Global existence of small amplitude solutions for a model quadratic quasi-linear coupled wave-Klein-Gordon system in two space dimension, with mildly decaying Cauchy data

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Abstract

The aim of this paper is to study the global existence of solutions to a coupled wave-Klein-Gordon system in space dimension two when initial data are small, smooth and mildly decaying at infinity. Some physical models strictly related to general relativity have shown the importance of studying such systems, but very few results are know at present in low space dimension. We study here a model two-dimensional system, in which the non-linearity writes in terms of "null forms", and show the global existence of small solutions. Our goal is to prove some energy estimates on the solution when a certain number of Klainerman vector fields is acting on it, and some optimal uniform estimates. The former ones are obtained using systematically quasi-linear normal forms, in their para-differential version; the latter ones are recovered by deducing a new coupled system of a transport equation and an ordinary differential equation from the starting PDE system, by means of a semi-classical micro-local analysis of the problem. We expect the strategy developed here to be robust enough to enable us, in the future, to treat the case of the most general non-linearities.

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Contents

Introduction

The result we present in this paper concerns the global existence of solutions to a quadratic quasi-linear coupled system of a wave equation and a Klein-Gordon equation in space dimension two, when initial data are small smooth and mildly decaying at infinity. We prove this result for a model non-linearity with the aim of extend it, in the future, to the most general case. Keeping this long term objective in mind, we shall try to develop a fairly general approach in spite of the fact that we are treating here a simple model. The Cauchy problem we consider is the following

$$\text{(1)} \qquad \begin{cases} (\partial_t^2 - \Delta_x)u(t,x) = Q_0(v, \partial_1 v)\,, \\ (\partial_t^2 - \Delta_x + 1)v(t,x) = Q_0(v, \partial_1 u)\,, \end{cases} \qquad (t,x) \in]1, +\infty[\times\mathbb{R}^2$$

with initial conditions

$$\text{(2)} \qquad \begin{cases} (u,v)(1,x) = \varepsilon(u_0(x), v_0(x))\,, \\ (\partial_t u, \partial_t v)(1,x) = \varepsilon(u_1(x), v_1(x))\,, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, and Q_0 is the null form:

$$Q_0(v,w) = (\partial_t v)(\partial_t w) - (\nabla_x v)\cdot(\nabla_x w)\,.$$

We also suppose that, for some $n \in \mathbb{N}$ sufficiently large, $(\nabla_x u_0, u_1)$ is in the unit ball of $H^n(\mathbb{R}^2,\mathbb{R}) \times H^n(\mathbb{R}^2,\mathbb{R})$, (v_0, v_1) in the unit ball of $H^{n+1}(\mathbb{R}^2,\mathbb{R}) \times H^n(\mathbb{R}^2,\mathbb{R})$, and that

$$\text{(3)} \qquad \sum_{1\le|\alpha|\le 3} \left(\|x^\alpha \nabla_x u_0\|_{H^{|\alpha|}} + \|x^\alpha v_0\|_{H^{|\alpha|+1}} + \|x^\alpha u_1\|_{H^{|\alpha|}} + \|x^\alpha v_1\|_{H^{|\alpha|}} \right) \le 1.$$

Some physical models, especially related to general relativity, have shown the importance of studying such systems to which several recent works have been dedicated. Most of the results known at present concern wave-Klein-Gordon systems in space dimension 3. One of the first ones goes back to Georgiev [9]. He observed that the vector fields' method developed by Klainerman was not well adapted to handle at the same time massless and massive wave equations because of the fact that the scaling vector field $S = t\partial_t + x\cdot\nabla_x$ is not a Killing vector field for the Klein-Gordon equation. To overcome this difficulty he adapted Klainerman's techniques, introducing a *strong null condition* to be satisfied by semi-linear nonlinearities that ensures global existence. In 2012 Katayama [18] showed the global existence of small amplitude solutions to coupled systems of wave and Klein-Gordon equations under certain suitable conditions on the non-linearity that include the *null condition* of Klainerman ([19]) on self-interactions between wave components, and are weaker than the *strong null condition* of Georgiev. Consequently, the result he obtains applies also to certain other physical systems such as Dirac-Klein-Gordon equations, Dirac-Proca equations and Klein-Gordon-Zakharov equations. Later, this problem was also studied by LeFloch, Ma [22] and Wang [31] as a model for the full Einstein-Klein-Gordon system (E-KG)

$$\begin{cases} Ric_{\alpha\beta} = \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi + \frac{1}{2}\psi^2 g_{\alpha\beta} \\ \Box_g\psi = \psi \end{cases}$$

The authors prove global existence of solutions to wave-Klein-Gordon systems with quasi-linear quadratic non-linearities satisfying suitable conditions, when initial data are small, smooth and compactly supported, using the so-called *hyperboloidal foliation method* introduced by Le Foch, Ma in [22]. Global stability for the full (E-KG) has been then proved by LeFloch-Ma [21, 20] in the case of small smooth perturbations that agree with a Scharzschild solution outside a compact set (see also Wang [30]). In a recent paper [17] Ionescu and Pausader prove global regularity and modified scattering in the case of small smooth initial data that decay at suitable rates at infinity, but not necessarily compactly supported. The quadratic quasi-linear problem they deal with is the following

$$\begin{cases} -\Box u = A^{\alpha\beta}\partial_\alpha v \partial_\beta v + Dv^2 \\ -(\Box + 1)v = uB^{\alpha\beta}\partial_\alpha\partial_b v \end{cases}$$

where $A^{\alpha\beta}, B^{\alpha\beta}, D$ are real constants. The system keeps the same linear structure as (E-KG) in harmonic gauge, but only keeps quadratic non-linearities that involve the massive scalar field v (semilinear in the wave equation, quasi-linear in the Klein-Gordon equation). Moreover, the non-linearity they consider does not present a null structure but shows a particular resonant pattern. Their result relies, on the one hand, on a combination of energy estimates to control high Sobolev norms and weighted norms involving the admissible vector fields; on the other hand, on a Fourier analysis, in connection with normal forms and analysis of resonant sets, to prove dispersive estimates and decay in suitable lower regularity norms. The only results we know about global existence of small amplitude solutions in lower space dimension are due to Ma. In space dimension 2 he considers the case of compactly supported Cauchy data and adapts the hyperboloidal foliation method mentioned above to $2+1$ spacetime wave-Klein-Gordon systems (see [26]). In particular, in [25] he combines this method with a normal form argument to treat some quasi-linear quadratic non-linearities, while in [24] he studies the case of some semi-linear quadratic interactions. In a very recent paper [23] he also tackles the one-dimensional problem, studying a model semi-linear cubic wave-Klein-Gordon system. In this work he finally overcomes the restriction on the support of initial data and generalizes the hyperboloidal foliation method, combining the hyperboloidal foliation of the translated light cone with the standard time-constant foliation outside of it. The analysis of the problem and the deduction of the estimates of interest is then made separately inside and outside the mentioned light cone.

The result we prove in this paper is the following:

Theorem 1. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$, system (1) with initial data satisfying (2), (3) admits a unique global solution defined on $[1, +\infty[$, with $\partial_{t,x} u \in C^0([1, +\infty[; H^n(\mathbb{R}^2))$ and $(v, \partial_t v) \in C^0([1, +\infty[; H^{n+1}(\mathbb{R}^2) \times H^n(\mathbb{R}^2))$.*

We describe below the strategy of the theorem's proof. First of all, we rewrite system (1) in terms of unkowns

$$u_\pm = (D_t \pm |D_x|)\, u, \quad v_\pm = (D_t \pm \langle D_x \rangle)\, v, \tag{4}$$

where $D_{t,x} = -i\partial_{t,x}$, and introduce the admissible Klainerman vector fields for this problem, i.e.

$$\Omega = x_1\partial_2 - x_2\partial_1, \quad Z_j = x_j\partial_t + t\partial_j, \quad j = 1, 2.$$

We also denote by $\mathcal{Z} = \{\Gamma_1, \dots, \Gamma_5\}$ the family made by above vector fields together with the two spatial derivatives, and if $I = (i_1, \dots, i_p)$ is an element of $\{1, \dots, 5\}^p$, $\Gamma^I w$ is the function obtained letting $\Gamma_{i_1}, \dots, \Gamma_{i_p}$ act successively on w. We then set

$$u_\pm^I = (D_t \pm |D_x|)\, \Gamma^I u, \quad v_\pm^I = (D_t \pm \langle D_x \rangle)\, \Gamma^I v, \tag{5}$$

and introduce the following energies:

$$E_0(t;u_\pm,v_\pm)=\int_{\mathbb{R}^2}\left(|u_+(t,x)|^2+|u_-(t,x)|^2+|v_+(t,x)|^2+|v_-(t,x)|^2\right)dx,$$

then for $n\geq 3$,

$$E_n(t;u_\pm,v_\pm)=\sum_{|\alpha|\leq n}E_0(t;D_x^\alpha u_\pm,D_x^\alpha v_\pm),$$

which controls the H^n regularity of $u_\pm,v_\pm$, and finally, for any integer k between 0 and 2,

$$E_3^k(t;u_\pm,v_\pm)=\sum_{\substack{|\alpha|+|I|\leq 3\\ |I|\leq 3-k}}E_0(t;D_x^\alpha u_\pm^I,D_x^\alpha v_\pm^I)$$

that takes into account the decay in space of $u_\pm,v_\pm$ and of at most three of their spatial derivatives. By a local existence argument, an a-priori estimate on E_n on a certain time interval will be enough to ensure the extension of the solution to that interval. For this reason, we are led to prove a result as the following one, in which $\mathrm{R}=(\mathrm{R}_1,\mathrm{R}_2)$ denotes the Riesz transform:

Theorem 2. *Let K_1,K_2 two constants strictly bigger than 1. There exist two integers $n\gg\rho\gg 1$, $\varepsilon_0\in]0,1[$ small enough, some small real $0<\delta\ll\delta_2\ll\delta_1\ll\delta_0\ll 1$ and two constants A,B sufficiently large such that, if functions $u_\pm,v_\pm$, defined by (4) from a solution to (1), satisfy*

$$\begin{aligned}
&\|\langle D_x\rangle^{\rho+1}u_\pm(t,\cdot)\|_{L^\infty}+\|\langle D_x\rangle^{\rho+1}\mathrm{R}u_\pm(t,\cdot)\|_{L^\infty}\leq A\varepsilon t^{-\frac{1}{2}}\\
&\|\langle D_x\rangle^{\rho}v_\pm\|_{L^\infty}\leq A\varepsilon t^{-1}\\
&E_n(t;u_\pm,v_\pm)\leq B^2\varepsilon^2t^{2\delta}\\
&E_3^k(t;u_\pm,v_\pm)\leq B^2\varepsilon^2t^{2\delta_k},\ 0\leq k\leq 2,
\end{aligned}\tag{6}$$

for every $t\in[1,T]$, then on the same interval $[1,T]$ we have

$$\begin{aligned}
&\|\langle D_x\rangle^{\rho+1}u_\pm(t,\cdot)\|_{L^\infty}+\|\langle D_x\rangle^{\rho+1}\mathrm{R}u_\pm(t,\cdot)\|_{L^\infty}\leq \frac{A}{K_1}\varepsilon t^{-\frac{1}{2}}\\
&\|\langle D_x\rangle^{\rho}v_\pm\|_{L^\infty}\leq \frac{A}{K_1}\varepsilon t^{-1}\\
&E_n(t;u_\pm,v_\pm)\leq \frac{B^2}{K_2^2}\varepsilon^2t^{2\delta}\\
&E_3^k(t;u_\pm,v_\pm)\leq \frac{B^2}{K_2^2}\varepsilon^2t^{2\delta_k},\ 0\leq k\leq 2.
\end{aligned}\tag{7}$$

The proof of the theorem consists, on the one hand, to prove that (6) implies the latter two estimates in (7) by means of an energy inequality. On the other hand, by reduction of the starting problem to a coupled system of an ordinary differential equation and a transport equation, we prove that (6) implies the first two estimates in (7).

In order to recover the mentioned energy inequality that allows us to propagate the a-priori energy estimates, we let family Γ^I of vector fields act on (1) and then pass to unknowns (5). We obtain a new system of the form

$$\begin{aligned}
(D_t\mp|D_x|)u_\pm^I&=NL_{\mathrm{w}}(v_\pm^I,v_\pm^I)\\
(D_t\mp|D_x|)v_\pm^I&=NL_{\mathrm{kg}}(v_\pm^I,u_\pm^I)
\end{aligned}$$

where the non-linearities (whose explicit expression may be found in the right hand side of (2.1.2)) are bilinear quantities of their arguments. Because of the quasi-linear nature of our

problem, the first step towards the derivation of the mentioned inequality is to highlight the very quasi-linear contribution to above non-linearities and make sure that it does not lead to a loss of derivatives. For this reason, we write the above system in a vectorial fashion by introducing vectors

$$U^I = \begin{bmatrix} u_+^I \\ 0 \\ u_-^I \\ 0 \end{bmatrix}, \ V^I = \begin{bmatrix} 0 \\ v_+^I \\ 0 \\ v_-^I \end{bmatrix}, \ W^I = U^I + V^I,$$

and successively *para-linearize* the vectorial equation satisfied by W^I (using the tools introduced in subsection 1.2.1) to stress out the quasi-linear contribution to the non-linearity. Finally, we *symmetrize* it (in the sense of subsection 2.1.3) by introducing some new unknown W_s^I comparable to W^I. What we would need to show in order to prove the last two inequalities in (7) is that, using the estimates in (6), the derivative in time of the L^2 norm to the square of W_s^I is bounded by $\frac{C\varepsilon}{t}\|W^I\|_{L^2}$. By analysing the semi-linear contributions in the symmetrized equation satisfied by W_s^I, we find out that the L^2 norm of some of those ones can only be estimated making appear the L^∞ norm on the wave factor and the L^2 norm on the Klein-Gordon one. Because of the very slow decay in time of the wave solution (the decay rate being $t^{-1/2}$, as assumed in the first inequality of (6)), we are hence very far away from the wished estimate. Consequently, the second step for the derivation of the right energy inequality consists in performing a normal form argument to get rid of those quadratic terms and replace them with cubic ones. For that, we first use a Shatah' normal form adapted to quasi-linear equations (see subsection 2.2.1) as already used by several authors (we cite [28, 5, 4, 6] for quasi-linear Klein-Gordon equations, and [13, 12, 16, 1, 15] for quasi-linear equations arising in fluids mechanics), but also a semi-linear normal form argument to treat some other terms on which we are allowed to lose some derivatives (see subsection 2.2.2). These two normal forms' steps lead us to define some new energies $\widetilde{E}_n(t;u_\pm,v_\pm), \widetilde{E}_3^k(t;u_\pm,v_\pm)$, equivalent to the starting ones $E_n(t;u_\pm,v_\pm)$, $\widetilde{E}_3^k(t;u_\pm,v_\pm)$, that we are able to propagate. That concludes the first part of the proof.

The last thing that remains to prove is that (6) implies the first two estimates in (7). The strategy we employ is very similar to the one developed in [29]: we deduce from the starting system (1) a new coupled one of an ordinary differential equation, coming from the Klein-Gordon equation, and of a transport equation, derived from the wave one. The study of this system will provide us with the wished L^∞ estimates. We start our analysis by another normal form in order to replace almost all quadratic non-linear terms in the equations satisfied by $u_\pm, v_\pm$ with cubic ones. The only contributions that cannot be eliminated are those depending on (v_+, v_-) which are resonant and should be suitably treated. We do not use directly the normal forms obtained in the previous step. In fact, our aim is basically to obtain an L^∞ estimate for at most ρ derivatives of the solution, having a control on their H^s norm for $s \gg \rho$. This permits us to lose some derivatives in the normal form reduction, so the fact that the system is quasi-linear is no longer important.

We define two new unknowns u^{NF}, v^{NF} by adding some quadratic perturbations to u_-, v_-, in such a way that they are solution to

$$(D_t + |D_x|)u^{NF} = q_w + c_w + r_w^{NF}, \ (D_t + |D_x|)v^{NF} = r_{kg}^{NF}, \tag{8}$$

where $r_w^{NF}, c_w, r_{kg}^{NF}$ are cubic terms, whereas q_w is the mentioned bilinear expression in v_+, v_- that cannot be eliminated by normal forms but whose structure will successively provide us with remainder terms. Then, if we define

$$\widetilde{u}(t,x) = tu^{NF}(t,tx), \ \widetilde{v}(t,x) = tv^{NF}(t,tx), \tag{9}$$

and introduce $h := t^{-1}$ the *semi-classical parameter*, we obtain that $\widetilde{u}, \widetilde{v}$ verify

$$
\begin{aligned}
&(D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|))\widetilde{u} = h^{-1}\left[q_w(t,tx) + c_w(t,tx) + r_w^{NF}(t,tx)\right]\\
&(D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle))\widetilde{v} = h^{-1} r_{kg}^{NF}(t,tx)
\end{aligned}
\tag{10}
$$

where Op_h^w is the Weyl quantization introduced, along with the semi-classical pseudo-differential calculus, in subsection 1.2.1. We also consider the following operators

$$
\mathcal{M}_j = \frac{1}{h}\Big(x_j|\xi| - \xi_j\Big),\ \mathcal{L}_j = \frac{1}{h}\Big(x_j - \frac{\xi_j}{\langle\xi\rangle}\Big),
$$

whose symbols are given respectively (up to the multiplication by $|\xi|$ for the former case) by the derivative with respect to ξ of symbols $x\cdot\xi - |\xi|$ and $x\cdot\xi - \langle\xi\rangle$ in (10). Using the equation satisfied by u^{NF} (resp. v^{NF}), we can express $\mathcal{M}_j\widetilde{u}$ (resp. $\mathcal{L}_j\widetilde{v}$) in terms of $Z_j u^{NF}$ (resp. $Z_j v^{NF}$) and of q_w, c_w, r_w^{NF} (resp. r_{kg}^{NF}). As done in [29], we first introduce the lagrangian

$$
\Lambda_{kg} = \Big\{(x,\xi) : x - \frac{\xi}{\langle\xi\rangle} = 0\Big\}
$$

which is the graph of $\xi = -d\phi(x)$, with $\phi(x) = \sqrt{1-|x|^2}$, and decompose $\widetilde{v}$ into the sum of a contribution micro-localised on a neighbourhood of size $\sqrt{h}$ of Λ_{kg}, and another one micro-localised out of that neighbourhood (in the spirit of [14]). The second contribution can be basically estimated in L^∞ by $h^{\frac{1}{2}-0}$ times the L^2 norm of some iterates of operator $\mathcal{L}$ acting on $\widetilde{v}$ (which are controlled by the L^2 hypothesis in theorem 2). The main contribution to $\widetilde{v}$ is then represented by $\widetilde{v}_{\Lambda_{kg}}$, which appears to be solution to

$$
[D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle)]\widetilde{v}_{\Lambda_{kg}} = \text{ controlled terms.}
$$

Developing the symbol in the above left hand side on Λ_{kg} we finally obtain the wished ODE, which combined with the a-priori estimate of the "controlled terms" allows us to deduce from (6) the second estimate in (7) (with $\rho = 0$, the general case being treated in the same way up to few more technicalities).

The same strategy is employed to obtain some uniform estimates on $\widetilde{u}$. We introduce the lagrangian

$$
\Lambda_w = \Big\{(x,\xi) : x - \frac{\xi}{|\xi|} = 0\Big\}
$$

which, differently from Λ_{kg}, is not a graph but projects on the basis as an hypersurface. For this reason, the classical problem associated to the first equation in (10) is rather a transport equation than an ordinary differential equation. It is obtained in a similar way by decomposing $\widetilde{u}$ into two contributions: one denoted by $\widetilde{u}_{\Lambda_w}$ and micro-localised in a neighbourhood of size $h^{\frac{1}{2}-\sigma}$ (for some small $\sigma > 0$) of Λ_w; another one micro-localised away from this neighbourhood. As for the Klein-Gordon component, this latter contribution can be easily controlled thanks to the L^2 estimates that the last two inequalities in (6) infer on the iterates of $\mathcal{M}_j$ acting on $\widetilde{u}$. By micro-localisation we derive that $\widetilde{u}_{\Lambda_w}$ satisfies

$$
[D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|)]\widetilde{u}_{\Lambda_w} = \text{ controlled terms,}
$$

and by developing symbol $x\cdot\xi - |\xi|$ on Λ_w we obtain the wished transport equation. Integrating this equation by the method of characteristics, we finally recover the first estimate in (6) and conclude the proof of theorem 2.

Chapter 1

Main Theorem and Preliminary Results

1.1 Statement of the main theorem

NOTATIONS: We warn the reader that, throughout the paper, we will often denote ∂_t (resp. ∂_{x_j}, $j = 1, 2$) by ∂_0 (resp. ∂_j, $j = 1, 2$), while symbol ∂ without any subscript will stand for one of the three derivatives ∂_a, $a = 0, 1, 2$. $\nabla_x f$ is the classical spatial gradient of f, $D := -i\partial$ and R_j denotes the Riesz operator $D_j|D_x|^{-1}$, for $j = 1, 2$. We will also employ notation $\|\partial_{t,x} w\|$ with the meaning $\|\partial_t w\| + \|\partial_x w\|$ and $\|\mathrm{R}w\| = \sum_j \|\mathrm{R}_j w\|$.

We consider the following quadratic, quasi-linear, coupled wave-Klein-Gordon system

$$\left\{ \begin{array}{l} (\partial_t^2 - \Delta_x)u(t,x) = Q_0(v, \partial_1 v)\,, \\ (\partial_t^2 - \Delta_x + 1)v(t,x) = Q_0(v, \partial_1 u)\,, \end{array} \right. \qquad (t,x) \in]1, +\infty[\times\mathbb{R}^2 \tag{1.1.1}$$

with initial conditions

$$\left\{ \begin{array}{l} (u,v)(1,x) = \varepsilon(u_0(x), v_0(x))\,, \\ (\partial_t u, \partial_t v)(1,x) = \varepsilon(u_1(x), v_1(x))\,, \end{array} \right. \tag{1.1.2}$$

where $\varepsilon > 0$ is a small parameter, and Q_0 is the null form:

$$Q_0(v,w) = (\partial_t v)(\partial_t w) - (\nabla_x v)\cdot(\nabla_x w)\,. \tag{1.1.3}$$

Our aim is to prove that there is a unique solution to Cauchy problem (1.1.1)-(1.1.2) provided that ε is sufficiently small and u_0, v_0, u_1, v_1 decay rapidly enough at infinity. The theorem we are going to demonstrate is the following:

Theorem 1.1.1 (Main Theorem)**.** *There exist an integer n sufficiently large and $\varepsilon_0 \in]0,1[$ sufficiently small such that, for any $\varepsilon \in]0, \varepsilon_0[$, any real valued u_0, v_0, u_1, v_1 satisfying:*

$$\begin{array}{c} \|\nabla_x u_0\|_{H^n} + \|v_0\|_{H^{n+1}} + \|u_1\|_{H^n} + \|v_1\|_{H^n} \leq 1, \\ \displaystyle\sum_{|\alpha|=1}^{2} \left(\|x^\alpha \nabla_x u_0\|_{H^{|\alpha|}} + \|x^\alpha v_0\|_{H^{|\alpha|+1}} + \|x^\alpha u_1\|_{H^{|\alpha|}} + \|x^\alpha v_1\|_{H^{|\alpha|}}\right) \leq 1, \end{array} \tag{1.1.4}$$

system (1.1.1)-(1.1.2) admits a unique global solution (u,v) with $\partial_{t,x} u \in C^0\left([1,\infty[; H^n(\mathbb{R}^2)\right)$, $v \in C^0\left([1,\infty[; H^{n+1}(\mathbb{R}^2)\right) \cap C^1\left([1,\infty[; H^n(\mathbb{R}^2)\right)$.

The proof of the main theorem is based on the introduction of four new functions u_+, u_-, v_+, v_-, defined in terms of u, v as follows:

$$\text{(1.1.5)} \qquad \begin{cases} u_+ := (D_t + |D_x|)u\,, \\ u_- := (D_t - |D_x|)u\,, \end{cases} \qquad \begin{cases} v_+ := (D_t + \langle D_x\rangle)v\,, \\ v_- := (D_t - \langle D_x\rangle)v\,, \end{cases}$$

and on the propagation of some a-priori estimates made on them in some interval $[1, T]$, for a fixed $T > 1$. In order to state this result we consider the admissible Klainerman vector fields for the wave-Klein-Gordon system:

$$\text{(1.1.6)} \qquad \Omega := x_1\partial_2 - x_2\partial_1\,, \quad Z_j := x_j\partial_t + t\partial_j\,, j = 1, 2$$

and denote by Γ a generic vector field in $\mathcal{Z} = \{\Omega, Z_j, \partial_j, j = 1, 2\}$. If $\mathcal{Z}$ is assumed ordered, i.e.

$$\text{(1.1.7)} \qquad \begin{gathered} \mathcal{Z} = \{\Gamma_1, \dots, \Gamma_5\} \\ \text{with} \quad \Gamma_1 = \Omega, \quad \Gamma_j = Z_{j-1} \text{ for } j = 2, 3, \quad \Gamma_j = \partial_{j-3} \text{ for } j = 4, 5, \end{gathered}$$

then for a multi-index $I = (i_1, \dots, i_n)$, $i_j \in \{1, \dots, 5\}$ for $j = 1, \dots, n$, we define the *length* of I as $|I| := n$, and $\Gamma^I := \Gamma_{i_1} \cdots \Gamma_{i_n}$ the product of vector fields $\Gamma_{i_j} \in \mathcal{Z}$, $j = 1, \dots, n$.

Vector fields Γ have two relevant properties: they act like derivations on non-linear terms; they exactly commute with the linear part of both wave and Klein-Gordon equation. This is the reason why we exclude of our consideration the scaling vector field $S = t\partial_t + \sum_j x_j\partial_j$, which is always considered in the so-called *Klainerman vector fields' method* for the wave equation, as it does not commute with the Klein-Gordon operator.

We also introduce the energy of (u_+, u_-, v_+, v_-) at time $t \geq 1$ as

$$\text{(1.1.8)} \qquad E_0(t; u_\pm, v_\pm) := \int \left(|u_+(t,x)|^2 + |u_-(t,x)|^2 + |v_+(t,x)|^2 + |u_-(t,x)|^2\right) dx,$$

together with the generalized energies

$$\text{(1.1.9a)} \qquad E_n(t; u_\pm, v_\pm) := \sum_{|\alpha| \leq n} E_0(t; D_x^\alpha u_\pm, D_x^\alpha v_\pm), \quad \forall n \in \mathbb{N}, n \geq 3,$$

and

$$\text{(1.1.9b)} \qquad E_3^k(t; u_\pm, v_\pm) := \sum_{\substack{|\alpha| + |I| \leq 3 \\ |I| \leq 3-k}} E_0(t; D_x^\alpha u_\pm^I; D_x^\alpha v_\pm^I), \quad 0 \leq k \leq 2,$$

where, for any multi-index I,

$$\text{(1.1.10)} \qquad u_\pm^I := (D_t \pm |D_x|)\Gamma^I u, \quad v_\pm^I := (D_t \pm \langle D_x\rangle)\Gamma^I v.$$

Energy $E_n(t; u_\pm, v_\pm)$, for $n \geq 3$, is introduced with the aim of controlling the Sobolev norm H^n of $u_\pm, v_\pm$ for large values of n. The reason of dealing with $E_3^k(t; u_\pm, v_\pm)$ is, instead, to control the L^2 norm of $\Gamma^I u_\pm, \Gamma^I v_\pm$, for any general $\Gamma \in \mathcal{Z}$ and $|I| \leq 3$. In particular, superscript k indicates that we are considering only products Γ^I containing at most $3 - k$ vector fields in $\{\Omega, Z_m, m = 1, 2\}$. For instance, the L^2 norms of $\Omega^3 u_\pm, \Omega Z_1^2 v_\pm$ are bounded by $E_3^0(t; u_\pm, v_\pm)$ but not by $E_3^1(t; u_\pm, v_\pm)$, while the L^2 norms of $Z_1^2 u_\pm, \partial_2 \Omega Z_2 v_\pm$ are controlled by both $E_3^1(t; u_\pm, v_\pm), E_3^0(t; u_\pm, v_\pm)$, etc. The interest of distinguishing between $k = 0, 1, 2$, is to take into account the different growth in time of the L^2 norm of such terms depending on the number of vector fields Ω, Z_m acting on $u_\pm, v_\pm$, as emerges from a-priori estimate (1.1.11d).

Theorem 1.1.2 (Bootstrap Argument). *Let $K_1, K_2 > 1$ and $H^{\rho,\infty}$ be the space defined in 1.2.1 (iii). There exist two integers $n \gg \rho$ sufficiently large, some $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1$ small, two constants $A, B > 1$ sufficiently large and $\varepsilon_0 \in]0, (2A+B)^{-1}[$ such that, for any $0 < \varepsilon < \varepsilon_0$, if (u, v) is solution to (1.1.1)-(1.1.2) on some interval $[1, T]$, for a fixed $T > 1$, and $u_\pm, v_\pm$ defined in (1.1.5) satisfy:*

$$\|u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} \leq A\varepsilon t^{-\frac{1}{2}}, \tag{1.1.11a}$$

$$\|v_\pm(t,\cdot)\|_{H^{\rho,\infty}} \leq A\varepsilon t^{-1}, \tag{1.1.11b}$$

$$E_n(t; u_\pm, v_\pm)^{\frac{1}{2}} \leq B\varepsilon t^{\frac{\delta}{2}}, \tag{1.1.11c}$$

$$E_3^k(t; u_\pm, v_\pm)^{\frac{1}{2}} \leq B\varepsilon t^{\frac{\delta_k}{2}}, \quad \forall\ 0 \leq k \leq 2, \tag{1.1.11d}$$

for every $t \in [1, T]$, then in the same interval they verify also

$$\|u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} \leq \frac{A}{K_1}\varepsilon t^{-\frac{1}{2}}, \tag{1.1.12a}$$

$$\|v_\pm(t,\cdot)\|_{H^{\rho,\infty}} \leq \frac{A}{K_1}\varepsilon t^{-1}, \tag{1.1.12b}$$

$$E_n(t; u_\pm, v_\pm)^{\frac{1}{2}} \leq \frac{B}{K_2}\varepsilon t^{\frac{\delta}{2}}, \tag{1.1.12c}$$

$$E_3^k(t; u_\pm, v_\pm)^{\frac{1}{2}} \leq \frac{B}{K_2}\varepsilon t^{\frac{\delta_k}{2}}, \quad \forall\ 0 \leq k \leq 2. \tag{1.1.12d}$$

The a-priori estimates on the uniform norm of $u_\pm, \mathrm{R}u_\pm, v_\pm$ made in the above theorem translate in terms of $u_\pm, v_\pm$ the sharp decay in time we expect for the solution (u, v) to starting problem (1.1.1). Indeed, from definitions (1.1.5) it appears that

$$D_t u = \frac{u_+ + u_-}{2}, \quad D_x u = \mathrm{R}\left(\frac{u_+ - u_-}{2}\right),$$
$$D_t v = \frac{v_+ + v_-}{2}, \quad v = \langle D_x\rangle^{-1}\left(\frac{v_+ - v_-}{2}\right),$$

so (1.1.11a), (1.1.11b) imply

$$\|\partial_{t,x} u(t,\cdot)\|_{H^{\rho,\infty}} \leq A\varepsilon t^{-\frac{1}{2}}, \quad \|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + \|v(t,\cdot)\|_{H^{\rho+1,\infty}} \leq A\varepsilon t^{-1}.$$

Furthermore, the following quantity

$$\|\partial_t u(t,\cdot)\|_{H^n} + \|\nabla_x u(t,\cdot)\|_{H^n} + \|\partial_t v(t,\cdot)\|_{H^n} + \|\nabla_x v(t,\cdot)\|_{H^n} + \|v(t,\cdot)\|_{H^n}$$

is equivalent to the square root of $E_n(t; u_\pm, v_\pm)$, which implies that the propagation of a-priori energy estimate (1.1.11c) is equivalent to the propagation of a certain estimate on the above Sobolev norms. For this reason, the propagation of the a-priori estimate on $E_n(t; u_\pm, v_\pm)$ and a local existence argument will imply theorem 1.1.1.

Before ending this section and going into the core of the subject, we briefly remind the general definition of *null condition* for a multilinear form on $\mathbb{R}^{1+n}$ and a result by Hörmander (see [11]).

Definition 1.1.3. A k-linear form G on $\mathbb{R}^{1+n}$ is said to satisfy the *null condition* if and only if, for all $\xi \in \mathbb{R}^n, \xi = (\xi_0, \ldots, \xi_n)$ such that $\xi_0^2 - \sum_{j=1}^n \xi_j^2 = 0$,

$$G(\underbrace{\xi, \ldots, \xi}_{k}) = 0. \tag{1.1.13}$$

EXAMPLE: The trilinear form $\xi_0^2\xi_a - \sum_{j=1,2}\xi_j^2\xi_a$ associated to $Q_0(v,\partial_a w)$ satisfies the null condition (1.1.13), for any $a=0,1,2$. This is the most common example of null form.

Lemma 1.1.4 (Hörmander [10], Lemma 6.6.5.)**.** *Let G be a k-linear form on $\mathbb{R}^{1+n}$, $k=k_1+\cdots+k_r$, with k_j positive integers, and $\Gamma\in\mathfrak{Z}$. For all $u_j\in C^{k+1}(\mathbb{R}^{1+n})$, all $\alpha_j\in\mathbb{N}^{1+n}$, $|\alpha_j|=k_j$, and $u_j^{(k_j)}:=\partial^{\alpha_j}u_j$,*

$$\begin{aligned}\Gamma G(u_1^{(k_1)},\dots,u_r^{(k_r)}) &= G((\Gamma u_1)^{(k_1)},\dots,u_r^{(k_r)})+\dots\\ &\quad + G(u_1^{(k_1)},\dots,(\Gamma u_r)^{(k_r)}) + G_1(u_1^{(k_1)},\dots,u_r^{(k_r)}),\end{aligned}\tag{1.1.14}$$

where G_1 satisfies the null condition.

Remark 1.1.5. Previous lemma simplifies when the multi-linear form G satisfying the null condition is $Q_0(v,\partial_a w)$, for any $a=0,1,2$. Indeed, the structure of the null form is not modified by the action of vector field Γ in the sense that

$$\Gamma Q_0(v,\partial_a w)=Q_0(\Gamma v,\partial_a w)+Q_0(v,\partial_a\Gamma w)+G_1(v,\partial w),\tag{1.1.15}$$

where $G_1(v,\partial w)=0$ if $\Gamma=\partial_m$, $m=1,2$, and

$$G_1(v,\partial w)=\begin{cases}-Q_0(v,\partial_m w), & \text{if } a=0,\Gamma=Z_m, m\in\{1,2\},\\ 0, & \text{if } a=0,\Gamma=\Omega,\\ -Q_0(v,\partial_t w), & \text{if } a\neq 0,\Gamma=Z_a,\\ 0, & \text{if } a\neq 0,\Gamma=Z_m, m\in\{1,2\}\setminus\{a\},\\ (-1)^a Q_0(v,\partial_m w), & \text{with } m\in\{1,2\}\setminus\{a\},\ \text{if } a\neq 0,\Gamma=\Omega.\end{cases}\tag{1.1.16}$$

If Γ^I contains at least $k\leq|I|$ space derivatives then

$$\Gamma^I Q_0(v,\partial_1 w)=\sum_{|I_1|+|I_2|=|I|} Q_0(\Gamma^{I_1}v,\partial_1\Gamma^{I_2}w)+\sum_{k\leq|I_1|+|I_2|<|I|} c_{I_1,I_2}Q_0(\Gamma^{I_1}v,\partial\Gamma^{I_2}w),\tag{1.1.17}$$

with $c_{I_1,I_2}\in\{-1,0,1\}$. In the above equality we should think of multi-index I_1 (resp. I_2) as obtained by extraction of a $|I_1|$-tuple (resp. $|I_2|$-tuple) from $I=(i_1,\dots i_n)$, in such a way that each i_j appearing in I and corresponding to a spatial derivative (e.g. $\Gamma_{i_j}=D_m$, for $m\in\{1,2\}$), appears either in I_1 or in I_2, but not in both. For further references, we define

$$\mathfrak{I}(I):=\{(I_1,I_2)|I_1,I_2 \text{ multi-indices obtained as described above}\}.\tag{1.1.18}$$

1.2 Preliminary Results

The aim of this section is to introduce most of the technical tools that will be used throughout the paper. In particular, subsections 1.2.1 and 1.2.2 are devoted to recall some definitions and results about paradifferential and pseudo-differential calculus respectively; subsection 1.2.3 and 1.2.4 are dedicated to the introduction of some special operators that we will frequently use when dealing with the wave and the Klein-Gordon component. Subsections 1.2.1, 1.2.2 barely contain proofs (we refer for that to [3], [27], [8], [32]), whereas subsections 1.2.3, 1.2.4 are much longer and richer in proofs and technicalities.

1.2.1 Paradifferential calculus

In the current subsection we recall some definitions and properties that will be useful in chapter 2. We first recall the definition of some spaces (Sobolev, Lipschitz and Hölder spaces) in dimension $d \geq 1$ and afterwards some results concerning symbolic calculus and the action of paradifferential operators on Sobolev spaces (see for instance [27]). We warn the reader that we will use both notations $\hat{w}(\xi)$ and $\mathcal{F}_{x\mapsto\xi}w$ for the Fourier transform of a function $w = w(x)$.

Definition 1.2.1 (Spaces)**.** (i) Let $s \in \mathbb{R}$. $H^s(\mathbb{R}^d)$ denotes the space of tempered distributions $w \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{w} \in L^2_{loc}(\mathbb{R}^d)$ and

$$\|w\|^2_{H^s(\mathbb{R}^d)} := \frac{1}{(2\pi)^d}\int (1+|\xi|^2)^s |\hat{w}(\xi)|^2 d\xi < +\infty;$$

(ii) For $\rho \in \mathbb{N}$, $W^{\rho,\infty}(\mathbb{R}^d)$ denotes the space of distributions $w \in \mathcal{D}'(\mathbb{R}^d)$ such that $\partial_x^\alpha w \in L^\infty(\mathbb{R}^d)$, for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq \rho$, endowed with the norm

$$\|w\|_{W^{\rho,\infty}} := \sum_{|\alpha|\leq\rho} \|\partial_x^\alpha w\|_{L^\infty};$$

(iii) For $\rho \in \mathbb{N}$, we also introduce $H^{\rho,\infty}(\mathbb{R}^d)$ as the space of tempered distributions $w \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|w\|_{H^{\rho,\infty}} := \|\langle D_x\rangle^\rho w\|_{L^\infty} < +\infty.$$

Definition 1.2.2. An operator T is said of order $\leq m \in \mathbb{R}$ if it is a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.

Definition 1.2.3 (Smooth symbols)**.** Let $m \in \mathbb{R}$.

(i) $S^m_0(\mathbb{R}^d)$ denotes the space of functions $a(x,\eta)$ on $\mathbb{R}^d \times \mathbb{R}^d$ which are C^∞ with respect to η and such that, for all $\alpha \in \mathbb{N}^d$, there exists a constant $C_\alpha > 0$ and

$$\|\partial_\eta^\alpha a(\cdot,\eta)\|_{L^\infty} \leq C_\alpha(1+|\eta|)^{m-|\alpha|}, \qquad \forall \eta \in \mathbb{R}^d.$$

$\Sigma^m_0(\mathbb{R}^d)$ denotes the subclass of symbols $a \in S^m_0(\mathbb{R}^d)$ satisfying

$$\exists \varepsilon < 1 \ : \mathcal{F}_{x\to\xi} a(\xi,\eta) = 0 \quad \text{for } |\xi| > \varepsilon(1+|\eta|). \tag{1.2.1}$$

S^m_0 is equipped with seminorm $M^m_0(a;n)$ given by

$$M^m_0(a;n) = \sup_{|\beta|\leq n}\sup_{\eta\in\mathbb{R}^2} \left\|(1+|\eta|)^{|\beta|-m}\partial_\eta^\beta a(\cdot,\eta)\right\|_{L^\infty}. \tag{1.2.2}$$

(ii) For $r \in \mathbb{N}$, $S^m_r(\mathbb{R}^d)$ denotes more generally the space of symbols $a \in S^m_0(\mathbb{R}^d)$ such that, for all $\alpha \in \mathbb{N}^d$ and all $\eta \in \mathbb{R}^d$, function $x \to \partial_\eta^\alpha a(x,\eta)$ belongs to $W^{r,\infty}(\mathbb{R}^d)$ and there exists a constant $C_\alpha > 0$ such that

$$\|\partial_\eta^\alpha a(\cdot,\eta)\|_{W^{r,\infty}} \leq C_\alpha(1+|\eta|)^{m-|\alpha|}, \qquad \forall \eta \in \mathbb{R}^d.$$

$\Sigma^m_r(\mathbb{R}^d)$ denotes the subclass of symbols $a \in S^m_r(\mathbb{R}^d)$ satisfying the spectral condition (1.2.1). S^m_r is equipped with seminorm $M^m_r(a;n)$, given by

$$M^m_r(a;n) = \sup_{|\beta|\leq n}\sup_{\eta\in\mathbb{R}^2} \left\|(1+|\eta|)^{|\beta|-m}\partial_\eta^\beta a(\cdot,\eta)\right\|_{W^{r,\infty}}. \tag{1.2.3}$$

These definitions extend to matrix valued symbols $a \in S^m_r$ ($a \in \Sigma^m_r$), $m \in \mathbb{R}$, $r \in \mathbb{N}$. If $a \in S^m_r$ (resp. $a \in \Sigma^m_r$), it is said *of order* m.

Definition 1.2.4. An admissible cut-off function $\psi(\xi,\eta)$ is a C^∞ function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

(i) there are $0 < \varepsilon_1 < \varepsilon_2 < 1$ and

$$\begin{cases} \psi(\xi,\eta) = 1, & \text{for } |\xi| \le \varepsilon_1(1+|\eta|) \\ \psi(\xi,\eta) = 0, & \text{for } |\xi| \ge \varepsilon_2(1+|\eta|); \end{cases} \tag{1.2.4}$$

(ii) for all $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d$ there is a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial^\alpha_\xi \partial^\beta_\eta \psi(\xi,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{-|\alpha|-|\beta|}, \quad \forall (\xi,\eta). \tag{1.2.5}$$

EXAMPLE: If χ is a smooth cut-off function such that $\chi(z) = 1$ for $|z| \le \varepsilon_1$ and is supported in the open ball $B_{\varepsilon_2}(0)$, with $0 < \varepsilon_1 < \varepsilon_2 < 1$, function $\psi(\xi,\eta) := \chi\big(\frac{\xi}{\langle\eta\rangle}\big)$ is an admissible cut-off function in the sense of definition 1.2.4. We will only consider this type of admissible cut-off functions for the rest of the paper and refer (abusively) to χ itself as an admissible cut-off.

Definition 1.2.5. Let χ be an admissible cut-off function and $a(x,\eta) \in S^m_r$, $m \in \mathbb{R}, r \in \mathbb{N}$. The *Bony quantization* (or *paradifferential quantization*) $Op^B(a(x,\eta))$ associated to symbol a and acting on a test function w is defined as

$$\begin{aligned} &Op^B(a(x,\eta))w(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\eta}\sigma^\chi_a(x,\eta)\hat{w}(\eta)d\eta\,, \\ &\text{with } \sigma^\chi_a(x,\eta) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\zeta}\chi\left(\frac{\zeta}{\langle\eta\rangle}\right)a(y,\eta)dyd\zeta\,. \end{aligned}$$

The operator defined above depends on the choice of the admissible cut-off function χ. However, if $a \in S^m_r$ for some $m \in \mathbb{R}, r \in \mathbb{N}$, a change of χ modifies $Op^B(a)$ only by the addition of a r-smoothing operator (i.e. an operator which is bounded from H^s to H^{s+r}, see [3]), so the choice of χ will be substantially irrelevant as long as we can neglect r-smoothing operators. For this reason, we will not indicate explicitly the dependence of Op^B (resp. of σ^χ_a) on χ to keep notations as light as possible. Let us also observe that, with such a definition, the Fourier transform of $Op^B(a)w$ has the following simple expression

$$\mathcal{F}_{x\to\xi}\Big(Op^B(a(x,\eta))w(x)\Big)(\xi) = \frac{1}{(2\pi)^d}\int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)\hat{a}_y(\xi-\eta,\eta)\hat{w}(\eta)d\eta\,, \tag{1.2.6}$$

where $\hat{a}_y(\xi,\eta) := \mathcal{F}_{y\to\xi}\big(a(y,\eta)\big)$, and the product of two functions u, v can be developed as

$$uv = Op^B(u)v + Op^B(v)u + R(u,v), \tag{1.2.7}$$

where remainder $R(u,v)$ writes on the Fourier side as

$$\widehat{R(u,v)}(\xi) = \frac{1}{(2\pi)^d}\int\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)\widehat{u}(\xi-\eta)\widehat{v}(\eta)d\eta. \tag{1.2.8}$$

We remark that frequencies η and $\xi-\eta$ in the above integral are either bounded or equivalent, and $R(u,v) = R(v,u)$. With the aim of having uniform notations, we introduce the operator Op^B_R associated to a symbol $a(x,\eta)$ and acting on a function w as

$$\begin{aligned} &Op^B_R(a(x,\eta))w(x) := \frac{1}{(2\pi)^d}\int e^{ix\cdot\eta}\delta^\chi_a(x,\eta)\hat{w}(\eta)d\eta\,, \\ &\text{with } \delta^\chi_a(x,\eta) := \frac{1}{(2\pi)^d}\int e^{i(x-y)\cdot\zeta}\left(1-\chi\left(\frac{\zeta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\zeta\rangle}\right)\right)a(y,\eta)dyd\zeta\,. \end{aligned} \tag{1.2.9}$$

For future references, we recall the definition of the Littlewood-Paley decomposition of a function w.

Definition 1.2.6 (Littlewood-Paley decomposition)**.** Let $\chi : \mathbb{R}^2 \to [0,1]$ be a smooth decaying radial function, supported for $|x| \leq 2 - \frac{1}{10}$ and identically equal to 1 for $|x| \leq 1 + \frac{1}{10}$. Let also $\varphi(\xi) := \chi(\xi) - \chi(2\xi) \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, supported for $\frac{1}{2} < |\xi| < 2$, and $\varphi_k(\xi) := \varphi(2^{-k}\xi)$ for all $k \in \mathbb{N}^*$, with the convention that $\varphi_0 := \chi$. Then $\sum_{k\in\mathbb{N}} \varphi(2^{-k}\xi) = 1$, and for any $w \in \mathcal{S}'(\mathbb{R}^d)$

$$w = \sum_{k\in\mathbb{N}} \varphi_k(D_x)w \tag{1.2.10}$$

is the Littlewood-Paley decomposition of w.

The following proposition is a classical result about the action of para-differential operators on Sobolev spaces (see [3] for further details). Proposition 1.2.8 shows, instead, that some results of continuity over L^2 hold also for operators whose symbol $a(x,\eta)$ is not a smooth function of η, and that map $(u,v) \mapsto R(u,v)$ is continuous from $H^{4,\infty} \times L^2$ to L^2.

Proposition 1.2.7 (Action)**.** *Let $m \in \mathbb{R}$. For all $s \in \mathbb{R}$ and $a \in S_0^m$, $Op^B(a)$ is a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$. In particular,*

$$\|Op^B(a)w\|_{H^s} \lesssim M_0^m\Big(a; \Big[\frac{d}{2}\Big] + 1\Big)\|w\|_{H^{s+m}}. \tag{1.2.11}$$

Proposition 1.2.8. *(i) Let $a(x,\eta) = a_1(x)b(\eta)$, with $a_1 \in L^\infty(\mathbb{R}^2)$ and $b(\eta)$ bounded, supported in some ball centred in the origin and such that $|\partial^\alpha b(\eta)| \lesssim_\alpha |\eta|^{-|\alpha|+1}$ for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \geq 1$. Then $Op^B(a(x,\eta)) : L^2 \to L^2$ is bounded and for any $w \in L^2(\mathbb{R}^2)$*

$$\|Op^B(a(x,\eta))w\|_{L^2} \lesssim \|a_1\|_{L^\infty}\|w\|_{L^2}.$$

The same result is true for $Op_R^B(a(x,\eta))$;

(ii) Map $(u,v) \in H^{4,\infty} \times L^2 \mapsto R(u,v) \in L^2$ is well defined and continuous.

Proof. As concerns (i) we have that

$$Op^B(a(x,\eta))w(x) = \int K(x-z, x-y)a_1(y)w(z)dydz$$

with

$$K(x,y) := \frac{1}{(2\pi)^4}\int e^{ix\cdot\eta + iy\cdot\zeta}\chi\Big(\frac{\zeta}{\langle\eta\rangle}\Big)b(\eta)d\eta d\zeta$$

and χ is an admissible cut-off function. After the hypothesis on b we have that for every $\alpha, \beta \in \mathbb{N}^2$,

$$\Big|\partial_\zeta^\beta\Big[\chi\Big(\frac{\zeta}{\langle\eta\rangle}\Big)b(\eta)\Big]\Big| \lesssim \mathbb{1}_{\{|\eta|\lesssim 1\}}|g_\beta(\zeta)|,$$

$$\Big|\partial_\eta^\alpha\partial_\zeta^\beta\Big[\chi\Big(\frac{\zeta}{\langle\eta\rangle}\Big)b(\eta)\Big]\Big| \lesssim \mathbb{1}_{\{|\eta|\lesssim 1\}}|\eta|^{1-|\alpha|}|g_\beta(\zeta)|, \qquad |\alpha| \geq 1,$$

for some bounded and compactly supported functions g_β. Lemma A.1 (i) and corollary A.2 (i) of appendix A imply that $|K(x,y)| \lesssim |x|^{-1}\langle x\rangle^{-2}\langle y\rangle^{-3}$ for any (x,y), and statement (i) follows by an inequality such as (A.8) with $L = L^2$.

In order to prove assertion (ii) we consider a cut-off function $\psi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in some closed ball $\overline{B_C(0)}$, for a $C \gg 1$, and decompose $R(u,v)$ as follows, using (1.2.8):

$$R(u,v) = \int K_0(x-y, y-z)u(y)v(z)dydz + \int K_1(x-y, y-z)[\langle D_x\rangle^4 u](y)v(z)dydz,$$

with

$$K_0(x,y) = \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi+iy\cdot\eta}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)\psi(\eta)d\xi d\eta,$$
$$K_1(x,y) = \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi+iy\cdot\eta}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\psi)(\eta)\langle\xi-\eta\rangle^{-4}d\xi d\eta.$$

Since frequencies ξ,η are both bounded on the support of $\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)\psi(\eta)$, one can show through some integration by parts that $|K_0(x,y)|\lesssim\langle x\rangle^{-3}\langle y\rangle^{-3}$ for any (x,y), to then deduce that

$$\left\|\int K_0(x-y,y-z)u(y)v(z)dydz\right\|_{L^2(dx)}\lesssim\|u\|_{L^\infty}\|v\|_{L^2}.$$

Kernel $K_1(x,y)$ can be split using a Littlewood-Paley decomposition as follows

$$K_1(x,y)=\sum_{k\geq1}\frac{1}{(2\pi)^2}\underbrace{\int e^{ix\cdot\xi+iy\cdot\eta}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\psi)(\eta)\varphi(2^{-k}\eta)\langle\xi-\eta\rangle^{-4}d\xi d\eta}_{K_{1,k}(x,y)},$$

for a suitable $\varphi\in C_0^\infty(\mathbb{R}^2\backslash\{0\})$. On the support of $\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\psi)(\eta)\varphi(2^{-k}\eta)$, frequencies $\eta,\xi-\eta$ are either bounded or equivalent and of size 2^k (which implies in particular that $\langle\xi-\eta\rangle^{-4}\lesssim\langle\xi\rangle^{-3}\langle\eta\rangle^{-1}$). After a change of coordinates and some integration by parts one can show that $|K_{1,k}(x,y)|\lesssim 2^k\langle x\rangle^{-3}\langle 2^k y\rangle^{-3}$, for any $k\geq1$, and therefore that

$$\begin{aligned}&\left\|\int e^{i(x-y)\cdot\xi+i(y-z)\cdot\eta}K_1(x-y,y-z)[\langle D_x\rangle^4u](y)v(z)dydz\right\|_{L^2(dx)}\\ \lesssim&\sum_{k\geq1}2^k\left\|\int\langle x-y\rangle^{-3}\langle 2^k(y-z)\rangle^{-3}|\langle D_x\rangle^4u(y)||w(z)|dydz\right\|_{L^2(dx)}\\ \lesssim&\sum_{k\geq1}2^k\int\langle y\rangle^{-3}\langle 2^kz\rangle^{-3}\|[\langle D_x\rangle^4u](\cdot-y)w(\cdot-y-z)\|_{L^2(dx)}dydz\lesssim\|u\|_{H^{4,\infty}}\|w\|_{L^2},\end{aligned}$$

which concludes the proof of statement (ii). □

The last results of this subsection are stated without proofs. All the details can be found in chapter 6 of [27] (see theorems 6.1.1, 6.1.4, 6.2.1, 6.2.4).

Proposition 1.2.9 (Composition)**.** *Consider $a\in S^m_r$, $b\in S^{m'}_r$, $r\in\mathbb{N}^*$, $m,m'\in\mathbb{R}$.*

(i) Symbol $a\sharp b:=\sum_{|\alpha|<r}\frac{1}{\alpha!}\partial^\alpha_\xi a(x,\xi)D^\alpha_x b(x,\xi)$ is well defined in $\sum_{j<r}S^{m+m'-j}_{r-j}$;

(ii) $Op^B(a)Op^B(b)-Op^B(a\sharp b)$ is an operator of order $\leq m+m'-r$, and for all $s\in\mathbb{R}$, there exists a constant $C>0$ such that, for all $a\in S^m_r(\mathbb{R}^d)$, $b\in S^{m'}_r(\mathbb{R}^d)$, and $w\in H^{s+m+m'-r}(\mathbb{R}^d)$,

$$\begin{aligned}&\|Op^B(a)Op^B(b)w-Op^B(a\sharp b)w\|_{H^s}\\ &\qquad\leq C\big(M^m_r(a;n)M^{m'}_0(b;n_0)+M^m_0(a;n)M^{m'}_r(b;n_0)\big)\|w\|_{H^{s+m+m'-r}},\end{aligned}$$

where $n_0=\left[\frac{d}{2}\right]+1$, $n=n_0+r$. Moreover, $Op^B(a)Op^B(b)-Op^B(a\sharp b)=\widetilde{\sigma}_r(x,D_x)$ with

$$\begin{aligned}\widetilde{\sigma}_r(x,\xi)=&(\sigma_a\sharp\sigma_b)(x,\xi)-\sigma_{a\sharp b}(x,\xi)\\ &+\sum_{|\alpha|=r}\frac{1}{r!(2\pi)^d}\int e^{i(x-y)\cdot\zeta}\left(\int_0^1\partial^\alpha_\xi\sigma_a(x,\xi+t\zeta)(1-t)^{r-1}dt\right)\theta(\zeta,\xi)D^\alpha_x\sigma_b(y,\xi)dyd\zeta\end{aligned}$$

with $\theta \equiv 1$ in a neighbourhood of the support of $\mathcal{F}_{y\mapsto\eta}\sigma_b(\eta,\xi)$.

These results extend to matrix valued symbols and operators.

Remark 1.2.10. If symbol $a(x,\xi)$ only depends on ξ then $\sigma_a\sharp\sigma_b - \sigma_{a\sharp b} = 0$ and $\widetilde{\sigma}_r$ reduces to the only integral term. Moreover,

$$\mathcal{F}_{x\mapsto\eta}\widetilde{\sigma}_r(\eta,\xi) = \sum_{|\alpha|=r}\frac{1}{r!}\left(\int_0^1 \partial_\xi^\alpha a(\xi+t\eta)(1-t)^{[r]_+-1}dt\right)\chi\Big(\frac{\eta}{\langle\xi\rangle}\Big)\eta^\alpha\hat{b}_y(\eta,\xi), \tag{1.2.12}$$

where $\chi\big(\frac{\eta}{\langle\xi\rangle}\big)$ is the admissible cut-off function defining σ_b.

Corollary 1.2.11. *For $d = 2$ and all $s \in \mathbb{R}$, there exists a constant $C > 0$ such that, for $a \in S^m_r, b \in S^{m'}_r$, $r \in \mathbb{N}^*$, and $w \in H^{s+m+m'-1}$,*

$$\|Op^B(a)Op^B(b)w - Op(ab)w\|_{H^s} \leq C\big(M_1^m(a;3)M_0^{m'}(b;2) + M_0^m(a;3)M_1^{m'}(b;2)\big)\|w\|_{H^{s+m+m'-1}}.$$

Proposition 1.2.12 (Adjoint)**.** *Consider $a \in S^m_r(\mathbb{R}^d)$, denote by $Op^B(a)^*$ the adjoint operator of $Op^B(a)$ and by $a^*(x,\xi) = \overline{a}(x,\xi)$ the complex conjugate of $a(x,\xi)$.*

(i) Symbol $b(x,\xi) := \sum_{|\alpha|<r}\frac{1}{\alpha!}D_x^\alpha\partial_\xi^\alpha a^*(x,\xi)$ is well defined in $\sum_{j<r}S^{m-j}_{r-j}$;

(ii) Operator $Op^B(a)^* - Op^B(b)$ is of order $\leq m-r$. Precisely, for all $s \in \mathbb{R}$ there is a constant $C > 0$ such that, for all $a \in S^m_r(\mathbb{R}^d)$ and $w \in H^{s+m-r}(\mathbb{R}^d)$,

$$\big\|Op^B(a)^*w - Op^B(b)w\big\|_{H^s} \leq CM_r^m(a;n)\|w\|_{H^{s+m-r}},$$

with $n_0 = \big[\frac{d}{2}\big] + 1$, $n = n_0 + r$.

These results extend to matrix valued symbols a, with $a^*(x,\xi)$ denoting the adjoint of matrix $a(x,\xi)$.

Corollary 1.2.13. *For $d = 2$ and all $s \in \mathbb{R}$, there exists a constant $C > 0$ such that, for $a \in S^m_r$, $r \in \mathbb{N}^*$ and $w \in H^{s+m-1}$,*

$$\|Op^B(a)^*w - Op(a^*)w\|_{H^s} \leq CM_1^m(a;3)\|w\|_{H^{s+m-1}}.$$

1.2.2 Semi-classical pseudodifferential calculus

In this subsection we recall some definitions and results about semi-classical symbolic calculus in general space dimension $d \geq 1$ which will be used in section 3.2. We refer the reader to [8] and [32] for more details.

Definition 1.2.14. An order function on $\mathbb{R}^d \times \mathbb{R}^d$ is a smooth map from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}_+$: $(x,\xi) \to M(x,\xi)$ such that there exist $N_0 \in \mathbb{N}$, $C > 0$ and for any $(x,\xi),(y,\eta) \in \mathbb{R}^d \times \mathbb{R}^d$

$$M(y,\eta) \leq C\langle x-y\rangle^{N_0}\langle\xi-\eta\rangle^{N_0}M(x,\xi)\,, \tag{1.2.13}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$.

Definition 1.2.15. Let M be an order function on $\mathbb{R}^d \times \mathbb{R}^d$, $\delta, \sigma \geq 0$. One denotes by $S_{\delta,\sigma}(M)$ the space of smooth functions

$$(x,\xi,h) \to a(x,\xi,h)$$
$$\mathbb{R}^d \times \mathbb{R}^d \times]0,1] \to \mathbb{C}$$

satisfying for any $\alpha_1, \alpha_2 \in \mathbb{N}^d, k, N \in \mathbb{N}$

$$|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}(h\partial_h)^k a(x,\xi,h)| \lesssim M(x,\xi)\, h^{-\delta(|\alpha_1|+|\alpha_2|)}(1+\sigma h^\sigma|\xi|)^{-N}\,. \tag{1.2.14}$$

A key role in this paper will be played by symbols a verifying (1.2.14) with $M(x,\xi) = \langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-N}$, for $N \in \mathbb{N}$ and a certain smooth function $f(\xi)$. This function M is no longer an order function because of the term $h^{-\frac{1}{2}}$, but nevertheless we keep writing $a \in S_{\delta,\sigma}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-N})$.

Definition 1.2.16. In the semi-classical setting we say that $a(x,\xi,h)$ is a symbol *of order* r if $a \in S_{\delta,\sigma}(\langle\xi\rangle^r)$, for some $\delta, \sigma \geq 0$.

Let us observe that when $\sigma > 0$ the symbol decays rapidly in $h^\sigma|\xi|$, which implies the following inclusion for $r \geq 0$:

$$S_{\delta,\sigma}(\langle\xi\rangle^r) \subset h^{-\sigma r} S_{\delta,\sigma}(1).$$

This means that, up to a small loss in h, this type of symbols can be always considered as symbols of order zero. In the rest of the paper we will not indicate explicitly the dependence of symbols on h, referring to $a(x,\xi,h)$ simply as $a(x,\xi)$.

Definition 1.2.17. Let $a \in S_{\delta,\sigma}(M)$ for some order function M, some $\delta, \sigma \geq 0$.

(i) We can define the *Weyl quantization* of a to be the operator $\mathrm{Op}_h^w(a) = a^w(x, hD)$ acting on $u \in \mathcal{S}(\mathbb{R}^d)$ by the following formula:

$$\mathrm{Op}_h^w(a(x,\xi))u(x) = \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{i}{h}(x-y)\cdot\xi} a\Big(\frac{x+y}{2},\xi\Big)\, u(y)\, dyd\xi\,;$$

(ii) We define also the *standard quantization* of a:

$$Op_h(a(x,\xi))u(x) = \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{i}{h}(x-y)\cdot\xi} a(x,\xi)\, u(y)\, dyd\xi\,.$$

It is clear from the definition that the two quantizations coincide when the symbol does not depend on x. We also introduce a semi-classical version of Sobolev spaces on which the above operators act naturally.

Definition 1.2.18. (i) Let $\rho \in \mathbb{N}$. We define the semi-classical Sobolev space $H_h^{\rho,\infty}(\mathbb{R}^d)$ as the space of tempered distributions w such that $\langle hD\rangle^\rho w := Op_h(\langle\xi\rangle^\rho)w \in L^\infty$, endowed with norm

$$\|w\|_{H_h^{\rho,\infty}} = \|\langle hD\rangle^\rho w\|_{L^\infty};$$

(ii) Let $s \in \mathbb{R}$. We define the semi-classical Sobolev space $H_h^s(\mathbb{R}^d)$ as the space of tempered distributions w such that $\langle hD\rangle^s w := Op_h(\langle\xi\rangle^s)w \in L^2$, endowed with norm

$$\|w\|_{H^s} = \|\langle hD\rangle^s w\|_{L^2}.$$

For future references, we write down the semi-classical Sobolev injection in space dimension 2:

$$\|v_h\|_{H_h^{\rho,\infty}(\mathbb{R}^2)} \lesssim_\sigma h^{-1}\|v_h\|_{H_h^{\rho+1+\sigma}(\mathbb{R}^2)}\,, \qquad \forall\sigma > 0\,. \tag{1.2.15}$$

The following two propositions are stated without proof. They concern the adjoint and the composition of pseudo-differential operators. All related details are provided in chapter 7 of [8] or in chapter 4 of [32].

Proposition 1.2.19 (Self-Adjointness). *If $a(x,\xi)$ is a real symbol its Weyl quantization is self-adjoint, i.e.*

$$\big(\mathrm{Op}_h^w(a)\big)^* = \mathrm{Op}_h^w(a)\,.$$

Proposition 1.2.20 (Composition for Weyl quantization). *Let $a, b \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$\mathrm{Op}_h^w(a) \circ \mathrm{Op}_h^w(b) = \mathrm{Op}_h^w(a\sharp b)\,,$$

where

$$a\sharp b\,(x,\xi) := \frac{1}{(\pi h)^{2d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{2i}{h}\sigma(y,\eta;\,z,\zeta)} a(x+z,\xi+\zeta) b(x+y,\xi+\eta)\; dyd\eta dzd\zeta, \tag{1.2.16}$$

and

$$\sigma(y,\eta;\,z,\zeta) = \eta\cdot z - y\cdot\zeta\,.$$

It is often useful to derive an asymptotic expansion for $a\sharp b$, as it allows easier computations than integral formula (1.2.16). This expansion is usually obtained by applying the stationary phase argument when $a, b \in S_{\delta,\sigma}(M)$, $\delta \in [0, \frac{1}{2}[$ (as shown in [32]). Here we provide an expansion at any order even when one of two symbols belongs to $S_{\frac{1}{2},\sigma_1}(M)$ (still having the other in $S_{\delta,\sigma_2}(M)$ for $\delta < \frac{1}{2}$, and σ_1, σ_2 either equal or, if not, one of them equal to zero), whose proof is based on the Taylor development of symbols a, b, and can be found in the appendix of [29] (for $d = 1$).

Proposition 1.2.21. *Let M_1, M_2 be two order functions and $a \in S_{\delta_1,\sigma_1}(M_1)$, $b \in S_{\delta_2,\sigma_2}(M_2)$, $\delta_1, \delta_2 \in [0, \frac{1}{2}]$, $\delta_1 + \delta_2 < 1$, $\sigma_1, \sigma_2 \geq 0$ such that*

$$\sigma_1 = \sigma_2 \geq 0 \qquad or \qquad \big[\sigma_1 \neq \sigma_2 \text{ and } \sigma_i = 0\,, \sigma_j > 0\,, i \neq j \in \{1,2\}\big]\,. \tag{1.2.17}$$

Then $a\sharp b \in S_{\delta,\sigma}(M_1M_2)$, where $\delta = \max\{\delta_1,\delta_2\}$, $\sigma = \max\{\sigma_1,\sigma_2\}$. Moreover,

$$a\sharp b = \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha|=0,\dots,N-1}} \frac{(-1)^{|\alpha_1|}}{\alpha!}\Big(\frac{h}{2i}\Big)^{|\alpha|} \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)\;\partial_x^{\alpha_2}\partial_\xi^{\alpha_1} b(x,\xi) + r_N\,, \tag{1.2.18}$$

where $r_N \in h^{N(1-(\delta_1+\delta_2))} S_{\delta,\sigma}(M_1M_2)$ and

$$\begin{aligned} r_N(x,\xi) = \Big(\frac{h}{2i}\Big)^N \frac{N}{(\pi h)^{2d}} \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha|=N}} \frac{(-1)^{|\alpha_1|}}{\alpha!} \int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \\ \times \Big(\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+tz,\xi+t\zeta)(1-t)^{N-1}dt\Big)\partial_x^{\alpha_2}\partial_\xi^{\alpha_1} b(x+y,\xi+\eta)\, dyd\eta dzd\zeta\,, \end{aligned} \tag{1.2.19}$$

or

$$\begin{aligned} r_N(x,\xi) = \Big(\frac{h}{2i}\Big)^N \frac{N}{(\pi h)^{2d}} \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha|=N}} \frac{(-1)^{|\alpha_1|}}{\alpha!} \int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+z,\xi+\zeta) \\ \times \Big(\int_0^1 \partial_x^{\alpha_2}\partial_\xi^{\alpha_1} b(x+ty,\xi+t\eta)(1-t)^{N-1}dt\Big)\, dyd\eta dzd\zeta\,. \end{aligned} \tag{1.2.20}$$

More generally, if $h^{N\delta_1}\partial^\alpha a \in S_{\delta_1,\sigma_1}(M_1^N)$, $h^{N\delta_2}\partial^\alpha b \in S_{\delta_2,\sigma_2}(M_2^N)$, for $|\alpha| = N$ and some order functions M_1^N, M_2^N, then $r_N \in h^{N(1-(\delta_1+\delta_2))} S_{\delta,\sigma}(M_1^N M_2^N)$.

Remark 1.2.22. From the previous proposition it follows that, if symbols $a \in S_{\delta_1,\sigma_1}(M_1)$, $b \in S_{\delta_2,\sigma_2}(M_2)$ are such that $\mathrm{supp}a \cap \mathrm{supp}b = \emptyset$, then $a\sharp b = O(h^\infty)$, meaning that, for every $N \in \mathbb{N}$, $a\sharp b = r_N$ with $r_N \in h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(M_1M_2)$.

Remark 1.2.23. We draw the reader's attention to the fact that symbol $\sharp$ is used simultaneously in Bony calculus (see proposition 1.2.9) and in Weyl semi-classical calculus (as in (1.2.18)) with two different meaning. However, we avoid to introduce different notations as it will be clear by the context if we are dealing with the former or the latter one.

The result of proposition 1.2.21 and remark 1.2.22 are still true even when one of the two order functions, or both, has the form $\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-1}$, for a smooth function $f(\xi)$, $\nabla f(\xi)$ bounded, as stated below (see the appendix of [29]).

Lemma 1.2.24. *Let $f(\xi) : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, with $|\nabla f(\xi)|$ bounded. Consider $a \in S_{\delta_1,\sigma_1}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m})$, $m \in \mathbb{N}$, and $b \in S_{\delta_2,\sigma_2}(M)$, for M order function or $M(x,\xi) = \langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-n}$, $n \in \mathbb{N}$, some $\delta_1 \in [0,\frac{1}{2}]$, $\delta_2 \in [0,\frac{1}{2}[$, $\sigma_1,\sigma_2 \geq 0$ as in (1.2.17). Then $a\sharp b \in S_{\delta,\sigma}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m}M)$, where $\delta = \max\{\delta_1,\delta_2\}$, $\sigma = \max\{\sigma_1,\sigma_2\}$, and the asymptotic expansion (1.2.18) holds, with $r_N \in h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m}M)$ given by (1.2.19) (or equivalently (1.2.20)).*
More generally, if $h^{N\delta_1}\partial^\alpha a \in S_{\delta_1,\sigma_1}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m'})$ and $h^{N\delta_2}\partial^\alpha b \in S_{\delta_2,\sigma_2}(M^N)$, $|\alpha| = N$, M^N order function or $M^N(x,\xi) = \langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-n'}$, for some $m',n' \in \mathbb{N}$, then remainder r_N belongs to $h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m'}M^N)$.

1.2.3 Semi-classical Operators for the Wave Solution: Some Estimates

From now on we place ourselves in space dimension $d = 2$. This technical subsection focuses on the introduction and the analysis of some particular operators that we will use when dealing with the wave component in the semi-classical framework (subsection 3.2.2). More precisely, lemma 1.2.25 will be often recalled to prove that some operator belongs to $\mathcal{L}(L^2;L^\infty)$ and compute its norm; propositions 1.2.27, 1.2.30 concern the continuity of some important operators like $\Gamma^{w,k}$ defined in (3.2.44), while propositions 1.2.28, 1.2.31 are devoted to prove the continuity of some other operators often arising when considering the quantization of symbolic integral remainders. Finally, lemmas 1.2.33 and 1.2.35 deal with the development of some special symbolic products. While 1.2.33 will be used several times throughout the paper, lemma 1.2.35 is stated explicitly on purpose to prove lemma 3.2.13.

Lemma 1.2.25. *There exists a constant $C > 0$ such that, for any function $A(x,\xi)$ with $\partial_x^\alpha\partial_\xi^\beta A \in L^2(\mathbb{R}^2\times\mathbb{R}^2)$ for $|\alpha|,|\beta| \leq 3$, and any function $w \in L^2(\mathbb{R}^2)$,*

$$\big|\mathrm{Op}_h^w\big(A(x,\xi)\big)w(x)\big| \leq C\|w\|_{L^2}\int_{\mathbb{R}^2}\langle x-y\rangle^{-3}\sum_{|\alpha|,|\beta|\leq 3}\Big\|\partial_y^\alpha\partial_\xi^\beta\Big[A\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy. \tag{1.2.21}$$

Moreover, if $A(x,\xi)$ is compactly supported in x there exists a smooth function, supported in a neighbourhood of $\mathrm{supp}A$, such that

$$\big|\mathrm{Op}_h^w\big(A(x,\xi)\big)w(x)\big| \leq C\|w\|_{L^2}\int_{\mathbb{R}^2}\Big|\theta'\Big(\frac{x+y}{2}\Big)\Big|\sum_{|\alpha|\leq 3}\Big\|\partial_y^\alpha\Big[A\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy. \tag{1.2.22}$$

Proof. Let us prove the statement for $A \in \mathcal{S}(\mathbb{R}^2\times\mathbb{R}^2)$ and $w \in \mathcal{S}(\mathbb{R}^2)$. The density of $\mathcal{S}(\mathbb{R}^2\times\mathbb{R}^2)$ into $\{A \in L^2(\mathbb{R}^2\times\mathbb{R}^2) | \partial_x^\alpha\partial_\xi^\beta A \in L^2(\mathbb{R}^2\times\mathbb{R}^2), |\alpha|,|\beta| \leq 3\}$ and of $\mathcal{S}(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$ will then

justify the definition of $\mathrm{Op}_h^w(A(x,\xi))w$ for A and w as in the statement, together with inequalities (1.2.21), (1.2.22).

Using integration by parts, Cauchy-Schwarz inequality, and Young's inequality for convolutions, we can write the following:

$$
\begin{aligned}
|\mathrm{Op}_h^w(A(x,\xi))w(x)| &= \frac{1}{(2\pi)^2}\left|\int_{\mathbb{R}^4} e^{i(x-y)\cdot\xi} A\Big(\frac{x+y}{2}, h\xi\Big) w(y)\; dyd\xi\right| \\
&= \frac{1}{(2\pi)^4}\left|\int_{\mathbb{R}^2} \widehat{w}(\eta)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi+iy\cdot\eta} A\Big(\frac{x+y}{2}, h\xi\Big)\; dyd\xi\; d\eta\right| \\
&= \frac{1}{(2\pi)^4}\left|\int_{\mathbb{R}^2} \widehat{w}(\eta)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(\frac{1-i(x-y)\cdot\partial_\xi}{1+|x-y|^2}\right)^3 \left(\frac{1+i(\xi-\eta)\cdot\partial_y}{1+|\xi-\eta|^2}\right)^3 e^{i(x-y)\cdot\xi+iy\cdot\eta}\right. \\
&\qquad\qquad \left.\times A\Big(\frac{x+y}{2}, h\xi\Big)\; dyd\xi\; d\eta\right| \\
&\lesssim \int_{\mathbb{R}^2} |\hat{w}(\eta)| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \langle x-y\rangle^{-3}\langle \xi-\eta\rangle^{-3} \sum_{|\alpha|,|\beta|\le 3} \Big|\partial_y^\alpha\partial_\xi^\beta\Big[A\Big(\frac{x+y}{2}, h\xi\Big)\Big]\Big| dyd\xi\; d\eta \\
&\lesssim \|\hat{w}\|_{L^2(d\eta)} \|\langle\eta\rangle^{-3}\|_{L^1(d\eta)} \int_{\mathbb{R}^2} \langle x-y\rangle^{-3} \sum_{|\alpha|,|\beta|\le 3} \Big\|\partial_y^\alpha\partial_\xi^\beta\Big[A\Big(\frac{x+y}{2}, h\xi\Big)\Big]\Big\|_{L^2(d\xi)} dy \\
&\lesssim \|w\|_{L^2} \int_{\mathbb{R}^2} \langle x-y\rangle^{-3} \sum_{|\alpha|,|\beta|\le 3} \Big\|\partial_y^\alpha\partial_\xi^\beta\Big[A\Big(\frac{x+y}{2}, h\xi\Big)\Big]\Big\|_{L^2(d\xi)} dy\,.
\end{aligned}
$$

If symbol $A(x,\xi)$ is compactly supported in x we can consider a smooth function $\theta' \in C_0^\infty(\mathbb{R})$, identically equal to 1 on the support of $A(x,\xi)$, and write

$$
\begin{aligned}
|\mathrm{Op}_h^w(A(x,\xi))w(x)| &= \frac{1}{(2\pi)^2}\left|\int_{\mathbb{R}^2} \widehat{w}(\eta)d\eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(\frac{1+i(\xi-\eta)\cdot\partial_y}{1+|\xi-\eta|^2}\right)^3 e^{i(x-y)\cdot\xi+iy\cdot\eta}\right. \\
&\qquad\qquad \left.\times A\Big(\frac{x+y}{2}, h\xi\Big)\; dyd\xi\right| \\
&\lesssim \int_{\mathbb{R}^2} |\hat{w}(\eta)|\, d\eta \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big|\theta'\Big(\frac{x+y}{2}\Big)\Big| \langle\xi-\eta\rangle^{-3} \sum_{|\alpha|\le 3} \Big|\partial_y^\alpha\Big[A\Big(\frac{x+y}{2}, h\xi\Big)\Big]\Big| dyd\xi \\
&\lesssim \|w\|_{L^2} \int_{\mathbb{R}^2} \Big|\theta'\Big(\frac{x+y}{2}\Big)\Big| \sum_{|\alpha|\le 3} \Big\|\partial_y^\alpha\Big[A\Big(\frac{x+y}{2}, h\xi\Big)\Big]\Big\|_{L^2(d\xi)} dy\,.
\end{aligned}
$$

□

A very important role in this subsection and in subsection 3.2.2 will be played by functions of the form $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)$, where $\gamma \in C^\infty(\mathbb{R}^2)$ is such that $|\partial^\alpha\gamma(z)| \lesssim \langle z\rangle^{-|\alpha|}$, $\psi \in C_0^\infty(\mathbb{R}^2 - \{0\})$, $\sigma > 0$ is a small fixed constant and k is an integer belonging to set K, with

$$
K := \{k \in \mathbb{Z} \;:\; h \lesssim 2^k \lesssim h^{-\sigma}\}. \tag{1.2.23}
$$

In various results, such as proposition 1.2.30, we will need a more decaying smooth function γ_1 verifying that $|\partial^\alpha\gamma_1(z)| \lesssim \langle z\rangle^{-(1+|\alpha|)}$. We introduce here some notations we will keep throughout the whole paper:

Notation 1. For any $n \in \mathbb{N}$, γ_n denotes a smooth function in $\mathbb{R}^2$ such that $|\partial^\alpha\gamma_n(z)| \lesssim_\alpha \langle z\rangle^{-(n+|\alpha|)}$, for any $\alpha \in \mathbb{N}^2$. We use the simplest notation γ for γ_0;

Notation 2. For any integer $m \in \mathbb{Z}$, $b_m(\xi)$ will denote any function satisfying $|\partial^\beta b_m(\xi)| \lesssim_\beta |\xi|^{m-|\beta|}$, for any ξ in its domain, any $\beta \in \mathbb{N}^2$.

The following lemma is a useful reference when we need to deal with some derivatives of $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)$.

Lemma 1.2.26. *Let $\sigma \in \mathbb{R}$ and $n \in \mathbb{N}$. For any multi-indices $\alpha, \beta \in \mathbb{N}^2$ we have that*

$$\partial_x^\alpha \partial_\xi^\beta \Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=0}^{|\beta|} h^{-(|\alpha|+k)(\frac{1}{2}-\sigma)} \gamma_{n+|\alpha|+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_{|\alpha|+k-|\beta|}(\xi). \tag{1.2.24}$$

Furthermore, if $\theta = \theta(x) \in C_0^\infty(\mathbb{R}^2)$, there exists a set $\{\theta_k(x)\}_{1\le k\le|\beta|}$ of smooth compactly supported functions such that

$$\theta(x)\partial_x^\alpha \partial_\xi^\beta \Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=1}^{|\beta|} h^{-(|\alpha|+k)(\frac{1}{2}-\sigma)} \gamma_{n+|\alpha|+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) \theta_k(x) b_{|\alpha|+k-|\beta|}(\xi). \tag{1.2.25}$$

Proof. Let δ_{ij} be the Kronecker delta and $\sum'$ be a concise notation to indicate a linear combination. For $i = 1, 2$,

(1.2.26)

$$\begin{aligned}
\partial_{\xi_i}\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] &= h^{-(\frac{1}{2}-\sigma)} \sum_{j=1}^{2} (\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(x_j\xi_i|\xi|^{-1} - \delta_{ij}) \\
&= \sum_{j=1}^{2} (\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi|-\xi_j}{h^{1/2-\sigma}}\Big)\ \xi_i|\xi|^{-2} + \sum_{j=1}^{2} h^{-(\frac{1}{2}-\sigma)} (\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)[\xi_i\xi_j|\xi|^{-2} - \delta_{ij}],
\end{aligned}$$

which can be summarized saying that

$$\partial_{\xi_i}\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum{}' \gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_{-1}(\xi) + h^{-(\frac{1}{2}-\sigma)} \gamma_{n+1}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_0(\xi),$$

for some new functions $\gamma_n, \gamma_{n+1}, b_0, b_{-1}$. Iterating this argument one finds that, for all $\beta \in \mathbb{N}^2$,

$$\partial_\xi^\beta \Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=0,\dots,|\beta|}{}' h^{-k(\frac{1}{2}-\sigma)} \gamma_{n+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_{k-|\beta|}(\xi),$$

and obtains (1.2.24) using that, for any $m \in \mathbb{N}, \alpha \in \mathbb{N}^2$,

$$\partial_x^\alpha \Big[\gamma_m\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = h^{-|\alpha|(\frac{1}{2}-\sigma)} (\partial^\alpha\gamma_m)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)|\xi|^{|\alpha|}. \tag{1.2.27}$$

Equality (1.2.25) is obtained replacing (1.2.26) with

$$\begin{aligned}
\theta(x)\partial_{\xi_i}\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] &= h^{-(\frac{1}{2}-\sigma)} \sum_{j=1}^{2} (\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\theta(x)x_j\xi_i|\xi|^{-1} - \theta(x)\delta_{ij}) \\
&= \sum{}' h^{-(\frac{1}{2}-\sigma)} \gamma_{n+1}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) \theta_1(x) b_0(\xi),
\end{aligned}$$

where $\theta_1(x)$ is a new compactly supported function. By iteration one finds that, for any $\beta \in \mathbb{N}^2$, there is a set of $|\beta|$ compactly supported functions $\theta_k(x)$, $1 \le k \le |\beta|$, such that

$$\theta(x)\partial_\xi^\beta \Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=1}^{|\beta|} h^{-k(\frac{1}{2}-\sigma)} \gamma_{n+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) \theta_k(x) b_{k-|\beta|}(\xi),$$

which combined with (1.2.27) gives (1.2.25). □

In some of the following results we denote by Θ_h the operator of change of coordinates

$$\Theta_h w(x) = w(\sqrt{h}x),$$

for any $h \in]0,1]$, and use that for any symbol $a(x,\xi)$,

$$\mathrm{Op}_h^w\big(a(x,\xi)\big) = \Theta_h \mathrm{Op}_h^w\big(\widetilde{a}(x,\xi)\big)\Theta_h^{-1}, \tag{1.2.28}$$

with $\widetilde{a}(x,\xi) = a\Big(\frac{x}{\sqrt{h}}, \sqrt{h}\xi\Big)$.

Proposition 1.2.27 (Continuity on L^2)**.** *Let $\sigma > 0$ be sufficiently small, K be the set defined in (1.2.23), $k \in K$ and $p \in \mathbb{Z}$. Let also $\psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and $a(x)$ be a smooth function, bounded together with all its derivatives. Then $\mathrm{Op}_h^w\big(\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi)\big) : L^2 \to L^2$ is bounded and*

$$\Big\|\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)\Big\|_{\mathcal{L}(L^2)} \lesssim 2^{kp}. \tag{1.2.29}$$

Proof. Let $A(x,\xi) = \gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi)$. For indices $k \in K$ such that $h^{1/2-\sigma} \lesssim 2^k \lesssim h^{-\sigma}$ the statement follows from the fact that $A(x,\xi) \in 2^{kp}S_{\frac{1}{2},0}(1)$ and by theorem 7.11 of [8]. For $k \in K$ such that $h \le 2^k \le h^{1/2-\sigma}$, $\widetilde{A}(x,\xi) := A(\frac{x}{\sqrt{h}}, \sqrt{h}\xi) \in 2^{kp}S_{\frac{1}{2},0}(1)$ and the result follows by theorem 7.11 of [8] and equality (1.2.28). □

Proposition 1.2.28. *Let σ, k, p be as in the previous proposition. Let also $q \in \mathbb{Z}$, $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, $a'(x)$ be a smooth function, bounded together with all its derivatives, and $f \in C(\mathbb{R})$. Define*

$$I_{p,q}^k(x,\xi) := \frac{1}{(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}\Big[\int_0^1 \Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+tz,\xi+t\zeta)}f(t)dt \\ \times\widetilde{\psi}(2^{-k}(\xi+\eta))a'(x+y)b_q(\xi+\eta)\Big]\, dydzd\eta d\zeta \tag{1.2.30}$$

and

$$J_{p,q}^k(x,\xi) := \frac{1}{(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}\Big[\int_0^1 \widetilde{\psi}(2^{-k}(\xi+t\zeta))a'(x+tz)b_q(\xi+t\zeta)f(t)dt \\ \times\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+y,\xi+\eta)}\Big]\, dydzd\eta d\zeta. \tag{1.2.31}$$

Then $\mathrm{Op}_h^w(I_{p,q}^k(x,\xi))$ and $\mathrm{Op}_h^w(J_{p,q}^k(x,\xi))$ are bounded operators on L^2 and

$$\Big\|\mathrm{Op}_h^w(I_{p,q}^k(x,\xi))\Big\|_{\mathcal{L}(L^2)} + \Big\|\mathrm{Op}_h^w(J_{p,q}^k(x,\xi))\Big\|_{\mathcal{L}(L^2)} \lesssim 2^{k(p+q)}.$$

The same results holds also if $q = 0$ and $\widetilde{\psi}(2^{-k}\xi)b_q(\xi) \equiv 1$.

Proof. We show the result for $\mathrm{Op}_h^w(I_{p,q}^k)$, leaving the reader to check that a similar argument can be used for $\mathrm{Op}_h^w(J_{p,q}^k)$.

We distinguish between two ranges of frequencies. For indices $k \in K$ such that $h^{1/2-\sigma} \le 2^k \lesssim h^{-\sigma}$ we observe that $I_{p,q}^k(x,\xi) \in 2^{k(p+q)}S_{\frac{1}{2},0}(1)$. Indeed, $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi) \in 2^{kp}S_{\frac{1}{2},\sigma}(1)$

by lemma 1.2.26 while $\widetilde{\psi}(2^{-k}\xi)a'(x)b_q(\xi) \in 2^{kq}S_{\frac{1}{2}-\sigma,\sigma}(1)$. Hence performing a change of variables $y \mapsto \sqrt{h}y$, $z \mapsto \sqrt{h}z$, $\eta \mapsto \sqrt{h}\eta$, $\zeta \mapsto \sqrt{h}\zeta$, writing

(1.2.32)
$$e^{2i(\eta\cdot z - y\cdot\zeta)} = \left(\frac{1+2iy\cdot\partial_\zeta}{1+4|y|^2}\right)^3 \left(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\right)^3 \left(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\right)^3 \left(\frac{1+2i\zeta\cdot\partial_y}{1+4|\zeta|^2}\right)^3 e^{2i(\eta\cdot z - y\cdot\zeta)},$$

and integrating by parts in all variables, we get that

$$\left|I^k_{p,q}(x,\xi)\right| \lesssim 2^{k(p+q)} \int \langle y\rangle^{-3}\langle z\rangle^{-3}\langle\eta\rangle^{-3}\langle\zeta\rangle^{-3}\, dydzd\eta d\zeta \lesssim 2^{k(p+q)},$$

without any loss in $h^{-\delta}$ due to the fact that we are considering symbols $A(x,\xi) \in S_{\delta,\sigma}(1)$ with $\delta \in \{0, 1/2-\sigma, 1/2\}$, and the derivatives of $A(x+\sqrt{h}y, \xi+\sqrt{h}\eta)$ (resp. of $A(x+t\sqrt{h}z, \xi+t\sqrt{h}\zeta)$) with respect to y and η (resp. with respect to z and ζ). In a similar way one can also prove that $|\partial^\alpha_x\partial^\beta_\xi I^k_{p,q}(x,\xi)| \lesssim_{\alpha,\beta} h^{-\frac{1}{2}(|\alpha|+|\beta|)}2^{k(p+q)}$, for any $\alpha,\beta \in \mathbb{N}^2$. Theorem 7.11 of [8] implies then the statement in this case.

For indices $k \in K$ such that $h \lesssim 2^k \leq h^{1/2-\sigma}$ we observe that

$$\gamma\Big(\frac{x|\xi|}{h^{1/2-\sigma}} - h^\sigma\xi\Big)\psi(2^{-k}\sqrt{h}\xi)a\Big(\frac{x}{\sqrt{h}}\Big)b_p(\sqrt{h}\xi) \in 2^{kp}S_{\frac{1}{2},\sigma}(1),$$
$$\widetilde{\psi}(2^{-k}\sqrt{h}\xi)a'\Big(\frac{x}{\sqrt{h}}\Big)b_q(\sqrt{h}\xi) \in 2^{kq}S_{\frac{1}{2},\sigma}(1).$$

Then $\widetilde{I}^k_{p,q}(x,\xi) = I^k_{p,q}\Big(\frac{x}{\sqrt{h}}, \sqrt{h}\xi\Big) \in 2^{k(p+q)}S_{\frac{1}{2},0}(1)$ and theorem 7.11 of [8] along with equality (1.2.28) imply that $\mathrm{Op}^w_h(I^k_{p,q}) : L^2 \to L^2$ is bounded, uniformly in h.

