirichlet-Neumann bracketing for horn-shaped regions. *J. Funct. Anal.*, 104(1):110–120, 1992.
- [35] Hermann Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Math. Ann.*, 71(4):441–479, 1912.
- [36] Jeremy Wong. An extension procedure for manifolds with boundary. *Pacific J. Math.*, 235(1):173–199, 2008.

- [37] H.-C. Zhang and X.-P. Zhu. Weyl's law on  $RCD^*(K, N)$  metric measure spaces. *Comm. Anal. Geom. (in press)*, January 2017.