The last part of the statement can be proved following an analogous scheme, after having previously made an integration in $dzd\eta$ (or in $dyd\zeta$ if dealing with $J^k_{p,0}$). □

Proposition 1.2.29 (Continuity on L^p)**.** *Let $1 \leq p \leq +\infty$, $\gamma \in C^\infty_0(\mathbb{R}^2)$ be radial, $\psi \in C^\infty_0(\mathbb{R}^2 \setminus \{0\})$, $a(x)$ be a smooth function, bounded together with all its derivatives. Let also $\sigma > 0$ be small, $k \in K$ with K given by* (1.2.23) *and $q \in \mathbb{Z}$. Then $\mathrm{Op}^w_h\big(\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_q(\xi)\big) : L^p \to L^p$ is a bounded operator with*

$$\left\|\mathrm{Op}^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_q(\xi)\Big)\right\|_{\mathcal{L}(L^p)} \lesssim 2^{kq}.$$

Proof. In order to prove the result of the statement we need to show that kernel $K^k(x,y)$ associated to $\mathrm{Op}^w_h\big(\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_q(\xi)\big)$, i.e.

$$K^k(x,y) := \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}(x-y)\cdot\xi}\gamma\Big(\frac{\big(\frac{x+y}{2}\big)|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a\Big(\frac{x+y}{2}\Big)b_q(\xi)d\xi, \tag{1.2.33}$$

is such that

$$\sup_x \int |K^k(x,y)|dy \lesssim 2^{kq}, \quad \sup_y \int |K^k(x,y)|dx \lesssim 2^{kq}.$$

From the symmetry between variables x, y, it will be enough to show that one of the two above inequalities is satisfied. To do that we study K^k separately in different spatial regions, distinguishing also between indices $k \in K$ such that $2^k \leq h^{1/2-\sigma}$ and $2^k > h^{1/2-\sigma}$. We hence introduce three smooth cut-off functions $\theta_s, \theta_b, \theta$, supported respectively for $|x| \leq m \ll 1$, $|x| \geq M \gg 1$,

$0 < m' \leq |x| \leq M' < +\infty$, for some constants m, m', M, M', and such that $\theta_s + \theta_b + \theta \equiv 1$. Denoting concisely by $A^k(x,\xi)$ the multiplier in (1.2.33), we split it as follows

$$A^k(x,\xi) = A^k_s(x,\xi) + A^k_b(x,\xi) + A^k_1(x,\xi),$$

with $A^k_s(x,\xi) := A^k(x,\xi)\theta_s(x)$, $A^k_b(x,\xi) := A^k(x,\xi)\theta_b(x)$ and $A^k_1(x,\xi) := A^k(x,\xi)\theta(x)$.

Case I: Let us consider $k \in K$ such that $h \lesssim 2^k \leq h^{1/2-\sigma}$. According to the above decomposition we have that

$$K^k(x,y) = K^k_s(x,y) + K^k_b(x,y) + K^k_1(x,y),$$

with clear meaning of kernels K^k_s, K^k_b, K^k_1. Let us first prove that

$$\sup_x \int |K^k_s(x,y)|dy + \sup_x \int |K^k_b(x,y)|dy \lesssim 2^{kq}. \tag{1.2.34}$$

For $|x| \ll 1$ (resp. $|x| \gg 1$), $\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right| \gtrsim h^{-1/2+\sigma}|\xi|$ (resp. $\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right| \gtrsim h^{-1/2+\sigma}|\xi||x| \gtrsim h^{-1/2+\sigma}|\xi|$) so by lemma 1.2.26

$$\left|\partial^\beta_\xi\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big]\right| \lesssim \sum_{j=0}^{|\beta|} h^{-j(\frac{1}{2}-\sigma)}\left\langle \frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right\rangle^{-j}|b_{j-|\beta|}(\xi)| \lesssim |\xi|^{-|\beta|}. \tag{1.2.35}$$

Therefore

$$\left|\partial^\beta_\xi A^k_s(x,2^k\xi)\right| \lesssim \sum_{|\beta_1|\leq|\beta|} 2^{k|\beta|}|2^k\xi|^{-|\beta_1|}2^{-k(|\beta|-|\beta_1|)+kq}\mathbb{1}_{|\xi|\sim 1} \lesssim 2^{kq}\mathbb{1}_{|\xi|\sim 1}, \tag{1.2.36}$$

so making a change of coordinates $\xi \mapsto 2^k\xi$ and some integration by parts we derive that

$$|K^k_s(x,y)| \lesssim 2^{kq}(2^kh^{-1})^2\left\langle 2^kh^{-1}(x-y)\right\rangle^{-3},$$

for every $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$. The same argument applies to $K^k_b(x,y)$, hence taking the L^1 norm we obtain (1.2.34).

As concerns kernel $K^k_1(x,y)$, we deduce from lemma 1.2.26, the fact that $\theta_1(x)$ is supported for $|x| \sim 1$, and that $2^k \lesssim h^{1/2-\sigma}$, the following inequality:

$$\left|\partial^\beta_\xi\Big[A^k_1\Big(\frac{x+y}{2}, 2^k\xi\Big)\Big]\right| \lesssim 2^{k|\beta|}\Big[2^{k(q-|\beta|)} + \sum_{j=1}^{|\beta|} h^{-j(\frac{1}{2}-\sigma)}|b_{j-|\beta|+q}(2^k\xi)|\Big] \lesssim 2^{kq}.$$

Performing a change of coordinates $\xi \mapsto 2^k\xi$ and making some integration by parts one finds that

$$|K^k_1(x,y)| \lesssim 2^{kq}(2^kh^{-1})^2\left\langle 2^kh^{-1}(x-y)\right\rangle^{-3}, \qquad \forall (x,y),$$

and consequently that

$$\sup_x \int |K^k(x,y)|dy \lesssim 2^{kq}.$$

Summing up with (1.2.34), this gives us that

$$\mathrm{Op}^w_h(A^k(x,\xi)) = \mathrm{Op}^w_h(A^k_s(x,\xi)) + \mathrm{Op}^w_h(A^k_b(x,\xi)) + \mathrm{Op}^w_h(A^k_1(x,\xi))$$

is a bounded operator on L^p, for every $1 \leq p \leq +\infty$, with norm $O(2^{kq})$.

Case II: Let us now suppose that $k \in K$ is such that $h^{1/2-\sigma} < 2^k \leq h^{-\sigma}$. From (1.2.35) we have that $\widetilde{A}_s^k(x,\xi) = A_s^k\big(\frac{x}{\sqrt{h}}, \sqrt{h}\xi\big)$ satisfies

$$\Big|\partial_\xi^\beta \widetilde{A}_s^k(x,\xi)\Big| \lesssim \sum_{|\beta_1|\leq|\beta|} h^{\frac{|\beta|}{2}} |\sqrt{h}\xi|^{-|\beta_1|} 2^{-k(|\beta|-|\beta_1|)+kq} \mathbb{1}_{|\xi|\sim 2^k h^{-1/2}},$$

for every $(x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, and hence

$$\Big|\partial_\xi^\beta \widetilde{A}_s^k(x, 2^k h^{-1/2}\xi)\Big| \lesssim \sum_{|\beta_1|\leq|\beta|} 2^{k|\beta|} |2^k\xi|^{-|\beta_1|} 2^{-k(|\beta|-|\beta_1|)+kq} \mathbb{1}_{|\xi|\sim 1} \lesssim 2^{kq} \mathbb{1}_{|\xi|\sim 1}.$$

By making a change of coordinates $\xi \mapsto 2^k h^{-1/2}\xi$, some integrations by parts and using the above inequality, one can show that kernel $\widetilde{K}_s^k(x,y)$ associated to $\mathrm{Op}_h^w(\widetilde{A}_s^k(x,\xi))$, i.e.

$$\widetilde{K}_s^k(x,y) = \frac{1}{(2\pi h)^2} \int e^{\frac{i}{h}(x-y)\cdot\xi} \widetilde{A}_s^k\Big(\frac{x+y}{2}, \xi\Big) d\xi,$$

is such that

$$|\widetilde{K}_s^k(x,y)| \lesssim 2^{kq} (2^k h^{-\frac{3}{2}})^2 \Big\langle 2^k h^{-\frac{3}{2}}(x-y)\Big\rangle^{-3} \qquad \forall (x,y),$$

which implies that $sup_x \int |\widetilde{K}_s^k(x,y)| dy \lesssim 2^{kq}$. The same argument and result hold for $\widetilde{K}_b^k(x,y)$ so $\mathrm{Op}_h^w(A_s^k)$ and $\mathrm{Op}_h^w(A_b^k)$ verify the statement.

The last thing to prove is that $\mathrm{Op}_h^w(A_1(x,\xi)) \in \mathcal{L}(L^p)$ for every $1 \leq p \leq +\infty$. Let $K_1^k(x,y)$ be its associated kernel, i.e.

$$K_1^k(x,y) = \frac{1}{(2\pi h)^2} \int e^{\frac{i}{h}(x-y)\cdot\xi} \gamma\Big(\frac{\big(\frac{x+y}{2}\big)|\xi| - \xi}{h^{1/2-\sigma}}\Big) \psi(2^{-k}\xi) a\Big(\frac{x+y}{2}\Big) b_q(\xi) d\xi, \tag{1.2.37}$$

and assume, without loss of generality, that $\gamma(x) = \gamma(|x|^2)$. Set

$$\frac{x+y}{2} = r[\cos\theta, \sin\theta],$$

with $m' \leq r \leq M'$ on the support of $\theta_1\big(\frac{x+y}{2}\big)$, and for fixed r, θ let

$$\xi = \rho[\cos\theta, \sin\theta] + r\Omega[-\sin\theta, \cos\theta]. \tag{1.2.38}$$

We immediately notice that $[\frac{\partial(\xi_1,\xi_2)}{\partial(\rho,\Omega)}] = r \sim 1$ and that $|\xi|^2 = \rho^2 + r^2\Omega^2$. Moreover,

$$\Big|\Big(\frac{x+y}{2}\Big)|\xi| - \xi\Big|^2 = \Big[r\sqrt{\rho^2+r^2\Omega^2} - \rho\Big]^2 + r^2\Omega^2.$$

If the support of γ is of size $0 < \alpha \ll 1$ sufficiently small, from the above equality and the fact that $|\xi| \sim 2^k$ on the support of $\psi(2^{-k}\xi)$, with $h^{1/2-\sigma} < 2^k \lesssim h^{-\sigma}$, we deduce that

$$r\Omega \leq \sqrt{\alpha} h^{1/2-\sigma} \quad \text{and} \quad |\rho| \sim |\xi| \sim 2^k \quad \text{and} \quad \frac{r\Omega}{|\rho|} \leq \sqrt{\alpha}.$$

Consequently

$$\alpha h^{1-2\sigma} \geq \Big[r\sqrt{\rho^2+r^2\Omega^2} - \rho\Big]^2 \gtrsim \rho^2 |r-1|^2.$$

The above left inequality implies that $\rho > 0$, inferring so the right one. Moreover

$$\begin{aligned}\alpha h^{1-2\sigma} \geq \Big[r\sqrt{\rho^2+r^2\Omega^2} - \rho\Big]^2 + r^2\Omega^2 &= \rho^2 \left[(r-1) + r\left[\sqrt{1+\frac{r^2\Omega^2}{\rho^2}} - 1\right]\right]^2 + r^2\Omega^2 \\ &= \rho^2|r-1|^2 + r^2\Omega^2\left[1 + a(r,\Omega,\rho)\right],\end{aligned}$$

with $a(r,\Omega,\rho)$ bounded such that, for any $l,m,n\in\mathbb{N}$,

$$|\partial_r^l\partial_\Omega^m\partial_\rho^n a(r,\Omega,\rho)|=O(\rho^{-(m+n)}).$$

If

$$\Gamma_h:=\gamma\Big(\frac{\rho^2|r-1|^2}{h^{1-2\sigma}}+\frac{r^2\Omega^2}{h^{1-2\sigma}}\left[1+a(r,\Omega,\rho)\right]\Big)\psi(2^{-k}\sqrt{\rho^2+r^2\Omega^2})a(r,\theta)b_q(\rho),$$

from all the observations made above along with the fact that $h^{-1/2+\sigma}\lesssim\rho^{-1}$ we deduce that, for any $m,n\in\mathbb{N}$,

$$\big|\partial_\rho^m\Gamma_h\big|=O(2^{kq}\rho^{-m})\quad\text{and}\quad|\partial_\Omega^n\Gamma_h|=O(2^{kq}\rho^{-n}).\tag{1.2.39}$$

With the change of coordinates considered in (1.2.38), and setting $w:=x-y$, $e_\theta:=[\cos\theta,\sin\theta]$, kernel $K_1^k(x,y)$ transforms into

$$\frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}\rho w\cdot e_\theta+\frac{i}{h}r\Omega w\cdot e_\theta^\perp}\Gamma_h\,rd\rho d\Omega$$

and is restricted to $|\rho|\sim 2^k$, $|\Omega|\lesssim h^{1/2-\sigma}$, so by making some integrations by parts, using (1.2.39), and reminding that $|r-1|\ll 2^{-k}h^{1/2-\sigma}\ll 1$ on the support of Γ_h, we find that, for any $N\in\mathbb{N}$,

$$|K_1^k(x,y)|\lesssim h^{-\frac{3}{2}-\sigma}2^k\left\langle\frac{2^k}{h}w\cdot e_\theta\right\rangle^{-N}\left\langle\frac{2^k}{h}w\cdot e_\theta^\perp\right\rangle^{-N}\mathbb{1}_{||\frac{x+y}{2}|-1|\ll 1}.$$

Now, as $w=(x-y)$, $e_\theta=\frac{x+y}{|x+y|}$, and $|x+y|=2r\sim 1$ on the support of Γ_h, we have that $|w\cdot e_\theta|\sim||x|^2-|y|^2|$, $|w\cdot e_\theta^\perp|\sim|(x+y)(x+y)^\perp|\sim 2|x\cdot y^\perp|=2|x_1y_2-x_2y_1|$, and consequently

$$|K_1^k(x,y)|\lesssim h^{-\frac{3}{2}-\sigma}2^{k(1+q)}\left\langle\frac{2^k}{h}\big||x|^2-|y|^2\big|\right\rangle^{-N}\left\langle\frac{2^k}{h}(x_1y_2-x_2y_1)\right\rangle^{-N}\mathbb{1}_{||\frac{x+y}{2}|-1|\ll 1}.$$

Successively, taking the $L^1(dy)$ norm of $K_1^k(x,y)$ and using the above estimate we find that:

• if $|x|\ll|y|$ or $|x|\gg|x|$,

$$\left\langle\frac{2^k}{h}\big||x|^2-|y|^2\big|\right\rangle^{-N}\mathbb{1}_{||\frac{x+y}{2}|-1|\ll 1}\lesssim h^{N(\frac{1}{2}+\sigma)},$$

as follows from the fact that $h2^{-k}<h^{1/2+\sigma}$. We obtain that

$$\sup_x\int|K_1^k(x,y)|dy\lesssim h^{-\frac{3}{2}}2^{k(1+q)}h^{N(\frac{1}{2}+\sigma)}\lesssim 1$$

by taking $N\in\mathbb{N}$ sufficiently large (e.g. $N>3$) and $\sigma>0$ small.

• if $|x|\sim|y|$, we deduce that $|x|\geq c>0$ from the fact that $\big|\big|\frac{x+y}{2}\big|-1\big|\leq\sqrt{\alpha}h^{1/2-\sigma}2^{-k}$ on the support of Γ_h. Without loss of generality we can assume that $x=\lambda e_1$ (this always being possible by making a rotation) and $|\lambda|\geq c>0$. If $w:=x+y$,

$$|x|^2-|y|^2=w\cdot(x-y)=w\cdot(2x-w)=w\cdot(2\lambda e_1-w)=2\lambda w_1-w_1^2-w_2^2,$$

and then

$$\frac{\big||x|^2-|y|^2\big|}{h}=-\frac{(w_1-\lambda)^2-\lambda^2}{h}+\Big(\frac{w_2}{\sqrt{h}}\Big)^2$$

while

$$x_1y_2 - x_2y_1 = \lambda w_2.$$

This implies that

$$|K_1^k(x,y)| \lesssim h^{-\frac{3}{2}-\sigma} 2^{k(1+q)} \left\langle \frac{2^k}{h}\left((w_1-\lambda)^2-\lambda^2\right)\right\rangle^{-N} \left\langle \frac{2^k}{h} w_2 \right\rangle^{-N}.$$

Since $\int |K_1^k(x,y)|dy = \int |K_1^k(x,y)|dw$, from the above estimate (with a fixed $N \in \mathbb{N}$ sufficiently large) this integral is bounded by 2^{kq} when restricted to $|x| \sim |y|$. Indeed, when the integral is taken in a neighbourhood of $w_1 = 0$ or $w_1 = 2\lambda$, $(w_1-\lambda^2)-\lambda^2$ can be considered as the variable of integration, and by a change of coordinates along with the fact that $2^{-k} < h^{-1/2+\sigma}$ one deduces that

$$\int_{U_0 \cup U_{2\lambda}} |K_1^k(x,y)|dw \lesssim h^{-\frac{3}{2}-\sigma} 2^{k(1+q)} h^2 2^{-2k} \lesssim 2^{kq},$$

where U_0 (resp. $U_{2\lambda}$) is a neighbourhood of $w_1 = 0$ (resp. of $w_1 = 2\lambda$). Outside of $U_0 \cup U_{2\lambda}$,

$$\left\langle \frac{2^k}{h}\left((w_1-\lambda)^2-\lambda^2\right)\right\rangle^{-N} \lesssim (h2^{-k})^N \langle w_1\rangle^{-N} \lesssim h^{N(\frac{1}{2}+\sigma)} \langle w_1\rangle^{-N},$$

so

$$\int_{(U_0\cup U_{2\lambda})^{\complement}} |K_1^k(x,y)|dw \lesssim h^{-\frac{3}{2}-\sigma} 2^{k(1+q)} h 2^{-k} h^{N(\frac{1}{2}+\sigma)} \lesssim 2^{kq}.$$

This finally proves that also $\mathrm{Op}_h^w(A_1^k(x,\xi))$ is a bounded operator on L^p with norm $O(2^{kq})$. □

Let us introduce the Euclidean rotation in the semi-classical setting

$$\Omega_h := x_1 hD_2 - x_2 hD_1 = \mathrm{Op}_h^w(x_1\xi_2 - x_2\xi_1). \tag{1.2.40}$$

Proposition 1.2.30. *Under the same assumptions as in proposition 1.2.27, with γ replaced by γ_1, we have that for any $w \in L^2(\mathbb{R}^2)$ such that $\Omega_h w \in L^2_{loc}(\mathbb{R}^2)$*

$$\left\| \mathrm{Op}_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)w\right\|_{L^\infty} \lesssim 2^{kp}h^{-\frac{1}{2}-\sigma}\left(\|w\|_{L^2} + \|\theta_0\Omega_h w\|_{L^2}\right), \tag{1.2.41}$$

where θ_0 is a smooth function supported in some annulus centred in the origin.

Proof. We prove the statement distinguishing between three spatial regions. For that, we introduce three cut-off functions: $\theta_s(x)$ supported for $|x| \le m \ll 1$; $\theta_b(x)$ supported for $|x| \ge M' \gg 1$; $\theta(x)$ supported for $m' \le |x| \le M'$, for some $0 < m' \ll 1, M \gg 1$, such that $\theta_s + \theta_b + \theta \equiv 1$. We define respectively $A_s^k(x,\xi) := \gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\theta_s(x)$, $A_b^k(x,\xi) := \gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\theta_b(x)$, and $A^k(x,\xi) := \gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\theta(x)$, so that

$$\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi) = A_s^k(x,\xi) + A_b^k(x,\xi) + A^k(x,\xi).$$

The fact that $\mathrm{Op}_h^w(A_s^k), \mathrm{Op}_h^w(A_b^k) \in \mathcal{L}(L^2;L^\infty)$ and their norm is a $O(2^{kp}h^{-1/2-\sigma})$ follows from lemmas 1.2.25 and 1.2.26. Indeed, when $|x| \ll 1$ (resp. $|x| \gg 1$) we have that $\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right| \gtrsim h^{-1/2+\sigma}|\xi|$ (resp. $\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right| \gtrsim h^{-1/2+\sigma}|\xi||x| \gtrsim h^{-1/2+\sigma}|\xi|$), so from lemma 1.2.26 we derive that

$$\left|\partial_x^\alpha\partial_\xi^\beta\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big]\right| \lesssim \sum_{j=0}^{|\beta|} h^{-(|\alpha|+j)(\frac{1}{2}-\sigma)}\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right|^{-1-|\alpha|-j} |b_{|\alpha|+j-|\beta|}(\xi)| \lesssim h^{\frac{1}{2}-\sigma}|\xi|^{-1-|\beta|}.$$

Consequently, as $2^{-k}h \leq 1$, we deduce that $|\partial_x^\alpha \partial_\xi^\beta \big[A_s^k(\frac{x+y}{2}, h\xi)\big]| \lesssim 2^{kp} h^{-1/2-\sigma}|\xi|^{-1}$ for any $\alpha, \beta \in \mathbb{N}^2$. Therefore

$$\Big\| \partial_y^\alpha \partial_\xi^\beta \Big[A_s^k\Big(\frac{x+y}{2}, h\xi\Big)\Big]\Big\|_{L^2(d\xi)} \lesssim 2^{kp} h^{-\frac{1}{2}-\sigma} \left(\int_{|\xi|\sim 2^k h^{-1}} |\xi|^{-2} d\xi \right)^{\frac{1}{2}} \lesssim 2^{kp} h^{-\frac{1}{2}-\sigma}.$$

The same holds for $A_b^k(x,\xi)$ so, injecting these estimates in inequality (1.2.21), we derive that $\|\mathrm{Op}_h^w(A_s^k(x,\xi))w\|_{L^\infty} + \|\mathrm{Op}_h^w(A_b^k(x,\xi))w\|_{L^\infty} \leq C 2^{kp} h^{-\frac{1}{2}-\sigma}\|w\|_{L^2}$.

A different analysis is needed for $\mathrm{Op}_h^w(A^k(x,\xi))w$, since it is no longer true that there exists a positive constant C such that $|x|\xi| - \xi| \geq C|\xi|$ on the support of $A^k(x,\xi)$. In this case we exploit the fact that $A^k(x,\xi)$ is supported in an annulus to perform a change of variables. If $\theta_0 \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is a cut-off function equal to 1 on the support of θ we have that, for any $N \in \mathbb{N}$, $A^k(x,\xi) = \theta_0(x) \sharp A^k(x,\xi) + r_N^k(x,\xi)$ by means of proposition 1.2.21, where

$$r_N^k(x,\xi) = \left(\frac{h}{2i}\right)^N \frac{N}{(\pi h)^4} \sum_{|\alpha|=N} \frac{(-1)^{|\alpha|}}{\alpha!} \int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \int_0^1 \partial_x^\alpha \theta_0(x+tz)(1-t)^{N-1} dt \\ \times (\partial_\xi^\alpha A^k)(x, \xi+\eta)\, dy dz d\eta d\zeta.$$

If we take N sufficiently large it turns out that the quantization of r_N^k satisfies a better estimate than (1.2.41). Indeed, using lemma 1.2.26 and integrating in $dyd\zeta$, it can be rewritten as

$$r_N^k(x,\xi) = \sum_{j\leq N} \frac{h^{N-j(\frac{1}{2}-\sigma)}}{(\pi h)^2} \int e^{\frac{2i}{h}\eta\cdot z} \int_0^1 \theta_0(x+tz)(1-t)^{N-1} dt \\ \times \gamma_{1+j}\Big(\frac{x|\xi+\eta| - (\xi+\eta)}{h^{1/2-\sigma}}\Big) \psi(2^{-k}(\xi+\eta))\theta_j(x) a(x) b_{p+j-N}(\xi+\eta)\, dz d\eta, \tag{1.2.42}$$

for some new functions $\theta_0, \gamma_{1+j}, \psi, \theta_j, a, b_{p+j-N}$. As it is compactly supported in x, by lemma 1.2.25 there is a new cut-off function (that we still call θ) such that

$$|\mathrm{Op}_h^w(r_N^k(x,\xi))w| \lesssim \|w\|_{L^2} \int \Big|\theta\Big(\frac{x+y}{2}\Big)\Big| \sum_{|\alpha'|\leq 3} \Big\| \partial_y^{\alpha'} \Big[r_N^k\Big(\frac{x+y}{2}, h\xi\Big)\Big]\Big\|_{L^2(d\xi)} dy.$$

One can check that the action of $\partial_y^{\alpha'}$ on $r_N^k(\frac{x+y}{2}, h\xi)$ makes appear factors $(h^{-1/2+\sigma}h|\xi+\eta|)^i$, for $i \leq |\alpha'|$, without changing the underlining structure of r_N^k, and these are bounded by $(h^{-1/2+\sigma}2^k)^i$ on the support of $\psi(2^{-k}h(\xi+\eta))$. After a change of variables $\eta \mapsto h\eta$ in (1.2.42), we use that $e^{2i\eta\cdot z} = \Big(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\Big)^3 \Big(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\Big)^3 e^{2i\eta\cdot z}$, integrate by parts, apply Young's inequality for convolutions, and fix $N > 7$, in order to deduce the following chain of inequalities:

$$\Big\| \partial_y^{\alpha'} r_N^k\Big(\frac{x+y}{2}, h\xi\Big)\Big\|_{L^2(d\xi)}^2 \\ \lesssim \sum_{i\leq|\alpha'|, j\leq N} h^{2N-2j(\frac{1}{2}-\sigma)} \big(h^{-\frac{1}{2}+\sigma}2^k\big)^{2i} 2^{2k(p+j-N)} \int d\xi \left| \int \langle z\rangle^{-3}\langle\eta\rangle^{-3} |\psi(2^{-k}h(\xi+\eta))| dz d\eta \right|^2 \\ \lesssim \sum_{i\leq|\alpha'|, j\leq N} h^{2N-2j(\frac{1}{2}-\sigma)} \big(h^{-\frac{1}{2}+\sigma}2^k\big)^{2i} 2^{2k(p+j-N)} \int |\psi(2^{-k}h\xi)|^2 d\xi \\ \lesssim \sum_{i\leq|\alpha'|, j\leq N} h^{2N-2j(\frac{1}{2}-\sigma)} \big(h^{-\frac{1}{2}+\sigma}2^k\big)^{2i} 2^{2k(p+j-N)} \big(h^{-1}2^k\big)^2 \lesssim 2^{2kp},$$

and that $\|\mathrm{Op}_h^w(r_N^k)\|_{\mathcal{L}(L^2;L^\infty)} \lesssim 2^{kp}$. We can then focus on the analysis of the L^∞ norm of $\theta_0(x)\mathrm{Op}_h^w(A^k(x,\xi))w$. In polar coordinates $x = \rho e^{i\alpha}$ operator Ω_h reads as D_α, so using the classical one-dimensional Sobolev injection with respect to variable α, the one-dimensional semi-classical Sobolev injection with respect to variable ρ, and successively returning back to coordinates x, we deduce that

$$\begin{aligned}\left|\theta_0(x)\mathrm{Op}_h^w(A^k(x,\xi))w\right| &\lesssim h^{-\frac{1}{2}}\Big[\|\mathrm{Op}_h^w(A^k)w\|_{L^2(dx)} + \|\mathrm{Op}_h^w(\xi)\mathrm{Op}_h^w(A^k)w\|_{L^2(dx)} \\ &\qquad + \|\Omega_h\theta_0\mathrm{Op}_h^w(A^k)w\|_{L^2(dx)} + \|\mathrm{Op}_h^w(\xi)\Omega_h\theta_0\mathrm{Op}_h^w(A^k)w\|_{L^2(dx)}\Big] \\ &\lesssim 2^{kp}h^{-\frac{1}{2}-\sigma}\left[\|w\|_{L^2} + \|\theta_0\Omega_h w\|_{L^2}\right].\end{aligned}$$

The latter of above inequalities is derived observing that the commutator between Ω_h and $\mathrm{Op}_h^w(A^k)$ is a semi-classical pseudo-differential operator whose symbol is linear combination of terms of the form

$$\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)\theta(x)b_p(\xi),$$

for some new $\gamma_1, \psi, a, \theta, b_p$, and from the fact that operators $\mathrm{Op}_h^w(A^k(x,\xi))$, $\mathrm{Op}_h^w(\xi A^k(x,\xi))$ are bounded on L^2 (see proposition 1.2.27), with norm $O(2^{kp})$, $O(2^{k(p+1)})$ respectively, and that $2^k \le h^{-\sigma}$. □

Proposition 1.2.31. *Under the same hypothesis as proposition 1.2.28,* $\mathrm{Op}_h^w(I_{p,q}^k(x,\xi))$ *and* $\mathrm{Op}_h^w(J_{p,q}^k(x,\xi))$ *are bounded operators from* L^∞ *to* L^2*, with*

(1.2.43)
$$\Big\|\mathrm{Op}_h^w(I_{p,q}^k(x,\xi))\Big\|_{\mathcal{L}(L^2;L^\infty)} + \Big\|\mathrm{Op}_h^w(J_{p,q}^k(x,\xi))\Big\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{i\le 6} 2^{k(p+q)}(h^{-\frac{1}{2}+\sigma}2^k)^i(h^{-1}2^k).$$

The same result holds if $q = 0$ *and* $\widetilde{\psi}(2^{-k}\xi)b_q(\xi) \equiv 1$.

Proof. As in proposition 1.2.28, we prove the statement only for $\mathrm{Op}_h^w(I_{p,q}^k)$, leaving to the reader to check that the result is true also for $\mathrm{Op}_h^w(J_{p,q}^k)$.

Let $w \in L^2$. After lemma 1.2.25 we should prove that $\Big\|\partial_y^\alpha\partial_\xi^\beta\Big[I_{p,q}^k(\frac{x+y}{2}, h\xi)\Big]\Big\|_{L^2(d\xi)}$ is estimated by the right hand side of (1.2.43), for any $|\alpha|, |\beta| \le 3$. A change of variables $\eta \mapsto h\eta$, $\zeta \mapsto h\zeta$ allows us to write $I_{p,q}^k(\frac{x+y}{2}, h\xi)$ as

$$\begin{aligned}\frac{1}{\pi^4}\int e^{2i(\eta\cdot z - y'\cdot\zeta)}\Big[\int_0^1 \Big(\gamma\big(h^{\frac{1}{2}+\sigma}(x|\xi| - \xi)\big)\psi(2^{-k}h\xi)a(x)b_p(h\xi)\Big)|_{(\frac{x+y}{2}+tz,\xi+t\zeta)}f(t)dt \\ \times\widetilde{\psi}(2^{-k}h(\xi+\eta))a'\Big(\frac{x+y}{2}+y'\Big)b_q(h(\xi+\eta))\Big]\,dy'dzd\eta d\zeta.\end{aligned}$$

We observe that, while on the one hand the action of ∂_y^α on the above integral makes appear a factor $(h^{-\frac{1}{2}+\sigma}|h(\xi+t\zeta)|)^i$, with $i \le |\alpha|$, on the other hand that of ∂_ξ^β has basically no effect on the L^2 norm that we want to estimate as one can check using lemma 1.2.26 and the fact that $2^{-k}h \le 1$. With this in mind, we can reduce to the study of the $L^2(d\xi)$ norm of an integral function as

$$\begin{aligned}\sum_{i\le 3}(h^{-\frac{1}{2}+\sigma}2^k)^i\int e^{2i(\eta\cdot z - y'\cdot\zeta)}\Big[\int_0^1 \Big(\gamma\Big(h^{\frac{1}{2}+\sigma}(x|\xi| - \xi)\Big)\psi(2^{-k}h\xi)a(x)b_p(h\xi)\Big)|_{(\frac{x+y}{2}+tz,\xi+t\zeta)}f(t)dt \\ \times\,\widetilde{\psi}(2^{-k}h(\xi+\eta))a'\Big(\frac{x+y}{2}+y'\Big)b_q(h(\xi+\eta))\Big]dy'dzd\eta d\zeta,\end{aligned}$$

for some new functions $\gamma, \psi, a, b_p, \widetilde{\psi}, a', b_q$, with the same properties as their previous homonyms. We use that

$$e^{2i(\eta z - y'\cdot\zeta)} = \left(\frac{1+2iy'\cdot\partial_\zeta}{1+4|y'|^2}\right)^3 \left(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\right)^3 \left(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\right)^3 \left(\frac{1+2i\zeta\cdot\partial_{y'}}{1+4|\zeta|^2}\right)^3 e^{2i(\eta\cdot z - y'\cdot\zeta)}$$

and make some integration by parts to obtain the integrability in $dy'dzd\eta d\zeta$, up to new factors $(h^{-\frac{1}{2}+\sigma}|h(\xi+t\zeta)|)^j$, with $j \leq 3$, coming out from the derivation of the integrand with respect to z. Then, using that functions $h^j b_{p-j}(h(\xi+t\zeta))$ (resp. $h^j b_{q-j}(h(\xi+\eta))$), $j \leq 3$, appearing from the derivation of $b_p(h(\xi+t\zeta))$ with respect to ζ (resp. the derivation of $b_q(h(\xi+\eta))$ with respect to η), are such that $|h^j b_{p-j}(h(\xi+t\zeta))| \leq h^j 2^{k(p-j)} \lesssim 2^{kp}$ on the support of $\psi(2^{-k}h(\xi+t\zeta))$ (resp. $|h^j b_{q-j}(h(\xi+\eta))| \leq 2^{kq}$ on the support of $\widetilde{\psi}(2^{-k}h(\xi+\eta))$), and the fact that

$$\Big\| \int \langle\eta\rangle^{-3} |\widetilde{\psi}(2^{-k}h(\xi+\eta))| d\eta \Big\|_{L^2(d\xi)} \leq \|\widetilde{\psi}(2^{-k}h\cdot)\|_{L^2} \lesssim h^{-1}2^k,$$

we obtain the result of the statement.

The last part of the statement can be proved following an analogous scheme, after having previously made an integration in $dzd\eta$ (or in $dyd\zeta$ if dealing with $J^k_{p,0}$). □

Lemma 1.2.32. *Let $\sigma > 0$ be sufficiently small, $k \in K$ with K given by (1.2.23) and $p, q \in \mathbb{N}$. Let also $\psi, \widetilde{\psi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, $a(x)$ be either a smooth compactly supported function or $a \equiv 1$, and $f \in C(\mathbb{R})$. For a fixed integer $N > 2(p+q)+9$ we define*

$$r^k_{N,p}(x,\xi) := \frac{h^N}{(\pi h)^4} \sum_{|\alpha|=N} \int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \Big[\int_0^1 \partial_x^\alpha \Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+tz,\xi+t\zeta)} \times f(t)dt\Big] \partial_\xi^\alpha \big(b_q(\xi)\widetilde{\psi}(2^{-k}\xi)\big)|_{(x+y,\xi+\eta)}\, dydzd\eta d\zeta, \tag{1.2.44}$$

and

$$\widetilde{r}^k_{N,p}(x,\xi) := \frac{h^N}{(\pi h)^4} \sum_{|\alpha_1|+|\alpha_2|=N} \int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \Big[\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} \Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+tz,\xi+t\zeta)} \times f(t)dt\Big] \partial_x^{\alpha_2}\partial_\xi^{\alpha_1} \big(x_n b_q(\xi)\widetilde{\psi}(2^{-k}\xi)\big)|_{(x+y,\xi+\eta)}\, dydzd\eta d\zeta\,. \tag{1.2.45}$$

Then

$$\|\mathrm{Op}^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2)} + \|\mathrm{Op}^w_h(\widetilde{r}^k_{N,p})\|_{\mathcal{L}(L^2)} + \|\mathrm{Op}^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2;L^\infty)} + \|\mathrm{Op}^w_h(\widetilde{r}^k_{N,p})\|_{\mathcal{L}(L^2;L^\infty)} \lesssim h^{p+q}. \tag{1.2.46}$$

Proof. We remind definition (1.2.30) of integral $I^k_{p,q}(x,\xi)$ for general $k \in K, p, q \in \mathbb{Z}$. After an explicit development of the derivatives appearing in (1.2.44) we find that $r^k_{N,p}(x,\xi)$ may be written as

$$\sum_{j\leq N} h^{N-j(\frac{1}{2}-\sigma)} I^k_{p+j,q-N}(x,\xi)$$

where γ is replaced with γ_1 and $a' \equiv 1$ in $I^k_{p+j,q-N}$. Propositions 1.2.28 and 1.2.31, combined with the fact that $h \leq 2^k \leq h^{-\sigma}$, imply respectively that

$$\begin{aligned}\|\mathrm{Op}^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2)} &\lesssim \sum_{j\leq N} h^{N-j(\frac{1}{2}-\sigma)} 2^{k(p+j+q-N)} \\ &\lesssim \sum_{\substack{j\leq N \\ p+j+q\leq N}} h^{N-j(\frac{1}{2}-\sigma)+p+j+q-N} + \sum_{\substack{j\leq N \\ p+j+q>N}} h^{N-j(\frac{1}{2}-\sigma)-\sigma(p+j+q-N)} \lesssim h^{p+q}\end{aligned}$$

and

$$\|\mathrm{Op}_h^w(r^k_{N,p})\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{i\le 6, j\le N} h^{N-j(\frac12-\sigma)} 2^{k(p+j+q-N)} (h^{-\frac12+\sigma}2^k)^i (h^{-1}2^k)$$

$$\lesssim \sum_{\substack{i\le 6, j\le N\\ p+i+j+q\le N-1}} h^{N-1-(i+j)(\frac12-\sigma)+p+i+j+q-N+1} + \sum_{\substack{i\le 6, j\le N\\ p+i+j+q> N-1}} h^{N-1-(i+j)(\frac12-\sigma)-\sigma(p+i+j+q-N+1)}$$

$$\lesssim h^{p+q},$$

as $N > 2(p+q)+9$.

As regards (1.2.45), we first observe that index α_2 is such that $|\alpha_2| \le 1$ since $x_n b_q(\xi)\widetilde{\psi}(2^{-k}\xi)$ is linear in x_n. An explicit development of derivatives in (1.2.45), combined with lemma 1.2.26, shows that $\widetilde{r}^k_{N,p}(x,\xi)$ splits into two contributions:

$$J_0(x,\xi) = \frac{h^N}{(\pi h)^4} \sum_{i\le N} h^{-i(\frac12-\sigma)} \int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \int_0^1 \Big(\gamma_{1+i}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i}(\xi)\Big)|_{(x+tz,\xi+t\zeta)} f(t)dt$$

$$\times (x_n+y_n) b_{q-N}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))\, dydzd\eta d\zeta,$$

for some new functions $a, \psi, \widetilde{\psi}$ and clear meaning for γ_i, b_{p+i}, b_{q-N}, coming out when $|\alpha_2| = 0$;

$$J_1(x,\xi) = \frac{h^N}{(\pi h)^4} \sum_{i\le N-1, j\le 1} h^{-(i+j)(\frac12-\sigma)} \int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}$$

$$\times \int_0^1 \Big(\gamma_{1+i+j}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+j-1}(\xi)\Big)|_{(x+tz,\xi+t\zeta)} f(t)dt$$

$$\times b_{q-N+1}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))\, dydzd\eta d\zeta,$$

for some new other $a, \psi, \widetilde{\psi}$, corresponding instead to $|\alpha_2| = 1$. One has that

$$J_1(x,\xi) = \sum_{i\le N-1, j\le 1} h^{N-(i+j)(\frac12-\sigma)} I^k_{p+i+j-1,q-N+1}(x,\xi),$$

with γ replaced with γ_1 and $a' \equiv 1$, so propositions 1.2.28 and 1.2.31, along with the fact that $N > 2(p+q)+9$, imply

$$\|\mathrm{Op}_h^w(J_1(x,\xi))\|_{\mathcal{L}(L^2)} \lesssim \sum_{i\le N-1, j\le 1} h^{N-(i+j)(\frac12-\sigma)} 2^{k(p+i+j+q-N)} \lesssim h^{p+q},$$

$$\|\mathrm{Op}_h^w(J_1(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{\substack{i\le N-1, j\le 1\\ l\le 6}} h^{N-(i+j)(\frac12-\sigma)} 2^{k(p+i+j+q-N)} (h^{-\frac12+\sigma}2^k)^l (h^{-1}2^k) \lesssim h^{p+q}.$$

In order to derive the same estimates for $J_0(x,\xi)$ we split the sum $x_n + y_n$ and analyse separately the two out-coming integrals, that we denote $J_{0,x}(x,\xi), J_{0,y}(x,\xi)$. In the latter one, we use that $y_n e^{-\frac{2i}{h}y\cdot\zeta} = -\frac{h}{2i}\partial_{\zeta_n} e^{-\frac{2i}{h}y\cdot\zeta}$ and successively integrate by parts in $d\zeta_n$ obtaining, with the help of lemma 1.2.26, that

$$J_{0,y}(x,\xi) = \sum_{i\le N, j\le 1} h^{N+1-(i+j)(\frac12-\sigma)} \int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \tag{1.2.47}$$

$$\times \int_0^1 \Big(\gamma_{1+i+j}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+j-1}(\xi)\Big)|_{(x+tz,\xi+t\zeta)} f(t)dt$$

$$\times b_{q-N}(\xi-\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))\, dydzd\eta d\zeta$$

for some new functions $a, \psi, \widetilde{\psi}, f$. Again by propositions 1.2.28, 1.2.31 and the fact that $h \leq 2^k \leq h^{-\sigma}$, $N > 2(p+q)+9$, we deduce that:

$$\|\mathrm{Op}_h^w(J_{0,y}(x,\xi))\|_{\mathcal{L}(L^2)} \lesssim \sum_{i\leq N, j\leq 1} h^{N+1-(i+j)(\frac{1}{2}-\sigma)} 2^{k(p+i+j+q-N-1)} \lesssim h^{p+q}, \tag{1.2.48a}$$

(1.2.48b)
$$\|\mathrm{Op}_h^w(J_{0,y}(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{\substack{i\leq N, j\leq 1\\ l\leq 6}} h^{N+1-(i+j)(\frac{1}{2}-\sigma)} 2^{k(p+i+j+q-N-1)} (h^{-\frac{1}{2}-\sigma}2^k)^l (h^{-1}2^k) \lesssim h^{p+q}.$$

In $J_{0,x}(x,\xi)$ we first integrate in $dyd\zeta$ and then we split the occurring integral into two other contributions, called $J_{0,x+tz}(x,\xi), J_{0,tz}(x,\xi)$, by writing $x_n = (x_n + tz_n) - tz_n$. Similarly to what done above, we use that $z_n e^{\frac{2i}{h}\eta\cdot z} = \frac{h}{2i}\partial_{\eta_n} e^{\frac{2i}{h}\eta\cdot z}$ in $J_{0,tz}$, and successively integrate by parts in $d\eta_n$: as $2^{-k}h \leq 1$, we obtain that $J_{0,tz}$ has the same form as (1.2.47) for some new $b_{q-N}, \widetilde{\psi}$, and verifies (1.2.48). Finally, using that $x_n + tz_n = h^{\frac{1}{2}-\sigma}\big(\frac{(x_n+tz_n)|\xi|-\xi_n}{h^{1/2-\sigma}}\big)|\xi|^{-1} + \xi_n|\xi|^{-1}$, we derive that

$$\begin{aligned}
&J_{0,x+tz}(x,\xi)\\
&= \sum_{i\leq N} h^{N-(i-1)(\frac{1}{2}-\sigma)} \int e^{\frac{2i}{h}\eta\cdot z} \int \Big(\gamma_i\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i-1}(\xi)\Big)|_{(x+tz,\xi)} f(t)dt\\
&\qquad\qquad \times b_{q-N}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))dzd\eta,\\
&+ \sum_{i\leq N} h^{N-i(\frac{1}{2}-\sigma)} \int e^{\frac{2i}{h}\eta\cdot z} \int \Big(\gamma_{1+i}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i}(\xi)\Big)|_{(x+tz,\xi)} f(t)dt\\
&\qquad\qquad \times b_{q-N}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))dzd\eta,
\end{aligned}$$

so by propositions 1.2.28 and 1.2.31

$$\|\mathrm{Op}_h^w(J_{0,x+tz}(x,\xi))\|_{\mathcal{L}(L^2)} \lesssim \sum_{i\leq N} h^{N-i(\frac{1}{2}-\sigma)} 2^{k(p+i+q-N)} \lesssim h^{p+q},$$

$$\|\mathrm{Op}_h^w(J_{0,x+tz}(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{i\leq N, l\leq 3} h^{N-i(\frac{1}{2}-\sigma)} 2^{k(p+i+q-N)} (h^{-\frac{1}{2}+\sigma}2^k)^l (h^{-1}2^k) \lesssim h^{p+q}.$$

That concludes the proof as $\widetilde{r}_{N,p}^k = J_{0,x+tz} + J_{0,tz} + J_{0,y} + J_1$. □

We introduce the following operator:

$$\mathcal{M}_j := \frac{1}{h}\mathrm{Op}_h^w(x_j|\xi| - \xi_j), \quad j = 1,2 \tag{1.2.49}$$

and use the notation $\|\mathcal{M}^\gamma w\| = \|\mathcal{M}_1^{\gamma_1}\mathcal{M}_2^{\gamma_2} w\|$ for any $\gamma = (\gamma_1,\gamma_2) \in \mathbb{N}^2$. We have now all the ingredients to state and prove the following two results.

Lemma 1.2.33. *Let σ, k, p, ψ, a be as in lemma 1.2.32 and $\widetilde{a}(x)$ such that*

$$(a \equiv 1) \Rightarrow (\widetilde{a} \equiv 1),$$
$$(a \textit{ compactly supported }) \Rightarrow [(\widetilde{a} \equiv 1) \textit{ or } (\widetilde{a} \textit{ compactly supported and } \widetilde{a}a \equiv a)].$$

We have that

$$\begin{aligned}
&\mathrm{Op}_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi|-\xi_n)\Big)\\
&\qquad = \mathrm{Op}_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)\widetilde{a}(x)h\mathcal{M}_n + \mathrm{Op}_h^w(r_p^k(x,\xi)),
\end{aligned} \tag{1.2.50}$$

where

$$\big\|\mathrm{Op}_h^w(r_p^k(x,\xi))w\big\|_{L^2} \lesssim h^{1-\beta}\|w\|_{L^2}, \tag{1.2.51a}$$

$$\big\|\mathrm{Op}_h^w(r_p^k(x,\xi))w\big\|_{L^\infty} \lesssim h^{\frac{1}{2}-\beta}(\|w\|_{L^2}+\|\theta_0\Omega_h w\|_{L^2}), \tag{1.2.51b}$$

for some $\theta \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ *and a small* $\beta>0$, $\beta\to 0$ *as* $\sigma\to 0$. *Moreover*

$$\Big\|\mathrm{Op}_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi|-\xi_n)\Big)w\Big\|_{L^2} \lesssim h^{1-\beta}\big(\|w\|_{L^2}+\|\mathcal{M}_n w\|_{L^2}\big), \tag{1.2.52a}$$

$$\begin{aligned}\Big\|\mathrm{Op}_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi|-\xi_n)\Big)w\Big\|_{L^\infty}&\\ \lesssim h^{\frac{1}{2}-\beta}\sum_{\mu=0}^{1}\Big(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}_n w\|_{L^2}\Big).&\end{aligned} \tag{1.2.52b}$$

Proof. The proof of the statement is basically made of tedious calculations and the application of propositions 1.2.27, 1.2.30 along with lemma 1.2.32.

Let $\widetilde{\psi}\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ such that $\widetilde{\psi}\equiv 1$ on the support of ψ. From formulas (1.2.18), (1.2.19) and the hypothesis of the statement we derive that for a fixed $N\in\mathbb{N}$, and up to negligible multiplicative constants,

$$\begin{aligned}&\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\sharp\Big[(x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\Big]\\
&=\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi|-\xi_n)\\
&+h\Big\{\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi),(x_n|\xi|-\xi_n)\Big\}\\
&+\sum_{\substack{2\le|\alpha|<N\\ |\alpha_1|+|\alpha_2|=|\alpha|}}h^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\big[(x_n|\xi|-\xi_n)\big]+r_{N,p}^k(x,\xi),\end{aligned} \tag{1.2.53}$$

with

$$\begin{aligned}r_{N,p}^k(x,\xi)=\frac{h^N}{(\pi h)^4}\sum_{|\alpha_1|+|\alpha_2|=N}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\Big[\int_0^1\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]|_{(x+tz,\xi+t\zeta)}&\\ \times(1-t)^{N-1}dt\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\big[(x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\big]|_{(x+y,\xi+\eta)}\,dydzd\eta d\zeta\,.&\end{aligned} \tag{1.2.54}$$

If $\widetilde{a}\equiv 1$ above $r_{N,p}^k$ can be decomposed into the sum of integrals of the form (1.2.44) and (1.2.45) with $q=1$, so

$$\Big\|\mathrm{Op}_h^w(r_{N,p}^k)\Big\|_{\mathcal{L}(L^2)}+\Big\|\mathrm{Op}_h^w(r_{N,p}^k)\Big\|_{\mathcal{L}(L^2;L^\infty)}\lesssim h^{1+p} \tag{1.2.55}$$

if N is taken sufficiently large (e.g. $N>2p+11$). The same is true if functions $a,\widetilde{a}$ are compactly supported as follows by propositions 1.2.28 and 1.2.31, since from lemma 1.2.26 and definition (1.2.30) of $I_{p,q}^k$ for general $k\in K, p,q\in\mathbb{Z}$

$$r_{N,p}^k(x,\xi)=\sum_{\substack{|\alpha_1|+|\alpha_2|=N\\ i\le|\alpha_1|,1\le j\le|\alpha_2|}}h^{N-(i+j)(\frac{1}{2}-\sigma)}I_{p+i+j-|\alpha_2|,1-|\alpha_1|}^k(x,\xi).$$

An explicit computation of the Poisson bracket in (1.2.53) shows that it is equal to

$$h(\partial\gamma_1)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_1\xi_2-x_2\xi_1}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi) + \sum{}'h\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi), \tag{1.2.56}$$

where $\sum'$ is a concise notation to indicate a linear combination, and ψ, a, b_p are some new functions with the same features of their homonyms. After writing

$$(x_1\xi_2-x_2\xi_1) = (x_1|\xi|-\xi_1)\xi_2|\xi|^{-1} - (x_2|\xi|-\xi_2)\xi_1|\xi|^{-1}, \tag{1.2.57}$$

we recognize that the quantization of (1.2.56) verifies estimates (1.2.51) thanks to propositions 1.2.27, 1.2.30 and the fact that $2^{kp}\leq h^{-\sigma p}$.

Let us denote concisely by t^k_α the $|\alpha|$-order contributions in (1.2.53), for $2\leq|\alpha|<N$. As factor $x_n|\xi|-\xi_n$ is affine in x_n, the length of multi-index α_2 is less or equal than 1 and, using lemma 1.2.26, t^k_α appears to be the sum of two terms. The first one corresponds to $|\alpha_2|=0$ and has the form

$$\sum_{\substack{i\leq|\alpha|\\ \mu=0,1}}{}'h^{|\alpha|-i(\frac12-\sigma)}\gamma_{1+i}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+1-|\alpha|}(\xi)\,x_n^\mu,$$

for some new functions ψ, a. Observe that $\mu=0$ if $a\equiv1$ because the derivation of $\gamma_1\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)$ $|\alpha_1|$-times with respect to x makes appear, inter alia, a factor $|\xi|^{|\alpha_1|}$ that allows us to rewrite $\partial^{\alpha_1}_\xi(x_n|\xi|-\xi_n)$ from $(x_n|\xi|-\xi_n)+b_0(\xi)$, for some new b_0, and $\partial^{\alpha_1}_z\gamma_1(z)z_n$ is of the form $\gamma_{|\alpha_1|}(z))$. The second term, corresponding instead to $|\alpha_2|=1$, is given by

$$\sum_{i\leq|\alpha|-1,j\leq1}{}'h^{|\alpha|-(i+j)(\frac12-\sigma)}\gamma_{1+i+j}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+j+1-|\alpha|}(\xi),$$

for some new other functions ψ, a. From propositions 1.2.27, 1.2.30 we then deduce that

$$\|\mathrm{Op}^w_h(t^k_\alpha)w\|_{L^2}\lesssim(h^{\frac{|\alpha|}{2}-\beta}+h^{1+p})\|w\|_{L^2}, \tag{1.2.58a}$$

$$\|\mathrm{Op}^w_h(t^k_\alpha)w\|_{L^\infty}\lesssim(h^{\frac{|\alpha|-1}{2}-\beta}+h^{\frac12+p})(\|w\|_{L^2}+\|\theta\Omega_h w\|_{L^2}), \tag{1.2.58b}$$

which concludes that

$$\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\sharp\Big[(x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\Big] = \gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi|-\xi_n)+r^k_p(x,\xi),$$

with r^k_p satisfying (1.2.51).

Finally, by symbolic calculus we have that, up to some multiplicative constants,

$$\begin{aligned}\mathrm{Op}^w_h\big((x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\big) &= \widetilde{a}(x)\mathrm{Op}^w_h\big((x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)\big)+\mathrm{Op}^w_h(r^k(x,\xi))\\ &= \mathrm{Op}^w_h(\widetilde{\psi}(2^{-k}\xi))\widetilde{a}(x)h\mathcal{M}_n+h\widetilde{a}(x)\mathrm{Op}^w_h((\partial\widetilde{\psi})(2^{-k}\xi)(2^{-k}|\xi|))\\ &+\mathrm{Op}^w_h(\widetilde{r}^k(x,\xi))h\mathcal{M}_n+\mathrm{Op}^w_h(r^k(x,\xi)),\end{aligned}$$

where

$$r^k(x,\xi) = \frac{h}{(2\pi)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int \partial_x\widetilde{a}(x+tz)dt\ \partial_\xi\big[(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)\big]|_{(x,\xi+\eta)}dzd\eta,$$
$$\widetilde{r}^k(x,\xi) = \frac{h2^{-k}}{(2\pi)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int \partial_x\widetilde{a}(x+tz)dt\ (\partial_\xi\widetilde{\psi})(2^{-k}(\xi+\eta))dzd\eta,$$

are such that $\|\mathrm{Op}_h^w(r_1^k)\|_{\mathcal{L}(L^2)} = O(h)$, $\|\mathrm{Op}_h^w(\widetilde{r}_1^k)\|_{\mathcal{L}(L^2)} = O(1)$. An explicit computation shows also that $\|[\Omega_h, \mathrm{Op}_h^w(r^k)]\|_{\mathcal{L}(L^2)} = O(h)$ and $\|[\Omega_h, \mathrm{Op}_h^w(\widetilde{r}^k)]\|_{\mathcal{L}(L^2)} = O(1)$. Therefore, since $\widetilde{\psi}\equiv 1$ on the support of ψ, $\widetilde{a}\equiv 1$ on the support of a, one can use remark 1.2.22 together with propositions 1.2.28, 1.2.31, and also propositions 1.2.27, 1.2.30, to show that

$$\mathrm{Op}_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)\mathrm{Op}_h^w\big((x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\big)$$
$$= \mathrm{Op}_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)\widetilde{a}(x)h\mathcal{M}_n + \mathrm{Op}_h^w(r_p^k(x,\xi)),$$

for a new $\mathrm{Op}_h^w(r_p^k(x,\xi))$ satisfying (1.2.51a). This concludes the proof of (1.2.50) and of the entire statement by applying propositions 1.2.27, 1.2.30 to the first operator in the above right hand side. □

Lemma 1.2.34. *Let $\sigma > 0$ be small, $k \in K$ with K given by (1.2.23) and $p \in \mathbb{N}$. Let also $\gamma \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, $\psi \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, and $a \in C_0^\infty(\mathbb{R}^2)$. For any function $w \in L^2(\mathbb{R}^2)$ such that $\mathcal{M}w \in L^2(\mathbb{R}^2)$, any $m,n = 1,2$, we have that*

$$\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)(x_n|\xi|-\xi_n)\Big)w$$
$$= \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)\Big)[h\mathcal{M}_n w] + O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big),$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. Let $\widetilde{\gamma}(z) := \gamma(z)z_m$ and $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ be identically equal to 1 on the support of ψ. We saw in the proof of the previous lemma that the symbolic product

$$\Big[\widetilde{\gamma}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\sharp[(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)]$$

develops as in (1.2.53), (1.2.54), with γ_1 replaced with $\widetilde{\gamma}$ and $\widetilde{a}\equiv 1$. From (1.2.56), the fact that

$$\{x_m|\xi|-\xi_m, x_n|\xi|-\xi_n\} = \begin{cases} 0 & \text{if } m=n,\\ (-1)^{m+1}(x_1\xi_2-\xi_2x_1) & \text{if } m\neq n,\end{cases}$$

and that $(x_1\xi_2-\xi_2x_1) = (x_1|\xi|-\xi_1)\xi_2|\xi|^{-1} - (x_2|\xi|-\xi_2)\xi_1|\xi|^{-1}$, we derive that the first order term of the mentioned symbolic development is a linear combination of products of the form

$$h^{\frac{3}{2}}\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_j|\xi|-\xi_j),$$

for some new functions γ,ψ,a, and its quantization acting on w is a remainder as in the statement after estimate (1.2.52a).

The second order term is given, up to some negligible multiplicative constants, by

$$h^{1+2\sigma}\sum_{|\alpha|=2}(\partial^\alpha\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a_1(x)b_{p+1}(\xi)(x_m|\xi|-\xi_m)$$
$$+h^{\frac{3}{2}+\sigma}\sum_{|\alpha|=1}(\partial^\alpha\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi_2(2^{-k}\xi)a_2(x)b_{p+1}(\xi)$$
$$+h^2{\sum}'\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi_3(2^{-k}\xi)a_3(x)b_{p+1}(\xi),$$

for some new smooth compactly supported $\psi_2,\psi_3,a_1,a_2,a_3$, and as the derivatives of γ vanish in a neighbourhood of the origin we can replace $(\partial^\alpha\gamma)(z)$ with $\sum_j\gamma_1^j(z)z_j$, $\gamma_j^1(z):=(\partial^\alpha\gamma)(z)z_j|z|^{-2}$, when $|\alpha|=1$. The third order one is given by

$$h^{\frac{3}{2}+3\sigma}\sum_{|\alpha|=3}(\partial^\alpha\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a_1(x)b_{p+1}(\xi)(x_m|\xi|-\xi_m)$$
$$+h^2{\sum}'\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi_1(2^{-k}\xi)a_2(x)b_{p+1}(\xi),$$

for some other ψ_1,a_1,a_2 and a new $\gamma_1\in C_0^\infty(\mathbb{R}^2)$. Using estimate (1.2.52a) for the summations in α and proposition 1.2.27 for the remaining terms in the above expressions we obtain that the quantizations of the second and third order term are also a $O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big)$ when acting on w, for a small $\beta>0$, $\beta\to0$ as $\sigma\to0$.

In all the other $|\alpha|$-order terms, with $4\le|\alpha|\le N-1$, and in integral remainder $r_{N,p}^k$, we look at $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)$ as a symbol of the form

$$\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+1}(\xi)$$

for a new $a_1\in C_0^\infty(\mathbb{R}^2)$. From (1.2.58a) and (1.2.55) when $N>11$, we derive that the quantizations of these terms are also a $O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big)$ when acting on w.

We finally obtain that

$$\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)(x_n|\xi|-\xi_n)\Big)w$$
$$=\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)\Big)\mathrm{Op}_h^w\big((x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)\big)$$
$$+O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big),$$

and the conclusion of the proof comes then from the fact that, by symbolic calculus,

$$\mathrm{Op}_h^w\big((x_n|\xi|-\xi_n)\widetilde{\psi}_1(2^{-k}\xi)\big)=h\mathrm{Op}_h^w(\widetilde{\psi}_1(2^{-k}\xi))\mathcal{M}_n-\frac{h}{2i}\mathrm{Op}_h^w\big((\partial\widetilde{\psi}_1)(2^{-k}\xi)\cdot(2^{-k}\xi)\big),$$

and by remark 1.2.22 as all derivatives of $\widetilde{\psi}$ vanish on the support of ψ. □

The following lemma is introduced especially for the proof of lemma 3.2.13. Even if quite similar to lemma 1.2.33, we are going to see that the particular structure of symbolic product in the left hand side of (1.2.59) allows for a remainder r_p^k satisfying enhanced estimate (1.2.60b) rather than (1.2.51b).

Lemma 1.2.35. *Let us take $\sigma > 0$ sufficiently small, $k \in K$ and $p, q \in \mathbb{N}$. Let also $\gamma \in C_0^\infty(\mathbb{R}^2)$ such that $\gamma \equiv 1$ in a neighbourhood of the origin, $\psi, \widetilde{\psi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\psi \equiv 1$ on the support of $\widetilde{\psi}$, $a(x)$ be a smooth compactly supported function. Then*

$$\Big[(x_n|\xi| - \xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)\Big] \sharp \Big[\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big] = \gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi| - \xi_n) + r_p^k(x,\xi), \tag{1.2.59}$$

where

$$\Big\|\mathrm{Op}_h^w(r_p^k(x,\xi))w\Big\|_{L^2} \lesssim h^{\frac{3}{2}-\beta}(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}) + h^{1+p}\|w\|_{L^2}, \tag{1.2.60a}$$

$$\Big\|\mathrm{Op}_h^w(r_p^k(x,\xi))w\Big\|_{L^\infty} \lesssim h^{1-\beta}\sum_{\mu=0}^{1}\Big(\|(\theta_0\Omega_h)^\mu w\|_{L^2} + \|(\theta_0\Omega_h)^\mu \mathcal{M}w\|_{L^2}\Big), \tag{1.2.60b}$$

for some $\theta \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, and a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Proof. Using proposition 1.2.21, for a fixed $N \in \mathbb{N}$ and up to multiplicative constants independent of h, k, we have the following symbolic development:

$$\begin{aligned}
&\Big[(x_n|\xi| - \xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)\Big] \sharp \Big[\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big] \\
&= \gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k})a(x)b_p(\xi)(x_n|\xi| - \xi_n) \\
&+ h\left\{(x_n|\xi| - \xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi), \gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\right\} \\
&+ \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 2\le|\alpha|<N}} h^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[(x_n|\xi| - \xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\Big[\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\Big] + r_{N,p}^k(x,\xi),
\end{aligned} \tag{1.2.61}$$

with

$$r_{N,p}^k(x,\xi) = \frac{h^N}{(\pi h)^4}\sum_{|\alpha_1|+|\alpha_2|=N}\int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}\Big[\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\big[(x_n|\xi| - \xi_n)a(x)b_p(\xi)\widetilde{\psi}(2^{-k}\xi)\big]|_{(x+tz,\xi+t\zeta)} \times (1-t)^{N-1}dt\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\Big[\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big]|_{(x+y,\xi+\eta)}\, dydzd\eta d\zeta.$$

For sake of simplicity, we denote by t_1^k (resp. t_α^k, $|\alpha| = 2, \dots, N-1$) the Poisson brackets (resp. the $|\alpha|$-th contribution) in (1.2.61). An explicit computation of t_1^k, combined with the fact that $x_1\xi_2 - x_2\xi_1 = (x_1|\xi| - \xi_1)\xi_2|\xi|^{-1} - (x_2|\xi| - \xi_2)\xi_1|\xi|^{-1}$, shows that it is linear combination of terms of the form

$$h(\partial\gamma)\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi| - \xi_j}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi),$$

for $j \in \{1, 2\}$ and some new functions $\widetilde{\psi}, a, b_p$, so by inequalities (1.2.52) we derive that

$$\Big\|\mathrm{Op}_h^w(t_1^k)w\Big\|_{L^2} \lesssim h^{\frac{3}{2}-\beta}\left(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\right), \tag{1.2.62a}$$

(1.2.62b) $$\Big\|\mathrm{Op}_h^w(t_1^k)w\Big\|_{L^\infty} \lesssim h^{1-\beta}\sum_{\mu=0}^{1}(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu \mathcal{M}w\|_{L^2}).$$

The improvement of these estimates with respect to (1.2.51) is attributable to the choice of ψ identically equal to 1 on the support of $\widetilde{\psi}$. All derivatives of ψ vanish against $\widetilde{\psi}$, so in the development of t_1^k we avoid terms like $\gamma\big(\frac{x|\xi|-\xi|}{h^{1/2-\sigma}}\big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)(\partial\psi)(2^{-k}\xi)(2^{-k}|\xi|)$, coming out from $\{x_n|\xi|-\xi_n,\psi(2^{-k}\xi)\}\gamma\big(\frac{x|\xi|-\xi|}{h^{1/2-\sigma}}\big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)$, that do not enjoy estimates like (1.2.62).

Using formula (1.2.25) and looking at $(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)$ as a linear combination of terms $\widetilde{\psi}(2^{-k}\xi)a(x)b_{p+1}(\xi)$, for some new $\widetilde{\psi},a,b_{p+1}$, we realize that, for any $2\leq|\alpha|<N$,

$$t_\alpha^k=\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\leq j\leq|\alpha_1|}} h^{|\alpha|-(j+|\alpha_2|)(\frac12-\sigma)}\gamma_{j+|\alpha_2|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a_j(x)b_{p+j+1-|\alpha_1|}(\xi),$$

for some new other $\widetilde{\psi},a_j$, with a_j compactly supported, and then that

$$\|\mathrm{Op}_h^w(t_\alpha^k)w\|_{L^2}\lesssim\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\leq j\leq|\alpha_1|}} h^{|\alpha|-(j+|\alpha_2|)(\frac12-\sigma)}2^{k(p+j+1-|\alpha_1|)}\|w\|_{L^2},$$

$$\begin{aligned}&\|\mathrm{Op}_h^w(t_\alpha^k)w\|_{L^\infty}\\&\lesssim\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\leq j\leq|\alpha_1|}} h^{|\alpha|-(j+|\alpha_2|)(\frac12-\sigma)}2^{k(p+j+1-|\alpha_1|)}h^{-\frac12-\sigma}\sum_{\mu=0}^{1}(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2}),\end{aligned}$$

after propositions 1.2.27, 1.2.30. For $|\alpha|\geq3$, the above estimates imply $\|\mathrm{Op}_h^w(t_\alpha^k)\|_{\mathcal{L}(L^2)}\lesssim h^{\frac32-\beta}$ and $\|\mathrm{Op}_h^w(t_\alpha^k)w\|_{L^\infty}\lesssim h^{1-\beta}\sum_{\mu=0}^1(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2})$. For $|\alpha|=2$, we exploit the fact that functions $\gamma_{j+|\alpha_2|}$ vanish in a neighbourhood of the origin, as they come from γ's derivatives, and define $\gamma^i_{j+|\alpha_2|}(z):=\gamma_{j+|\alpha_2|}(z)z_i|z|^{-2}$, $i=1,2$, so that

$$t_\alpha^k=\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\leq j\leq|\alpha_1|,i=1,2}} h^{|\alpha|-(j+|\alpha_2|)(\frac12-\sigma)}\gamma^i_{j+|\alpha_2|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_i|\xi|-\xi_i}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a_j(x)b_{p+j+1-|\alpha_1|}(\xi),$$

to which we can then apply lemma 1.2.33. After inequalities (1.2.52), $\mathrm{Op}_h^w(t_\alpha^k)$ with $|\alpha|=2$ also satisfies (1.2.62).

Finally, reminding definition (1.2.31) of $J^k_{p,q}(x,\xi)$ for general $k\in K,p,q\in\mathbb{Z}$, and developing derivatives in $r^k_{N,p}$ using lemma 1.2.26, we observe that

$$r_{N,p}^k=\sum_{\substack{|\alpha_1|+|\alpha_2|=N\\0\leq j\leq|\alpha_1|}} h^{N-(|\alpha_2|+j)(\frac12-\sigma)}J^k_{p+1-|\alpha_2|,|\alpha_2|+j-|\alpha_1|}(x,\xi),$$

hence propositions 1.2.28 and 1.2.31 give that

$$\|\mathrm{Op}_h^w(r_{N,p}^k)\|_{\mathcal{L}(L^2)}\lesssim\sum_{\substack{|\alpha_1|+|\alpha_2|=N\\0\leq j\leq|\alpha_1|}} h^{N-(|\alpha_2|+j)(\frac12-\sigma)}2^{k(p+1+j-|\alpha_1|)}\lesssim h^{1+p},$$

$$\|\mathrm{Op}_h^w(r_{N,p}^k)\|_{\mathcal{L}(L^2;L^\infty)}\lesssim\sum_{\substack{|\alpha_1|+|\alpha_2|=N\\0\leq j\leq|\alpha_1|,i\leq6}} h^{N-(|\alpha_2|+j)(\frac12-\sigma)}2^{k(p+1+j-|\alpha_1|)}(h^{-\frac12+\sigma}2^k)^i(h^{-1}2^k)\lesssim h^{1+p},$$

if N is chosen sufficiently large (e.g. $N > 10 + 2p$). We should also highlight the fact that, at the difference of (1.2.60b), (1.2.60a) does not improve (1.2.51a): if we get a $h^{\frac{3}{2}-\beta}$ factor in front of the first term in the right hand side, the second term $h^{1+p}\|w\|_{L^2}$ is just a $O(h^{1-\beta})$ in the case $p = 0$, coming from $|\alpha_1| = N$, $j = |\alpha_2| = 0$, $p = 0$ above. □

1.2.4 Operators for the Klein-Gordon solution: some estimates

This subsection is mostly devoted to the introduction of some symbols and operators, along with their properties, that we will often use in the paper when dealing with the Klein-Gordon component of the solution to starting system (1.1.1). From now on we will use the notation $p(\xi) := \sqrt{1 + |\xi|^2}$ (thus, $p'(\xi)$ denotes the gradient of $p(\xi)$, $p''(\xi) = (\partial^2_{ij} p(\xi))_{ij}$ the 2×2 Hessian matrix of $p(\xi)$).

Proposition 1.2.36 is a general result about continuity on spaces $H^s_h(\mathbb{R}^2)$ of operators with symbols of order $r \in \mathbb{R}$ and generalises theorem 7.11 in [8]. Proposition 1.2.37 is a result of continuity from L^2 to $H^{\rho,\infty}_h$ of a particular class of operators that will act on the Klein-Gordon component. In the spirit of [14] for the Schrödinger equation, it allows to pass from uniform norms to the L^2 norm losing only a power $h^{-\frac{1}{2}-\beta}$ for a small $\beta > 0$ instead of a h^{-1} as for the semi-classical Sobolev injection. Proposition 1.2.39 is, instead, a result of uniform $L^p - L^p$ continuity of such operators, for every $1 \leq p \leq +\infty$.

Proposition 1.2.36 (Continuity on H^s_h)**.** *Let $s \in \mathbb{R}$. Let $a \in S_{\delta,\sigma}(\langle\xi\rangle^r)$, $r \in \mathbb{R}$, $\delta \in [0, \frac{1}{2}]$, $\sigma \geq 0$. Then $\mathrm{Op}^w_h(a)$ is uniformly bounded : $H^s_h(\mathbb{R}^2) \to H^{s-r}_h(\mathbb{R}^2)$ and there exists a positive constant C independent of h such that*

$$\|\mathrm{Op}^w_h(a)\|_{\mathcal{L}(H^s_h; H^{s-r}_h)} \leq C\,, \qquad \forall h \in]0, 1]\,.$$

Proposition 1.2.37 (Continuity from L^2 to $H^{\rho,\infty}_h$)**.** *Let $\rho \in \mathbb{N}$. Let $a \in S_{\delta,\sigma}(\langle \frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\delta \in [0, \frac{1}{2}]$, $\sigma > 0$. Then $\mathrm{Op}^w_h(a)$ is bounded : $L^2(\mathbb{R}^2) \to H^{\rho,\infty}_h(\mathbb{R}^2)$ and there exists a positive constant C independent of h such that*

$$\|\mathrm{Op}^w_h(a)\|_{\mathcal{L}(L^2; H^{\rho,\infty}_h)} \leq C h^{-\frac{1}{2}-\beta}\,, \qquad \forall h \in]0, 1]\,,$$

where $\beta > 0$ depends linearly on σ.

Proof. We first remark that, after definition 1.2.18 (i) of the $H^{\rho,\infty}_h$ norm,

$$\|\mathrm{Op}^w_h(a)w\|_{H^{\rho,\infty}_h} = \|\langle hD_x\rangle^\rho \mathrm{Op}^w_h(a)w\|_{L^\infty},$$

and that, by symbolic calculus of lemma 1.2.24, $\langle\xi\rangle^\rho \sharp a(x,\xi)$ belongs to $S_{\delta,\sigma}(\langle\xi\rangle^\rho \big\langle \frac{x-p'(\xi)}{\sqrt{h}}\big\rangle^{-1}) \subset h^{-\rho\sigma} S_{\delta,\sigma}(\big\langle \frac{x-p'(\xi)}{\sqrt{h}}\big\rangle^{-1})$. This means that estimating the $H^{\rho,\infty}_h$ norm of an operator whose symbol is rapidly decaying in $|h^\sigma\xi|$ corresponds actually to estimate the L^∞ norm of an operator associated to another symbol (namely, $\tilde{a}(x,\xi) = \langle\xi\rangle^\rho \sharp a(x,\xi)$) which is still in the same class as a, up to a small loss $h^{-\rho\sigma}$.

From definition 1.2.17 (i) of $\mathrm{Op}^w_h(a)w$, and using a change of coordinates $y \mapsto \sqrt{h}y$, $\xi \mapsto \sqrt{h}\xi$, integration by part, Cauchy-Schwarz inequality, and Young's inequality for convolutions, we

derive what follows:

(1.2.63)

$$
\begin{aligned}
&|\mathrm{Op}_h^w(a)w| = \\
&= \Big| \frac{1}{(2\pi)^2} \int\int e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi} a\Big(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi\Big) w(\sqrt{h}y)\, dyd\xi \Big| \\
&= \Big| \frac{1}{(2\pi)^4 h} \int \hat{w}\Big(\frac{\eta}{\sqrt{h}}\Big) d\eta \int\int e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi + i\eta\cdot y} a\Big(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi\Big)\, dyd\xi \Big| \\
&= \Big| \frac{1}{(2\pi)^4 h} \int \hat{w}\Big(\frac{\eta}{\sqrt{h}}\Big) \int\int \left(\frac{1 - i\big(\frac{x}{\sqrt{h}} - y\big)\cdot\partial_\xi}{1 + |\frac{x}{\sqrt{h}} - y|^2}\right)^3 \left(\frac{1 + i(\xi-\eta)\cdot\partial_y}{1+|\xi-\eta|^2}\right)^3 e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi + i\eta\cdot y} \\
&\qquad \times a\Big(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi\Big)\, dyd\xi d\eta \Big| \\
&\lesssim \frac{1}{h} \int \Big|\hat{w}\Big(\frac{\eta}{\sqrt{h}}\Big)\Big| \int\int \Big\langle \frac{x}{\sqrt{h}} - y\Big\rangle^{-3} \langle \xi-\eta\rangle^{-3} \langle h^\sigma \sqrt{h}\xi\rangle^{-N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1} dyd\xi d\eta \\
&\lesssim \frac{1}{h} \Big\|\hat{w}\Big(\frac{\cdot}{\sqrt{h}}\Big)\Big\|_{L^2} \|\langle\eta\rangle^{-3}\|_{L^1(\eta)} \Big\| \int \Big\langle \frac{x}{\sqrt{h}} - y\Big\rangle^{-3} \langle h^\sigma\sqrt{h}\xi\rangle^{-N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1} dy \Big\|_{L^2(d\xi)} \\
&\lesssim h^{-\frac{1}{2}} \|w\|_{L^2} \int \Big\langle \frac{x}{\sqrt{h}} - y\Big\rangle^{-3} \Big\| \langle h^\sigma\sqrt{h}\xi\rangle^{-N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1} \Big\|_{L^2(\xi)} dy\,,
\end{aligned}
$$

where $N > 0$ will be properly chosen later. We draw attention to two facts: in the third equality in (1.2.63) we use that

$$
\left(\frac{1 - i(\frac{x}{\sqrt{h}} - y)\cdot\partial_\xi}{1 + (\frac{x}{\sqrt{h}} - y)^2}\right)^3 \left(\frac{1 + i(\xi-\eta)\cdot\partial_y}{1+(\xi-\eta)^2}\right)^3 \Big[e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi + i\eta\cdot y}\Big] = e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi + i\eta\cdot y}
$$

so, integrating by part, derivatives ∂_y, ∂_ξ fall on $\langle \frac{x}{\sqrt{h}} - y\rangle^{-1}$, $\langle \xi - \eta\rangle^{-1}$ (giving rise to more decreasing factors) and/or on $a\left(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi\right)$; symbol a belongs to $S_{\delta,\sigma}(1)$ with $\delta \le \frac{1}{2}$, but the loss of $h^{-\delta}$ is offset by the factor $\sqrt{h}$ coming from the derivation of $a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi)$ with respect to y and ξ.

In order to estimate $\big\|\langle h^\sigma\sqrt{h}\xi\rangle^{-N}\big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\big\rangle^{-1}\big\|_{L^2_\xi}$ we first introduce a smooth cut-off function $\chi(\frac{x+\sqrt{h}y}{2})$, with χ supported in some ball $B_C(0)$, to distinguish between the case when $\frac{x+\sqrt{h}y}{2}$ is bounded from the one where $|\frac{x+\sqrt{h}y}{2}| \to +\infty$. In the latter situation, say for $|\frac{x+\sqrt{h}y}{2}| > 2$, we have $\big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\big\rangle^{-1} \lesssim \sqrt{h}$ and

$$
\Big|(1-\chi)\Big(\frac{x+\sqrt{h}y}{2}\Big)\Big| \Big\| \langle h^\sigma\sqrt{h}\xi\rangle^{-N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1} \Big\|_{L^2(d\xi)} \lesssim h^{-\sigma}.
$$

On the other hand, when $\frac{x+\sqrt{h}y}{2}$ is bounded we consider a Littlewood-Paley decomposition and

write

(1.2.64)
$$\Big\| \langle h^\sigma \sqrt{h}\xi\rangle^{-N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}} \Big\rangle^{-1} \Big\|^2_{L^2(\xi)} = h^{-1} \sum_{k\geq 0} \int \langle h^\sigma \xi\rangle^{-2N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\xi)}{\sqrt{h}} \Big\rangle^{-2} \varphi_k(\xi) d\xi$$
$$= h^{-1} \sum_{k\geq 0} I_k$$

where

$$I_0 = \int \langle h^\sigma \xi\rangle^{-2N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\xi)}{\sqrt{h}} \Big\rangle^{-2} \varphi_0(\xi) d\xi$$

and

$$\begin{aligned} I_k &= \int \langle h^\sigma \xi\rangle^{-2N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\xi)}{\sqrt{h}} \Big\rangle^{-2} \varphi(2^{-k}\xi) d\xi \\ &= 2^{2k} \int \langle h^\sigma 2^k \xi\rangle^{-2N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(2^k\xi)}{\sqrt{h}} \Big\rangle^{-2} \varphi(\xi) d\xi \qquad k \geq 1 \\ &\lesssim 2^{(-2N+2)k} h^{-2\sigma N} \int \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(2^k\xi)}{\sqrt{h}} \Big\rangle^{-2} \varphi(\xi) d\xi \,. \end{aligned} \tag{1.2.65}$$

For a fixed k_0 and any $k \leq k_0$, $|\det(p''(2^k\xi))| \geq C > 0$ on the support of φ. For $k \geq k_0$, function $\xi \to g_k(\xi) = 2^{3k}(\frac{x+\sqrt{h}y}{2}) - 2^{3k}p'(2^k\xi)$ is such that $\det(g_k'(\xi)) = \frac{2^{4k}}{(1+|2^k\xi|^2)^2}$ and $|\det(g_k'(\xi))| \sim 1$ on the support of φ. We may thus split the $d\xi$ integral in a finite number (independent of k) of integrals, computed on compact domains, on which $\xi \mapsto g_k(\xi)$ is a change of variables with Jacobian of size 1. We are then reduced to estimate $2^{(-2N+2)k}h^{-2\sigma N}\int_{|z|\leq C}\langle \frac{z+g_k(\xi_0)}{2^{3k}\sqrt{h}}\rangle^{-2}dz$, where C is a positive constant and ξ_0 is in $supp\varphi$. Since we assumed that $\frac{x+\sqrt{h}y}{2}$ is bounded, $|g_k(\xi_0)| = O(2^{3k})$ and we get

$$\begin{aligned} I_k &\lesssim 2^{(-2N+2)k} h^{-2\sigma N} \int_{|z|\lesssim 2^{3k}} \Big\langle \frac{z}{2^{3k}\sqrt{h}} \Big\rangle^{-2} dz \\ &\lesssim 2^{(-2N+8)k} h^{-2\sigma N} h \int_{|z|\lesssim h^{-1/2}} \langle z\rangle^{-2} dz \\ &\lesssim 2^{(-2N+8)k} h^{-2\sigma N+1} \log(h^{-1}) \,. \end{aligned}$$

Taking the sum of all I_k for $k \geq 0$ we then deduce that

$$\Big\| \langle h^\sigma \sqrt{h}\xi\rangle^{-N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}} \Big\rangle^{-1} \Big\|_{L^2(\xi)} \lesssim h^{-\sigma N-\delta} \Big(\sum_{k\geq 0} 2^{(-2N+8)k} \Big)^{\frac{1}{2}} \lesssim h^{-\sigma N-\delta} \,,$$

for $\delta > 0$ as small as we want, if we choose $N > 0$ such that $-2N + 8 < 0$ (e.g. $N = 5$). Finally

$$\|\mathrm{Op}_h^w(a)\|_{\mathcal{L}(L^2;H_h^{\rho,\infty})} = O(h^{-\frac{1}{2}-\beta}) \,,$$

where $\beta(\sigma) = (N + \rho)\sigma + \delta$. □

The following lemma is as simple as useful and will be largely recalled from subsection 3.2.1 on. It is also useful to introduce now the following manifold

$$\Lambda_{kg} := \{(x, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 : x - p'(\xi) = 0\} \tag{1.2.66}$$

which appears to be the graph of function $\xi = -d\phi(x)$, with $\phi(x) = \sqrt{1 - |x|^2}$ (see picture 3.1).

Lemma 1.2.38. *Let $\gamma, \chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin and with sufficiently small support, and $\sigma > 0$ be small. There exists a family of smooth functions $\theta_h(x)$, real valued, equal to 1 for $|x| \leq 1 - ch^{2\sigma}$ and supported for $|x| \leq 1 - c_1 h^{2\sigma}$, for some $0 < c_1 < c$, with $\|\partial^\alpha \theta_h\|_{L^\infty} = O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k \theta_h$ bounded for every $k \in \mathbb{N}$, such that*

$$\gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma \xi) = \theta_h(x)\gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma \xi). \tag{1.2.67}$$

Proof. Straightforward after observing that function $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$ is localized around manifold Λ_{kg}, meaning that its support is included in $\{(x,\xi)||\xi| \lesssim h^{-\sigma}, |x| \leq 1 - ch^{2\sigma}\}$, for a small $c > 0$. □

Proposition 1.2.39 (Continuity from L^p to L^p)**.** *Let $\gamma, \chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin and with sufficiently small support, $\Sigma(\xi) = \langle \xi \rangle^\rho$ with $\rho \in \mathbb{N}$, and $\sigma > 0$. Then $\mathrm{Op}_h^w\big(\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)\Sigma(\xi)\big) : L^p \to L^p$ is bounded and its $\mathcal{L}(L^p)$ norm is estimated by $h^{-\sigma\rho-\beta}$, for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$, for every $1 \leq p \leq +\infty$.*

Proof. From lemma 1.2.38 and the fact that $\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)$ is supported in a neighbourhood of Λ_{kg} introduced above, we can find a new smooth cut-off function γ_1, suitably supported, so that

$$\mathrm{Op}_h^w\left(\gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi)\right) = \mathrm{Op}_h^w\left(\gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi)\gamma_1\Big(\frac{\xi + d\phi(x)}{h^{1/2-\beta}}\Big)\theta_h(x)\right)$$

where $\beta > 0$ is a small constant, $\beta \to 0$ as $\sigma \to 0$, that takes into account the degeneracy of the equivalence between the two equations of Λ_{kg} when approaching the boundary of $supp\theta_h$. Denoting $\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi)$ concisely by $A(x,\xi)$ and looking at the kernel associated to the above operator

$$\begin{aligned} K(x,y) &:= \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}(x-y)\cdot\xi} A\left(\frac{x+y}{2},\xi\right)\gamma_1\Big(\frac{\xi + d\phi(\frac{x+y}{2})}{h^{1/2-\beta}}\Big)\theta_h\Big(\frac{x+y}{2}\Big)d\xi \\ &= \frac{e^{-\frac{i}{h}(x-y)\cdot d\phi(\frac{x+y}{2})}}{(2\pi h)^2}\theta_h\Big(\frac{x+y}{2}\Big)\int e^{\frac{i}{h}(x-y)\cdot\xi} A\left(\frac{x+y}{2},\xi - d\phi\Big(\frac{x+y}{2}\Big)\right)\gamma_1\Big(\frac{\xi}{h^{1/2-\beta}}\Big)d\xi, \end{aligned}$$

we observe that, since

$$\Big(\frac{x}{\sqrt{h}}\Big)^\alpha e^{\frac{i}{h}(x-y)\cdot\xi} = \Big(\frac{\sqrt{h}}{i}\Big)^{|\alpha|}\partial_\xi^\alpha e^{\frac{i}{h}(x-y)\cdot\xi}$$

and $h^{|\alpha|/2}\partial_\xi^\alpha A(\frac{x+y}{2},\xi)$ is bounded by $h^{-\sigma\rho}$ for any $\alpha \in \mathbb{N}^2$, by making some integration by parts

$$\left|\Big(\frac{x}{\sqrt{h}}\Big)^\alpha K(x,y)\right| \lesssim h^{-2-\sigma\rho}\int_{|\xi|\lesssim h^{1/2-\beta}} d\xi \lesssim h^{-1-\sigma\rho-2\beta}, \quad \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

This means in particular that

$$|K(x,y)| \lesssim h^{-1-\sigma\rho-2\beta}\Big\langle\frac{x}{\sqrt{h}}\Big\rangle^{-3}, \quad |K(x,y)| \lesssim h^{-1-\sigma\rho-2\beta}\Big\langle\frac{y}{\sqrt{h}}\Big\rangle^{-3}, \quad \forall (x,y)$$

implying that

$$\sup_x \int |K(x,y)|dy \lesssim h^{-\sigma\rho-2\beta}, \quad \sup_y \int |K(x,y)|dx \lesssim h^{-\sigma\rho-2\beta}.$$

The operator associated to $K(x,y)$ is hence bounded on L^p with norm $O(h^{-\sigma\rho-2\beta})$, for every $1 \leq p \leq +\infty$. □

The following lemma shows that we have nice upper bounds for operators whose symbol is supported for large frequencies $|\xi| \geq h^{-\sigma}$, $\sigma > 0$, when acting on functions w that belong to H^s_h, for some large s. We state it in space dimension 2 but it is clear that it holds true in general space dimension $d \geq 1$. This result is useful when we want to reduce to symbols rapidly decaying in $|h^\sigma \xi|$, for example in the intention of using proposition 1.2.37 or when we want to pass from a symbol of a certain positive order to another one of order zero, up to small losses of order $O(h^{-\beta})$, $\beta > 0$ depending linearly on σ. We can always split a symbol using that $1 = \chi(h^\sigma\xi)+(1-\chi)(h^\sigma\xi)$, for a smooth χ equal to 1 close to the origin, and consider as remainders all contributions coming from the latter.

Lemma 1.2.40. *Let $s' \geq 0$ and $\chi \in C^\infty_0(\mathbb{R}^2)$, $\chi \equiv 1$ in a neighbourhood of zero. Then*

$$\|\mathrm{Op}^w_h((1-\chi)(h^\sigma\xi))w\|_{H^{s'}_h} \leq Ch^{\sigma(s-s')}\|w\|_{H^s_h}\,, \qquad \forall s > s'\,.$$

Proof. The result is a simple consequence of the fact that $(1-\chi)(h^\sigma\xi)$ is supported for $|\xi| \gtrsim h^{-\sigma}$, because

$$\begin{aligned}\|\mathrm{Op}^w_h((1-\chi)(h^\sigma\xi))w\|^2_{H^{s'}_h} &= \int (1+|h\xi|^2)^{s'}|(1-\chi)(h^\sigma h\xi)|^2|\hat{w}(\xi)|^2 d\xi \\ &= \int (1+|h\xi|^2)^{s}(1+|h\xi|^2)^{s'-s}|(1-\chi)(h^\sigma h\xi)|^2|\hat{w}(\xi)|^2 d\xi \\ &\leq Ch^{2\sigma(s-s')}\|w\|^2_{H^s_h}\,,\end{aligned}$$

where the last inequality follows from an integration on $|h\xi| \gtrsim h^{-\sigma}$ and from the fact that $s'-s<0$, $(1+|h\xi|^2)^{s'-s} \leq Ch^{-2\sigma(s'-s)}$. □

We introduce the following operator:

$$\mathcal{L}_j := \frac{1}{h}\mathrm{Op}^w_h(x - p'_j(\xi)), \quad j = 1,2, \tag{1.2.68}$$

and use the notation $\|\mathcal{L}^\gamma w\| = \|\mathcal{L}_1^{\gamma_1}\mathcal{L}_2^{\gamma_2} w\|$ for any $\gamma = (\gamma_1,\gamma_2) \in \mathbb{N}^2$.

Lemma 1.2.41. *Let $\gamma \in C^\infty_0(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, $c(x,\xi) \in S_{\delta,\sigma}(1)$ with $\delta \in [0,\frac{1}{2}[$ and $\sigma > 0$. Then $\gamma(\frac{x-p'(\xi)}{\sqrt{h}})c(x,\xi)$ belongs to $S_{\frac{1}{2},\sigma}(1)\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-N}\big)$, for all $N \geq 0$.*

Proof. Straightforward. □

Lemma 1.2.42. *Let $n \in \mathbb{N}$ and $\gamma_n(z)$ be a smooth function such that $|\partial^\alpha\gamma_n(z)| \lesssim \langle z\rangle^{-|\alpha|-n}$ for all $\alpha \in \mathbb{N}^2$. Let also $c(x,\xi) \in S_{\delta,\sigma}(1)$, with $\delta \in [0,\frac{1}{2}[$, $\sigma > 0$, be supported for $|\xi| \lesssim h^{-\sigma}$. Up to some multiplicative constants independent of h, we have the following equality:*

$$\begin{aligned}&\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big]\sharp\big(x_j - p'_j(\xi)\big) = c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\big(x_j - p'_j(\xi)\big) \\ &+ h\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\big[(\partial_{\xi_j}c) + (\partial_x c)\cdot(\partial_\xi p'_j)\big] + h\sum_{|\alpha|=2}(\partial^\alpha\gamma_n)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)(\partial^\alpha_\xi p'_j)(\xi) + r(x,\xi),\end{aligned} \tag{1.2.69}$$

with $r \in h^{3/2-\delta}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n}\big)$, and if $\chi \in C^\infty_0(\mathbb{R}^2)$ is such that $\chi(h^\sigma\xi) \equiv 1$ on the support of $c(x,\xi)$,

$$\Big\|\mathrm{Op}^w_h\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_j - p'_j(\xi))\Big)\widetilde{v}\Big\|_{L^2} \lesssim \sum_{|\gamma|=0}^{1} h^{1-\beta}\|\mathrm{Op}^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\,, \tag{1.2.70a}$$

(1.2.70b) $$\Big\|\mathrm{Op}_h^w\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_j-p_j'(\xi))\Big)\widetilde{v}\Big\|_{L^\infty}\lesssim\sum_{|\gamma|=0}^{1}h^{\frac12\delta_n-\beta}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\,,$$

where $\delta_n=1$ if $n>0$, 0 otherwise, and $\beta>0$ is small, $\beta\to0$ as $\delta,\sigma\to0$.

Moreover, if $n\in\mathbb{N}^*$ and $\partial^\alpha\gamma_n$ vanishes in a neighbourhood of the origin whenever $|\alpha|\geq1$, we also have that

(1.2.71a) $$\Big\|\mathrm{Op}_h^w\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p_i'(\xi))(x_j-p_j'(\xi))\Big)\widetilde{v}\Big\|_{L^2}\lesssim\sum_{0\leq|\gamma|\leq2}h^{2-\beta}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2},$$

(1.2.71b) $$\Big\|\mathrm{Op}_h^w\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p_i'(\xi))(x_j-p_j'(\xi))\Big)\widetilde{v}\Big\|_{L^\infty}\lesssim\sum_{0\leq|\gamma|\leq2}h^{\frac32-\beta}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}.$$

Proof. As $c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\in S_{\frac12,\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n}\big)$ and $\partial^\alpha_{x,\xi}(x_j-p_j'(\xi))\in S_{0,0}(1)$ for any $|\alpha|\geq1$, equality (1.2.69) follows from the last part of lemma 1.2.24 and symbolic development (1.2.18) until order 2, after having observed that

(1.2.72) $$\Big\{c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big),x_j-p_j'(\xi)\Big\}=\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\big[(\partial_{\xi_j}c)+(\partial_xc)\cdot(\partial_\xi p_j')\big],$$

and that, up to some multiplicative negligible,

$$h^2\sum_{|\alpha|=2}\partial_x^\alpha\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big](\partial_\xi^\alpha p_j')(\xi)=h\sum_{|\alpha|=2}(\partial^\alpha\gamma_n)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)(\partial_\xi^\alpha p_j')(\xi)$$
$$\underbrace{+h^{\frac32}\sum_{\substack{|\alpha|=2\\|\alpha_1|,|\alpha_2|=1}}(\partial^{\alpha_1}\gamma_n)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial_x^{\alpha_2}c)(x,\xi)(\partial_\xi^\alpha p_j')(\xi)+h^2\sum_{|\alpha|=2}\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial_x^\alpha c)(x,\xi)(\partial_\xi^\alpha p_j')(\xi)\,.}_{\in h^{\frac32-\delta}S_{\frac12,\sigma}\left(\left\langle\frac{x-p'(\xi)}{\sqrt{h}}\right\rangle^{-n}\right)}$$

If χ is a cut-off function as in the statement, its derivatives vanish on the support of $c(x,\xi)$, and from remark 1.2.22

(1.2.73) $$c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)=\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big]\sharp\chi(h^\sigma\xi)+r_\infty(x,\xi)$$

with $r_\infty\in h^NS_{\frac12,\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n})$, $N\in\mathbb{N}$ as large as we want. Estimates (1.2.70) follow then as a straight consequence of (1.2.69), definition (1.2.68) of $\mathcal{L}_j$, proposition 1.2.36 and semi-classical Sobolev's injection (1.2.15) (resp. proposition 1.2.37) when $n=0$ (resp. $n>0$).

In order to prove the last part of the statement (estimates (1.2.71)) we use equality (1.2.69) with γ_n replaced by $\widetilde{\gamma}_{n-1}(z)=\gamma_n(z)z_i$, where $|\partial^\alpha\widetilde{\gamma}_{n-1}(z)|\lesssim\langle z\rangle^{-|\alpha|-(n-1)}$, which gives that

$$c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p_i'(\xi))(x_j-p_j'(\xi))=\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p_i'(\xi))\Big]\sharp(x_j-p_j'(\xi))$$
$$-h\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p_i'(\xi))\big[(\partial_{\xi_j}c)+(\partial_xc)\cdot(\partial_\xi p_j')\big]$$
$$-h^{\frac32}\sum_{|\alpha|=2}(\partial^\alpha\widetilde{\gamma}_{n-1})\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)(\partial_\xi^\alpha p_j')(\xi)-\sqrt{h}r(x,\xi),$$

with $r \in h^{\frac{3}{2}-\delta} S_{\frac{1}{2},\sigma}\big(\langle \frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-(n-1)}\big)$. As $\partial^\alpha \widetilde{\gamma}_{n-1}$ vanishes in a neighbourhood of the origin for $|\alpha| = 2$ by the hypothesis made on γ_n, we can rewrite it as $\sum_{l=1}^2 \widetilde{\gamma}^l_{n+2}(z) z_l$, where $\widetilde{\gamma}^l_{n+2}(z) := (\partial^\alpha \widetilde{\gamma}_{n-1})(z) z_l |z|^{-2}$ is such that $|\partial^\beta \widetilde{\gamma}^l_{n+2}(z)| \lesssim \langle z\rangle^{-|\beta|-(n+2)}$. Then, using again equality (1.2.69) for all products different from $r(x,\xi)$ in the above right hand side (with c replaced with $h^\delta[(\partial_{\xi_j} c) - (\partial_x c)\cdot(\partial_\xi p'_j)]$ in the second addend, and γ_n and c replaced with $\widetilde{\gamma}^l_{n+2}$ and $c(\partial^\alpha_\xi p'_j)$ respectively in the third one, $l = 1, 2$) we find that

$$c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i - p'_i(\xi))(x_j - p'_j(\xi)) = \\ \Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big] \sharp (x_i - p'_i(\xi))\sharp(x_j - p'_j(\xi)) + h r_1(x,\xi)\sharp(x_j - p'_j(\xi)) - \sqrt{h} r(x,\xi),$$

for a new $r_1 \in h^{-\delta} S_{\frac{1}{2},\sigma}\big(\langle \frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n}\big)$. Estimates (1.2.71) are then obtained using (1.2.73) and propositions 1.2.36, 1.2.37). □

We will also need the following result, which is detailed in lemma 1.2.6 in [7] for the one-dimensional case.

Lemma 1.2.43. *Let $\gamma \in C_0^\infty(\mathbb{R}^2)$, and $\phi(x) = \sqrt{1-|x|^2}$. If the support of γ is sufficiently small,*

$$(x_k - p'_k(\xi))\gamma\big(\langle\xi\rangle^2(x - p'(\xi))\big) = \sum_{l=1}^2 e^k_l(x,\xi)(\xi_l + d_l\phi(\xi)), \tag{1.2.74a}$$

$$(\xi_k + d_k\phi(x))\gamma\big(\langle\xi\rangle^2(x - p'(\xi))\big) = \sum_{l=1}^2 \widetilde{e}^k_l(x,\xi)(x_l - p'_l(\xi)), \tag{1.2.74b}$$

for any $k = 1, 2$, where functions $e^k_l(x,\xi), \widetilde{e}^k_l(x,\xi)$ are such that, for any $\alpha, \beta \in \mathbb{N}^2$,

$$|\partial^\alpha_x \partial^\beta_\xi e^k_l(x,\xi)| \lesssim_{\alpha\beta} \langle\xi\rangle^{-3+2|\alpha|-|\beta|}, \tag{1.2.75a}$$

$$|\partial^\alpha_x \partial^\beta_\xi \widetilde{e}^k_l(x,\xi)| \lesssim_{\alpha\beta} \langle\xi\rangle^{3+2|\alpha|-|\beta|}, \tag{1.2.75b}$$

for any $k, l = 1, 2$.

Chapter 2

Energy Estimates

The aim of this chapter is to write an energy inequality for $E_n(t;u_\pm,v_\pm)$ and $E_3^k(t;u_\pm,v_\pm)$ respectively, which allows us to propagate the a-priori energy estimates made in theorem 1.1.2, i.e. to pass from (1.1.11) to (1.1.12c), (1.1.12d). Such an inequality is in general derived by computing and estimating the derivative in time of the energy, i.e. of the L^2 norm to the square of $u_\pm^I, v_\pm^I$. As this computation makes use of the system of equations satisfied by $(u_\pm^I, v_\pm^I)$ (see (2.1.2)), two main difficulties arise due to the quasi-linear nature of the starting problem and the very slow decay in time (1.1.11a) of the wave solution.

On the one hand, among all quadratic terms appearing in the right hand side of (2.1.2) we find the quasi-linear ones $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm^I)$ and $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm^I)$, whose L^2 norm is bounded by $\|v_\pm(t,\cdot)\|_{H^{1,\infty}}(\|u_\pm^I(t,\cdot)\|_{H^1}+\|v_\pm^I(t,\cdot)\|_{H^1})$, as usual for this kind of terms. This means that they are at the wrong energy level, in the sense that they cannot be controlled in L^2 by $E_n(t;u_\pm,v_\pm)$ or $E_3^k(t;u_\pm,v_\pm)$. This causes a "loss of derivatives" in the energy inequality if we roughly estimate

$$\frac{1}{2}\partial_t\Big(\|u_\pm^I(t,\cdot)\|_{L^2}^2+\|v_\pm^I(t,\cdot)\|_{L^2}^2\Big)=-\Im\Big[\langle Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm^I),u_\pm^I\rangle+\langle Q_0^{\mathrm{kg}}(v_\pm,D_1u_\pm^I),v_\pm^I\rangle+\dots\Big]$$

using the Cauchy-Schwarz inequality. This issue is however only technical. In fact, by writing system (2.1.2) in a vectorial fashion and para-linearising it in order to stress out the very troublesome terms (see subsection 2.1.1) we are able to *symmetrize* it, i.e. to derive an equivalent system in which the quasi-linear contribution is represented by a self-adjoint operator of order 1 (see subsection 2.1.3, proposition 2.1.5). As this operator is self-adjoint it essentially disappears in the energy inequality, replaced with an operator of order 0 whose action on $u_\pm^I, v_\pm^I$ is bounded in L^2 by $E_n(t;u_\pm,v_\pm)$ or $E_3^k(t;u_\pm,v_\pm)$, depending on the multi-index I we are dealing with.

On the other hand, the L^2 norm of some semi-linear contributions to the right hand side of (2.1.2) decays very slowly in time. It is the case, for instance, of $Q_0^{\mathrm{kg}}(v_\pm^I, D_1 u_\pm)$, whose L^2 norm is bounded by $\|u_\pm(t,\cdot)\|_{H^{2,\infty}}\|v_\pm^I(t,\cdot)\|_{L^2}$ and only has the slow decay (1.1.11a) of the wave component $u_\pm$. Since we want to prove that

$$\partial_t E_n(t;u_\pm,v_\pm)=O\big(\varepsilon t^{-1+\frac{\delta}{2}}E_n(t;u_\pm,v_\pm)^{\frac12}\big),\quad \partial_t E_3^k(t;u_\pm,v_\pm)=O\big(\varepsilon t^{-1+\frac{\delta_k}{2}}E_3^k(t;u_\pm,v_\pm)^{\frac12}\big)$$

we need to get rid of such terms by means of normal forms (see section 3.1). Because of the quasi-linear nature of our problem, some of them will be eliminated by an adapted quasi-linear normal form argument (see subsection 2.2.1), while the remaining ones can be treated with an usual semi-linear one (see subsection 2.2.2). At that point we will be able to prove proposition 2.2.13 and to derive estimates (1.1.12c), (1.1.12d).

2.1 Paralinearization and Symmetrization

As anticipated above, the first step towards the derivation of the right energy inequality is to handle the quasi-linear terms appearing in the right hand side of (2.1.2) in order to avoid any loss of derivatives. We realize that the very quasi-linear contribution to our system appears in equation (2.1.20) through a para-differential operator whose symbol is a real *non symmetric* matrix. As we need this operator to be self-adjoint (up to an operator of order 0), we *symmetrize* equation (2.1.20) by defining a new function W_s^I in terms of W^I, that will be solution to a new equation in which the symbol of the quasi-linear contribution is a real symmetric matrix (see subsection 2.1.3). Also, we set aside subsection 2.1.2 to the estimate of the L^2 norms of the non-linear terms in the right hand side of (2.1.20).

2.1.1 Paralinearization

Let us remind definitions (1.1.10) and (1.1.18). Since admissible vector fields considered in $\mathcal{Z} = \{\Omega, Z_j, \partial_j, j = 1, 2\}$ exactly commute with the linear part of system (1.1.1), we deduce from remark 1.1.5 and (1.1.17) that, for any multi-index I, $(\Gamma^I u, \Gamma^I v)$ is solution to

$$\begin{cases} \big(\partial_t^2 - \Delta_x\big)\,\Gamma^I u = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0(\Gamma^{I_1} v, \partial_1 \Gamma^{I_2} v) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2} Q_0(\Gamma^{I_1} v, \partial \Gamma^{I_2} v), \\ \big(\partial_t^2 - \Delta_x + 1\big)\,\Gamma^I v = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0(\Gamma^{I_1} v, \partial_1 \Gamma^{I_2} u) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2} Q_0(\Gamma^{I_1} v, \partial \Gamma^{I_2} u), \end{cases}$$

with coefficients $c_{I_1,I_2} \in \{-1, 0, 1\}$ such that $c_{I_1,I_2} = 1$ for $|I_1| + |I_2| = |I|$, in which case the derivative ∂ acting on $\Gamma^{I_2} v$ (resp. on $\Gamma^{I_2} u$) is equal to ∂_1, and ∂ representing one of the partial derivatives ∂_a, $a \in \{0, 1, 2\}$. Let us remind that, if Γ^I contains at least k ($\leq |I|$) space derivatives, above summations are taken over indices I_1, I_2 such that $k \leq |I_1| + |I_2| \leq |I|$. Hence, introducing from (1.1.3), (1.1.5),

$$\begin{aligned} Q_0^{\mathrm{w}}(v_\pm, D_a v_\pm) &:= \frac{i}{4}\left[(v_+ + v_-)D_a(v_+ + v_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{D_x D_a}{\langle D_x\rangle}(v_+ - v_-)\right], \\ Q_0^{\mathrm{kg}}(v_\pm, D_a u_\pm) &:= \frac{i}{4}\left[(v_+ + v_-)D_a(u_+ + u_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{D_x D_a}{|D_x|}(u_+ - u_-)\right], \end{aligned} \tag{2.1.1}$$

for any $a = 0, 1, 2$, we deduce that $(u_+^I, v_+^I, u_-^I, v_-^I)$ is solution to

$$\begin{cases} (D_t - |D_x|)u_+^I(t,x) = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_1 v_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D v_\pm^{I_2}) \\ (D_t - \langle D_x\rangle)v_+^I(t,x) = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D u_\pm^{I_2}) \\ (D_t + |D_x|)u_-^I(t,x) = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_1 v_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D v_\pm^{I_2}) \\ (D_t + \langle D_x\rangle)v_-^I(t,x) = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D u_\pm^{I_2}) \end{cases} \tag{2.1.2}$$

The quasi-linear structure of the above system can be emphasized by using (1.2.7) and decomposing $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm^I)$, $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm^I)$ as follows:

$$Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm^I) = (QL)_1 + (SL)_1, \quad Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm^I) = (QL)_2 + (SL)_2, \tag{2.1.3}$$

with

$$
\begin{aligned}
(QL)_1 &:= \frac{i}{4}\Big[Op^B\big((v_+ + v_-)\eta_1\big)(v_+^I + v_-^I) - Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta\eta_1}{\langle\eta\rangle}\Big)(v_+^I - v_-^I)\Big],\\
(SL)_1 &:= \frac{i}{4}\Big[Op^B\big(D_1(v_+^I + v_-^I)\big)(v_+ + v_-) - Op^B\Big(\frac{D_xD_1}{\langle D_x\rangle}(v_+^I - v_-^I)\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+ - v_-)\\
&\qquad + Op_R^B\big((v_+ + v_-)\eta_1\big)(v_+^I + v_-^I)\big) - Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta\eta_1}{\langle\eta\rangle}\Big)(v_+^I - v_-^I)\Big],\\
(QL)_2 &:= \frac{i}{4}\Big[Op^B\big((v_+ + v_-)\eta_1\big)(u_+^I + u_-^I) - Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u_+^I - u_-^I)\Big],\\
(SL)_2 &:= \frac{i}{4}\Big[Op^B\big(D_1(u_+^I + u_-^I)\big)(v_+ + v_-) - Op^B\Big(\frac{D_xD_1}{|D_x|}(u_+^I - u_-^I)\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+ - v_-)\\
&\qquad + Op_R^B\big((v_+ + v_-)\eta_1\big)(u_+^I + u_-^I) - Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u_+^I - u_-^I)\Big],
\end{aligned}
$$

where the Bony quantization Op^B (resp. Op_R^B) has been defined in 1.2.5 (resp. in (1.2.9)). We do a similar decomposition also for the semi-linear contribution $Q_0^{\mathrm{kg}}(v_\pm^I, D_1u_\pm)$, for this term will thereafter be the object of the two normal forms mentioned at the beginning of this section:

(2.1.4)

$$
\begin{aligned}
Q_0^{\mathrm{kg}}(v_\pm^I, D_1u_\pm) &= \frac{i}{4}\Big[Op^B\big((v_+^I + v_-^I)\eta_1\big)(u_+ + u_-) - Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+^I - v_-^I)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u_+ - u_-)\Big]\\
&+ \frac{i}{4}\Big[Op^B\big(D_1(u_+ + u_-)\big)(v_+^I + v_-^I) - Op^B\Big(\frac{D_xD_1}{|D_x|}(u_+ - u_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+^I - v_-^I)\Big]\\
&+ \frac{i}{4}\Big[Op_R^B\big((v_+^I + v_-^I)\eta_1\big)(u_+ + u_-) - Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+^I - v_-^I)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u_+ - u_-)\Big].
\end{aligned}
$$

In order to handle system (2.1.2) in the most efficient way we proceed to write it in a vectorial fashion. To this purpose, we introduce the following matrices:

$$
A(\eta) = \begin{bmatrix} |\eta| & 0 & 0 & 0\\ 0 & \langle\eta\rangle & 0 & 0\\ 0 & 0 & -|\eta| & 0\\ 0 & 0 & 0 & -\langle\eta\rangle \end{bmatrix}, \quad A'(V;\eta) := \begin{bmatrix} 0 & a_k\eta_1 & 0 & b_k\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0\\ 0 & a_k\eta_1 & 0 & b_k\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0 \end{bmatrix}, \tag{2.1.5}
$$

$$
A''(V^I;\eta) := \begin{bmatrix} 0 & 0 & 0 & 0\\ a_0^I\eta_1 & 0 & b_0^I\eta_1 & 0\\ 0 & 0 & 0 & 0\\ a_0^I\eta_1 & 0 & b_0^I\eta_1 & 0 \end{bmatrix}, \tag{2.1.6}
$$

$$
C'(W^I;\eta) := \begin{bmatrix} 0 & c_0^I & 0 & d_0^I\\ 0 & e_0^I & 0 & f_0^I\\ 0 & c_0^I & 0 & d_0^I\\ 0 & e_0^I & 0 & f_0^I \end{bmatrix}, \quad C''(U;\eta) := \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & e_0 & 0 & f_0\\ 0 & 0 & 0 & 0\\ 0 & e_0 & 0 & f_0 \end{bmatrix} \tag{2.1.7}
$$

where

$$
\begin{cases}
a_k = a_k(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\\
b_k = b_k(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) + \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\\
a_0 = a_0(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{|\eta|}\big]\\
b_0 = b_0(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) + \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{|\eta|}\big]
\end{cases} \tag{2.1.8}
$$

$$
\left\{\begin{array}{l}
c_0 = c_0(v_\pm;\eta) := \frac{i}{4}\big[D_1(v_+ + v_-) - \frac{D_x D_1}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big] \\
d_0 = d_0(v_\pm;\eta) := \frac{i}{4}\big[D_1(v_+ + v_-) + \frac{D_x D_1}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big] \\
e_0 = e_0(u_\pm;\eta) := \frac{i}{4}\big[D_1(u_+ + u_-) - \frac{D_x D_1}{|D_x|}(u_+ - u_-)\cdot\frac{\eta}{\langle\eta\rangle}\big] \\
f_0 = f_0(u_\pm;\eta) := \frac{i}{4}\big[D_1(u_+ + u_-) + \frac{D_x D_1}{|D_x|}(u_+ - u_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]
\end{array}\right. \tag{2.1.9}
$$

$$
\begin{array}{c}
a_0^I = a_0(v_\pm^I;\eta), \quad b_0^I = b_0(v_\pm^I;\eta), c_0^I = c_0(v_\pm^I;\eta), \quad d_0^I = d_0(v_\pm^I;\eta), \\
e_0^I = e_0(u_\pm^I;\eta), \quad f_0^I(u_\pm^I;\eta),
\end{array} \tag{2.1.10}
$$

vectors W, U, V:

$$
W := \begin{bmatrix} u_+ \\ v_+ \\ u_- \\ v_- \end{bmatrix}, \quad V := \begin{bmatrix} 0 \\ v_+ \\ 0 \\ v_- \end{bmatrix}, \quad U := \begin{bmatrix} u_+ \\ 0 \\ u_- \\ 0 \end{bmatrix}, \tag{2.1.11}
$$

along with W^I (resp. V^I, U^I) defined from W (resp. V, U) by replacing $u_\pm, v_\pm$ with $u_\pm^I, v_\pm^I$; and finally

$$
Q_0^I(V,W) = \begin{bmatrix}
\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{w}}(v_\pm^{I_1}, Dv_\pm^{I_2}) \\
\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2}) \\
\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{w}}(v_\pm^{I_1}, Dv_\pm^{I_2}) \\
\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}} c_{I_1,I_2} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2})
\end{bmatrix} \tag{2.1.12}
$$

The quantization Op^B (resp. Op_R^B) of a matrix $A = (a_{ij})_{1\le i,j\le n}$ is meant as a matrix of operators $Op^B(A) = (Op^B(a_{ij}))_{1\le i,j\le n}$ (resp. $Op_R^B(A) = (Op_R^B(a_{ij}))_{1\le i,j\le n}$), and for a vector $Y = [y_1, \dots, y_n]$,

$$
Op^B(A)Y^\dagger = \begin{bmatrix} \sum_{j=1}^n Op^B(a_{1j})y_j \\ \vdots \\ \sum_{j=1}^n Op^B(a_{nj})y_j \end{bmatrix},
$$

$Y^\dagger$ being the transpose of Y. We also remind that

$$
\|A\|_{L^2} = \Big(\sum_{i,j} |a_{ij}|^2\Big)^{\frac{1}{2}}, \quad \|A\|_{L^\infty} = \sup_{ij} |a_{ij}|.
$$

With notations introduced above, system (2.1.2) writes in the following compact fashion which has the merit to well highlight, among all non-linear terms, the very quasi-linear contributions $(QL)_1, (QL)_2$, represented below by $Op^B(A'(V;\eta))W^I$:

$$
\begin{array}{l}
D_t W^I = A(D)W^I + Op^B(A'(V;\eta))W^I + Op^B(C'(W^I;\eta))V + Op_R^B(A'(V;\eta))W^I \\
\qquad + Op^B(A''(V^I;\eta))U + Op^B(C''(U;\eta))V^I + Op_R^B(A''(V^I;\eta))U + Q_0^I(V,W).
\end{array} \tag{2.1.13}
$$

The energies defined in (1.1.9) take the form

$$E_n(t;u_\pm,v_\pm)=\sum_{|\alpha|\le n}\|D_x^\alpha W(t,\cdot)\|_{L^2},\quad \forall\, n\in\mathbb{N},n\ge 3, \tag{2.1.14a}$$

$$E_3^k(t;u_\pm,v_\pm)=\sum_{\substack{|\alpha|+|I|\le 3\\ |I|\le 3-k}}\|D_x^\alpha W^I(t,\cdot)\|_{L^2}^2,\quad \forall\, 0\le k\le 2, \tag{2.1.14b}$$

and we can refer to them, respectively, as $E_n(t;W), E_3^k(t;W)$. We also notice that, since

$$[\Gamma, D_t\pm|D_x|]=\begin{cases}0 & \text{if } \Gamma\in\{\Omega,\partial_j, j=1,2\},\\ \mp\frac{D_m}{|D_x|}(D_t\pm|D_x|) & \text{if } \Gamma=Z_m, m=1,2,\end{cases} \tag{2.1.15a}$$

$$[\Gamma, D_t\pm\langle D_x\rangle]=\begin{cases}0 & \text{if } \Gamma\in\{\Omega,\partial_j, j=1,2\},\\ \mp\frac{D_m}{\langle D_x\rangle}(D_t\pm\langle D_x\rangle) & \text{if } \Gamma=Z_m, m=1,2,\end{cases} \tag{2.1.15b}$$

and operators $D_m|D_x|^{-1}, D_m\langle D_x\rangle^{-1}$ are continuous on L^2 for $m=1,2$, there exists a constant $C>0$ such that

$$C^{-1}\sum_{I\in\mathfrak{I}_3^k}\|\Gamma^I W(t,\cdot)\|_{L^2}^2\le E_3^k(t;W)\le C\sum_{I\in\mathfrak{I}_3^k}\|\Gamma^I W(t,\cdot)\|_{L^2}^2, \tag{2.1.16}$$

where, for any integer $0\le k\le 2$,

$$\mathfrak{I}_3^k:=\left\{|I|\le 3:\Gamma^I=D_x^\alpha\Gamma^J \text{ with } |\alpha|+|J|=|I|, |J|\le 3-k\right\}. \tag{2.1.17}$$

For convenience, we also introduce the following set:

$$\mathfrak{I}_n:=\left\{|I|\le n:\Gamma^I=D_x^\alpha \text{ with } |\alpha|=|I|\right\},\quad n\in\mathbb{N}, n\ge 3. \tag{2.1.18}$$

Matrices $A(\eta), A'(V;\eta), A''(V^I;\eta)$ are of order 1 and $A'(V;\eta), A''(V^I;\eta)$ are singular at $\eta=0$ (i.e. some of their elements are singular at $\eta=0$), while $C'(W^I;\eta), C''(U;\eta)$ are of order 0. Since we will need to do some symbolic calculus on $A'(V;\eta)$, we need to isolate the mentioned singularity. We hence define

(2.1.19)

$$A_1'(V;\eta):=\begin{bmatrix}0 & a_0\eta_1 & 0 & b_0\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0\\ 0 & a_0\eta_1 & 0 & b_0\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0\end{bmatrix},\quad A_{-1}'(V;\eta):=\begin{bmatrix}0 & (a_k-a_0)\eta_1 & 0 & (b_k-b_0)\eta_1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & (a_k-a_0)\eta_1 & 0 & (b_k-b_0)\eta_1\end{bmatrix},$$

$A_1'(V;\eta)$ being a matrix of order 1, $A_{-1}'(V;\eta)$ of order -1, both singular at $\eta=0$, and write $A_1'(V;\eta)=A_1'(V;\eta)(1-\chi)(\eta)+A_1'(V;\eta)\chi(\eta)$, where $\chi\in C_0^\infty(\mathbb{R}^2)$ is equal to 1 in the unit ball. Equation (2.1.13) can be the rewritten as follows

(2.1.20)

$$\begin{aligned}D_tW^I=&A(D)W^I+Op^B(A_1'(V;\eta)(1-\chi)(\eta))W^I+Op^B(A_1'(V;\eta)\chi(\eta))W^I\\ &+Op^B(A_{-1}'(V;\eta))W^I+Op^B(C'(W^I;\eta))V+Op_R^B(A'(V;\eta))W^I+Op^B(A''(V^I;\eta))U\\ &+Op^B(C''(U;\eta))V^I+Op_R^B(A''(V^I;\eta))U+Q_0^I(V,W),\end{aligned}$$

and the symbol $A_1'(V;\eta)(1-\chi)(\eta)$ associated to the quasi-linear contribution is no longer singular at $\eta=0$. We observe that this matrix is real since $i(v_+ + v_-)=2\partial_t v$, $i\frac{D_x}{\langle D_x\rangle}(v_+-v_-)=2\partial_x v$ and v is a real solution, but it is not symmetric and such a lack of symmetry could lead to a loss of derivatives when writing an energy inequality for W^I. The issue is however only technical, in the sense that $A_1(V;\eta)(1-\chi)(\eta)$ can be replaced with a real, symmetric matrix, as explained in subsection 2.1.3 (see proposition 2.1.5). Before proving such result, we need to derive some L^2 estimates for the semi-linear terms in the right hand side of (2.1.20).

2.1.2 Estimates of quadratic terms

In this subsection we recover some estimates for the L^2 norm of the non-linear terms in the right hand side of equation (2.1.20).

Lemma 2.1.1. *Let I be a fixed multi-index and $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin. The following estimates hold:*

$$\Big\| \Big[Op^B\big(A_1'(V;\eta)\chi(\eta)\big) + Op^B\big(A_{-1}'(V;\eta)\big)\Big] W^I(t,\cdot)\Big\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}} \|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21a}$$

$$\big\|Op^B(C'(W^I;\eta))V(t,\cdot)\big\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{6,\infty}} \|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21b}$$

$$\|Op_R^B(A'(V;\eta))W^I(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{7,\infty}} \|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21c}$$

$$\begin{aligned}\|Op^B(A''(V^I;\eta))U(t,\cdot)\|_{L^2} + \|Op_R^B(A''(V^I;\eta))U(t,\cdot)\|_{L^2} \\ \lesssim \big(\|R_1U(t,\cdot)\|_{H^{6,\infty}} + \|U(t,\cdot)\|_{H^{6,\infty}}\big)\|V^I(t,\cdot)\|_{L^2};\end{aligned} \tag{2.1.21d}$$

$$\|Op^B(C''(U;\eta))V^I(t,\cdot)\|_{L^2} \lesssim \big(\|R_1U(t,\cdot)\|_{H^{2,\infty}} + \|U(t,\cdot)\|_{H^{2,\infty}}\big)\|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21e}$$

Proof. • Inequality (2.1.21a) follows applying proposition 1.2.7 to $Op^B\left(A_{-1}'(V;\eta)(1-\chi)(\eta)\right)W^I$ whose symbol $A_{-1}'(V;\eta)(1-\chi)(\eta)$ is of order -1 and has M_0^{-1} seminorm bounded from above by $\|V(t,\cdot)\|_{H^{1,\infty}}$, after definitions (1.2.2), (2.1.8) and (2.1.19).

• Since from definition (2.1.7) of matrix $C'(W^I;\eta)$

$$\begin{aligned}\big\|Op^B(C'(W^I;\eta))V\big\|_{L^2} \lesssim \big\|Op^B(D_1(v_+^I+v_-^I))v_\pm\big\|_{L^2} + \Big\|Op^B\Big(\frac{D_xD_1}{\langle D_x\rangle}(v_+^I-v_-^I)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_\pm\Big\|_{L^2} \\ + \big\|Op^B\big(D_1(u_+^I+u_-^I)\big)v_\pm\big\|_{L^2} + \Big\|Op^B\Big(\frac{D_xD_1}{|D_x|}(u_+^I-u_-^I)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_\pm\Big\|_{L^2},\end{aligned}$$

we reduce to prove inequality (2.1.21b) for $Op^B\big(\frac{D_xD_1}{\langle D_x\rangle}(v_+^I-v_-^I)\cdot\frac{\eta}{\langle\eta\rangle}\big)v_+$, the same argument being applicable to all other L^2 norms appearing in the above right hand side. Using equality (1.2.6), and considering a new admissible cut-off function χ_1 identically equal to 1 on the support of χ, we first derive that

$$\begin{aligned}&\widehat{Op^B\Big(\tfrac{D_xD_1}{\langle D_x\rangle}(v_+^I+v_-^I)\cdot\tfrac{\eta}{\langle\eta\rangle}\Big)v_+}(\xi) = \frac{1}{(2\pi)^2}\int \chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\widehat{\tfrac{D_xD_1}{\langle D_x\rangle}(v_+^I+v_-^I)}(\xi-\eta)\cdot\widehat{\tfrac{D_x}{\langle D_x\rangle}v_+}(\eta)d\eta\\
&= \frac{1}{(2\pi)^2}\int \chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\Big(\frac{\xi_1-\eta_1}{\langle\eta\rangle}\Big)\widehat{\tfrac{D_x}{\langle D_x\rangle}(v_+^I+v_-^I)}(\xi-\eta)\cdot\widehat{D_xv_+}(\eta)d\eta\\
&= \frac{1}{(2\pi)^2}\int \chi_1\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\widehat{\Big[\chi\Big(\tfrac{D_x}{\langle\eta\rangle}\Big)\tfrac{D_1}{\langle\eta\rangle}\tfrac{D_x}{\langle D_x\rangle}(v_+^I+v_-^I)\Big]}(\xi-\eta)\cdot\widehat{D_xv_+}(\eta)d\eta\\
&= \widehat{Op^B\Big(\chi\Big(\tfrac{D_x}{\langle\eta\rangle}\Big)\tfrac{D_1}{\langle\eta\rangle}\tfrac{D_x}{\langle D_x\rangle}(v_+^I+v_-^I)\Big)D_xv_+}(\xi).\end{aligned}$$

Successively, by decomposition (1.2.7) and the fact that $R(u,v)$ is symmetric in (u,v), we have that

$$\begin{aligned}Op^B\Big(\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)\frac{D_1}{\langle\eta\rangle}\frac{D_x}{\langle D_x\rangle}(v_+^I+v_-^I)\Big)D_xv_+ = \chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)\frac{D_1}{\langle\eta\rangle}\frac{D_x}{\langle D_x\rangle}(v_+^I+v_-^I)\cdot D_xv_+ \\ - \big[Op^B(D_xv_+) + Op_R^B(D_xv_+)\big]\left[\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)\frac{D_1}{\langle\eta\rangle}\frac{D_x}{\langle D_x\rangle}(v_+^I+v_-^I)\right],\end{aligned}$$

so propositions 1.2.7, 1.2.8 (ii), and the fact that $\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)\frac{D_1}{\langle\eta\rangle}\frac{D_x}{\langle D_x\rangle}$ is an operator uniformly bounded on L^2, imply that

$$\Big\|Op^B\Big(\frac{D_xD_1}{\langle D_x\rangle}(v_+^I+v_-^I)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_+\Big\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{6,\infty}}\|V^I(t,\cdot)\|_{L^2}.$$

• By definition (2.1.5) of $A'(V;\eta)$,

$$\begin{aligned}\big\|Op_R^B\big(A'(V;\eta)\big)W^I(t,\cdot)\big\|_{L^2}\lesssim\big\|Op_R^B(v_++v_-)v_\pm^I\big\|_{L^2}+\Big\|Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_\pm^I\Big\|_{L^2}\\+\big\|Op_R^B(v_++v_-)u_\pm^I\big\|_{L^2}+\Big\|Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta}{|\eta|}\Big)u_\pm^I\Big\|_{L^2}.\end{aligned}$$

Let us only show that inequality (2.1.21c) holds for $Op^B\big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\big)u_+^I$. For a smooth cut-off function ϕ equal to 1 in the unit ball we write

$$\begin{aligned}Op_R^B\left(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\right)u_+^I=Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\phi(\eta)\Big)u_+^I\\+Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}(1-\phi)(\eta)\Big)u_+^I,\end{aligned}$$

where by proposition 1.2.8 (i)

$$\begin{aligned}\Big\|Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\phi(\eta)\Big)u_+^I\Big\|_{L^2}\lesssim\Big\|\frac{D_x}{\langle D_x\rangle}(v_+-v_-)(t,\cdot)\Big\|_{L^\infty}\|u_+^I(t,\cdot)\|_{L^2}\\\lesssim\|V(t,\cdot)\|_{H^{1,\infty}}\|W^I(t,\cdot)\|_{L^2}.\end{aligned}$$

On the other hand

$$Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}(1-\phi)(\eta)\Big)u_+^I=\int e^{ix\cdot\xi}m(\xi,\eta)\left[\langle D_x\rangle^7(\hat v_+-\hat v_-)(\xi-\eta)\right]\hat u_+^I(\eta)d\xi d\eta,$$

where

$$m(\xi,\eta):=\frac{1}{(2\pi)^2}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\phi)(\eta)\frac{\xi-\eta}{\langle\xi-\eta\rangle^8}\cdot\frac{\eta\eta_1}{|\eta|}$$

and frequencies $\xi-\eta$ and η are either bounded or equivalent on the support of $m(\xi,\eta)$. Therefore $m(\xi,\eta)$ satisfies the hypothesis of lemma A.1 (i) $|\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)|\lesssim\langle\xi\rangle^{-3}\langle\eta\rangle^{-3}$ for any $\alpha,\beta\in\mathbb{N}^2$, and by inequality (A.4a)

$$\Big\|Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}(1-\phi)(\eta)\Big)u_+^I\Big\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{7,\infty}}\|W^I(t,\cdot)\|_{L^2}.$$

• From definition (2.1.6) of $A''(V;\eta)$,

$$\big\|Op^B\big(A''(V;\eta)\big)U(t,\cdot)\big\|_{L^2}\lesssim\big\|Op^B\big((v_+^I+v_-^I)\eta_1\big)u_\pm\big\|_{L^2}+\Big\|Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+^I-v_-^I)\cdot\frac{\eta\eta_1}{|\eta|}\Big)u_\pm\Big\|_{L^2},$$

(the same inequality holds evidently when Op^B is replaced by Op_R^B). As done for previous cases, we reduce to show (2.1.21d) for $Op^B\big(\frac{D_x}{\langle D_x\rangle}(v_+^I-v_-^I)\cdot\frac{\eta\eta_1}{|\eta|}\big)u_+$ (resp. for Op^B replaced with Op_R^B). Using decomposition (1.2.7) and the fact that $R(u,v)$ is symmetric in (u,v) we find that

$$\begin{aligned}Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+^I-v_-^I)\cdot\frac{\eta\eta_1}{|\eta|}\Big)u_+=\frac{D_x}{\langle D_x\rangle}(v_+^I-v_-^I)\cdot\frac{D_xD_1}{|D_x|}u_+\\-Op^B\Big(\frac{D_xD_1}{|D_x|}u_+\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+^I-v_-^I)-Op_R^B\Big(\frac{D_xD_1}{|D_x|}u_+\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+^I-v_-^I),\end{aligned}$$

and

$$Op_R^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+^I - v_-^I)\cdot\frac{\eta\eta_1}{|\eta|}\Big)u_+ = Op_R^B\Big(\frac{D_xD_1}{|D_x|}u_+\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+^I - v_-^I),$$

so a direct application of propositions 1.2.7 and 1.2.8 (ii) gives that the L^2 norm of the above right hand sides is bounded by $\Big\|\frac{D_xD_1}{|D_x|}u_+\Big\|_{H^{4,\infty}}\|V^I(t,\cdot)\|_{L^2}$, and hence by $\|R_1U(t,\cdot)\|_{H^{6,\infty}}\|V^I(t,\cdot)\|_{L^2}$, which gives inequality (2.1.21d).

• From definition (2.1.7) of matrix $C''(U;\eta)$,

$$\|Op^B(C''(U;\eta))V^I\|_{L^2} \lesssim \big\|Op^B(D_1(u_+ + u_-))(v_+^I + v_-^I)\big\|_{L^2} + \left\|Op^B\left(\frac{D_xD_1}{|D_x|}(u_+ - u_-)\cdot\frac{\eta}{\langle\eta\rangle}\right)(v_+^I - v_-^I)\right\|_{L^2},$$

so estimate (2.1.21e) follows immediately from proposition 1.2.7. □

Lemmas 2.1.2 and 2.1.3 below are introduced with the aim of deriving an estimate of the L^2 norm of vector $Q_0^I(V,W)$ defined in (2.1.12) (see corollary 2.1.4). We remind that the summations defining $Q_0^I(V,W)$ come from the action of family Γ^I of admissible vector fields on the quadratic non-linearities $Q_0(v,\partial_1 v)$ and $Q_0(v,\partial_1 u)$ in (1.1.1) (or, in terms of $u_\pm, v_\pm$, on $Q_0^{\rm w}(v_\pm, D_1v_\pm)$ and $Q_0^{\rm kg}(v_\pm, D_1u_\pm)$). According to remark 1.1.5, if $I\in\mathcal{I}_n$ and Γ^I is a product of spatial derivatives only the action of Γ^I on $Q_0^{\rm w}(v_\pm, D_1v_\pm)$ (resp. on $Q_0^{\rm kg}(v_\pm, D_1u_\pm)$) "distributes" entirely on its factors, meaning that

$$\Gamma^I Q_0^{\rm w}(v_\pm, D_1v_\pm) = \sum_{\substack{(I_1,I_2)\in\mathcal{J}(I)\\ |I_1|+|I_2|=|I|}} Q_0^{\rm w}(v_\pm^{I_1}, D_1v_\pm^{I_2}),$$

(the same for $\Gamma^I Q_0^{\rm kg}(v_\pm, D_1u_\pm)$), and all coefficients c_{I_1,I_2} in the right hand side of (2.1.2) are equal to 0. On the contrary, if $I\in\mathcal{I}_3^k$ for $0\le k\le 2$ and Γ^I contains some Klainerman vector fields $\Omega, Z_m, m=1,2$, the commutation between Γ^I and the null structure gives rise to new quadratic contributions in which the derivative D_1 is eventually replaced with D_2, D_t. As already seen in (1.1.17), in this case we have

$$\Gamma^I Q_0^{\rm w}(v_\pm, D_1v_\pm) = \sum_{\substack{(I_1,I_2)\in\mathcal{J}(I)\\ |I_1|+|I_2|=|I|}} Q_0^{\rm w}(v_\pm^{I_1}, D_1v_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathcal{J}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2}Q_0^{\rm w}(v_\pm^{I_1}, Dv_\pm^{I_2}), \tag{2.1.22}$$

with some of the coefficients c_{I_1,I_2} being equal to 1 or -1, and $D\in\{D_1,D_2,D_t\}$ depending on the addend we are considering (similarly for $\Gamma^I Q_0^{\rm kg}(v_\pm, D_1u_\pm)$). For our scopes, there will be no difference between the case $D=D_1$ and $D=D_2$, the two associated quadratic contributions enjoying the same L^2 and L^∞ estimates. When $D=D_t$, we should make use of the equation satisfied by $v_\pm^{I_2}$ (resp. by $u_\pm^{I_2}$) in system (2.1.2) to replace $Q_0^{\rm w}(v_\pm^{I_1}, D_tv_\pm^{I_2})$ (resp. $Q_0^{\rm kg}(v_\pm^{I_1}, D_tu_\pm^{I_2})$) with

$$\begin{gathered} Q_0^{\rm w}(v_\pm^{I_1}, \langle D_x\rangle v_\pm^{I_2}) + Q_0^{\rm w}\left(v_\pm^{I_1}, \Gamma^{I_2}Q_0^{\rm kg}(v_\pm, D_1u_\pm)\right),\\ \Big(\text{resp. with } Q_0^{\rm kg}(v_\pm^{I_1}, |D_x|u_\pm^{I_2}) + Q_0^{\rm kg}\left(v_\pm^{I_1}, \Gamma^{I_2}Q_0^{\rm w}(v_\pm, D_1v_\pm)\right)\Big),\end{gathered} \tag{2.1.23}$$

where the left hand side quadratic terms are given by

(2.1.24)

$$Q_0^{\rm w}(v_\pm^{I_1}, \langle D_x\rangle v_\pm^{I_2}) = (v_+^{I_1} + v_-^{I_1})\langle D_x\rangle(v_+^{I_2} - v_-^{I_2}) - \frac{D_x}{\langle D_x\rangle}(v_+^{I_1} - v_-^{I_1})\cdot D_x(v_+^{I_2} + v_-^{I_2}),$$

$$\left(\text{resp. } Q_0^{\rm kg}(v_\pm^{I_1}, |D_x|u_\pm^{I_2}) = (v_+^{I_1} + v_-^{I_1})|D_x|(u_+^{I_2} - u_-^{I_2}) - \frac{D_x}{\langle D_x\rangle}(v_+^{I_1} - v_-^{I_1})\cdot D_x(u_+^{I_2} + u_-^{I_2})\right),$$

while the right hand side ones in (2.1.23) are cubic. On the Fourier side, these new quadratic contributions write as

$$\sum_{j_1,j_2\in\{+,-\}}\int j_2\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\right)\langle\eta\rangle\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{v}^{I_2}_{j_2}(\eta)d\xi d\eta,$$

$$\left(\text{resp.}\sum_{j_1,j_2\in\{+,-\}}\int j_2\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)|\eta|\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)d\xi d\eta\right),$$

and have basically the same nature of the starting ones, as

$$\widehat{Q_0^{\mathrm{w}}(v^{I_1}_\pm,D_1v^{I_2}_\pm)}(\xi)=\sum_{j_1,j_2\in\{+,-\}}\int\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\right)\eta_1\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{v}^{I_2}_{j_2}(\eta)d\xi d\eta,$$

$$\left(\text{resp. }\widehat{Q_0^{\mathrm{kg}}(v^{I_1}_\pm,D_1u^{I_2}_\pm)}(\xi)=\sum_{j_1,j_2\in\{+,-\}}\int\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_1\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)d\xi d\eta\right).$$

For this reason, as long as we can neglect the cubic terms in (2.1.23), we will not pay attention to the value of $D\in\{D_1,D_2,D_t\}$ in the second sum in the right hand side of (2.1.22). Lemma 2.1.3 is meant to show that the mentioned cubic terms are, indeed, remainders.

Before proving lemmas 2.1.2, 2.1.3, we need to introduce a new set of indices. According to the order established in $\mathcal{Z}$ at the beginning of section 1.1 (see (1.1.7)), we define

$$\mathcal{K}:=\{I=(i_1,i_2):i_1,i_2=1,2,3\}\tag{2.1.25}$$

as the set of indices I such that Γ^I is the product of two Klainerman vector fields only, together with

$$\mathcal{V}^k:=\{I\in\mathcal{I}^k_3:\exists(I_1,I_2)\in\mathcal{I}(I)\text{ with }I_1\in\mathcal{K}\},\tag{2.1.26}$$

which is evidently empty when $k=2$. We also warn the reader that, in inequality (2.1.30) with $k=2$, $E^3_3(t;W)$ stands for $E_3(t;W)$, this double notation allowing us to combine in one line all cases $k=0,1,2$.

Lemma 2.1.2. *(i) Let $n\in\mathbb{N},n\geq3$ and $I\in\mathcal{I}_n$. Then*

(2.1.27)
$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_2|<n}}\left\|Q_0^{\mathrm{w}}(v^{I_1}_\pm,D_xv^{I_2}_\pm)\right\|_{L^2}+\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|\leq[\frac{n}{2}],|I_2|<n}}\left\|Q_0^{\mathrm{kg}}(v^{I_1}_\pm,D_xu^{I_2}_\pm)\right\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}}E_n(t;W)^{\frac12},$$

(2.1.28)
$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|>[\frac{n}{2}]}}\left\|Q_0^{\mathrm{kg}}(v^{I_1}_\pm,D_xu^{I_2}_\pm)\right\|_{L^2}\lesssim\left(\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}}\right)E_n(t;W)^{\frac12}.$$

(ii) Let $0\leq k\leq2$ and $I\in\mathcal{I}^k_3$. There exists a constant $C>0$ such that, if we assume a-priori estimates (1.1.11a), (1.1.11b) *satisfied and $0<\varepsilon_0<(2A+B)^{-1}$ small, for any $\chi\in C^\infty_0(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin and $\sigma>0$ small we have*

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_2|<3}}Q_0^{\mathrm{w}}(v^{I_1}_\pm,D_xv^{I_2}_\pm)=\mathfrak{R}^k_3(t,x),\tag{2.1.29a}$$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|,|I_2|<3}}Q_0^{\mathrm{kg}}(v^{I_1}_\pm,D_xu^{I_2}_\pm)=\delta_{\mathcal{V}^k}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\I_1\in\mathcal{K},|I_2|\leq1}}Q_0^{\mathrm{kg}}\left(v^{I_1}_\pm,\chi(t^{-\sigma}D_x)D_xu^{I_2}_\pm\right)+\mathfrak{R}^k_3(t,x),\tag{2.1.29b}$$

where $\delta_{\mathcal{V}^k} = 1$ *if* $I \in \mathcal{V}^k$*, 0 otherwise, and*

$$\|\mathfrak{R}_3^k(t,\cdot)\|_{L^2} \leq C(A+B)\varepsilon t^{-1} E_3^k(t,W)^{\frac{1}{2}} + CB\varepsilon t^{-\frac{5}{4}}, \tag{2.1.30}$$

with $\beta > 0$ *small,* $\beta \to 0$ *as* $\sigma \to 0$*, for all* $t \in [1,T]$*. The same result holds with* $D_x v_\pm^{I_2}$ *(resp.* $D_x u_\pm^{I_2}$*) replaced with* $\langle D_x \rangle v_\pm^{I_2}$ *(resp.* $|D_x| u_\pm^{I_2}$*).*

Proof. (i) The proof of follows straightly from (2.1.1) with $a = 1, 2$, by bounding the L^2 norm of each product with the L^∞ norm of the factor indexed in $J \in \{I_1, I_2\}$ such that $|J| \leq \left[\frac{|I|}{2}\right]$, times the L^2 norm of the remaining one.

(ii) Let $I \in \mathcal{I}_3^k$. One immediately sees that:

$$\begin{aligned}\sum_{(J,0)\in\mathcal{I}(I)} \|Q_0^{\mathrm{w}}(v_\pm^J, D_x v_\pm)\|_{L^2} + \sum_{\substack{(J,0)\in\mathcal{I}(I)\\ |J|<3}} \Big(\|Q_0^{\mathrm{w}}(v_\pm, D_x v_\pm^J)\|_{L^2} + \|Q_0^{\mathrm{kg}}(v_\pm, D_x u_\pm^J)\|_{L^2}\Big) \\ \lesssim \|V(t,\cdot)\|_{H^{2,\infty}} E_3^k(t;W)^{\frac{1}{2}};\end{aligned} \tag{2.1.31}$$

if $(I_1, I_2) \in \mathcal{I}(I)$ is such that $|I_2| < 3$ and either Γ^{I_1} or Γ^{I_2} is a product of spatial derivatives only

$$\|Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_x v_\pm^{I_2})\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{4,\infty}} E_3^k(t;W)^{\frac{1}{2}}; \tag{2.1.32}$$

if $(I_1, I_2) \in \mathcal{I}(I)$ is such that $|I_2| < 3$ and Γ^{I_1} is a product of spatial derivatives only

$$\|Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_x u_\pm^{I_2})\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{3,\infty}} E_3^k(t;W)^{\frac{1}{2}}. \tag{2.1.33}$$

Hence, the remaining quadratic contributions to be estimated are those corresponding to indices $(I_1, I_2) \in \mathcal{I}(I)$, with $|I_2| < 3$, such that: both Γ^{I_1} and Γ^{I_2} contain at least one Klainerman vector field, in the left hand side of (2.1.29a); Γ^{I_1} contains one or two Klainerman vector fields, in the left hand side of (2.1.29b).

The idea to estimate the L^2 norm of the $Q_0^{\mathrm{w}}(v_\pm^{I_1}, Dv_\pm^{I_2})$, for indices I_1, I_2 just mentioned above, is to decompose the Klein-Gordon component carrying exactly one Klainerman vector field in frequencies, by means of a truncation $\chi(t^{-\sigma}D_x)$ for some smooth cut-off function χ and $\sigma > 0$ small. Basically, the L^∞ norm of the contribution truncated for large frequencies $|\xi| \gtrsim t^\sigma$ can be bounded by making appear a power of t as negative as we want, while that of the remaining one, localized for $|\xi| \lesssim t^\sigma$, enjoys the sharp Klein-Gordon decay t^{-1} as proved in lemma B.4.14 in appendix B. The same argument can be applied to $Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_x u_\pm^{I_2})$ with I_1 such that Γ^{I_1} contains exactly one Klainerman vector field. Then, by lemma B.2.4 in appendix B with $L = L^2$ we find that, for some $\chi \in C_0^\infty(\mathbb{R}^2)$, the following: if Γ^{I_1} contains exactly one Klainerman vector field,

$$\begin{aligned}\Big\|Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_x v_\pm^{I_2})(t,\cdot)\Big\|_{L^2} \lesssim \Big\|\chi(t^{-\sigma}D_x) v_\pm^{I_1}(t,\cdot)\Big\|_{H^{1,\infty}} \|v_\pm^{I_2}(t,\cdot)\|_{H^1} \\ + t^{-N(s)} \left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s}\right) \Big(\sum_{|\mu|=0}^{1} \|x^\mu v_\pm^{I_2}(t,\cdot)\|_{H^1} + t\|v_\pm^{I_2}(t,\cdot)\|_{H^1}\Big)\end{aligned}$$

and

$$\begin{aligned}\Big\|Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_x u_\pm^{I_2})(t,\cdot)\Big\|_{L^2} \lesssim \Big\|\chi(t^{-\sigma}D_x) v_\pm^{I_1}(t,\cdot)\Big\|_{H^{1,\infty}} \|u_\pm^{I_2}(t,\cdot)\|_{H^1} \\ + t^{-N(s)} \left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s}\right) \Big(\sum_{|\mu|=0}^{1} \|x^\mu D_x u_\pm^{I_2}(t,\cdot)\|_{L^2} + t\|u_\pm^{I_2}(t,\cdot)\|_{H^1}\Big);\end{aligned}$$

if Γ^{I_2} contains exactly one Klainerman vector field,

$$\Big\|Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_x v_\pm^{I_2})(t,\cdot)\Big\|_{L^2} \lesssim \Big\|\chi(t^{-\sigma}D_x)v_\pm^{I_2}(t,\cdot)\Big\|_{H^{2,\infty}} \|v_\pm^{I_1}(t,\cdot)\|_{L^2} \\ + t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{|\mu|=0}^{1}\|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2} + t\|v_\pm^{I_1}(t,\cdot)\|_{L^2}\Big),$$

where, in all above inequalities, $N(s) \geq 3$ if $s > 0$ is large enough. From inequalities (B.1.5a), (B.1.6a), estimates (B.1.17), lemma B.4.14 and the boostrap assumptions (1.1.11), together with the fact $\delta, \delta_j \ll 1$ are small, for $j = 0, 1, 2$, we derive that there is a positive constant C such that, for multi-indices I_1, I_2 considered in above inequalities,

$$\Big\|Q_0^{\mathrm{w}}(v_\pm^{I_1}, Dv_\pm^{I_2})(t,\cdot)\Big\|_{L^2} + \Big\|Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2})(t,\cdot)\Big\|_{L^2} \leq CB\varepsilon t^{-1}E_3^k(t;W)^{\frac{1}{2}} + CB\varepsilon t^{-\frac{5}{4}}.$$

The remaining quadratic terms are $Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_x u_\pm^{I_2})$ with $I_1 \in \mathcal{K}$ (and hence $|I_2| \leq 1$) if $\mathcal{V}^k$ is non empty. Applying lemma B.2.4 with $L = L^2$, $w = u$ and the same s as before, and making use of estimates (1.1.11), (B.1.17), together with inequality (B.1.5a), we see that

$$\Big\|Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_x u_\pm^{I_2})(t,\cdot)\Big\|_{L^2} \lesssim \Big\|Q_0^{\mathrm{kg}}\Big(v_\pm^{I_1}, \chi(t^{-\sigma}D_x)D_x u_\pm^{I_2}\Big)(t,\cdot)\Big\|_{L^2} \\ + t^{-3}\Big(\sum_{|\mu|=0}^{1}\|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2} + t\|v_\pm^{I_1}(t,\cdot)\|_{L^2}\Big)\left(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\right) \\ \lesssim \Big\|Q_0^{\mathrm{kg}}\Big(v_\pm^{I_1}, \chi(t^{-\sigma}D_x)D_x u_\pm^{I_2}\Big)(t,\cdot)\Big\|_{L^2} + CB\varepsilon t^{-\frac{5}{4}},$$

which hence concludes the proof of (ii). We should highlight the fact that the quadratic contribution in the above left hand side is treated differently from the previous ones, because we do not have a sharp decay $O(t^{-1})$ for $v_\pm^{I_1}$ when $I_1 \in \mathcal{K}$ (neither when truncated for moderate frequencies), but only a control in $O(t^{-1+\beta'})$, for some small $\beta' > 0$ (see lemma B.4.2). Moreover, the decay enjoyed by the uniform norm of $\chi(t^{-\sigma}D_x)D_x u_\pm^{I_2}$, appearing in the quadratic term in the above right hand side, is very weak (only $t^{-1/2+\beta'}$, see lemma B.2.10). Such terms, that contribute to the energy and decay slowly in time, will be successively eliminated by a normal form argument (see subsection 2.2.2). □

Lemma 2.1.3. *Let $0 \leq k \leq 2$ and $I \in \mathcal{I}_3^k$. For any $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin and $\sigma > 0$ small*

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|\leq 2}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_t v_\pm^{I_2}) = \mathfrak{R}_3^k(t,x), \tag{2.1.34a}$$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|\leq 2}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_t u_\pm^{I_2}) = \delta_{\mathcal{V}^k}\sum_{\substack{(J,0)\in\mathcal{I}(I)\\ J\in\mathcal{K}}} Q_0^{\mathrm{kg}}(v_\pm^{J}, \chi(t^{-\sigma}D_x)|D_x|u_\pm) + \mathfrak{R}_3^k(t,x), \tag{2.1.34b}$$

with $\delta_{\mathcal{V}^k} = 1$ if $I \in \mathcal{V}^k$, 0 otherwise, and $\mathfrak{R}_3^k(t,x)$ satisfying (2.1.30).

Proof. Using the equation satisfied by $v_\pm^{I_2}$ and $u_\pm^{I_2}$ respectively in system (2.1.2) with $I = I_2$ we

see that

$$\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_t v_\pm^{I_2}) = \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, \langle D_x\rangle v_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2}} \sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} c_{J_1,J_2} Q_0^{\mathrm{w}}\left(v_\pm^{I_1}, Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})\right), \tag{2.1.35a}$$

$$\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_t u_\pm^{I_2}) = \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, |D_x| u_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2}} \sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} c_{J_1,J_2} Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, Q_0^{\mathrm{w}}(v_\pm^{J_1}, Dv_\pm^{J_2})\right), \tag{2.1.35b}$$

with coefficients $c_{J_1,J_2} \in \{-1, 0, -1\}$ such that $c_{J_1,J_2} = 1$ whenever $|J_1| + |J_2| = |I_2|$, and $Q_0^{\mathrm{w}}(v_\pm^{I_1}, \langle D_x\rangle v_\pm^{I_2})$ (in which case $D = D_1$), $Q_0^{\mathrm{kg}}(v_\pm^{I_1}, |D_x| u_\pm^{I_2})$ given explicitly by (2.1.24). After lemma 2.1.2 (ii) we know that

$$\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2}} \left[Q_0^{\mathrm{w}}(v_\pm^{I_1}, \langle D_x\rangle v_\pm^{I_2}) + Q_0^{\mathrm{kg}}(v_\pm^{I_1}, |D_x| u_\pm^{I_2})\right] = \sum_{\substack{(J,0)\in\mathfrak{I}(I)\\ J\in\mathcal{K}}} Q_0^{\mathrm{kg}}(v_\pm^{J}, |D_x| u_\pm) + \mathfrak{R}_3^k(t,x),$$

with $\mathfrak{R}_3^k$ verifying (2.1.30). The only thing to prove is that the cubic terms in the right hand side of (2.1.35) are remainders $\mathfrak{R}_3^k$. We focus on those in the right hand side of (2.1.35a) as the same argument applies to the ones in (2.1.35b).

First, let us consider cubic terms corresponding to indices I_1, I_2 such that $|I_1| = 2$ and $|I_2| = 0$. In this case we evidently have that $|J_1| = |J_2| = 0$, and by (B.1.4e) with $s = 1$ and $\theta \ll 1$ small, together with a-priori estimate (1.1.11),

$$\left\| Q_0^{\mathrm{w}}\left(v_\pm^{I_1}, Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)\right)\right\|_{L^2} \lesssim \|v_\pm^{I_1}(t,\cdot)\|_{L^2} \|Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)\|_{H^{1,\infty}} \leq CB\varepsilon t^{-\frac{3}{2}+\beta'},$$

for some $\beta' > 0$ small as long as σ, δ_0 are small.

Let us now consider indices I_1, I_2 such that $\Gamma^{I_1} \in \{\Omega, Z_m, m = 1, 2\}$. As we also require that $(I_1, I_2) \in \mathfrak{I}(I)$ with $|I_2| \leq 2$, we have in this case that $|I_2| \leq 1$ and consequently, for each $(J_1, J_2) \in \mathfrak{I}(I_2)$, either $|J_1| = 0$ or $|J_2| = 0$. Using lemma B.2.4 in appendix B with $L = L^2$ and $w = v$, we derive that for any $\chi \in C_0^\infty(\mathbb{R}^2)$ as in the statement and $\sigma > 0$ small

$$\begin{aligned}
&\sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} \left\| Q_0^{\mathrm{w}}\left(v_\pm^{I_1}, Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})\right)(t,\cdot)\right\|_{L^2} \\
&\qquad \lesssim \sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} \left\|\chi(t^{-\sigma}D_x) v_\pm^{I_1}(t,\cdot)\right\|_{L^\infty} \left\|Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})(t,\cdot)\right\|_{L^2} \\
&\qquad + \sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} t^{-N(s)} \left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s}\right) \\
&\qquad\qquad \times \Big(\sum_{|\mu|=0}^{1} \left\|x^\mu Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})(t,\cdot)\right\|_{L^2} + t \left\|Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})(t,\cdot)\right\|_{L^2}\Big),
\end{aligned}$$

with $N(s) \geq 3$ is $s > 0$ is sufficiently large. Here

$$\sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} \Big\| x Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})(t,\cdot)\Big\|_{L^2}$$
$$\lesssim \sum_{\substack{|\mu|=0,1\\ |J|\leq 1}} \Big[\Big\| x\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty} \big(\|u_\pm^J(t,\cdot)\|_{H^1} + \|D_t u_\pm^J(t,\cdot)\|_{L^2}\big)$$
$$+\|x v_\pm^J(t,\cdot)\|_{L^2}\big(\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|D_t\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{1,\infty}}\big)\Big] \leq C(A+B)B\varepsilon^2 t^{\frac12+\frac{\delta_2}{2}}$$

and

$$\sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} \Big\| Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})(t,\cdot)\Big\|_{L^2} \tag{2.1.36}$$
$$\lesssim \sum_{|J|\leq 1} \Big[\|v_\pm^J(t,\cdot)\|_{L^2}\Big(\sum_{|\mu|=0}^{1} \|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|D_t\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{1,\infty}}\Big)$$
$$+ \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\big(\|u_\pm^J(t,\cdot)\|_{H^1} + \|D_t u_\pm^J(t,\cdot)\|_{L^2}\big)\Big] \leq C(A+B)B\varepsilon^2 t^{-\frac12+\frac{\delta_2}{2}}$$

by (B.1.5a), (B.1.5b), (B.1.5c), (B.1.7) and estimates (1.1.11), (B.1.10), (B.1.17), so together with lemma B.4.14 and (B.1.6a), these inequalities give

$$\sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} \Big\| Q_0^{\mathrm{w}}\Big(v_\pm^{I_1}, Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})\Big)(t,\cdot)\Big\|_{L^2} \leq CB\varepsilon t^{-\frac32+\beta'},$$

for some new $\beta' > 0$ small, $\beta' \to 0$ as $\sigma, \delta_0 \to 0$.

Finally, for indices I_1, I_2 such that $\Gamma^{I_1} \in \{D_x^\alpha, |\alpha| \leq 1\}$

$$\sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} \Big\| Q_0^{\mathrm{w}}\Big(v_\pm^{I_1}, Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})\Big)\Big\|_{L^2} \lesssim \sum_{(J_1,J_2)\in\mathfrak{I}(I_2)} \|v_\pm(t,\cdot)\|_{H^{2,\infty}} \Big\| Q_0^{\mathrm{kg}}(v_\pm^{J_1}, Du_\pm^{J_2})\Big\|_{L^2}. \tag{2.1.37}$$

For $(J_1, J_2) \in \mathfrak{I}(I_2)$ such that $|J_1| + |J_2| = |I_2|$ we have by lemma 2.1.2 (ii) and a-priori estimates (1.1.11) that

$$\|Q_0^{\mathrm{kg}}(v_\pm^{J_1}, D_1 u_\pm^{J_2})\|_{L^2} \lesssim \|\mathfrak{R}_3^k(t,\cdot)\|_{L^2} + \sum_{J\in\mathcal{K}} \|Q_0^{\mathrm{kg}}(v_\pm^J, D_1\chi(t^{-\sigma}D_x)u_\pm)\|_{L^2}$$
$$\lesssim \|\mathfrak{R}_3^k(t,\cdot)\|_{L^2} + t^\beta \sum_{|\mu|=0}^{1} \|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty} E_3^1(t;W)^{\frac12}$$
$$\leq CB\varepsilon t^{-\frac12+\beta+\frac{\delta_1}{2}},$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$, while for $(J_1, J_2) \in \mathfrak{I}(I_2)$ such that $|J_1| + |J_2| < |I_2|$ (hence < 2) an estimate such as (2.1.36) holds. These estimates, together with (1.1.11b), imply that the right hand side of (2.1.37) is bounded by $CAB\varepsilon^2 t^{-\frac32+\beta'}$, for a new small $\beta' > 0$, $\beta' \to 0$ as $\sigma, \delta_0 \to 0$, and that concludes the proof of the statement. □

Corollary 2.1.4. *Let $Q_0^I(V,W)$ be the vector defined in* (2.1.12)*. There exists a constant $C > 0$ such that, if we assume that a-priori estimates* (1.1.11) *are satisfied in interval $[1,T]$, for some fixed $T > 1$, with $\varepsilon_0 < (2A+B)^{-1}$ small:*

(i) if $I \in \mathfrak{I}_n$ with $n \geq 3$:

$$\|Q_0^I(V,W)\|_{L^2} \leq CA\varepsilon t^{-\frac12+\frac{\delta}{2}}; \tag{2.1.38}$$

(ii) if $I \in \mathfrak{I}_3^k$, with $0 \leq k \leq 2$,

$$\|Q_0^I(V,W)\|_{L^2} \leq C(A+B)\varepsilon t^{-\frac{1}{2}+\frac{\delta_k}{2}}. \tag{2.1.39}$$

Proof. *(i)* Inequality (2.1.38) is straightforward after definition (2.1.12) (all coefficients c_{I_1,I_2} are equal to 0 when $I \in \mathfrak{I}_n$), lemma 2.1.2 *(i)*, and a-priori estimates (1.1.11a), (1.1.11b).

(ii) If $I \in \mathfrak{I}_3^k$ for a fixed $0 \leq k \leq 2$ we have by definition (2.1.12) and lemmas 2.1.2, 2.1.3 that

$$\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_2|<|I|}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_x v_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2, |I_2|<|I|}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_t v_\pm^{I_2}) = \mathfrak{R}_3^k(t,x), \tag{2.1.40}$$

with $\mathfrak{R}_3^k(t,x)$ satisfying (2.1.30). Moreover, for some smooth $\chi \in C_0^\infty(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin and $\sigma > 0$ small,

$$\begin{aligned}
&\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_2|<|I|}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_x u_\pm^{I_2}) = \delta_{\mathcal{V}^k} \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ I_1\in\mathcal{K}, |I_2|\leq 1}} Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, \chi(t^{-\sigma}D_x) D_x u_\pm^{I_2}\right) + \mathfrak{R}_3^k(t,x),\\
&\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|\leq 2, |I_2|<|I|}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_t u_\pm^{I_2}) = \delta_{\mathcal{V}^k} \sum_{\substack{(J,0)\in\mathfrak{I}(I)\\ J\in\mathcal{K}}} Q_0^{\mathrm{kg}}\left(v_\pm^{J}, \chi(t^{-\sigma}D_x)|D_x| u_\pm\right) + \mathfrak{R}_3^k(t,x),
\end{aligned} \tag{2.1.41}$$

with sets $\mathcal{K}, \mathcal{V}^k$ given, respectively, by (2.1.25), (2.1.26), $\delta_{\mathcal{V}^k} = 1$ if $I \in \mathcal{V}^k$, 0 otherwise (remind that $\mathcal{V}^2$ is empty). Observe that, if $k = 0, 1$, $I \in \mathfrak{I}_3^k$ and $(I_1, I_2) \in \mathfrak{I}(I)$ with $I_1 \in \mathcal{K}$, two situations may occur: if $\Gamma^{I_2} \in \{D_x^\alpha, |\alpha| \leq 1\}$ then product Γ^{I_1} contains exactly the same number of Klainerman vector fields as in Γ^I and V^{I_1} would be at the same energy level as V^I (i.e. its L^2 norm being controlled by $E_3^k(t;W)^{1/2}$). In this case, from a-priori estimates (1.1.11a)

$$\|v_\pm^{I_1}(t,\cdot)\|_{L^2}\left(\|\chi(t^{-\sigma}D_x)u_\pm^{I_2}(t,\cdot)\|_{H^{\rho,\infty}} + \|\chi(t^{-\sigma}D_x)\mathrm{R}u_\pm^{I_2}(t,\cdot)\|_{H^{\rho,\infty}}\right) \leq A\varepsilon t^{-\frac{1}{2}} E_3^k(t;W)^{\frac{1}{2}}. \tag{2.1.42}$$

If instead I_2 is such that $\Gamma^{I_2} \in \{\Omega, Z_m, m = 1, 2\}$ is a Klainerman vector field, we automatically have that Γ^I is a product of three Klainerman vector fields and that V^{I_1} is at an energy level strictly lower than V^I (i.e. its L^2 norm is controlled by $E_3^1(t;W)^{1/2}$ whereas that of V^I is bounded by $E_3^0(t;W)^{1/2}$). From lemma B.2.10 we deduce that

$$\begin{aligned}
\|v_\pm^{I_1}(t,\cdot)\|_{L^2}\left(\|\chi(t^{-\sigma}D_x)u_\pm^{I_2}(t,\cdot)\|_{H^{\rho,\infty}} + \|\chi(t^{-\sigma}D_x)\mathrm{R}u_\pm^{I_2}(t,\cdot)\|_{H^{\rho,\infty}}\right)&\\
\leq C(A+B)\varepsilon t^{-\frac{1}{2}+\beta+\frac{\delta_1}{2}} E_3^1(t;W)^{\frac{1}{2}},&
\end{aligned} \tag{2.1.43}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$. Summing up (2.1.40) to (2.1.43) and using (2.1.30) we obtain that there is a positive constant C such that

$$\|Q_0^I(V,W)\|_{L^2} \leq \delta_k C(A+B)\varepsilon t^{-\frac{1}{2}}\left[E_3^k(t;W)^{\frac{1}{2}} + \delta_0 t^{\beta+\frac{\delta_1}{2}} E_3^1(t;W)^{\frac{1}{2}}\right] + CB\varepsilon t^{-\frac{5}{4}}, \tag{2.1.44}$$

with $\delta_k = 1$ for $k = 0, 1$, equal to 0 when $k = 2$, and $\delta_0 = 1$ only when $k = 0$, 0 otherwise. Finally, taking $\sigma > 0$ small so that $\beta + \delta_1/2 \ll \delta_0/2$ and using a-priori estimates (1.1.11d) we deduce estimate (2.1.39) from (2.1.44). □

2.1.3 Symmetrization

Proposition 2.1.5. *As long as $H^{1,\infty}$ norm of $V(t,\cdot)$ is sufficiently small, there exists a real matrix $P(V;\eta)$ of order 0 and a real symmetric matrix $\widetilde{A}_1(V;\eta)$ of order 1, vanishing at order 1 at $V=0$, such that*

$$W_s^I := Op^B\big(P(V;\eta)\big)W^I \tag{2.1.45}$$

is solution to

$$\begin{aligned} D_t W_s^I = {} & A(D)W_s^I + Op^B(\widetilde{A}_1(V;\eta))W_s^I + Op^B(A''(V^I;\eta))U \\ & + Op^B(C''(U;\eta))V^I + Op_R^B(A''(V^I;\eta))U + Q_0^I(V,W) + \mathfrak{R}(U,V), \end{aligned} \tag{2.1.46}$$

where $\mathfrak{R}(U,V)$ satisfies, for any $\theta\in]0,1[$,

$$\begin{aligned} \|\mathfrak{R}(U,V)(t,\cdot)\|_{L^2} \lesssim {} & \Big[\|V(t,\cdot)\|_{H^{7,\infty}} + \|V(t,\cdot)\|_{H^{1,\infty}}^{1-\theta}\|V(t,\cdot)\|_{H^3}^{\theta}\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right) \\ & + \|V(t,\cdot)\|_{L^\infty}\left(\|U(t,\cdot)\|_{H^{2,\infty}}^{1-\theta} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}^{1-\theta}\right)\|U(t,\cdot)\|_{H^4}^{\theta}\Big]\|W^I(t,\cdot)\|_{L^2} \\ & + \|V(t,\cdot)\|_{H^{1,\infty}}\Big(\|W(t,\cdot)\|_{H^{7,\infty}} + \|\mathrm{R}U(t,\cdot)\|_{H^{6,\infty}}\Big)\|W^I(t,\cdot)\|_{L^2} \\ & + \|V(t,\cdot)\|_{H^{1,\infty}}\|Q_0^I(V,W)\|_{L^2}. \end{aligned} \tag{2.1.47}$$

Moreover, for any $n,r\in\mathbb{N}$,

$$M_r^0\left(P(V;\eta)-I_4;n\right) \lesssim \|V(t,\cdot)\|_{H^{1+r,\infty}}, \tag{2.1.48}$$

$$M_r^1\big(\widetilde{A}_1(V;\eta);n\big) \lesssim \|V(t,\cdot)\|_{H^{1+r,\infty}}, \tag{2.1.49}$$

and as long as the $H^{2,\infty}$ norm of $V(t,\cdot)$ is small there is a constant $C>0$ such that

$$C^{-1}\|W^I(t,\cdot)\|_{L^2} \le \|W_s^I(t,\cdot)\|_{L^2} \le C\|W^I(t,\cdot)\|_{L^2}. \tag{2.1.50}$$

In order to prove proposition 2.1.5, we first need to introduce the following lemma.

Lemma 2.1.6. *Let $\alpha,\beta\in\mathbb{R}$, $L\in M_2(\mathbb{R})$ and $M_0, N(\alpha,\beta)\in M_4(\mathbb{R})$ given by*

$$L=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad M_0=\begin{bmatrix}I_2&0\\0&-I_2\end{bmatrix},\quad N(\alpha,\beta)=\begin{bmatrix}\alpha L&\beta L\\ \alpha L&\beta L\end{bmatrix}=\begin{bmatrix}0&\alpha&0&\beta\\ \alpha&0&\beta&0\\ 0&\alpha&0&\beta\\ \alpha&0&\beta&0\end{bmatrix}.$$

There exist a small $\delta>0$ and a smooth function defined on open ball $B_\delta(0)$ of radius δ,

$$(\alpha,\beta)\in B_\delta(0)\to P(\alpha,\beta)\in Sym_4(\mathbb{R}),$$

with values in the space of real, symmetric, 4×4 matrices $Sym_4(\mathbb{R})$, such that $P(0,0)=I_4$, $P(\alpha,\beta)=I_4+O(|\alpha|+|\beta|)$ and $P(\alpha,\beta)^{-1}\big(M_0+N(\alpha,\beta)\big)P(\alpha,\beta)$ is symmetric for any $(\alpha,\beta)\in B_\delta(0)$. Furthermore $P^{-1}(\alpha,\beta)=I_4+O(|\alpha|+|\beta|)$.

Proof. Let $\mathcal{E}$ be the vector space of 2×2 matrices $B(\alpha,\beta)=\alpha I_2+\beta L$ and $\mathcal{F}$ be the set of 4×4 matrices of the form

$$\begin{bmatrix}F_{11}&F_{12}\\F_{21}&F_{22}\end{bmatrix}$$

with $F_{ij} \in \mathcal{E}$. We look for a matrix P of the form

$$P(B) = (I_2 - B^2)^{-\frac{1}{2}} \begin{bmatrix} I_2 & -B \\ -B & I_2 \end{bmatrix} \tag{2.1.51}$$

with $B \in \mathcal{E}$ close to zero (so that in particular $(I_2 - B^2)^{1/2}$ is well defined). We remark that matrix $P(B)^{-1}$ has the form

$$P(B)^{-1} = (I_2 - B^2)^{-\frac{1}{2}} \begin{bmatrix} I_2 & B \\ B & I_2 \end{bmatrix}$$

and that $P(0) = P^{-1}(0) = I_4$. We consider $\Phi : \mathbb{R}^2 \times \mathcal{E} \to \mathcal{F}$ defined by $\Phi(\alpha, \beta, B) := P(B)^{-1}\big[M_0 + N(\alpha, \beta)\big]P(B) = \big(\Phi_{ij}(\alpha, \beta, B)\big)_{1 \leq i,j \leq 2}$, where $\Phi_{ij} \in \mathcal{E}$ as $\mathcal{E}$ is a commutative sub-algebra of $M_2(\mathbb{R})$. We also define $\Psi(\alpha, \beta, B) := \Phi_{12}(\alpha, \beta, B) - \Phi_{21}^{\dagger}(\alpha, \beta, B)$ with $\Phi_{21}^{\dagger}$ denoting the transpose of Φ_{21}. We have that $\Psi(0, 0, 0) = 0$ and

$$D_B\Phi(0, 0, 0) \cdot B = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} M_0 - M_0 \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} = 2 \begin{bmatrix} 0 & -B \\ B & 0 \end{bmatrix}$$

from which follows that $D_B\Psi(0, 0, 0) \cdot B = -4B$, i.e. $D_B\Psi(0, 0, 0) = -4I$. Therefore, there exist a small $\delta > 0$ and a smooth function $(\alpha, \beta) \in B_\delta(0) \to B(\alpha, \beta) \in \mathcal{E}$ such that $B(0, 0) = 0$ (which implies $P(B(0, 0)) = I_4$), and $\Psi(\alpha, \beta, B(\alpha, \beta)) = 0 \ \forall (\alpha, \beta) \in B_\delta(0)$. This is equivalent to say that $\Phi(\alpha, \beta, B(\alpha, \beta))$ is symmetric and moreover $P(B(\alpha, \beta)), P(B(\alpha, \beta))^{-1} = I_4 + O(|\alpha| + |\beta|)$. Defining $P(\alpha, \beta) := P(B(\alpha, \beta))$ concludes the proof of the statement. □

Proof of proposition 2.1.5. With notations introduced in lemma 2.1.6 and in (2.1.5), (2.1.19), $A(\eta) = \langle \eta \rangle M_0 + S(\eta)$ and $A_1'(V; \eta)(1 - \chi)(\eta) = \langle \eta \rangle N(\alpha, \beta)$, with

$$S(\eta) = \begin{bmatrix} |\eta| - \langle \eta \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(|\eta| - \langle \eta \rangle) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ whose elements are } O(|\eta|^{-1}), |\eta| \to +\infty,$$

and $\alpha = a_0(v_\pm; \eta)\frac{\eta_1}{\langle \eta \rangle}(1 - \chi)(\eta)$, $\beta = b_0(v_\pm; \eta)\frac{\eta_1}{\langle \eta \rangle}(1 - \chi)(\eta)$, a_0, b_0 defined in (2.1.8). Since $\sup_\eta \big(|\alpha| + |\beta|\big) \lesssim \|V(t, \cdot)\|_{H^{1,\infty}}$, by lemma 2.1.6 we have that, as long as $\|V(t, \cdot)\|_{H^{1,\infty}}$ is sufficiently small, there exists a real symmetric matrix $P = P(V; \eta)$ of the form (2.1.51) such that $P(V; \eta)^{-1}\big[M_0 + N(\alpha, \beta)\big]P(V; \eta)$ is real and symmetric. Moreover $P = I_4 + Q(V; \eta)$ and $P^{-1} = I_4 + Q'(V; \eta)$, where $Q(V; \eta)$, $Q'(V; \eta)$ are matrices depending smoothly on α, β (which are symbols of order 0), null at order 1 at $V = 0$, verifying for any $n, r \in \mathbb{N}$

$$M_r^0\left(Q(V; \eta); n\right) + M_r^0\left(Q'(V; \eta); n\right) \lesssim \|V(t, \cdot)\|_{H^{1+r,\infty}}.$$

We define the following matrix of order 1

$$\widetilde{A}_1(V; \eta) := P(V; \eta)^{-1}\big[\langle \eta \rangle \big(M_0 + N(\alpha, \beta)\big)\big]P(V; \eta) - \langle \eta \rangle M_0$$

and $W_s^I := Op^B(P^{-1}(V; \eta))W^I$. From the fact that $\widetilde{A}_1(V; \eta)$ also writes as

$$\langle \eta \rangle \left[Q'(V; \eta)M_0 + P^{-1}(V; \eta)M_0Q(V; \eta) + P^{-1}(V; \eta)N(\alpha, \beta)P(V; \eta)\right]$$

we see that it vanishes at order 1 at $V = 0$ and is such that $M_r^1(\widetilde{A}_1(V; \eta); n) \lesssim \|V(t, \cdot)\|_{H^{1+r,\infty}}$. Moreover, from proposition 1.2.9 (ii) with $r = 1$ it follows that

$$I = Op^B(P(V; \eta))Op^B(P^{-1}(V; \eta)) + T_{-1}(V), \tag{2.1.52}$$

where operator $T_{-1}(V)$ is of order less or equal than -1 whose $\mathcal{L}(L^2)$ norm is a $O(\|V(t,\cdot)\|_{H^{2,\infty}})$. Therefore $W^I = Op^B(P(V;\eta))W_s^I + T_{-1}(V)W^I$ and from proposition 1.2.7 the L^2 norms of W^I, W_s^I are equivalent as long as the $H^{2,\infty}$ norm of V is small. Using equation (2.1.20) we find that:

(2.1.53)
$$\begin{aligned}
D_t W_s^I &= Op^B(P^{-1}(V;\eta))Op^B\big(A(\eta) + A_1'(V;\eta)(1-\chi)(\eta)\big)W^I \\
&+ Op^B(P^{-1}(V;\eta))\Big[Op^B\big(A_1'(V;\eta)\chi(\eta)\big) + Op^B\big(A_{-1}'(V;\eta)\big)\Big]W^I \\
&+ Op^B(P^{-1}(V;\eta))\Big[Op^B(C'(W^I;\eta))V + Op_R^B(A'(V;\eta))W^I\Big] \\
&+ Op^B(P^{-1}(V;\eta))\Big[Op^B(A''(V^I;\eta))U + Op^B(C''(U;\eta))V^I + Op_R^B(A''(V^I;\eta))U\Big] \\
&+ Op^B(P^{-1}(V;\eta))Q_0^I(V,W) + Op^B(D_tP^{-1}(V;\eta))W^I
\end{aligned}$$

where

(2.1.54)
$$\begin{aligned}
&Op^B(P^{-1}(V;\eta))Op^B\big(A(\eta) + A_1'(V;\eta)(1-\chi)(\eta)\big)W^I \\
&= Op^B(P^{-1}(V;\eta))Op^B\left(\langle\eta\rangle\big(M_0 + N(\alpha,\beta)\big)\right)W^I + Op^B(S(\eta))W^I + Op^B(Q'(V;\eta))Op^B(S(\eta))W^I \\
&= Op^B(P^{-1}(V;\eta))Op^B\left(\langle\eta\rangle\big(M_0 + N(\alpha,\beta)\big)\right)Op^B(P(V;\eta))W_s^I \\
&+ Op^B(P^{-1}(V;\eta))Op^B\left(\langle\eta\rangle\big(M_0 + N(\alpha,\beta)\big)\right)T_{-1}(V)W^I + Op^B(S(\eta))W_s^I \\
&+ Op^B(S(\eta))Op^B(Q(V;\eta))W_s^I + Op^B(S(\eta))T_{-1}(V)W^I + Op^B(Q'(V;\eta))Op^B(S(\eta))W^I \\
=&Op^B(A(\eta) + \widetilde{A}_1(V;\eta))W_s^I + \widetilde{T}_0(V)W_s^I + \widetilde{T'}_0(V)W^I
\end{aligned}$$

with $\widetilde{T}_0(V), \widetilde{T'}_0(V)$ operators of order 0 and $\mathcal{L}(L^2)$ norm $O(\|V(t,\cdot)\|_{H^{2,\infty}})$. Last equality follows indeed from the fact that, by proposition 1.2.9 (ii) with $r=1$ and proposition 1.2.7,

$$\begin{aligned}
&Op^B(P^{-1}(V;\eta))Op^B\big[\langle\eta\rangle\big(M_0 + N(\alpha,\beta)\big)\big]Op^B(P(V;\eta)) \\
&\qquad = Op^B\big(P(V;\eta)^{-1}\big[\langle\eta\rangle\big(M_0 + N(\alpha,\beta)\big)\big]P(V;\eta)\big) + \widetilde{T}_0(V)
\end{aligned}$$

and $Op^B(S(\eta))Op^B(Q(V;\eta))$, $Op^B(Q'(V;\eta))Op^B(S(\eta))$ are operator of order 0, too (the former of the form $T_0(V)$, the latter of the form $T_0(V)$), while $Op^B(S(\eta))T_{-1}(V)$ is of order -1 (and can be included in $T_0(V)$). The equivalence between the L^2 norms of W_s^I and W^I implies that $\widetilde{T}_0(V)W_s^I + \widetilde{T'}_0(V)W^I$ in (2.1.54) is a remainder $\mathfrak{R}(U,V)$.

All operators appearing in the second and third line of (2.1.53) are also remainders $\mathfrak{R}(U,V)$ because, from proposition 1.2.7, the fact that $M_0^0(P^{-1}(V;\eta);2) = O(1)$ and lemma 2.1.1, their L^2 norm is bounded by $\|V(t,\cdot)\|_{H^{7,\infty}}\|W^I(t,\cdot)\|_{L^2}$. Last term in (2.1.53) also contributes to $\mathfrak{R}(U,V)$ for matrix $D_tP^{-1}(V;\eta)$ is of order 0, its $M_0^0(\cdot,2)$ seminorm is bounded by $\|D_tV(t,\cdot)\|_{H^{1,\infty}}$ and for any $\theta\in[0,1]$

$$\begin{aligned}
\|D_tV(t,\cdot)\|_{H^{1,\infty}} \lesssim \|V(t,\cdot)\|_{H^{2,\infty}} + \|V(t,\cdot)\|_{H^{1,\infty}}^{1-\theta}\|V(t,\cdot)\|_{H^3}^{\theta}\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right) \\
+ \|V(t,\cdot)\|_{L^\infty}\left(\|U(t,\cdot)\|_{H^{2,\infty}}^{1-\theta} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}^{1-\theta}\right)\|U(t,\cdot)\|_{H^4}^{\theta},
\end{aligned}$$

as follows from (B.1.6b) with $s=1$. Finally, in remaining contributions in (2.1.53) we replace $Op^B(P^{-1}(V;\eta))$ with $I + Op^B(Q'(V;\eta))$ and observe that the terms on which $Op^B(Q'(V;\eta))$ acts are remainders $\mathfrak{R}(U,V)$ after proposition 1.2.7, the fact that $M_0^0\left(Q'(V;\eta);2\right) = O(\|V(t,\cdot)\|_{H^{1,\infty}})$ and lemma 2.1.1. Interchanging the notation of $P(V;\eta)$ and $P^{-1}(V;\eta)$, we obtain the result of the statement. □

2.2 Normal forms and energy estimates

Before going further in writing an energy inequality for W_s^I we should make few remarks. As we previously anticipated, the L^2 norm of some of the semi-linear terms appearing in equation (2.1.46) have a very slow decay in time. On the one hand, it is the case of $Op^B(A''(V^I;\eta))U$, $Op^B(C''(U;\eta))V^I$ and $Op_R^B(A''(V;\eta))U$, whose L^2 norms are estimated in (2.1.21d), (2.1.21e) in terms of the uniform norms of $U, \mathrm{R}_1 U$. On the other hand, also some of the contributions to $Q_0^I(V,W)$ are only a $O_{L^2}(t^{-1/2+\beta'})$, for some small $\beta' > 0$, after corollary 2.1.4. Nevertheless, we are going to see that $Op^B(A''(V^I;\eta))U$, $Op_R^B(A''(V;\eta))U$ and the mentioned contributions to $Q_0^I(V,W)$ can be easily eliminated by performing a semi-linear normal form argument in the energy inequality (see subsection 2.2.2). Such an argument is however not well adapted to handle $Op^B(C''(U;\eta))V^I$, for it leads to a loss of derivatives linked to the quasi-linear nature of the problem, i.e. to the fact that matrix $\widetilde{A}_1(V;\eta)$ in the right hand side of (2.1.46) is of order 1. This latter contribution should instead be eliminated through a suitable normal form applied directly on equation (2.1.46), which is the object of the subsection 2.2.1.

2.2.1 A first normal forms transformation and the energy inequality

First of all, we replace $Op^B(C''(U;\eta))V^I$ in equation (2.1.46) with $Op^B(C''(U;\eta))V_s^I$, having defined $V_s^I := Op^B(P^{-1}(V;\eta))V^I$, and remind that from (2.1.48) with $r = 0$ and (2.1.52) the L^2 norm of V^I and V_s^I are equivalent as long as the $H^{2,\infty}$ norm of $V(t,\cdot)$ is small (assumption compatible with (1.1.11b) if $\rho \geq 2$). We will rather deal with

$$\begin{aligned}(D_t - A(D))W_s^I = {} & Op^B(\widetilde{A}_1(V;\eta))W_s^I + Op^B(A''(V^I;\eta))U + Op^B(C''(U;\eta))V_s^I \\ & + Op_R^B(A''(V^I;\eta))U + Q_0^I(V,W) + \mathfrak{R}(U,V),\end{aligned} \tag{2.2.1}$$

for a new $\mathfrak{R}(U,V)$ satisfying (2.1.47) and show how to get rid of $Op^B(C''(U;\eta))V_s^I$ in the above right hand side. More precisely, we are going to prove the following result:

Proposition 2.2.1. *Let $N \in \mathbb{N}^*$. There exist three matrices $E_d^0(U;\eta), E_d^{-1}(U;\eta)$, $E_{nd}(U;\eta)$ linear in (u_+,u_-), with $E_d^0(U;\eta)$ real diagonal of order 0 and $E_d^{-1}(U;\eta), E_{nd}(U;\eta)$ of order -1, and, as long as $\|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}$ is small, a real diagonal matrix $F_d^0(U;\eta)$ of order 0 such that, if*

$$\begin{aligned}&\widetilde{W}_s^I := Op^B(I_4 + E(U;\eta))W_s^I, \\ &\text{with } E(U;\eta) := E_d^0(U;\eta) + E_d^{-1}(U;\eta) + E_{nd}(U;\eta),\end{aligned} \tag{2.2.2}$$

then

$$\begin{aligned}(D_t - A(D))\widetilde{W}_s^I = {} & Op^B\left((I_4 + E_d^0(U;\eta))\widetilde{A}_1(V;\eta)(I_4 + F_d^0(U;\eta))\right)\widetilde{W}_s^I \\ & + Op^B(A''(V^I;\eta))U + Op_R^B(A''(V^I;\eta))U + Q_0^I(V,W) + T_{-N}(U)W_s^I + \mathfrak{R}'(U,V).\end{aligned} \tag{2.2.3}$$

In the above right hand side $T_{-N}(U) = (\sigma_{ij}(U,D_x))_{ij}$ is a pseudo-differential operator of order less or equal than $-N$, with

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim \|\mathrm{R}_1 U(t,\cdot)\|_{H^{N+2,\infty}} + \|U(t,\cdot)\|_{H^{N+6,\infty}}, \tag{2.2.4}$$

for any $s \in \mathbb{R}$ and such that

$$\mathcal{F}_{x\mapsto\xi}(\sigma_{ij}(U,\eta))(\xi) = \begin{cases}\sigma_{ij}^+(\xi,\eta)\hat{u}_+(\xi) + \sigma_{ij}^-(\xi,\eta)\hat{u}_-(\xi), & i,j \in \{2,4\}, \\ 0, & \text{otherwise},\end{cases} \tag{2.2.5a}$$

with $\sigma_{ij}^{\pm}(\xi,\eta)$ *supported for* $|\xi|\leq\varepsilon\langle\eta\rangle$ *for a small* $\varepsilon>0$*, and for any* $\alpha,\beta\in\mathbb{N}^2$

$$|\partial_\xi^\alpha\partial_\eta^\beta\sigma_{ij}^{\pm}(\xi,\eta)|\lesssim_{\alpha,\beta}|\xi|^{N+1-|\alpha|}\langle\eta\rangle^{-N-|\beta|},\quad i,j\in\{2,4\}. \tag{2.2.5b}$$

Also, $\mathfrak{R}'(U,V)$ *is a remainder satisfying, for any* $\theta\in]0,1[$

$$\begin{aligned}\|\mathfrak{R}'(U,V)(t,\cdot)\|_{L^2}\lesssim{}&(1+\|U(t,\cdot)\|_{H^{5,\infty}})\|\mathfrak{R}(U,V)\|_{L^2}\\&+(\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}+\|U(t,\cdot)\|_{H^{5,\infty}})\Big[\|Q_0^I(V,W)\|_{L^2}\\&+(\|\mathrm{R}U(t,\cdot)\|_{H^{6,\infty}}+\|U(t,\cdot)\|_{H^{6,\infty}})\|W^I(t,\cdot)\|_{L^2}\Big]\\&+\|V(t,\cdot)\|_{H^{5,\infty}}^{2-\theta}\|V(t,\cdot)\|_{H^7}^{\theta}\|W^I(t,\cdot)\|_{L^2},\end{aligned} \tag{2.2.6}$$

with $\mathfrak{R}(U,V)$ *verifying* (2.1.47)*.*

For any $n,r\in\mathbb{N}$*, any* $\chi\in C_0^\infty(\mathbb{R}^2)$ *equal to 1 close to the origin and supported in open ball* $B_\varepsilon(0)$*, with* $\varepsilon>0$ *sufficiently small, we have that*

$$M_r^0\left(E_d^0\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right)\lesssim\|\mathrm{R}_1U(t,\cdot)\|_{H^{1+r,\infty}}, \tag{2.2.7a}$$

$$M_r^{-1}\left(E_d^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right)\lesssim\|U(t,\cdot)\|_{H^{5+r,\infty}}, \tag{2.2.7b}$$

$$M_r^{-1}\left(E_{nd}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right)\lesssim\|U(t,\cdot)\|_{H^{5+r,\infty}}; \tag{2.2.7c}$$

and

$$M_r^0\left(F_d^0\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right)\lesssim\|\mathrm{R}_1U(t,\cdot)\|_{H^{1+r,\infty}}. \tag{2.2.8}$$

Finally, as long as $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}+\|U(t,\cdot)\|_{H^{5,\infty}}$ *is small, there is a constant* $C>0$ *such that*

$$C^{-1}\|W_s^I(t,\cdot)\|_{L^2}\leq\|\widetilde{W}_s^I(t,\cdot)\|_{L^2}\leq C\|W_s^I(t,\cdot)\|_{L^2}. \tag{2.2.9}$$

Remark 2.2.2. From propositions 2.1.5 and 2.2.1 it follows that, as long as $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}$, $\|U(t,\cdot)\|_{H^{5,\infty}}$ and $\|V(t,\cdot)\|_{H^{2,\infty}}$ are small, there is a constant $C>0$ such that

$$C^{-1}\|W^I(t,\cdot)\|_{L^2}\leq\|\widetilde{W}_s^I(t,\cdot)\|_{L^2}\leq C\|W^I(t,\cdot)\|_{L^2}. \tag{2.2.10}$$

This implies that, if

$$\widetilde{E}_n(t;W):=\sum_{|\alpha|\leq n}\left\|Op^B(I_4+E(U;\eta))Op^B(P(V;\eta))D_x^\alpha W(t,\cdot)\right\|_{L^2},\quad\forall\, n\in\mathbb{N},n\geq3, \tag{2.2.11a}$$

$$\widetilde{E}_3^k(t;W):=\sum_{\substack{|\alpha|+|I|\leq3\\|I|\leq3-k}}\left\|Op^B(I_4+E(U;\eta))Op^B(P(V;\eta))D_x^\alpha W^I(t,\cdot)\right\|_{L^2},\forall\,0\leq k\leq2, \tag{2.2.11b}$$

there exists a constant $C_1>0$ such that

$$\begin{aligned}C_1^{-1}E_n(t;W)\leq\widetilde{E}_n(t;W)\leq C_1E_n(t;W),\quad\forall\, n\geq3,\\C_1^{-1}E_3^k(t;W)\leq\widetilde{E}_3^k(t;W)\leq C_1E_3^k(t;W),\quad\forall\,0\leq k\leq2.\end{aligned} \tag{2.2.12}$$

Thanks to the above equivalence, the propagation of some suitable estimates on $\widetilde{E}_n(t;W)$ and $\widetilde{E}_3^k(t;W)$ will provide us with (1.1.12c) and (1.1.12d) respectively, so we can rather focus on the derivation of an energy inequality for $\widetilde{E}_n(t;W),\widetilde{E}_3^k(t;W)$.

In order to get rid of $Op^B(C_d''(U;\eta))V_s^I$ in (2.2.1)we introduce matrices

$$C_d''(U;\eta)=\begin{bmatrix}0&0&0&0\\0&e_0&0&0\\0&0&0&0\\0&0&0&f_0\end{bmatrix},\quad C_{nd}''(U;\eta)=\begin{bmatrix}0&0&0&0\\0&0&0&f_0\\0&0&0&0\\0&e_0&0&0\end{bmatrix} \tag{2.2.13}$$

so that

$$C''(U;\eta)=C_d''(U;\eta)+C_{nd}''(U;\eta),$$

and proceed to eliminate $Op^B(C_d''(U;\eta))V_s^I$ and $Op^B(C_{nd}''(U;\eta))V_s^I$ separately.

Lemma 2.2.3. *Let $N\in\mathbb{N}^*$. There exists a diagonal matrix $E_d(U;\eta)$ of order 0, linear in (u_+,u_-), such that*

$$\begin{aligned}Op^B(C_d''(U;\eta))V_s^I+Op^B(D_tE_d(U;\eta))W_s^I-[A(D),Op^B(E_d(U;\eta))]W_s^I\\ =T_{-N}(U)W_s^I+\mathfrak{R}'(V,V),\end{aligned} \tag{2.2.14}$$

where $\mathfrak{R}'(V,V)$ satisfies, for any $\theta\in]0,1[$,

$$\|\mathfrak{R}'(V,V)(t,\cdot)\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{5,\infty}}^{2-\theta}\|V(t,\cdot)\|_{H^7}^{\theta}\|V^I(t,\cdot)\|_{L^2}, \tag{2.2.15}$$

and $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$ such that, for any $s\in\mathbb{R}$,

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)}\lesssim\|\mathrm{R}_1U(t,\cdot)\|_{H^{N+2,\infty}}+\|U(t,\cdot)\|_{H^{N+6,\infty}}, \tag{2.2.16}$$

whose symbol $\sigma(U,\eta)=(\sigma_{ij}(U,\eta))_{1\leq i,j\leq 4}$ is such that

$$\mathcal{F}_{x\mapsto\xi}(\sigma_{ij}(U,\eta))(\xi)=\begin{cases}\sigma_{ii}^+(\xi,\eta)\hat{u}_+(\xi)+\sigma_{ii}^-(\xi,\eta)\hat{u}_-(\xi), & i=j\in\{2,4\},\\ 0, & \text{otherwise},\end{cases} \tag{2.2.17a}$$

with $\sigma_{ii}^{\pm}(\xi,\eta)$ supported for $|\xi|\leq\varepsilon\langle\eta\rangle$ for a small $\varepsilon>0$, and verifying, for any $\alpha,\beta\in\mathbb{N}^2$,

$$|\partial_\xi^\alpha\partial_\eta^\beta\sigma_{ii}^{\pm}(\xi,\eta)|\lesssim_{\alpha,\beta}|\xi|^{N+1-|\alpha|}\langle\eta\rangle^{-N-|\beta|},\quad \text{for } i=2,4. \tag{2.2.17b}$$

Moreover, if $\chi\in C_0^\infty(\mathbb{R}^2)$ is equal to 1 close to the origin and has a sufficiently small support,

$$E_d\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right)=E_d^0\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right)+E_d^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right), \tag{2.2.18}$$

the former matrix in the above right hand side being real of order 0 and satisfying (2.2.7a)*, the latter being of order -1 and verifying* (2.2.7b)*.*

Proof. Because of the diagonal structure of $A(\eta)$ and $C_d''(U;\eta)$ we look for a matrix $E_d=(e_{ij})_{1\leq i,j\leq 4}$ satisfying (2.2.14) such that $e_{ij}=0$ for all i,j but $i=j\in\{2,4\}$, and we also require symbols e_{22},e_{44} to be of order 0 and linear in (u_+,u_-). If we remind that matrix $A(\eta)$ in (2.1.5) is of order 1 and make the ansatz that E_d is of order 0, then by symbolic calculus of proposition 1.2.9 we have that

$$-[A(D),Op^B(E_d(U;\eta))]=-\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}Op^B\left(\partial_\eta^\alpha A(\eta)D_x^\alpha E_d(U;\eta)\right)+T_{-N}(U) \tag{2.2.19}$$

with $T_{-N}(U)$ pseudo-differential operator of order less or equal than $-N$ such that, for any $s \in \mathbb{R}$,

(2.2.20)
$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim M^1_{N+1}(A(\eta);N+3)M^0_0(E_d(U;\eta);2) + M^1_0(A(\eta);N+3)M^0_{N+1}(E_d(U;\eta);2)$$

and whose symbol $\sigma(U,\eta) = (\sigma_{ij}(U,\eta))_{ij}$ is such that $\sigma_{ij}(U,\eta) = 0$ for all i,j but $i = j \in \{2,4\}$. Therefore, for any fixed $\chi \in C^\infty_0(\mathbb{R}^2)$ equal to 1 in $\overline{B_{\varepsilon_1}(0)}$ and supported in $B_{\varepsilon_2}(0)$, for some $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, we look for $E_d(U;\eta)$ such that

$$\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\left[C_d''(U;\eta) + D_t E_d(U;\eta) - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha A(\eta) D_x^\alpha E_d(U;\eta)\right] = 0.$$

Since $E_d(U;\eta)$ is required to be linear in (u_+,u_-), we should write it rather as $E_d(u_+,u_-;\eta)$ to then realize that, as u_+ (resp. u_-) is solution to the first (resp. to the third) equation in (2.1.2) with $|I| = 0$,

$$\begin{aligned}
&D_t E_d(u_+,u_-;\eta) = E_d(|D_x|u_+, -|D_x|u_-;\eta) + E_d\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm);\eta\big),\\
&D_x^\alpha E_d(u_+,u_-;\eta) = E_d(D_x^\alpha u_+, D_x^\alpha u_-;\eta), \quad \forall \alpha \in \mathbb{N}^2.
\end{aligned}$$

If we temporarily neglecting contribution $E_d\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm);\eta\big)$, we are lead to solve the following equation

$$\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\left[C_d''(U;\eta) + E_d(|D_x|u_+, -|D_x|u_-;\eta) - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha A(\eta) E_d(D_x^\alpha u_+, D_x^\alpha u_-;\eta)\right] = 0,$$

which is equivalent to system

$$\begin{cases}
e_{22}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x| - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_+, -\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x| + \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_-;\eta\right)\\
\qquad\qquad = -\chi\left(\frac{D_x}{\langle\eta\rangle}\right)e_0\\
e_{44}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x| + \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_+, -\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x| - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_-;\eta\right)\\
\qquad\qquad = -\chi\left(\frac{D_x}{\langle\eta\rangle}\right)f_0,
\end{cases}$$

with e_0, f_0 defined in (2.1.9). Then, if we look for e_{ii} of the form

$$e_{ii}(u_+,u_-;\eta) = \int e^{ix\cdot\xi}\alpha_{ii}(\xi,\eta)\hat{u}_+(\xi)d\xi + \int e^{ix\cdot\xi}\beta_{ii}(\xi,\eta)\hat{u}_-(\xi)d\xi, \tag{2.2.21}$$

this system implies, inter alia, that

$$\begin{aligned}
\int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\Big(|\xi| - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)\xi^\alpha\Big)\alpha_{22}(\xi,\eta)\hat{u}_+(\xi)d\xi =&\\
-\frac{i}{4}\int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\Big(1 - \frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\Big)\xi_1\hat{u}_+(\xi)d\xi.&
\end{aligned}$$

As

$$\left(1 \mp \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)\frac{\xi^\alpha}{|\xi|}\right) = 1 \mp \sum_{k=1}^{N}\frac{1}{k!}(\xi\cdot\nabla_\eta)^k(\langle\eta\rangle)$$

and

$$(\partial_{\eta_1}\xi_1+\partial_{\eta_2}\xi_2)^k\langle\eta\rangle=\frac{|\xi|^k}{\langle\eta\rangle^{k-1}}\left(1-\left(\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)^2\right)b_k(\xi,\eta),\quad 2\le k\le N,$$

with $b_k(\xi,\eta)$ polynomial of degree $k-2$ in $\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}$, we derive that

$$\left(1\mp\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)\frac{\xi^\alpha}{|\xi|}\right)=\left(1\mp\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)(1\mp b_\pm(\xi,\eta)) \tag{2.2.22}$$

with

$$\begin{gathered}b_\pm(\xi,\eta):=\sum_{k=2}^{N}\frac{1}{k!}|\xi|^{k-1}\langle\eta\rangle^{-(k-1)}\Big(1\pm\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\Big)b_k(\xi,\eta),\\ |\partial_\xi^\mu\partial_\eta^\nu b_\pm(\xi,\eta)|\lesssim_{\mu,\nu}|\xi|^{1-|\mu|}\langle\eta\rangle^{-1-|\nu|},\quad\forall\mu,\nu\in\mathbb{N}^2,\end{gathered} \tag{2.2.23}$$

and we can then choose $\alpha_{22}(\xi,\eta)$ in (2.2.21) such that, when $|\xi|\le\varepsilon_2\langle\eta\rangle$,

$$\alpha_{22}(\xi,\eta)=-\frac{i}{4}\left(1-b_+(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|}. \tag{2.2.24}$$

Similarly, we choose multipliers $\beta_{22},\alpha_{44},\beta_{44}$ such that, as long as $|\xi|\le\varepsilon_2\langle\eta\rangle$,

$$\begin{aligned}\beta_{22}(\xi,\eta)&=\frac{i}{4}\left(1+b_-(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|},\\ \alpha_{44}(\xi,\eta)&=-\frac{i}{4}\left(1+b_-(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|},\quad\beta_{44}(\xi,\eta)=\frac{i}{4}\left(1-b_+(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|}.\end{aligned}$$

These multipliers are all well defined for $|\xi|\le\varepsilon_2\langle\eta\rangle$ as $b_\pm(\xi,\eta)=O(|\xi|\langle\eta\rangle^{-1})$. Moreover, using that $(1\pm b_\mp(\xi,\eta))^{-1}=1\mp b_\mp(\xi,\eta)+O(|\xi|^2\langle\eta\rangle^{-2})$ as long as $|\xi|\le\varepsilon_2\langle\eta\rangle$, we have that

$$\alpha_{22}(\xi,\eta)=-\frac{i}{4}\frac{\xi_1}{|\xi|}+\alpha_{22}^{-1}(\xi,\eta),\quad\beta_{22}(\xi,\eta)=\frac{i}{4}\frac{\xi_1}{|\xi|}+\beta_{22}^{-1}(\xi,\eta),$$

$$\alpha_{44}(\xi,\eta)=-\frac{i}{4}\frac{\xi_1}{|\xi|}+\alpha_{44}^{-1}(\xi,\eta),\quad\beta_{44}(\xi,\eta)=\frac{i}{4}\frac{\xi_1}{|\xi|}+\beta_{44}^{-1}(\xi,\eta),$$

with $|\partial_\xi^\mu\partial_\eta^\nu\alpha_{ii}^{-1}|+|\partial_\xi^\mu\partial_\eta^\nu\beta_{ii}^{-1}|\lesssim_{\mu,\nu}|\xi|^{1-|\mu|}\langle\eta\rangle^{-1-|\nu|}$ for any $\mu,\nu\in\mathbb{N}^2$. Injecting the above α_{ii},β_{ii}, $i\in\{2,4\}$, in (2.2.21) we find that

$$\begin{aligned}e_{22}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right)&=-\frac{i}{4}R_1(u_+-u_-)+e_{22}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right),\\ e_{44}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right)&=-\frac{i}{4}R_1(u_+-u_-)+e_{44}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right),\end{aligned}$$

where, for $i\in\{2,4\}$,

$$\begin{aligned}&e_{ii}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right)=\\ &\qquad\int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\alpha_{ii}^{-1}(\xi,\eta)\hat{u}_+(\xi)d\xi+\int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\beta_{ii}^{-1}(\xi,\eta)\hat{u}_-(\xi)d\xi.\end{aligned}$$

After lemma A.1 (i) and above estimates for $\alpha_{ii}^{-1}, \beta_{ii}^{-1}$, kernels

$$K_+^i(x,\eta) := \int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\alpha_{ii}^{-1}(\xi,\eta)\langle\xi\rangle^{-4}d\xi, \quad K_-^i(x,\eta) := \int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\beta_{ii}^{-1}(\xi,\eta)\langle\xi\rangle^{-4}d\xi$$

are such that, for any $\beta \in \mathbb{N}^2$, $|\partial_\eta^\beta K_\pm^i(x,\eta)| \lesssim |x|^{-1}\langle x\rangle^{-2}\langle\eta\rangle^{-1-|\beta|}$ for every (x,η). This implies that

$$\left|\partial_\eta^\beta e_{ii}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+, \chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right)\right| \leq$$
$$\left|\int \partial_\eta^\beta K_+^i(x-y,\eta)[\langle D_x\rangle^4 u_+](y)dy\right| + \left|\int \partial_\eta^\beta K_-^i(x-y,\eta)[\langle D_x\rangle^4 u_-](y)dy\right| \lesssim \|U(t,\cdot)\|_{H^{4,\infty}}\langle\eta\rangle^{-1-|\beta|}$$

and e_{ii}^{-1} is a symbol of order -1, for $i = 2, 4$. Moreover, using definition (1.2.3) and the fact that space $W^{r,\infty}$ injects in $H^{r+1,\infty}$, one can check that for any $r, n \in \mathbb{N}$,

$$M_r^{-1}\left(e_{ii}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+, \chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right);n\right) \lesssim \|U(t,\cdot)\|_{H^{5+r,\infty}}$$

and therefore that

$$M_r^0\left(e_{ii}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+, \chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right);n\right) \lesssim \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1+r,\infty}} + \|U(t,\cdot)\|_{H^{5+r,\infty}}.$$

Defining

$$E_d^0(U;\eta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{4}R_1(u_+ - u_-) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{4}R_1(u_+ - u_-) \end{bmatrix}, \quad E_d^{-1}(U;\eta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{22}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{44}^{-1} \end{bmatrix},$$

decomposition (2.2.18) and estimate (2.2.7a), (2.2.7b) hold. Consequently, as

$$E_d\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm);\eta\big) = E_d^{-1}\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm);\eta\big)$$

for any $n \in \mathbb{N}$ and $\theta \in]0,1[$, we derive from (B.1.3d) with $s = 4$ that

$$M_0^0\left(E_d\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm);\eta\big);n\right) \lesssim \|Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\|_{H^{4,\infty}}$$
$$\lesssim \|V(t,\cdot)\|_{H^{5,\infty}}^{2-\theta}\|V(t,\cdot)\|_{H^7}^{\theta},$$

and hence that the quantization of $E_d\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm);\eta\big)$ acting on V_s^I verifies (2.2.15) after proposition 1.2.7. Also, (2.2.16) is deduced from (2.2.20) while properties (2.2.17) are obtained using essentially (1.2.12). □

Lemma 2.2.4. *Let $N \in \mathbb{N}^*$. There exists a purely imaginary matrix $E_{nd}(U;\eta)$, linear in (u_+, u_-) and of order -1, satisfying estimate* (2.2.7c)*, such that*

$$Op^B(C''_{nd}(U;\eta))V_s^I + Op^B(D_t E_{nd}(U;\eta))W_s^I - [A(D), Op^B(E_{nd}(U;\eta))]W_s^I$$
$$= T_{-N}(U)W_s^I + \mathfrak{R}'(V,V), \tag{2.2.25}$$

where $\mathfrak{R}'(V,V)$ is a remainder satisfying (2.2.15) *and $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$ such that, for any $s \in \mathbb{R}$,*

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim \|U(t,\cdot)\|_{H^{N+6,\infty}}. \tag{2.2.26}$$

Moreover, its symbol $\sigma(U,\eta)=(\sigma_{ij}(U,\eta))_{1\leq i,j\leq 4}$ *is such that*

$$\mathcal{F}_{x\mapsto\xi}(\sigma_{ij}(U,\eta))(\xi)=\begin{cases}\sigma_{ij}^{+}(\xi,\eta)\hat{u}_{+}(\xi)+\sigma_{ij}^{-}(\xi,\eta)\hat{u}_{-}(\xi), & (i,j)\in\{(2,4),(4,2)\},\\ 0, & \textit{otherwise},\end{cases} \tag{2.2.27a}$$

with $\sigma_{ij}^{\pm}$ *supported for* $|\xi|\leq\varepsilon\langle\eta\rangle$ *for a small* $\varepsilon>0$*, and verifying, for any* $\alpha,\beta\in\mathbb{N}^2$*,*

$$|\partial_{\xi}^{\alpha}\partial_{\eta}^{\beta}\sigma_{ij}^{\pm}(\xi,\eta)|\lesssim_{\alpha,\beta}|\xi|^{N+2-|\alpha|}\langle\eta\rangle^{-N-1-|\beta|}, \tag{2.2.27b}$$

for $(i,j)\in\{(2,4),(4,2)\}$*.*

Proof. Because of the structure of $C_{nd}''(U;\eta)$, we seek for a matrix $E_{nd}(U;\eta)$ satisfying (2.2.25), of the form $E_{nd}(U;\eta)=(e_{ij})_{1\leq i,j\leq 4}$ with $e_{ij}=0$ for all i,j, except $(i,j)\in\{(2,4),(4,2)\}$. If we make the ansatz that $E_{nd}(U;\eta)$ is linear in (u_+,u_-), of order -1, and remind that $A(\eta)$ in (2.1.5) is of order 1, from symbolic calculus of proposition 1.2.9 we have that

$$\begin{aligned}-[A(D),Op^{B}(E_{nd}(U;\eta))]=&-Op^{B}(A(\eta)E_{nd}(U;\eta)-E_{nd}(U;\eta)A(\eta))\\&-\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}Op^{B}(\partial_{\eta}^{\alpha}A(\eta)\cdot D_{x}^{\alpha}E_{nd}(U;\eta))+T_{-N}(U),\end{aligned}$$

where $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$, such that, for any $s\in\mathbb{R}$,

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^{s})}\lesssim M_{N+1}^{1}(A(\eta);N+3)M_{0}^{-1}(E_{nd}(U;\eta);2)+M_{0}^{1}(A(\eta);N+3)M_{N+1}^{-1}(E_{nd}(U;\eta);2), \tag{2.2.28}$$

and whose symbol $\sigma(U,\eta)=(\sigma_{ij}(U,\eta))_{ij}$ is such that $\sigma_{ij}=0$ for all i,j but $(i,j)\in\{(2,4),(4,2)\}$. Hence, for any fixed $\chi\in\mathbb{R}^2$ equal to 1 in $\overline{B_{\varepsilon_1}(0)}$ and supported in $B_{\varepsilon_2}(0)$, for some $0<\varepsilon_1<\varepsilon_2\ll 1$, we look for $E_{nd}(U;\eta)$ such that

$$\begin{aligned}\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big[&C_{nd}''(U;\eta)+D_tE_{nd}(U;\eta)-A(\eta)E_{nd}(U;\eta)+E_{nd}(U;\eta)A(\eta)\\&-\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_{\eta}^{\alpha}A(\eta)\cdot D_{x}^{\alpha}E_{nd}(U;\eta)\Big]=0.\end{aligned} \tag{2.2.29}$$

Furthermore, as $E_{nd}(U;\eta)=E_{nd}(u_+,u_-;\eta)$ is linear in (u_+,u_-) and u_+ (resp. u_-) is solution to the first (resp. the third) equation in (2.1.2) with $|I|=0$, we have that

$$\begin{aligned}D_tE_{nd}(u_+,u_-;\eta)&=E_{nd}(|D_x|u_+,-|D_x|u_-;\eta)+E_{nd}\big(Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm),Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm);\eta\big),\\D_x^{\alpha}E_{nd}(u_+,u_-;\eta)&=E_{nd}(D_x^{\alpha}u_+,D_x^{\alpha}u_-;\eta),\quad\forall\alpha\in\mathbb{N}^2\end{aligned}$$

while

$$-A(\eta)E_{nd}(U;\eta)+E_{nd}(U;\eta)A(\eta)=\begin{bmatrix}0&0&0&0\\0&0&0&-2\langle\eta\rangle e_{24}\\0&0&0&0\\0&2\langle\eta\rangle e_{42}&0&0\end{bmatrix}.$$

Then we rather search for symbols e_{24} and e_{42} such that

$$\begin{cases}\chi\left(\frac{D_x}{\langle\eta\rangle}\right)e_{2,4}\left(\Big(|D_x|-\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^{\alpha}(\langle\eta\rangle)D_x^{\alpha}-2\langle\eta\rangle\Big)u_+,-\Big(|D_x|+\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^{\alpha}(\langle\eta\rangle)D_x^{\alpha}+2\langle\eta\rangle\Big)u_-;\eta\right)\\\qquad\qquad=-\chi\left(\frac{D_x}{\langle\eta\rangle}\right)f_0,\\\chi\left(\frac{D_x}{\langle\eta\rangle}\right)e_{4,2}\left(\Big(|D_x|+\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^{\alpha}(\langle\eta\rangle)D_x^{\alpha}+2\langle\eta\rangle\Big)u_+,-\Big(|D_x|-\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^{\alpha}(\langle\eta\rangle)D_x^{\alpha}-2\langle\eta\rangle\Big)u_-;\eta\right)\\\qquad\qquad=-\chi\left(\frac{D_x}{\langle\eta\rangle}\right)e_0,\end{cases}$$

with e_0, f_0 given by (2.1.9), neglecting contribution $E_{nd}\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm); \eta\big)$ whose quantization acting on W_s^I gives rise to a remainder $\mathfrak{R}'(V,V)$, as we will see at the end of the proof. We look for e_{ij} of the form

$$e_{ij}(u_+, u_-; \eta) = \int e^{ix\cdot\xi}\alpha_{ij}(\xi,\eta)\hat{u}_+(\xi)d\xi + \int e^{ix\cdot\xi}\beta_{ij}(\xi,\eta)\hat{u}_-(\xi)d\xi,$$

for $(i,j) \in \{(2,4),(4,2)\}$, and reminding (2.2.22), (2.2.23) we choose the above multipliers such that, as long as $|\xi| \leq \varepsilon_2\langle\eta\rangle$,

$$\alpha_{24}(\xi,\eta) = -\frac{i}{4}\left(1 + \frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)\left(\left(1 - \frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)(1 - b_+(\xi,\eta)) - 2\frac{\langle\eta\rangle}{|\xi|}\right)^{-1}\frac{\xi_1}{|\xi|},$$
$$\beta_{24}(\xi,\eta) = -\frac{i}{4}\left(1 - \frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)\left(\left(1 + \frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)(1 + b_-(\xi,\eta)) + 2\frac{\langle\eta\rangle}{|\xi|}\right)^{-1}\frac{\xi_1}{|\xi|},$$
$$\alpha_{42}(\xi,\eta) = \beta_{24}, \quad \beta_{42}(\xi,\eta) = \alpha_{24}(\xi,\eta).$$

One can check that, on the support of $\chi\big(\frac{\xi}{\langle\eta\rangle}\big)$ and for any $\mu,\nu \in \mathbb{N}^2$, $|\partial_\xi^\mu\partial_\eta^\nu\alpha_{ij}| + |\partial_\xi^\mu\partial_\eta^\nu\beta_{ij}| \lesssim_{\mu,\nu} |\xi|^{1-|\mu|}\langle\eta\rangle^{-1-|\nu|}$, and then that, if

$$K_+^{ij}(x,\eta) := \int e^{ix\cdot\eta}\chi\Big(\frac{\xi}{\langle\eta\rangle}\Big)\alpha_{ij}(\xi,\eta)\langle\xi\rangle^{-4}d\xi, \quad K_-^{ij}(x,\eta) := \int e^{ix\cdot\eta}\chi\Big(\frac{\xi}{\langle\eta\rangle}\Big)\beta_{ij}(\xi,\eta)\langle\xi\rangle^{-4}d\xi,$$

for $(i,j) \in \{(2,4),(4,2)\}$, $|\partial_\eta^\beta K_\pm^{ij}(x,\eta)| \lesssim |x|^{-1}\langle x\rangle^{-2}\langle\eta\rangle^{-1-|\beta|}$, for any $\beta \in \mathbb{N}^2$ and $(x,\eta) \in \mathbb{R}^2\times\mathbb{R}^2$, as a consequence of lemma A.1. Therefore

$$\left|\partial_\eta^\beta e_{ij}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+, \chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-; \eta\right)\right| \leq$$
$$\left|\int \partial_\eta^\beta K_+^{ij}(x-y,\eta)[\langle D_x\rangle^4 u_+](y)dy\right| + \left|\int \partial_\eta^\beta K_-^{ij}(x-y,\eta)[\langle D_x\rangle^4 u_-](y)dy\right| \lesssim \|U(t,\cdot)\|_{H^{4,\infty}}\langle\eta\rangle^{-1-|\beta|},$$

which implies that e_{24}, e_{42} are symbols of order -1. Also, for $(i,j) \in \{(2,4),(4,2)\}$ and any $n, r \in \mathbb{N}$, one can prove that

$$M_r^{-1}\left(e_{ij}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+, \chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-; \eta\right); n\right) \lesssim \|U(t,\cdot)\|_{H^{5+r,\infty}}$$

using definition (1.2.3) and the fact that space $W^{r,\infty}$ injects in H^{r+1} for any $r \in \mathbb{N}$. Estimate (2.2.26) follows from (2.2.28) and symbol $\sigma(U;\eta)$ associated to $T_{-N}(U)$ satisfies (2.2.27), as one can check using (1.2.12) and the estimates derived above for α_{ij}, β_{ij}. Finally, from (B.1.3d) with $s = 4$ we deduce that, for any $\theta \in]0,1[$,

$$M_0^{-1}\big(E_{nd}\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm); \eta\big); n\big) \lesssim \|V(t,\cdot)\|_{H^{5,\infty}}^{2-\theta}\|V(t,\cdot)\|_{H^7}^{\theta},$$

and the quantization of $E_{nd}\big(Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$ acting on W_s^I is a remainder verifying (2.2.15) by proposition 1.2.7. □

Proof of Proposition 2.2.1. Lemmas 2.2.3 and 2.2.4 show that there exist two matrices $E_d(U;\eta)$ and $E_{nd}(U;\eta)$, linear in (u_+,u_-), satisfying equations (2.2.14) and (2.2.25) respectively. After definition (2.2.2) of $\widetilde{W}_s^I$ and equalities (2.2.1), (2.2.14) and (2.2.25) we deduce that

$$\begin{aligned}(D_t - A(D))\widetilde{W}_s^I &= Op^B(\widetilde{A}_1(V;\eta))W_s^I + Op^B(A''(V^I;\eta))U + Op_R^B(A''(V^I;\eta))U\\ &+ Q_0^I(V,W) + \mathfrak{R}(U,V) + Op^B(E(U;\eta))\Big[Op^B(\widetilde{A}_1(V;\eta))W_s^I + Op^B(A''(V^I;\eta))U\\ &+ Op^B(C''(U;\eta))V_s^I + Op_R^B(A''(V^I;\eta))U + Q_0^I(V,W) + \mathfrak{R}(U,V)\Big] + T_{-N}(U)W_s^I + \mathfrak{R}'(V,V)\end{aligned}$$

where $\mathfrak{R}(U,V)$ satisfies (2.1.47), $\mathfrak{R}'(V,V)$ satisfies (2.2.15), and $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$ verifying (2.2.5), (2.2.4). Contribution

$$Op^B(E(U;\eta))\Big[Op^B(A''(V^I;\eta))U + Op^B(C''(U;\eta))V_s^I + Op_R^B(A''(V^I;\eta))U \\ + Q_0^I(V,W) + \mathfrak{R}(U,V)\Big]$$

is a remainder of the form $\mathfrak{R}'(U,V)$ satisfying estimate (2.2.6) as a consequence of proposition 1.2.7, estimates (2.2.7) with $r=0$, lemma 2.1.1, and the fact that the L^2 norms of V_s^I and V^I are equivalent as long as $\|V(t,\cdot)\|_{H^{2,\infty}}$ is small.

According to the definition of $E(U;\eta)$ and decomposition (2.2.18)

$$Op^B(E(U;\eta))Op^B(\widetilde{A}_1(V;\eta)) = Op^B(E_d^0(U;\eta))Op^B(\widetilde{A}_1(V;\eta)) \\ + Op^B\big(E_d^{-1}(U;\eta) + E_{nd}(U;\eta)\big)Op^B(\widetilde{A}_1(V;\eta)).$$

Proposition 1.2.7 and estimates (2.1.49), (2.2.7b), (2.2.7c) with $r=0$, imply that the latter addend in the above right hand side is a bounded operator on L^2 whose $\mathcal{L}(L^2)$ norm is estimated by $\|U(t,\cdot)\|_{H^{5,\infty}}\|V(t,\cdot)\|_{H^{1,\infty}}$. The former one writes instead as $Op^B(E_d^0(U;\eta)\widetilde{A}_1(V;\eta)) + T_0(U,V)$, for an operator $T_0(U,V)$ of order less or equal than 0 and $\mathcal{L}(L^2)$ norm controlled by $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\|V(t,\cdot)\|_{H^{2,\infty}}$, as follows from corollary 1.2.11 and estimates (2.1.49), (2.2.7a) with $r=1$. Hence,

$$Op^B(E(U;\eta))Op^B(\widetilde{A}_1(V;\eta))W_s^I = Op^B(E_d^0(U;\eta)\widetilde{A}_1(V;\eta))W_s^I + \mathfrak{R}'(U,V),$$

for a new $\mathfrak{R}'(U,V)$ satisfying (2.2.6).

After (2.2.7a) matrix $I_4 + E_d^0\big(\chi(\frac{D_x}{\langle\eta\rangle})U;\eta\big)$ is invertible as long as $\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}$ is small and $F_d^0(U;\eta) := \big[I_4 + E_d^0\big(\chi(\frac{D_x}{\langle\eta\rangle})U;\eta\big)\big]^{-1} - I_4$ is such that, for any $n, r \in \mathbb{N}$,

$$M_r^0\left(F_d^0\Big(\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)U;\eta\Big);n\right) \lesssim \|\mathrm{R}_1U(t,\cdot)\|_{H^{1+r,\infty}}.$$

Moreover, $F_d^0(U;\eta)$ is a real diagonal matrix of order 0, and by corollary 1.2.11 with $r=1$

$$Op^B(I_4 + F_d^0(U;\eta))Op^B(I_4 + E_d^0(U;\eta)) = Id + T_{-1}(U),$$

with $T_{-1}(U)$ of order less or equal than 0 and $\mathcal{L}(H^{s-1};H^s)$ norm bounded by $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}$, for any $s\in\mathbb{R}$. This implies that

$$Op^B(I_4 + F_d^0(U;\eta))\widetilde{W}_s^I = W_s^I + \widetilde{T}_{-1}(U)W_s^I, \widetilde{T}_{-1}(U) = T_{-1}(U) + Op^B(E_d^{-1}(U;\eta) + E_{nd}(U;\eta))$$

with $\widetilde{T}_{-1}(U)$ of order less or equal than -1 and

$$\|\widetilde{T}_{-1}(U)\|_{\mathcal{L}(H^{s-1};H^s)} \lesssim \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}} + \|U(t,\cdot)\|_{H^{5,\infty}} \tag{2.2.30}$$

for any $s\in\mathbb{R}$. Hence, as long as this quantity is small, there exists a positive constant C such that (2.2.9) holds. Also,

$$Op^B(I_4 + E_d^0(U;\eta))Op^B(\widetilde{A}_1(V;\eta))W_s^I \\ = Op^B(I_4 + E_d^0(U;\eta))Op^B(\widetilde{A}_1(V;\eta))Op^B(I_4 + F_d^0(U;\eta))\widetilde{W}_s^I \\ - Op^B(I_4 + E_d^0(U;\eta))Op^B(\widetilde{A}_1(V;\eta))\widetilde{T}_{-1}(U)W_s^I,$$

where from proposition 1.2.7, (2.1.49), (2.2.7a), (2.2.30) and (2.1.50) the L^2 norm of the latter term in the above right hand side is estimated by

$$\|V(t,\cdot)\|_{H^{1,\infty}}\left(\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}+\|U(t,\cdot)\|_{H^{5,\infty}}\right)\|W^I(t,\cdot)\|_{L^2}. \tag{2.2.31}$$

On the other hand, by corollary 1.2.11 with $r=1$ we get that

$$\begin{aligned}Op^B(I_4+E_d^0(U;\eta))Op^B(\widetilde{A}_1(V;\eta))Op^B(I_4+F_d^0(U;\eta))\widetilde{W}_s^I&\\ =Op^B\big((I_4+E_d^0(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F_d^0(U;\eta))\big)\widetilde{W}_s^I&\\ +Op^B(I_4+E_d^0(U;\eta))T_0(U,V)\widetilde{W}_s^I+\widetilde{T}_0(U,V)\widetilde{W}_s^I,&\end{aligned}$$

with $T_0(U,V),\widetilde{T}_0(U,V)$ operators of order less or equal than 0 and $\mathcal{L}(L^2)$ norm controlled, respectively, by $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\|V(t,\cdot)\|_{H^{2,\infty}}$ and (2.2.31), so the last two terms in above right hand side are also remainders $\mathfrak{R}'(U,V)$ by proposition 1.2.7 and estimate (2.2.7a). That concludes the proof of the statement. □

2.2.2 A second normal forms transformation.

In proposition 2.2.1 in previous subsection we showed that one can get rid of the slow-decaying-in-time semi-linear contribution $Op^B\big(C''(U;\eta)\big)V_s^I$ in (2.1.46) by introducing a new function $\widetilde{W}_s^I$, defined in (2.2.2) in terms of W_s^I and solution to equation (2.2.3). That naturally led us to the introduction of new energies $\widetilde{E}_n(t;W)$, for $n\in\mathbb{N}, n\geq 3$, and $\widetilde{E}_3^k(t;W)$, for $k\in\mathbb{N}, 0\leq k\leq 2$, (see (2.2.11a)) which are respectively equivalent to starting $E_n(t;W)$ and $E_3^k(t;W)$ whenever some uniform norms of U,V are sufficiently small. However, these new energies do not allow us yet to recover enhanced estimates (1.1.12c) and (1.1.12d) as it is not true that

$$\left|\partial_t\widetilde{E}_n(t;W)\right|=O\left(\varepsilon t^{-1+\frac{\delta}{2}}E_n(t;W)^{\frac{1}{2}}\right),\quad \left|\partial_t\widetilde{E}_3^k(t;W)\right|=O\left(\varepsilon t^{-1+\frac{\delta}{2}}E_3^k(t;W)^{\frac{1}{2}}\right). \tag{2.2.32}$$

This is do to the fact that we still have to deal with semi-linear slow-decaying contributions $Op^B(A''(V^I;\eta))U$, $Op_R^B(A''(V^I;\eta))U$, $Q_0^I(V,W)$ to the right hand side of (2.2.3), together with the new $T_{-N}(U)W_s^I$ whose L^2 norm is also a $O(t^{-\frac{1}{2}}\|W^I(t,\cdot)\|_{L^2})$ after (2.2.4) and (1.1.11a). The aim of the current subsection is hence to perform a new normal form argument to replace the mentioned terms with more decaying ones. This is actually done at the energy level, meaning that we are going to add some suitable cubic perturbations to $\widetilde{E}_n(t;W)$ and $\widetilde{E}_3^k(t;W)$ so that the new energies so defined satisfy estimates as in (2.2.32).

Let us first focus on the slow decaying terms that appear when computing

$$\partial_t\widetilde{E}_n(t;W)=\sum_{I\in\mathcal{J}_n}\left\langle\partial_t\widetilde{W}_s^I,\widetilde{W}_s^I\right\rangle$$

for any integer $n\geq 3$. Using equation (2.2.3) and rewriting $\widetilde{W}_s^I$ in terms of W^I we find, on the one hand, the contribution

$$-\sum_{I\in\mathcal{J}_n}\Im\Big[\langle Op^B(A''(V^I;\eta))U+Op_R^B(A''(V^I;\eta))U,W^I\rangle+\langle T_{-N}(U)W^I,W^I\rangle\Big], \tag{2.2.33}$$

which is a $O(\varepsilon t^{-1/2}E_n(t;W))$ after Cauchy-Schwarz inequality, lemma 2.1.1 and a-priori estimates (1.1.11). But we also have

$$-\sum_{I\in\mathcal{J}_n}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}}\Im\left[\langle Q_0^{\mathrm{kg}}(v_\pm^{I_1},D_1u_\pm^{I_2}),v_+^I+v_-^I\rangle\right], \tag{2.2.34}$$

which enjoys the same decay as the previous one, as can be immediately seen using again Cauchy-Schwarz inequality along with (2.1.28) and (1.1.11a). From definition (2.1.6) of matrix $A''(V^I,\eta)$, Plancherel's formula, (1.2.6) and the fact that $\overline{v_+^I} = -v_-^I$

$$\begin{aligned}\langle Op^B(A''(V^I;\eta))U, W^I\rangle &= \langle Op^B(a_0(v_\pm^I;\eta)\eta_1)u_+ + Op^B(b_0(v_\pm^I;\eta)\eta_1)u_-, v_+^I + v_-^I\rangle\\ &= -\frac{i}{4(2\pi)^2}\int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)\Big[\widehat{(v_+^I+v_-^I)}(\xi-\eta)\widehat{(u_++u_-)}(\eta) - \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\widehat{(v_+^I-v_-^I)}(\xi-\eta)\\ &\qquad\times\widehat{(u_+-u_-)}(\eta)\Big]\eta_1\widehat{(v_-^I+v_-^I)}(-\xi)d\xi d\eta,\end{aligned}$$

with χ denoting a smooth function equal to 1 in $\overline{B_{\varepsilon_1}(0)}$ and supported in $B_{\varepsilon_2}(0)$, for some $0<\varepsilon_1<\varepsilon_2\ll 1$. Hence

$$-\Im\left[\langle Op^B(A''(V^I;\eta))U, W^I\rangle\right] = \sum_{j_k\in\{+,-\}} C^I_{(j_1,j_2,j_3)} \tag{2.2.35}$$

with

$$C^I_{(j_1,j_2,j_3)} = \frac{1}{4(2\pi)^2}\int \chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\left(1 - j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_1\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta, \tag{2.2.36}$$

for any $j_1,j_2,j_3\in\{+,-\}$. Analogously, from equality (1.2.8)

$$-\Im\left[\langle Op^B_R(A''(V^I;\eta))U, W^I\rangle\right] = \sum_{j_k\in\{+,-\}} C^{I,R}_{(j_1,j_2,j_3)} \tag{2.2.37}$$

with

$$\begin{aligned}C^{I,R}_{(j_1,j_2,j_3)} = \frac{1}{4(2\pi)^2}\int \left[1-\chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)-\chi\Big(\frac{\eta}{\langle\xi-\eta\rangle}\Big)\right]\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_1\\ \times\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta.\end{aligned} \tag{2.2.38}$$

After proposition 2.2.1, $T_{-N}(U) = (\sigma_{ij}(U,D_x))_{ij}$ with symbols $\sigma_{ij}(U,\eta)$ satisfying (2.2.5). Introducing $\rho:\{+,-\}\to\{2,4\}$ such that $\rho(+)=2,\rho(-)=4$ and using the convention that $-j_k\in\{+,-\}\setminus\{j_k\}$, we have that

$$\begin{aligned}\langle T_{-N}(U)W^I, W^I\rangle &= \sum_{i,j\in\{+,-\}}\langle\sigma_{\rho(i)\rho(j)}(U,D_x)v_j^I, v_i^I\rangle\\ &= -\frac{1}{(2\pi)^2}\sum_{j_k\in\{+,-\}}\int\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta,\end{aligned} \tag{2.2.39}$$

where multipliers $\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)$ are supported for $|\eta|\le\varepsilon|\xi-\eta|$ and such that, for any $\alpha,\beta\in\mathbb{N}^2$,

$$\left|\partial_\xi^\alpha\partial_\eta^\beta\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)\right| \lesssim_{\alpha,\beta} |\eta|^{N+1-|\beta|}\langle\xi-\eta\rangle^{-N-|\alpha|},$$

for any $(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2$, any $j_1,j_2,j_3\in\{+,-\}$. Moreover, by (2.1.1) we have that

$$-\Im\left[\langle Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1u_\pm^{I_2}), v_+^I + v_-^I\rangle\right] = \sum_{j_k\in\{+,-\}} C^{I_1,I_2}_{(j_1,j_2,j_3)} \tag{2.2.40}$$

with

$$C^{I_1,I_2}_{(j_1,j_2,j_3)} := \frac{1}{4(2\pi)^2}\int \left(1 - j_1 j_2 \frac{\xi-\eta}{\langle \xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_1\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)\hat{v}^{I}_{j_3}(-\xi)d\xi d\eta. \tag{2.2.41}$$

The above equalities lead us to introduce the following multipliers

$$B^k_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{1}{j_1\langle\xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}\left(1 - j_1 j_2 \frac{\xi-\eta}{\langle \xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_k, \quad k=1,2 \tag{2.2.42}$$

and

$$\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)}{j_1\langle\xi-\eta\rangle + j_2|\eta| - j_3\langle\xi\rangle}, \tag{2.2.43}$$

together with the following integrals

$$D^I_{(j_1,j_2,j_3)} := \frac{i}{4(2\pi)^2}\int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\,d\xi d\eta, \tag{2.2.44a}$$

$$D^{I,R}_{(j_1,j_2,j_3)} := \frac{i}{4(2\pi)^2}\int \left[1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) - \chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right] B^1_{(j_1,j_2,j_3)}(\xi,\eta) \times \hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta, \tag{2.2.44b}$$

$$D^{I,T_{-N}}_{(j_1,j_2,j_3)} := \mathrm{Re}\left[\frac{1}{(2\pi)^2}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta\right] \tag{2.2.44c}$$

and

$$D^{I_1,I_2}_{(j_1,j_2,j_3)} := \frac{i}{4(2\pi)^2}\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\,d\xi d\eta \tag{2.2.45}$$

for any triplet $(j_1,j_2,j_3)\in\{+,-\}^3$. We warn the reader that definitions (2.2.44) and (2.2.45) are given here for any general multi-indices I, I_1, I_2.

Definition 2.2.5. For every integer $n\geq 3$ we define the second modified energy $\widetilde{E}^\dagger_n(t;W)$ as

$$\widetilde{E}^\dagger_n(t;W) := \widetilde{E}_n(t;W) + \sum_{\substack{I\in\mathfrak{I}_n\\ j_i\in\{+,-\}}}\left(D^I_{(j_1,j_2,j_3)} + D^{I,R}_{(j_1,j_2,j_3)} + D^{I,T_{-N}}_{(j_1,j_2,j_3)}\right) + \sum_{\substack{I\in\mathfrak{I}_n\\ j_i\in\{+,-\}}}\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}} D^{I_1,I_2}_{(j_1,j_2,j_3)}. \tag{2.2.46}$$

Let us now analyse the time derivative of $\widetilde{E}^k_3(t;W)$ for integers $0\leq k\leq 2$. As in the previous case, from equation (2.2.3) we see appear the same contribution as in (2.2.33), but with the sum over $\mathfrak{I}_n$ replaced with that on $\mathfrak{I}^k_3$. We also find

$$-\sum_{I\in\mathfrak{I}^k_3}\Im[\langle Q^I_0(V,W),W^I\rangle] \tag{2.2.47}$$

which is a $O(\varepsilon t^{-(1+\delta_k)/2}E^k_3(t;W)^{1/2})$ from Cauchy-Schwarz inequality and estimate (2.1.39). To be more precise, the slow decay in time of the above scalar product is due to some particular

quadratic term appearing in $Q_0^I(V,W)$. In fact, according to definition (2.1.12) and to (2.1.29a), (2.1.30), (2.1.34a), (1.1.11d), for any $I \in \mathcal{I}_3^k$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}} \left|\langle Q_0^{\rm w}(v_\pm^{I_1}, D_x v_\pm^{I_2}), u_+^I + u_-^I\rangle\right| + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|\le 2, |I_2|<|I|}} \left|\langle Q_0^{\rm w}(v_\pm^{I_1}, D_t v_\pm^{I_2}), u_+^I + u_-^I\rangle\right| \\ \lesssim \|\mathfrak{R}_3^k(t,\cdot)\|_{L^2}\|U^I(t,\cdot)\|_{L^2} \le C(A+B)\varepsilon t^{-1+\frac{\delta_k}{2}} E_3^k(t;W)^{\frac12}. \tag{2.2.48}$$

Also, after (2.1.29b) and (2.1.34b) we have that for all $I \notin \mathcal{V}^k$, with $\mathcal{V}^k$ defined in (2.1.26),

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}} \left|\left\langle Q_0^{\rm kg}(v_\pm^{I_1}, D_x u_\pm^{I_2}), v_+^I + v_-^I\right\rangle\right| + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|\le 2, |I_1|,|I_2|<|I|}} \left|\left\langle Q_0^{\rm kg}(v_\pm^{I_1}, D_t u_\pm^{I_2}), v_+^I + v_-^I\right\rangle\right| \\ \lesssim \|\mathfrak{R}_3^k(t,\cdot)\|_{L^2}\|V^I(t,\cdot)\|_{L^2} \le C(A+B)\varepsilon t^{-1+\frac{\delta_k}{2}} E_3^k(t;W)^{\frac12}. \tag{2.2.49}$$

Observe that the decay rate $O(t^{-1+\delta_k/2})$ in the right hand side of the two above inequalities is the slowest one that allows us to propagate a-priori estimate (1.1.11d) and it gives us back exactly the slow growth in time $t^{\delta_k/2}$ enjoyed by $E_3^k(t;W)^{1/2}$, for $0 \le k \le 2$. On the other hand, for $I \in \mathcal{V}^k$ with $k = 0, 1$, we have that, for some smooth cut-off function χ and some $\sigma > 0$ small,

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}} c_{I_1,I_2} Q_0^{\rm kg}(v_\pm^{I_1}, D_x u_\pm^{I_2}) = \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K}, |I_2|\le 1}} c_{I_1,I_2} Q_0^{\rm kg}\left(v_\pm^{I_1}, \chi(t^{-\sigma}D_x) D_x u_\pm^{I_2}\right) + \mathfrak{R}_3^k(t,x),$$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|\le 2, |I_1|,|I_2|<|I|}} c_{I_1,I_2} Q_0^{\rm kg}(v_\pm^{I_1}, D_t u_\pm^{I_2}) = \sum_{\substack{(J,0)\in\mathcal{I}(I)\\ J\in\mathcal{K}}} c_{J,0} Q_0^{\rm kg}(v_\pm^J, \chi(t^{-\sigma}D_x)|D_x| u_\pm) + \mathfrak{R}_3^k(t,x).$$

The L^2 norms of the summations in the above right hand sides are bounded by

$$\sum_{J|\le 1} \left(\|\chi(t^{-\sigma}D_x) u_\pm^J(t,\cdot)\|_{H^{2,\infty}} + \|\chi(t^{-\sigma}D_x) {\rm R} u_\pm^J(t,\cdot)\|_{H^{2,\infty}}\right) E_3^k(t;W)^{\frac12}$$

and hence by $\varepsilon t^{-1/2} E_3^k(t;W)^{1/2}$ as follows by sharp estimate (B.2.57) derived in appendix B. Therefore, the very contribution to (2.2.47) that has to be eliminated from $\partial_t \widetilde{E}_3^k(t;W)$ appears only for $k = 0, 1$ and is

$$-\sum_{I\in\mathcal{V}^k} \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K}, |I_2|\le 1}} c_{I_1,I_2} \Im\left[\left\langle Q_0^{\rm kg}\left(v_\pm^{I_1}, \chi(t^{-\sigma}D_x) D_x u_\pm^{I_2}\right), v_+^I + v_-^I\right\rangle\right] \\ - \sum_{I\in\mathcal{V}^k} \sum_{\substack{(J,0)\in\mathcal{I}(I)\\ J\in\mathcal{K}}} c_{J,0} \Im\left[\left\langle Q_0^{\rm kg}\left(v_\pm^J, \chi(t^{-\sigma}D_x)|D_x| u_\pm\right), v_+^I + v_-^I\right\rangle\right]. \tag{2.2.50}$$

As

$$-\Im\left[\langle Q_0^{\rm kg}(v_\pm^{I_1}, \chi(t^{-\sigma}D_x) D_l u_\pm^{I_2}), v_+^I + v_-^I\rangle\right] = \sum_{j_i\in\{+,-\}} F^{I_1,I_2,l}_{(j_1,j_2,j_3)}, \quad l = 1,2 \\ -\Im\left[\langle Q_0^{\rm kg}(v_\pm^{I_1}, \chi(t^{-\sigma}D_x)|D_x| u_\pm^{I_2}), v_+^I + v_-^I\rangle\right] = \sum_{j_i\in\{+,-\}} F^{I_1,I_2,3}_{(j_1,j_2,j_3)}, \tag{2.2.51}$$

with

$$F^{I_1,I_2,l}_{(j_1,j_2,j_3)} = \frac{1}{4(2\pi)^2}\int\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_l\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\hat{v}^{I}_{j_3}(-\xi)d\xi d\eta, \tag{2.2.52}$$

for any $j_i\in\{+,-\}$, $l=1,2,3$, and $\eta_3 := j_2|\eta|$, we introduce a new multiplier

$$B^3_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{j_2}{j_1\langle\xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)|\eta| \tag{2.2.53}$$

together with integrals

$$G^{I_1,I_2,l}_{(j_1,j_2,j_3)} = \frac{i}{4(2\pi)^2}\int B^l_{(j_1,j_2,j_3)}(\xi,\eta)\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\hat{v}^{I}_{j_3}(-\xi)d\xi d\eta \tag{2.2.54}$$

for any $l=1,2,3$, $(j_1,j_2,j_3)\in\{+,-\}^3$, with multipliers $B^l_{(j_1,j_2,j_3)}$ given by (2.2.42) when $l=1,2$, and by (2.2.53) when $l=3$. We warn the reader that in what follows we will sometimes refer to multipliers $B^l_{(j_1,j_2,j_3)}$ (resp. integrals $F^{I_1,I_2,l}_{(j_1,j_2,j_3)}$ and $G^{I_1,I_2,l}_{(j_1,j_2,j_3)}$) simply as $B_{(j_1,j_2,j_3)}$ (resp. $F^{I_1,I_2}_{(j_1,j_2,j_3)}$ and $G^{I_1,I_2}_{(j_1,j_2,j_3)}$) forgetting about superscript l. This choice reveals to be convenient when we do not need to distinguish between $l=1,2,3$.

Definition 2.2.6. For every integer $0\leq k\leq 2$ we define the second modified energy $\widetilde{E}^{k,\dagger}_3(t;W)$ as

$$\begin{aligned}\widetilde{E}^{k,\dagger}_3(t;W) := \widetilde{E}^k_3(t;W) + \sum_{\substack{I\in\mathcal{I}^k_3\\ j_i\in\{+,-\}}}\left(D^I_{(j_1,j_2,j_3)} + D^{I,R}_{(j_1,j_2,j_3)} + D^{I,T_{-N}}_{(j_1,j_2,j_3)}\right)\\ + \delta_{k<2}\sum_{\substack{I\in\mathcal{V}^k\\ j_i\in\{+,-\}}}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\leq 1}} c_{I_1,I_2}G^{I_1,I_2}_{(j_1,j_2,j_3)},\end{aligned} \tag{2.2.55}$$

with $\delta_{k<2}=1$ if $k=0,1$, 0 otherwise, and coefficients $c_{I_1,I_2}\in\{-1,0,1\}$.

In view of the lemmas to follow it is useful to remind that, after system (2.1.2), for any multi-index I vector $(\hat{u}^I_+,\hat{v}^I_+,\hat{u}^I_-,\hat{v}^I_-)$ is solution to

$$\begin{cases}(D_t-|\xi|)\hat{u}^I_+(t,\xi) = \sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1v^{I_2}_\pm)} + \sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)}\\ (D_t-\langle\xi\rangle)\hat{v}^I_+(t,\xi) = \sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1u^{I_2}_\pm)} + \sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)}\\ (D_t+|\xi|)\hat{u}^I_-(t,x) = \sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1v^{I_2}_\pm)} + \sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)}\\ (D_t+\langle\xi\rangle)\hat{v}^I_-(t,x) = \sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1u^{I_2}_\pm)} + \sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)}\end{cases} \tag{2.2.56}$$

with coefficients $c_{I_1,I_2}\in\{-1,0,1\}$ and indices I_1,I_2 in above right hand side such that $(I_1,I_2)\in\mathcal{I}(I)$. In lemmas 2.2.9 and 2.2.10 we will check that, with definitions 2.2.5, 2.2.6, the slow decaying contributions highlighted in (2.2.33) are replaced in $\partial_t\widetilde{E}^\dagger_n(t;W)$, $\partial_t\widetilde{E}^{k,\dagger}_3(t;W)$ by some new quartic terms. These latter ones are obtained from integrals (2.2.44) by replacing each factor $\hat{v}^I_{j_1},\hat{u}_{j_2},\hat{v}^I_{j_3}$ at a time with the non-linearity appearing in the equation that factor satisfies in (2.2.56). Lemma 2.2.11 (resp. lemma 2.2.12) shows that the same is for troublesome contributions (2.2.34) in $\partial_t\widetilde{E}^\dagger_n(t;W)$ (resp. for (2.2.50) in $\partial_t\widetilde{E}^{k,\dagger}_3(t;W)$). We are also going to see that, if $N\in\mathbb{N}^*$ is chosen sufficiently large (e.g. $N=18$), all these quartic terms suitably decay in time, and that modified energies $\widetilde{E}^\dagger_n(t;W),\widetilde{E}^{k,\dagger}_3(t;W)$ are equivalent, respectively, to $E_n(t;W),E^k_3(t,W)$. We point out the fact that the normal form's step performed in previous section was necessary to avoid here some problematic quartic contributions coming from quasi-linear terms in (2.2.56) and that could lead to some loss of derivatives. Before proving the mentioned lemmas, we need to introduce two preliminary results, that will be useful in the proof of lemmas 2.2.9, 2.2.11.

Lemma 2.2.7. *For any $j_i \in \{+,-\}$, $i=1,2,3$, let $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ be the multiplier defined in (2.2.42) when $k=1,2$ and in (2.2.53) when $k=3$, and ψ_1,ψ_2,ψ_3 be three smooth cut-off functions such that $\psi_1(x)$ is supported for $|x|\le c$, $\psi_2(x)$ is supported for $c'\le|x|\le C'$, $\psi_3(x)$ is supported for $|x|\ge C$, for some $0<c,c'\ll 1$, $C,C'\gg 1$, and $\psi_1+\psi_2+\psi_3\equiv 1$. Let also δ_k be equal to 1 for $k=1,2$, 0 for $k=3$.*

(i) For any $j_1,\dots,j_5\in\{+,-\}$, $i=1,2$, and any u_1,u_2,u_3,u_4 such that $u_1\in H^{4,\infty}(\mathbb{R}^2)$, $u_2,u_4\in L^2(\mathbb{R}^2)$, $u_3\in H^{11,\infty}(\mathbb{R}^2)$ and $\delta_k\mathrm{R}_k u_3\in H^{7,\infty}(\mathbb{R}^2)$,

(2.2.57)
$$\Big|\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\Big| \lesssim \|u_1\|_{H^{4,\infty}}\|u_2\|_{L^2}\left(\|u_3\|_{H^{11,\infty}}+\delta_k\|\mathrm{R}_k u_3\|_{H^{7,\infty}}\right)\|u_4\|_{L^2};$$

(ii) For any $j_1,\dots,j_5\in\{+,-\}$, and any u_1,u_2,u_3,u_4 such that $u_1\in H^{7,\infty}(\mathbb{R}^2)$, $u_2\in H^1(\mathbb{R}^2)$, $u_4\in L^2(\mathbb{R}^2)$, $u_3\in H^{4,\infty}(\mathbb{R}^2)$ and $\delta_k\mathrm{R}_k u_3\in L^\infty(\mathbb{R}^2)$,

(2.2.58)
$$\Big|\int \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\Big| \lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{H^1}\left(\|u_3\|_{H^{4,\infty}}+\delta_k\|\mathrm{R}_k u_3\|_{L^\infty}\right)\|u_4\|_{L^2}.$$

Proof. Let $k=1,2$. We are going to refer to $B^k_{(j_1,j_2,j_3)}$ (resp. η_k and R_k) simply as $B_{(j_1,j_2,j_3)}$ (resp. η and R) and rather use a superscript to define a decomposition of this multiplier (see (2.2.60))

Let us observe that, as

$$B_{(j_1,j_2,j_3)}(\xi,\eta)=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta|}\eta,$$

we can write

$$B_{(j_1,j_2,j_3)}(\xi,\eta)=B^0_{(j_1,j_2,j_3)}(\xi,\eta)\frac{\eta}{|\eta|}+B^1_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^4, \tag{2.2.59}$$

where for any smooth cut-off function ϕ, equal to 1 in a neighbourhood of the origin,

$$\begin{aligned} B^0_{(j_1,j_2,j_3)}(\xi,\eta)&:=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle}\phi(\eta),\\ B^1_{(j_1,j_2,j_3)}(\xi,\eta)&:=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta|}\eta\langle\eta\rangle^{-4}(1-\phi)(\eta).\end{aligned} \tag{2.2.60}$$

According to decomposition (2.2.59) we have that, for any $i=1,2,3$,

(2.2.61)
$$\begin{aligned}&\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\\ &=\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^0_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\mathrm{R}u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\\ &+\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^4 u_3}(\eta)\hat{u}_4(-\xi)\\ &\qquad\qquad d\xi d\eta d\zeta\\ &=: I^0_i+I^1_i.\end{aligned}$$

(i) The first thing we observe concerning integral I_i^k for $k=0,1$, $i=1,2$, is that $|\xi-\eta|, |\xi| \lesssim \langle\eta\rangle$ on the support of $\psi_i\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)$ and that $|\zeta| \le \langle\xi-\eta-\zeta\rangle\langle\eta\rangle$. Therefore, introducing the following multipliers

$$B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta) := \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big) B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1 - j_4 j_5 \frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\eta\rangle^{-7}\langle\xi-\eta-\zeta\rangle^{-4},$$

for any $j_1,\dots j_5 \in \{+,-\}$, $k=0,1$, $i=1,2$, a straight computation shows that, for any $\alpha,\beta,\gamma\in\mathbb{N}^2$,

$$\begin{aligned} &\Big|\partial_\xi^\alpha\partial_\eta^\beta B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\Big| \lesssim \langle\zeta\rangle^{-3}|g_{\alpha,\beta}(\xi,\eta)|,\\ &\Big|\partial_\xi^\alpha\partial_\eta^\beta\partial_\zeta^\gamma B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\Big| \lesssim (|\zeta|\langle\zeta\rangle^{-1})^{1-|\gamma|}\langle\zeta\rangle^{-3}|g_{\alpha,\beta}(\xi,\eta)|,\ |\gamma|\ge 1, \end{aligned} \tag{2.2.62}$$

with

$$\begin{aligned} &|g_{\alpha,0}(\xi,\eta)| \lesssim_\alpha \langle\eta\rangle^{-3}\langle\xi\rangle^{-3},\\ &|g_{\alpha,\beta}(\xi,\eta)| \lesssim_{\alpha,\beta} (|\eta|\langle\eta\rangle^{-1})^{1-|\beta|}\langle\eta\rangle^{-3}\langle\xi\rangle^{-3},\ |\beta|\ge 1. \end{aligned} \tag{2.2.63}$$

If

$$K^{i,k}_{(j_1,\dots,j_5)}(x,y,z) := \int e^{ix\cdot\xi+iy\cdot\eta+iz\cdot\zeta} B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)d\xi d\eta d\zeta,$$

by lemma A.1 (i) we first find that, for any $\alpha,\beta\in\mathbb{N}^2$,

$$\Big|\partial_\xi^\alpha\partial_\eta^\beta\int e^{iz\cdot\zeta}B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)d\zeta\Big| \lesssim |z|^{-1}\langle z\rangle^{-2}|g_{\alpha,\beta}(\xi,\eta)|$$

and successively that

$$\Big|\partial_\xi^\alpha\int e^{iy\cdot\eta+iz\cdot\zeta}B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)d\eta d\zeta\Big| \lesssim |y|^{-1}\langle y\rangle^{-2}|z|^{-1}\langle z\rangle^{-2}\langle\xi\rangle^{-3},$$

for every $\xi\in\mathbb{R}^2$, $(y,z)\in\mathbb{R}^2\times\mathbb{R}^2$. Corollary A.2 (i) hence implies that

$$|K^{i,k}_{(j_1,\dots,j_5)}(x,y,z)| \lesssim \langle x\rangle^{-3}|y|^{-1}\langle y\rangle^{-2}|z|^{-1}\langle z\rangle^{-2},\quad \forall (x,y,z)\in(\mathbb{R}^2)^3.$$

As for $i=1,2$

$$\begin{aligned} I_i^0 &= \int B^{i,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^4 u_1}(\xi-\eta-\zeta)\hat u_2(\zeta)\widehat{\langle D_x\rangle^7\mathrm{R}u_3}(\eta)\hat u_4(-\xi)\,d\xi d\eta d\zeta,\\ &= \int K^{i,0}_{(j_1,\dots,j_5)}(t-x,x-z,x-y)[\langle D_x\rangle^4 u_1](x)u_2(y)[\langle D_x\rangle^7\mathrm{R}u_3](z)u_4(t)dxdydzdt,\end{aligned}$$

$$\begin{aligned} I_i^1 &= \int B^{i,1}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^4 u_1}(\xi-\eta-\zeta)\hat u_2(\zeta)\widehat{\langle D_x\rangle^{11}u_3}(\eta)\hat u_4(-\xi)\,d\xi d\eta d\zeta\\ &= \int K^{i,1}_{(j_1,\dots,j_5)}(t-x,x-z,x-y)[\langle D_x\rangle^4 u_1](x)u_2(y)[\langle D_x\rangle^{11}u_3](z)u_4(t)dxdydzdt,\end{aligned}$$

inequality (2.2.57) follows by the fact that, for any $\widetilde{u}_1,\dots\widetilde{u}_4\in L^2\cap L^\infty$, any $f,g,h\in L^1$, integrals such as

$$\int f(t-x)g(x-z)h(x-y)|\widetilde{u}_1(x)||\widetilde{u}_2(y)||\widetilde{u}_3(z)||\widetilde{u}_4(t)|dxdydzdt \tag{2.2.64}$$

can be bounded from above by the product of the L^2 norm of any two functions $\widetilde{u}_k$ times the L^∞ norm of the remaining ones.

(ii) For a cut-off function ϕ as the one introduced at the beginning of the proof we decompose integral I_3^k, $k=0,1$, distinguishing between $|\zeta|\lesssim 1$ and $|\zeta|\gtrsim 1$. On the one hand, for any $j_1,\dots,j_5, k=0,1$, we consider

$$B^{3,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta):=\psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\phi(\zeta)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\xi-\eta-\zeta\rangle^{-3}$$

and observe that, since $|\xi|\leq\langle\xi-\eta-\zeta\rangle$ on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)\phi(\zeta)$, the above multiplier satisfies estimates (2.2.62), (2.2.63). From the same argument as before this implies that

$$\left|J_3^0:=\int B^{3,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^3u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\mathrm{R}u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \tag{2.2.65}$$
$$\lesssim \|u_1\|_{H^{3,\infty}}\|u_2\|_{L^2}\|\mathrm{R}u_3\|_{L^\infty}\|u_4\|_{L^2}$$

together with

$$\left|J_3^1:=\int B^{3,1}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^3u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^4u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \tag{2.2.66}$$
$$\lesssim \|u_1\|_{H^{3,\infty}}\|u_2\|_{L^2}\|u_3\|_{H^{4,\infty}}\|u_4\|_{L^2}.$$

On the other hand, we make a further decomposition on the integral restricted to $|\zeta|\gtrsim 1$ by means of functions $\psi_i, i=1,2,3$, distinguishing between three regions: for $|\zeta|\leq c\langle\xi-\eta\rangle$, for $c'\langle\xi-\eta\rangle\leq|\zeta|\leq C'\langle\xi-\eta\rangle$ and $|\zeta|>C\langle\xi-\eta\rangle$. For any $j_1,\dots,j_5\in\{+,-\}$, $k=0,1$, we hence introduce the following multipliers

$$\widetilde{B}^{3,i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta):=\psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)(1-\phi)(\zeta)\psi_i\Big(\frac{\zeta}{\langle\xi-\eta\rangle}\Big)$$
$$\times B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\xi-\eta-\zeta\rangle^{-7};$$

for $i=1m,3$, and

$$\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta):=\psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)(1-\phi)(\zeta)\psi_2\Big(\frac{\zeta}{\langle\xi-\eta\rangle}\Big) \tag{2.2.67}$$
$$\times B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\zeta\rangle^{-1}.$$

Since $|\xi|\sim|\xi-\eta|\sim|\xi-\eta-\zeta|$ on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)(1-\phi)(\zeta)\psi_1\big(\frac{\zeta}{\langle\xi-\eta\rangle}\big)$ (resp. $|\xi|\sim|\xi-\eta|\lesssim|\zeta|\sim|\xi-\eta-\zeta|$ on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)(1-\phi)(\zeta)\psi_3\big(\frac{\zeta}{\langle\xi-\eta\rangle}\big)$), a straight computation shows that above multipliers verify (2.2.62), (2.2.63), from which follows that

$$\left|\widetilde{J}_3^{i,0}:=\int \widetilde{B}^{3,i,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^7u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\mathrm{R}u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \tag{2.2.68}$$
$$\lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{L^2}\|\mathrm{R}u_3\|_{L^\infty}\|u_4\|_{L^2}$$

along with

$$\left|\widetilde{J}_3^{i,1}:=\int \widetilde{B}^{3,i,1}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^7u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^4u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \tag{2.2.69}$$
$$\lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{L^2}\|u_3\|_{H^{4,\infty}}\|u_4\|_{L^2},$$

for $i=1,3$. Finally, on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)(1-\phi)(\zeta)\psi_2\big(\frac{\zeta}{\langle\xi-\eta\rangle}\big)$ we have that $|\xi|\sim|\xi-\eta|\sim|\zeta|$ and $|\xi-\eta-\zeta|\lesssim|\zeta|$. Replacing ζ with $\xi-\zeta$ by a change of coordinates we find that, for any $\alpha,\beta,\gamma\in\mathbb{N}^2$,

$$\begin{aligned}\Big|\partial_\xi^\alpha\partial_\zeta^\gamma \widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\Big|&\lesssim_{\alpha,\gamma}\langle\eta\rangle^{-3}\langle\xi\rangle^{-|\alpha|},\\ \Big|\partial_\xi^\alpha\partial_\eta^\beta\partial_\zeta^\gamma \widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\Big|&\lesssim(|\eta|\langle\eta\rangle^{-1})^{1-|\beta|}\langle\eta\rangle^{-3}\langle\xi\rangle^{-|\alpha|},\ |\beta|\geq1.\end{aligned}\tag{2.2.70}$$

If we introduce a Littlewood-Paley decomposition such that

$$\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)=\sum_{l\geq1}\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\varphi(2^{-l}\xi),$$

one can check, using lemma A.1 (i) to obtain the decay in y, making a change of coordinates $\xi\mapsto 2^l\xi$, some integration by parts, and using inequalities (2.2.70), that

$$K^{k,l}_{(j_1,\dots,j_5)}(x,y,z):=\int e^{ix\cdot\xi+iy\cdot\eta+iz\cdot\zeta}\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\varphi(2^{-l}\xi)d\xi d\eta d\zeta$$

is such that

$$|K^{k,l}_{(j_1,\dots,j_5)}(x,y,z)|\lesssim 2^{2l}\langle 2^l x\rangle^{-3}|y|^{-1}\langle y\rangle^{-2}\langle z\rangle^{-3},\quad\forall(x,y,z)\in(\mathbb{R}^2)^3.\tag{2.2.71}$$

Moreover, since $|\xi|\sim|\xi-\zeta|$ on the support of $\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)$ there are two other suitably supported cut-off functions φ_1,φ_2 such that $\varphi(2^{-l}\xi)=\varphi(2^{-l}\xi)\varphi_1(2^{-l}\xi)\varphi_2(2^{-l}(\xi-\zeta))$, for any $l\geq1$. If $\Delta_j^l w:=\varphi_j(2^{-l}D_x)w$, we finally obtain that

$$\begin{aligned}\widetilde{J}_3^{2,0}&:=\int\widetilde{B}^{3,2,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\hat{u}_1(\xi-\eta-\zeta)\widehat{\langle D_x\rangle u}_2(\zeta)\widehat{\mathrm{R}u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\\&=\int\widetilde{B}^{3,2,0}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\hat{u}_1(\zeta-\eta)\widehat{\langle D_x\rangle u}_2(\xi-\zeta)\widehat{\mathrm{R}u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\\&=\sum_{l\geq1}\int K^{0,l}_{(j_1,\dots,j_5)}(t-y,x-z,y-x)u_1(x)[\Delta_1^l\langle D_x\rangle u_2](y)[\mathrm{R}u_3](z)[\Delta_2^l u_4](t)dxdydzdt,\end{aligned}$$

and by (2.2.71) together with Cauchy-Schwarz inequality we derive that

$$|\widetilde{J}_3^{2,0}|\lesssim\|u_1\|_{L^\infty}\|\mathrm{R}_1u_3\|_{L^\infty}\sum_{l\geq1}\|\Delta_1^l\langle D_x\rangle u_2\|_{L^2}\|\Delta_2^l u_4\|_{L^2}\lesssim\|u_1\|_{L^\infty}\|u_2\|_{H^1}\|\mathrm{R}_1u_3\|_{L^\infty}\|u_4\|_{L^2}.\tag{2.2.72}$$

In a similar way we obtain that

$$\widetilde{J}_3^{2,1}:=\int\widetilde{B}^{3,2,1}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\hat{u}_1(\zeta-\eta)\widehat{\langle D_x\rangle u}_2(\xi-\zeta)\widehat{\langle D_x\rangle^4 u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta$$

satisfies

$$|\widetilde{J}_3^{2,1}|\lesssim\|u_1\|_{L^\infty}\|u_2\|_{H^1}\|u_3\|_{H^{4,\infty}}\|u_4\|_{L^2}.\tag{2.2.73}$$

The result of statement (ii) follows then from inequalities (2.2.65), (2.2.66), (2.2.68), (2.2.69), (2.2.72), (2.2.73), after having recognized that

$$\begin{aligned}\int\psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\\=\sum_{k=0}^1 J_3^k+\sum_{k=0}^1\sum_{i=1}^3\widetilde{J}_3^{i,k}.\end{aligned}$$

In conclusion, the same proof of above applies to multiplier $B^3_{(j_1,j_2,j_3)}$ introduced in (2.2.53), which can be decomposed as

$$j_2 B^0_{(j_1,j_2,j_3)}(\xi,\eta) + \widetilde{B}^1_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^4$$

with the same $B^0_{(j_1,j_2,j_3)}$ as in (2.2.60) and

$$\widetilde{B}^1_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{j_1\langle\xi-\eta\rangle + j_2|\eta| - j_3\langle\xi\rangle}{2j_1\langle\xi-\eta\rangle}\langle\eta\rangle^{-4}(1-\phi)(\eta).$$

The lack of factor $\eta_1|\eta|^{-1}$ against $B^0_{(j_1,j_2,j_3)}$, in comparison to decomposition (2.2.59), is the reason why inequality (2.2.57) (resp. (2.2.58)) holds with $\|u_3\|_{H^{11,\infty}} + \|\mathrm{R}u_3\|_{H^{7,\infty}}$ (resp. $\|u_3\|_{H^{4,\infty}} + \|\mathrm{R}u_3\|_{L^\infty}$) replaced with $\|u_3\|_{H^{11,\infty}}$ (resp. with $\|u_3\|_{H^{4,\infty}}$). □

Lemma 2.2.8. *Under the same assumptions as in lemma 2.2.7 we have that:*
(i) for any $j_1,\dots,j_5 \in \{+,-\}$, $i=1,2$, and any u_1,u_2,u_3,u_4 such that $u_1 \in H^{4,\infty}(\mathbb{R}^2)$, $u_2,u_4 \in L^2(\mathbb{R}^2)$, $u_3 \in H^{11,\infty}(\mathbb{R}^2)$ and $\delta_k\mathrm{R}_k u_3 \in H^{7,\infty}(\mathbb{R}^2)$,

$$\begin{aligned}(2.2.74)\quad &\left|\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1+j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(-\xi-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(\xi-\eta)d\xi d\eta d\zeta\right| \\ &\lesssim \|u_1\|_{H^{4,\infty}}\|u_2\|_{L^2}\left(\|u_3\|_{H^{11,\infty}} + \delta_k\|\mathrm{R}_k u_3\|_{H^{7,\infty}}\right)\|u_4\|_{L^2};\end{aligned}$$

(ii) for any $j_1,\dots,j_5 \in \{+,-\}$, and any u_1,u_2,u_3,u_4 such that $u_1 \in H^{7,\infty}(\mathbb{R}^2)$, $u_2 \in L^2(\mathbb{R}^2)$, $u_4 \in H^1(\mathbb{R}^2)$, $u_3 \in H^{4,\infty}(\mathbb{R}^2)$ and $\delta_k\mathrm{R}_k u_3 \in L^\infty(\mathbb{R}^2)$,

$$\begin{aligned}(2.2.75)\quad &\left|\int \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1+j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(-\xi-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(\xi-\eta)d\xi d\eta d\zeta\right| \\ &\lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{L^2}\left(\|u_3\|_{H^{4,\infty}} + \delta_k\|\mathrm{R}_k u_3\|_{L^\infty}\right)\|u_4\|_{H^1}.\end{aligned}$$

Proof. The proof of the statement is analogous to that of lemma 2.2.7 after a change of coordinates $-\xi \mapsto \xi-\eta$. In (2.2.75) we take the H^1 norm on u_4 instead of u_2, as done in (2.2.58), by replacing multiplier $\widetilde{B}^{3,2,k}_{(j_1,j_2,j_3)}$ in (2.2.67) with

$$\psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)(1-\phi)(\zeta)\psi_2\Big(\frac{\zeta}{\langle\xi-\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\xi\rangle^{-1}.$$

□

Lemma 2.2.9 (Analysis of quartic terms. I)**.** *For any general multi-index I, any $j_k \in \{+,-\}$, $k=1,2,3$, let $C^I_{(j_1,j_2,j_3)}$, $C^{I,R}_{(j_1,j_2,j_3)}$ be the integrals defined in* (2.2.36), (2.2.38) *respectively, and $D^I_{(j_1,j_2,j_3)}$, $D^{I,R}_{(j_1,j_2,j_3)}$ introduced in* (2.2.44a), (2.2.44b)*. Then*

$$(2.2.76)\qquad \partial_t\left[D^I_{(j_1,j_2,j_3)} + D^{I,R}_{(j_1,j_2,j_3)}\right] = -C^I_{(j_1,j_2,j_3)} - C^{I,R}_{(j_1,j_2,j_3)} + \mathfrak{D}^I_{quart},$$

where $\mathfrak{D}^I_{quart}$ satisfies

$$\begin{aligned}&|\mathfrak{D}^I_{quart}(t)| \\ &\lesssim \left[\|V(t,\cdot)\|^{\frac{7}{4}}_{H^{10,\infty}}\|V(t,\cdot)\|^{\frac{1}{4}}_{H^{12}} + \|V(t,\cdot)\|_{H^{4,\infty}}\left(\|U(t,\cdot)\|_{H^{11,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{7,\infty}}\right)\right]\|W^I(t,\cdot)\|^2_{L^2} \\ &+ \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}}\|Q_0^{\mathrm{kg}}(v^{I_1}_\pm, Du^{I_2}_\pm)(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{11,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{7,\infty}}\right)\|V^I(t,\cdot)\|_{L^2}.\end{aligned} \tag{2.2.77}$$

Proof. Using definitions (2.2.36), (2.2.44a), (2.2.42) with $k = 1$, and system (2.2.56), we find that

(2.2.78)
$$
\begin{aligned}
&- 4(2\pi)^2 \left[\partial_t D^I_{(j_1,j_2,j_3)} + C^I_{(j_1,j_2,j_3)}\right] \\
&= \int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta) \left[\sum_{(I_1,I_2)\in\mathcal{J}(I)} c_{I_1,I_2} \widehat{Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2})}\right](\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta \\
&+ \int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\ \hat{v}^I_{j_1}(\xi-\eta)\widehat{Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta \\
&+ \int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta) \left[\sum_{(I_1,I_2)\in\mathcal{J}(I)} c_{I_1,I_2} \widehat{Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2})}\right](-\xi)d\xi d\eta \\
&=: S_1 + S_2 + S_3,
\end{aligned}
$$

where coefficients $c_{I_1,I_2} \in \{-1,0,1\}$ are such that $c_{I_1,I_2} = 1$ when $|I_1| + |I_2| = |I|$ (in which case $D = D_1$) and $\chi \in C_0^\infty(\mathbb{R}^2)$ is equal to 1 close to the origin and has a sufficiently small support. All integrals in the above right hand side are quartic terms for they involve the quadratic non-linearities of (2.2.56).

The fact that S_2 is a remainder $\mathfrak{D}^I_{\mathrm{quart}}$ satisfying (2.2.77) follows by inequalities (A.17), (B.1.3d) with $s = 7$, and the fact that

$$
\|\mathrm{R}_1 Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\|_{H^{7,\infty}} \lesssim \|V(t,\cdot)\|^{2-(2-\theta)\theta}_{H^{10,\infty}} \|V(t,\cdot)\|^{(2-\theta)\theta}_{H^{12}}, \tag{2.2.79}
$$

for any $\theta \in]0,1[$. The above inequality is justified by the fact that, for any function $w \in W^{1,\infty} \cap H^1$, $\rho \in \mathbb{N}$ and any $\theta \in]0,1[$, setting $p = \frac{2}{\theta} \in]2,\infty[$,

$$
\begin{aligned}
\|\langle D_x\rangle^\rho \mathrm{R}_1 w\|_{L^\infty} \lesssim \|\langle D_x\rangle^\rho \mathrm{R}_1 w\|_{W^{1,p}} \lesssim \|\langle D_x\rangle^\rho w\|_{W^{1,p}} &\lesssim \|\langle D_x\rangle^\rho w\|^{1-\theta}_{W^{1,\infty}} \|\langle D_x\rangle^\rho w\|^\theta_{H^1} \\
&\lesssim \|\langle D_x\rangle^\rho w\|^{1-\theta}_{H^{2,\infty}} \|\langle D_x\rangle^\rho w\|^\theta_{H^1},
\end{aligned} \tag{2.2.80}
$$

as a consequence of Morrey's inequality, continuity of $\mathrm{R}_1 : L^p \to L^p$ for $p < +\infty$, interpolation inequality, and the injection of $W^{1,\infty}$ into $H^{2,\infty}$. This implies that

$$
\|\mathrm{R}_1 Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\|_{H^{\rho,\infty}} \lesssim \|Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\|^{1-\theta}_{H^{\rho+2,\infty}} \|Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\|^\theta_{H^{\rho+1}}, \tag{2.2.81}
$$

for any $\rho \in \mathbb{N}$, and gives (2.2.79) when $\rho = 7$ after inequalities (B.1.3c) with $s = 8$, (B.1.3d) with $s = 9$. Therefore, for any $\theta \in]0,1[$,

$$
|S_2| \lesssim \left(\|V(t,\cdot)\|^{2-\theta}_{H^{8,\infty}} \|V(t,\cdot)\|^\theta_{H^{10}} + \|V(t,\cdot)\|^{2-(2-\theta)\theta}_{H^{10,\infty}} \|V(t,\cdot)\|^{(2-\theta)\theta}_{H^{12}}\right) \|V^I(t,\cdot)\|^2_{L^2},
$$

so choosing $\theta \ll 1$ small (e.g. $\theta \leq 1/8$) and keeping in mind estimates (1.1.11b), (1.1.11c) we deduce that S_2 is controlled by the first term in the right hand side of (2.2.77).

Inequality (A.17) allows also to estimate all integrals in summations S_1, S_3 corresponding to indices $(I_1, I_2) \in \mathcal{J}(I)$ with $|I_2| < |I|$, and to bound them with the latter term in the right hand side of (2.2.77). This is not the case for integrals with $I_2 = I$ involving quasi-linear term $Q_0^{\mathrm{kg}}(v_\pm, D_1 u^I_\pm)$, because a straight application of that inequality would give a bound at the wrong energy level $n+1$, as $\|Q_0^{\mathrm{kg}}(v_\pm, D_1 u^I_\pm)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}} \|D_1 U^I(t,\cdot)\|_{L^2}$. Instead, since

$$
\widehat{Q_0^{\mathrm{kg}}(v_\pm, D_1 u^I_\pm)}(\xi) = \frac{i}{4} \sum_{j_4,j_5\in\{+,-\}} \int \Big(1 - j_4 j_5 \frac{\xi-\zeta}{\langle\xi-\zeta\rangle} \cdot \frac{\zeta}{|\zeta|}\Big) \zeta_1 \hat{v}_{j_4}(\xi-\zeta)\hat{u}^I_{j_5}(\zeta)d\zeta, \tag{2.2.82}
$$

we can rather write those integrals as the sum over $j_k \in \{+,-\}, k = 1, \dots 4$, of the following:

(2.2.83a)
$$\int \chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1 - j_4 j_5 \frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \hat{v}_{j_4}(\xi-\eta-\zeta)\hat{u}^I_{j_5}(\zeta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\, d\xi d\eta d\zeta,$$

(2.2.83b)
$$\int \chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1 + j_4 j_5 \frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}_{j_4}(-\xi-\zeta)\hat{u}^I_{j_5}(\zeta)\, d\xi d\eta d\zeta,$$

and estimate them by using inequalities (2.2.57) and (2.2.74) respectively. We hence obtain that

$$\begin{aligned}|S_1| + |S_3| \lesssim{}& \|V(t,\cdot)\|_{H^{4,\infty}} \left(\|U(t,\cdot)\|_{H^{11,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{7,\infty}}\right) \|W^I(t,\cdot)\|^2_{L^2} \\ &+ \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_2|<|I|}} \|Q_0^{\mathrm{kg}}(v^{I_1}_\pm, Du^{I_2}_\pm)(t,\cdot)\|_{L^2} \left(\|U(t,\cdot)\|_{H^{11,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{7,\infty}}\right) \|V^I(t,\cdot)\|_{L^2},\end{aligned}$$

and, since the same argument applies to $\partial_t D^{I,R}_{(j_1,j_2,j_3)}$, this also concludes the proof of the statement. □

Lemma 2.2.10 (Analysis of quartic terms. II)**.** *For any general multi-index I, any $j_k \in \{+,-\}$, $k = 1,2,3$, let $D^{I,T_{-N}}_{(j_1,j_2,j_3)}$ be defined as in* (2.2.44c)*. Then*

$$\partial_t D^{I,T_{-N}}_{(j_1,j_2,j_3)} = \Im\left[\langle T_{-N}(U)W^I, W^I\right] + \mathfrak{D}^{I,N}_{quart} \tag{2.2.84}$$

and if $N \geq 18$ $\mathfrak{D}^{I,N}_{quart}$satisfies

$$\begin{aligned}\left|\mathfrak{D}^{I,N}_{quart}\right| \lesssim{}& \|V(t,\cdot)\|^{\frac{7}{4}}_{H^{N+4,\infty}} \|V(t,\cdot)\|^{\frac{1}{4}}_{H^{N+6}} \|W^I(t,\cdot)\|^2_{L^2} \\ &+ \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_2|<|I|}} \left\|Q_0^{\mathrm{kg}}(v^{I_1}_\pm, Du^{I_2}_\pm)\right\|_{L^2} \|U(t,\cdot)\|_{H^{N+3,\infty}} \|V^I(t,\cdot)\|_{L^2}.\end{aligned} \tag{2.2.85}$$

Proof. For any triplet (j_1,j_2,j_3), we compute the time derivative of $D^{I,T_{-N}}$ by making use of system (2.2.56). Recalling (2.2.39) and (2.2.43), we find that

(2.2.86)
$$\begin{aligned}&\partial_t\left[\sum_{j_k\in\{+,-\}} D^{I,T_{-N}}_{(j_1,j_2,j_3)}\right] - \Im[\langle T_{-N}(U)W^I, W^I\rangle] = \\ &= \mathrm{Re}\left[\frac{1}{(2\pi)^2}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\left[\sum_{(I_1,I_2)\in\mathfrak{I}(I)} c_{I_1,I_2}\widehat{Q_0^{\mathrm{kg}}(v^{I_1}_\pm, Du^{I_2}_\pm)}(\xi-\eta)\right]\hat{u}_{j_2}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta\right. \\ &\quad + \frac{1}{(2\pi)^2}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\widehat{Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta \\ &\quad \left. + \frac{1}{(2\pi)^2}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\left[\sum_{(I_1,I_2)\in\mathfrak{I}(I)} c_{I_1,I_2}\widehat{Q_0^{\mathrm{kg}}(v^{I_1}_\pm, Du^{I_2}_\pm)}(-\xi)\right]d\xi d\eta\right] \\ &=: S_1^{T_{-N}} + S_2^{T_{-N}} + S_3^{T_{-N}},\end{aligned}$$

with coefficients $c_{I_1,I_2} \in \{-1,0,1\}$ such that $c_{I_1,I_2} = 1$ whenever $|I_1| + |I_2| = |I|$ (in which case $D = D_1$). After lemma A.6 and inequality (B.1.3d) with $s = N+3$ we deduce that, if $N \geq 15$, for any $\theta \in]0,1[$

$$|S_2^{T_{-N}}| \lesssim \|V(t,\cdot)\|^{2-\theta}_{H^{N+4,\infty}} \|V(t,\cdot)\|^{\theta}_{H^{N+6}} \|V^I(t,\cdot)\|^2_{L^2}.$$

Choosing $\theta \ll 1$ small (e.g. $\theta \leq 1/8$) we then obtain that $S_2^{T_{-N}}$ is a remainder $\mathfrak{D}_{\text{quart}}^{I,N}$ satisfying (2.2.85). Also, the same lemma implies that each contribution in $S_1^{T_{-N}}, S_3^{T_{-N}}$ corresponding to $(I_1, I_2) \in \mathfrak{I}(I)$ with $|I_2| < |I|$ is bounded by

$$\Big\| Q_0^{\text{kg}}(v_\pm^{I_1}, Du_\pm^{I_2}) \Big\|_{L^2} \|U(t,\cdot)\|_{H^{N+3,\infty}} \|V^I(t,\cdot)\|_{L^2}.$$

Reminding instead (2.2.82), we find that the remaining contribution to $S_1^{T_{-N}}$, corresponding to $I_2 = I$, is equal to the sum over $j_1, \dots, j_5 \in \{+,-\}$ of the (imaginary part) of the following integrals:

$$\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1 - j_4 j_5 \frac{\xi-\eta-\zeta}{\langle \xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \hat{v}_{j_4}(\xi-\eta-\zeta)\hat{u}^I_{j_5}(\zeta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\, d\xi d\eta d\zeta. \tag{2.2.87}$$

Analogously, the contribution corresponding to $I_2 = I$ in $S_3^{T_{-N}}$ is the sum over $j_k \in \{+,-\}, k = 1,\dots,5$ of

$$\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1 + j_4 j_5 \frac{\xi+\zeta}{\langle \xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}_{j_4}(-\xi-\zeta)\hat{u}^I_{j_5}(\zeta)\, d\xi d\eta d\zeta. \tag{2.2.88}$$

Since $\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)$ satisfies (A.23) and is supported for $|\eta| \leq \varepsilon|\xi-\eta|$, for a small $0 < \varepsilon \ll 1$, we rewrite above integrals, respectively, as

$$\begin{aligned}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^{-N-3}&\Big(1 - j_4 j_5 \frac{\xi-\eta-\zeta}{\langle \xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1\langle\xi-\eta-\zeta\rangle^{-4}\\ &\times \widehat{\langle D_x\rangle^4 v}_{j_4}(\xi-\eta-\zeta)\hat{u}^I_{j_5}(\zeta)\widehat{\langle D_x\rangle^{N+3} u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\, d\xi d\eta d\zeta,\end{aligned} \tag{2.2.89}$$

and

$$\begin{aligned}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^{-N-7}&\Big(1 + j_4 j_5 \frac{\xi+\zeta}{\langle \xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1\langle\xi+\zeta\rangle^{-4}\\ &\times \hat{v}^I_{j_1}(\xi-\eta)\widehat{\langle D_x\rangle^{N+7} u}_{j_2}(\eta)\widehat{\langle D_x\rangle^4 v}_{j_4}(-\xi-\zeta)\hat{u}^I_{j_5}(\zeta)\, d\xi d\eta d\zeta.\end{aligned} \tag{2.2.90}$$

With such a choice, the new multipliers, that we denote concisely by $\widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)$, $k = 0, 1$, verify, for any $\alpha, \beta, \gamma \in \mathbb{N}^2$,

$$\begin{aligned}\Big|\partial_\xi^\alpha \partial_\eta^\beta \widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\Big| &\lesssim \langle\zeta\rangle^{-3}|g^N_{\alpha,\beta}(\xi)|,\\ \Big|\partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma \widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\Big| &\lesssim (|\zeta|\langle\zeta\rangle^{-1})^{1-|\gamma|}\langle\zeta\rangle^{-3}|g^N_{\alpha,\beta}(\xi)|, \quad |\gamma| \geq 1,\end{aligned}$$

with $g^N_{\alpha,\beta}(\xi,\eta)$ supported for $|\eta| \leq \varepsilon|\xi-\eta|$ and such that

$$|g^N_{\alpha,\beta}(\xi,\eta)| \lesssim \langle\xi-\eta\rangle^{6-N+|\alpha|+2|\beta|}|\eta|^{N-|\beta|}\langle\eta\rangle^{-N-3}, \quad \forall (\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

If $N \in \mathbb{N}^*$ is sufficiently large (e.g. $N \geq 18$), the above estimate implies that, for any $\alpha, \beta \in \mathbb{N}^2$ of length less or equal than 3,

$$|g^N_{\alpha,\beta}(\xi,\eta)| \lesssim \langle\eta\rangle^{-3}\langle\xi\rangle^{-3},$$

so by lemma A.1 (i) together with corollary A.2 (i) we obtain that, for any $k = 0, 1$,

$$K^{N,k}_{(j_1,\dots,j_5)}(x,y,z) := \int e^{ix\cdot\xi + iy\cdot\eta + iz\cdot\zeta}\widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)d\xi d\eta d\zeta$$

is such that

$$|K^{N,k}_{(j_1,\dots,j_5)}(x,y,z)| \lesssim \langle x\rangle^{-3}|y|^{-1}\langle y\rangle^{-2}|z|^{-1}\langle z\rangle^{-2}, \quad \forall (x,y,z)\in(\mathbb{R}^2)^3. \tag{2.2.91}$$

By (2.2.89), (2.2.90), integrals (2.2.87), (2.2.88) are respectively equal to

$$\int K^{N,0}_{(j_1,\dots,j_5)}(t-x,x-z,x-y)[\langle D_x\rangle^4 v_{j_4}](x)u^I_{j_5}(y)[\langle D_x\rangle^{N+3}u_{j_2}](z)v^I_{j_3}(t)dxdydzdt$$

and

$$\int K^{N,1}_{(j_1,\dots,j_5)}(z-x,x-y,z-t)v^I_{j_1}(x)[\langle D_x\rangle^{N+7}u_{j_2}](y)[\langle D_x\rangle^4 v_{j_4}](z)u^I_{j_5}(t)dxdydzdt.$$

Using (2.2.91) and the fact that integrals such as (2.2.64) can be bounded from above by the product of the L^2 norm of any two functions $\widetilde{u}_k$ times the L^∞ norm of the remaining ones, they are estimated by

$$\|V(t,\cdot)\|_{H^{4,\infty}}\|U(t,\cdot)\|_{H^{N+7,\infty}}\|W^I(t,\cdot)\|^2_{L^2},$$

which concludes the proof of the statement. □

Lemma 2.2.11 (Analysis of quartic terms. III)**.** *Let $n\in\mathbb{N}, n\geq 3$, $I\in\mathcal{I}_n$ and $(I_1,I_2)\in\mathcal{I}(I)$ be such that $[\frac{|I|}{2}]<|I_1|<|I|$. Let also $C^{I_1,I_2}_{(j_1,j_2,j_3)}$, $D^{I_1,I_2}_{(j_1,j_2,j_3)}$ be the integrals defined, respectively, in (2.2.41), (2.2.45), for any $j_k\in\{+,-\}, k=1,2,3$. Then*

$$\partial_t D^{I_1,I_2}_{(j_1,j_2,j_3)} = -C^{I_1,I_2}_{(j_1,j_2,j_3)} + \mathfrak{D}^{I_1,I_2}_{quart}, \tag{2.2.92}$$

where $\mathfrak{D}^{I_1,I_2}_{quart}$ satisfies

$$\begin{aligned}\left|\mathfrak{D}^{I_1,I_2}_{quart}(t)\right| \lesssim \Bigg[&\Big(\|W(t,\cdot)\|_{H^{[\frac{n}{2}]+12,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}\Big)^2\\ &+\|V(t,\cdot)\|^{\frac{7}{4}}_{H^{[\frac{n}{2}]+11,\infty}}\|V(t,\cdot)\|^{\frac{1}{4}}_{H^{[\frac{n}{2}]+12}}\Bigg]E_n(t;W).\end{aligned} \tag{2.2.93}$$

Proof. We compute the time derivative of $D^{I_1,I_2}_{(j_1,j_2,j_3)}$ by making use of system (2.2.56). We remind that, after remark 1.1.5 and definition (1.1.18), if Γ^I is a product of spatial derivatives then all couples of indices (I_1,I_2) belonging to $\mathcal{I}(I)$ are such that $|I_1|+|I_2|=|I|$ and $\Gamma^{I_1},\Gamma^{I_2}$ are also products of spatial derivatives. Therefore, all coefficients c_{I_1,I_2} appearing in the right hand side of (2.2.56) are equal to 0. By definitions (2.2.42) with $k=1$, (2.2.41), (2.2.45), we find that

$$\begin{aligned}-4(2\pi)^2&\Big[\partial_t D^{I_1,I_2}_{(j_1,j_2,j_3)} + C^{I_1,I_2}_{(j_1,j_2,j_3)}\Big] =\\ &\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left[\sum_{(J_1,J_2)\in\mathcal{I}(I_1)}\widehat{Q^{\mathrm{kg}}_0(v^{J_1}_\pm, D_1u^{J_2}_\pm)}(\xi-\eta)\right]\hat{u}^{I_2}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta\\ &+\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\left[\sum_{(J_1,J_2)\in\mathcal{I}(I_2)}\widehat{Q^{\mathrm{w}}_0(v^{J_1}_\pm, D_1v^{J_2}_\pm)}(\eta)\right]\hat{v}^I_{j_3}(-\xi)d\xi d\eta\\ &+\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)\left[\sum_{(J_1,J_2)\in\mathcal{I}(I)}\widehat{Q^{\mathrm{kg}}_0(v^{J_1}_\pm, D_1u^{J_2}_\pm)}\right](-\xi)d\xi d\eta\\ &=: S^{I_1,I_2}_1 + S^{I_1,I_2}_2 + S^{I_1,I_2}_3.\end{aligned} \tag{2.2.94}$$

Since $|J_1|+|J_2|=|I_1|<|I|\leq n$ in $S_1^{I_1,I_2}$, we can estimate all its contributions using inequality (A.17). Using lemma 2.1.2 (i), the fact that $|I_2|\leq[\frac{n}{2}]$ by the hypothesis and, hence, that

$$\|u_\pm^{I_2}(t,\cdot)\|_{H^{7,\infty}}+\|\mathrm{R}_1 u_\pm^{I_2}(t,\cdot)\|_{H^{7,\infty}}\lesssim\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}},$$

we deduce that

$$\left|S_1^{I_1,I_2}\right|\lesssim\left(\|W(t,\cdot)\|_{H^{[\frac{n}{2}]+2}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}}\right)\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}E_n(t;W),$$

and above estimate holds also for all integrals in $S_3^{I_1,I_2}$ corresponding to $|J_2|<|I|$. The same inequality (A.17), combined with (2.2.81) applied to $Q_0^{\mathrm{w}}(v_\pm^{J_1},D_1v_\pm^{J_2})$ and with corollary A.4 in appendix A, gives that, for any $\theta\in]0,1[$,

$$\begin{aligned}&|S_2^{I_1,I_2}|\\&\lesssim\sum_{|J_1|+|J_2|=|I_2|}\left[\left\|Q_0^{\mathrm{w}}(v_\pm^{J_1},D_1v_\pm^{J_2})\right\|_{H^{7,\infty}}+\left\|Q_0^{\mathrm{w}}(v_\pm^{J_1},D_1v_\pm^{J_2})\right\|_{H^{9,\infty}}^{1-\theta}\left\|Q_0^{\mathrm{w}}(v_\pm^{J_1},D_1v_\pm^{J_2})\right\|_{H^8}^{\theta}\right]E_n(t;W)\\&\lesssim\|V(t,\cdot)\|_{H^{[\frac{n}{2}]+11,\infty}}^{2-(2-\theta)\theta}\|V(t,\cdot)\|_{H^{[\frac{n}{2}]+12}}^{(2-\theta)\theta}E_n(t;W).\end{aligned}$$

Finally, the last remaining integral in $S_3^{I_1,I_2}$, corresponding to indices $J_1=0,J_2=I$, can be written using (2.2.82) as

$$\sum_{j_4,j_4\in\{+,-\}}\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left(1+j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{v}_{j_1}^{I_1}(\xi-\eta)\hat{u}_{j_2}^{I_2}(\eta)\hat{v}_{j_4}(-\xi-\zeta)\hat{u}_{j_5}^{I}(\zeta)d\xi d\eta d\zeta,$$

and is estimated, after lemma 2.2.8 and the fact that $|I_1|<|I|$, by

$$\|V(t,\cdot)\|_{H^{7,\infty}}\left(\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+12,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}\right)E_n(t;W).$$

This gives that

$$\left|S_3^{I_1,I_2}\right|\lesssim\left(\|W(t,\cdot)\|_{H^{[\frac{n}{2}]+12,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}\right)^2E_n(t;W)$$

and concludes the proof of the statement. □

Lemma 2.2.12 (Analysis of quartic terms. IV)**.** *Let $k=0,1$, $\mathcal{K},\mathcal{V}_k$ be the sets introduced in* (2.1.25), (2.1.26) *respectively, $I\in\mathcal{V}^k$ and $(I_1,I_2)\in\mathcal{I}(I)$ be such that $I_1\in\mathcal{K}$, $|I_2|\leq 1$. Let also $F^{I_1,I_2,l}_{(j_1,j_2,j_3)}$, $G^{I_1,I_2,l}_{(j_1,j_2,j_3)}$ be the integrals defined in* (2.2.52), (2.2.54)*, for any $l=1,2,3$, $j_i\in\{+,-\}, i=1,2,3$. For any $l=1,2,3$, any triplet (j_1,j_2,j_3), we have that*

$$\partial_tG^{I_1,I_2,l}_{(j_1,j_2,j_3)}=-F^{I_1,I_2,l}_{(j_1,j_2,j_3)}+\mathfrak{G}^{I_1,I_2}_{quart},\tag{2.2.95}$$

and there is a constant $C>0$ such that, if a-priori estimates (1.1.11) *are satisfied in interval $[1,T]$ for a fixed $T>1$, with $\varepsilon_0<(2A+B)^{-1}$ small,*

$$|\mathfrak{G}^{I_1,I_2}_{quart}(t)|\leq C(A+B)^2\varepsilon^2t^{-1+\frac{\delta_k}{2}}\left[E_3^k(t;W)^{\frac{1}{2}}+\delta_{\mathcal{V}^0}t^{\beta+\frac{\delta_1}{2}}E_3^1(t;W)^{\frac{1}{2}}+t^{-\frac{1}{4}-\frac{\delta_k}{2}}\right],\tag{2.2.96}$$

for every $t\in[1,T]$, with $\delta_{\mathcal{V}^0}=1$ if $I\in\mathcal{V}^0$, 0 otherwise, and $\beta>0$ as small as we want.

Proof. First of all, it is useful to remind that from (2.1.42), (2.1.43) and a-priori estimate (1.1.11d), for any $k = 0, 1$, $I \in \mathcal{I}_3^k$, $(I_1, I_2) \in \mathcal{I}(I)$ such that $I_1 \in \mathcal{K}$, $|I_2| \leq 1$, and $\sigma > 0$ sufficiently small

(2.2.97)
$$\|V^{I_1}(t,\cdot)\|_{L^2}\left(\|\chi(t^{-\sigma}D_x)U^{I_2}(t,\cdot)\|_{H^{\rho,\infty}} + \|\chi(t^{-\sigma}D_x)\mathrm{R}U^{I_2}(t,\cdot)\|_{H^{\rho,\infty}}\right) \leq C(A+B)B\varepsilon^2 t^{-\frac{1}{2}+\frac{\delta_k}{2}},$$

for every $t \in [1, T]$.

For any fixed (j_1, j_2, j_3), any $l = 1, 2, 3$, we compute $\partial_t G^{I_1,I_2,l}_{(j_1,j_2,j_3)}$ recurring to system (2.2.56) along with its compact form

$$\begin{cases} (D_t \mp \langle D_x\rangle)v_\pm^I = \Gamma^I Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), \\ (D_t \mp |D_x|)u_\pm^I = \Gamma^I Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm), \end{cases}$$

and using that $[D_t, \chi(t^{-\sigma}D_x)] = t^{-1}\chi_1(t^{-\sigma}D_x)$ with $\chi_1(\xi) := i\sigma(\partial\chi)(\xi)\cdot\xi$. We find that

$$\begin{aligned}
&-4(2\pi)^2\left[\partial_t G^{I_1,I_2,l}_{(j_1,j_2,j_3)} + F^{I_1,I_2,l}_{(j_1,j_2,j_3)}\right] \\
&= \int B^l_{(j_1,j_2,j_3)}(\xi,\eta)\Big[\widehat{\Gamma^{I_1}Q_0^{\mathrm{kg}}(v_\pm, D_1u_\pm)}(\xi-\eta)\Big]\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta \\
&+ \int B^l_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\Big[\widehat{\chi(t^{-\sigma}D_x)\Gamma^{I_2}Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)}(\eta) + t^{-1}\widehat{\chi_1(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\Big]\hat{v}^I_{j_3}(-\xi)d\xi d\eta \\
&+ \int B^l_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\Big[\sum_{(J_1,J_2)\in\mathcal{I}(I)} c_{J_1,J_2}\widehat{Q_0^{\mathrm{kg}}(v^{J_1}_\pm, Du^{J_2}_\pm)}(-\xi)\Big]d\xi d\eta \\
&=: S_1^{I_1,I_2,l} + S_2^{I_1,I_2,l} + S_3^{I_1,I_2,l},
\end{aligned}$$

with $B^l_{(j_1,j_2,j_3)}$ given by (2.2.42) when $l = 1, 2$ or (2.2.53) when $l = 3$.

Applying (A.17) to $S_2^{I_1,I_2,l}$, using (2.2.80) with $w = \Gamma^{I_2}Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)$ and $\rho = 7$, together with the fact that operators $\chi(t^{-\sigma}D_x)$, $\chi_1(t^{-\sigma}D_x)$ are bounded from L^∞ to $H^{\rho,\infty}$ for any $\rho \geq 0$ with norm $O(t^{\sigma\rho})$, and from L^2 to H^s for any $s \geq 0$ with norm $O(t^{\sigma s})$, we deduce that, for any $\theta \in]0, 1[$,

$$\begin{aligned}
(2.2.98)\quad |S_2^{I_1,I_2,l}| &\lesssim t^\beta \|V^{I_1}(t,\cdot)\|_{L^2}\|V^I(t,\cdot)\|_{L^2} \\
&\times \Big[\|\Gamma^{I_2}Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)\|_{L^\infty} + \delta_l\|\Gamma^{I_2}Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)\|_{L^\infty}^{1-\theta}\|\Gamma^{I_2}Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)\|_{L^2}^{\theta} \\
&\qquad + t^{-1}\left(\|\chi_1(t^{-\sigma}D_x)u^{I_2}_\pm(t,\cdot)\|_{L^\infty} + \|\chi_1(t^{-\sigma}D_x)\mathrm{R}u^{I_2}_\pm(t,\cdot)\|_{L^\infty}\right)\Big],
\end{aligned}$$

for some $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$, and with $\delta_l = 1$ if $l = 1, 2$, 0 otherwise. When $|I_2| = 0$ the above right hand side can be estimated using (B.1.3a), (B.1.3b) and a-priori estimates (1.1.11). When $|I_2| = 1$ we derive from (1.1.15) that

$$\Gamma^{I_2}Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm) = Q_0^{\mathrm{w}}(v^{I_2}_\pm, D_1v_\pm) + Q_0^{\mathrm{w}}(v_\pm, D_1v^{I_2}_\pm) + G_1^w(v_\pm, Dv_\pm)$$

with $G_1^w(v_\pm, Dv_\pm) = G_1(v, \partial v)$ given by (1.1.16). Using lemma B.2.4 in appendix B with $L = L^\infty$ to estimate the L^∞ norm of the first two quadratic terms in the above right hand side, we find that, for some new $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\sigma > 0$ small, there is a constant $C > 0$ such that

$$\begin{aligned}
\|\Gamma^{I_2}Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)\|_{L^\infty} &\lesssim \Big\|\chi(t^{-\sigma}D_x)v^{I_2}_\pm(t,\cdot)\Big\|_{H^{2,\infty}}\|v_\pm(t,\cdot)\|_{H^{2,\infty}} \\
&+ t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_tv_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{|\mu|=0}^{1}\|x^\mu v_\pm(t,\cdot)\|_{H^1} + t\|v_\pm(t,\cdot)\|_{H^1}\Big) \\
&+ \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|v_\pm(t,\cdot)\|_{H^{2,\infty}} + \|D_tv_\pm(t,\cdot)\|_{H^{1,\infty}}\right) \\
&\leq CAB\varepsilon^2 t^{-2},
\end{aligned}$$

where last inequality is obtained by picking $s > 0$ sufficiently large so that $N(s) \geq 4$ and using (B.1.6a), (B.1.6b), (B.1.10a), lemma B.4.14, together with a-priori estimates. Also, by (B.1.6a) with $s = 0$ and a-priori estimates

$$\|\Gamma^{I_2} Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{2,\infty}} \left(\|V^{I_2}(t,\cdot)\|_{H^1} + \|D_t V(t,\cdot)\|_{L^2}\right) \leq CAB\varepsilon^2 t^{-1+\frac{\delta_2}{2}}.$$

Therefore, using lemma B.2.10 and taking $\theta, \sigma > 0$ sufficiently small we deduce from (2.2.98) and the above estimates that, for any $l = 1, 2, 3$ and a new $C > 0$,

$$|S_2^{I_1,I_2,l}| \leq CAB\varepsilon^2 t^{-\frac{5}{4}} E_3^k(t;W)^{\frac{1}{2}}. \tag{2.2.99}$$

We make use of inequality (A.17) to estimate $S_1^{I_1,I_2,l}$, too. From (1.1.17) we have that

$$\Gamma^{I_1} Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm) = Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm) + \sum_{\substack{(J_1,J_2)\in \mathcal{J}(I_1)\\ |J_1|<|I_1|}} c_{J_1,J_2} Q_0^{\mathrm{kg}}(v_\pm^{J_1}, D u_\pm^{J_2})$$

with $c_{J_1,J_2} \in \{-1, 0, 1\}$, and then from (2.1.29b), (2.1.34b) and the fact that $I_1 \in \mathcal{K}$,

$$\Gamma^{I_1} Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm) = Q_0^{\mathrm{kg}}(v_\pm^{I_1}, \chi(t^{-\sigma} D_x) D_1 u_\pm) + \mathfrak{R}_3^k(t,x),$$

with $\mathfrak{R}_3^k$ satisfying (2.1.30) and

$$\|Q_0^{\mathrm{kg}}(v_\pm^{I_1}, \chi(t^{-\sigma} D_x) D_1 u_\pm)\|_{L^2} \leq \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}U(t,\cdot)\|_{H^{2,\infty}}\right) \|V^{I_1}(t,\cdot)\|_{L^2}.$$

So from (2.1.30), (2.2.97), lemma B.2.10 and priori estimates (1.1.11)

$$\begin{aligned} |S_1^{I_1,I_2,l}| &\lesssim \Big[\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}U(t,\cdot)\|_{H^{2,\infty}}\right) \|V^{I_1}(t,\cdot)\|_{L^2} + \|\mathfrak{R}_3^k(t,\cdot)\|_{L^2}\Big] \\ &\quad \times \left(\|\chi(t^{-\sigma} D_x) U^{I_2}(t,\cdot)\|_{H^{7,\infty}} + \|\chi(t^{-\sigma} D_x)\mathrm{R}U^{I_2}(t,\cdot)\|_{H^{7,\infty}}\right) \|V^I(t,\cdot)\|_{L^2} \\ &\leq CAB\varepsilon^2 t^{-1+\frac{\delta_k}{2}} E_3^k(t;W)^{\frac{1}{2}}. \end{aligned} \tag{2.2.100}$$

Let us now consider all the addends in $S_3^{I_1,I_2,l}$ with $|J_2| < |I|$, which by inequality (A.17) are bounded by

$$\|V^{I_1}(t,\cdot)\|_{L^2} \Big(\sum_{|\mu|=0}^{1} \|\chi(t^{-\sigma} D_x)\mathrm{R}^\mu U^{I_2}(t,\cdot)\|_{H^{7,\infty}} \Big) \sum_{\substack{(J_1,J_2)\in \mathcal{J}(I)\\ |J_2|<|I|}} \left\| c_{J_1,J_2} Q_0^{\mathrm{kg}}(v_\pm^{J_1}, D u_\pm^{J_2}) \right\|_{L^2}.$$

As the latter above factor is bounded by the L^2 norm of $Q_0^I(V,W)$ (see definition (2.1.12)), inequalities (2.1.44) and (2.2.97) imply that those integrals are remainders $\mathfrak{G}_{\mathrm{quart}}^{I_1,I_2}$ satisfying (2.2.96). Finally, the last contribution to $S_3^{I_1,I_2,l}$, corresponding to $|J_1| = 0, J_2 = I$, for which $D = D_1$, can be rewritten using (2.2.82) as the sum over $j_4, j_5 \in \{+,-\}$ of

$$\int B^1_{(j_1,j_2,j_3)}(\xi,\eta) \left(1 + j_4 j_5 \frac{\xi+\zeta}{\langle \xi+\zeta\rangle} \cdot \frac{\zeta}{|\zeta|}\right) \zeta_1 \hat{v}_4(-\xi-\zeta) \hat{u}^I_{j_5}(\zeta) \widehat{\chi(t^{-\sigma} D_x) u^{I_2}_{j_2}}(\eta) \hat{v}^{I_1}_{j_1}(\xi-\eta) d\xi d\eta.$$

By means of lemma 2.2.8 it is bounded by

$$\|V(t,\cdot)\|_{H^{7,\infty}} \Big(\sum_{|\mu|=0}^{1} \|\chi(t^{-\sigma} D_x) D_1 \mathrm{R}^\mu U^{J_2}(t,\cdot)\|_{H^{11,\infty}} \Big) \|V^{I_1}(t,\cdot)\|_{H^1} \|U^I(t,\cdot)\|_{L^2}$$

for every $t \in [1, T]$, and hence by $CA(A+B)\varepsilon^2 t^{-\frac{3}{2}+\beta'} E_3^k(t;W)$, with $\beta' > 0$ small as long as σ, δ_0 are small, as follows by a-priori estimate (1.1.11b) and lemma B.2.10. □

2.2.3 Propagation of the energy estimate

Proposition 2.2.13 (Propagation of the energy estimate)**.** *Let us fix $K_2 > 0$. There exist two integers $n \gg \rho \gg 1$ sufficiently large, two constants $A, B > 1$ sufficiently large, $\varepsilon_0 \in]0, (2A+B)^{-1}[$ sufficiently small, and some $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1$ small such that, for any $0 < \varepsilon < \varepsilon_0$, if (u, v) is solution to* (1.1.1)-(1.1.2) *in some interval $[1, T]$ for a fixed $T > 1$, and $u_\pm, v_\pm$ defined in* (1.1.5) *satisfy a-priori estimates* (1.1.11) *for every $t \in [1, T]$, then they also verify* (1.1.12c), (1.1.12d) *on the same interval $[1, T]$.*

Proof. For any integer $k, n \in \mathbb{N}$, with $n \geq 3$ and $0 \leq k \leq 2$, let $\widetilde{E}_n(t; W)$, $\widetilde{E}_3^k(t; W)$ be the first modified energies introduced in (2.2.11) and $\widetilde{E}_n^\dagger(t; W)$, $\widetilde{E}_3^{k,\dagger}(t; W)$ be the second modified energies, introduced in (2.2.46) and (2.2.55) respectively. Let also $D^I_{(j_1,j_2,j_3)}, D^{I,R}_{(j_1,j_2,j_3)}, D^{I,T_{-N}}_{(j_1,j_2,j_3)}$ be the integrals defined in (2.2.44), $D^{I_1,I_2}_{(j_1,j_2,j_3)}$ in (2.2.45), and $G^{I_1,I_2,l}_{(j_1,j_2,j_3)}$ in (2.2.54). Fix $N = 18$. The first thing we observe is that, as long as estimates (1.1.11a), (1.1.11b) are satisfied and $\rho \in \mathbb{N}$ is sufficiently large (e.g. $\rho \geq \max\{[\frac{n}{2}] + 8, 21\}$), there is a constant $C > 0$ such that for every $t \in [1, T]$

$$C^{-1} E_n(t; W) \leq \widetilde{E}_n^\dagger(t; W) \leq C E_n(t; W), \tag{2.2.101a}$$

$$C^{-1} E_3^k(t; W) \leq \widetilde{E}_3^{k,\dagger}(t; W) \leq C E_3^k(t; W). \tag{2.2.101b}$$

Above equivalences follow from (2.2.12), a-priori estimates (1.1.11a), (1.1.11b), the fact that for a general multi-index I ($I \in \mathcal{I}_n$ or $I \in \mathcal{I}_3^k$ for $0 \leq k \leq 2$)

$$\sum_{j_i \in \{+,-\}} \left| D^I_{(j_1,j_2,j_3)} \right| + \left| D^{I,R}_{(j_1,j_2,j_3)} \right| \lesssim \left(\|U(t,\cdot)\|_{H^{7,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{7,\infty}} \right) \|V^I(t,\cdot)\|_{L^2}^2$$

by inequality (A.17),

$$\sum_{j_k \in \{+,-\}} \left| D^{I,T_{-18}}_{(j_1,j_2,j_3)} \right| \lesssim \|U(t,\cdot)\|_{H^{21,\infty}} \|W^I(t,\cdot)\|_{L^2}^2$$

by inequality (A.24), and:

• as concerns especially (2.2.101a), from the fact that for any $I \in \mathcal{I}_n$, any $(I_1, I_2) \in \mathcal{I}(I)$ with $[\frac{|I|}{2}] < |I_1| < |I|$, by (A.17)

$$\begin{aligned} \sum_{j_i \in \{+,-\}} \left| D^{I_1,I_2}_{(j_1,j_2,j_3)} \right| &\lesssim \left(\|U^{I_2}(t,\cdot)\|_{H^{7,\infty}} + \|\mathrm{R}_1 U^{I_2}(t,\cdot)\|_{H^{7,\infty}} \right) \|V^{I_1}(t,\cdot)\|_{L^2} \|V^I(t,\cdot)\|_{L^2} \\ &\lesssim \left(\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}} \right) E_n(t; W); \end{aligned}$$

• as concerns especially (2.2.101b), the fact that for any $I \in \mathcal{V}^k$ (see definition (2.1.26)), any $(I_1, I_2) \in \mathcal{I}(I)$ with $I_1 \in \mathcal{K}$ (see (2.1.25)) and $|I_2| \leq 1$, and any $l = 1, 2, 3$, by (A.17) and (2.2.97)

$$\begin{aligned} \sum_{j_i \in \{+,-\}} \left| G^{I_1,I_2,l}_{(j_1,j_2,j_3)} \right| &\lesssim \sum_{|\mu|=0}^{1} \|\chi(t^{-\sigma} D_x) \mathrm{R}^\mu U^{I_2}(t,\cdot)\|_{H^{7,\infty}} \|V^{I_1}(t,\cdot)\|_{L^2} \|V^I(t,\cdot)\|_{L^2} \\ &\leq C(A+B) B \varepsilon^2 t^{-\frac{1}{2}+\frac{\delta_k}{2}} E_3^k(t; W)^{\frac{1}{2}}. \end{aligned} \tag{2.2.102}$$

Let us now consider a general multi-index I. From equation (2.2.3) we deduce the following equality:

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\widetilde{W}_s^I(t,\cdot)\|_{L^2}^2 = -\Im\Big[\langle D_t\widetilde{W}_s^I,\widetilde{W}_s^I\rangle\Big]\\
&= -\Im\Big[\langle A(D)\widetilde{W}_s^I,\widetilde{W}_s^I\rangle + \Big\langle Op^B\Big((I_4+E_d^0(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F_d^0(U;\eta))\Big)\widetilde{W}_s^I,\widetilde{W}_s^I\Big\rangle\\
&\quad +\langle Op^B(A''(V^I;\eta))U + Op_R^B(A''(V^I;\eta))U,\widetilde{W}_s^I\rangle + \langle Q_0^I(V,W),\widetilde{W}_s^I\rangle\\
&\quad +\langle T_{-18}(U)W_s^I,\widetilde{W}_s^I\rangle + \langle\mathfrak{R}'(U,V),\widetilde{W}_s^I\rangle\Big]
\end{aligned}
\tag{2.2.103}
$$

and immediately notice that $\Im[\langle A(D)\widetilde{W}_s^I,\widetilde{W}_s^I\rangle]=0$ because of the fact that $A(\eta)$, introduced in (2.1.5), is real diagonal matrix and its quantization is a self-adjoint operator.

Matrix $(I_4+E_d^0(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F_d^0(U;\eta))$ is real, symmetric, of order 1, with semi-norm

$$
M_1^1\Big(\big(I_4+E_d^0(U;\eta)\big)\widetilde{A}_1(V;\eta)\big(I_4+F_d^0(U;\eta)\big),3\Big)\lesssim(1+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}})^2\|V(t,\cdot)\|_{H^{2,\infty}}
$$

as follows by estimate (2.2.7a) on E_d^0, (2.2.8) of F_d^0, and (2.1.49) on $\widetilde{A}_1(V;\eta)$. Corollary 1.2.13 and a-priori estimates (1.1.11a), (1.1.11b) imply then that the second term in the right hand side of (2.2.103) reduces to $\langle T_0(U,V)\widetilde{W}_s^I,\widetilde{W}_s^I\rangle$, with $T_0(U,V)$ operator of order less or equal than 0 such that

$$
\|T_0(U,V)\|_{\mathcal{L}(L^2)}\lesssim M_1^1\Big(\big(I_4+E_d^0(U;\eta)\big)\widetilde{A}_1(V;\eta)\big(I_4+F_d^0(U;\eta)\big),3\Big)\leq CA\varepsilon t^{-1},
$$

so after Cauchy-Schwarz inequality and equivalence (2.2.10) it is a remainder $R(t)$ satisfying, for every $t\in[1,T]$

$$
|R(t)|\leq CA\varepsilon t^{-1}\|W^I(t,\cdot)\|_{L^2}^2. \tag{2.2.104}
$$

Observe that, by the definition of $\widetilde{W}_s^I$ in (2.2.2) and of W_s^I in (2.1.45), we have that

$$
\begin{aligned}
\Big\|(\widetilde{W}_s^I-W^I)(t,\cdot)\Big\|_{L^2} &\leq \|Op^B(P(V;\eta)-I_4)W^I\|_{L^2}+\|Op^B(E(U;\eta))W_s^I\|_{L^2}\\
&\lesssim\big(\|V(t,\cdot)\|_{H^{1,\infty}}+\|U(t,\cdot)\|_{H^{5,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\big)\|W^I(t,\cdot)\|_{L^2},
\end{aligned}
\tag{2.2.105}
$$

the latter inequality following from proposition 1.2.7, estimate (2.1.48), the fact that $E(U;\eta)$ verifies, after (2.2.7) and for any admissible cut-off function χ,

$$
M_0^0\left(E\Big(\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)U;\eta\Big);n\right)\lesssim\|U(t,\cdot)\|_{H^{5,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}},
$$

and equivalence (2.1.50). Therefore, third and fifth brackets in the right hand side of (2.2.103) can be replaced with

$$
\langle Op^B(A''(V^I;\eta))U+Op_R^B(A''(V^I;\eta))U,W^I\rangle+\langle T_{-18}(U)W^I,W^I\rangle
$$

up to some new remainders $R(t)$, satisfying (2.2.104) after Cauchy-Schwarz inequality, estimates (2.1.21d), (2.2.4), (2.2.105) and (1.1.11a), (1.1.11b).

Summing up, equality (2.2.103) reduces to:

$$
\begin{aligned}
\frac{1}{2}\partial_t\|\widetilde{W}_s^I(t,\cdot)\|_{L^2} = -\Im\Big[&\langle Op^B(A''(V^I;\eta))U+Op_R^B(A''(V^I;\eta))U,W^I\rangle\\
&+\langle Q_0^I(V,W),\widetilde{W}_s^I\rangle+\langle T_{-18}(U)W_s^I,\widetilde{W}_s^I\rangle+\langle\mathfrak{R}'(U,V),\widetilde{W}_s^I\rangle\Big]+R(t).
\end{aligned}
\tag{2.2.106}
$$

In order to analyse the behaviour of the second and fourth brakets in above right hand side we need, at this point, to distinguishing between indices $I \in \mathfrak{I}_n$ and $I \in \mathfrak{I}_3^k$.

Propagation of a-priori estimate (1.1.11c): Let us suppose that $I \in \mathfrak{I}_n$. Using (2.2.105) and estimate (2.1.38) we find that

$$\langle Q_0^I(V,W), \widetilde{W}_s^I\rangle = \langle Q_0^I(V,W), W^I\rangle + R_n(t) \tag{2.2.107}$$

where, for a new constant $C>0$ and every $t\in[1,T]$,

$$|R_n(t)| \leq CA\varepsilon t^{-1+\frac{\delta}{2}} E_n(t;W)^{\frac{1}{2}}. \tag{2.2.108}$$

Reminding definition (2.1.12) of $Q_0^I(V,W)$ and the fact that coefficients c_{I_1,I_2} are all equal to 0 when $I\in\mathfrak{I}_n$, we notice that some of the contributions to the scalar product in the right hand side of (2.2.107) are also remainders $R_n(t)$. These are precisely the following ones:

$$\sum_{(I_1,I_2)\in\mathfrak{I}(I)} \langle Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_1 v_\pm^{I_2}), u_+^I + u_-^I\rangle + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|\leq[\frac{|I|}{2}]}} \langle Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2}), v_+^I + v_-^I\rangle$$

in consequence of Cauchy-Schwarz inequality and estimates (2.1.27), (1.1.11b), (1.1.11c). Moreover, $\langle \mathfrak{R}'(U,V), \widetilde{W}^I\rangle$ in the right hand side of (2.2.106) is also a remainder $R_n(t)$ because of Cauchy-Schwarz, (2.2.105), a-priori estimates (1.1.11a), (1.1.11b), and the fact that

$$\|\mathfrak{R}'(U,V)\|_{L^2} \leq CA\varepsilon t^{-1+\frac{\delta}{2}},$$

which follows choosing $\theta \ll 1$ in (2.2.6), using (2.1.38) and (1.1.11a)-(1.1.11c).

Since remainder $R(t)$ in (2.2.106) (verifying (2.2.104)) can be enclosed in $R_n(t)$ after (1.1.11c), we obtain that equality (2.2.106) can be further reduced to

$$\begin{aligned}\frac{1}{2}\partial_t\|\widetilde{W}_s^I(t,\cdot)\|_{L^2}^2 = -\Im\Big[&\langle Op^B(A''(V^I;\eta))U + Op_R^B(A''(V^I;\eta))U, W^I\rangle\\ &+ \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}} \langle Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2}), v_+^I + v_-^I\rangle + \langle T_{-18}(U)W^I, W^I\rangle\Big] + R_n(t).\end{aligned}$$

From definition (2.2.46), equalities (2.2.35), (2.2.37), (2.2.39) with $N=18$, (2.2.40), together with (2.2.76), (2.2.84) with $N=18$, (2.2.92), we deduce that

$$\frac{1}{2}\left|\partial_t \widetilde{E}_n^\dagger(t;W)\right| \lesssim |R_n(t)| + \sum_{I\in\mathfrak{I}_n}\left(\left|\mathfrak{D}_{\mathrm{quart}}^I(t)\right| + \left|\mathfrak{D}_{\mathrm{quart}}^{I,18}(t)\right|\right) + \sum_{I\in\mathfrak{I}_n}\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}}\left|\mathfrak{D}_{\mathrm{quart}}^{I_1,I_2}(t)\right|,$$

where quartic terms $\mathfrak{D}_{\mathrm{quart}}^I, \mathfrak{D}_{\mathrm{quart}}^{I,18}, \mathfrak{D}_{\mathrm{quart}}^{I_1,I_2}$ satisfy, respectively, (2.2.77), (2.2.85) with $N=18$, (2.2.93). These latter ones can also be considered as remainders $R_n(t)$ thanks to lemma 2.1.2 (i) and a-priori estimates (1.1.11), which implies that, for some new $C>0$ and every $t\in[1,T]$,

$$\left|\partial_t\widetilde{E}_n^\dagger(t;W)\right| \leq CA\varepsilon t^{-1+\frac{\delta}{2}} E_n(t;W)^{\frac{1}{2}}.$$

Then

$$\widetilde{E}_n^\dagger(t;W)^{\frac{1}{2}} \leq \widetilde{E}_n^\dagger(1;W)^{\frac{1}{2}} + \int_1^t CA\varepsilon\tau^{-1+\frac{\delta}{2}}d\tau,$$

so after equivalence (2.2.101a) and a-priori estimate (1.1.11c)

$$\begin{aligned} E_n(t;W)^{\frac{1}{2}} &\leq CE_n(1;W)^{\frac{1}{2}} + \int_1^t CA\varepsilon\tau^{-1+\frac{\delta}{2}}d\tau \\ &\leq CE_n(1;W)^{\frac{1}{2}} + \frac{2CA\varepsilon}{\delta}t^{\frac{\delta}{2}}, \end{aligned}$$

again for a new $C > 0$. Taking $B > 1$ sufficiently large so that $E_n(1;W)^{\frac{1}{2}} \leq \frac{B\varepsilon}{2CK_2}$ and $\frac{2CA}{\delta} < \frac{B}{2K_2}$ we finally obtain (1.1.12c).

Propagation of a-priori estimate (1.1.11d): Let us now suppose that $I \in \mathcal{I}_3^k$ for $0 \leq k \leq 2$. After (2.1.39) and (2.2.105) we have that

$$\langle Q_0^I(V,W), \widetilde{W}_s^I\rangle = \langle Q_0^I(V,W), W^I\rangle + R_3^k(t)$$

with

$$|R_3^k(t)| \leq CA(A+B)\varepsilon^2 t^{-1+\frac{\delta_k}{2}} E_3^k(t;W)^{\frac{1}{2}}, \tag{2.2.109}$$

and moreover

(2.2.110)

$$\begin{aligned} -\Im\left[\langle Q_0^I(V,W), W^I\rangle\right] =& -\delta_{\mathcal{V}^k} \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\leq 1}} c_{I_1,I_2}\Im\left[\left\langle Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, \chi(t^{-\sigma}D_x)D_x u_\pm^{I_2}\right), v_+^I + v_-^I\right\rangle\right] \\ &- \delta_{\mathcal{V}^k} \sum_{\substack{(J,0)\in\mathcal{I}(I)\\ J\in\mathcal{K}}} c_{J,0}\Im\left[\left\langle Q_0^{\mathrm{kg}}\left(v_\pm^{J}, \chi(t^{-\sigma}D_x)|D_x| u_\pm\right), v_+^I + v_-^I\right\rangle\right] + R_3^k(t), \end{aligned}$$

with $\delta_{\mathcal{V}^k} = 1$ if $I \in \mathcal{V}^k$, 0 otherwise, as already seen in (2.2.50). Also, $\langle \mathfrak{R}'(U,V), \widetilde{W}_s^I\rangle$ in the right hand side of (2.2.106) and $R(t)$ are remainders $R_3^k(t)$ in consequence of the same argument used in the previous case, but with estimate (2.1.38) replaced with (2.1.39). Therefore, we can further reduce (2.2.106) to the following equality:

$$\begin{aligned} \frac{1}{2}\partial_t\|\widetilde{W}_s^I(t,\cdot)\|_{L^2}^2 =& -\Im\left[\langle Op^B(A''(V^I;\eta))U + Op_R^B(A''(V^I;\eta))U, \widetilde{W}^I\rangle + \langle T_{-18}(U)W^I, W^I\rangle\right] \\ &- \delta_{\mathcal{V}^k} \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\leq 1}} c_{I_1,I_2}\Im\left[\left\langle Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, \chi(t^{-\sigma}D_x)Du_\pm^{I_2}\right), v_+^I + v_-^I\right\rangle\right] \\ &- \delta_{\mathcal{V}^k} \sum_{\substack{(J,0)\in\mathcal{I}(I)\\ J\in\mathcal{K}}} c_{J,0}\Im\left[\left\langle Q_0^{\mathrm{kg}}\left(v_\pm^{J}, \chi(t^{-\sigma}D_x)|D_x| u_\pm\right), v_+^I + v_-^I\right\rangle\right] + R_3^k(t), \end{aligned}$$

and deduce from definition (2.2.55), equalities (2.2.35), (2.2.37), (2.2.39) with $N = 18$, (2.2.51), together with (2.2.76), (2.2.84) with $N = 18$, and (2.2.95), that

$$\left|\partial_t \widetilde{E}_3^{k,\dagger}(t;W)\right| \lesssim |R_3^k(t)| + \sum_{I\in\mathcal{I}_3^k}\left(\left|\mathfrak{D}_{\mathrm{quart}}^I(t)\right| + \left|\mathfrak{D}_{\mathrm{quart}}^{I,18}(t)\right|\right) + \delta_{k<2}\sum_{\substack{I\in\mathcal{V}^k\\ j_i\in\{+,-\}}}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\leq 1}}\left|\mathfrak{G}_{(j_1,j_2,j_3)}^{I_1,I_2}\right|$$

with $\delta_{k<2} = 1$ for $k < 2$, 0 otherwise. On the one hand, quartic terms $\mathfrak{D}_{\mathrm{quart}}^I, \mathfrak{D}_{\mathrm{quart}}^{I,18}$ satisfy, respectively, (2.2.77) and (2.2.85) with $N = 18$, and are remainders $R_3^k(t)$ after (2.1.39) and

a-priori estimates. On the other hand, $\mathfrak{G}^{I_1,I_2}_{(j_1,j_2,j_3)}$ verifies estimate (2.2.96). Consequently, there is a constant $C>0$ such that

$$\widetilde{E}_3^{k,\dagger}(t;W) \leq \widetilde{E}_3^{k,\dagger}(1;W) + C(A+B)^2\varepsilon^2 \int_1^t \tau^{-1+\frac{\delta_k}{2}} E_3^k(t\tau;W)^{\frac{1}{2}} d\tau \\ + \delta_{k<2}\, C(A+B)^2\varepsilon^2 \left[\delta_{k=0}\int_1^t \tau^{-1+\frac{\delta_0}{2}+\beta+\frac{\delta_1}{2}} E_3^1(\tau;W)^{\frac{1}{2}} d\tau + \int_1^t \tau^{-\frac{5}{4}} d\tau\right]$$

with $\delta_{k=0}=1$ if $k=0$, 0 otherwise, $\beta>0$ as small as we want, and after equivalence (2.2.101b)

$$E_3^k(t;W) \leq CE_3^k(1;W) + C(A+B)^2\varepsilon^2 \int_1^t \tau^{-1+\frac{\delta_k}{2}} E_3^k(\tau;W)^{\frac{1}{2}} d\tau \\ + \delta_{k<2}\, C(A+B)^2\varepsilon^2 \left[\delta_{k=0}\int_1^t \tau^{-1+\frac{\delta_0}{2}+\beta+\frac{\delta_1}{2}} E_3^1(\tau;W)^{\frac{1}{2}} d\tau + \int_1^t \tau^{-\frac{5}{4}} d\tau\right],$$

for a new $C>0$. Injecting (1.1.11d) in the above inequality and integrating in $d\tau$, we obtain that

$$E_3^k(t;W) \leq CE_3^k(1;W) + C(A+B)^2B\varepsilon^3 \left[\frac{1}{\delta_k}t^{\delta_k} + \delta_{k=0}\ \frac{1}{\frac{\delta_0}{2}+\beta+\delta_1} t^{\frac{\delta_0}{2}+\beta+\delta_1}\right],$$

and taking β sufficiently small so that $\beta+\delta_1 \leq \delta_0/2$, $B>1$ sufficiently large so that $E_3^k(1;W) \leq \frac{B^2\varepsilon^2}{2CK_2^2}$ and $B \geq A$, and $\varepsilon_0>0$ sufficiently small so that

$$\varepsilon_0 \leq \frac{1}{8BCK_2^2}\Big[\frac{1}{\delta_k} + \delta_{k=0}\frac{1}{\frac{\delta_0}{2}+\beta+\delta_1}\Big]^{-1},$$

we finally derive enhanced estimate (1.1.12d) and the conclusion of the proof. □

Chapter 3

Uniform Estimates

3.1 Semilinear normal forms

In proposition 2.2.13 of the previous chapter we proved the propagation of the a-priori the energy estimates, i.e. that there exist some constants $A, B > 1$ large and $\varepsilon_0 > 0$ small, such that (1.1.11) implies (1.1.12c), (1.1.12d). To conclude the proof of theorem 1.1.2 it only remains to show that (1.1.11) also implies (1.1.12a), (1.1.12b). In particular, as $u_+ = -\overline{u_-}$ and $v_+ = -\overline{u_-}$, it will be enough to prove this result for (u_-, v_-), which is solution to

$$\begin{cases} (D_t + |D_x|)\, u_- = Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), \\ (D_t + \langle D_x \rangle)\, v_- = Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm), \end{cases} \tag{3.1.1}$$

with $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ given by (2.1.1).

As for the simpler case of the one-dimensional Klein-Gordon equation (see [29]), the main idea is to reformulate system (3.1.1) in terms of two new functions $\widetilde{u}, \widetilde{v}$, defined from u_-, v_- and living in a new framework (the *semi-classical framework*), and to deduce a new simpler system, made of a transport equation and an ODE. Through this new system we will be able to recover the required enhanced estimates (1.1.12a), (1.1.12b).

Before introducing the semi-classical framework in which we will work for the rest of the paper, we need to replace almost all quadratic non-linearities in (3.1.1) with cubic ones by a normal forms. This is the object of the following two subsections. We highlight the fact that we do not make use directly of the normal forms obtained in the proof of the energy inequality, because our goals and constraints are henceforth different. In fact, we want to obtain a L^∞ estimate for essentially ρ derivatives of our solution, having a control on its H^s norm for $s \gg \rho$. Therefore, we are allowed to lose some derivatives in the normal form reduction, which means that we do not care any more about the quasi-linear nature of our problem.

We warn the reader that, for seek of compactness, we will often use the notation NL_w (resp. NL_{kg}) when referring to $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$ (resp. to $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$).

3.1.1 Normal forms for the Klein-Gordon equation

The aim of this subsection is to introduce a new unknown v^{NF}, defined in terms of v_-, in such a way it is solution to a cubic half Klein-Gordon equation instead of the quadratic one satisfied by v_- in (3.1.1). This normal form is motivated by the fact that the L^2 norm of $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ decays too slowly in time (only $t^{-1+\delta/2}$), as follows from (B.1.4a) and a-priori estimates (1.1.11b),

(1.1.11c), and this decay is not enough in view of proposition 3.2.7 (the required one being strictly faster than $t^{-3/2}$).

It is fundamental to observe that, after (1.1.11) and inequality (3.1.7b) below with $\theta \ll 1$ small enough (e.g. $\theta < (2+\delta)^{-1}$), v^{NF} and v_- are comparable, in the sense that there is a positive constant C such that

$$\big| \|v_-(t,\cdot)\|_{H^{\rho,\infty}} - \|v^{NF}(t,\cdot)\|_{H^{\rho,\infty}} \big| \leq C\varepsilon^2 t^{-1}. \tag{3.1.2}$$

Then bootstrap assumption (1.1.11b) implies that the new unknown v^{NF} disperses in time at the same rate t^{-1} as v_-, and the propagation of a suitable estimate of the $H^{\rho,\infty}$ norm of v^{NF} will provide us with enhanced (1.1.12b).

Proposition 3.1.1. *Assume that (u,v) is solution to (1.1.1) in $[1,T]$ for a fixed $T>1$, consider (u_+,v_+,u_-,v_-) defined in (1.1.5) and solution to (2.1.2) with $|I|=0$, and remind definition (2.1.11) of vectors U,V. Let*

$$v^{NF} := v_- - \frac{i}{4(2\pi)^2} \sum_{j_1,j_2\in\{+,-\}} \int e^{ix\cdot\xi} B^1_{(j_1,j_2,+)}(\xi,\eta)\hat{v}_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)d\xi d\eta, \tag{3.1.3}$$

with $B^1_{(j_1,j_2,+)}(\xi,\eta)$ given by (2.2.42) with $k=1$ and $j_3=+$. Then for every $t\in[1,T]$ v^{NF} is solution to

$$(D_t + \langle D_x\rangle)\, v^{NF}(t,x) = r^{NF}_{kg}(t,x), \tag{3.1.4}$$

where

$$\begin{aligned} r^{NF}_{kg}(t,x) = -\frac{i}{4(2\pi)^2} \sum_{j_1,j_2\in\{+,-\}} \int e^{ix\cdot\xi} B^1_{(j_1,j_2,+)}(\xi,\eta) \\ \times \Big[\widehat{NL_{kg}}(\xi-\eta)\hat{u}_{j_2}(\eta) + \hat{v}_{j_1}(\xi-\eta)\widehat{NL_w}(\eta)\Big]\, d\xi d\eta \end{aligned} \tag{3.1.5}$$

satisfies

$$\|r^{NF}_{kg}(t,\cdot)\|_{L^2} \lesssim \sum_{\mu=0}^{1} \|V(t,\cdot)\|_{H^{1,\infty}} \|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty} \|U(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|^2_{H^{2,\infty}} \|V(t,\cdot)\|_{H^2}, \tag{3.1.6a}$$

$$\|\chi(t^{-\sigma}D_x) r^{NF}_{kg}(t,\cdot)\|_{L^\infty} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}} \Big(\sum_{\mu=0}^{1} \|\mathrm{R}_1^\mu U(t,\cdot)\|_{H^{2,\infty}} \Big)^2 + t^\sigma \|V(t,\cdot)\|^3_{H^{2,\infty}}, \tag{3.1.6b}$$

for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$. Moreover, for every $s,\rho\geq 0$, any $\theta\in]0,1[$,

$$\big\|(v^{NF}-v_-)(t,\cdot)\big\|_{H^s} \lesssim \sum_{\mu=0}^{1} \|V(t,\cdot)\|_{H^s} \|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty} + \|V(t,\cdot)\|_{L^\infty} \|U(t,\cdot)\|_{H^{s+1}}, \tag{3.1.7a}$$

$$\begin{aligned} \big\|(v^{NF}-v_-)(t,\cdot)\big\|_{H^{s,\infty}} \lesssim \sum_{\mu=0}^{1} \|V(t,\cdot)\|^{1-\theta}_{H^{s,\infty}} \|V(t,\cdot)\|^{\theta}_{H^{s+2}} \|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty} \\ + \sum_{\mu=0}^{1} \|V(t,\cdot)\|_{L^\infty} \|\mathrm{R}_1^\mu U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}} \|U(t,\cdot)\|^{\theta}_{H^{s+3}}, \end{aligned} \tag{3.1.7b}$$

(3.1.7c)

$$\big\|\Omega(v^{NF}-v_-)(t,\cdot)\big\|_{L^2}\lesssim\sum_{\mu,\nu=0}^{1}[\|\Omega^\mu V(t,\cdot)\|_{L^2}\|\mathrm{R}_1^\nu U(t,\cdot)\|_{L^\infty}+\|V(t,\cdot)\|_{L^\infty}\|\Omega^\mu U(t,\cdot)\|_{H^1}]$$
$$+\|\Omega V(t,\cdot)\|_{H^2}\|U(t,\cdot)\|_{H^1}+\|V(t,\cdot)\|_{L^2}\|\Omega U(t,\cdot)\|_{H^2},$$

and

$$\big\|\chi(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\big\|_{L^2}\lesssim t^\sigma\|V(t,\cdot)\|_{H^{1,\infty}}\|U(t,\cdot)\|_{L^2},\tag{3.1.8a}$$

(3.1.8b)
$$\big\|\chi(t^{-\sigma}D_x)\Omega(v^{NF}-v_-)(t,\cdot)\big\|_{L^2}$$
$$\lesssim t^\sigma\left[\sum_{\mu=0}^{1}\|\Omega V(t,\cdot)\|_{L^2}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}+\|V(t,\cdot)\|_{H^{1,\infty}}\|\Omega^\mu U(t,\cdot)\|_{L^2}\right].$$

Proof. From definition (3.1.3) of v^{NF}, system (2.1.2) with $|I|=0$, and the fact that

$$Q_0^{\mathrm{kg}}(v_\pm,D_1u_\pm)=\frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}\Big(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\Big)\eta_1\hat v_{j_1}(\xi-\eta)\hat u_{j_2}(\eta)d\xi d\eta,\tag{3.1.9}$$

it immediately follows that v^{NF} is solution to (3.1.4) with r_{kg}^{NF} given by (3.1.5). We observe that, after formula (A.15), we have the following explicit expressions:

(3.1.10)
$$v^{NF}-v_-=-\frac{i}{8}\Big[(v_++v_-)\mathrm{R}_1(u_+-u_-)-\frac{D_1}{\langle D_x\rangle}(v_+-v_-)(u_++u_-)$$
$$+D_1\big[[\langle D_x\rangle^{-1}(v_+-v_-)](u_++u_-)\big]-\langle D_x\rangle\big[[\langle D_x\rangle^{-1}(v_+-v_-)]\mathrm{R}_1(u_+-u_-)\big]\Big]$$

and

$$r_{kg}^{NF}=-\frac{i}{4}\left[NL_{kg}\,\mathrm{R}_1(u_+-u_-)-\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\,NL_w+D_1\big[\langle D_x\rangle^{-1}(v_+-v_-)\,NL_w\big]\right].\tag{3.1.11}$$

Inequalities (3.1.7a), (3.1.7b) are straightforward from (3.1.10) and corollary A.4 in appendix A. Inequality (3.1.7c) is also obtained from corollary A.4, after having applied Ω to (3.1.10) and used the Leibniz rule, and from bounding the L^∞ norm of $\Omega u_\pm,\Omega v_\pm$ with their H^2 norm by means of the classical Sobolev injection. Inequalities (3.1.8a), (3.1.8b) are also straightforward if one observes that operator $\chi(t^{-\sigma}D_x)$, with $\chi\in C_0^\infty(\mathbb{R}^2)$ and $\sigma>0$, is L^2-H^1 continuous with norm $O(t^\sigma)$.

As concerns r_{kg}^{NF}, from (3.1.11) and corollary A.4 we find that

$$\|r_{kg}^{NF}(t,\cdot)\|_{L^2}\lesssim\sum_{\mu=0}^{1}\|NL_{kg}(t,\cdot)\|_{L^2}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}+\|V(t,\cdot)\|_{L^2}\|NL_w(t,\cdot)\|_{L^\infty}$$
$$+\|V(t,\cdot)\|_{L^\infty}\|NL_w(t,\cdot)\|_{H^1}$$

and

$$\|\chi(t^{-\sigma}D_x)r_{kg}^{NF}(t,\cdot)\|_{L^\infty}\lesssim\sum_{\mu=0}^{1}\|NL_{kg}(t,\cdot)\|_{L^\infty}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}+t^\sigma\|V(t,\cdot)\|_{H^{1,\infty}}\|NL_w(t,\cdot)\|_{L^\infty}.$$

Inequalities (3.1.6a) and (3.1.6b) follow then by (B.1.3c) with $s=1$, (B.1.3b), (B.1.4a) and (B.1.4b). □

3.1.2 Normal forms for the wave equation

We now focus on the wave equation satisfied by u_-:

$$(D_t + |D_x|)u_-(t,x) = Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm),$$

and perform a normal form argument in order to replace (a part of) the quadratic non-linearity in the above right hand side with a cubic non-local one. The fundamental reason for that is to be found in lemma 3.2.14, where we have to prove that the L^2 norm of some operator, acting on the non-linearity of the equation satisfied by u_-, decays like $t^{-1/2+\beta}$, for a small $\beta > 0$. That decay is obtained if the L^2 norm of the mentioned non-linearity is a $O(t^{-3/2+\beta'})$, for some new small $\beta' > 0$, which is not the case for $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$, as follows from (B.1.3a), (1.1.11b), (1.1.11c). This normal form can be actually performed only on contributions depending on (v_+, v_+) and (v_-, v_-) but not on (v_+, v_-), which are resonant. Nevertheless, the structure of these latter contributions allows us to recover the right mentioned time decay for their L^2 norm (see lemmas 3.2.15 and 3.2.16).

Thanks to inequalities (3.1.20b), (3.1.20c) and a-priori estimates (1.1.11), the new unknown u^{NF} we define in (3.1.15) below is equivalent to the former u_-, meaning that there exists a positive constant C such that

$$\sum_{\kappa=0}^{1} \left| \|\mathrm{R}_1^\kappa u_-(t,\cdot)\|_{H^{\rho+1,\infty}} - \|\mathrm{R}_1^\kappa u^{NF}(t,\cdot)\|_{H^{\rho,\infty}} \right| \leq C\varepsilon^2 t^{-1+\frac{\delta}{2}}. \tag{3.1.12}$$

After (1.1.11a) this means that u^{NF} and $\mathrm{R}_1 u^{NF}$ decay in the $H^{\rho+1,\infty}$ norm at the same rate $t^{-1/2}$ as $u_-, \mathrm{R}_1 u_-$, and the propagation of a suitable estimate of this norm will provide us with enhanced (1.1.12a).

Let us rewrite $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$ as follows

$$\begin{aligned} Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm) = &-\frac{1}{2}\Im\left[v_+\, D_1 v_- + \frac{D_x}{\langle D_x\rangle} v_+ \cdot \frac{D_x D_1}{\langle D_x\rangle} v_-\right] \\ &+ \frac{i}{4(2\pi)^2} \sum_{j\in\{+,-\}} \int e^{ix\cdot\xi} \left(1 - \frac{\xi-\eta}{\langle \xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\right) \eta_1 \hat{v}_j(\xi-\eta)\hat{v}_j(\eta) d\xi d\eta, \end{aligned} \tag{3.1.13}$$

and introduce, for any $j \in \{+,-\}$,

$$D_j(\xi,\eta) := \frac{\left(1 - \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\right)\eta_1}{j\langle\xi-\eta\rangle + j\langle\eta\rangle + |\xi|}. \tag{3.1.14}$$

Proposition 3.1.2. *Assume that (u,v) is solution to (1.1.1) in $[1,T]$ for a fixed $T > 1$, consider (u_+, v_+, u_-, v_-) defined in (1.1.5) and solution to (2.1.2) with $|I| = 0$, remind definition (2.1.11) of vectors U, V and (3.1.3) of v^{NF}. Let*

$$u^{NF} := u_- - \frac{i}{4(2\pi)^2} \sum_{j\in\{+,-\}} \int e^{ix\cdot\xi} D_j(\xi,\eta)\hat{v}_j(\xi-\eta)\hat{v}_j(\eta) d\xi d\eta, \tag{3.1.15}$$

with multiplier D_j defined in (3.1.14). For every $t \in [1,T]$ u^{NF} is solution to

$$(D_t + |D_x|)u^{NF}(t,x) = q_w(t,x) + c_w(t,x) + r_w^{NF}(t,x), \tag{3.1.16}$$

where quadratic term q_w is given by

$$q_w(t,x) = \frac{1}{2}\Im\left[\overline{v^{NF}}\, D_1 v^{NF} - \overline{\frac{D_x}{\langle D_x\rangle} v^{NF}} \cdot \frac{D_x D_1}{\langle D_x\rangle} v^{NF}\right], \tag{3.1.17}$$

while cubic terms c_w, r_w^{NF} *are equal, respectively, to*

$$
\begin{aligned}
c_w(t,x) = \frac{1}{2}\Im\Big[&\overline{(v_- - v^{NF})}\, D_1 v_- + \overline{v^{NF}}\, D_1(v_- - v^{NF}) \\
&- \overline{\frac{D_x}{\langle D_x\rangle}(v_- - v^{NF})}\cdot \frac{D_x D_1}{\langle D_x\rangle} v_- - \overline{\frac{D_x}{\langle D_x\rangle} v^{NF}}\cdot \frac{D_x D_1}{\langle D_x\rangle}(v_- - v^{NF})\Big],
\end{aligned}
\tag{3.1.18}
$$

and

$$
r_w^{NF}(t,x) = -\frac{i}{4(2\pi)^2}\sum_{j\in\{+,-\}}\int e^{ix\cdot\xi} D_j(\xi,\eta)\Big[\widehat{NL_{kg}}(\xi-\eta)\hat{v}_j(\eta) + \hat{v}_j(\xi-\eta)\widehat{NL_{kg}}(\eta)\Big]\, d\xi d\eta.
\tag{3.1.19}
$$

For any $s, \rho \geq 0$, *any* $t \in [1,T]$,

$$
\|u^{NF}(t,\cdot) - u_-(t,\cdot)\|_{H^s} \lesssim \|V(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{s+15}},
\tag{3.1.20a}
$$

$$
\|u^{NF}(t,\cdot) - u_-(t,\cdot)\|_{H^{\rho+1,\infty}} \lesssim \|V(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{\rho+18}},
\tag{3.1.20b}
$$

$$
\|\mathrm{R}_j u^{NF}(t,\cdot) - \mathrm{R}_j u_-(t,\cdot)\|_{H^{\rho+1,\infty}} \lesssim \|V(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{\rho+8}}, \quad j = 1,2.
\tag{3.1.20c}
$$

Moreover, for any cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$ *and* $\sigma > 0$ *there exists some* $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ *and* $s > 0$ *such that*

$$
\begin{aligned}
\big\|\chi(t^{-\sigma}D_x)c_w(t,\cdot)\big\|_{L^2} \lesssim\ & t^\sigma \big\|\chi_1(t^{-\sigma}D_x)(v^{NF} - v_-)(t,\cdot)\big\|_{L^2}\big(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\big) \\
&+ t^{-N(s)}\big\|(v^{NF} - v_-)(t,\cdot)\big\|_{H^1}\big(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\big),
\end{aligned}
\tag{3.1.21a}
$$

$$
\begin{aligned}
\big\|\chi(t^{-\sigma}D_x)c_w(t,\cdot)\big\|_{L^\infty} \lesssim\ & t^\sigma \big\|\chi_1(t^{-\sigma}D_x)\left(v^{NF} - v_-\right)(t,\cdot)\big\|_{L^\infty}\big(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\big) \\
&+ t^{-N(s)}\big\|(v^{NF} - v_-)(t,\cdot)\big\|_{H^1}\big(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\big)
\end{aligned}
\tag{3.1.21b}
$$

$$
\begin{aligned}
\big\|\chi(t^{-\sigma}D_x)\Omega c_w(t,\cdot)\big\|_{L^2} \lesssim\ & t^\sigma \big\|\chi_1(t^{-\sigma}D_x)\Omega(v^{NF} - v_-)(t,\cdot)\big\|_{L^2}\big(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\big) \\
&+ t^{-N(s)}\big\|\Omega(v^{NF} - v_-)(t,\cdot)\big\|_{L^2}\big(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\big) \\
&+ t^\sigma \big\|(v^{NF} - v_-)(t,\cdot)\big\|_{H^{1,\infty}} \times \sum_{\mu=0}^{1}\big(\|\Omega^\mu V(t,\cdot)\|_{H^1} + \|\Omega^\mu v^{NF}(t,\cdot)\|_{L^2}\big)
\end{aligned}
\tag{3.1.21c}
$$

with $N(s) > 0$ *as large as we want as long as* $s > 0$ *is large, and*

$$
\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{13,\infty}}^2\|U(t,\cdot)\|_{H^1},
\tag{3.1.22a}
$$

$$
\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{L^\infty} \lesssim \|V(t,\cdot)\|_{H^{13,\infty}}^2\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right),
\tag{3.1.22b}
$$

and for any $\theta \in]0,1[$,

$$
\begin{aligned}
\|\chi(t^{-\sigma}D_x)\Omega r_w^{NF}(t,\cdot)\|_{L^2} \lesssim\ & t^\beta\Big[\|V(t,\cdot)\|_{H^{15,\infty}}^{1-\theta}\|V(t,\cdot)\|_{H^{17}}^{\theta}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right) \\
&+ \|V(t,\cdot)\|_{L^\infty}\Big(\|U(t,\cdot)\|_{H^{16,\infty}}^{1-\theta} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{16,\infty}}^{1-\theta}\Big)\|U(t,\cdot)\|_{H^{18}}^{\theta}\Big]\|\Omega V(t,\cdot)\|_{L^2} \\
&+ t^\beta\Big[\|V(t,\cdot)\|_{H^{1,\infty}}\left(\|U(t,\cdot)\|_{H^1} + \|\Omega U(t,\cdot)\|_{H^1}\right) \\
&+ \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right)\left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{L^2}\right)\Big]\|V(t,\cdot)\|_{H^{17,\infty}}.
\end{aligned}
\tag{3.1.22c}
$$

Proof. By definition (3.1.15) of u^{NF}, system (2.1.2) with $|I| = 0$, (3.1.13) and (3.1.14), it follows that u^{NF} is solution to

$$(D_t + |D_x|)u^{NF}(t,x) = -\frac{1}{2}\Im\left[v_+\, D_1 v_- + \frac{D_x}{\langle D_x\rangle}v_+ \cdot \frac{D_x D_1}{\langle D_x\rangle}v_-\right] + r_w^{NF}(t,x),$$

with r_w^{NF} given by (3.1.19). Reminding that $v_+ = -\overline{v_-}$ and replacing each occurrence of v_- in the quadratic contribution to the above right hand side, we find that u^{NF} is solution to (3.1.16).

The first part of lemma A.8 and the fact that any $H^{\rho+1,\infty}$ injects into $H^{\rho+3}$ by Sobolev inequality immediately imply estimates (3.1.20) and

$$\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{L^2} \lesssim \|NL_{kg}(t,\cdot)\|_{L^2}\|V(t,\cdot)\|_{H^{13,\infty}},$$

$$\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{L^\infty} \lesssim \|NL_{kg}(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{13,\infty}},$$

for any $s, \rho \geq 0$. Moreover, from (A.37a) we derive that

$$\begin{aligned}\|\chi(t^{-\sigma}D_x)\Omega r_w^{NF}(t,\cdot)\|_{L^2} \lesssim t^\beta\left(\|NL_{kg}(t,\cdot)\|_{L^2} + \|\Omega NL_{kg}(t,\cdot)\|_{L^2}\right)\|V(t,\cdot)\|_{H^{17,\infty}} \\ + t^\beta\|NL_{kg}(t,\cdot)\|_{H^{15,\infty}}\|\Omega V(t,\cdot)\|_{L^2},\end{aligned}$$

so estimates (3.1.22) are obtained using (B.1.4a), (B.1.4e) with $s = 15$, and (B.1.4f).

Finally, inequality (3.1.21a) (resp. (3.1.21b)) is obtained using lemma B.2.2 in appendix B with $L = L^2$ (resp. $L = L^\infty$), $w = v_- - v^{NF}$, and the fact that $\chi_1(t^{-\sigma}D_x)$ is continuous from L^2 to H^1 (resp. from L^∞ to $H^{1,\infty}$) with norm $O(t^\sigma)$. Inequality (3.1.21c) is deduced applying Ω to (3.1.18) and using the Leibniz rule. The L^2 norm of products in which Ω is acting on $v_- - v^{NF}$ is estimated by means of lemma B.2.2 with $L = L^2$, $w = v_- - v^{NF}$, whereas the L^2 norm of the remaining products is simply estimated by taking the L^∞ norm on $v_- - v^{NF}$ times the L^2 norm of the remaining factor. □

3.2 From PDEs to ODEs

In the previous section we showed that, if (u_-, v_-) is solution to system (3.1.1) in some interval $[1,T]$, for a fixed $T > 1$, one can define two new functions, v^{NF} as in (3.1.3) and u^{NF} as in (3.1.15), respectively comparable to v_- and u_- in the sense of (3.1.2) and (3.1.12), such that (u^{NF}, v^{NF}) is solution to a new wave-Klein-Gordon system:

$$\begin{cases}(D_t + |D_x|)\, u^{NF}(t,x) = q_w(t,x) + c_w(t,x) + r_w^{NF}(t,x),\\ (D_t + \langle D_x\rangle)\, v^{NF}(t,x) = r_{kg}^{NF}(t,x),\end{cases} \tag{3.2.1}$$

for every $(t,x) \in [1,T]\times\mathbb{R}^2$, where quadratic inhomogeneous term q_w is given by (3.1.17) and cubic ones c_w, r_w^{NF} and r_{kg}^{NF} respectively by (3.1.18), (3.1.19) and (3.1.5).

As anticipated before, our aim is to deduce from (3.2.1) a system made of a transport equation and an ODE, from which it will be possible to deduce suitable estimates on (u^{NF}, v^{NF}) (and consequently on (u_-, v_-)). Thanks to (3.1.2) and (3.1.12) these estimates will allow us to close the bootstrap argument and prove theorem 1.1.2.

In subsection 3.2.1 we focus on the deduction of the mentioned ODE starting from the Klein-Gordon equation satisfied by v^{NF}, while in subsection 3.2.2 we show how to derive a transport equation from the wave equation satisfied by u^{NF}. The framework in which this plan takes place is the *semi-classical framework*, introduced below.

Let us introduce the *semi-classical parameter* $h := t^{-1}$ together with the following two new functions:

$$\widetilde{u}(t,x) := tu^{NF}(t,tx), \qquad \widetilde{v}(t,x) := tv^{NF}(t,tx), \tag{3.2.2}$$

and observe that, from definition (3.2.2) and inequalities (3.1.12), (3.1.2), a-priori estimates (1.1.11a), (1.1.11b) are equivalent respectively to

$$\|\widetilde{u}(t,\cdot)\|_{H_h^{\rho+1,\infty}} + \left\|\mathrm{Op}_h^w(\xi|\xi|^{-1})\widetilde{u}(t,\cdot)\right\|_{H_h^{\rho+1,\infty}} \leq C\varepsilon h^{-\frac{1}{2}}, \tag{3.2.3a}$$

$$\|\widetilde{v}(t,\cdot)\|_{H_h^{\rho,\infty}} \leq C\varepsilon, \tag{3.2.3b}$$

for some positive constant C. A suitable propagation of the above estimates will therefore provide us with (1.1.12a) and (1.1.12b).

A straight computation shows that $(\widetilde{u},\widetilde{v})$ satisfies the following coupled system of semi-classical pseudo-differential equations:

$$\begin{cases} \left[D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|)\right]\widetilde{u}(t,x) = h^{-1}\left[q_w(t,tx) + c_w(t,tx) + r_w^{NF}(t,tx)\right] \\ \left[D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle)\right]\widetilde{v}(t,x) = h^{-1}r_{kg}^{NF}(t,tx), \end{cases} \tag{3.2.4}$$

where Op_h^w denotes the semi-classical Weyl quantization introduced in 1.2.17 (i). Moreover, if $\mathcal{M}_j$ (resp. $\mathcal{L}_j$), $j=1,2$, is the operator introduced in (1.2.49) (resp. (1.2.68)), $\mathcal{M}_j\widetilde{u}$ (resp. $\mathcal{L}_j\widetilde{v}$) can be expressed in term of $Z_j u^{NF}$ (resp. $Z_j v^{NF}$). We have the following general result:

Lemma 3.2.1. (i) *Let $w(t,x)$ be a solution to the inhomogeneous half wave equation*

$$\left[D_t + |D_x|\right] w(t,x) = f(t,x), \tag{3.2.5}$$

and $\widetilde{w}(t,x) = tw(t,tx)$. For any $j=1,2$,

$$Z_j w(t,y) = ih\left[-\mathcal{M}_j\widetilde{w}(t,x) + \frac{1}{2i}\mathrm{Op}_h^w\left(\frac{\xi_j}{|\xi|}\right)\widetilde{w}(t,x)\right]|_{x=\frac{y}{t}} + iy_j f(t,y); \tag{3.2.6}$$

(ii) *If $w(t,x)$ is solution to the inhomogeneous half Klein-Gordon equation*

$$\left[D_t + \langle D_x\rangle\right] w(t,x) = f(t,x), \tag{3.2.7}$$

then

$$Z_j w(t,y) = ih\left[-\mathrm{Op}_h^w(\langle\xi\rangle)\mathcal{L}_j\widetilde{w}(t,x) + \frac{1}{i}\mathrm{Op}_h^w\Big(\frac{\xi_j}{\langle\xi\rangle}\Big)\widetilde{w}(t,x)\right]|_{x=\frac{y}{t}} + iy_j f(t,y). \tag{3.2.8}$$

Proof. (i) If w is solution to half wave equation (3.2.5) then $\widetilde{w}(t,x)$ satisfies

$$\left[D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|)\right]\widetilde{w}(t,x) = h^{-1}f(t,tx),$$

so, for any $i=1,2$,

$$\begin{aligned}
&Z_j w(t,y) = \\
&ih^{-1}\left[x_jD_t + \mathrm{Op}_h^w(\xi_j - x_jx\cdot\xi) + \frac{3h}{2i}x_j\right]\left(\frac{1}{t}\widetilde{w}(t,x)\right)\Big|_{x=\frac{y}{t}} \\
&= i\left[x_jD_t + \mathrm{Op}_h^w(\xi_j - x_jx\cdot\xi) + \frac{h}{2i}x_j\right]\widetilde{w}(t,x)\Big|_{x=\frac{y}{t}} \\
&= i\left[x_j\mathrm{Op}_h^w(x\cdot\xi - |\xi|)\widetilde{w}(t,x) + \mathrm{Op}_h^w(\xi_j - x_jx\cdot\xi)\widetilde{w}(t,x) + \frac{h}{2i}x_j\widetilde{u}(t,x) + h^{-1}x_jf(t,tx)\right]|_{x=\frac{y}{t}} \\
&= ih\left[-\mathcal{M}_j\widetilde{w}(t,x) + \frac{1}{2i}\mathrm{Op}_h^w\left(\frac{\xi_j}{|\xi|}\right)\widetilde{w}(t,x)\right]|_{x=\frac{y}{t}} + iy_jf(t,y).
\end{aligned}$$

We should specify that last equality is obtained by a trivial version of symbolic calculus (1.2.18), that applies also to symbols $b(\xi)$ singular at $\xi = 0$. Indeed, if symbol $a = a(x,\xi)$ is linear in x, and $b(\xi)$ is lipschitz, the development $a\sharp b$ is actually finite:

$$a\sharp b(x,\xi) = a(x,\xi)b(\xi) - \frac{h}{2i}\partial_x a(x,\xi)\cdot\partial_\xi b(\xi).$$

(ii) The result is analogous to the previous one, after observing that $\widetilde{w}$ satisfies

$$\big[D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle)\big]\widetilde{w}(t,x) = h^{-1}f(t,tx).$$

□

As a straight consequence of lemma 3.2.1 and system (3.2.4) we have that

(3.2.9a)
$$Z_j u^{NF}(t,y) = ih\left[-\mathcal{M}_j\widetilde{u}(t,x) + \frac{1}{2i}\mathrm{Op}_h^w\left(\frac{\xi_j}{|\xi|}\right)\widetilde{u}(t,x)\right]|_{x=\frac{y}{t}} + iy_j\left[q_w + c_w + r_w^{NF}\right](t,y),$$

(3.2.9b)
$$Z_j v^{NF}(t,y) = ih\left[-\mathrm{Op}_h^w(\langle\xi\rangle)\mathcal{L}_j\widetilde{v}(t,x) + \frac{1}{i}\mathrm{Op}_h^w\Big(\frac{\xi_j}{\langle\xi\rangle}\Big)\widetilde{v}(t,x)\right]\Big|_{x=\frac{y}{t}} + iy_j r_{kg}^{NF}(t,y).$$

In view of lemma 3.2.14, it is also useful to write down the analogous relation between $(Z_mZ_nu)_-$ and $\mathcal{M}[t(Z_nu)_-(t,tx)]$. As $(Z_nu)_-$ is solution to

$$\big(D_t + |D_x|\big)(Z_nu)_- = Z_nNL_w(t,x),$$

from equality (3.2.6) with $w = (Z_nu)_-$ and the commutation between Z_m and $D_t - |D_x|$ (see (2.1.15a)) we find that

$$(Z_mZ_nu)_-(t,y) = ih\Big[-\mathcal{M}_m\widetilde{u}^J(t,x) + \frac{1}{2i}\mathrm{Op}_h^w\Big(\frac{\xi_m}{|\xi|}\Big)\widetilde{u}^J(t,x)\Big]\big|_{x=\frac{y}{t}} + iy_mZ_nNL_w(t,y) - \frac{D_m}{|D_y|}(Z_nu)_-(t,y), \tag{3.2.10}$$

where J is the index such that $\Gamma^J = Z_n$ and $\widetilde{u}^J(t,x) := t(Z_nu)_-(t,tx)$. Also, observe that from (1.1.15), (1.1.16), (1.1.5) and (1.1.10)

$$Z_nNL_w = Q_0^{\mathrm{w}}\big((Z_nv)_\pm, D_1v_\pm\big) + Q_0^{\mathrm{w}}\big(v_\pm, D_1(Z_nv)_\pm\big) - \delta_n^1 Q_0^{\mathrm{w}}(v_\pm, D_tv_\pm)$$

with $\delta_n^1 = 1$ for $n = 1$, and that from inequality (B.1.6a) with $s = 0$,

$$\begin{aligned}\|Z_nNL_w(t,\cdot)\|_{L^2} \lesssim{}& \|Z_nV(t,\cdot)\|_{H^1}\|V(t,\cdot)\|_{H^{2,\infty}} + \big[\|V(t,\cdot)\|_{H^1}\\ &+ \|V(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\right) + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\big]\|V(t,\cdot)\|_{H^{1,\infty}}.\end{aligned} \tag{3.2.11}$$

Moreover, from the definition of $\mathcal{M}_j$ and $\mathcal{L}_j$ we see that

$$h\mathcal{M}_j\widetilde{w}(t,x) = \left[y_j|D_y| - tD_j + \frac{1}{2i}\frac{D_j}{|D_y|}\right]w(t,y)|_{y=tx},$$

$$h\mathrm{Op}_h^w(\langle\xi\rangle)\mathcal{L}_j\widetilde{w}(t,x) = \left[y_j\langle D_y\rangle - tD_j - i\frac{D_j}{\langle D_y\rangle}\right]w(t,y)|_{y=tx},$$

so lemma 3.2.1 implies that, if w is solution to half wave equation (3.2.5) (resp. to half Klein-Gordon (3.2.7)),

$$\left[y_j|D_y| - tD_j + \frac{1}{2i}\frac{D_j}{|D_y|}\right]w(t,y) = iZ_jw(t,y) + \frac{1}{2i}\frac{D_j}{|D_y|}w(t,y) + y_jf(t,y), \tag{3.2.12a}$$

$$\left(\text{resp. } \left[\langle D_y\rangle y_j - tD_j\right]w(t,y) = iZ_jw(t,y) - i\frac{D_j}{\langle D_y\rangle}w(t,y) + y_jf(t,y)\right). \tag{3.2.12b}$$

3.2.1 Derivation of the ODE and propagation of the uniform estimate on the Klein-Gordon component

Let us firstly deal with the semi-classical Klein-Gordon equation satisfied by $\widetilde{v}$:

$$\big[D_t - \mathrm{Op}_h^w(x\cdot\xi - p(\xi))\big]\widetilde{v}(t,x) = h^{-1}r_{kg}^{NF}(t,tx), \tag{3.2.13}$$

where $p(\xi) = \langle\xi\rangle$ and r_{kg}^{NF} is given by (3.1.5) and satisfies (3.1.6). We remind that $p'(\xi)$ denotes the gradient of $p(\xi)$ while $p''(\xi)$ is its 2×2 Hessian matrix, and that $\mathcal{L}_j$ is the operator introduced in (1.2.68) for $j = 1, 2$. We also remind definition (1.2.66) of manifold Λ_{kg}, represented in dimension 1 by picture 3.1 below, and decompose $\widetilde{v}$ into the sum of two contributions: one localized in a neighbourhood of Λ_{kg} of size $\sqrt{h}$ (in the spirit of [14]), the other localized out of this neighbourhood. The contribution localized away from Λ_{kg} appears to be a $O(h^{1/2-0})$ if we

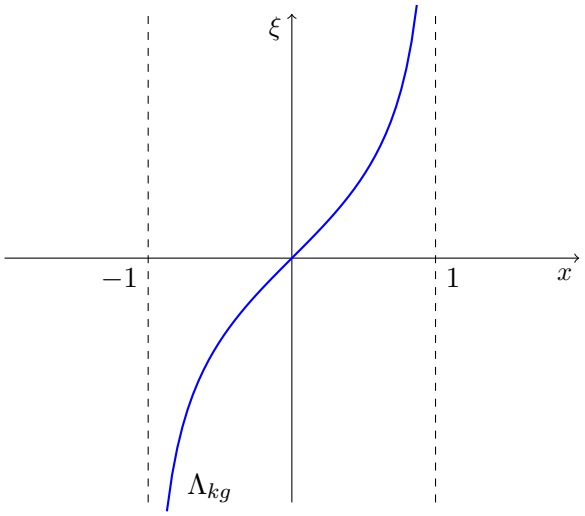

Figure 3.1: Lagrangian for the Klein-Gordon equation

assume a moderate growth for the L^2 norm of $\mathcal{L}^\mu\widetilde{v}$, with $0 \le |\mu| \le 2$, and has hence a better decay in time than the one expected for $\widetilde{v}$ (remind $h = t^{-1}$). Thus the main contribution to $\widetilde{v}$ is the one localized around Λ_{kg}. We are going to show that this latter one is solution to an ODE (see proposition 3.2.6) and that its $H_h^{\rho,\infty}$ norm is uniformly bounded in time, which will finally enable us to propagate (3.2.3b) and obtain (1.1.11b) (see proposition 3.2.7).

For any fixed $\rho \in \mathbb{Z}$ let $\Sigma(\xi) := \langle\xi\rangle^\rho$, and for some $\gamma, \chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 close to the origin, $\sigma > 0$ small (e.g. $\sigma < \frac{1}{4}$) let

$$\Gamma^{kg} := \mathrm{Op}_h^w\left(\gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\right). \tag{3.2.14}$$

We also introduce the following notations:

$$\widetilde{v}^\Sigma := \mathrm{Op}_h^w(\Sigma(\xi))\widetilde{v}, \tag{3.2.15}$$

together with

$$\widetilde{v}_{\Lambda_{kg}}^\Sigma := \Gamma^{kg}\widetilde{v}^\Sigma, \tag{3.2.16a}$$

$$\widetilde{v}_{\Lambda_{kg}^c}^\Sigma := \mathrm{Op}_h^w\left(1 - \gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\right)\widetilde{v}^\Sigma, \tag{3.2.16b}$$

so that $\widetilde{v}^\Sigma = \widetilde{v}_{\Lambda_{kg}}^\Sigma + \widetilde{v}_{\Lambda_{kg}^c}^\Sigma$, and remind that $\|\mathcal{L}^\gamma w\| = \|\mathcal{L}_1^{\gamma_1}\mathcal{L}_2^{\gamma_2}w\|$, for any $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$.

Lemma 3.2.2. *Let $\widetilde{\gamma} \in C^\infty(\mathbb{R}^2)$ vanish in a neighbourhood of the origin and be such that $|\partial_z^\alpha \widetilde{\gamma}(z)| \lesssim \langle z\rangle^{-|\alpha|}$. Let $c(x,\xi) \in S_{\delta,\sigma}(1)$ with $\delta \in [0, \frac{1}{2}]$, $\sigma > 0$, be supported for $|\xi| \lesssim h^{-\sigma}$. For any $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi(h^\sigma \xi) \equiv 1$ on the support of $c(x,\xi)$,*

$$\left\| \mathrm{Op}_h^w\Big(\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^2} \lesssim \sum_{|\mu|=0}^{1} h^{\frac{1}{2}-\beta} \left\| \mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.17a}$$

$$\left\| \mathrm{Op}_h^w\Big(\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{1} h^{-\beta} \left\| \mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.17b}$$

and

$$\left\| \mathrm{Op}_h^w\Big(\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^2} \lesssim \sum_{|\mu|=0}^{2} h^{1-\beta} \left\| \mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.18a}$$

$$\left\| \mathrm{Op}_h^w\Big(\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{2} h^{\frac{1}{2}-\beta} \left\| \mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.18b}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Proof. The proof of (3.2.17) (resp. of (3.2.18)) follows straightly by inequalities (1.2.70) (resp. (1.2.71)), after observing that, as $\widetilde{\gamma}$ vanishes in a neighbourhood of the origin,

$$\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi) = \sum_{j=1}^{2} \widetilde{\gamma}_1^j\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x_j-p_j'(\xi)}{\sqrt{h}}\Big)c(x,\xi),$$

where $\widetilde{\gamma}_1^j(z) := \widetilde{\gamma}(z) z_j |z|^{-2}$ is such that $|\partial_z^\alpha \widetilde{\gamma}_1^j(z)| \lesssim \langle z\rangle^{-1-|\alpha|}$ (resp.

$$\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi) = \sum_{j=1}^{2} \widetilde{\gamma}_2\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)^2 c(x,\xi),$$

where $\widetilde{\gamma}_2(z) := \widetilde{\gamma}(z)|z|^{-2}$ is such that $|\partial_z^\alpha \widetilde{\gamma}(z)| \lesssim \langle z\rangle^{-2-|\alpha|}$). □

Corollary 3.2.3. *There exists $s > 0$ sufficiently large such that*

$$\left\| \widetilde{v}_{\Lambda_{kg}^c}^{\Sigma}(t,\cdot)\right\|_{L^2} \lesssim h^{1-\beta}\left(\|\widetilde{v}(t,\cdot)\|_{H_h^s} + \sum_{1\le|\mu|\le 2} \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2}\right), \tag{3.2.19a}$$

$$\left\| \widetilde{v}_{\Lambda_{kg}^c}^{\Sigma}(t,\cdot)\right\|_{L^\infty} \lesssim h^{\frac{1}{2}-\beta}\left(\|\widetilde{v}(t,\cdot)\|_{H_h^s} + \sum_{1\le|\mu|\le 2} \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2}\right). \tag{3.2.19b}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Proof. Since symbol $1 - \gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$ is supported for $|\frac{x-p'(\xi)}{\sqrt{h}}| \ge d_1 > 0$ or $|h^\sigma\xi| \ge d_2 > 0$, for some small $d_1, d_2 > 0$, we may consider a smooth cut-off function $\widetilde{\chi}$ equal to 1 close to the origin and such that $\widetilde{\chi}\chi \equiv \widetilde{\chi}$, so that $1 - \gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$ writes as

$$\left[1 - \gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\right]\widetilde{\chi}(h^\sigma\xi) + \left[1 - \gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\right](1-\widetilde{\chi})(h^\sigma\xi),$$

the first symbol being supported in $\{(x,\xi) : |\frac{x-p'(\xi)}{\sqrt{h}}| \geq d_1, |\xi| \lesssim h^{-\sigma}\}$, the second one for large frequencies $|\xi| \gtrsim h^{-\sigma}$.

Using lemma 1.2.24 and the fact that $\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi) \in S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big)$, for any $M \in \mathbb{N}$, we have that, for a fixed $N \in \mathbb{N}^*$,

$$\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big]\big(1-\widetilde{\chi}(h^\sigma\xi)\big) = \big(1-\widetilde{\chi}(h^\sigma\xi)\big)\sharp\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big] + \sum_{1\leq j<N}\widetilde{\chi}_j(h^\sigma\xi)\sharp a_j(x,\xi) + r_N(x,\xi),$$

where function $\widetilde{\chi}_j(h^\sigma\xi)$ is still supported for large frequencies $|\xi| \gtrsim h^{-\sigma}$, for every $1 \leq j < N$, up to negligible multiplicative constants,

$$a_j(x,\xi) = h^{j(\frac{1}{2}+\sigma)}\sum_{|\alpha|=j}(\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi) \in h^{j(\frac{1}{2}+\sigma)}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big),$$

and $r_N \in h^{N(\frac{1}{2}+\sigma)}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big)$. Lemma 1.2.40, proposition 1.2.36, and the semi-classical Sobolev injection imply that

$$\Big\|\mathrm{Op}_h^w\Big(\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big](1-\widetilde{\chi})(h^\sigma\xi)\Big)\widetilde{v}^\Sigma(t,\cdot)\Big\|_{L^2} \lesssim h^{N(s)}\|\widetilde{v}(t,\cdot)\|_{H^s_h},$$
$$\Big\|\mathrm{Op}_h^w\Big(\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big](1-\widetilde{\chi})(h^\sigma\xi)\Big)\widetilde{v}^\Sigma(t,\cdot)\Big\|_{L^\infty} \lesssim h^{N'(s)}\|\widetilde{v}(t,\cdot)\|_{H^s_h},$$

where $N(s), N'(s) \geq 1$ if $s > 2$ is sufficiently large.

On the other hand, as function $(1-\gamma)\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)$ vanishes in a neighbourhood of the origin and is such that $|\partial_z^\alpha(1-\gamma)(z)| \lesssim \langle z\rangle^{-|\alpha|}$, by inequalities (3.2.18) and the fact that, using symbolic calculus to commute $\mathcal{L}$ with $\Sigma(\xi)$,

$$\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}^\Sigma(t,\cdot)\|_{L^2} \lesssim h^{-\nu}\sum_{|\mu_1|\leq|\mu|}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^{\mu_1}\widetilde{v}(t,\cdot)\|_{L^2} \tag{3.2.20}$$

with $\nu = \rho\sigma$ if $\rho \geq 0$, 0 otherwise, we have that

$$\Big\|\mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}^\Sigma(t,\cdot)\Big\|_{L^2} \lesssim \sum_{|\mu|\leq 2}h^{1-\beta}\,\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\,,$$
$$\Big\|\mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}^\Sigma(t,\cdot)\Big\|_{L^\infty} \lesssim \sum_{|\mu|\leq 2}h^{\frac{1}{2}-\beta}\,\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\,,$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$. □

In the following lemma we show how to develop the symbol $a(x,\xi)$ associated to an operator acting on $\Gamma^{kg}w$, for some suitable function w, at $\xi = -d\phi(x)$, where $\phi(x) = \sqrt{1-|x|^2}$.

Lemma 3.2.4. *Let $a(x,\xi)$ be a real symbol in $S_{\delta,0}(\langle\xi\rangle^q)$, $q \in \mathbb{R}$, for some $\delta > 0$ small, $\Sigma(\xi) = \langle\xi\rangle^\rho$ for some fixed $\rho \in \mathbb{Z}$, Γ^{kg} the operator introduced in (3.2.14) and $w = w(t,x)$ such that $\mathcal{L}^\mu w(t,\cdot) \in L^2(\mathbb{R}^2)$ for any $|\mu| \leq 2$. Let us also introduce $w^\Sigma_{\Lambda_{kg}} := \Gamma^{kg}\mathrm{Op}_h^w(\Sigma)w$. There exists a family $(\theta_h(x))_h$ of C_0^∞ functions real valued, equal to 1 on the closed ball $\overline{B_{1-ch^{2\sigma}}(0)}$ and supported*

in $\overline{B_{1-c_1h^{2\sigma}}(0)}$*, for some small* $0 < c_1 < c, \sigma > 0$*, with* $\|\partial_x^\alpha \theta_h\|_{L^\infty} = O(h^{-2|\alpha|\sigma})$ *and* $(h\partial_h)^k\theta_h$ *bounded for every* k*, such that*

$$\mathrm{Op}_h^w(a)w_{\Lambda_{kg}}^\Sigma = \theta_h(x)a(x,-d\phi(x))w_{\Lambda_{kg}}^\Sigma + R_1(w)\,, \tag{3.2.21}$$

where $R_1(w)$ *satisfies*

$$\|R_1(w)(t,\cdot)\|_{L^2} \lesssim h^{1-\beta}\left(\|w(t,\cdot)\|_{H_h^s} + \sum_{|\gamma|=1}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma w(t,\cdot)\|_{L^2}\right), \tag{3.2.22a}$$

$$\|R_1(w)(t,\cdot)\|_{L^\infty} \lesssim h^{\frac{1}{2}-\beta}\left(\|w(t,\cdot)\|_{H_h^s} + \sum_{|\gamma|=1}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma w(t,\cdot)\|_{L^2}\right), \tag{3.2.22b}$$

with $\beta = \beta(\sigma,\delta) > 0$*,* $\beta \to 0$ *as* $\sigma, \delta \to 0$*. Moreover, if* $\partial_\xi a(x,\xi)$ *vanishes at* $\xi = -d\phi(x)$*, the above estimates can be improved and* $R_1(w)$ *is rather a remainder* $R_2(w)$ *such that*

$$\|R_2(w)(t,\cdot)\|_{L^2} \lesssim h^{2-\beta}\left(\|w(t,\cdot)\|_{H_h^s} + \sum_{1\le|\gamma|\le 2}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma w(t,\cdot)\|_{L^2}\right), \tag{3.2.23a}$$

$$\|R_2(w)(t,\cdot)\|_{L^\infty} \lesssim h^{\frac{3}{2}-\beta}\left(\|w(t,\cdot)\|_{H_h^s} + \sum_{1\le|\gamma|\le 2}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma w(t,\cdot)\|_{L^2}\right). \tag{3.2.23b}$$

Proof. After lemma 1.2.38 we know that there exists a family of functions $\theta_h(x)$ as in the statement such that equality (1.2.67) holds. We highlight the fact that any derivative of θ_h vanishes on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$ and its derivatives. After remark 1.2.22, this implies that

$$w_{\Lambda_{kg}}^\Sigma = \theta_h(x)w_{\Lambda_{kg}}^\Sigma + r_\infty, \quad r_\infty \in h^N S_{\frac{1}{2},\sigma}(\langle x\rangle^{-\infty})$$

and hence that

$$\mathrm{Op}_h^w(a)w_{\Lambda_{kg}}^\Sigma = \mathrm{Op}_h^w(a)\theta_h(x)w_{\Lambda_{kg}}^\Sigma + \mathrm{Op}_h^w(r_\infty^a)w_{\Lambda_{kg}}^\Sigma,$$

with $r_\infty^a = a\sharp r_\infty \in h^{N-\gamma}S_{\frac{1}{2},\sigma}(\langle x\rangle^{-\infty})$ and $\gamma = q\sigma$ if $q \ge 0$, 0 otherwise. From proposition 1.2.36 and the semi-classical Sobolev injection it follows at once that $\mathrm{Op}_h^w(r_\infty^a)w_{\Lambda_{kg}}^\Sigma$ satisfies enhanced estimates (3.2.23) if N is taken sufficiently large. Up to negligible multiplicative constants, a further application of symbolic calculus gives also that

$$\mathrm{Op}_h^w(a(x,\xi))\theta_h(x)w_{\Lambda_{kg}}^\Sigma = \mathrm{Op}_h^w(a(x,\xi)\theta_h(x))w_{\Lambda_{kg}}^\Sigma + \sum_{|\alpha|=1}^{N-1} h^{|\alpha|}\mathrm{Op}_h^w\big(\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha\theta_h(x)\big)w_{\Lambda_{kg}}^\Sigma + \mathrm{Op}_h^w(r_N(x,\xi))w_{\Lambda_{kg}}^\Sigma,$$

where $r_N \in h^{N-\beta}S_{\delta',0}(\langle\xi\rangle^{q-N}\langle x\rangle^{-\infty})$ for a new small $\beta = \beta(\delta,\sigma)$ and $\delta' = \max\{\delta,\sigma\}$. From the same argument as above $\mathrm{Op}_h^w(r_N)w_{\Lambda_{kg}}^\Sigma$ verifies enhanced estimates (3.2.23) if N is suitably chosen. Also, since the support of $\partial_\xi^\alpha a(x,\xi)\cdot\partial_x^\alpha\theta_h(x)$ has empty intersection with that of $\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)$ for any $|\alpha| \ge 1$, all the $|\alpha|$-order terms in the above equality are remainders $R_1(w)$.

Now, as symbol $a(x,\xi)\theta_h(x)$ is supported for $|x| \leq 1 - c_1 h^{2\sigma} < 1$, we are allowed to develop it at $\xi = -d\phi(x)$:

$$a(x,\xi)\theta_h(x) = a(x,-d\phi(x))\theta_h(x) + \sum_{|\alpha|=1} \int_0^1 (\partial_\xi^\alpha a)(x, t\xi + (1-t)d\phi(x))dt\, \theta_h(x)(\xi + d\phi(x))^\alpha$$

$$= a(x,-d\phi(x))\theta_h(x) + \sum_{j=1}^{2} b_j(x,\xi)(x_j - p_j'(\xi)), \tag{3.2.24}$$

with

$$b_j(x,\xi) = \sum_{|\alpha|=1} \int_0^1 (\partial_\xi^\alpha a)(x, t\xi + (1-t)d\phi(x))dt\, \theta_h(x) \frac{(\xi + d\phi(x))^\alpha (x_j - p_j'(\xi))}{|x - p'(\xi)|^2}, \quad j = 1,2. \tag{3.2.25}$$

If $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ is a new cut-off function equal to 1 close to the origin, we can reduce ourselves to the analysis of symbol $b_j(x,\xi)(x_j - p_j'(\xi))\chi_1(h^\sigma\xi)$. In fact, as $b_j(x,\xi)(x_j - p_j'(\xi))(1 - \chi_1)(h^\sigma\xi)$ is supported for large frequencies, one can prove that its operator acting on $w_{\Lambda_{kg}}^\Sigma$ is a $O_{L^2\cap L^\infty}(h^N \|w(t,\cdot)\|_{H_h^s})$ with $N > 0$ large as long as $s > 0$ is large, by using the semi-classical Sobolev injection, symbolic calculus of proposition 1.2.21, lemma 1.2.40 and proposition 1.2.36. Furthermore, if we consider a smooth cut-off function $\widetilde{\gamma} \in C_0^\infty(\mathbb{R}^2)$, equal to 1 close to the origin and such that $\widetilde{\gamma}\big(\langle\xi\rangle^2(x - p'(\xi))\big) \equiv 1$ on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$ (which is possible if $\sigma < 1/4$), we have that

$$b_j(x,\xi)(x_j - p_j'(\xi))\chi_1(h^\sigma\xi) = b_j(x,\xi)(x_j - p_j'(\xi))\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x - p'(\xi))\big)$$
$$+ b_j(x,\xi)(x_j - p_j'(\xi))\chi_1(h^\sigma\xi)(1 - \widetilde{\gamma})\big(\langle\xi\rangle^2(x - p'(\xi))\big).$$

Since $b_j(x,\xi)(x_j - p_j'(\xi))\chi_1(h^\sigma\xi)(1-\widetilde{\gamma})\big(\langle\xi\rangle^2(x-p'(\xi))\big) \in h^{-\beta}S_{\delta,\sigma}(1)$, for some new small $\beta, \delta > 0$, and its support has empty intersection with that of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)$ (which instead belongs to class $S_{\frac{1}{2},0}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M})$, for $M \in \mathbb{N}$ as large as we want), its quantization acting on $w_{\Lambda_{kg}}^\Sigma$ is also an enhanced remainder $R_2(w)$.

The very contribution that only enjoys estimates (3.2.22) is $\mathrm{Op}_h^w\big(c(x,\xi)(x_j - p_j'(\xi))\big)w_{\Lambda_{kg}}^\Sigma$, with $c(x,\xi) := b_j(x,\xi)\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x - p'(\xi))\big) \in h^{-\beta}S_{2\sigma,\sigma}(1)$ and β depending linearly on σ. In fact, if we assume that the support of χ_1 is sufficiently small so that $\chi_1\chi \equiv \chi_1$ and all derivatives of χ vanish on that support, by using symbolic development (1.2.18) until a sufficiently large order N and observing that

$$\Big\{c(x,\xi)(x_j - p_j'(\xi)), \gamma\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)\Big\} = \Big\{c(x,\xi), \gamma\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)\Big\}(x_j - p_j'(\xi))$$
$$= \Big[(\partial_\xi c)\cdot(\partial\gamma)\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big) + (\partial_x c)\cdot(\partial\gamma)\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)p''(\xi)\Big]\Big(\frac{x_j - p_j'(\xi)}{\sqrt{h}}\Big)$$

does not lose any power $h^{-1/2}$, we derive that, up to negligible constants,

$$\Big[c(x,\xi)(x_j - p_j'(\xi))\Big]\sharp\Big[\gamma\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big] = \gamma\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)c(x,\xi)(x_j - p_j'(\xi))$$
$$+ \sum{}' h\widetilde{\gamma}\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)\widetilde{c}(x,\xi) + r_N(x,\xi).$$

In the above equality $\sum'$ is a concise notation to indicate a linear combination, $\widetilde{\gamma} \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, $\widetilde{c} \in h^{-\beta}S_{\delta,\sigma}(1)$ for some new small $\beta,\delta>0$, and $r_N \in h^{(N+1)/2-\beta}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-(M-1)}\big)$ as $c(x,\xi)(x_j-p_j'(\xi)) \in h^{1/2-\beta}S_{2\sigma,\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle\big)$. From inequalities (1.2.70) and (3.2.20) we deduce that $\mathrm{Op}_h^w\big(\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)c(x,\xi)(x_j-p_j'(\xi))\big)\mathrm{Op}_h^w(\Sigma)w$ is a remainder $R_1(w)$ satisfying (3.2.22). The quantization of all the addends in $\sum'$ acting on $\mathrm{Op}_h^w(\Sigma)w$ is estimated by using that $\widetilde{\gamma}(z)$ vanishes in a neighbourhood of the origin and can be rewritten as $\sum_{j=1,2}\widetilde{\gamma}_2(z)z_j^2$, with $\widetilde{\gamma}_2(z):=\widetilde{\gamma}(z)|z|^{-2}$ such that $|\partial_z^\alpha\widetilde{\gamma}_2(z)|\lesssim\langle z\rangle^{-2-|\alpha|}$. Inequalities (1.2.71) and the successive commutation of $\mathcal{L}^\gamma$ with Σ, for $|\gamma|=1,2$, give then that $h\mathrm{Op}_h^w\big(\widetilde{\gamma}\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\widetilde{c}(x,\xi)\big)\mathrm{Op}_h^w(\Sigma)w$ is a remainder $R_2(w)$. Finally, as

$$r_N(x,\xi)\sharp\Sigma(\xi)\in h^{\frac{N}{2}-\beta-\mu}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-(M-1)})$$

with $\mu=\sigma\rho$ if $\rho\geq 0$, 0 otherwise, $\mathrm{Op}_h^w(r_N)\mathrm{Op}_h^w(\Sigma)w$ is also a remainder $R_2(w)$ just from 1.2.36, 1.2.37, fixing $N\in\mathbb{N}$ sufficiently large (e.g. $N=3$).

If symbol $a(x,\xi)$ is such that $\partial_\xi a|_{\xi=-d\phi}=0$, instead of equality (3.2.24) with b_j given by (3.2.25), we have

$$a(x,\xi)\theta_h(x)=a(x,-d\phi(x))\theta_h(x)+\sum_{j=1,2}b(x,\xi)(x_j-p_j'(\xi))^2,$$

with

$$b(x,\xi)=\sum_{|\alpha|=2}\frac{2}{\alpha!}\int_0^1(\partial_\xi^\alpha a)(t\xi-(1-t)d\phi(x))(1-t)dt\,\theta_h(x)\frac{(\xi+d\phi(x))^\alpha}{|x-p'(\xi)|^2}.$$

The same argument as before can be applied to $\mathrm{Op}_h^w\big(b(x,\xi)\theta_h(x)(x_j-p_j'(\xi))^2\big)w_{\Lambda_{kg}}^\Sigma$ to show that it reduces to

$$\mathrm{Op}_h^w\Big(b(x,\xi)\theta_h(x)(x_j-p_j'(\xi))^2\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x-p'(\xi))\big)\Big)w_{\Lambda_{kg}}^\Sigma+R_2(w),$$

with $R_2(w)$ satisfying (3.2.23). If

$$B(x,\xi):=b(x,\xi)\theta_h(x)\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x-p'(\xi))\big)$$

then $B(x,\xi)(x_j-p_j'(\xi))^2\in h^{-\beta}S_{\delta',\sigma}(1)$ by lemma 1.2.43, for some new small $\beta,\delta'>0$ depending on σ,δ. Using lemma 1.2.24, symbolic development (1.2.18) until order 4, and assuming that the support of χ_1 is sufficiently small so that $\chi\chi_1\equiv\chi$, we derive that

$$\begin{aligned}\Big[B(x,\xi)(x_j-p_j'(\xi))^2\Big]\sharp\Big[\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big]&=B(x,\xi)\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_j-p_j'(\xi))^2\\&+\frac{h}{i}\sum_{i=1}^2(\partial_i\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x_j-p_j'(\xi)}{\sqrt{h}}\Big)\left[(\partial_{\xi_i}B)+\sum_j(\partial_{x_j}B)p_{ij}''(\xi)\right](x_j-p_j'(\xi))\\&+{\sum_{2\leq|\alpha|\leq 3}}'h^{\frac{|\alpha|}{2}-2\delta'-\beta}\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)B_\alpha(x,\xi)+r_4(x,\xi),\end{aligned}$$

where $\gamma_\alpha\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, $B_\alpha(x,\xi)\in S_{\delta',\sigma}(1)$, and $r_4(x,\xi)\in h^{2-4\delta'-\beta}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big)$. As $r_4(x,\xi)\sharp\Sigma(\xi)\in h^{2-\beta'}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big)$, for $\beta'=2-4\delta'-\beta-\rho\sigma$ if $\rho\geq 0$, $\beta'=2-4\delta'-\beta$ otherwise, it immediately follows from propositions 1.2.36 and 1.2.37 that $\mathrm{Op}_h^w(r_4)\widetilde{v}^\Sigma$ is a remainder $R_2(w)$. After inequalities (1.2.71) with $\gamma_n=\gamma$ and $c=B$ (resp. inequalities (1.2.70) with $\gamma_n(z)=\partial_i\gamma(z)z_j$ and $c=h^{\delta'}[(\partial_{\xi_i}B)+(\partial_xB)\cdot(\partial_\xi p_1'+\partial_\xi p_2')]\in S_{\delta',\sigma}(1)$, for $i,j=1,2$), and

(3.2.20), we deduce that the quantization of the first (resp. the second) contribution in above symbolic development is a remainder $R_2(w)$, when acting on $\mathrm{Op}_h^w(\Sigma)w$. Finally, as γ_α vanishes in a neighbourhood of the origin, we write

$$\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)=\sum_{k=1}^{2}h^{-1}\underbrace{\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big|\frac{x-p'(\xi)}{\sqrt{h}}\Big|^{-2}}_{\widetilde{\gamma}_\alpha\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)}\times(x_k-p_k'(\xi))^2,\quad |\alpha|=2,$$

$$\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)=\sum_{k=1}^{2}h^{-\frac{1}{2}}\underbrace{\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x_k-p_k'(\xi)}{\sqrt{h}}\Big)\Big|\frac{x-p'(\xi)}{\sqrt{h}}\Big|^{-2}}_{\widetilde{\gamma}_\alpha^k\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)}\times(x_k-p_k'(\xi)),\quad |\alpha|=3$$

and obtain that the quantization of α-th order term with $|\alpha|=2$ (resp. $|\alpha|=3$) is a remainder $R_2(w)$ when acting on $\mathrm{Op}_h^w(\Sigma)w$, after inequalities (1.2.71) (resp. (1.2.70)) with $\gamma_n=\widetilde{\gamma}_\alpha$ (resp. $\gamma_n=\widetilde{\gamma}_\alpha^k$, $k=1,2$) and $c=B_\alpha$. □

The following two results allow us to finally derive the ODE satisfied by $\widetilde{v}_{\Lambda_{kg}}^{\Sigma}$.

Lemma 3.2.5. *We have that*

$$\big[D_t-\mathrm{Op}_h^w(x\cdot\xi-p(\xi)),\Gamma^{kg}\big]=\mathrm{Op}_h^w(b),\tag{3.2.26}$$

where

(3.2.27)
$$\begin{aligned}b(x,\xi)=&-\frac{h}{2i}(\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)-\frac{\sigma h}{i}\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\\&+\frac{i}{24}h^{\frac{3}{2}}\sum_{|\alpha|=3}(\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial_\xi^\alpha p'(\xi))\chi(h^\sigma\xi)+r(x,\xi)\end{aligned}$$

and $r\in h^{5/2}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-N})$ for any $N\geq 0$. Therefore, function $\widetilde{v}_{\Lambda_{kg}}^{\Sigma}$ is solution to

$$\big[D_t-\mathrm{Op}_h^w(x\cdot\xi-p(\xi))\big]\widetilde{v}_{\Lambda_{kg}}^{\Sigma}=\Gamma^{kg}\mathrm{Op}_h^w(\Sigma(\xi))\big[h^{-1}r_{kg}^{NF}(t,tx)\big]+R_2(\widetilde{v})\tag{3.2.28}$$

with $R_2(\widetilde{v})$ satisfying estimates (3.2.23).

Proof. Recalling the definition (3.2.14) of Γ^{kg}, one can prove by a straight computation that

$$\begin{aligned}\big[D_t,\Gamma^{kg}\big]=&\frac{h}{i}\mathrm{Op}_h^w\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\frac{p''(\xi)\xi}{\sqrt{h}}\chi(h^\sigma\xi)\Big)\\&+\frac{h}{2i}\mathrm{Op}_h^w\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)-\frac{(1+\sigma)h}{i}\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\Big).\end{aligned}$$

Since the development of a commutator's symbol only contains odd-order terms, lemma 1.2.24 gives that the symbol associated to $\big[\Gamma^{kg},\mathrm{Op}_h^w(x\cdot\xi-p(\xi))\big]$ writes as

$$\frac{h}{i}\Big\{\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi),x\cdot\xi-p(\xi)\Big\}+\frac{i}{24}h^{\frac{3}{2}}\sum_{|\alpha|=3}(\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\partial_\xi^\alpha p(\xi))+r_5(x,\xi)$$

with $r_5 \in h^{5/2} S_{\frac{1}{2},\sigma}(\langle \frac{x-p'(\xi)}{\sqrt{h}} \rangle^{-N})$ for any $N \geq 0$. Developing the above Poisson bracket one finds that

$$\begin{aligned}\big[\Gamma^{kg}, \mathrm{Op}_h^w(x\cdot\xi - p(\xi))\big] &= -\frac{h}{i}\mathrm{Op}_h^w\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\frac{p''(\xi)\xi}{\sqrt{h}}\chi(h^\sigma\xi)\Big)\\ -\frac{h}{i}\mathrm{Op}_h^w\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big) &+ \frac{h}{i}\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\Big)\\ &+ \frac{i}{24}h^{\frac{3}{2}}\sum_{|\alpha|=3}\mathrm{Op}_h^w\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial_\xi^\alpha p'(\xi))\chi(h^\sigma\xi)\Big) + \mathrm{Op}_h^w(r_5(x,\xi)),\end{aligned}$$

which summed to the previous commutator gives (3.2.27).

The last part of the statement follows applying to equation (3.2.13) operators $\mathrm{Op}_h^w(\Sigma(\xi))$ (which commutes exactly with the linear part of the equation, evident in non semi-classical coordinates) and Γ^{kg}. Since

$$\begin{aligned}h\mathrm{Op}_h^w\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}^\Sigma&\\ = \sum_{k=1}^{2}\mathrm{Op}_h^w\Big(\gamma^k\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot(x-p'(\xi))(x_k-p_k'(\xi))\Big)\widetilde{v}^\Sigma&\end{aligned}$$

with $\gamma^k(z) := (\partial\gamma)(z)z_k|z|^{-2}$, and

$$h^{\frac{3}{2}}\mathrm{Op}_h^w\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial_\xi^\alpha p'(\xi))\Big) = h\mathrm{Op}_h^w\Big(\gamma_\alpha^k\Big(\frac{\xi-p'(\xi)}{\sqrt{h}}\Big)(\partial_\xi^\alpha p'(\xi))(x_k-p_k'(\xi))\Big)\widetilde{v}^\Sigma$$

with $\gamma_\alpha^k(z) := (\partial^\alpha\gamma)(z)z_k|z|^{-2}$, we obtain from inequalities (1.2.71) (resp. (1.2.70)) and (3.2.20) that $h\mathrm{Op}_h^w\big((\partial\gamma)\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\cdot\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)\big)\widetilde{v}^\Sigma$ (resp. $h^{3/2}\mathrm{Op}_h^w\big((\partial^\alpha\gamma)\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)(\partial_\xi^\alpha p'(\xi))\big)$, $|\alpha|=3$) is a remainder $R_2(\widetilde{v})$. The same holds true for $\mathrm{Op}_h^w\big(\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\big)\widetilde{v}^\Sigma$, as follows combining symbolic calculus and lemma 1.2.40, because its symbol is supported for large frequencies $|\xi| \gtrsim h^{-\sigma}$. From propositions 1.2.36 and 1.2.37 it immediately follows that $\mathrm{Op}_h^w(r_5)\widetilde{v}^\Sigma$ satisfies (3.2.23a) and (3.2.23b). □

Proposition 3.2.6 (Deduction of the ODE)**.** *There exists a family $(\theta_h(x))_h$ of C_0^∞ functions, real valued, equal to 1 on the closed ball $\overline{B_{1-ch^{2\sigma}}(0)}$ and supported in $\overline{B_{1-c_1h^{2\sigma}}(0)}$, for some small $0 < c_1 < c$, $\sigma > 0$, with $\|\partial_x^\alpha\theta_h\|_{L^\infty} = O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every k, such that*

$$\mathrm{Op}_h^w(x\cdot\xi - p(\xi))\widetilde{v}_{\Lambda_{kg}}^\Sigma = -\phi(x)\theta_h(x)\widetilde{v}_{\Lambda_{kg}}^\Sigma + R_2(\widetilde{v}), \tag{3.2.29}$$

where $\phi(x) = \sqrt{1-|x|^2}$ and $R_2(\widetilde{v})$ satisfies estimates (3.2.23)*. Therefore, $\widetilde{v}_{\Lambda_{kg}}^\Sigma$ is solution of the following non-homogeneous ODE:*

$$D_t\widetilde{v}_{\Lambda_{kg}}^\Sigma = -\phi(x)\theta_h(x)\widetilde{v}_{\Lambda_{kg}}^\Sigma + \Gamma^{kg}\mathrm{Op}_h^w(\Sigma(\xi))\big[h^{-1}r_{kg}^{NF}(t,tx)\big] + R_2(\widetilde{v}), \tag{3.2.30}$$

with r_{kg}^{NF} given by (3.1.5)*.*

Proof. The proof of the statement follows directly from lemma 3.2.4 if we observe that $\partial_\xi(x\cdot\xi - p(\xi)) = 0$ at $\xi = -d\phi(x)$ and $x\cdot(-d\phi(x)) - p(-d\phi(x)) = -\phi(x)$. Therefore, (3.2.29) holds and, injecting it in (3.2.28), we obtain (3.2.30). □

Proposition 3.2.7 (Propagation of the uniform estimate on V)**.** *Let us fix $K_1 > 0$. There exist two integers $n \gg \rho \gg 1$ sufficiently large, two constants $A, B > 1$ sufficiently large, $\varepsilon_0 \in]0, (2A+B)^{-1}[$ sufficiently small, and $0 \ll \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1$ small, such that, for any $0 < \varepsilon < \varepsilon_0$, if (u, v) is solution to (1.1.1)-(1.1.2) in some interval $[1, T]$ for a fixed $T > 1$, and $u_\pm, v_\pm$ defined in (1.1.5) satisfy a-priori estimates (1.1.11) for every $t \in [1, T]$, then it also verify (1.1.12b) in the same interval $[1, T]$.*

Proof. We warn the reader that, throughout the proof, we will denote by C, β (resp. β') two positive constants such that $\beta \to 0$ as $\sigma \to 0$ (resp. $\beta' \to 0$ as $\delta_0, \sigma \to 0$). These constants may change line after line. We also remind that $h = 1/t$.

In proposition 3.1.1 we introduced function v^{NF}, defined from v_- through (3.1.3), and proved that its $H^{\rho,\infty}$ norm differs from that of v_- by a quantity satisfying (3.1.7b). Hence, from a-priori estimates (1.1.11a), (1.1.11b), (1.1.11c) and for $\theta \in]0, 1[$ sufficiently small (e.g. $\theta < 1/4$)

$$\|v_-(t,\cdot)\|_{H^{\rho,\infty}} \leq \|v^{NF}(t,\cdot)\|_{H^{\rho,\infty}} + CA^{2-\theta}B^\theta\varepsilon^2 t^{-\frac{5}{4}}. \tag{3.2.31}$$

We successively introduced $\widetilde{v}$ in (3.2.2) and decomposed it into the sum of functions $\widetilde{v}^{\Sigma}_{\Lambda_{kg}}$ and $\widetilde{v}^{\Sigma}_{\Lambda^c_{kg}}$ (see (3.2.16)). We will show in lemma B.2.14 of appendix B that, for any $s \leq n$,

$$\|\widetilde{v}(t,\cdot)\|_{H^s_h} + \sum_{|\gamma|=1}^{2} \|\mathrm{Op}^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}(t,\cdot)\|_{L^2} \leq CB\varepsilon h^{-\beta'} \tag{3.2.32}$$

for all $t \in [1, T]$, so inequality (3.2.19b) gives that

$$\|\widetilde{v}^{\Sigma}_{\Lambda^c_{kg}}(t,\cdot)\|_{L^\infty} \leq CB\varepsilon h^{\frac{1}{2}-\beta'}. \tag{3.2.33}$$

As concerns $\widetilde{v}^{\Sigma}_{\Lambda_{kg}}$, we proved in proposition 3.2.6 that it is solution to ODE (3.2.30), with r^{NF}_{kg} given by (3.1.5) and satisfying (3.1.6), and $R_2(\widetilde{v})$ verifying (3.2.23). From (3.2.32), we then have that

$$\|R_2(\widetilde{v})(t,\cdot)\|_{L^\infty} \leq CB\varepsilon t^{-\frac{3}{2}+\beta'}.$$

We also have that

$$\Big\|\Gamma^{kg}\mathrm{Op}^w_h(\Sigma(\xi))[tr^{NF}_{kg}(t,tx)]\Big\|_{L^\infty(dx)} \leq C(A+B)AB\varepsilon^3 t^{-\frac{3}{2}+\beta'}. \tag{3.2.34}$$

In fact, by symbolic calculus of lemma 1.2.24 we derive that, for a fixed $N \in \mathbb{N}$ (e.g. $N > \rho$) and up to negligible multiplicative constants,

$$\Gamma^{kg}\mathrm{Op}^w_h(\Sigma(\xi)) = \sum_{|\alpha|=0}^{N-1} h^{\frac{|\alpha|}{2}}\mathrm{Op}^w_h\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\partial^\alpha\Sigma)(\xi)\Big) + \mathrm{Op}^w_h(r_N(x,\xi)),$$

where $r_N \in h^{\frac{N}{2}}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. Choosing N sufficiently large, we deduce from proposition 1.2.37, the fact that $\|tw(t,t\cdot)\|_{L^2} = \|w(t,\cdot)\|_{L^2}$, inequality (3.1.6a) and a-priori estimates, that for every $t \in [1, T]$

$$\Big\|\mathrm{Op}^w_h(r_N(x,\xi))[tr^{NF}_{kg}(t,tx)]\Big\|_{L^\infty(dx)} \leq CA^2B\varepsilon^3 t^{-2}.$$

Using, instead, proposition 1.2.39 with $p=+\infty$, inequality (B.3.25) in appendix B, and that $h=t^{-1}$, we deduce that

$$\sum_{|\alpha|=0}^{N-1} h^{\frac{|\alpha|}{2}} \left\| \mathrm{Op}_h^w\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\partial^\alpha\Sigma)(\xi)\Big)\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[tr_{kg}^{NF}(t,tx)]\right\|_{L^\infty}$$
$$\lesssim t^{1+\beta}\left\|\chi(t^{-\sigma}D_x)r_{kg}^{NF}(t,\cdot)\right\|_{L^\infty} \leq C(A+B)AB\varepsilon^3 t^{-\frac{3}{2}+\beta'}.$$

Summing up, $\Gamma^{kg}\mathrm{Op}_h^w(\Sigma(\xi))[t^{-1}r_{kg}^{NF}(t,tx)]+R_2(\widetilde{v})=F_{kg}(t,x)$ with

$$\|F_{kg}(t,\cdot)\|_{L^\infty}\leq [C(A+B)AB\varepsilon^3+CB\varepsilon]t^{-\frac{3}{2}+\beta'},$$

Using equation (3.2.30) we deduce that

$$\frac{1}{2}\partial_t|\widetilde{v}_{\Lambda_{kg}}^\Sigma(t,x)|^2=\Im\left(\widetilde{v}_{\Lambda_{kg}}^\Sigma\overline{D_t\widetilde{v}_{\Lambda_{kg}}^\Sigma}\right)\leq|\widetilde{v}_{\Lambda_{kg}}^\Sigma(t,x)||F_{kg}(t,x)| \tag{3.2.35}$$

and hence that

$$\begin{aligned}\|\widetilde{v}_{\Lambda_{kg}}^\Sigma(t,\cdot)\|_{L^\infty}&\leq\|\widetilde{v}_{\Lambda_{kg}}^\Sigma(1,\cdot)\|_{L^\infty}+\int_1^t\|F_{kg}(\tau,\cdot)\|_{L^\infty}d\tau\\&\leq\|\widetilde{v}_{\Lambda_{kg}}^\Sigma(1,\cdot)\|_{L^\infty}+C(A+B)AB\varepsilon^3+CB\varepsilon.\end{aligned}$$

As $\|\widetilde{v}_{\Lambda_{kg}}^\Sigma(1,\cdot)\|_{L^\infty}\lesssim\|\widetilde{v}(1,\cdot)\|_{L^2}\leq CB\varepsilon$ by proposition 1.2.37 and a-priori estimate (1.1.11c), the above inequality together with(3.2.33) and definition (3.2.2) of $\widetilde{v}$, gives that

$$\|v^{NF}(t,\cdot)\|_{L^\infty}\leq(C(A+B)AB\varepsilon^3+CB\varepsilon)t^{-1},$$

which injected in (3.2.31) leads finally to (1.1.12b) if we take $A>1$ sufficiently large such that $CB<\frac{A}{3K_1}$, and $\varepsilon_0>0$ sufficiently small to verify $C(A+B)B\varepsilon_0^2+CA^{1-\theta}B^\theta\varepsilon_0\leq\frac{1}{3K_1}$. □

3.2.2 The derivation of the transport equation

We now focus on the semi-classical wave equation satisfied by $\widetilde{u}$:

$$\left[D_t-\mathrm{Op}_h^w(x\cdot\xi-|\xi|)\right]\widetilde{u}(t,x)=h^{-1}\left[q_w(t,tx)+c_w(t,tx)+r_w^{NF}(t,tx)\right], \tag{3.2.36}$$

with q_w, c_w, r_w^{NF} given by (3.1.17), (3.1.18), (3.1.19) respectively, and on the derivation of the mentioned transport equation. As we will make use several times of proposition 1.2.30 and inequalities (1.2.52), we remind the reader about definition (1.2.40) of Ω_h and (1.2.49) of $\mathcal{M}_j$. Also, $\theta_0(x)$ denotes a smooth radial cut-off function (often coming with operator Ω_h) while $\chi\in C_0^\infty(\mathbb{R}^2)$ is equal to 1 in a neighbourhood of the origin and suitably supported.

In order to recover a sharp estimate for $\widetilde{u}$ such as (3.2.3a), we study the behaviour of this function separately in different regions of the phase space $(x,\xi)\in\mathbb{R}^2\times\mathbb{R}^2$. We start by fixing $\rho\in\mathbb{Z}$, and by introducing

$$\Sigma_j(\xi):=\begin{cases}\langle\xi\rangle^\rho, & \text{for } j=0,\\ \langle\xi\rangle^\rho\xi_j|\xi|^{-1}, & \text{for } j=1,2.\end{cases} \tag{3.2.37}$$

Taking a smooth cut-off function χ_0 equal to 1 in a neighbourhood of the origin, a Littlewood-Paley decomposition, and a small $\sigma>0$, we write the following for any $j\in\{0,1,2\}$:

(3.2.38)
$$\mathrm{Op}_h^w(\Sigma_j(\xi))\widetilde{u}=\mathrm{Op}_h^w(\Sigma_j(\xi)\chi_0(h^{-1}\xi))\widetilde{u}+\sum_k\mathrm{Op}_h^w\big(\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\widetilde{u}$$
$$+\mathrm{Op}_h^w(\Sigma_j(\xi)(1-\chi_0)(h^\sigma\xi))\widetilde{u},$$

observing that the sum over k is actually finite and restricted to set of indices $K := \{k \in \mathbb{Z} : h \lesssim 2^k \lesssim h^{-\sigma}\}$. From the classical Sobolev injection and the continuity on L^2 of the Riesz operator

$$\|\mathrm{Op}_h^w(\Sigma_j(\xi)\chi_0(h^{-1}\xi))\widetilde{u}(t,\cdot)\|_{L^\infty} = \|\Sigma_j(hD)\chi_0(D)\widetilde{u}(t,\cdot)\|_{L^\infty} \lesssim \|\widetilde{u}(t,\cdot)\|_{L^2}, \tag{3.2.39}$$

while from the semi-classical Sobolev injection along with lemma 1.2.40

$$\|\mathrm{Op}_h^w(\Sigma_j(\xi)(1-\chi_0)(h^\sigma\xi))\|_{L^\infty} \lesssim h^N\|\widetilde{u}(t,\cdot)\|_{H_h^s}, \tag{3.2.40}$$

where $N = N(s) \geq 0$ if $s > 0$ is sufficiently large. The remaining terms in the right hand side of (3.2.38), localised for frequencies $|\xi| \sim 2^k$, need a sharper analysis because a direct application of semi-classical Sobolev injection only gives that

$$\Big\|\mathrm{Op}_h^w\big(\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\widetilde{u}\Big\|_{L^\infty} \leq 2^k h^{-1-\mu}\|\widetilde{u}\|_{L^2},$$

with $\mu = \sigma\rho$ if $\rho \geq 0$, 0 otherwise, and factor $2^k h^{-1-\mu}$ may grow too much when $h \to 0$.

For any fixed $k \in K$, $\rho \in \mathbb{Z}$ and $j \in \{0,1,2\}$, let us introduce

$$\widetilde{u}^{\Sigma_j,k}(t,x) := \mathrm{Op}_h^w\big(\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\widetilde{u}(t,x) \tag{3.2.41}$$

and observe that, from the commutation of the above operator with the linear part of equation (3.2.36), we get that $\widetilde{u}^{\Sigma_j,k}$ is solution to

(3.2.42)
$$\begin{aligned}
&[D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|)]\widetilde{u}^{\Sigma_j,k}(t,x)\\
&= h^{-1}\mathrm{Op}_h^w\big(\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\left[q_w(t,tx) + c_w(t,tx) + r_w^{NF}(t,tx)\right]\\
&- ih\,\mathrm{Op}_h^w\big(\Sigma_j(\xi)(\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\varphi(2^{-k}\xi)\big)\widetilde{u} - i\sigma h\,\mathrm{Op}_h^w\big(\Sigma_j(\xi)\varphi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi))\cdot(h^\sigma\xi)\big)\widetilde{u}.
\end{aligned}$$

We introduce the following manifold (see picture 3.2)

$$\Lambda_w := \left\{(x,\xi) : x - \frac{\xi}{|\xi|} = 0\right\}, \tag{3.2.43}$$

together with operator

$$\Gamma^{w,k} := \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big), \tag{3.2.44}$$

for some $\gamma \in C_0^\infty(\mathbb{R}^2)$ equal to 1 close to the origin and $\psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ equal to 1 on $supp\varphi$, whose symbol is localized in a neighbourhood of $\Lambda_w \cap \{|\xi| \sim 2^k\}$ of size $h^{1/2-\sigma}$. We also define

$$\widetilde{u}_{\Lambda_w}^{\Sigma_j,k} := \Gamma^{w,k}\widetilde{u}^{\Sigma_j,k}, \tag{3.2.45a}$$

$$\widetilde{u}_{\Lambda_w^c}^{\Sigma_j,k} := \mathrm{Op}_h^w\Big(\big(1-\gamma\big)\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big)\widetilde{u}^{\Sigma_j,k}, \tag{3.2.45b}$$

so that $\widetilde{u}^{\Sigma_j,k} = \widetilde{u}_{\Lambda_w}^{\Sigma_j,k} + \widetilde{u}_{\Lambda_w^c}^{\Sigma_j,k}$. We are going to prove that, if we suitably control the L^2 norm of $(\theta_0\Omega_h)^\mu\mathcal{M}^\nu\widetilde{u}^{\Sigma_j,k}$, for any $\mu, |\nu| \leq 1$, then $\widetilde{u}_{\Lambda_w{}^c}^{\Sigma_j,k}$ is a $O_{L^\infty}(h^{-0})$ (see proposition 3.2.8). As $h = t^{-1}$, this means that $\widetilde{u}_{\Lambda_w^c}^{\Sigma_j,k}$ grows in time at a rate slower than the one expected for $\widetilde{u}^{\Sigma_j,k}$ (that is $t^{1/2}$ after (3.2.3a)). Analogously to the Klein-Gordon case discussed in the previous subsection, the main contribution to $\widetilde{u}^{\Sigma_j,k}$ is hence the one localized around Λ_w and represented by $\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}$. We will show that this function is solution to a transport equation (see proposition 3.2.17), from which we will be able to derive a suitable estimate of its uniform norm and to finally propagate (3.1.12) (see proposition 3.3.7).

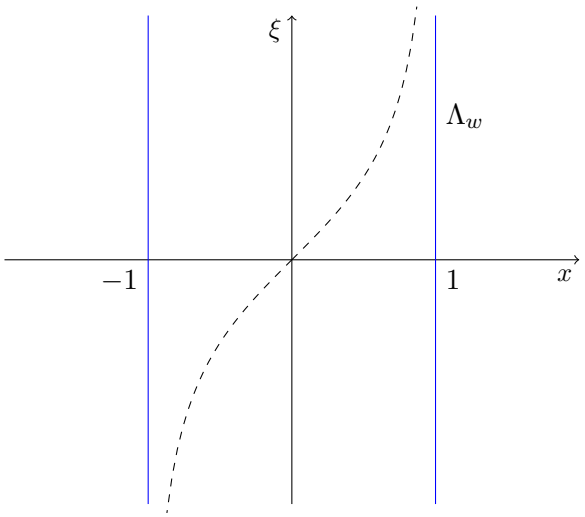

Figure 3.2: Lagrangian for the wave equation

Proposition 3.2.8. *There exists a constant $C > 0$ such that, for any $h \in]0,1]$, $k \in K$,*

$$\|\widetilde{u}^{\Sigma_j,k}_{\Lambda_w^c}(t,\cdot)\|_{L^2} \leq Ch^{\frac{1}{2}-\beta}\big(\|\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} + \|\mathcal{M}\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2}\big)\,, \tag{3.2.46a}$$

$$\|\widetilde{u}^{\Sigma_j,k}_{\Lambda_w^c}(t,\cdot)\|_{L^\infty} \leq Ch^{-\beta}\sum_{\mu=0}^{1}\big(\|(\theta_0\Omega_h)^\mu\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} + \|(\theta_0\Omega_h)^\mu\mathcal{M}\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2}\big)\,, \tag{3.2.46b}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Proof. The proof is straightforward if one writes

$$\widetilde{u}^{\Sigma_j,k}_{\Lambda_w^c} = \sum_{j=1}^{2}\mathrm{Op}_h^w\Big(\gamma_1^j\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi|-\xi_j}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big)\widetilde{u}^{\Sigma_j,k},$$

where $\gamma_1^j(z) := \frac{(1-\gamma)(z)z_j}{|z|^2}$ is such that $|\partial_z^\alpha\gamma_1^j(z)| \lesssim \langle z\rangle^{-(|\alpha|+1)}$, and uses inequalities (1.2.52) with $a(x) = b_p(\xi) \equiv 1$. □

Lemma 3.2.9. *Let $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ be such that $\widetilde{\varphi} \equiv 1$ on suppφ and have a sufficiently small support so that $\psi\widetilde{\varphi} \equiv \psi$. Then for any $k \in K$*

$$\Big[\Gamma^{w,k}, D_t - \mathrm{Op}_h^w\big((x\cdot\xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\Big]\,\mathrm{Op}_h^w(\varphi(2^{-k}\xi)) = \mathrm{Op}_h^w(b(x,\xi)), \tag{3.2.47}$$

where, for any $w \in L^2$ such that $\theta_0\Omega_h w, (\theta_0\Omega_h)^\mu\mathcal{M}w \in L^2(\mathbb{R}^2)$, for $\mu = 0,1$,

$$\|\mathrm{Op}_h^w(b(x,\xi))w\|_{L^2} \lesssim h^{1-\beta}\left(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\right), \tag{3.2.48a}$$

$$\|\mathrm{Op}_h^w(b(x,\xi))w\|_{L^\infty} \lesssim h^{1-\beta}\sum_{\mu=0}^{1}\big(\|(\theta_0\Omega_h)^\mu w\|_{L^2} + \|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2}\big), \tag{3.2.48b}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. We warn the reader that most of the terms arising from the development of the commutator in the left hand side of (3.2.47) satisfy a better L^2 estimate than (3.2.48a), namely

$$\|\cdot\|_{L^2} \lesssim h^{\frac{3}{2}-\beta}\big(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\big). \tag{3.2.49}$$

The only contribution whose L^2 norm is only a $O(h\|w\|_{L^2})$ is the integral remainder called $\widetilde{r}_N^k$, appearing in symbolic development (3.2.51).

Since $\partial_t = -h^2\partial_h$, an easy computation shows that

$$\begin{aligned}[\Gamma^{w,k}, D_t] =& \Big(\frac{1}{2}+\sigma\Big)\frac{h}{i}\mathrm{Op}_h^w\Big((\partial\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\cdot\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big)\\ &+\frac{h}{i}\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)\Big).\end{aligned} \tag{3.2.50}$$

The first term in the above right hand side satisfies (3.2.49) and (3.2.48b) after inequalities (1.2.52). The same estimates hold also for the latter one when it acts on $\mathrm{Op}_h^w(\varphi(2^{-k}\xi))w$, for the derivatives of ψ vanish on the support of $\widetilde{\varphi}$ (and then of φ) as a consequence of our assumptions. In fact, if we introduce a smooth function $\widetilde{\psi}\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, equal to 1 on the support of $\partial\psi$ and such that $supp\widetilde{\psi}\cap supp\varphi=\emptyset$, and use symbolic calculus we find that, for any fixed $N\in\mathbb{N}$,

$$\begin{aligned}&\mathrm{Op}_h^w\left(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)\right)\mathrm{Op}_h^w(\varphi(2^{-k}\xi))\\ &\qquad=\mathrm{Op}_h^w\left(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)\right)\mathrm{Op}_h^w\big(\widetilde{\psi}(2^{-k}\xi)\varphi(2^{-k}\xi)\big)-\mathrm{Op}_h^w(r_N^k),\end{aligned}$$

where the first term in the above right hand side is 0, and integral remainder r_N^k is given by

$$\begin{aligned}r_N^k=\Big(\frac{h}{2i}\Big)^N\sum_{|\alpha|=N}\frac{N(-1)^{|\alpha|}}{\alpha!(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\int_0^1\partial_x^\alpha\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)\Big]|_{(x+tz,\xi+t\zeta)}dt\\ \times\partial_\xi^\alpha\big(\widetilde{\psi}(2^{-k}\xi)\big)|_{(\xi+\eta)}\,dydzd\eta d\zeta.\end{aligned}$$

Developing explicitly the above derivatives and reminding definition (1.2.30) of integrals $I_{p,q}^k$, for general $k\in K$, $p,q\in\mathbb{Z}$, one recognizes that, up to some multiplicative constants, r_N^k has the form

$$h^{N-N(\frac{1}{2}-\sigma)}2^{-kN}I_{N,0}^k(x,\xi),$$

with $a,a',b_q\equiv 1$, $p=N$ and $\psi(2^{-k}\xi)$ replaced with $(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)$. Propositions 1.2.28 and 1.2.31 imply then that

$$\|\mathrm{Op}_h^w(r_N^k)\|_{\mathcal{L}(L^2)}+\|\mathrm{Op}_h^w(r_N^k)\|_{\mathcal{L}(L^2;L^\infty)}\lesssim h$$

if $N\in\mathbb{N}$ is chosen sufficiently large (e.g. $N>9$), which implies that the $\mathcal{L}(L^2)$ and $\mathcal{L}(L^2;L^\infty)$ norms of the latter operator in the right hand side of (3.2.50) is bounded by h^2.

As regards $[\Gamma^{w,k},\mathrm{Op}_h^w((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi))]$, we first remind that the symbolic development of a commutator's symbol only contains odd order terms. Consequently, for a new fixed $N\in\mathbb{N}$ and up to multiplicative constants independent of h,k, the symbol of the considered commutator writes as

$$\begin{aligned}&h\Big\{\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big),(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\Big\}\\ &\qquad+\sum_{\substack{3\le|\alpha|<N\\ |\alpha|=|\alpha_1|+|\alpha_2|}}h^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\big[(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big]+\widetilde{r}_N^k(x,\xi),\end{aligned} \tag{3.2.51}$$

with

$$\widetilde{r}_N^k(x,\xi)=\Big(\frac{h}{2i}\Big)^N\sum_{|\alpha_1|+|\alpha_2|=N}\frac{N(-1)^{|\alpha_1|}}{\alpha!(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\int_0^1\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big]\big|_{(x+tz,\xi+t\zeta)}dt$$
$$\times\,\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\big[(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big]\big|_{(x+y,\xi+\eta)}\,dydzd\eta d\zeta\,.$$

Since $\big\{\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big),x\cdot\xi-|\xi|\big\}=0$ the Poisson braket in the above sum reduces to

$$h\sum_{j,l}(\partial_j\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial_j\widetilde{\varphi})(2^{-k}\xi)(\frac{x_l|\xi|-\xi_l}{h^{1/2-\sigma}})(2^{-k}\xi_l)$$

and its quantization acting on $\mathrm{Op}_h^w(\varphi(2^{-k}\xi))w$ satisfies (3.2.49), (3.2.48b) because $\partial\widetilde{\varphi}$ vanishes on the support of φ.

An explicit calculation of terms of order $3\leq|\alpha|<N$, with the help of lemma 1.2.26 and the observation that $|\alpha_2|\leq 1$ because $(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)$ is affine in x, shows that they are linear combination of products

$$h^{|\alpha|-|\alpha|(\frac12-\sigma)}\gamma_{|\alpha|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)x^\nu b_1(\xi)$$

and

$$h^{|\alpha|-(|\alpha|-1)(\frac12-\sigma)}\widetilde{\gamma}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)b_0(\xi)$$

for two new cut-off functions $\widetilde{\gamma},\widetilde{\varphi}$, $|\partial^\beta b_0(\xi)|\lesssim_\beta|\xi|^{-|\beta|}$, and $\nu\in\mathbb{N}^2$ of length at most 1. Furthermore, for $j=1,2$,

$$h^{|\alpha|-|\alpha|(\frac12-\sigma)}\gamma_{|\alpha|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)x_jb_1(\xi)=h^{|\alpha|-(|\alpha|-1)(\frac12-\sigma)}\widetilde{\gamma}^j_{|\alpha|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)b_0(\xi)$$
$$+\,h^{|\alpha|-|\alpha|(\frac12-\sigma)}\gamma_{|\alpha|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)\xi_jb_0(\xi),$$

with $\widetilde{\gamma}^j_{|\alpha|}(z):=\gamma_{|\alpha|}(z)z_j$. From propositions 1.2.27, 1.2.30, the fact that $|\alpha|\geq3$ and $2^k\leq h^{-\sigma}$ we deduce that the quantization of these $|\alpha|$-order terms acting on $\mathrm{Op}_h^w(\varphi(2^{-k}\xi))w$ satisfies (3.2.49), (3.2.48b).

Finally, we notice that integral remainder $\widetilde{r}_N^k$ can be actually seen as the sum of two contributions, one of the form (1.2.44), the other like (1.2.45), with $a\equiv1$ and $p=1$. Lemma 1.2.32 implies then that the $\mathcal{L}(L^2)$ and $\mathcal{L}(L^2;L^\infty)$ norms of $\mathrm{Op}_h^w(\widetilde{r}_N^k)$ are bounded by h as foretold, which concludes the proof of the statement. □

Lemma 3.2.10. *Function $\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}$ is solution to the following equation:*

(3.2.52)
$$\begin{aligned}&\big[D_t-\mathrm{Op}_h^w\big((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\big]\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,x)=f_k^w(t,x)\\&+h^{-1}\Gamma^{w,k}\mathrm{Op}_h^w\big(\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\left[q_w(t,tx)+c_w(t,tx)+r_w^{NF}(t,tx)\right]\\&-ih\,\Gamma^{w,k}\mathrm{Op}_h^w\big(\Sigma_j(\xi)(\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\varphi(2^{-k}\xi)\big)\widetilde{u}\\&-i\sigma h\,\Gamma^{w,k}\mathrm{Op}_h^w\big(\Sigma_j(\xi)\varphi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi))\cdot(h^\sigma\xi)\big)\widetilde{u},\end{aligned}$$

where $\widetilde{\varphi}\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ is equal to 1 on suppφ, and there exist two constants $C,C'>0$ such that, for any $h\in]0,1]$, $k\in K$,

$$\|f_k^w(t,\cdot)\|_{L^2}\leq Ch^{1-\beta}\big(\|\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2}+\|\mathcal{M}\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2}\big)\,,\tag{3.2.53a}$$

(3.2.53b) $$\|f_k^w(t,\cdot)\|_{L^\infty} \leq C' h^{1-\beta} \sum_{\mu=0}^{1} \left(\|(\theta_0 \Omega_h)^\mu \widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} + \|(\theta_0 \Omega_h)^\mu \mathcal{M} \widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} \right),$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. If we consider a cut-off function $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\widetilde{\varphi} \equiv 1$ on the support of φ (φ being the truncation on $\widetilde{u}^{\Sigma_j,k}$'s frequencies), we have the exact equality

$$\mathrm{Op}_h^w(x \cdot \xi - |\xi|)\widetilde{u}^{\Sigma_j,k} = \mathrm{Op}_h^w((x \cdot \xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}^{\Sigma_j,k}.$$

Moreover, if we assume that its support is sufficiently small so that $\psi\widetilde{\varphi} \equiv \widetilde{\varphi}$, and apply operator $\Gamma^{w,k}$ to equation (3.2.42), lemma 3.2.9 gives us the result of the statement. □

The transport equation we talked about at the beginning of this section will be deduced from equation (3.2.52) by suitably developing symbol $(x \cdot \xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi)$. To do that, we first need to restrict the support of that symbol to bounded values of x through the introduction of a new cut-off function $\theta(x)$. We remind that Σ' is a concise notation that we use to indicate a linear combination of a finite number of terms of the same form.

Lemma 3.2.11. *Let $0 < D_1 < D_2$ and $\theta = \theta(x)$ be a smooth function equal to 1 for $|x| \leq D_1$ and supported for $|x| \leq D_2$. Then,*

(3.2.54)
$$\begin{aligned}\mathrm{Op}_h^w\big((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big) = \mathrm{Op}_h^w\big(\theta(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big) + (1-\theta)(x)\mathrm{Op}_h^w((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)) \\ + \sum{}'\widetilde{\theta}(x)\mathrm{Op}_h^w(\widetilde{\varphi}_1(2^{-k}\xi)) + \mathrm{Op}_h^w(r(x,\xi)),\end{aligned}$$

where $\widetilde{\theta}$ is a smooth function supported for $D_1 < |x| < D_2$, $\widetilde{\varphi}_1 \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$\|\mathrm{Op}_h^w(r)\|_{\mathcal{L}(L^2)} + \|\mathrm{Op}_h^w(r)\|_{\mathcal{L}(L^2;L^\infty)} \lesssim h.$$

Therefore, $\widetilde{u}_{\Lambda_{kg}}^{\Sigma_j,k}$ verifies

(3.2.55)
$$\begin{aligned}&\Big[D_t - \mathrm{Op}_h^w\big(\theta(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\Big]\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,x) = f_k^w(t,x) \\ &+ (1-\theta)(x)\mathrm{Op}_h^w((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k} + \sum{}'\widetilde{\theta}(x)\mathrm{Op}_h^w(\widetilde{\varphi}_1(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k} \\ &+ h^{-1}\Gamma^{w,k}\mathrm{Op}_h^w\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\left[q_w(t,tx) + c_w(t,tx) + r_w^{NF}(t,tx)\right] \\ &- ih\,\Gamma^{w,k}\mathrm{Op}_h^w\big(\Sigma(\xi)(\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\varphi(2^{-k}\xi)\big)\widetilde{u} \\ &- i\sigma h\,\Gamma^{w,k}\mathrm{Op}_h^w\big(\Sigma(\xi)\varphi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi))\cdot(h^\sigma\xi)\big)\widetilde{u},\end{aligned}$$

where f_k^w satisfies estimates (3.2.53).

Proof. Let $\theta(x)$ be the cut-off function of the statement. By proposition 1.2.21 we have that

(3.2.56)
$$\begin{aligned}&(1-\theta)(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi) = (1-\theta)(x)\sharp\left[(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\right] \\ &-\frac{h}{2i}\partial\theta(x)\cdot\Big(x-\frac{\xi}{|\xi|}\Big)\widetilde{\varphi}(2^{-k}\xi) - \frac{2^{-k}h}{2i}(x\cdot\xi-|\xi|)\partial\theta(x)\cdot(\partial\widetilde{\varphi})(2^{-k}\xi) + r_2^k(x,\xi) \\ &= (1-\theta)(x)\sharp\left[(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\right] - \frac{h}{2i}\Big[\partial\theta(x)\cdot x\Big]\sharp\,\widetilde{\varphi}(2^{-k}\xi) + \frac{h}{2i}\sum_{l=1}^{2}\partial_l\theta(x)\sharp\Big[\frac{\xi_l}{|\xi|}\widetilde{\varphi}(2^{-k}\xi)\Big] \\ &-\frac{h}{2i}\sum_{j,l=1}^{2}\Big[\partial_j\theta(x)x_l\Big]\sharp\Big[(\partial_j\widetilde{\varphi})(2^{-k}\xi)(2^{-k}\xi_l)\Big] + \frac{h}{2i}\sum_{l=1}^{2}\partial_l\theta(x)\sharp\Big[(2^{-k}|\xi|)(\partial_l\widetilde{\varphi})(2^{-k}\xi)\Big] + r_2^k(x,\xi) + \widetilde{r}_2^k(x,\xi),\end{aligned}$$

where $\partial\theta$ is supported for $D_1 < |x| < D_2$, and $r_2^k(t,x)$ (resp. $\widetilde{r}_2^k(t,x)$) is a linear combination of integrals of the form

$$\frac{h^2 2^{-k}}{(\pi h)^2} \int e^{\frac{2i}{h}\eta\cdot z} \int_0^1 \theta(x+tz)(1-t)^2 dt\ x^\nu \widetilde{\varphi}(2^{-k}(\xi+\eta)) dz d\eta,$$

with $|\nu| = 0, 1$ (resp. $|\nu| = 0$), for some new $\theta, \widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. By writing x as $(x+tz) - tz$, using that $ze^{\frac{2i}{h}\eta\cdot z} = \big(\frac{h}{2i}\big)\partial_\xi e^{\frac{2i}{h}\eta\cdot z}$, and making an integration by parts, one can express$r_2^k(t,x)$ as the sum over $|\nu| = 0, 1$ of integrals such as

$$\frac{h^2 2^{-k}(h2^{-k})^\nu}{(\pi h)^2} \int e^{\frac{2i}{h}\eta\cdot z} \int_0^1 \theta(x+tz) f(t) dt\ \widetilde{\varphi}(2^{-k}(\xi+\eta)) dz d\eta,$$

for some new smooth $\theta, f, \widetilde{\varphi}$, and show that for any $\alpha, \beta \in \mathbb{N}^2$

$$\big|\partial_x^\alpha \partial_\xi^\beta \big[(r_2^k + \widetilde{r}_2^k)(x, h\xi)\big]\big| \lesssim_{\alpha,\beta} h^2 2^{-k} \lesssim_{\alpha,\beta} h.$$

Thus $(r_2^k + \widetilde{r}_2^k)(x, h\xi) \in hS_0(1)$, which means, by classical results on pseudo-differential operators (see for instance [11]), that

$$\mathrm{Op}_h^w((r_2^k + \widetilde{r}_2^k)(x,\xi)) = Op^w((r_2^k + \widetilde{r}_2^k)(x, h\xi)) \in \mathcal{L}(L^2)$$

with norm $O(h)$. Furthermore, one can also show that $\|\mathrm{Op}_h^w(r_2^k + \widetilde{r}_2^k)\|_{\mathcal{L}(L^2;L^\infty)} \lesssim h$ using lemma 1.2.25 and the fact that, by making some integrations by parts, for any multi-indices $\alpha, \beta \in \mathbb{N}^2$ and a new $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$

$$\left\| \partial_y^\alpha \partial_\xi^\beta \left[(r_2^k + \widetilde{r}_2^k)\Big(\frac{x+y}{2}, h\xi\Big)\right] \right\|_{L^2(d\xi)} \lesssim h^2 2^{-k} \left\| \int \langle\eta\rangle^{-3} |\widetilde{\varphi}(2^{-k}h(\xi+\eta))| d\eta \right\|_{L^2(d\xi)} \lesssim h.$$

These considerations, along with the continuity of $\Gamma^{w,k}$ on L^2, uniformly in h and k (see proposition 1.2.27), imply that $\mathrm{Op}_h^w(r_2^k + \widetilde{r}_2^k)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}$ is a remainder f_k^w. □

Lemma 3.2.12. *We have that*

$$|\xi| - x\cdot\xi = \frac{1}{2}(1-|x|^2)x\cdot\xi + e(x,\xi)$$

with

$$e(x,\xi) = \frac{1}{2}|\xi|\Big|x - \frac{\xi}{|\xi|}\Big|^2 + \frac{1}{2}\Big(\Big(x - \frac{\xi}{|\xi|}\Big)\cdot\xi\Big)\Big(x - \frac{\xi}{|\xi|}\Big)\cdot\Big(x + \frac{\xi}{|\xi|}\Big). \tag{3.2.57}$$

Proof.

$$\begin{aligned}
|\xi| - x\xi &= \frac{1}{2}|\xi|\left|x - \frac{\xi}{|\xi|}\right|^2 + \frac{1}{2}|\xi|(1-|x|^2) \\
&= \frac{1}{2}|\xi|\left|x - \frac{\xi}{|\xi|}\right|^2 + \frac{1}{2}(|\xi| - x\cdot\xi)(1-|x|^2) + \frac{1}{2}(1-|x|^2)x\cdot\xi \\
&= \underbrace{\frac{1}{2}|\xi|\left|x - \frac{\xi}{|\xi|}\right|^2 + \frac{1}{2}\left(\left(\frac{\xi}{|\xi|} - x\right)\cdot\xi\right)\left(\frac{\xi}{|\xi|} - x\right)\cdot\left(\frac{\xi}{|\xi|} + x\right)}_{e(x,\xi)} + \frac{1}{2}(1-|x|^2)x\cdot\xi\,.
\end{aligned}$$

Lemma 3.2.13. *Let $\gamma, \theta \in C_0^\infty(\mathbb{R}^2)$ and $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ be such that $\widetilde{\varphi} \equiv 1$ on the support of φ and have a sufficiently small support so that $\psi\widetilde{\varphi} \equiv \widetilde{\varphi}$. Let also*

$$B(x,\xi) := \gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)\theta(x)\Big(x_m - \frac{\xi_m}{|\xi|}\Big), \qquad m \in \{1,2\}. \tag{3.2.58}$$

For any function $w \in L^2(\mathbb{R}^2)$ such that $\mathcal{M}w \in L^2(\mathbb{R}^2)$, any $m, n \in \{1,2\}$,

$$\Big\|\mathrm{Op}_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_n|\xi| - \xi_n)\Big)\Gamma^{w,k}w\Big\|_{L^2} \lesssim h^{1-\beta}\big(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\big), \tag{3.2.59a}$$

$$\Big\|\mathrm{Op}_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_n|\xi| - \xi_n)\Big)\Gamma^{w,k}w\Big\|_{L^\infty} \lesssim h^{1-\beta}\big(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\big) + h^{-\beta}\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathcal{M}w\|_{L^2}, \tag{3.2.59b}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. After lemma 1.2.35 with $p = 0$ we have that

$$\mathrm{Op}_h^w\left(\theta(x)\widetilde{\varphi}(2^{-k}\xi)\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_n|\xi| - \xi_n)\right)\Gamma^{w,k}w = \mathrm{Op}_h^w\left(B(x,\xi)(x_n|\xi| - \xi_n)\right)w + \mathrm{Op}_h^w(r_0^k(x,\xi))w,$$

and the L^2 (resp. L^∞) norm of the latter term in the above right hand side is bounded by the right hand side of (3.2.59a) (resp. of (3.2.59b)) after inequality (1.2.60a) (resp. (1.2.60b)). Moreover, the L^2 norm of $\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w$ is also bounded by the right hand side of (3.2.59a) as straightly follows from emma 1.2.33. It only remains to prove that the L^∞ norm of this term is bounded by the right hand side of (3.2.59b).

We first consider a new cut-off function $\widetilde{\varphi}_1 \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, equal to 1 on $supp\widetilde{\varphi}$ so that its derivatives vanish against φ, and use symbolic calculus to write

$$\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big) = \mathrm{Op}_h^w(\widetilde{\varphi}_1(2^{-k}\xi))\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big) + \mathrm{Op}_h^w(r_{N,1}^k(x,\xi)),$$

where $r_{N,1}^k(x,\xi)$ is obtained using (1.2.20). Up to interchange the role of variables y and z (resp. η and ζ) and to consider $e^{\frac{2i}{h}(y\cdot\zeta - \eta\cdot\zeta)}$ instead of $e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}$ (which does not affect estimate (1.2.46)), $r_{N,1}^k$ is analogous to integral (1.2.45) with $p = 1$. Therefore, if $N \in \mathbb{N}$ is chosen sufficiently large (e.g. $N > 11$), lemma 1.2.32 implies that $\|\mathrm{Op}_h^w(r_{N,1}^k)\|_{\mathcal{L}(L^2;L^\infty)} = O(h)$.

Since $\widetilde{\varphi}_1$ localises frequencies ξ in an annulus, the classical Sobolev injection gives that

$$\begin{aligned}\Big\|\mathrm{Op}_h^w(\widetilde{\varphi}_1(2^{-k}\xi))\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w\Big\|_{L^\infty} \\ \lesssim \log h\,\Big\|\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w\Big\|_{L^2} + \Big\|D_x\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w\Big\|_{L^2}.\end{aligned}$$

As previously said, the former norm in the above right hand side satisfies inequality (3.2.59a). As concerns the latter one, we remark that thanks to the specific structure of symbol $B(x,\xi)$ its first derivative with respect to x does not lose any factor $h^{-1/2+\sigma}$, as

$$\partial_x\left[\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\right]\widetilde{\varphi}(2^{-k}\xi)\theta(x)\Big(x_m - \frac{\xi_m}{|\xi|}\Big) = (\partial\gamma)\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)\theta(x)\Big(\frac{x_m|\xi| - \xi_m}{h^{1/2-\sigma}}\Big). \tag{3.2.60}$$

Consequently, by using symbolic calculus we derive that

$$\begin{aligned}D_x\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w = h^{-1}\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\xi\big)w \\ + {\sum}'\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)a(x)b_0(\xi)(x_j|\xi| - \xi_j)\Big)w,\end{aligned}$$

where $\sum'$ is a concise notation to indicate linear combinations, $j \in \{m, n\}$ and $\gamma, \widetilde{\varphi}, a$ are some new smooth functions with $a(x)$ compactly supported. Again by lemma 1.2.33 the L^2 norms of latter contributions in the above right hand side are bounded by $h^{1-\beta}(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2})$.

Finally, we observe that symbol $B(x,\xi)\xi$ can be seen as

$$\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)(x_m|\xi| - \xi_m)\widetilde{\varphi}(2^{-k}\xi)\theta(x)b_0(x), \tag{3.2.61}$$

which implies, after lemma 1.2.34, that

$$h^{-1}\mathrm{Op}_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\xi\big)w = \mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathcal{M}_n w + O_{L^2}(h^{1-\beta}(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2})).$$

□

Lemma 3.2.14. *Let $e(x,\xi)$ be the symbol defined in (3.2.57), $\theta \in C_0^\infty(\mathbb{R}^2)$, and $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2\backslash\{0\})$ with sufficiently small support so that $\psi\widetilde{\varphi} \equiv \widetilde{\varphi}$. If a-priori estimates (1.1.11) are satisfied for every $t \in [1,T]$, for some fixed $T > 1$, there exists a constant $C > 0$ such that*

$$\Big\|\mathrm{Op}_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\Big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,\cdot)\Big\|_{L^2} + \Big\|\mathrm{Op}_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\Big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,\cdot)\Big\|_{L^\infty} \leq CB\varepsilon h^{1-\beta'} \tag{3.2.62}$$

for every $t \in [1,T]$, with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. We warn the reader that, throughout this proof, C, β and β' will denote three positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$ (resp. $\beta' \to 0$ as $\sigma, \delta_1 \to 0$).

Since symbol $e(x,\xi)$ writes as

$$e(x,\xi) = \frac{1}{2}\sum_{m=1}^{2}\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_m|\xi| - \xi_m) + \frac{1}{2}\sum_{m,n=1}^{2}\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_n|\xi| - \xi_n)\Big(\frac{\xi_m}{|\xi|}\frac{\xi_n}{|\xi|} + x_n\frac{\xi_m}{|\xi|}\Big),$$

it follows that the L^2 norm of $\mathrm{Op}_h^w\big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}$ satisfies inequality (3.2.62) after lemmas 3.2.13 and B.2.1 in appendix B. Moreover, from lemma 3.2.13

$$\begin{aligned}\Big\|\mathrm{Op}_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\Big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}\Big\|_{L^\infty} \lesssim\ & h^{1-\beta}\Big(\|\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} + \|\mathcal{M}\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2}\Big)\\ & + h^{-\beta}\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathcal{M}\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2},\end{aligned}$$

with $B(x,\xi)$ defined in (3.2.58). The aim of the proof is then to show that the L^2 norm of $\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathcal{M}\widetilde{u}^{\Sigma_j,k}$ is estimated by the right hand side of (3.2.62).

First of all, we remind that $B(x,\xi)\xi$ can be seen as a symbol of the form (3.2.61). From proposition 1.2.27 we hence have that

$$\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\|_{\mathcal{L}(L^2)} = O(h^{\frac{1}{2}-\beta}), \tag{3.2.63a}$$

while from inequality (1.2.52a)

$$\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)w\|_{L^2} \lesssim h^{1-\beta}(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}). \tag{3.2.63b}$$

We also recall definition (3.2.41) of $\widetilde{u}^{\Sigma_j,k}$, use the concise notation $\phi_k^j(\xi)$ for its symbol $\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)$, and observe that

$$\begin{aligned}\Big[\mathcal{M}_n, \mathrm{Op}_h^w(\phi_k^j(\xi))\Big] &= -\frac{1}{2i}\mathrm{Op}_h^w\big(|\xi|\partial_n\phi_k^j(\xi)\big),\\ \Big\|\Big[\mathcal{M}_n, \mathrm{Op}_h^w(\phi_k^j(\xi))\Big]\Big\|_{\mathcal{L}(L^2)} &= O(h^{-\sigma}),\end{aligned} \tag{3.2.64}$$

after propositions 1.2.21 and 1.2.27.

Using (3.2.64) and recalling relation (3.2.9a), we find that for any $n = 1, 2$,

$$\begin{aligned}\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathcal{M}_n\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} &\lesssim \|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathrm{Op}_h^w(\phi_k^j(\xi))[t(Z_nu^{NF})(t,tx)]\|_{L^2(dx)}\\ &+ \Big\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathrm{Op}_h^w\big(\xi_n|\xi|^{-1}\phi_k^j(\xi)\big)\widetilde{u}(t,\cdot)\Big\|_{L^2} + \Big\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathrm{Op}_h^w\big(|\xi|\partial_n\phi_k^j(\xi)\big)\widetilde{u}(t,\cdot)\Big\|_{L^2}\\ &+ \Big\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathrm{Op}_h^w(\phi_k^j(\xi))\left[t(tx_n)\left[q_w(t,tx)+c_w(t,tx)+r_w^{NF}(t,tx)\right]\right]\Big\|_{L^2(dx)},\end{aligned}$$

with u^{NF} defined in (3.1.15), q_w, c_w and r_w^{NF} given by (3.1.17), (3.1.18) and (3.1.19) respectively. Evidently, after (3.2.63b) and a further commutation of $\mathcal{M}$ with $\mathrm{Op}_h^w\big(\xi_n|\xi|^{-1}\phi_k^j(\xi)\big)$ and $\mathrm{Op}_h^w\big(|\xi|\partial_n\phi_k^j(\xi)\big)$ respectively, the second and third L^2 norm in the above right hand side are estimated by

$$h^{1-\beta}(\|\widetilde{u}(t,\cdot)\|_{L^2} + \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}),$$

for some $\chi \in C_0^\infty(\mathbb{R}^2)$. They are hence bounded by $CB\varepsilon h^{1-\beta'}$ by lemma B.2.1.

• **Estimate of $\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\mathrm{Op}_h^w(\phi_k^j(\xi))[t(Z_nu^{NF})(t,tx)]\|_{L^2}$:** This L^2 norm is basically estimated in terms of the L^2 norm of $(Z^\mu u)_-$, for $|\mu| \leq 2$. In fact, after definition (3.1.15) and equality (2.1.15a)

$$\begin{aligned}(3.2.65)\quad (Z_nu^{NF})(t,tx) = (Z_nu)_-(t,tx) + \Big(\frac{D_n}{|D_x|}u_-\Big)(t,tx)\\ - \frac{i}{4(2\pi)^2}\sum_{l\in\{+,-\}}\Big[Z_n\int e^{iy\cdot\xi}D_l(\xi,\eta)\hat{v}_l(\xi-\eta)\hat{v}_l(\eta)d\xi d\eta\Big]\big|_{y=tx},\end{aligned}$$

with D_l given by (3.1.14). On the one hand, taking a new smooth cut-off function θ_1 equal to 1 on the support of θ, using (1.2.50) with $\widetilde{a} = \theta_1$, together with (1.2.51a), proposition 1.2.27, and (3.2.64), we deduce that

$$\begin{aligned}&\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\mathrm{Op}_h^w(\phi_k^j(\xi))[t(Z_nu)_-(t,tx)]\|_{L^2(dx)}\\ &\qquad\lesssim \sum_{m=1}^{2} h\|\theta_1(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\mathcal{M}_m[t(Z_nu)_-(t,tx)]\|_{L^2(dx)} + h^{1-\beta}\|(Z_nu)_-(t,\cdot)\|_{L^2}.\end{aligned}$$

After relation (3.2.10),

$$\begin{aligned}\|\theta_1(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\mathcal{M}_m[t(Z_nu)_-(t,tx)]\|_{L^2} &\lesssim \|(Z_mZ_nu)_-(t,\cdot)\|_{L^2} + \|(Z_nu)_-(t,\cdot)\|_{L^2}\\ &+ \Big\|\theta_1\Big(\frac{x}{t}\Big)\phi_k^j(D_x)\left[x_mZ_nNL_w\right](t,\cdot)\Big\|_{L^2}.\end{aligned}$$

Moreover,

$$\theta_1\Big(\frac{x}{t}\Big)\phi_k^j(D_x)x_m = t\theta_{1,m}\Big(\frac{x}{t}\Big)\phi_k^j(D_x) + \theta_1\Big(\frac{x}{t}\Big)[\phi_k^j(D_x), x_m],$$

where $\theta_{1,m}(z) = \theta_1(z)z_m$, and $[\phi_k^j(D_x), x_m]$ is a bounded operator on L^2 with norm $O(t)$, as one can check computing its associated symbol and using that $2^{-k} \lesssim h^{-1} = t$. Therefore, using also

inequality (3.2.11) together with a-priori estimates (1.1.11) we deduce that

(3.2.66)
$$\begin{aligned}
&\Big\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\mathrm{Op}_h^w(\phi_k^j(\xi))\Big[t(Z_nu)_-(t,tx)\Big]\Big\|_{L^2(dx)}\\
&\lesssim \sum_{|\mu|=1}^{2} h\|(Z^\mu u)_-(t,\cdot)\|_{L^2}+\|Z_nV(t,\cdot)\|_{H^1}\|V(t,\cdot)\|_{H^{2,\infty}}+\big[\|V(t,\cdot)\|_{H^1}\\
&+\|V(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{1,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\right)+\|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\big]\|V(t,\cdot)\|_{H^{1,\infty}}\\
&\leq CB\varepsilon h^{1-\frac{\delta_1}{2}}.
\end{aligned}$$

On the other hand, it is a straight consequence of (3.2.63b), (3.2.64) and lemma B.2.1 that

$$\begin{aligned}
(3.2.67)\quad &\Big\|\mathrm{Op}_h^w\big(B(x,\xi)\xi\mathrm{Op}_h^w(\phi_k^j(\xi))[t(D_n|D_x|^{-1}u)_-(t,tx)]\Big\|_{L^2}\\
&\qquad\lesssim h^{1-\beta}(\|\widetilde{u}(t,\cdot)\|_{L^2}+\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2})\leq CB\varepsilon h^{1-\frac{\delta_2}{2}}.
\end{aligned}$$

Finally, by symbolic calculus and (3.2.60) we have that

$$(3.2.68)\qquad \mathrm{Op}_h^w(B(x,\xi)\xi)=\mathrm{Op}_h^w(B(x,\xi))(hD_x)+\frac{h}{2i}\mathrm{Op}_h^w\big(\partial_xB(x,\xi)\big),$$

where ∂_xB is of the form

$$(3.2.69)\qquad \gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)\theta(x)b_0(\xi)$$

for some new $\gamma,\theta\in C_0^\infty(\mathbb{R}^2)$. Consequently, by proposition 1.2.27

$$\begin{aligned}
(3.2.70)\quad &\Big\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))\Big[tZ_n\int e^{iy\cdot\xi}D_l(\xi,\eta)\hat{v}_l(\xi-\eta)\hat{v}_l(\eta)d\xi d\eta\Big]\big|_{y=tx}\Big\|_{L^2(dx)}\\
&\lesssim \Big\|\chi(t^{-\sigma}D_x)D_xZ_n\int e^{ix\cdot\xi}D_l(\xi,\eta)\hat{v}_l(\xi-\eta)\hat{v}_l(\eta)d\xi d\eta\Big\|_{L^2(dx)}\\
&\qquad+h\Big\|\chi(t^{-\sigma}D_x)Z_n\int e^{ix\cdot\xi}D_l(\xi,\eta)\hat{v}_l(\xi-\eta)\hat{v}_l(\eta)d\xi d\eta\Big\|_{L^2(dx)}
\end{aligned}$$

and the above right hand side is bounded by

$$\begin{aligned}
&h^{-\beta}\left(\|V(t,\cdot)\|_{H^1}+\|V(t,\cdot)\|_{L^2}(\|U(t,\cdot)\|_{H^{1,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}})+\|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\right)\\
&\qquad\times(\|V(t,\cdot)\|_{H^{14,\infty}}+h\|V(t,\cdot)\|_{H^{13}})+h^{-\beta}\|Z_nV(t,\cdot)\|_{L^2}\|V(t,\cdot)\|_{H^{17,\infty}}
\end{aligned}$$

after inequalities (A.37b), (A.37c) and (B.1.6a) with $s=0$. From a-priori estimates (1.1.11) we then deduce that the left hand side of (3.2.70) is bounded by $CB\varepsilon h^{1-\beta'}$, which implies, together with equality (3.2.65) and estimates(3.2.66), (3.2.67), that the L^2 norm of contribution $\mathrm{Op}_h^w\big(B(x,\xi)\xi\mathrm{Op}_h^w(\phi_k^j(\xi))[t(Z_nu^{NF})(t,tx)]$ is estimated with the right hand side of (3.2.62).

• **Estimate of** $\|\mathrm{Op}_h^w(B(x,\xi)\xi)\left[t(tx_n)q_w(t,tx)\right]\|_{L^2(dx)}$: After definition (3.1.17) of $q_w(t,x)$ and (3.2.2) of $\widetilde{v}$, we first notice that

$$(3.2.71)\qquad tq_w(t,tx)=\frac{h}{2}\Im\left[\overline{\widetilde{v}}\,\mathrm{Op}_h^w(\xi_1)\widetilde{v}-\overline{\mathrm{Op}_h^w\Big(\frac{\xi_1}{\langle\xi\rangle}\Big)\widetilde{v}}\cdot\mathrm{Op}_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)\widetilde{v}\right](t,x)=:\widetilde{q}_w(t,x),$$

where

$$(3.2.72)\qquad \|\widetilde{q}_w(t,\cdot)\|_{L^2}\lesssim h\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}}\|\widetilde{v}(t,\cdot)\|_{H^1}.$$

Then

$$\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))\,[t(tx_n)q_w(t,tx)]\,\|_{L^2(dx)} = h^{-1}\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))\,[x_n\widetilde{q}_w(t,x)]\,\|_{L^2(dx)}.$$

Since $B(x,\xi)$ is compactly supported in x and

$$\Big\|\Big[\mathrm{Op}_h^w\big(B(x,\xi)\xi\big)\mathrm{Op}_h^w(\phi_k^j(\xi)),x_n\Big]\Big\|_{\mathcal{L}(L^2)} = O(h^{\frac{1}{2}-\beta}),$$

as follows from symbolic calculus, (3.2.63a), equality (1.2.25) and proposition 1.2.27, we can morally reduce ourselves to the study of the L^2 norm of

$$h^{-1}\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))\widetilde{q}_w(t,x)$$

up to a $O_{L^2}(h^{-1/2-\beta}\|\widetilde{q}_w\|_{L^2})$. Using (3.2.68), (3.2.69), together with proposition 1.2.27, we deduce that

$$h^{-1}\Big\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))\widetilde{q}_w(t,\cdot)\Big\|_{L^2} \lesssim h^{-1}\|\mathrm{Op}_h^w(\phi_k^j(\xi))(hD_x)\widetilde{q}_w(t,\cdot)\|_{L^2} + \|\widetilde{q}_w(t,\cdot)\|_{L^2},$$

so from lemma 3.2.15 below, estimates (3.2.72), (3.2.3b), and lemmas B.2.14, B.3.7 in appendix B, we conclude that

$$\begin{aligned}(3.2.73)\quad & h^{-1}\|\mathrm{Op}_h^w(\phi_k^j(\xi))(hD_x)\widetilde{q}_w(t,\cdot)\|_{L^2}\\ &\lesssim h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H^s} + \sum_{|\mu|=1}^{2}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}} \leq CB\varepsilon h^{1-\beta'};\end{aligned}$$

• **Estimate of** $\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))(t(tx_n)c_w(t,tx)\|_{L^2(dx)}$: As for the previous estimate, we can reduce to the study of the L^2 norm of

$$\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))[t^2c_w(t,tx)],$$

up to a $O_{L^2}\big(h^{-1/2-\beta}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))[tc_w(t,tx)]\|_{L^2(dx)}\big)$ for some $\chi\in C_0^\infty(\mathbb{R}^2)$. So using (3.2.63a), the fact that $\|tw(t,t\cdot)\|_{L^2} = \|w(t,\cdot)\|_{L^2}$, and (3.1.21a) with $s>0$ sufficiently large so that $N(s)>2$, we obtain that for a new $\chi_1\in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned}&\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))[t^2c_w(t,t\cdot)]\|_{L^2} \lesssim h^{-\frac{1}{2}-\beta}\big\|\chi(t^{-\sigma}D_x)c_w(t,\cdot)\big\|_{L^2}\\ &\lesssim h^{-\frac{1}{2}-\beta}\big\|\chi_1(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\big\|_{L^2}\big(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\big)\\ &+ h^{\frac{3}{2}}\big\|(v^{NF}-v_-)(t,\cdot)\big\|_{H^1}\big(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\big).\end{aligned}\tag{3.2.74}$$

Then inequalities (3.1.7a) with $s=1$ and (3.1.8a), together with a-priori estimates, give that

$$\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))[t^2c_w(t,t\cdot)]\|_{L^2} \leq CB\varepsilon h^{1-\beta'}.$$

• **Estimate of** $\|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))(t(tx_n)r_w^{NF}(t,tx)\|_{L^2(dx)}$: Analogously, from (3.1.22a) and (1.1.11) we obtain that

$$\begin{aligned}(3.2.75)\quad \|\mathrm{Op}_h^w(B(x,\xi)\xi)\mathrm{Op}_h^w(\phi_k^j(\xi))[t^2r_w^{NF}(t,t\cdot)]\|_{L^2} &\lesssim h^{-\frac{1}{2}-\beta}\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{L^2}\\ &\lesssim h^{-\frac{1}{2}-\beta}\|V(t,\cdot)\|_{H^{13,\infty}}^2\|U(t,\cdot)\|_{H^1} \leq CB\varepsilon h^{\frac{3}{2}-\beta'}.\end{aligned}$$

□

Lemma 3.2.15. *Let $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, $k \in K$ and $a_j(\xi)$ be two smooth real symbols of order $j = 0, 1$. Then*

$$\left\| \mathrm{Op}_h^w(\varphi(2^{-k}\xi))(hD_x) \left[\overline{a_0(hD_x)\widetilde{v}}\, a_1(hD_x)\widetilde{v} \right](t,\cdot) \right\|_{L^2} \lesssim h^{1-\beta} \Big(\|\widetilde{v}(t,\cdot)\|_{H_h^s} + \sum_{|\mu|=1}^{2} \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2} \Big) \|\widetilde{v}(t,\cdot)\|_{H_h^{1,\infty}}. \tag{3.2.76}$$

Proof. Let us split both $\widetilde{v}$ in the left hand side of (3.2.76) into the sum $\widetilde{v}_{\Lambda_{kg}} + \widetilde{v}_{\Lambda_{kg}^c}$, with $\widetilde{v}_{\Lambda_{kg}}, \widetilde{v}_{\Lambda_{kg}^c}$ introduced in (3.2.16) with $\Sigma_j \equiv 1$. Remind that $\widetilde{v}_{\Lambda_{kg}^c}$ satisfies inequality (3.2.19a) and that

$$\|a_0(hD_x)\widetilde{v}(t,\cdot)\|_{L^\infty} + \|a_0(hD_x)\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\|_{L^\infty} \lesssim h^{-\beta}\|\widetilde{v}(t,\cdot)\|_{H_h^{1,\infty}},$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$, as follows from lemma 1.2.39 with $p = +\infty$ and the uniform continuity of $a_0(hD_x)$ from $H^{1,\infty}$ to L^∞. Therefore, using the continuity on L^2 of $\mathrm{Op}_h^w(\varphi(2^{-k}\xi))(hD_x)$ with norm $O(2^k)$ and the fact that $2^k \lesssim h^{-\sigma}$ we deduce that, for any $w_1, w_2 \in \{\widetilde{v}, \widetilde{v}_{\Lambda_{kg}}, \widetilde{v}_{\Lambda_{kg}^c}\}$ with at least one w_j equal to $\widetilde{v}_{\Lambda_{kg}^c}$,

$$\left\| \mathrm{Op}_h^w(\varphi(2^{-k}\xi))(hD_x) \left[\overline{a_0(hD_x)w_1} a_1(hD_x)w_2 \right] \right\|_{L^2} \lesssim h^{1-\beta} \Big(\|\widetilde{v}(t,\cdot)\|_{H_h^s} + \sum_{|\mu|=1}^{2} \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2} \Big) \|\widetilde{v}(t,\cdot)\|_{H_h^{1,\infty}}.$$

We are thus reduced to proving inequality (3.2.76) for

$$\left\| \mathrm{Op}_h^w(\varphi(2^{-k}\xi))(hD_x) \left[\overline{a_0(hD_x)\widetilde{v}_{\Lambda_{kg}}}\, a_1(hD_x)\widetilde{v}_{\Lambda_{kg}} \right](t,\cdot) \right\|_{L^2}.$$

Furthermore, by means of lemma 3.2.4 we can replace the action of $a_j(hD_x)$ in the above L^2 norm, for $j = 0, 1$, with the multiplication operator by a real function, up to new remainders bounded in L^2 by the right hand side of (3.2.76). In fact,

$$a_j(hD_x)\widetilde{v}_{\Lambda_{kg}} = \theta_h(x) a_j(-d\phi(x))\widetilde{v}_{\Lambda_{kg}} + R_1(\widetilde{v}), \quad j = 0, 1,$$

where θ_h is a smooth cut-off function as in the statement of lemma 3.2.4 and $R_1(\widetilde{v})$ satisfies (3.2.22a). Now

$$hD_x|\widetilde{v}_{\Lambda_{kg}}|^2 = \left[\mathrm{Op}_h^w(\xi + d\phi(x)\theta_h(x))\widetilde{v}_{\Lambda_{kg}}\right]\overline{\widetilde{v}_{\Lambda_{kg}}} - \widetilde{v}_{\Lambda_{kg}} \left[\overline{\mathrm{Op}_h^w(\xi + d\phi(x)\theta_h(x))\widetilde{v}_{\Lambda_{kg}}}\right],$$

and from lemma 3.2.16 below

$$\|\mathrm{Op}_h^w(\xi + d\phi(x)\theta_h(x))\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\|_{L^2} \lesssim h^{1-\beta} \sum_{|\mu|=0}^{1} \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2}.$$

This implies, after having applied the Leibniz rule and proposition 1.2.39, that

$$\left\| hD_x \left[a_0(-d\phi(x)) a_1(-d\phi(x))\theta_h^2(x)|\widetilde{v}_{\Lambda_{kg}}|^2(t,\cdot) \right] \right\|_{L^2} \lesssim h^{1-\beta} \Big(\|\widetilde{v}(t,\cdot)\|_{H_h^s} + \sum_{|\mu|=1}^{2} \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2} \Big) \|\widetilde{v}(t,\cdot)\|_{L^\infty}$$

and the conclusion of the statement. □

Lemma 3.2.16. *Let $\gamma, \chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, $\sigma > 0$ small, $(\theta_h(x))_h$ be a family of $C_0^\infty(B_1(0))$ functions, equal to 1 on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$, with $\|\partial_x^\alpha\theta_h\|_{L^\infty} = O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every k. Let also $\phi(x) = \sqrt{1-|x|^2}$. Then for every $j = 1, 2$*

$$\Big\|\mathrm{Op}_h^w(\xi_j + d_j\phi(x)\theta_h(x))\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}(t,\cdot)\Big\|_{L^2} \lesssim h^{1-\beta}\sum_{|\mu|=0}^{2}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2},$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. By symbolic calculus of lemma 1.2.24 and the fact that $\theta_h \equiv 1$ on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$, we have that, for any $j = 1, 2$,

(3.2.77)

$$\begin{aligned}\mathrm{Op}_h^w(\xi_j+d_j\phi(x)\theta_h(x))\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v} &= \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\xi_j + d_j\phi(x))\Big)\widetilde{v}\\
&+ \frac{\sqrt{h}}{2i}\mathrm{Op}_h^w\Big((\partial_j\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}\\
&- \frac{\sqrt{h}}{2i}\sum_{k,l=1}^{2}\mathrm{Op}_h^w\Big((\partial_l\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)p''_{k,l}(\xi)\partial_k(d_j\phi(x)\theta_h(x))\chi(h^\sigma\xi)\Big)\widetilde{v}\\
&+ \frac{h^{1+\sigma}}{2i}\sum_{k=1}^{2}\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\partial_k(d_j\phi(x)\theta_h(x))(\partial_k\chi)(h^\sigma\xi)\Big)\widetilde{v} + \mathrm{Op}_h^w(r_2(x,\xi))\widetilde{v},\end{aligned}$$

with $r_2 \in h^{1-4\sigma}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. On the one hand, as

$$\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\xi_j - d_j\phi(x))\Big)\widetilde{v} = \sum_{k=1}^{2}\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\widetilde{e}_k^j(x,\xi)(x_k - p'_k(\xi))\Big)\widetilde{v},$$

with $\widetilde{e}_k^j$ satisfying (1.2.75b) on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$, the L^2 norm of the first term in the right hand side of (3.2.77) can be estimated using (1.2.71a).

On the other hand, as $\partial\gamma$ vanishes in a neighbourhood of the origin, the L^2 norm of the second and third term in the right hand side of (3.2.77) can be estimated using (3.2.17a).

The two remaining contributions to the right hand side of (3.2.77), that already carry the right power of h, can be estimated with $h^{1-\beta}\|\widetilde{v}(t,\cdot)\|_{L^2}$ simply by proposition 1.2.36. □

We can finally state the following result:

Proposition 3.2.17 (Deduction of the transport equation)**.** *For any fixed $T > 1$, $D > 0$, let $\mathcal{C}_D^T := \{(t,x) : 1 \le t \le T, |x| \le D\}$ be the truncated cylinder, and assume that estimates* (1.1.11) *are satisfied in time interval $[1,T]$. Then function*

$$\widetilde{u}_{\Lambda_w}^{\Sigma_j}(t,x) := \sum_k \widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,x) \tag{3.2.78}$$

is solution to the following transport equation:

$$\left[D_t+\frac{1}{2}(1-|x|^2)x\cdot(hD_x)+\frac{h}{2i}(1-2|x|^2)\right]\widetilde{u}^{\Sigma}_{\Lambda_w}(t,x)=F_w(t,x),\quad\forall(t,x)\in\mathcal{C}^T_D, \tag{3.2.79}$$

and there exists some constant $C>0$ such that

$$\|F_w(t,\cdot)\|_{L^\infty}\leq CB\varepsilon h^{1-\beta'} \tag{3.2.80}$$

for some $\beta'>0$ small, $\beta'\to 0$ as $\sigma,\delta_1\to 0$.

Proof. By the assumption in the statement, all that we are going to say is to be meant in time interval $[1,T]$. We remind the reader that, by the definition of $\widetilde{u}^{\Sigma_j,k}_{\Lambda_w}$ in (3.2.45a) and of $\widetilde{u}^{\Sigma_j,k}$ in (3.2.41), the sum defining $\widetilde{u}^{\Sigma_j}_{\Lambda_w}$ is finite and restricted to indices $k\in K:=\{k\in\mathbb{Z}:h\lesssim 2^k\lesssim h^{-\sigma}\}$. Also, we warn the reader that, throughout the proof, C and β will denote two positive constants that may change line after line, with $\beta\to 0$ as $\sigma\to 0$.

In lemma 3.2.11 we proved that function $\widetilde{u}^{\Sigma_j,k}_{\Lambda_w}$ is solution to (3.2.55) with f^w_k verifying (3.2.53). Hence, by lemma B.2.1 we derive that f^w_k is a remainder of the form F_w satisfying (3.2.80).

For seek of compactness, we denote symbol $\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)$ in the right hand side of (3.2.55) by $\phi^j_k(\xi)$. On the one hand, reminding (3.2.71) and using the $L^\infty-L^\infty$ continuity of operator $\Gamma^{w,k}$ (see proposition 1.2.29), together with the classical Sobolev injection, the fact that

$$\left\|\mathrm{Op}^w_h(\phi^j_k(\xi))\right\|_{\mathcal{L}(L^2)}=O(h^{-\mu}), \tag{3.2.81}$$

with $\mu=\sigma\rho$ if $\rho\geq 0$, 0 otherwise, estimates (3.2.72),(3.2.73) and (3.2.3b), we find that

$$\begin{aligned}&\left\|\Gamma^{w,k}\mathrm{Op}^w_h(\Sigma_j(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi))\,[tq_w(t,tx)]\right\|_{L^\infty}\\&\lesssim h^{-\beta}\|\widetilde{q}_w(t,\cdot)\|_{L^2}+h^{-1-\beta}\|\mathrm{Op}^w_h(\varphi(2^{-k}\xi))(hD_x)\widetilde{q}_w(t,\cdot)\|_{L^2}\leq CB\varepsilon h^{1-\beta'}.\end{aligned} \tag{3.2.82}$$

On the other hand, using proposition 1.2.30, estimate (3.2.81), the fact that the commutator between $\mathrm{Op}^w_h(\phi^j_k(\xi))$ and Ω_h is also continuous on L^2 with norm $O(h^{-\mu})$, equality $\|tw(t,t\cdot)\|_{L^2}=\|w(t,\cdot)\|_{L^2}$, and (3.1.21a), (3.1.21c) (in which we choose $s>0$ large enough to have, say, $N(s)\geq 2$), we deduce that there is a $\chi\in C^\infty_0(\mathbb{R}^2)$ such that

$$\begin{aligned}&\|\Gamma^{w,k}\mathrm{Op}^w_h\big(\phi^j_k(\xi)\big)(h^{-1}c_w(t,tx))\|_{L^\infty(dx)}\\&\lesssim t^{\frac{1}{2}+\beta}\left\|\chi(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{2,\infty}}+\|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right)\\&+t^{-\frac{3}{2}+\beta}\left\|(v^{NF}-v_-)(t,\cdot)\right\|_{H^1}\left(\|V(t,\cdot)\|_{H^s}+\|v^{NF}(t,\cdot)\|_{H^s}\right)\\&+t^{\frac{1}{2}+\beta}\left\|\chi(t^{-\sigma}D_x)\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{2,\infty}}+\|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right)\\&+t^{-\frac{3}{2}+\beta}\left\|\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^s}+\|v^{NF}(t,\cdot)\|_{H^s}\right)\\&+t^{\frac{1}{2}+\beta}\left\|(v^{NF}-v_-)(t,\cdot)\right\|_{H^{1,\infty}}\times\sum_{\mu=0}^{1}\left(\|\Omega^\mu V(t,\cdot)\|_{H^1}+\|\Omega^\mu v^{NF}(t,\cdot)\|_{L^2}\right).\end{aligned} \tag{3.2.83}$$

Also, from (3.1.22a), (3.1.22c) we get that for every $\theta \in]0,1[$

$$
\begin{aligned}
&\|\Gamma^{w,k}\mathrm{Op}_h^w\big(\phi_k^j(\xi)\big)(h^{-1}r_w^{NF}(t,tx))\|_{L^\infty} \lesssim t^{\frac{1}{2}+\beta}\|V(t,\cdot)\|_{H^{13,\infty}}^2\|U(t,\cdot)\|_{H^1}\\
&+t^{\frac{1}{2}+\beta}\Big[\|V(t,\cdot)\|_{H^{15,\infty}}^{1-\theta}\|V(t,\cdot)\|_{H^{17}}^{\theta}\left(\|U(t,\cdot)\|_{H^{1,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\right)\\
&\qquad+\|V(t,\cdot)\|_{L^\infty}\left(\|U(t,\cdot)\|_{H^{16,\infty}}^{1-\theta}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{16,\infty}}^{1-\theta}\right)\|U(t,\cdot)\|_{H^{18}}^{\theta}\Big]\|\Omega V(t,\cdot)\|_{L^2}\\
&+t^{\frac{1}{2}+\beta}\Big[\|V(t,\cdot)\|_{H^{1,\infty}}\left(\|U(t,\cdot)\|_{H^1}+\|\Omega U(t,\cdot)\|_{H^1}\right)\\
&\qquad+\left(\|U(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\|\Omega V(t,\cdot)\|_{L^2}\Big]\|V(t,\cdot)\|_{H^{17,\infty}}.
\end{aligned}
\tag{3.2.84}
$$

Therefore, using (3.1.7) with $s=1$, (3.1.8a), (1.1.11), and choosing $\theta \ll 1$ sufficiently small, we derive that $h^{-1}\Gamma^{w,k}\mathrm{Op}_h^w\big(\phi_k^j(\xi)\big)(c_w(t,tx)+r_w^{NF}(t,tx))$ is a remainder $F_w(t,x)$ satisfying (3.2.80). Since function $(\partial\chi_0)(h^{-1}\xi)$ is localized for frequencies of size h, its product with $\psi(2^{-k}\xi)$ is non-zero only for values of $k\in\mathbb{Z}$ such that $2^k\sim h$. In that case, by commutating $\Gamma^{w,k}$ with $\mathrm{Op}_h^w\big((\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\psi(2^{-k}\xi)\big)$ and using the classical Sobolev injection, together with proposition 1.2.27, we find that

$$
\Big\|ih\,\Gamma^{w,k}\mathrm{Op}_h^w\big((\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\psi(2^{-k}\xi)\big)\widetilde{u}(t,\cdot)\Big\|_{L^\infty}\lesssim h\|\widetilde{u}(t,\cdot)\|_{L^2}.
\tag{3.2.85}
$$

Since $(\partial\chi_0)(h^\sigma\xi)$ is, instead, localized for frequencies larger than $h^{-\sigma}$, by applying the semi-classical Sobolev injection and lemma 1.2.40 we find that

$$
\Big\|i\sigma h\,\Gamma^{w,k}\mathrm{Op}_h^w\big(\psi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi))\cdot(h^\sigma\xi)\big)\widetilde{u}(t,\cdot)\Big\|_{L^\infty}\lesssim h^N\|\widetilde{u}(t,\cdot)\|_{H^s_h},
\tag{3.2.86}
$$

with $N=N(s)>1$ as long as $s>0$ is sufficiently large. By lemma B.2.1 we obtain that also the fifth and sixth addend in the right hand side of (3.2.55) are remainders $F_w(t,x)$.

Finally, after lemma 3.2.12

$$
\begin{aligned}
-\mathrm{Op}_h^w\big(\theta(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k} &= \frac{1}{2}\mathrm{Op}_h^w\big(\theta(x)(1-|x|^2)x\cdot\xi\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}\\
&\quad+\mathrm{Op}_h^w(\theta(x)e(x,\xi)\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}
\end{aligned}
$$

with $e(x,\xi)$ given by (3.2.57), and latter term in the above right hand side satisfies (3.2.62). Using symbolic calculus of proposition 1.2.21 until order $N\in\mathbb{N}$ we find that

$$
\begin{aligned}
&\frac{1}{2}\mathrm{Op}_h^w\big(\theta(x)(1-|x|^2)x\cdot\xi\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}=\theta(x)\Big[\frac{1}{2}(1-|x|^2)x\cdot(hD_x)+\frac{h}{2i}(1-2|x|^2)\Big]\mathrm{Op}_h^w(\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}\\
&+\frac{h}{4i}(\partial\theta)(x)\cdot x(1-|x|^2)\mathrm{Op}_h^w(\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}+\sum{}'h\theta_1(x)\mathrm{Op}_h^w(\widetilde{\varphi}_1(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}+\mathrm{Op}_h^w(r_{(N}x,\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k},
\end{aligned}
$$

with $\sum'$ being a concise notation to indicate a linear combination, $\partial\theta(x)$ supported for $|x|>D_1$, $\theta_1\in C_0^\infty(\mathbb{R}^2)$, $\widetilde{\varphi}_1\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ coming out from the derivatives of $\widetilde{\varphi}$, and $r_N(x,\xi)$ integral remainder of the form

$$
\frac{h^N}{(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int_0^1\theta_N(x+tz)(1-t)^{N-1}dt\;\widetilde{\varphi}_N(2^{-k}(\xi+\eta))dzd\eta,
$$

for some other $\theta_N\in C_0^\infty(\mathbb{R}^2)$, $\widetilde{\varphi}_N\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, verifying that

$$
\|\mathrm{Op}_h^w(r_N(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)}=O(h)
\tag{3.2.87}
$$

if N is taken sufficiently large. Therefore, from proposition 1.2.27, (3.2.81) and (B.2.1a)

$$
\Big\|\mathrm{Op}_h^w(r(x,\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,\cdot)\Big\|_{L^\infty}\lesssim h^{1-\beta}\|\widetilde{u}(t,\cdot)\|_{L^2}\leq CB\varepsilon h^{1-\beta'}.
$$

Moreover, since $\widetilde{\varphi} \equiv 1$ on the support of φ (which defines $\widetilde{u}^{\Sigma_j,k}$), by commutating $\mathrm{Op}_h^w(\widetilde{\varphi}(2^{-k}\xi))$ with $\Gamma^{w,k}$ and using remark 1.2.22 we find that, for any $N \in \mathbb{N}$ as large as we want,

$$\mathrm{Op}_h^w(\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k} = \widetilde{u}_{\Lambda_w}^{\Sigma_j,k} + O_{L^\infty}(h^N\|\widetilde{u}\|_{L^2}).$$

Also, since $\widetilde{\varphi}_1$ is obtained from the derivatives of $\widetilde{\varphi}$ and vanishes on the support of φ,

$$\theta_1(x)\mathrm{Op}_h^w(\widetilde{\varphi}_1(2^{-k}\xi))\widetilde{u}_{\Lambda_w}^{\Sigma_j,k} = O_{L^\infty}(h^N\|\widetilde{u}\|_{L^2}).$$

Therefore, again from (B.2.1a) we deduce that

$$\begin{aligned}-\mathrm{Op}_h^w\big(\theta(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k} &= \theta(x)\Big[\frac{1}{2}(1-|x|^2)x\cdot(hD_x)+\frac{h}{2i}(1-2|x|^2)\Big]\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}\\ &+\frac{h}{4i}(\partial\theta)(x)\cdot x(1-|x|^2)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}+\mathrm{Op}_h^w\big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\big)\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}+O_{L^\infty}(h^{1-\beta'}),\end{aligned}$$

which implies, summed up with estimates from (3.2.82) to (3.2.86), that $\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}$ is solution to

$$\begin{aligned}&\left[D_t+\theta(x)\frac{1}{2}(1-|x|^2)x\cdot(hD_x)+\theta(x)\frac{h}{2i}(1-2|x|^2)\right]\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,x)=F_w^k(t,x)\\ &+\Big[(1-\theta)(x)\mathrm{Op}_h^w((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi))+\widetilde{\theta}(x)\mathrm{Op}_h^w(\widetilde{\varphi}_1(2^{-k}\xi))-\frac{h}{4i}(\partial\theta)(x)\cdot x(1-|x|^2)\Big]\widetilde{u}_{\Lambda_w}^{\Sigma_j,k}(t,x)\,,\end{aligned}$$

where $F_w^k(t,x)$ satisfies (3.2.80). Choosing $D_1 = D$, we obtain that $\widetilde{u}_{\Lambda_w}^{\Sigma_j}$ is solution to (3.2.79) in cylinder $\mathcal{C}_D^T$, with $F_w(t,x) := \sum_k F_w^k(t,x)$ (this sum being finite and restricted to indices $k \in K$) satisfying the same L^∞ estimate as F_w^k, up to an additional factor $h^{-\sigma}$. □

3.3 Analysis of the transport equation and end of the proof

In previous section (see proposition 3.2.7) we firstly showed how to propagate a-priori uniform estimate (1.1.11b) on the Klein-Gordon component v_-, in the sense of deducing (1.1.12b) from estimates (1.1.11). We then passed to the study of the wave equation and proved that, if (u_-, v_-) is solution to (3.1.1) in some interval $[1,T]$, function $\widetilde{u}_{\Lambda_w}^{\Sigma_j}$ defined in (3.2.78) is solution to transport equation (3.2.79) in truncated cylinder $\mathcal{C}_D^T := \{(t,x) : 1 \le t \le T, |x| \le D\}$, for any $D > 0$. The aim of this section is to study such a transport equation in order to deduce some information on the uniform norm of its solutions. This will allow us to finally propagate a-priori estimate (1.1.11a) on the wave component u_- and to close the bootstrap argument. A short proof of main theorem 1.1.1 is given at the end of this section.

3.3.1 The inhomogeneous transport equation

The aim of this subsection is to study the behaviour of a solution w to the following transport equation

$$\left[D_t+\frac{1}{2}(1-|x|^2)x\cdot(hD_x)-\frac{i}{2t}(1-2|x|^2)\right]w = f\,, \tag{3.3.1}$$

in a cylinder $\mathcal{C} = \{(t,x) : t \ge 1, |x| \le D\}$ for a large constant $D \gg 1$, where the inhomogeneous term f is a $O_{L^\infty}(\varepsilon t^{-1+\beta})$, for some $\varepsilon > 0$ small and $0 \le \beta < 1/2$. We distinguish in $\mathcal{C}$ two subregions:

$$I_1 := \Big\{(t,x) : t \ge 1, |x| < \Big(\frac{t}{t-1}\Big)^{\frac{1}{2}}, |x| \le D\Big\}\,, \qquad I_2 := \Big\{(t,x) : t > 1, \Big(\frac{t}{t-1}\Big)^{\frac{1}{2}} \le |x| \le D\Big\},$$

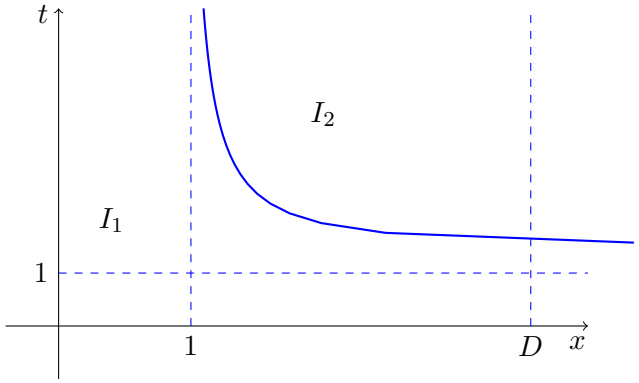

Figure 3.3: Regions I_1 and I_2 in space dimension 1

and denote by $I_{1,t}, I_{2,t}$ their sections at a fixed time $t \geq 1$,

$$I_{1,t} := \left\{ x : |x| < \left(\frac{t}{t-1}\right)^{\frac{1}{2}}, |x| \leq D \right\}, \qquad I_{2,t} := \left\{ x : \left(\frac{t}{t-1}\right)^{\frac{1}{2}} \leq |x| \leq D \right\}.$$

The result we prove is the following.

Proposition 3.3.1. *Let $\varepsilon > 0$ be small and w be the solution to the following Cauchy problem*

$$\begin{cases} \left[D_t + \frac{1}{2}(1-|x|^2)x \cdot (hD_x) - \frac{i}{2t}(1-2|x|^2)\right] w = f\,, \\ w(1,x) = \varepsilon w_0(x)\,, \end{cases} \tag{3.3.2}$$

with $f = O_{L^\infty}(\varepsilon t^{-1+\beta})$, for some fixed $0 \leq \beta < 1/2$. Let us suppose that $|w_0(x)| \lesssim \langle x \rangle^{-2}$ and that $|w(t,x)| \lesssim \varepsilon t^{\beta'}$ for some $\beta' > 0$ whenever $|x| > D \gg 1$. Therefore,

$$|w(t,x)| \lesssim \varepsilon \|w_0\|_{L^\infty} t^{\beta''} (1+|x|)^{-\frac{1}{2}} (t^{-1} + |1-|x||)^{-\frac{1}{2}+\beta''}\,, \tag{3.3.3}$$

for every $(t,x) \in \mathcal{C}_D = \{(t,x) | t \geq 1, |x| \leq D\}$, with $\beta'' = \max\{\beta, \beta'\}$.

We observe that, if $W(t,x) = t^{-1} w(t, t^{-1}x)$, the above inequality implies that

$$|W(t,x)| \lesssim \varepsilon \|w_0\|_{L^\infty} (t+|x|)^{-\frac{1}{2}} (1+|t-|x||)^{-\frac{1}{2}+\beta''},$$

showing that the uniform norm of $W(t,\cdot)$ decays in time at a rate $t^{-1/2}$, enhanced to $t^{-1+\beta''}$ out of the light cone $t = |x|$.

In order to prove the result of proposition 3.3.1 we fix $T \geq 1$, $x \in B_D(0)$, and look for the characteristic curve of (3.3.2) with initial point (T,x), i.e. map $t \mapsto X(t;T,x)$ solution of

$$\begin{cases} \frac{d}{dt} X(t;T,x) = \frac{1}{2t}\big(1 - |X(t;T,x)|^2\big) X(t;T,x) \\ X(T;T,x) = x \end{cases} \qquad t \geq T. \tag{3.3.4}$$

Lemma 3.3.2. *Solution $X(t;T,x)$ to (3.3.4) writes explicitly as*

$$X(t;T,x) = \frac{\sqrt{t}x}{(T-(T-t)|x|^2)^{\frac{1}{2}}} \tag{3.3.5}$$

and it is well defined for all $t > T(1-|x|^{-2})$. Moreover, for any fixed $t > T$, map $x \in \mathbb{R}^2 \mapsto X(t;T,x) \in \left\{ |x| < \left(\frac{t}{t-T}\right)^{\frac{1}{2}} \right\}$ is a diffeomorphism of inverse $Y(t,y) = \frac{\sqrt{T}y}{(t+(T-t)|y|^2)^{\frac{1}{2}}}$.

Proof. Multiplying equation (3.3.4) by $2X(t;T,x)$ we deduce that $|X(t;T,x)|^2$ satisfies the equation

$$\frac{d}{dt}|X(t;T,x)|^2 = \frac{1}{t}\big(1-|X(t;T,x)|^2\big)|X(t;T,x)|^2\,,$$

from which follows that $1-|X(t;T,x)|^2 = \frac{T(1-|x|^2)}{T-(T-t)|x|^2}$. Injecting this result in (3.3.4) and integrating in time, we obtain expression (3.3.5) and observe that the obtained map is well defined for all $t > T(1-|x|^{-2})$.

In order to prove the second part of the statement, we fix $t>T$, $y \in \left\{|x| \le \left(\frac{t}{t-T}\right)^{\frac{1}{2}}\right\}$ and look for $Y(t,y)$ such that $X(t;T,Y(t,y)) = y$. In other words,

$$y = \frac{\sqrt{t}Y(t,y)}{(T-(T-t)|Y(t,y)|^2)^{\frac{1}{2}}},$$

which implies that $Y(t,y) = \frac{\sqrt{T}y}{(t+(T-t)|y|^2)^{\frac{1}{2}}}$. This map is well defined as long as $|y| < \left(\frac{t}{t-T}\right)^{\frac{1}{2}}$. □

Along the characteristic curve $X(t;T,x)$ function w satisfies

$$\begin{aligned}\frac{d}{dt}w\big(t,X(t;T,x)\big) &= -\frac{1}{2t}\big(1-2|X(t;T,x)|^2\big)\; w\big(t,X(t;T,x)\big) + if\big(t,X(t;T,x)\big)\\ &= -\frac{1}{2t}\frac{T-T|x|^2-t|x|^2}{T-(T-t)|x|^2}\; w\big(t,X(t;T,x)\big) + if\big(t,X(t;T,x)\big)\end{aligned}$$

and hence

$$\begin{aligned}\frac{d}{dt}\left[\left(\exp\int_T^t \frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2}d\tau\right) w\big(t,X(t;T,x)\big)\right]\\ = i\left(\exp\int_T^t \frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2}d\tau\right) f\big(t,X(t;T,x)\big)\,.\end{aligned} \tag{3.3.6}$$

Lemma 3.3.3.

$$\exp\int_T^t \frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2}\,d\tau = \Big(\frac{t}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T|x|^2+t|x|^2}{T}\Big)^{-1}. \tag{3.3.7}$$

Proof. The result follows writing

$$\frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2} = \frac{1}{2\tau} - \frac{|x|^2}{T-T|x|^2+\tau|x|^2}\,,$$

taking the integral over $\tau\in[T,t]$ and then passing to its exponential. □

Let us first study the behaviour of w, solution to (3.3.2), in region I_1. We fix $T=1$ and, integrating equality (3.3.6) over $[1,t]$, we find that

$$\begin{aligned}\left(\exp\int_1^t \frac{1}{2\tau}\frac{1-|x|^2-\tau|x|^2}{1-(1-\tau)|x|^2}d\tau\right) w\big(t,X(t;1,x)\big)\\ = w(1,x) + i\int_1^t \left(\exp\int_1^s \frac{1}{2\tau}\frac{1-|x|^2-s|x|^2}{1-(1-s)|x|^2}ds\right) f\big(s,X(s;1,x)\big)ds.\end{aligned} \tag{3.3.8}$$

Using (3.3.7) and the fact that $f = O_{L^\infty}(\varepsilon t^{-1+\beta})$, we then obtain that

$$\begin{aligned}(3.3.9)\quad \big|w(t,X(t;1,x))\big| &\le t^{-\frac{1}{2}}(1-|x|^2+t|x|^2)|w(1,x)| \\ &\quad + C\varepsilon t^{-\frac{1}{2}}(1-|x|^2+t|x|^2)\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}}\,,\end{aligned}$$

for some positive constant C.

Lemma 3.3.4. *For any fixed $0 \le \beta < 1/2$*

$$(3.3.10)\qquad \int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} \lesssim \frac{t^{\frac{1}{2}+\beta}}{(1+\sqrt{t}|x|)^{1+2\beta}}(1+|x|)^{-1+2\beta+\beta'}\,,$$

for all $t \ge 1$ and $\beta' > 0$ as small as we want.

Proof. For $\sqrt{t}|x| \le 1$, we have that

$$\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} \lesssim t^{\frac{1}{2}+\beta} \lesssim \frac{t^{\frac{1}{2}+\beta}}{(1+\sqrt{t}|x|)^{1+2\beta}}(1+|x|)^{-1+2\beta+\beta'},$$

for any $\beta' \ge 0$. Suppose then that $\sqrt{t}|x| > 1$. For $t \le 2$

$$\begin{aligned}&\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} \lesssim (1+|x|)^{-2}\log(1+|x|^2) \\ \text{and}\quad &|x|^{-2}\log(1+|x|^2)\frac{(1+\sqrt{t}|x|)^{1+2\beta}}{t^{\frac{1}{2}+\beta}} \lesssim (1+|x|)^{-1+2\beta}\log(1+|x|^2),\end{aligned}$$

which immediately implies inequality (3.3.10). For $t \ge 2$

$$\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} = \int_1^2 \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} + \int_2^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}}\,,$$

where the first integral is bounded from the right hand side of (3.3.10). The second one is less or equal than $\int_1^{t-1} \frac{ds}{(1+s|x|^2)s^{\frac{1}{2}-\beta}}$, so for $|x| \ge 1$ it follows that

$$\int_1^{t-1} \frac{ds}{(1+s|x|^2)s^{\frac{1}{2}-\beta}} \le |x|^{-2}\int_1^{t-1} \frac{ds}{s^{\frac{3}{2}-\beta}} \lesssim (1+|x|)^{-2}.$$

Since $\frac{(1+\sqrt{t}|x|)^{1+2\beta}}{t^{\frac{1}{2}+\beta}} \le (1+|x|)^{1+2\beta}$, from the above inequality we deduce the right bound of the statement. For $|x| < 1$, a change of variables gives that

$$\int_1^{t-1} \frac{ds}{(1+s|x|^2)s^{\frac{1}{2}-\beta}} = |x|^{-1-2\beta}\int_{|x|^2}^{(t-1)|x|^2} \frac{ds}{(1+s)s^{\frac{1}{2}-\beta}} \lesssim |x|^{-1-2\beta}\frac{(t|x|^2)^{\frac{1}{2}+\beta}}{(1+t|x|^2)^{\frac{1}{2}+\beta}} \le \frac{t^{\frac{1}{2}+\beta}}{(1+t|x|^2)^{\frac{1}{2}+\beta}}.$$

□

If initial condition $w_0(x)$ is sufficiently decaying in space, e.g. $|w_0(x)| \lesssim \langle x\rangle^{-2}$, we deduce from inequalities (3.3.9) and (3.3.10) the following bound for w along the characteristic curve $X(t;1,x)$:

$$(3.3.11)\qquad \big|w(t,X(t;1,x))\big| \lesssim \varepsilon\|w_0\|_{L^\infty}t^\beta(1+\sqrt{t}|x|)^{1-2\beta}(1+|x|)^{-1+2\beta+\beta'}\,,$$

for any $\beta' > 0$ as small as we want.

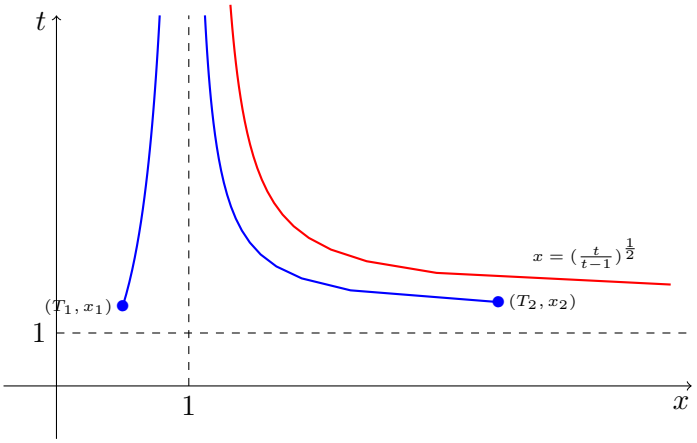

Figure 3.4: Characteristic curves of initial point $(T_i, x_i) \in I_1$, $i = 1, 2$, in space dimension 1

Proposition 3.3.5. *Let w be the solution to transport equation (3.3.2), with $\|f(t,\cdot)\|_{L^\infty} \lesssim \varepsilon t^{-1+\beta}$ for some fixed $0 \le \beta < 1/2$, and initial condition $|w_0(x)| \lesssim \langle x\rangle^{-2}$, $\forall x \in \mathbb{R}^2$. Then*

$$|w(t,x)| \lesssim \varepsilon t^{\beta}\big[t^{-1} + |1-|x||\big]^{-\frac{1}{2}+\beta} \tag{3.3.12}$$

for every $(t,x) \in I_1 = \{(t,x) : t \ge 1, |x| < \big(\frac{t}{t-1}\big)^{\frac{1}{2}}, |x| \le D\}$.

Proof. In lemma 3.3.2 we proved that, for any fixed $t > T = 1$, map $x \in \mathbb{R}^2 \mapsto X(t;1,x) \in \big\{x : |x| < (\frac{t}{t-1})^{\frac{1}{2}}\big\}$ is a diffeomorphism with inverse $Y(t,y) = y(t + (1-t)|y|^2)^{-1/2}$. From inequality (3.3.11) we hence deduce that, for any y such that $|y| < \big(\frac{t}{t-1}\big)^{\frac{1}{2}}$,

$$|w(t,y)| \lesssim \varepsilon t^{\beta}\big(1 + \sqrt{t}|Y(t,y)|\big)^{1-2\beta}\big(1 + |Y(t,y)|\big)^{-1+2\beta+\beta'}.$$

In particular, as $t(1-|y|^2) + |y|^2) \sim t|1-|y|^2| + |y|^2$ when $|y| < \big(\frac{t}{t-1}\big)^{\frac{1}{2}}$ and $t \ge t_0 > 1$, and $t|1-|y|^2| + |y|^2 \sim t|1-|y|| + |y|$ when $|y| \le D$, we find for those values of (t,y) that

$$|w(t,y)| \lesssim \varepsilon t^{\beta}\left(1 + \frac{\sqrt{t}|y|}{(t|1-|y|| + |y|)^{\frac{1}{2}}}\right)^{1-2\beta} \lesssim \varepsilon t^{\beta}\big[t^{-1} + |1-|y||\big]^{-\frac{1}{2}+\beta},$$

simply using that $(1 + |Y(t,y)|)^{-1+2\beta+\beta'} \le 1$. Moreover, for $t \to 1$ and $|y| \le D$,

$$|w(t,y)| \lesssim \varepsilon \lesssim \varepsilon t^{\beta}\big[t^{-1} + |1-|y||\big]^{-\frac{1}{2}+\beta}.$$

□

Proposition 3.3.6. *Let $\varepsilon > 0$ be small and w be the solution to transport equation (3.3.2), with $\|f(t,\cdot)\|_{L^\infty} \lesssim \varepsilon t^{-1+\beta}$ for some fixed $0 \le \beta < 1/2$, and suppose that $|w(t,x)| \lesssim \varepsilon t^{\beta'}$ for some $\beta' > 0$ whenever $|x| \ge D$. Then*

$$|w(t,x)| \lesssim \varepsilon t^{\beta''}(|x|^2 - 1)^{\beta''-\frac{1}{2}},$$

for every $(t,x) \in I_2 = \{(t,x) : t > 1, \big(\frac{t}{t-1}\big)^{\frac{1}{2}} \le |x| \le D\}$, where $\beta'' = \max\{\beta, \beta'\}$.

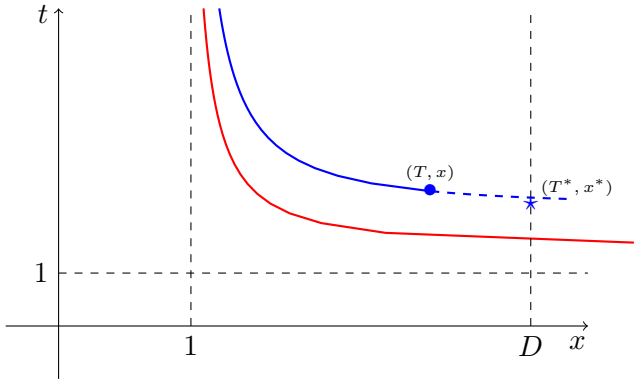

Figure 3.5: Characteristic curve of initial point $(T,x) \in I_2$

Proof. For a fixed $(T,x) \in I_2$ we look at $X(t;T,x)$, solution to (3.3.4) and given by the explicit expression (3.3.5). We observe that there exists a time T^*, $1 < T^* < T$, such that $X(t;T,x)$ hits the boundary $|y| = D$ at $t = T^*$. In other words, $t = T^*$ is the first time when $X(t;T,x)$ enters in the region $\{(t,x) : t \geq 1, |x| \leq D\}$, to never leave it again for function $t \mapsto |X(t;T,x)|$ is strictly decreasing. A simple computation shows that

$$T^* = \frac{D^2}{D^2-1}(1-|x|^{-2})T < T\,. \tag{3.3.13}$$

Integrating expression (3.3.6) over $[T^*,T]$ and using (3.3.7), we find that

$$\begin{aligned} w(T,x) = \Big(\frac{T^*}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T(1-|x|^{-2})}{T^*-T(1-|x|^{-2})}\Big)w(T^*,X(T^*;T,x)) \\ + i\int_{T^*}^{T}\Big(\frac{t}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T(1-|x|^{-2})}{t-T(1-|x|^{-2})}\Big)f\big(t,X(t;T,x)\big)dt\,. \end{aligned} \tag{3.3.14}$$

From (3.3.13)

$$T^*-T(1-|x|^{-2}) = \frac{1}{D^2-1}(1-|x|^{-2})T \quad \text{and} \quad \frac{T^*}{T} = \frac{D^2}{D^2-1}(1-|x|^{-2})$$

so since $|w(t,x)| \lesssim \varepsilon t^{\beta'}$ whenever $|x| \geq D$, for some $\beta' > 0$ by the hypothesis, we find that the first term in right hand side of (3.3.14) is bounded by $C\varepsilon(|x|^2-1)^{-\frac{1}{2}}(T^*)^{\beta'}$, for some constant $C > 0$. Setting $c = \frac{1}{D^2-1}$, by the hypothesis on f we derive that

$$\begin{aligned} \Big|\int_{T^*}^{T}\Big(\frac{t}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T(1-|x|^{-2})}{t-T(1-|x|^{-2})}\Big)f\big(t,X(t;T,x)\big)dt\Big| &\lesssim \varepsilon T^{\frac{1}{2}}\int_{T^*}^{T}\big(t-T(1-|x|^{-2})\big)^{-1}t^{-\frac{1}{2}+\beta}dt \\ &= \varepsilon T^{\frac{1}{2}}\int_{T^*}^{T}\big(t-T^*+c(1-|x|^{-2})T\big)^{-1}t^{-\frac{1}{2}+\beta}dt \\ &\leq \varepsilon T^{\frac{1}{2}}\int_{0}^{T-T^*}\frac{dt}{\big(t+c(1-|x|^{-2})T\big)t^{\frac{1}{2}-\beta}} \\ &\lesssim \varepsilon T^{\frac{1}{2}}\big((1-|x|^{-2})T\big)^{\beta-\frac{1}{2}} = \varepsilon T^{\beta}(1-|x|^{-2})^{\beta-\frac{1}{2}}\,. \end{aligned}$$

□

3.3.2 Propagation of the uniform estimate on the wave component

Proposition 3.3.7 (Propagation of the a-priori estimate on U, RU)**.** *Let us fix $K_1 > 0$. There exist two integers $n \gg \rho \gg 1$ sufficiently large,, two constants $A, B > 1$ sufficiently large, some small $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0$, and $\varepsilon_0 \in]0,1[$ sufficiently small, such that, for any $0 < \varepsilon < \varepsilon_0$, if (u, v) is solution to* (1.1.1)-(1.1.2) *in some interval $[1, T]$, for a fixed $T > 1$, and $u_\pm, v_\pm$ defined in* (1.1.5) *satisfy a-priori estimates* (1.1.11)*, for every $t \in [1, T]$, then it also verify* (1.1.12a) *in the same interval $[1, T]$.*

Proof. We warn the reader that, throughout the proof, C, β, β' will denote some positive constants that may change line after line, such that $\beta \to 0$ as $\sigma \to 0$ (resp. $\beta' \to 0$ as $\delta_1, \sigma \to 0$). We also remind that $h = 1/t$.

In proposition 3.1.2 we introduced function u^{NF}, defined from u_- through (3.1.15), and showed that its $H^{\rho+1,\infty}$ norm (resp. the $H^{\rho+1,\infty}$ norm of $\mathrm{R}u^{NF}$) differs from that of u_- (resp. of $\mathrm{R}u_-$) by a quantity satisfying (3.1.20b) (resp. (3.1.20c)). If n is sufficiently large with respect to ρ (at least $n \geq \rho + 18$), a-priori estimates (1.1.11b), (1.1.11c) give that, for every $t \in [1, T]$,

$$\begin{aligned}(3.3.15)\quad &\|u_-(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2} \|\mathrm{R}_j u_-(t,\cdot)\|_{H^{\rho+1,\infty}} \\ &\qquad \leq \|u^{NF}(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2} \|\mathrm{R}_j u^{NF}(t,\cdot)\|_{H^{\rho,\infty}} + 2AB\varepsilon^2 t^{-1+\frac{\delta}{2}}.\end{aligned}$$

We successively considered $\widetilde{u}(t,x) := t\widetilde{u}^{NF}(t,tx)$ and decomposed it as in (3.2.38), with Σ_j given by (3.2.37), showing that it satisfies (3.2.39) (resp. (3.2.40)) when restricted to small frequencies $|\xi| \lesssim t^{-1}$ (resp. large frequencies $|\xi| \gtrsim t^{\sigma}$). We then focused on $\widetilde{u}^{\Sigma_j,k}$ defined in (3.2.41), which is localized for frequencies supported in an annulus of size 2^k with $k \in K = \{k \in \mathbb{Z} : h \lesssim 2^k \lesssim h^{-\sigma}\}$, and further split it into the sum of functions $\widetilde{u}^{\Sigma_j,k}_{\Lambda_w}$, $\widetilde{u}^{\Sigma_j,k}_{\Lambda_w^c}$ (see (3.2.45)). On the one hand, from inequality (3.2.46b) and lemma B.2.1 we have that there is a positive constant C such that, for every $t \in [1, T]$,

$$\|\widetilde{u}^{\Sigma_j,k}_{\Lambda_w^c}(t,\cdot)\|_{L^\infty} \leq C\varepsilon t^{\beta'}.$$

On the other hand, we proved in proposition 3.2.17 that, for any $D > 0$ and any (t, x) in truncated cylinder $\mathcal{C}_D^T = \{(t,x) : 1 \leq t \leq T, |x| \leq D\}$, $\widetilde{u}^{\Sigma_j}_{\Lambda_w}(t,x)$ defined in (3.2.78) is solution to inhomogeneous transport equation (3.2.79), with inhomogeneous term $F_w(t,x)$ satisfying (3.2.80). Observe that, by definition (1.2.49) of $\mathcal{M}$, symbolic calculus, and proposition 1.2.36, we have that

$$\|\widetilde{u}^{\Sigma_j}_{\Lambda_w}(1,\cdot)\|_{L^2} + \|x\widetilde{u}^{\Sigma_j}_{\Lambda_w}(1,\cdot)\|_{L^2} \lesssim \|\widetilde{u}(1,\cdot)\|_{L^2} + \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(1,\cdot)\|_{L^2} \leq C\varepsilon,$$

which means that $\varepsilon^{-1}\langle x\rangle \widetilde{u}^{\Sigma_j}_{\Lambda_w}(1,x) \in L^2$. That hence implies that $|\widetilde{u}^{\Sigma_j}_{\Lambda_w}(1,x)| \lesssim \varepsilon\langle x\rangle^{-2}$ for every $x \in \mathbb{R}^2$ (if not, we would have $\|\langle\cdot\rangle^{-1}\|_{L^2} \leq \varepsilon^{-1}\|\langle\cdot\rangle\widetilde{u}^{\Sigma_j}_{\Lambda_w}(1,\cdot)\|_{L^2}$). Moreover, if $D \gg 1$ is sufficiently large, from lemma 3.3.9 below and B.2.1 in appendix B we deduce that

$$\begin{aligned}(3.3.16)\quad |\mathbb{1}_{|x|\geq D}\widetilde{u}^{\Sigma_j}_{\Lambda_w}(t,x)| &\leq C\frac{\log|x|}{|x|}h^{-\beta}\big(\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{u}(t,\cdot)\|_{L^2} + \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}\big) \\ &\leq C\varepsilon\frac{\log|x|}{|x|}t^{\beta'}.\end{aligned}$$

Therefore, from proposition 3.3.1 we obtain that

$$|\widetilde{u}^{\Sigma_j}_{\Lambda_w}(t,x)| \lesssim C\varepsilon t^{\beta'}(1+|x|)^{-\frac{1}{2}}\big(t^{-1}+|1-|x||\big)^{-\frac{1}{2}+\beta'}, \quad \forall (t,x)\in \mathcal{C}^T_D.$$

Summing up, denoting by $\mathbb{1}_{\mathcal{C}^T_D}$ the characteristic function of cylinder $\mathcal{C}^T_D$,

$$|\widetilde{u}^{\Sigma}(t,x)| \leq C\varepsilon \mathbb{1}_{\mathcal{C}^T_D} t^{\beta'}(1+|x|)^{-\frac{1}{2}}\big(t^{-1}+|1-|x||\big)^{-\frac{1}{2}+\beta'} + C\varepsilon t^{\beta'}, \quad \forall (t,x)\in [1,T]\times\mathbb{R}^2.$$

Returning back to function u^{NF} via (3.2.2), this means that, for every $(t,x)\in[1,T]\times\mathbb{R}^2$,

$$\begin{aligned}(3.3.17)\quad &\big|\langle D_x\rangle^{\rho} u^{NF}(t,x)\big| + \sum_{j=1}^{2}\big|\langle D_x\rangle^{\rho} R_j u^{NF}(t,x)\big| \\ &\qquad\qquad \leq C\varepsilon \mathbb{1}_{\{|x|\leq Dt\}}(t+|x|)^{-\frac{1}{2}}(1+|t-|x||)^{-\frac{1}{2}+\beta'} + C\varepsilon t^{-1+\beta'}.\end{aligned}$$

Finally, reminding definition 1.2.1 (iii) of space $H^{\rho,\infty}$, injecting the above inequality in (3.3.15), and choosing $A>1$ sufficiently large such that $C<\frac{A}{3K_1}$, $\varepsilon_0>0$ sufficiently small so that $CB\varepsilon_0<(3K_1)^{-1}$, we deduce enhanced estimate (1.1.12a). □

Remark 3.3.8. Beside the propagation of estimate (1.1.11a), by combining inequalities (3.3.15), (3.3.17), and (1.1.5), we also deduce the following inequality

$$|\partial_t u(t,x)| + |\nabla_x u(t,x)| \leq C\varepsilon \mathbb{1}_{\{|x|\leq Dt\}}(t+|x|)^{-\frac{1}{2}}(1+|t-|x||)^{-\frac{1}{2}+\beta'} + C\varepsilon t^{-1+\beta'},$$

with $\beta'>0$ small as long as σ,δ_1 are small, which almost corresponds to the optimal decay in time and space enjoyed by the linear wave in space dimension two.

Lemma 3.3.9. *Let $\chi\in C^\infty_0(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin and $\sigma>0$ be small. Let also $\varphi\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$. There exists a constant $C>0$ such that for every $h\in]0,1[$, $R\gg 1$, and any function $w(t,x)$ with $w(t,\cdot), \mathrm{Op}^w_h(\chi(h^\sigma\xi))\mathcal{M}w(t,\cdot)\in L^2(\mathbb{R}^2)$,*
(3.3.18)

$$\Big\|\varphi\Big(\frac{\cdot}{R}\Big)\mathrm{Op}^w_h(\chi(h^\sigma\xi))w(t,\cdot)\Big\|_{L^\infty} \leq CR^{-1}(\log R + |\log h|)\sum_{|\gamma|=0}^{1}\|\mathrm{Op}^w_h(\chi(h^\sigma\xi))\mathcal{M}^\gamma w(t,\cdot)\|_{L^2}.$$

Proof. Let us fix $R\gg 1$ and, for seek of compactness, denote $\mathrm{Op}^w_h(\chi(h^\sigma\xi))w$ by w^χ. For a new smooth cut-off function χ_1 equal to 1 on the support of χ, we have that

$$\varphi\Big(\frac{x}{R}\Big)\mathrm{Op}^w_h(\chi(h^\sigma\xi))w = \mathrm{Op}^w_h(\chi_1(h^\sigma\xi))\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big] + \Big[\varphi\Big(\frac{x}{R}\Big), \mathrm{Op}^w_h(\chi_1(h^\sigma\xi))\Big]w^\chi,$$

where the symbol associated to above commutator is given by

$$r_R(x,\xi) = -\frac{h^{1+\sigma}R^{-1}}{i(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\left[\int_0^1(\partial\varphi)\Big(\frac{x+tz}{R}\Big)dt\right](\partial\chi_1)(h^\sigma(\xi+\eta))dzd\eta,$$

as follows from (1.2.19) and integration in $dy, d\zeta$. Since $(\partial\chi_1)(h^\sigma\xi)$ is supported for frequencies $|\xi|\leq h^{-\sigma}$, and $R^{-1}, h^{1+\sigma}\leq 1$, by making a change of coordinates $\eta/h\mapsto\eta$ and using that $e^{2i\eta\cdot z} = \big(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\big)\big(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\big)e^{2i\eta\cdot z}$, together with some integration by parts, one can check that

$$\Big\|\partial_y^\alpha\partial_\xi^\beta\big[r_R(\frac{x+y}{2},h\xi)\big]\Big\|_{L^2(d\xi)} \lesssim R^{-1}$$

for any $\alpha, \beta \in \mathbb{N}^2$, and hence obtain from lemma 1.2.25 that

$$\|\mathrm{Op}_h^w(r_R^k(x,\xi))w^\chi(t,\cdot)\|_{L^\infty} \lesssim R^{-1}\|w^\chi(t,\cdot)\|_{L^2}.$$

Successively, taking a Littlewood-Paley decomposition such that

$$\chi_1(h^\sigma\xi) \equiv \left[\phi\Big(\frac{R}{h}\xi\Big) + \sum_{hR^{-1}\leq 2^j\leq h^{-\sigma}} (1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\right]\chi_1(h^\sigma\xi),$$

with $\phi \in C_0^\infty(\mathbb{R}^2)$, equal to 1 close to the origin and $\psi \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, we derive that

$$\begin{aligned}(3.3.19)\quad \Big\|\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\Big\|_{L^\infty} &\lesssim \Big\|\mathrm{Op}_h^w\Big(\phi\Big(\frac{R}{h}\xi\Big)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\Big\|_{L^\infty}\\ &+ \sum_{hR^{-1}\leq 2^j\leq h^{-\sigma}} \Big\|\mathrm{Op}_h^w\Big((1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\Big\|_{L^\infty},\end{aligned}$$

and immediately notice that

$$\begin{aligned}(3.3.20)\quad &\Big\|\mathrm{Op}_h^w\Big(\phi\Big(\frac{R}{h}\xi\Big)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\Big\|_{L^\infty}\\ &\qquad= \Big\|\phi(RD_x)\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\Big\|_{L^\infty} \lesssim R^{-1}\|w^\chi(t,\cdot)\|_{L^2},\end{aligned}$$

just by the classical Sobolev injection and the uniform continuity of $\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\varphi\Big(\frac{x}{R}\Big)$ on L^2. Introducing operators Θ_R, Θ_R^{-1}, where $\Theta_R u(x) := u(Rx)$, $\Theta_R^{-1}u(x) := u\big(\frac{x}{R}\big)$, we have the following equality

$$\begin{aligned}(3.3.21)\quad &\mathrm{Op}_h^w\Big((1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big]\\ &\qquad= \Big[\Theta_R^{-1}Op_{h_{Rj}}^w\Big((1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\Big)\varphi(x)\Theta_R\Big]w^\chi\end{aligned}$$

with $h_{Rj} := \frac{h}{R2^j} \leq 1$, and by h_{Rj}-symbolic calculus (that is proposition 1.2.21 with h replaced by h_{Rj}),

$$\begin{aligned}&Op_{h_{Rj}}^w\Big((1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\Big)\varphi(x) =\\ &\qquad Op_{h_{Rj}}^w\Big((1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x)\Big) + Op_{h_{Rj}}^w(r(x,\xi))\end{aligned}$$

with

$$r(x,\xi) = \frac{h_{Rj}}{2i(\pi h_{Rj})^2}\int e^{-\frac{2i}{h_{Rj}}y\cdot\zeta}\left[\int_0^1 \partial_\xi\Big[(1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\Big]\big|_{(\xi+t\zeta)}dt\right](\partial\varphi)(x+y)dyd\zeta.$$

Similarly as before, one can prove that

$$\Big\|\partial_x^\alpha\partial_\xi^\beta\big[r(\frac{x+y}{2},h\xi)\big]\Big\|_{L^2(d\xi)} \lesssim 1$$

for any $\alpha, \beta \in \mathbb{N}^2$, observing that no negative power of h_{Rj} appears in the right hand side of this inequality for the product of $\psi(\xi)$ with any derivative of $(1-\phi)(\frac{\xi}{h_{Rj}})$ is supported for

$h_{Rj} \sim |\xi| \sim 1$. Hence lemma 1.2.25 gives that operator $Op^w_{h_{Rj}}(r(x,\xi))$ is uniformly bounded from L^2 to L^∞ and

$$\big\|Op^w_{h_{Rj}}(r(x,\xi))\Theta_R w^\chi(t,\cdot)\big\|_{L^\infty} \lesssim \|\Theta_R w^\chi(t,\cdot)\|_{L^2} \lesssim R^{-1}\|w^\chi(t,\cdot)\|_{L^2}\,.$$

Since symbol $(1-\phi)\big(\frac{\xi}{h_{Rj}}\big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x)$ is supported for $|x|\sim|\xi|\sim 1$,

$$\begin{aligned}(1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)&\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x)\\ &= \sum_{l=1}^{2} \underbrace{\frac{(1-\phi)\big(\frac{\xi}{h_{Rj}}\big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x)(Rx_l|2^j\xi|-2^j\xi_l)}{|Rx|2^j\xi|-2^j\xi|^2}}_{a_l(x,\xi)}\big(Rx_l|2^j\xi|-2^j\xi_l\big),\end{aligned}$$

with $a_l(x,\xi)\in R^{-1}2^{-j}S_{0,0}(1)$ as long as $R\gg 1$, and by h_{Rj}-symbolic calculus

$$(1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x) = \sum_{l=1}^{2} a_l(x,\xi)\sharp\Big[(Rx_l|2^j\xi|-2^j\xi_l)\widetilde{\psi}(\xi)\Big] + r_{Rj}(x,\xi),$$

with $\widetilde{\psi}\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ such that $\widetilde{\psi}\psi\equiv\psi$, and $r_{Rj}\in h_{Rj}S_{0,0}(1)$. From semi-classical Sobolev injection

$$\|Op^w_{h_{Rj}}(r_{Rj}(x,\xi))\Theta_R w^\chi(t,\cdot)\|_{L^\infty} \lesssim \|\Theta_R w^\chi(t,\cdot)\|_{L^2} \le R^{-1}\|w^\chi(t,\cdot)\|_{L^2}$$

while

(3.3.22)

$$\begin{aligned}&Op^w_{h_{Rj}}(a_l(x,\xi))Op^w_{h_{Rj}}\big((Rx_l|2^j\xi|-2^j\xi)\widetilde{\psi}(\xi)\big)\Theta_R w^\chi\\ &= Op^w_{h_{Rj}}(a_l(x,\xi))\Theta_R\Big[\mathrm{Op}^w_h\big((x_l|\xi|-\xi)\widetilde{\psi}(2^{-j}\xi)\big)w^\chi\Big]\\ &= Op^w_{h_{Rj}}(a_l(x,\xi))\Theta_R\Big[\mathrm{Op}^w_h(\widetilde{\psi}(2^{-j}\xi))\mathrm{Op}^w_h(x_l|\xi|-\xi)w^\chi - \frac{h}{2i}\mathrm{Op}^w_h((2^{-j}\xi)\cdot(\partial\widetilde{\psi})(2^{-j}\xi))w^\chi\Big].\end{aligned}$$

The last thing to do to conclude the proof of the statement is to study continuity of operator $Op^w_{h_{Rj}}(a_l(x,\xi))$.

Lemma 3.3.10. *We have that $Op^w_{h_{Rj}}(a_l(x,\xi)) : L^2\to L^\infty$ is bounded with norm*

$$\Big\|Op^w_{h_{Rj}}(a_l(x,\xi))\Big\|_{\mathcal{L}(L^2;L^\infty)} \lesssim h^{-1}.$$

Proof. The result comes straightly from lemma 1.2.25. Indeed, since symbol $a_l(x,\xi)$ is compactly supported in x there is a smooth cut-off function $\varphi_1\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, with $\varphi_1\varphi\equiv\varphi$, such that

$$\Big|Op^w_{h_{Rj}}(a_l(x,\xi))w\Big| \lesssim \|w\|_{L^2(dx)}\int\Big|\varphi_1\Big(\frac{x+y}{2}\Big)\Big|\sum_{|\alpha|\le 3}\Big\|\partial_y^\alpha\Big[a_l\Big(\frac{x+y}{2},h_{R_j}\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy,$$

and for $|\alpha|\le 3$

$$\begin{aligned}&\Big\|\partial_y^\alpha\Big[a_l\Big(\frac{x+y}{2},h_{R_j}\xi\Big)\Big]\Big\|_{L^2(d\xi)}\\ &\quad\lesssim \frac{R}{h}\Big\|\partial_y^\alpha\Big[\frac{(1-\phi)(\xi)\psi(h_{Rj}\xi)\chi_1(h_{Rj}h^\sigma 2^j\xi)\varphi_1(\frac{x+y}{2})}{|R(\frac{x+y}{2})|\xi|-\xi|^2}\Big(R\Big(\frac{x_l+y_l}{2}\Big)|\xi|-\xi_l\Big)\Big]\Big\|_{L^2(d\xi)}\\ &\quad\lesssim \frac{|\widetilde{\varphi}(\frac{x+y}{2})|}{h}\left(\int\frac{|\psi(h_{Rj}\xi)|^2}{|\xi|^2}d\xi\right)^{\frac12}\lesssim\frac{|\widetilde{\varphi}(\frac{x+y}{2})|}{h},\end{aligned}$$

where $\widetilde{\varphi}\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$. □

Finally, summing up all formulas from (3.3.21) to (3.3.22) and using lemma 3.3.10, we obtain that

$$\Big\|\mathrm{Op}_h^w\Big((1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi(t,\cdot)\Big]\Big\|_{L^\infty}\lesssim R^{-1}(\|w^\chi(t,\cdot)\|_{L^2}+\|\mathcal{M}w^\chi(t,\cdot)\|_{L^2}),$$

for any index $j\in\mathbb{Z}$ such that $hR^{-1}\leq 2^j\leq h^{-\sigma}$. Injecting (3.3.20) and the above inequality in (3.3.19), and using that $[\mathcal{M},\mathrm{Op}_h^w(\chi(h^\sigma\xi))]=i\mathrm{Op}_h^w((\partial\chi)(h^\sigma\xi)(h^\sigma|\xi|))$ is uniformly continuous on L^2, we deduce (3.3.18) (the loss in $\log R+|\log h|$ arising from the fact that we are considering a sum over indices j, with $\log h-\log R\lesssim j\lesssim\log(h^{-1})$). □

3.3.3 Proof of the main theorems

Proof of theorem 1.1.2. Straightforward after propositions 2.2.13, 3.2.7, 3.3.7. □

Proof of theorem 1.1.1. Let us prove that, for small enough data satisfying (1.1.4), Cauchy problem (1.1.1)-(1.1.2) has a unique global solution. This result follows by a local existence argument, after having proved that there exist two integers $n\gg\rho\gg 1$, two constants $A',B'>1$ sufficiently large, $\varepsilon_0>0$ sufficiently small, and $0<\delta\ll\delta_2\ll\delta_1\ll\delta_0$ small, such that, for any $0<\varepsilon<\varepsilon_0$, if (u,v) is solution to (1.1.1)-(1.1.2) in $[1,T]\times\mathbb{R}^2$, for some $T>1$, with $\partial_{t,x}u\in C^0([1,T];H^n(\mathbb{R}^2))$, $v\in C^0([1,T];H^{n+1}(\mathbb{R}^2))\cap C^1([1,T];H^n(\mathbb{R}^2))$, and satisfies

(3.3.23a)
$$\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}+\|\nabla_x u(t,\cdot)\|_{H^{\rho+1,\infty}}+\||D_x|u(t,\cdot)\|_{H^{\rho+1,\infty}}+\sum_{j=1}^2\|\mathrm{R}_j\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}\leq A'\varepsilon t^{-\frac12},$$

(3.3.23b)
$$\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}}+\|v(t,\cdot)\|_{H^{\rho+1,\infty}}\leq A'\varepsilon t^{-1},$$

(3.3.23c)
$$\|\partial_t u(t,\cdot)\|_{H^n}+\|\nabla_x u(t,\cdot)\|_{H^n}+\|\partial_t v(t,\cdot)\|_{H^n}+\|\nabla_x v(t,\cdot)\|_{H^n}+\|v(t,\cdot)\|_{H^n}\leq B'\varepsilon t^{\frac{\delta}{2}},$$

(3.3.23d)
$$\sum_{|I|=k}\big[\|\partial_t\Gamma^I u(t,\cdot)\|_{L^2}+\|\nabla_x\Gamma^I u(t,\cdot)\|_{L^2}+\|\partial_t\Gamma^I v(t,\cdot)\|_{L^2}+\|\nabla_x\Gamma^I v(t,\cdot)\|_{L^2}+\|\Gamma^I v(t,\cdot)\|_{L^2}\big]\leq B'\varepsilon t^{\frac{\delta_{3-k}}{2}},\quad 1\leq k\leq 3,$$

for every $t\in[1,T]$, then in the same interval it satisfies

(3.3.24a)
$$\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}+\|\nabla_x u(t,\cdot)\|_{H^{\rho+1,\infty}}+\||D_x|u(t,\cdot)\|_{H^{\rho+1,\infty}}+\sum_{j=1}^2\|\mathrm{R}_j\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}\leq\frac{A'}{2}\varepsilon t^{-\frac12},$$

(3.3.24b)
$$\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}}+\|v(t,\cdot)\|_{H^{\rho+1,\infty}}\leq\frac{A'}{2}\varepsilon t^{-1},$$

(3.3.24c)
$$\|\partial_t u(t,\cdot)\|_{H^n}+\|\nabla_x u(t,\cdot)\|_{H^n}+\|\partial_t v(t,\cdot)\|_{H^n}+\|\nabla_x v(t,\cdot)\|_{H^n}+\|v(t,\cdot)\|_{H^n}\leq\frac{B'}{2}\varepsilon t^{\frac{\delta}{2}},$$

(3.3.24d)

(3.3.24e)
$$\sum_{|I|=k}\big[\|\partial_t\Gamma^I u(t,\cdot)\|_{L^2}+\|\nabla_x\Gamma^I u(t,\cdot)\|_{L^2}+\|\partial_t\Gamma^I v(t,\cdot)\|_{L^2}+\|\nabla_x\Gamma^I v(t,\cdot)\|_{L^2}+\|\Gamma^I v(t,\cdot)\|_{L^2}\big]\leq\frac{B'}{2}\varepsilon t^{\frac{\delta_{3-k}}{2}},\quad 1\leq k\leq 3.$$

We remind that, if $I = (i_1, \dots, i_n)$ is a multi-index of length $|I| = n$, with $i_j \in \{1, \dots, 5\}$, $\Gamma^I = \Gamma_{i_1} \cdots \Gamma_{i_n}$ is a product of vector fields in family $\mathcal{Z} = \{\Omega, Z_j, \partial_j | j = 1, 2\}$.

We can immediately observe that the above bounds are verified at time $t = 1$ after (1.1.4) and Sobolev injection. By definition (1.1.5) we also notice that

$$\|u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2} \|\mathrm{R}_j u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} \leq 2\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} + 2\||D_x| u(t,\cdot)\|_{H^{\rho+1,\infty}} + 2\sum_{j=1}^{2} \left(\|\partial_j u(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}_j \partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}\right), \tag{3.3.25a}$$

$$\|v_\pm(t,\cdot)\|_{H^{\rho,\infty}} \leq 2\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + 2\|v(t,\cdot)\|_{H^{\rho+1,\infty}}, \tag{3.3.25b}$$

and, conversely,

$$\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} + \||D_x| u(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2} \left(\|\partial_j u(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}_j \partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}\right) \leq \|u_+(t,\cdot)\|_{H^{\rho+1,\infty}} + \|u_-(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2} \left(\|\mathrm{R}_j u_+(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}_j u_-(t,\cdot)\|_{H^{\rho+1,\infty}}\right), \tag{3.3.26a}$$

$$\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + \|v(t,\cdot)\|_{H^{\rho+1,\infty}} \leq \|v_+(t,\cdot)\|_{H^{\rho,\infty}} + \|v_-(t,\cdot)\|_{H^{\rho,\infty}}. \tag{3.3.26b}$$

Moreover, reminding definition (1.1.9) of generalized energies $E_n(t; u_\pm, v_\pm)$, $E_3^k(t; u_\pm, v_\pm)$, for $n \geq 3$ and $0 \leq k \leq 2$, and of set $\mathcal{I}_3^k$ in (2.1.17), there is a constant $C > 0$ such that

$$C^{-1} E_n(t; u_\pm, v_\pm) \leq \big[\|\partial_t u(t,\cdot)\|_{H^n}^2 + \|\nabla_x u(t,\cdot)\|_{H^n}^2 + \|\partial_t v(t,\cdot)\|_{H^n}^2 + \|\nabla_x v(t,\cdot)\|_{H^n}^2 + \|v(t,\cdot)\|_{H^n}^2\big] \leq C E_n(t; u_\pm, v_\pm), \tag{3.3.27a}$$

and for any $0 \leq k \leq 2$,

$$C^{-1} E_3^k(t; u_\pm, v_\pm) \leq \sum_{I \in \mathcal{I}_3^k} \big[\|\partial_t \Gamma^I u(t,\cdot)\|_{L^2}^2 + \|\nabla_x \Gamma^I u(t,\cdot)\|_{L^2}^2 + \|\partial_t \Gamma^I v(t,\cdot)\|_{L^2}^2 + \|\nabla_x \Gamma^I v(t,\cdot)\|_{L^2}^2 + \|\Gamma^I v(t,\cdot)\|_{L^2}^2\big] \leq C E_3^k(t; u_\pm, v_\pm). \tag{3.3.27b}$$

Therefore, after (3.3.25), (3.3.27), and (3.3.23), we deduce that estimates (1.1.11) are satisfied with $A = 2A'$, $B = C_1 B'$, for some new $C_1 > 0$, so choosing for instance $K_1 = 4$ and K_2 sufficiently large, theorem 1.1.2 and inequalities (3.3.26), (3.3.27) imply (3.3.24). □

Appendix A

The aim of this appendix is to prove the continuity of some trilinear integral operators (see lemmas A.5 and A.6) that arise in subsection 2.2.2 when performing a normal form argument at the energy level, and of some bilinear integral operators (see lemma A.8) that instead appear in subsection 3.1.2 when we perform a normal form the wave equation (see proposition 3.1.2). All the other results of this chapter are stated and proved in view of the above mentioned lemmas.

Lemma A.1. *Let $\check{a}(x)$ denote the inverse transform of a function $a(\xi)$.*
(i) If $a : \mathbb{R}^2 \to \mathbb{C}$ is such that, for any $\alpha \in \mathbb{N}^2$ with $1 \leq |\alpha| \leq 4$,

$$|a(\xi)| \lesssim \langle \xi \rangle^{-3} \quad \text{and} \quad |\partial^\alpha a(\xi)| \lesssim_\alpha (|\xi|\langle \xi \rangle^{-1})^{1-|\alpha|}\langle \xi \rangle^{-3} \quad \forall \xi \in \mathbb{R}^2$$

then

$$|\check{a}(x)| \lesssim |x|^{-1}\langle x \rangle^{-2}, \quad \forall x \in \mathbb{R}^2.$$

(ii) If a is such that, for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 3$,

$$|\partial^\alpha a(\xi)| \lesssim (|\xi|\langle \xi \rangle^{-1})^{-|\alpha|}\langle \xi \rangle^{-3}, \quad \forall \xi \in \mathbb{R}^2$$

then

$$|\check{a}(x)| \lesssim \langle x \rangle^{-2}, \quad \forall x \in \mathbb{R}^2;$$

(iii) Let $N \in \mathbb{N}$. If for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq N$ there exists $f_\alpha \in L^1(\mathbb{R}^2)$ such that $|\partial^\alpha a(\xi)| \lesssim_\alpha |f_\alpha(\xi)|$ then

$$|\check{a}(x)| \lesssim \langle x \rangle^{-N}, \quad \forall x \in \mathbb{R}^2.$$

Proof. (i) We consider a cut-off function $\phi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in the unit ball and write

$$\check{a}(x) = K_0(x) + K_1(x)$$
$$\text{with} \quad K_0(x) := \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi}a(\xi)\phi(\xi)d\xi, \quad K_1(x) := \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi}a(\xi)(1-\phi)(\xi)d\xi. \tag{A.1}$$

On the one hand, since $|\partial^\alpha a(\xi)| \lesssim_\alpha \langle \xi \rangle^{-3}$ on the support of $(1-\phi)(\xi)$ for any $|\alpha| \leq 4$, we immediately deduce by integration by parts that $|K_1(x)| \lesssim \langle x \rangle^{-4}$ for any $x \in \mathbb{R}^2$. On the other hand, again an integration by parts gives that

$$xK_0(x) = \int e^{ix\cdot\xi}a_1(\xi)d\xi$$

with $a_1(\xi)$ supported for $|\xi| \lesssim 1$ and such that $|\partial^\alpha a_1(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|}$ for any $\xi \in \mathbb{R}^2$, any $|\alpha| \leq 3$. This implies that $|xK_0(x)| \lesssim 1$ for any $x \in \mathbb{R}^2$. Moreover, $|x^\alpha x\, K_0(x)| \lesssim_\alpha 1$ for any $|\alpha| \leq 3$. This

is obvious in the unit ball. Out of the unit ball we consider a Littlewood-Paley decomposition in frequencies so that

$$\phi(\xi) = \phi(\xi)\left[\varphi_0(2^{-L_0}\xi) + \sum_{k=L_0+1}^{0} \varphi(2^{-k}\xi)\right],$$

with $supp\varphi_0 \subset B_1(0)$, $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and $L_0 < 0$ such that $2^{L_0} \sim |x|^{-1}$, and write

$$xK_0(x) = K_0^0(x) + \sum_{k=L_0+1}^{0} K_0^k(x)$$

$$\text{with} \quad K_0^0(x) := \int e^{ix\cdot\xi} a_1(\xi)\varphi_0(2^{-L_0}\xi)d\xi, \quad K_0^k(x) := \int e^{ix\cdot\xi} a_1(\xi)\varphi_k(2^{-k}\xi)d\xi.$$

Performing a change of coordinates and making some integrations by parts we deduce that

$$|K_0^0(x)| \lesssim 2^{2L_0} \quad \text{and} \quad |K_0^k(x)| \lesssim 2^{2k}\langle 2^k x\rangle^{-3}, \quad L_0 + 1 \le k \le 0$$

for any $x \in \mathbb{R}^2$, which finally implies $|xK_0(x)| \lesssim 2^{2L_0} \sim |x|^{-2}$.

(ii) The result follows splitting $\check{a}$ as in (A.1) and applying to $K_0(x)$ the same argument previously used for $xK_0(x)$.

(iii) The result follows straightly from integration by parts and the fact that $f_\alpha \in L^1(\mathbb{R}^2)$ for any $|\alpha| \le N$. □

Corollary A.2. *Let $d \in \mathbb{N}^*$, $N \in \mathbb{N}$ and $g_\beta \in L^1(\mathbb{R}^d)$ for every $|\beta| \le N$.*
(i) If $a(\xi,\eta) : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{C}$ is such that, for any $\beta \in \mathbb{N}^d$ with $|\beta| \le N$,

$$\begin{aligned} |\partial_\eta^\beta a(\xi,\eta)| &\lesssim_\beta \langle\xi\rangle^{-3}|g_\beta(\eta)|, \\ |\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)| &\lesssim_{\alpha,\beta} (|\xi|\langle\xi\rangle^{-1})^{1-|\alpha|}\langle\xi\rangle^{-3}|g_\beta(\eta)|, \quad 1 \le |\alpha| \le 4. \end{aligned} \tag{A.2}$$

for any $(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^d$, then

$$\left|\int e^{ix\cdot\xi+iy\cdot\eta} a(\xi,\eta)d\xi d\eta\right| \lesssim |x|^{-1}\langle x\rangle^{-2}\langle y\rangle^{-N}, \quad \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^d. \tag{A.3}$$

Moreover, if $d = 2$ and $N = 3$, for any $u, v \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$

$$\left\|\int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L^2(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^\infty} \,(or\ \lesssim \|u\|_{L^\infty}\|v\|_{L^2}) \tag{A.4a}$$

and

$$\left\|\int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L^\infty(dx)} \lesssim \|u\|_{L^\infty}\|v\|_{L^\infty}. \tag{A.4b}$$

(ii) If $a(\xi,\eta)$ is such that, for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \le 3$, $\beta \in \mathbb{N}^d$ with $|\beta| \le N$,

$$|\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)| \lesssim_{\alpha,\beta} (|\xi|\langle\xi\rangle^{-1})^{-|\alpha|}\langle\xi\rangle^{-3}|g_\beta(\eta)|, \tag{A.5}$$

for any $(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^d$, then

$$\left|\int e^{ix\cdot\xi+iy\cdot\eta} a(\xi,\eta)d\xi d\eta\right| \lesssim \langle x\rangle^{-2}\langle y\rangle^{-N}, \quad \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^d. \tag{A.6}$$

Moreover, if $d = 2, N = 3$, for any $u, v \in L^2(\mathbb{R}^2)$

$$\left\| \int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L^2(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^2} \tag{A.7a}$$

while if $u \in L^2(\mathbb{R}^2), v \in L^\infty(\mathbb{R}^2)$,

$$\left\| \int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L^\infty(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^\infty}. \tag{A.7b}$$

Proof. Let

$$K(x,\eta) := \int e^{ix\cdot\xi} a(\xi,\eta)d\xi \quad \text{and} \quad \widetilde{K}(x,y) := \int e^{ix\cdot\xi} K(x,\eta)d\eta.$$

By the hypothesis on $a(\xi,\eta)$ and lemma A.1 (i) (resp. (ii)) we derive that, for any $\beta \in \mathbb{N}^d$ with $|\beta| \le N$,

$$|\partial_\eta^\beta K(x,\eta)| \lesssim |x|^{-1}\langle x\rangle^{-2}|g_\beta(\eta)| \quad \Big(\text{resp. } |\partial_\eta^\beta K(x,\eta)| \lesssim \langle x\rangle^{-2}|g_\beta(\eta)|\Big) \quad \forall (x,\eta) \in \mathbb{R}^2 \times \mathbb{R}^d.$$

Hence (A.3) (resp. (A.6)) follows applying lemma A.1 (iii) to $\widetilde{K}(x,y)$.

(i) If $d = 2, N = 3$, inequality (A.4a) from the fact that

$$\int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta = \int \widetilde{K}(x-y, y-z)u(y)v(z)dydx,$$

and by (A.3), for $L = L^2$ or $L = L^\infty$,

(A.8)

$$\begin{aligned}\left\| \int \widetilde{K}(x-y,y-z)\widetilde{u}(y)\widetilde{v}(z)dydz \right\|_{L(dx)} &\lesssim \left\| \int |x-y|^{-1}\langle x-y\rangle^{-2}\langle y-z\rangle^{-3}\widetilde{u}(y)\widetilde{v}(z)dydz \right\|_{L(dx)} \\ &\lesssim \int |y|^{-1}\langle y\rangle^{-2}\langle z\rangle^{-3}\|\widetilde{u}(\cdot-y)\widetilde{v}(\cdot-y-z)\|_{L(dx)}dydz \\ &\lesssim \|\widetilde{u}\|_{L^\infty}\|\widetilde{v}\|_L \ (\text{or } \lesssim \|\widetilde{u}\|_L\|\widetilde{v}\|_{L^\infty}).\end{aligned}$$

(ii) By inequality (A.6)

$$\begin{aligned}&\left\| \int \widetilde{K}(x-y,y-z)u(y)v(z)dydz \right\|_{L^2(dx)} \lesssim \left\| \int \langle x-y\rangle^{-2}\langle y-z\rangle^{-3}|u(y)||v(z)|dydz \right\|_{L^2(dx)} \\ &\lesssim \int \langle y-z\rangle^{-3}|u(y)||v(z)|dydz \lesssim \int |v(z)| \left(\int \langle y-z\rangle^{-3}dy \right)^{\frac{1}{2}} \left(\int \langle y-z\rangle^{-3}|u(y)|^2dy \right)^{\frac{1}{2}} dz \\ &\lesssim \|v\|_{L^2} \left(\int \langle y-z\rangle^{-3}|u(y)|^2dydz \right)^{\frac{1}{2}} \lesssim \|u\|_{L^2}\|v\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}\left\| \int \widetilde{K}(x-y,y-z)u(y)v(z)dydz \right\|_{L^\infty(dx)} &\lesssim \left\| \int \langle x-y\rangle^{-2}\langle y-z\rangle^{-3}|u(y)||v(z)|dydz \right\|_{L^\infty(dx)} \\ &\lesssim \|v\|_{L^\infty} \left\| \int \langle x-y\rangle^{-2}|u(y)|dy \right\|_{L^\infty(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^\infty}.\end{aligned}$$

□

Lemma A.3 (Sobolev norm of a product)**.** *Let $s \in \mathbb{N}^*$. For any $u, v \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,*

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s}; \tag{A.9}$$

for any $u, v \in H^{s,\infty}(\mathbb{R}^2) \cap H^{s+2}(\mathbb{R}^2)$, any $\theta \in]0,1[$,

$$\|uv\|_{H^{s,\infty}} \lesssim \|u\|^{1-\theta}_{H^{s,\infty}}\|u\|^{\theta}_{H^{s+2}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|^{1-\theta}_{H^{s,\infty}}\|v\|^{\theta}_{H^{s+2}}. \tag{A.10}$$

Proof. Inequality (A.9) is a classical result (see, for instance, [2]).
In order to deduce (A.10) we decompose product uv as follows:

$$uv = T_u v + T_v u + R(u,v), \tag{A.11}$$

where $T_u v$ is the para-product of u times v defined by

$$T_u v := S_{-3}u S_0 v + \sum_{k\geq 1} S_{k-3}u\Delta_k v,$$

with $S_k = \chi(2^{-k}D_x)$, $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1/2$, $\chi(\xi) = 0$ for $|\xi| \geq 1$, $\Delta_0 = S_0$ and $\Delta_k = S_k - S_{k-1}$ for $k \geq 1$, and $R(u,v) = \sum_k \Delta_k u \widetilde{\Delta}_k v$, with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Since

$$T_u v = \sum_{j\geq 0} \Delta_j(T_u v) = \sum_{\substack{j,k \\ |j-k|\leq N_0}} \Delta_j[S_{k-3}u\Delta_k v]$$

for a certain $N_0 \in \mathbb{N}$, by definition 1.2.1 (*iii*) of the $H^{s,\infty}$ norm and the fact that $\|\Delta_k v\|_{L^\infty} \lesssim 2^k\|\Delta_k v\|_{L^2}$ we deduce that, for any fixed $\theta \in]0,1[$,

(A.12)

$$\begin{aligned}
&\|T_u v\|_{H^{s,\infty}} = \|\langle D_x\rangle^s T_u v\|_{L^\infty} \leq \sum_{\substack{j,k \\ |j-k|\leq N_0}} 2^{js}\|\Delta_j[S_{k-3}u\Delta_k v]\|_{L^\infty} \\
&\leq \sum_{\substack{j,k \\ |j-k|\leq N_0}} 2^{js}\|S_{k-3}u\|_{L^\infty}\|\Delta_k v\|_{L^\infty} \leq \sum_{\substack{j,k \\ |j-k|\leq N_0}} 2^{js}\|u\|_{L^\infty}(2^{-ks}\|\Delta_k\langle D_x\rangle^s v\|_{L^\infty})^{1-\theta}(2^k\|\Delta_k v\|_{L^2})^{\theta} \\
&\lesssim \sum_{\substack{j,k \\ |j-k|\leq N_0}} 2^{(j-k)s}\|u\|_{L^\infty}\|\Delta_k\langle D_x\rangle^s v\|^{1-\theta}_{L^\infty}\left(2^{-k}\|\Delta_k\langle D_x\rangle^{s+2} v\|_{L^2}\right)^{\theta} \\
&\lesssim \|u\|_{L^\infty}\|v\|^{1-\theta}_{H^{s,\infty}}\|v\|^{\theta}_{H^{s+2}}.
\end{aligned}$$

Similarly,

$$\|T_v u\|_{H^{s,\infty}} + \|R(u,v)\|_{H^{s,\infty}} \lesssim \|u\|^{1-\theta}_{H^{s,\infty}}\|u\|^{\theta}_{H^{s+2}}\|v\|_{L^\infty}.$$

□

Corollary A.4. *Let $s \in \mathbb{N}^*$, $a_1(\xi) \in S_0^{m_1}(\mathbb{R}^2)$, $a_2(\xi) \in S_0^{m_2}(\mathbb{R}^2)$, for some $m_1, m_2 \geq 0$. For any $u \in H^{s+m_1}(\mathbb{R}^2) \cap H^{m_1,\infty}(\mathbb{R}^2)$, $v \in H^{s+m_2}(\mathbb{R}^2) \cap H^{m_2,\infty}(\mathbb{R}^2)$,*

$$\|[a_1(D_x)u]\,[a_2(D_x)v]\|_{H^s} \lesssim \|u\|_{H^{s+m_1}}\|v\|_{H^{m_2,\infty}} + \|u\|_{H^{m_1,\infty}}\|v\|_{H^{s+m_2}}; \tag{A.13}$$

for any $u \in H^{s+m_1,\infty}(\mathbb{R}^2) \cap H^{s+m_1+2}(\mathbb{R}^2)$, $v \in H^{s+m_2,\infty}(\mathbb{R}^2) \cap H^{s+m_2+2}(\mathbb{R}^2)$, any $\theta \in]0,1[$,

$$\begin{aligned}
&\|[a_1(D_x)u]\,[a_2(D_x)v]\|_{H^{s,\infty}} \\
&\qquad \lesssim \|u\|^{1-\theta}_{H^{s+m_1,\infty}}\|u\|^{\theta}_{H^{s+m_1+2}}\|v\|_{H^{m_2,\infty}} + \|u\|_{H^{m_1,\infty}}\|v\|^{1-\theta}_{H^{s+m_2,\infty}}\|v\|^{\theta}_{H^{s+m_2+2}}.
\end{aligned} \tag{A.14}$$

Proof. The result of the statement follows writing $[a_1(D_x)u]\,[a_2(D_x)v]$ in terms of para-products as in (A.11), and using that $T_{a_1(D)u}(a_2(D)v)$, $T_{a_2(D)v}(a_1(D)u)$ and remainder $R\big(a_1(D)u, a_2(D)v\big)$ can be written from $\widetilde{u} = \langle D_x\rangle^{m_1}u$, $\widetilde{v} = \langle D_x\rangle^{m_2}v$, as done below for the former of these terms:

$$\begin{aligned}T_{a_1(D)u}(a_2(D)v) &= [S_{-3}a_1(D)\langle D_x\rangle^{-m_1}\widetilde{u}][S_0a_2(D)\langle D_x\rangle^{-m_2}\widetilde{v}]\\ &\quad+\sum_k[S_{k-3}a_1(D)\langle D_x\rangle^{-m_1}\widetilde{u}][\Delta_ka_2(D)\langle D_x\rangle^{-m_2}\widetilde{v}].\end{aligned}$$

Since $a_j(\xi)\langle\xi\rangle^{-m_j}\in S^0_0(\mathbb{R}^2)$, $j=1,2$, operators $S_ka_j(D)\langle D_x\rangle^{-m_j}$, $\Delta_ka_j(D)\langle D_x\rangle^{-m_j}$ have the same spectrum (i.e. the support of the Fourier transform) up to a negligible constant of S_k and Δ_k respectively. □

In the following lemma we prove a result of continuity for a trilinear integral operator defined from multiplier $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ given by (2.2.42) (resp. by (2.2.53)) for $k=1,2$ (resp. $k=3$), any $j_1,j_2,j_3\in\{+,-\}$. It is useful to observe that, since

$$B^k_{(j_1,j_2,j_3)}(\xi,\eta)=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta|}\eta_k,\quad k=1,2$$

from (2.2.42), while

$$B^3_{(j_1,j_2,j_3)}(\xi,\eta)=j_1\,\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2\langle\xi-\eta\rangle}$$

(2.2.53), we have that

$$\begin{aligned}(\text{A.15})\quad \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi}B^k_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta &= \frac{j_2}{2}(u\mathrm{R}_kv)(x)-\frac{j_1}{2}\Big[\Big(\frac{D_1}{\langle D_x\rangle}u\Big)v\Big](x)\\ &\quad+\frac{j_1}{2}D_1\left[(\langle D_x\rangle^{-1}u)v\right](x)-\frac{j_3}{2j_1j_2}\langle D_x\rangle[(\langle D_x\rangle^{-1}u)(\mathrm{R}_kv)](x)\end{aligned}$$

for $k=1,2$, while for $k=3$

$$\begin{aligned}(\text{A.16})\quad \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi}B^3_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta &= \frac{1}{2}(uv)(x)+\frac{j_1j_2}{2}[(\langle D_x\rangle^{-1}u)|D_x|v](x)\\ &\quad-\frac{j_1j_3}{2}\langle D_x\rangle\left[(\langle D_x\rangle^{-1}u)v\right](x).\end{aligned}$$

Lemma A.5. *Let $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ be given by (2.2.42) when $k=1,2$, and by (2.2.53) when $k=3$, for any $j_1,j_2,j_3\in\{+,-\}$. Let also $\delta_k=1$ if $k\in\{1,2\}$, $\delta_k=0$ if $k=3$. For any $u,w\in L^2(\mathbb{R}^2), v\in H^{2,\infty}(\mathbb{R}^2)$ such that $\delta_k\mathrm{R}_kv\in H^{2,\infty}(\mathbb{R}^2)$,*

$$(\text{A.17})\quad \left|\int B^k_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta\right|\lesssim\|u\|_{L^2}\left(\|v\|_{H^{7,\infty}}+\delta_k\|\mathrm{R}_kv\|_{H^{7,\infty}}\right)\|w\|_{L^2}.$$

Proof. First of all we observe that for $k\in\{1,2\}$

(A.18a)

$$\begin{aligned}\int B^k_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta &= \frac{j_2}{2}\int\widehat{u(\mathrm{R}_kv)}(\xi)\hat{w}(-\xi)d\xi-\frac{j_1}{2}\int\widehat{\left[\left(\tfrac{D_x}{\langle D_x\rangle}u\right)v\right]}(\xi)\hat{w}(-\xi)d\xi\\ &\quad+\frac{j_1}{2}\int\frac{\xi_1}{\langle\xi-\eta\rangle}\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta-\frac{j_3}{2j_1j_2}\int\frac{\langle\xi\rangle}{\langle\xi-\eta\rangle}\hat{u}(\xi-\eta)\widehat{\mathrm{R}_kv}(\eta)\hat{w}(-\xi)d\xi d\eta,\end{aligned}$$

while for $k=3$,

$$\text{(A.18b)}\quad \int B^3_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta = \frac{1}{2}\int \widehat{uv}(\xi)\hat{w}(-\xi)d\xi + \frac{j_1j_2}{2}\int \widehat{\left[(\langle D_x\rangle^{-1}u)|D_x|v\right]}(\xi)\hat{w}(-\xi)d\xi - \frac{j_1j_3}{2}\int \frac{\langle\xi\rangle}{\langle\xi-\eta\rangle}\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi.$$

Hölder's inequality shows immediately that the first two addends in both above right hand sides are bounded by the right hand side of (A.17). Then the result of the statement follows by proving that inequality (A.17) is satisfied by integrals such as

$$\int a(\xi,\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta$$

with $a(\xi,\eta)=\xi_1\langle\xi-\eta\rangle^{-1}$ or $a(\xi,\eta)=\langle\xi\rangle\langle\xi-\eta\rangle^{-1}$, and some general functions $u_1,u_3\in L^2(\mathbb{R}^2), u_2\in L^\infty(\mathbb{R}^2)$. By taking a Littlewood-Paley decomposition we can split the above integral as

$$\text{(A.19)}\qquad \sum_{k,l\geq 0}\int a(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta,$$

with $\varphi_0\in C_0^\infty(\mathbb{R}^2)$, $\varphi\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ and $\varphi_k(\zeta)=\varphi(2^{-k}\zeta)$ for any $k\in\mathbb{N}^*$. Since frequencies ξ,η are bounded on the support of $\varphi_0(\xi)\varphi_0(\eta)$, kernel

$$K_0(x,y):=\int e^{ix\cdot\xi+iy\cdot\eta}a(\xi,\eta)\varphi_0(\xi)\varphi_0(\eta)d\xi d\eta$$

is such that $|K_0(x,y)|\lesssim\langle x\rangle^{-3}\langle y\rangle^{-3}$ for any (x,y), after the first part of corollary A.2 (i). Therefore

$$\begin{aligned}
&\left|\int a(\xi,\eta)\varphi_0(\xi)\varphi_0(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right|\\
&=\left|\int K_0(z-x,x-y)u_1(x)u_2(y)u_3(z)dxdydz\right|\\
&\lesssim\int\langle z-x\rangle^{-3}\langle x-y\rangle^{-3}|u_1(x)||u_2(y)||u_3(z)|dxdydz\\
&\lesssim\|u_2\|_{L^\infty}\int\langle x\rangle^{-3}|u_1(z-x)||u_3(z)|dxdz\lesssim\|u_1\|_{L^2}\|u_2\|_{L^\infty}\|u_3\|_{L^2},
\end{aligned}$$

where last inequality obtained by Hölder inequality.

For positive indices l,k such that $l>k+N_0\geq 0$ (resp. $|l-k|\leq N_0$), for a suitably large integer $N_0>1$, we have that $|\xi|<|\eta|\sim|\xi-\eta|$ (resp. $|\xi|\sim|\eta|$) on the support of $\varphi_k(\xi)\varphi_l(\eta)$. If we define $a_{l>k+N_0}(\xi,\eta):=a(\xi,\eta)\langle\eta\rangle^{-1}$ and $a_{|l-k|\leq N_0}(\xi,\eta):=a(\xi,\eta)\langle\eta\rangle^{-7}$, it is a computation to check that, for any $\alpha,\beta\in\mathbb{N}^2$ with $|\alpha|,|\beta|\leq 3$,

$$|\partial_\xi^\alpha\partial_\eta^\beta[a_{l>k+N_0}(2^k\xi,2^l\eta)]|+|\partial_\xi^\alpha\partial_\eta^\beta a_{|l-k|\leq N_0}(\xi,\eta)|\lesssim 2^{-l}.$$

Hence, their associated kernels $K_{l>k+N_0}(x,y)$ and $K_{|l-k|\leq N_0}(x,y)$ are such that

$$|K_{l>k+N_0}(x,y)|+|K_{|l-k|\leq N_0}(x,y)|\lesssim 2^{2k}2^l\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3},\quad\forall(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$$

as follows after a change of coordinates and some integrations by parts, and for any $l>k+N_0$

$$\begin{aligned}
&\left|\int a(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right|\\
&=\left|\int K_{l>k+N_0}(z-x,x-y)u_1(x)[\langle D_x\rangle u_2](y)u_3(z)dxdydz\right|\\
&\lesssim 2^{2k}2^l\left|\int\langle 2^k(z-x)\rangle^{-3}\langle 2^l(x-y)\rangle^{-3}|u_1(x)||\langle D_x\rangle u_2(y)||u_3(z)|dxdydz\right|\\
&\lesssim 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u_1\|_{L^2}\|u_2\|_{H^{1,\infty}}\|u_3\|_{L^2},
\end{aligned}\qquad\text{(A.20)}$$

while for $|l-k| \leq N_0$

$$\begin{aligned}
&\left|\int a(\xi,\eta)\varphi_l(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right| \\
=&\left|\int K_{|l-k|\leq N_0}(z-x,x-y)u_1(x)[\langle D_x\rangle^7 u_2](y)u_3(z)dxdydz\right| \\
\lesssim& 2^{3l}\left|\int \langle 2^l(z-x)\rangle^{-3}\langle 2^l(x-y)\rangle^{-3}|u_1(x)||\langle D_x\rangle^7 u_2(y)||u_3(z)|dxdydz\right| \\
\lesssim& 2^{-l}\|u_1\|_{L^2}\|u_2\|_{H^{7,\infty}}\|u_3\|_{L^2}.
\end{aligned} \tag{A.21}$$

Finally, when positive indices l,k are such that $k > l - N_0$ we observe that frequencies ξ and $\xi-\eta$ are equivalent and of size 2^k on the support of $\varphi_k(\xi)\varphi_l(\eta)$. If we take $a_{k>l-N_0}(\xi,\eta)$ equal to $a_{l>k+N_0}(\xi,\eta)$, denote by $K_{k>l-N_0}(x,y)$ its associated kernel (which is hence equal to $K_{l>k+N_0}(x,y)$), and introduce two new smooth cut-off function $\varphi^1,\varphi^2 \in C_0^\infty(\mathbb{R}^2)$ equal to 1 on the support of φ, together with operators $\Delta_k^1 := \varphi^1(2^{-k}D_x), \Delta_k^2 := \varphi^2(2^{-k}D_x)$, we deduce that

$$\begin{aligned}
&\left|\int a(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right| \\
=&\left|\int K_{k>l-N_0}(z-x,x-y)[\Delta_k^1 u_1](x)[\langle D_x\rangle u_2](y)[\Delta_k^2 u_3](z)dxdydz\right| \\
\lesssim& 2^{2k}2^l\left|\int \langle 2^k(z-x)\rangle^{-3}\langle 2^l(x-y)\rangle^{-3}|[\Delta_k^1 u_1](x)||\langle D_x\rangle u_2(y)||[\Delta_k^2 u_3](z)|dxdydz\right| \\
\lesssim& 2^{-l}\|\Delta_k^1 u_1\|_{L^2}\|u_2\|_{H^{1,\infty}}\|\Delta_k^2 u_3\|_{L^2}.
\end{aligned}$$

Combining decomposition (A.19) together with (A.20), (A.21) and Cauchy-Schwarz inequality we finally obtain that

$$\left|\int a(\xi,\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right| \lesssim \|u_1\|_{L^2}\|u_2\|_{H^{7,\infty}}\|u_3\|_{L^2},$$

□

Lemma A.6. *Let $\varepsilon > 0$ be small, $N \in \mathbb{N}^*$, and $\sigma^N(\xi,\eta) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ be supported for $|\xi| \leq \varepsilon\langle\eta\rangle$ and such that, for any $\alpha,\beta \in \mathbb{N}^2$,*

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma^N(\xi,\eta)| \lesssim |\xi|^{N+1-|\alpha|}\langle\eta\rangle^{-N-|\beta|}, \quad \forall(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

For any $(j_1,j_2,j_3) \in \{+,-\}^3$ let also

$$\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{\sigma^N(\eta,\xi-\eta)}{j_1\langle\xi-\eta\rangle + j_2|\eta| - j_3\langle\xi\rangle}. \tag{A.22}$$

Then for any $\alpha,\beta \in \mathbb{N}^2$

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\right| \lesssim_{\alpha,\beta} \langle\xi-\eta\rangle^{2-N+|\alpha|+2|\beta|}|\eta|^{N-|\beta|}, \quad \forall(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 \tag{A.23}$$

and if $N \geq 15$, for any $u,w \in L^2(\mathbb{R}^2)$, $v \in H^{N+3,\infty}(\mathbb{R}^2)$,

$$\left|\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta\right| \lesssim \|u\|_{L^2}\|v\|_{H^{N+3,\infty}}\|w\|_{L^2}. \tag{A.24}$$

Proof. From definition (A.22) function $\widetilde{\sigma}^N_{(j_1,j_2,j_3)}$ can be written as follows

$$\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta) = \frac{j_1\langle\xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta| - 2(\xi-\eta)\cdot\eta}\sigma^N_{(j_1,j_2,j_3)}(\eta,\xi-\eta).$$

We observe that

$$[j_1 j_2 \langle \xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta]^{-1} \lesssim \langle \xi-\eta\rangle |\eta|^{-1}, \quad \forall (\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2$$

and that for any multi-indices $\alpha,\beta\in\mathbb{N}^2$ of positive length

$$\begin{aligned}\left|\partial_\xi^\alpha \left[(j_1 j_2\langle \xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta)^{-1}\right]\right| \\ \lesssim \sum_{1\le|\alpha_1|\le|\alpha|} |j_1 j_2\langle \xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta|^{-1-|\alpha_1|}\,|\eta|^{|\alpha_1|}\langle \xi-\eta\rangle^{-(|\alpha|-|\alpha_1|)},\end{aligned}$$

$$\begin{aligned}\left|\partial_\eta^\beta \left[(j_1 j_2\langle \xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta)^{-1}\right]\right| \\ \lesssim \sum_{0\le|\beta_1|<|\beta|} |j_1 j_2\langle \xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta|^{-1-(|\beta|-|\beta_1|)} \sum_{\substack{i+j=|\beta|-2|\beta_1|\\ i,j\le|\beta|-|\beta_1|}} \langle \xi-\eta\rangle^i|\eta|^j.\end{aligned}$$

From above inequalities we hence deduce that on the support of $\sigma^N_{(j_1,j_2,j_3)}(\eta,\xi-\eta)$ (i.e. for $|\eta|\le\varepsilon|\xi-\eta|$), for any $\alpha,\beta\in\mathbb{N}^2$,

$$\left|\partial_\xi^\alpha\partial_\eta^\beta \left[(j_1 j_2\langle \xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta)^{-1}\right]\right| \lesssim_{\alpha,\beta} \langle \xi-\eta\rangle^{1+|\alpha|+2|\beta|}|\eta|^{-1-|\beta|},$$

and therefore that

$$\left|\partial_\xi^\alpha\partial_\eta^\beta \left[\frac{j_1\langle \xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}{2j_1 j_2\langle \xi-\eta\rangle|\eta| - 2(\xi-\eta)\cdot\eta}\right]\right| \lesssim_{\alpha,\beta} \langle \xi-\eta\rangle^{2+|\alpha|+2|\beta|}|\eta|^{-1-|\beta|} + \langle\xi\rangle\langle \xi-\eta\rangle^{1+|\alpha|+2|\beta|}|\eta|^{-1-|\beta|}.$$

The above estimates, summed up with the fact

$$|\partial_\xi^\alpha\partial_\eta^\beta[\sigma^N_{(j_1,j_2,j_3)}(\eta,\xi-\eta)]| \lesssim_{\alpha,\beta} \langle \xi-\eta\rangle^{-N-|\alpha|}|\eta|^{N+1-|\beta|},$$

gives the first part of the statement.

Let us now suppose that $N\ge 15$ and take $\chi\in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin. We have that

$$\begin{aligned}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta = \int K_0^N(z-x,x-y)u(x)v(y)w(z)dxdydz \\ + \int K_1^N(z-x,x-y)u(x)[\langle D_x\rangle^{N+3}v](y)w(z)dxdydz,\end{aligned}$$

with

$$K_k^N(x,y) := \int e^{ix\cdot\xi+iy\cdot\eta}\widetilde{\sigma}^{N,k}_{(j_1,j_2,j_3)}(\xi,\eta)d\xi d\eta,$$

$$\widetilde{\sigma}^{N,0}_{(j_1,j_2,j_3)}(\xi,\eta) = \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\chi(\eta) \quad\text{and}\quad \widetilde{\sigma}^{N,1}_{(j_1,j_2,j_3)}(\xi,\eta) = \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^{-N-3}(1-\chi)(\eta).$$

Then inequality (A.23) is obtained using the fact that, for any $\widetilde{u},\widetilde{w}\in L^2,\widetilde{v}\in L^\infty$,

$$\begin{aligned}\int \langle z-x\rangle^{-3}\langle x-y\rangle^{-3}|\widetilde{u}(x)||\widetilde{v}(y)||\widetilde{w}(z)|dxdydz \lesssim \|v\|_{L^\infty}\int \langle z\rangle^{-3}|\widetilde{u}(x)||\widetilde{w}(z-x)|dxdz \\ \lesssim \|u\|_{L^2}\|v\|_{L^\infty}\|w\|_{L^2}.\end{aligned}$$

□

In the following lemma we derive some results on the Sobolev continuity of the bilinear integral operator

$$(u,v)\mapsto \int e^{ix\cdot\xi}D_{(j_1,j_2)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta,$$

with $D_{(j_1,j_2)}$ defined in (3.1.14). We warn the reader that we are not going to take advantage of factor $(1-\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle})$ in $D_{(j_1,j_2)}(\xi,\eta)$ when deriving the estimates mentioned below, since the Sobolev continuity of the above integral operator does not depend on the null structure $Q_0(v,\partial_1 v)$ we chose for the Klein-Gordon self-interaction in the wave equation in system (1.1.1).

Lemma A.7. *Let $\rho\in\mathbb{N}$ and $D(\xi,\eta)$ a function satisfying, for any multi-indices $\alpha,\beta\in\mathbb{N}^2$, the following:*

(i) if $|\xi|\lesssim 1$,

$$|\partial_\eta^\beta D(\xi,\eta)|\lesssim_\beta \langle\eta\rangle^{\rho+|\beta|},$$
$$|\partial_\xi^\alpha\partial_\eta^\beta D(\xi,\eta)|\lesssim_{\alpha,\beta}\langle\eta\rangle^{\rho+|\alpha|+|\beta|}+\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}|\xi|^{1-|\alpha_1|}\langle\eta\rangle^{\rho+|\alpha_2|+|\beta|},\quad |\alpha|\geq 1;$$

(ii) for $|\xi|\gtrsim 1$, $|\eta|\lesssim\langle\xi-\eta\rangle$,

$$|\partial_\xi^\alpha\partial_\eta^\beta D(\xi,\eta)|\lesssim_{\alpha,\beta}\langle\xi-\eta\rangle^{\rho+|\alpha|+|\beta|};$$

(iii) for $|\xi|\gtrsim 1$, $|\eta|\gtrsim\langle\xi-\eta\rangle$:

$$|\partial_\xi^\alpha\partial_\eta^\beta D(\xi,\eta)|\lesssim_{\alpha,\beta}\langle\eta\rangle^{\rho+|\alpha|+|\beta|}.$$

Then for any $s\geq 0$, any $u,v\in H^{s+\rho+13}(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ (resp. $u,v\in H^{s+\rho+13,\infty}(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$)

$$\begin{aligned}\text{(A.25a)}\quad \left\|\int e^{ix\cdot\xi}D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{H^s(dx)}&\lesssim \|u\|_{H^{s+\rho+13}}\|v\|_{L^\infty}+\|u\|_{L^\infty}\|v\|_{H^{s+\rho+13}}\\ &(or\ \lesssim \|u\|_{H^{s+\rho+13,\infty}}\|v\|_{L^2}+\|u\|_{L^2}\|v\|_{H^{s+\rho+13,\infty}}),\end{aligned}$$

and for any $u,v\in H^{s+\rho+13,\infty}(\mathbb{R}^2)$

$$\text{(A.25b)}\quad \left\|\int e^{ix\cdot\xi}D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{H^{s,\infty}(dx)}\lesssim \|u\|_{H^{s+\rho+13,\infty}}\|v\|_{L^\infty}+\|u\|_{L^\infty}\|v\|_{H^{s+\rho+13,\infty}}.$$

Furthermore, if $\phi\in C_0^\infty(\mathbb{R}^2)$, $t\geq 1$, $\sigma>0$ small, there exists $\delta>0$ depending linearly on σ, such that

$$\begin{aligned}\text{(A.26a)}\quad \left\|\phi(t^{-\sigma}D_x)\int e^{ix\cdot\xi}D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{H^s(dx)}&\lesssim t^\delta\|u\|_{H^{\rho+13}}\|v\|_{L^\infty}\\ &(or\ \lesssim t^\delta\|u\|_{H^{\rho+13,\infty}}\|v\|_{L^2})\\ &(or\ \lesssim t^\delta\|u\|_{L^\infty}\|v\|_{H^{\rho+13}}),\\ &(or\ \lesssim t^\delta\|u\|_{L^2}\|v\|_{H^{\rho+13,\infty}}),\end{aligned}$$

$$\begin{aligned}\text{(A.26b)}\quad \left\|\phi(t^{-\sigma}D_x)\int e^{ix\cdot\xi}D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{H^{s,\infty}}&\lesssim t^\delta\|u\|_{H^{\rho+13,\infty}}\|v\|_{L^\infty}\\ &(or\ \lesssim t^\delta\|u\|_{L^\infty}\|v\|_{H^{\rho+13,\infty}}).\end{aligned}$$

Finally, if for any $\alpha,\beta\in\mathbb{N}^2$ $D(\xi,\eta)$ satisfies (ii), (iii) when $|\xi|\gtrsim 1$, together with:

$(\widetilde{i})$ *if* $|\xi| \lesssim 1$

$$|\partial_\xi^\alpha \partial_\eta^\beta \widetilde{D}(\xi,\eta)| \lesssim_{\alpha,\beta} \langle\eta\rangle^{\rho+|\alpha|+|\beta|} + \sum_{|\alpha_1|+|\alpha_2|=|\alpha|} |\xi|^{-|\alpha_1|+1} \langle\eta\rangle^{\rho+|\alpha_2|+|\beta|},$$

then, for any $u, v \in H^{s+\rho+13}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,

(A.27a)
$$\left\| \int e^{ix\cdot\xi} D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{H^s(dx)} \lesssim \|u\|_{H^{\rho+10}}\|v\|_{L^2} + \|u\|_{H^{s+\rho+13}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{s+\rho+13}}$$
$$(or \ \lesssim \|u\|_{L^2}\|v\|_{H^{\rho+10}} + \|u\|_{H^{s+\rho+13,\infty}}\|v\|_{L^2} + \|u\|_{L^2}\|v\|_{H^{s+\rho+13,\infty}}),$$

and for any $u, v \in H^{s+\rho+13,\infty}(\mathbb{R}^2)$, *with* $u \in H^{\rho+10}(\mathbb{R}^2)$ *(or* $u \in L^2(\mathbb{R}^2)$*),*

(A.27b)
$$\left\| \int e^{ix\cdot\xi} D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{H^{s,\infty}(dx)} \lesssim$$
$$\|u\|_{H^{\rho+10}}\|v\|_{L^\infty} + \|u\|_{H^{s+\rho+13,\infty}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{s+\rho+13,\infty}}$$
$$(or \ \lesssim \|u\|_{L^2}\|v\|_{H^{\rho+10,\infty}} + \|u\|_{H^{s+\rho+13,\infty}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{s+\rho+13,\infty}}).$$

Proof. Let $L(\mathbb{R}^2)$ denote either the $L^2(\mathbb{R}^2)$ space or the $L^\infty(\mathbb{R}^2)$ one. After definition 1.2.1 (i) of space H^s (resp. (iii) of $H^{s,\infty}$), we should prove that the L^2 norm (resp. the L^∞) norm of

$$\int e^{ix\cdot\xi} D^s(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta, \tag{A.28}$$

with $D^s(\xi,\eta) := D(\xi,\eta)\langle\xi\rangle^s$, is bounded by the right hand side of (A.25a) and (A.26a) (resp. (A.25b) and (A.26b)). Let us first take $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin and split the above integral, distinguishing between bounded and unbounded frequencies ξ, as

$$\int e^{ix\cdot\xi} D^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta + \int e^{ix\cdot\xi} D^s(\xi,\eta)(1-\chi)(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta. \tag{A.29}$$

On the support of $\chi(\xi)$ frequencies $\xi-\eta, \eta$ are either bounded or equivalent, thus if

$$a_0^s(\xi,\eta) := \begin{cases} D^s(\xi,\eta)\chi(\xi)\langle\xi-\eta\rangle^{-\rho-10} \\ \quad \text{or} \\ D^s(\xi,\eta)\chi(\xi)\langle\eta\rangle^{-\rho-10} \end{cases}$$

$a_0^s(\xi,\eta)$ satisfies (A.2) with $g_\beta(\eta) = \langle\eta\rangle^{-3}$ for any $|\beta| \le 3$, after hypothesis (i) on $D(\xi,\eta)$. Then by (A.4) and depending on the choice of $a_0^s(\xi,\eta)$, we have that

(A.30a)
$$\left\| \int e^{ix\cdot\xi} D^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} = \left\| \int e^{ix\cdot\xi} a_0^s(\xi,\eta)\widehat{\langle D_x\rangle^{\rho+10}u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)}$$
$$\lesssim \|\langle D_x\rangle^{\rho+10}u\|_L\|v\|_{L^\infty} (\text{or } \|\langle D_x\rangle^{\rho+10}u\|_{L^\infty}\|v\|_L),$$

or

(A.30b)
$$\left\| \int e^{ix\cdot\xi} D^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} = \left\| \int e^{ix\cdot\xi} a_0^s(\xi,\eta)\hat{u}(\xi-\eta)\widehat{\langle D_x\rangle^{\rho+10}v}(\eta)d\xi d\eta \right\|_{L(dx)}$$
$$\lesssim \|u\|_{L^\infty}\|\langle D_x\rangle^{\rho+10}v\|_L (\text{or } \|u\|_L\|\langle D_x\rangle^{\rho+10}v\|_{L^\infty}).$$

Successively, we consider a Littlewood-Paley decomposition in order to write

$$\begin{aligned}(\text{A.31})\quad &\int e^{ix\cdot\xi}D^s(\xi,\eta)(1-\chi)(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\\ &\qquad=\sum_{k\geq 1,l\geq 0}\int e^{ix\cdot\xi}D^s(\xi,\eta)(1-\chi)(\xi)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta,\end{aligned}$$

where $\varphi_0\in C_0^\infty(\mathbb{R}^2)$, $\varphi_k(\zeta)=\varphi(2^{-k}\zeta)$ with $\varphi\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ for any $k\in\mathbb{N}^*$. When positive indices l,k are such that $k>l+N_0$ for a certain large $N_0\in\mathbb{N}^*$, we have that $|\eta|<|\xi-\eta|$ and $|\xi-\eta|\sim|\xi|\sim 2^k$ on the support of $\varphi_k(\xi)\varphi_l(\eta)$. If

$$a^s_{k>l+N_0}(\xi,\eta):=D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\langle\xi-\eta\rangle^{-s-\rho-13},$$

by hypothesis (ii) we deduce that, for any $\alpha,\beta\in\mathbb{N}^2$ of length less or equal than 3,

$$|\partial_\xi^\alpha\partial_\eta^\beta[a^s_{k>l+N_0}(2^k\xi,2^l\eta)]|\lesssim 2^{-k},\quad\forall(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2$$

and its associated kernel

$$K^s_{k>l+N_0}(x,y):=\int e^{ix\cdot\xi+iy\cdot\eta}a^s_{k>l+N_0}(\xi,\eta)d\xi d\eta$$

verifies that

$$|K^s_{k>l+N_0}(x,y)|\lesssim 2^k2^{2l}\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3},\quad\forall(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$$

as one can check doing some integration by parts. Therefore

$$\begin{aligned}&\left\|\int e^{ix\cdot\xi}D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)}\\ &=\left\|\int K^s_{k>l+N_0}(x-y,y-z)[\langle D_x\rangle^{s+\rho+13}u](y)v(z)dydz\right\|_{L(dx)}\\ &\lesssim\ 2^k2^{2l}\left\|\int\langle 2^k(x-y)\rangle^{-3}\langle 2^l(y-z)\rangle^{-3}|\langle D_x\rangle^{s+\rho+13}u(y)||v(z)|dydz\right\|_{L(dx)}\\ &\lesssim\ 2^k2^{2l}\int\langle 2^ky\rangle^{-3}\langle 2^lz\rangle^{-3}\|[\langle D_x\rangle^{s+\rho+13}u](\cdot-y)v(\cdot-y-z)\|_{L(dx)}dydz\\ &\lesssim 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|\langle D_x\rangle^{s+\rho+13}u\|_L\|v\|_{L^\infty}\ (\text{ or } 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|\langle D_x\rangle^{s+\rho+13}u\|_{L^\infty}\|v\|_L).\end{aligned}\tag{A.32}$$

For indices l,k such that $1\leq k\leq l+N_0$ we have that $|\xi-\eta|\lesssim|\eta|$ on the support of $\varphi_k(\xi)\varphi_l(\eta)$. If

$$a^s_{k\leq l+N_0}(\xi,\eta):=D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\langle\eta\rangle^{-s-\rho-13}$$

by hypothesis (iii) for any multi-indices α,β of length less or equal than 3,

$$|\partial_\xi^\alpha\partial_\eta^\beta[a^s_{k\leq l+N_0}(2^k\xi,2^l\eta)]|\lesssim_{\alpha,\beta}2^{-l},$$

and its associated kernel $K^s_{k\leq l+N_0}(x,y)$ is such that

$$K^s_{k\leq l+N_0}(x,y)|\lesssim 2^{2k}2^l\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3},\quad\forall(x,y)\in\mathbb{R}^2\times\mathbb{R}^2.$$

Consequently

$$\begin{aligned}&\left\|\int e^{ix\cdot\xi}D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)}\\ &\lesssim\ 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u\|_{L^\infty}\|\langle D_x\rangle^{s+\rho+13}v\|_L\ (\text{or } 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u\|_L\|\langle D_x\rangle^{s+\rho+13}v\|_{L^\infty}),\end{aligned}\tag{A.33}$$

and inequality (A.25a) (resp. (A.25b)) is hence obtained by combining inequalities (A.30), (A.32), (A.33) with $L = L^2$ (resp. $L = L^\infty$), and taking the sum over $k \geq 1, l \geq 0$.

In order to derive inequalities (A.26), we first observe that we can reduce to study the L^2 and L^∞ norm of (A.28) with $s = 0$ and $D(\xi, \eta)$ multiplied by $\phi(t^{-\sigma}\xi)$, up to a factor $t^{s\sigma}$. Here we use again decompositions (A.29), (A.31), and only need to modify some of the multipliers defined above, depending on if we want derivatives falling entirely on u or rather on v. In fact, in order to prove the first two inequalities in (A.26a) and the first one in (A.26b) we introduce

$$a^\phi_{l\leq k+N_0}(\xi,\eta) := D(\xi,\eta)\chi(t^{-\sigma}\xi)\varphi_k(\xi)\varphi_l(\eta)$$
$$a^\phi_{l>k+N_0}(\xi,\eta) := D(\xi,\eta)\chi(t^{-\sigma}\xi)\varphi_k(\xi)\varphi_l(\eta)\langle \xi-\eta\rangle^{-\rho-13}$$

and deduce from hypothesis $(ii)-(iii)$ on $D(\xi,\eta)$ and the fact that $|\xi| \lesssim t^\sigma$ on the support of $\phi(t^{-\sigma}\xi)$ that, for any $\alpha, \beta \in \mathbb{N}^2$ of length less or equal than 3,

$$|\partial^\alpha_\xi \partial^\beta_\eta [a^\phi_{l\leq k+N_0}(2^k\xi, 2^l\eta)]| \lesssim t^\delta 2^{-k} \quad \text{and} \quad |\partial^\alpha_\xi \partial^\beta_\eta [a^\phi_{l>k+N_0}(2^k\xi, 2^l\eta)]| \lesssim 2^{-l}$$

with $\delta > 0$, $\delta \to 0$ as $\sigma \to 0$. On the one hand, kernel $K^\phi_{l\leq k+N_0}(x,y)$ associated to $a^\phi_{l\leq k+N_0}(\xi,\eta)$ verifies

$$|K^\phi_{l\leq k+N_0}(x,y)| \lesssim t^\delta 2^k 2^{2l}\langle 2^k x\rangle^{-3}\langle 2^l y\rangle^{-3}, \quad \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$$

and then for any l, k such that $l \leq k + N_0$

$$\begin{aligned} &\left\| \int e^{ix\cdot\xi} D(\xi,\eta)\phi(t^\sigma\xi)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} \\ &= \left\| \int K^\phi_{l\leq k+N_0}(x-y, y-z)u(y)v(z)dydz \right\|_{L(dx)} \lesssim t^\delta 2^{-\frac{k}{2}} 2^{-\frac{l}{2}} \|u\|_L \|v\|_{L^\infty} \\ &\qquad\qquad (\text{or } \lesssim t^\delta 2^{-\frac{k}{2}} 2^{-\frac{l}{2}} \|u\|_{L^\infty} \|v\|_L). \end{aligned} \tag{A.34}$$

On the other hand, kernel $K^\phi_{l>k+N_0}(x,y)$ associated to $a^\phi_{l>k+N_0}(\xi,\eta)$ satisfies

$$|K^\phi_{l>k+N_0}(x,y)| \lesssim 2^{2k} 2^l \langle 2^k x\rangle^{-3}\langle 2^l y\rangle^{-3}, \quad \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$$

so for indices l, k such that $l > k + N_0$

$$\begin{aligned} &\left\| \int e^{ix\cdot\xi} D(\xi,\eta)\phi(t^\sigma\xi)\varphi_l(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} \\ &= \left\| \int K^\phi_{l>k+N_0}(x-y, y-z)[\langle D_x\rangle^{\rho+13}u](y)v(z)dydz \right\|_{L(dx)} \lesssim 2^{-\frac{k}{2}} 2^{-\frac{l}{2}} \|\langle D_x\rangle^{\rho+13}u\|_L \|v\|_{L^\infty} \\ &\qquad\qquad (\text{or } \lesssim 2^{-\frac{k}{2}} 2^{-\frac{l}{2}} \|\langle D_x\rangle^{\rho+13}u\|_{L^\infty} \|v\|_L)\Big). \end{aligned}$$

Combining these two inequalities with (A.30a) and taking the sum over $k \geq 1, l \geq 0$ we obtain the wished estimates.

Last two inequalities in (A.26a) and last one in (A.26b) are instead obtained combining (A.30b) with (A.33) (that evidently holds for $D^s(\xi,\eta)$ replaced with $D(\xi,\eta)\phi(t^\sigma\xi)$) and (A.34).

Finally, last part of the statement follows from the same argument of above, with the only difference that, after hypothesis $(\widetilde{i})$, multiplier $\widetilde{a}^s_0(\xi,\eta) := \widetilde{D}(\xi,\eta)\chi(\xi)\langle\eta\rangle^{-\rho-10}$ satisfies (A.5) with $|g_\beta(\eta)| \lesssim \langle\eta\rangle^{-3}$ for any $|\beta| \leq 3$, then by (A.7) we have that

$$\begin{aligned} &\left\| \int e^{ix\cdot\xi} \widetilde{D}^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} \\ &\qquad = \left\| \int e^{ix\cdot\xi} \widetilde{a}^s_0(\xi,\eta)\widehat{\langle D_x\rangle^{\rho+10}u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} \lesssim \|\langle D_x\rangle^{\rho+10}u\|_{L^2}\|v\|_L, \end{aligned}$$

or

$$\Big\| \int e^{ix\cdot\xi} \widetilde{D}^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{L(dx)}$$
$$= \Big\| \int e^{ix\cdot\xi} \widetilde{a}_0^s(\xi,\eta)\hat{u}(\xi-\eta)\widehat{\langle D_x\rangle^{\rho+10}v}(\eta)d\xi d\eta \Big\|_{L(dx)} \lesssim \|u\|_{L^2}\|\langle D_x\rangle^{\rho+10}v\|_L.$$

□

Lemma A.8. *Let $j \in \{+,-\}$, $\phi \in C_0^\infty(\mathbb{R}^2)$, $t \geq 1, \sigma > 0$, and $D_j(\xi,\eta)$ be the multiplier introduced in* (3.1.14). *For any $s \geq 0$, $i = 1,2$, $D_j(\xi,\eta)$ and $\frac{\xi_i}{|\xi|}D_j(\xi,\eta)$ satisfy inequalities* (A.25), (A.26) *with $\rho = 2$, and*

(A.35a)
$$\Big\| \int e^{ix\cdot\xi}\partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{H^s(dx)} \lesssim \|u\|_{H^{13}}\|v\|_{L^2} + \|u\|_{H^{s+16}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{s+16}}$$
$$(\textit{resp.} \lesssim \|u\|_{H^{13}}\|v\|_{L^2} + \|u\|_{H^{s+16,\infty}}\|v\|_{L^2} + \|u\|_{L^2}\|v\|_{H^{s+16,\infty}}),$$

(A.35b)
$$\Big\| \int e^{ix\cdot\xi}\partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{H^{s,\infty}(dx)} \lesssim \|u\|_{H^{13}}\|v\|_{L^\infty} + \|u\|_{H^{s+16,\infty}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{s+16,\infty}},$$

together with

(A.36a)
$$\Big\| \phi(t^{-\sigma}D_x)\int e^{ix\cdot\xi}\partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{H^s(dx)} \lesssim t^\delta\|u\|_{H^{13}}\left(\|v\|_{L^2} + \|v\|_{L^\infty}\right)$$
$$(\textit{or } \lesssim t^\delta\|u\|_{L^2}\left(\|v\|_{H^{10}} + \|v\|_{H^{13,\infty}}\right)),$$

(A.36b)
$$\Big\| \phi(t^{-\sigma}D_x)\int e^{ix\cdot\xi}\partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{H^{s,\infty}(dx)} \lesssim t^\delta\left(\|u\|_{H^{13}} + \|u\|_{H^{16,\infty}}\right)\|v\|_{L^\infty}$$
$$(\textit{or } \lesssim t^\delta\left(\|u\|_{L^2} + \|u\|_{L^\infty}\right)\|v\|_{H^{16,\infty}}).$$

Moreover, if $\Omega = x_1\partial_2 - x_2\partial_1$ and $Z_n = x_n\partial_t + t\partial_n$, $n = 1,2$,

(A.37a)
$$\Big\| \phi(t^{-\sigma}D_x)\Omega\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{L^2(dx)} \lesssim t^\delta\left[\left(\|u\|_{L^2} + \|\Omega u\|_{L^2}\right)\|v\|_{H^{17,\infty}} + \|u\|_{H^{15,\infty}}\|\Omega v\|_{L^2}\right],$$

(A.37b)
$$\Big\| \phi(t^{-\sigma}D_x)Z_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{L^2(dx)} \lesssim t^\delta\left[\|\partial_t u\|_{L^2}\|v\|_{H^{13}} + \|u\|_{H^{13}}\|\partial_t v\|_{L^2} + \|Z_n u\|_{L^2}\|v\|_{H^{15,\infty}} + \|u\|_{H^{15,\infty}}\|Z_n v\|_{L^2}\right],$$

(A.37c)
$$\Big\| \phi(t^{-\sigma}D_x)D_jZ_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \Big\|_{L^2(dx)} \lesssim t^\delta\left[\|\partial_t u\|_{L^2}\|v\|_{H^{14,\infty}} + \|u\|_{H^{14,\infty}}\|\partial_t v\|_{L^2} + \|Z_n u\|_{L^2}\|v\|_{H^{17,\infty}} + \|u\|_{H^{17,\infty}}\|Z_n v\|_{L^2}\right],$$

with $\delta > 0$, $\delta \to 0$ as $\sigma \to 0$.

Proof. The statement follows essentially from the observation that, for $j \in \{+,-\}$, functions $D_j(\xi,\eta)$ and $[(\xi_i\partial_{\xi_j})^{k_1}(\eta_i\partial_{\eta_j})^{k_2}D_j](\xi,\eta)$ satisfy hypothesis $(i)-(iii)$ of lemma A.7 with $\rho=2$ and $\rho=2+2(k_1+k_2)$ respectively, while $\partial_\xi D_j(\xi,\eta)$ satisfies $(\widetilde{i}),(ii),(iii)$ with $\rho=3$. In fact, we first remark that, for every ξ,η, denominator $1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta$ is bounded from below by a positive constant; secondly, the derivation of that denominator gives rise to losses in $\langle\xi-\eta\rangle,\langle\eta\rangle$, as

$$\partial_{\xi_k}(1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta)=\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\langle\eta\rangle+\eta_k,$$
$$\partial_{\eta_k}(1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta)=\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\langle\eta\rangle+\langle\xi-\eta\rangle\frac{\eta_k}{\langle\eta\rangle}+\eta_k-(\xi_k-\eta_k).$$

For $|\xi|\lesssim 1$ we have that $\langle\xi-\eta\rangle\lesssim\langle\eta\rangle$, so for any $\alpha,\beta\in\mathbb{N}^2$

$$\left|\partial_\xi^\alpha\partial_\eta^\beta\left[\frac{j\langle\xi-\eta\rangle+j\langle\eta\rangle}{1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta}\eta_1\right]\right|\lesssim_{\alpha,\beta}\langle\eta\rangle^{2+|\alpha|+|\beta|},$$

while

$$\left|\partial_\eta^\beta\left[\frac{|\xi|}{1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta}\eta_1\right]\right|\lesssim_\beta\langle\eta\rangle^{1+|\beta|},$$
$$\left|\partial_\xi^\alpha\partial_\eta^\beta\left[\frac{|\xi|}{1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta}\eta_1\right]\right|\lesssim_{\alpha,\beta}\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}|\xi|^{1-|\alpha_1|}\langle\eta\rangle^{1+|\alpha_2|+|\beta|},\quad|\alpha|\geq 1.$$

For $|\xi|\gtrsim 1$ and $|\eta|\lesssim\langle\xi-\eta\rangle$ (resp. $|\eta|\gtrsim\langle\xi-\eta\rangle$) we have that $|\xi|\lesssim|\xi-\eta|$ (resp.$|\xi|\lesssim|\eta|$), so each time a derivative hits the denominator of $D_j(\xi,\eta)$ we lose a factor $\langle\xi-\eta\rangle$ (resp. $\langle\eta\rangle$). Hence lemma A.7 immediately implies inequalities (A.25), (A.26) with $D=D_j$ and $\rho=2$, together with (A.35), (A.36), while inequalities (A.37) follow from (A.26) and the fact that, after some integration by parts,

$$\begin{aligned}
&\Omega\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\\
&\quad=\sum_{k_1+k_2+k_3+k_4=1}\int e^{ix\cdot\xi}[(\xi_1\partial_{\xi_2}-\xi_2\partial_{\xi_1})^{k_1}(\eta_1\partial_{\eta_2}-\eta_2\partial_{\eta_1})^{k_2}D_j](\xi,\eta)\widehat{\Omega^{k_3}u}(\xi-\eta)\widehat{\Omega^{k_4}v}(\eta)d\xi d\eta,\\
&Z_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\\
&\quad=\int e^{ix\cdot\xi}[\partial_{\xi_n}D_j](\xi,\eta)D_t\big[\hat{u}(\xi-\eta)\hat{v}(\eta)\big]d\xi d\eta+\int e^{ix\cdot\xi}[\partial_{\eta_n}D_j](\xi,\eta)\hat{u}(\xi-\eta)\widehat{D_tv}(\eta)d\xi d\eta\\
&\quad+\int e^{ix\cdot\xi}D_j(\xi,\eta)\widehat{Z_nu}(\xi-\eta)\hat{v}(\eta)d\xi d\eta+\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\widehat{Z_nv}(\eta)d\xi d\eta,
\end{aligned}$$

and, if δ_{jn} denotes the Kronecker delta,

$$\begin{aligned}
&D_jZ_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\\
&\quad=\delta_{jn}\int e^{ix\cdot\xi}D_j(\xi,\eta)D_t\left[\hat{u}(\xi-\eta)\hat{v}(\eta)\right]d\xi d\eta\\
&\quad+\int e^{ix\cdot\xi}\partial_{\xi_n}[\xi_jD_j](\xi,\eta)D_t\big[\hat{u}(\xi-\eta)\hat{v}(\eta)\big]d\xi d\eta+\int e^{ix\cdot\xi}\partial_{\eta_n}[\xi_jD_j](\xi,\eta)\hat{u}(\xi-\eta)\widehat{D_tv}(\eta)d\xi d\eta\\
&\quad+\int e^{ix\cdot\xi}\xi_jD_j(\xi,\eta)\widehat{Z_nu}(\xi-\eta)\hat{v}(\eta)d\xi d\eta+\int e^{ix\cdot\xi}\xi_jD_j(\xi,\eta)\hat{u}(\xi-\eta)\widehat{Z_nv}(\eta)d\xi d\eta.
\end{aligned}$$

. □

Appendix B

The aim of this chapter is to show how, from the bootstrap assumptions (1.1.11), it is possible to derive a moderate growth in time for the L^2 norm of $\mathcal{L}^\mu\widetilde{v}$, with $0 \leq |\mu| \leq 2$, and of $\Omega_h^\mu \mathcal{M}^\nu \widetilde{u}^{\Sigma,k}$, with $\mu, |\nu| = 0, 1$. These estimates are fundamentally used in propositions 3.2.7 and 3.3.7. Moreover, we also prove in lemma B.4.14 a sharp decay estimate for the uniform norm of the Klein-Gordon solution when one Klainerman vector field is acting on it (and when considered for frequencies less or equal than t^σ, with $\sigma > 0$ small). We are hence going to assume for the rest of this chapter that a-priori estimates (1.1.11) are satisfied in interval $[1, T]$, for some fixed $T > 1$, and that $\varepsilon_0 < (2A+B)^{-1}$. We remind that Γ generally denotes one of the admissible vector fields belonging to $\mathfrak{Z}$ (see (1.1.7)) and that, for a multi-index $I = (i_1, \dots, i_n)$ with $i_j \in \{1, \dots, 5\}$ for $j = 1, \dots, n$, $\Gamma^I = \Gamma_{i_1} \cdots \Gamma_{i_n}$. Also, we warn the reader that any norm X ($X = L^\infty, H^s, H_h^s ...$) of $w = w(t, x)$ is here considered with respect to spatial variable x. We will often write $\| \cdot \|_X$ in place of $\| \cdot \|_{X(dx)}$.

B.1 Some preliminary lemmas

In the current section we list, on the one hand, some inequalities concerning the H^s and $H^{s,\infty}$ norm of the quadratic non-linearities $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ (see lemmas B.1.1, B.1.2), as they are very frequently recalled in the second part of the paper. On the other hand, we introduce some preliminary small results that will be useful in sections B.2 and B.3.

For seek of compactness, we denote $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$ and $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ by NL_w and NL_{kg} respectively, i.e.

$$NL_w := \frac{i}{4}\left[(v_+ + v_-)D_1(v_+ + v_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{D_x D_1}{\langle D_x\rangle}(v_+ - v_-)\right], \tag{B.1.1a}$$

$$NL_{kg} := \frac{i}{4}\left[(v_+ + v_-)D_1(u_+ + u_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{D_x D_1}{|D_x|}(u_+ - u_-)\right]. \tag{B.1.1b}$$

We recall the result of lemma 1.2.40, that can be also stated in the classical setting and says that, for any real positive $s > s'$ and $w \in H^s(\mathbb{R}^2)$,

$$\|(1-\chi)(t^{-\sigma}D_x)w\|_{H^{s'}} \leq Ct^{-\sigma(s-s')}\|w\|_{H^s}, \quad \forall s > s'. \tag{B.1.2}$$

It is also useful to remind, in view of upcoming lemmas, that the L^2 norm of $(\Gamma^I u)_\pm$ and $(\Gamma^I v)_\pm$ is estimated with:

- $E_n(t; W)^{\frac{1}{2}}$, whenever $|I| \leq n$ and Γ^I is a product of spatial derivatives;
- $E_3^k(t; W)^{\frac{1}{2}}$, whenever $|I| \leq 3$ and at most $3 - k$ vector fields in Γ^I, with $0 \leq k \leq 2$ belong to $\{\Omega, Z_m, m = 1, 2\}$.

As assumed in (1.1.11c), (1.1.11d), such energies have a moderate growth in time and a hierarchy is established among them in the sense that

$$0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1.$$

We warn the reader that this hierarchy is often implicitly used throughout this chapter.

Lemma B.1.1. *For any $s \geq 0$, any $\theta \in]0,1[$, NL_w satisfies the following inequalities:*

$$\|NL_w(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}} \|V(t,\cdot)\|_{H^1}, \tag{B.1.3a}$$

$$\|NL_w(t,\cdot)\|_{L^\infty} \lesssim \|V(t,\cdot)\|^2_{H^{2,\infty}}, \tag{B.1.3b}$$

$$\|NL_w(t,\cdot)\|_{H^s} \lesssim \|V(t,\cdot)\|_{H^{s+1}} \|V(t,\cdot)\|_{H^{1,\infty}}, \tag{B.1.3c}$$

$$\|NL_w(t,\cdot)\|_{H^{s,\infty}} \lesssim \|V(t,\cdot)\|^{2-\theta}_{H^{s+1,\infty}} \|V(t,\cdot)\|^{\theta}_{H^{s+3}}, \tag{B.1.3d}$$

$$\|\Omega NL_w(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{2,\infty}} \left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{H^1}\right), \tag{B.1.3e}$$

while for NL_{kg} we have that:

$$\|NL_{kg}(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}} \|U(t,\cdot)\|_{H^1}, \tag{B.1.4a}$$

$$\|NL_{kg}(t,\cdot)\|_{L^\infty} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}} \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right), \tag{B.1.4b}$$

$$\|NL_{kg}(t,\cdot)\|_{H^s} \lesssim \|V(t,\cdot)\|_{H^s} \left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right) + \|V(t,\cdot)\|_{L^\infty} \|U(t,\cdot)\|_{H^{s+1}}, \tag{B.1.4c}$$

$$\begin{aligned} \|NL_{kg}(t,\cdot)\|_{H^{s,\infty}} \lesssim{}& \|V(t,\cdot)\|^{1-\theta}_{H^{s,\infty}} \|V(t,\cdot)\|^{\theta}_{H^{s+2}} \left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right) \\ &+ \|V(t,\cdot)\|_{L^\infty} \left(\|U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}}\right) \|U(t,\cdot)\|^{\theta}_{H^{s+3}}, \end{aligned} \tag{B.1.4d}$$

$$\begin{aligned} \|\Omega NL_{kg}(t,\cdot)\|_{L^2} \lesssim{}& \left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{L^2}\right) \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right) \\ &+ \|V(t,\cdot)\|_{H^{1,\infty}} \|\Omega U(t,\cdot)\|_{H^1}. \end{aligned} \tag{B.1.4e}$$

$$\|D_t U(t,\cdot)\|_{H^s} \lesssim \|U(t,\cdot)\|_{H^{s+1}} + \|V(t,\cdot)\|_{H^{s+1}} \|V(t,\cdot)\|_{H^{1,\infty}}, \tag{B.1.4f}$$

Proof. Inequalities (B.1.3a), (B.1.3b), (B.1.4a), and (B.1.4b) are straightforward. The same is for (B.1.3e) and (B.1.4f) after commutation of Ω with the operators appearing in (2.1.1). All other inequalities in the statement are rather derived using corollary A.4. □

Lemma B.1.2. *For any $s \geq 0$, any $\theta \in]0,1[$,*

$$\|D_t U(t,\cdot)\|_{H^{s,\infty}} \lesssim \|U(t,\cdot)\|_{H^{s+2,\infty}} + \|V(t,\cdot)\|^{2-\theta}_{H^{s+1,\infty}} \|V(t,\cdot)\|^{\theta}_{H^{s+3}}, \tag{B.1.5b}$$

$$\|D_t \mathrm{R}_1 U(t,\cdot)\|_{H^{s,\infty}} \lesssim \|\mathrm{R}_1 U(t,\cdot)\|_{H^{s+1,\infty}} + \|V(t,\cdot)\|_{H^{s+3}} \|V(t,\cdot)\|_{H^{1,\infty}}, \tag{B.1.5c}$$

$$\|D_t \Omega U(t,\cdot)\|_{L^2} \leq \|\Omega U(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{H^{2,\infty}} \left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{H^1}\right), \tag{B.1.5d}$$

while

$$\begin{aligned} \|D_t V(t,\cdot)\|_{H^s} \lesssim{}& \|V(t,\cdot)\|_{H^{s+1}} + \|V(t,\cdot)\|_{H^s} \left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right) \\ &+ \|V(t,\cdot)\|_{L^\infty} \|U(t,\cdot)\|_{H^{s+1}}, \end{aligned} \tag{B.1.6a}$$

$$\begin{aligned} \|D_t V(t,\cdot)\|_{H^{s,\infty}} \lesssim{}& \|V(t,\cdot)\|_{H^{s+1,\infty}} + \|V(t,\cdot)\|^{1-\theta}_{H^{s,\infty}} \|V(t,\cdot)\|^{\theta}_{H^{s+1}} \left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right) \\ &+ \|V(t,\cdot)\|_{L^\infty} \left(\|U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}}\right) \|U(t,\cdot)\|^{\theta}_{H^{s+3}}, \end{aligned} \tag{B.1.6b}$$

$$\begin{aligned} \|D_t \Omega V(t,\cdot)\|_{L^2} \leq{}& \|\Omega V(t,\cdot)\|_{H^1} + \left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{L^2}\right) \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right) \\ &+ \|V(t,\cdot)\|_{H^{1,\infty}} \|\Omega U(t,\cdot)\|_{H^1}. \end{aligned} \tag{B.1.6c}$$

Proof. Straight consequence of the previous lemma and the fact that (u_+, v_+, u_-, v_-) is solution to system (3.1.1). Observe that inequality (B.1.5c) is derived using that

$$\|\mathrm{R}_1 NL_w(t,\cdot)\|_{H^{s,\infty}} \lesssim \|NL_w(t,\cdot)\|_{H^{s+2}}$$

after classical Sobolev injection and continuity of $\mathrm{R}_1 : H^s \to H^s$, for any $s \geq 0$. □

Lemma B.1.3. *Let $|I| = 1$ be such that $\Gamma^I \in \{\Omega, Z_m, m = 1, 2\}$. Then*

$$\begin{aligned}\|D_t U^I(t,\cdot)\|_{L^2} \lesssim \|U^I(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{H^{2,\infty}}\Big[\|V^I(t,\cdot)\|_{H^1} \\ + \|V(t,\cdot)\|_{H^1}\Big(1 + \sum_{\mu=0}^{1}\|\mathrm{R}_1^\mu U(t,\cdot))\|_{H^{1,\infty}}\Big) + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\Big],\end{aligned} \tag{B.1.7}$$

$$\begin{aligned}\|D_t V^I(t,\cdot)\|_{L^2} \lesssim \|V^I(t,\cdot)\|_{H^1} + \sum_{\mu=0}^{1}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{H^{2,\infty}}\|V^I(t,\cdot)\|_{L^2} \\ + \|V(t,\cdot)\|_{H^{1,\infty}}\left(\|U^I(t,\cdot)\|_{H^1} + \|U(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{H^{1,\infty}}\|V(t,\cdot)\|_{H^1}\right).\end{aligned} \tag{B.1.8}$$

Proof. The result of the statement follows using the equation satisfied, respectively, by $u_\pm^I$ and $v_\pm^I$, together with (B.1.5a), (B.1.6a) with $s = 0$. In fact, by (1.1.15) with $|I| = 1$,

$$\begin{aligned}D_t u_\pm^I &= \pm|D_x|u_\pm^I + Q_0^{\mathrm{w}}(v_\pm^I, D_1 v_\pm) + Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm^I) + G_1^{\mathrm{w}}(v_\pm, Dv_\pm),\\ D_t v_\pm^I &= \pm\langle D_x\rangle v_\pm^I + Q_0^{\mathrm{kg}}(v_\pm^I, D_1 u_\pm) + Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm^I) + G_1^{\mathrm{kg}}(v_\pm, Du_\pm),\end{aligned}$$

with $G_1^{\mathrm{w}}(v_\pm, \partial v_\pm) = G_1(v, \partial v)$, $G_1^{\mathrm{kg}}(v_\pm, Du_\pm) = G_1(v, \partial u)$ and G_1 given by (1.1.16). Hence one can estimate the L^2 norm of the first two quadratic terms in above equalities with the L^2 norm of factors indexed in I times the L^∞ norm of the remaining one, while the L^2 norm of the latter quadratic terms can be instead bounded by taking the L^2 norm of one of the two factors times the L^∞ norm of the remaining one, indifferently. We choose here to consider the L^2 norm of factors $Du_\pm, Dv_\pm$, and use (B.1.5a), (B.1.6a) if the derivative D is a time derivative. □

It is useful to remind that, if $w(t,x)$ is solution to inhomogeneous half wave equation (3.2.5) from (3.2.12a) we have that for any $j, k \in \{1, 2\}$ and $|\mu| \leq 1$

$$\begin{aligned}x_j D_k\Big(\frac{D_x}{|D_x|}\Big)^\mu w =& \frac{D_k}{|D_x|}\Big(\frac{D_x}{|D_x|}\Big)^\mu\Big[x_j|D_x| - tD_j + \frac{1}{2i}\frac{D_j}{|D_x|}\Big]w + t\frac{D_jD_k}{|D_x|}\Big(\frac{D_x}{|D_x|}\Big)^\mu w\\ &- \frac{1}{2i}\frac{D_jD_k}{|D_x|^2}\Big(\frac{D_x}{|D_x|}\Big)^\mu w + iOp\Big(\partial_j\Big(\frac{\xi_k}{|\xi|}\Big(\frac{\xi}{|\xi|}\Big)^\mu\Big)|\xi|\Big)w\\ =& i\frac{D_k}{|D_x|}\Big(\frac{D_x}{|D_x|}\Big)^\mu Z_j w + \frac{D_k}{|D_x|}\Big(\frac{D_x}{|D_x|}\Big)^\mu[x_j f(t,x)]\\ &+ t\frac{D_jD_k}{|D_x|}\Big(\frac{D_x}{|D_x|}\Big)^\mu w + iOp\Big(\partial_j\Big(\frac{\xi_k}{|\xi|}\Big(\frac{\xi}{|\xi|}\Big)^\mu\Big)|\xi|\Big)w.\end{aligned} \tag{B.1.9a}$$

Analogously, if $w(t,x)$ is solution to inhomogeneous half Klein-Gordon (3.2.7), from (3.2.12b) we have that

$$\begin{aligned}x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu w &= \frac{1}{\langle D_x\rangle}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu[\langle D_x\rangle x_j - tD_j]\,w + t\frac{D_j}{\langle D_x\rangle}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu w + i\mathrm{Op}_h^w\Big(\partial_j\Big(\frac{\xi}{\langle\xi\rangle}\Big)^\mu\Big)w\\ &= i\frac{1}{\langle D_x\rangle}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu Z_j w - i\frac{D_j}{\langle D_x\rangle^2}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu w + \frac{1}{\langle D_x\rangle}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu[x_j f(t,x)]\\ &+ t\frac{D_j}{\langle D_x\rangle}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu w + i\mathrm{Op}_h^w\Big(\partial_j\Big(\frac{\xi}{\langle\xi\rangle}\Big)^\mu\Big)w.\end{aligned} \tag{B.1.9b}$$

We also remind the reader about equivalence (2.1.16), so we won't particularly care if we are dealing with $\Gamma^I u_\pm, \Gamma^I v_\pm$ instead of $(\Gamma^I u)_\pm, (\Gamma^I v)_\pm$, when we bound the L^2 norm of those terms with the energy defined in (1.1.9).

Lemma B.1.4. *There exists a positive constant $C > 0$ such that, for every $j = 1, 2$, $t \in [1, T]$,*

$$\sum_{|\mu|=0}^{1} \left\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_\pm(t,\cdot) \right\|_{H^1} \leq CB\varepsilon t^{1+\frac{\delta}{2}}, \tag{B.1.10a}$$

$$\sum_{|\mu|=0}^{1} \left\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_\pm(t,\cdot) \right\|_{H^{1,\infty}} \leq C(A+B)\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.1.10b}$$

and

$$\sum_{|\mu|=0}^{1} \left\| x_j D_x \Big(\frac{D_x}{|D_x|} \Big)^\mu u_\pm(t,\cdot) \right\|_{L^2} \leq CB\varepsilon t^{1+\frac{\delta}{2}}. \tag{B.1.11}$$

Proof. We warn the reader that, throughout the proof, C will denote a positive constant that may change line after line. As $v_+ = -\overline{v_-}$ (resp. $u_+ = -\overline{u_-}$), it is enough to prove the statement for v_- (resp. for u_-).

Since v_- is solution to equation (3.2.7) with $f = NL_{kg}$, from (B.1.9b) it immediately follows that, for any $|\mu| \leq 1$,

$$\left\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_-(t,\cdot) \right\|_{H^1} \lesssim \|Z_j v_-(t,\cdot)\|_{L^2} + t\|v_-(t,\cdot)\|_{H^1} + \|x_j NL_{kg}(t,\cdot)\|_{L^2(dx)} \tag{B.1.12a}$$

along with

$$\left\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_-(t,\cdot) \right\|_{H^{1,\infty}} \leq \|Z_j v_-(t,\cdot)\|_{H^2} + t\|v_-(t,\cdot)\|_{H^{2,\infty}} + \|x_j NL_{kg}(t,\cdot)\|_{L^\infty(dx)}, \tag{B.1.12b}$$

derived by using the classical Sobolev injection. Observe that

$$\|x_j NL_{kg}(t,\cdot)\|_{L^\infty} \lesssim \left(\|x_j v_-(t,\cdot)\|_{L^\infty} + \left\| x_j \frac{D_x}{\langle D_x \rangle} v_-(t,\cdot) \right\|_{L^\infty} \right) \sum_{\mu=0}^{1} \|\mathrm{R}_1^\mu U(t,\cdot)\|_{H^{2,\infty}}, \tag{B.1.13a}$$

but also

$$\|x_j NL_{kg}(t,\cdot)\|_{L^2} \lesssim \sum_{|\mu|=0}^{1} \left\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_\pm(t,\cdot) \right\|_{L^2} \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}} \right). \tag{B.1.13b}$$

Thus, if $\varepsilon_0 > 0$ is assumed sufficiently small to verify $\varepsilon_0 < (2A)^{-1}$, by injecting (B.1.13b) (resp. (B.1.13a)) into (B.1.12a) (resp. in (B.1.12b)), and using a-priori estimates (1.1.11), we obtain that

$$\begin{aligned} \sum_{|\mu|=0}^{1} \left\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_\pm(t,\cdot) \right\|_{H^1} &\leq C \left[E_3^2(t;W)^{\frac{1}{2}} + tE_3(t;W)^{\frac{1}{2}} \right] + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}} E_0(t;W)^{\frac{1}{2}} \\ &\leq CB\varepsilon t^{1+\frac{\delta}{2}} \end{aligned}$$

$$\Big(\text{resp.} \sum_{|\mu|=0}^{1} \left\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_\pm(t,\cdot) \right\|_{H^{1,\infty}} \leq CE_3^2(t;W)^{\frac{1}{2}} + t\|V(t,\cdot)\|_{H^{2,\infty}} \leq C(A+B)\varepsilon t^{\frac{\delta_2}{2}} \Big),$$

and the conclusion of the proof of (B.1.10).

Analogously, from (B.1.9a) with $w = u_-$ and $f = NL_w$,

$$\sum_{|\mu|=0}^{1} \Big\| x_j D_k \Big(\frac{D_x}{|D_x|} \Big)^{\mu} u_-(t,\cdot) \Big\|_{L^2} \lesssim \|Z_j u_\pm(t,\cdot)\|_{L^2} + t\|u_\pm(t,\cdot)\|_{L^2} + \|x_j NL_w(t,\cdot)\|_{L^2(dx)}$$
$$\leq CB\varepsilon t^{1+\frac{\delta}{2}},$$

as follows (1.1.11c), (1.1.11d), (B.1.10b) and the fact that

$$\|x_j NL_w(t,\cdot)\|_{L^2} \lesssim \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^{\mu} v_\pm(t,\cdot) \Big\|_{L^\infty} \|v_\pm(t,\cdot)\|_{H^1}. \tag{B.1.14}$$

□

Corollary B.1.5. *There exists a constant $C > 0$ such that, for every $j = 1, 2$, $t \in [1, T]$,*

$$\|x_j NL_{kg}(t,\cdot)\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{\frac{\delta+\delta_2}{2}}, \tag{B.1.15a}$$

$$\|x_j NL_{kg}(t,\cdot)\|_{L^\infty} \leq C(A+B)B\varepsilon^2 t^{-\frac{1}{2}+\frac{\delta_2}{2}}, \tag{B.1.15b}$$

and

$$\|x_j NL_w(t,\cdot)\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{\frac{\delta+\delta_2}{2}}, \tag{B.1.16a}$$

$$\|x_j NL_w(t,\cdot)\|_{L^\infty} \leq C(A+B)B\varepsilon^2 t^{-1+\frac{\delta_2}{2}}. \tag{B.1.16b}$$

Proof. From

$$\|x_j NL_{kg}(t,\cdot)\|_{L^2} \lesssim \sum_{\mu=0}^{1} \big\| x_j (D_x \langle D_x \rangle^{-1})^{\mu} v_\pm(t,\cdot) \big\|_{L^\infty} \|u_\pm(t,\cdot)\|_{H^1},$$

and (B.1.13a), together with (B.1.14) and

$$\|x_j NL_w(t,\cdot)\|_{L^\infty} \lesssim \sum_{\mu=0}^{1} \big\| x_j (D_x \langle D_x \rangle^{-1})^{\mu} v_\pm(t,\cdot) \big\|_{L^\infty} \|v_\pm(t,\cdot)\|_{H^{2,\infty}},$$

we immediately derive the estimates of the statement using (B.1.10b) and a-priori estimates. □

Lemma B.1.6. *There exists a positive constant $C > 0$ such that, for any multi-index I of length k, with $1 \leq k \leq 2$, any $j = 1, 2$, $t \in [1, T]$,*

$$\sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^{\mu} (\Gamma^I v)_\pm(t,\cdot) \Big\|_{H^1} + \Big\| x_j D_x \Big(\frac{D_x}{|D_x|} \Big)^{\mu} (\Gamma^I u)_\pm(t,\cdot) \Big\|_{L^2} \leq CB\varepsilon t^{1+\frac{\delta_{3-k}}{2}}. \tag{B.1.17}$$

Proof. We warn the reader that, throughout the proof, C will denote a positive constant that may change line after line. As $\Gamma^I w_+ = -\overline{\Gamma^I w_-}$, for any I and $w \in \{v, u\}$, it is enough to prove the statement for $\Gamma^I v_-$, $\Gamma^I u_-$.

From equalities (B.1.9) together with the fact that, for any multi-index I, $(\Gamma^I v)_-$, $(\Gamma^I u)_-$ are solution to

$$[D_t + \langle D_x \rangle](\Gamma^I v)_-(t,x) = \Gamma^I NL_{kg} \tag{B.1.18a}$$

and

$$[D_t + \langle D_x\rangle](\Gamma^I u)_-(t,x) = \Gamma^I NL_w \tag{B.1.18b}$$

respectively, we derive that, for any $j,k \in \{1,2\}$,

$$\sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} (\Gamma^I v)_\pm(t,\cdot)\Big\|_{H^1} \leq \|Z_j(\Gamma^I v)_-(t,\cdot)\|_{L^2} + t\|(\Gamma^I v)_-(t,\cdot)\|_{L^2} + \|x_j\Gamma^I NL_{kg}(t,\cdot)\|_{L^2} \tag{B.1.19a}$$

together with

$$\sum_{|\mu|=0}^{1} \Big\| x_j D_x\Big(\frac{D_x}{|D_x|}\Big)^{\mu} (\Gamma^I u)_\pm(t,\cdot)\Big\|_{L^2} \leq \|Z_j(\Gamma^I u)_-(t,\cdot)\|_{L^2} + t\|(\Gamma^I u)_-(t,\cdot)\|_{L^2} + \|x_j\Gamma^I NL_w(t,\cdot)\|_{L^2}. \tag{B.1.19b}$$

The first two quantities in above right hand sides are bounded by $CB\varepsilon t^{1+\delta_{3-k}/2}$ after (1.1.11d), so the quantities that need to be estimated in order to prove the statement are the L^2 norms of $x_j\Gamma^I NL_{kg}$, $x_j\Gamma^I NL_w$, for $1 \leq |I| \leq 2$.

We first prove (B.1.17) for $|I| = 1$ and $\Gamma^I = \Gamma$, reminding that from (1.1.15),

$$\Gamma NL_{kg} = Q_0^{\mathrm{kg}}\big((\Gamma v)_\pm, D_1 u_\pm\big) + Q_0^{\mathrm{kg}}\big(v_\pm, D_1(\Gamma u)_\pm\big) + G_1^{\mathrm{kg}}\big(v_\pm, Du_\pm\big) \tag{B.1.20a}$$

and

$$\Gamma NL_w = Q_0^{\mathrm{w}}\big((\Gamma v)_\pm, D_1 v_\pm\big) + Q_0^{\mathrm{w}}\big(v_\pm, D_1(\Gamma v)_\pm\big) + G_1^{\mathrm{w}}\big(v_\pm, Dv_\pm\big), \tag{B.1.20b}$$

with $G_1^{\mathrm{kg}}\big(v_\pm, Du_\pm\big) = G_1(v,\partial u)$, $G_1^{\mathrm{w}}\big(v_\pm, Dv_\pm\big) = G_1(v,\partial v)$, and G_1 given by (1.1.16).

By multiplying x_j against the Klein-Gordon component in each product of ΓNL_{kg} we find that

$$\begin{aligned}\|x_j\Gamma NL_{kg}(t,\cdot)\|_{L^2} \lesssim{}& \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} (\Gamma v)_-(t,\cdot)\Big\|_{L^2} \big(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\big)\\ &+ \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_\pm(t,\cdot)\Big\|_{L^\infty} \big(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1} + \|u_\pm(t,\cdot)\|_{H^1} + \|D_t u_\pm(t,\cdot)\|_{L^2}\big),\end{aligned} \tag{B.1.21}$$

which injected into (B.1.19a) with $\Gamma^I = \Gamma$, together with (B.1.5a) with $s = 0$, (B.1.10b), and a-priori estimates (1.1.11), gives that

$$\sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} (\Gamma^I v)_\pm(t,\cdot)\Big\|_{H^1} \leq CB\varepsilon t^{1+\frac{\delta_2}{2}}.$$

Similarly, using the above estimate together with (B.1.6a) with $s = 0$, (B.1.10b) and a-priori estimates, we derive that

$$\begin{aligned}\|x_j\Gamma NL_w(t,\cdot)\|_{L^2} \lesssim{}& \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} (\Gamma v)_-(t,\cdot)\Big\|_{L^2} \|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\ &+ \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_\pm(t,\cdot)\Big\|_{L^\infty} \big(\|(\Gamma v)_\pm(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1} + \|D_t v_\pm(t,\cdot)\|_{L^2}\big)\\ &\leq C(A+B)B\varepsilon^2 t^{\delta_2}.\end{aligned} \tag{B.1.22}$$

Plugging the above inequality in (B.1.19b) for $\Gamma^I = \Gamma$ and using again a-priori estimates we deduce that

$$\sum_{|\mu|=0}^{1} \Big\| x_j D_k \Big(\frac{D_x}{|D_x|} \Big)^{\mu} (\Gamma u)_-(t,\cdot) \Big\|_{L^2} \leq CB\varepsilon t^{1+\frac{\delta_2}{2}},$$

and conclude the proof of (B.1.17) when $|I| = 1$.

When $|I| = 2$ we observe that, from (1.1.17),

$$\Gamma^I NL_{kg} = Q_0^{\mathrm{kg}}(v_\pm^I, D_1 u_\pm) + Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm^I) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|=|I_2|=1}} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|\leq 1}} c_{I_1,I_2} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D u_\pm^{I_2}), \tag{B.1.23}$$

with $c_{I_1,I_2} \in \{-1, 0, 1\}$. Since the L^2 norm of terms indexed in I_1, I_2 with $|I_1| = |I_2| = 1$ can be estimated using the Sobolev injection as follows:

$$\Big\| x_j Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2}) \Big\|_{L^2} \lesssim \sum_{|\mu|=0}^{1} \|v_\pm^{I_1}(t,\cdot)\|_{H^2} \Big\| x_j D_1 \Big(\frac{D_x}{|D_x|} \Big)^{\mu} u_\pm^{I_2}(t,\cdot) \Big\|_{L^2}, \tag{B.1.24}$$

from (B.1.23) we derive that

$$\begin{aligned}
\|x_j \Gamma^I NL_{kg}\|_{L^2} \lesssim{}& \sum_{\substack{|J|\leq 2\\ |\mu|,\nu=0}}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle} \Big)^{\mu} (\Gamma^J v)_-(t,\cdot) \Big\|_{L^2} \Big(\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|D_t \mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{1,\infty}} \Big) \\
&+ \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle} \Big)^{\mu} v_\pm(t,\cdot) \Big\|_{L^\infty} \Big[\|u_\pm^I(t,\cdot)\|_{H^1} + \sum_{|J|\leq 1} \big(\|u_\pm^J(t,\cdot)\|_{H^1} + \|D_t u_\pm^J(t,\cdot)\|_{L^2} \big) \Big] \\
&+ \sum_{\substack{|I_1|=|I_2|=1\\ |\mu|=0,1}} \|v_\pm^{I_1}(t,\cdot)\|_{H^2} \Big\| x_j D_1 \Big(\frac{D_x}{|D_x|} \Big)^{\mu} u_\pm^{I_2}(t,\cdot) \Big\|_{L^2}.
\end{aligned}$$

As before, injecting the above inequality into (B.1.19a), using a-priori estimates (1.1.11) and the fact that $\varepsilon_0 < (2A)^{-1}$, together with (B.1.5a) with $s = 0$, (B.1.5b), (B.1.5c) with $s = 1$, (B.1.7), (B.1.10b), and (B.1.17) with $k = 1$, we obtain that

$$\sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x\rangle} \Big)^{\mu} (\Gamma^I v)_-(t,\cdot) \Big\|_{H^1} \leq CB\varepsilon t^{1+\frac{\delta_1}{2}}. \tag{B.1.25}$$

Analogously, since

$$\Gamma^I NL_w = Q_0^{\mathrm{w}}(v_\pm^I, D_1 v_\pm) + Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm^I) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|=|I_2|=1}} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D_1 v_\pm^{I_2}) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<2}} c_{I_1,I_2} Q_0^{\mathrm{w}}(v_\pm^{I_1}, D v_\pm^{I_2}),$$

we have that

$$\|x_j\Gamma^I NL_w\|_{L^2} \lesssim \sum_{\substack{|J|\le 2\\ |\mu|=0,1}} \Big\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma^J v)_\pm(t,\cdot)\Big\|_{L^2}\Big(\|v_\pm(t,\cdot)\|_{H^{2,\infty}}+\|D_t v_\pm(t,\cdot)\|_{H^{1,\infty}}\Big)$$
$$+\sum_{|\mu|=0}^{1}\Big\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{H^{1,\infty}}\left(\sum_{|J|\le 2}\|(\Gamma^J v)_\pm(t,\cdot)\|_{H^1}+\sum_{|J|\le 1}\|D_t v_\pm^J(t,\cdot)\|_{L^2}\right)$$
$$+\sum_{\substack{|I_1|=|I_2|=1\\ |\mu|=0,1}}\|(\Gamma^{I_1}v)_\pm(t,\cdot)\|_{H^2}\Big\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma^{I_2}v)_\pm(t,\cdot)\Big\|_{L^2},$$

so from (B.1.6a) with $s=0$, (B.1.6b) with $s=1$, (B.1.8), (B.1.10b), (B.1.17) with $|I|=1$, (B.1.25) and a-priori estimates (1.1.11), we deduce

$$\sum_{|\mu|=0}^{1}\Big\|x_j D_k\Big(\frac{D_x}{|D_x|}\Big)^\mu(\Gamma^I u)_-(t,\cdot)\Big\|_{L^2}\lesssim CB\varepsilon t^{1+\frac{\delta_1}{2}},$$

and hence conclude the proof of inequality (B.1.17) also for the case $|I|=2$. □

Corollary B.1.7. *There exists a positive constant $C>0$ such that, for any $\Gamma\in\mathcal{Z}$, $j=1,2$, and every $t\in[1,T]$,*

$$\|x_j\Gamma NL_{kg}(t,\cdot)\|_{L^2}\le C(A+B)B\varepsilon^2 t^{\frac12+\frac{\delta_2}{2}}, \tag{B.1.26a}$$
$$\|x_j\Gamma NL_w(t,\cdot)\|_{L^2}\le C(A+B)B\varepsilon^2 t^{\delta_2}. \tag{B.1.26b}$$

Proof. Estimate (B.1.26a) follows straightly from (B.1.21), (B.1.5a) with $s=0$, and estimates (1.1.11), (B.1.10b), and (B.1.17) with $k=1$, while (B.1.26b) has already been proved in (B.1.22). □

Lemma B.1.8. *There exists a constant $C>0$ such that, for every $i,j=1,2$, every $t\in[1,T]$,*

$$\sum_{|\mu|=0}^{1}\Big\|x_j x_k\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^2}\le CB\varepsilon t^{2+\frac{\delta_2}{2}}, \tag{B.1.27a}$$
$$\sum_{|\mu|=0}^{1}\Big\|x_j x_k\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty}\le C(A+B)\varepsilon t^{1+\frac{\delta_2}{2}}. \tag{B.1.27b}$$

Moreover, for any $\Gamma\in\mathcal{Z}$,

$$\sum_{|\mu|=0}^{1}\Big\|x_i x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_\pm(t,\cdot)\Big\|_{L^2}\le CB\varepsilon t^{2+\frac{\delta_2}{2}}. \tag{B.1.28}$$

Proof. The proof of the statement follows from the fact that, by multiplying (B.1.9b) by x_i and using that

$$\|x_i x_j NL_{kg}(t,\cdot)\|_{L^2}\lesssim\sum_{|\mu|=0}^{1}\Big\|x_i x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_-(t,\cdot)\Big\|_{L^2}(\|u_\pm(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}})$$

together with

$$\|x_i x_j NL_{kg}(t,\cdot)\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{1} \Big\| x_j x_k \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_-(t,\cdot)\Big\|_{L^\infty} \big(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}\big)\,,$$

we derive that

$$\begin{aligned}\sum_{|\mu|=0}^{1} \Big\| x_j x_k \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^2} &\lesssim \sum_{|\mu|=0}^{1} \big(\|x_i^\mu (Z_j v)_-(t,\cdot)\|_{L^2} + t\|x_i^\mu v_-(t,\cdot)\|_{L^2}\big)\\ &\quad + \sum_{\mu=0}^{1} \Big\| x_i x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_-(t,\cdot)\Big\|_{L^2} \big(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}\big)\,,\end{aligned}$$

and using that operator $\langle D_x\rangle^{-1}$ is bounded from H^1 to L^∞

$$\begin{aligned}\sum_{|\mu|=0}^{1} \Big\| x_i x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_-(t,\cdot)\Big\|_{L^\infty} &\lesssim \sum_{k=0}^{1} \Big(\|x_i^k (Z_j v)_-(t,\cdot)\|_{H^1} + t\|x_i^k v_-(t,\cdot)\|_{H^{1,\infty}}\Big)\\ &\quad + \sum_{k,|\mu|=0}^{1} \Big\| x_i^k x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_-(t,\cdot)\Big\|_{L^\infty} \big(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}\big)\,.\end{aligned}$$

As $\varepsilon_0 > 0$ verifies that $\varepsilon_0 < (2A)^{-1}$, inequality (B.1.10a), (B.1.17) with $k = 1$, and a-priori estimates (1.1.11) imply that

$$\sum_{|\mu|=0}^{1} \Big\| x_i x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_-(t,\cdot)\Big\|_{L^2} \lesssim CB\varepsilon t^{2+\frac{\delta_2}{2}},$$

while from (B.1.10b), (B.1.17) with $k = 1$ and a-priori estimates,

$$\sum_{|\mu|=0}^{1} \Big\| x_i x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_-(t,\cdot)\Big\|_{L^\infty} \leq C(A+B)\varepsilon t^{1+\frac{\delta_2}{2}}.$$

As $v_+ = -\overline{v_-}$, that implies the first part of the statement.
Analogously, using (B.1.9b) with $w = (\Gamma v)_-$ and multiplying that relation by x_i we find that

$$\begin{aligned}(\text{B.1.29})\quad &\sum_{|\mu|=0}^{1} \Big\| x_i x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu (\Gamma v)_-(t,\cdot)\Big\|_{L^2}\\ &\qquad \lesssim \sum_{\mu=0}^{1} \big(\|x_i^\mu Z_j(\Gamma v)_-(t,\cdot)\|_{L^2} + t\|x_i^\mu(\Gamma v)_-(t,\cdot)\|_{L^2} + \|x_i^\mu x_j \Gamma NL_{kg}(t,\cdot)\|_{L^2}\big),\end{aligned}$$

and after (B.1.17), (B.1.26a) and a-priori estimates,

$$(\text{B.1.30})\qquad \sum_{\mu=0}^{1} \big(\|x_i^\mu Z_j(\Gamma v)_-(t,\cdot)\|_{L^2} + t\|x_i^\mu(\Gamma v)_-(t,\cdot)\|_{L^2}\big) + \|x_j \Gamma NL_{kg}(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{2+\frac{\delta_2}{2}}.$$

By multiplying both x_i, x_j against each Klein-Gordon factor in ΓNL_{kg} (see equality (B.1.20a)) we derive that

$$\begin{aligned}\|x_ix_j\Gamma NL_{kg}(t,\cdot)\|_{L^2} \lesssim \sum_{|\mu|\nu=0}^{1} \Big\|x_ix_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}(\Gamma v)_-(t,\cdot)\Big\|_{L^2} \|\mathrm{R}_1^{\nu}u_\pm(t,\cdot)\|_{H^{2,\infty}} \\ + \sum_{|\mu|=0}^{1} \Big\|x_ix_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\Big\|_{L^\infty} \left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1} + \|u_\pm(t,\cdot)\|_{H^1} + \|D_tu_\pm(t,\cdot)\|_{L^2}\right),\end{aligned}$$

so by (B.1.5a) with $s=0$, (B.1.27b), a-priori estimates and the fact that $\varepsilon_0 < (2A)^{-1}$,

$$\|x_ix_j\Gamma NL_{kg}(t,\cdot)\|_{L^2} \leq \frac{1}{2}\|x_ix_j(\Gamma v)_-(t,\cdot)\|_{L^2} + C(A+B)B\varepsilon^2t^{1+\delta_2},$$

which injected in (B.1.29), together with (B.1.30), implies (B.1.28). □

Corollary B.1.9. *There exists a constant $C>0$ such that, for every $i,j=1,2$, every $t\in[1,T]$, B.1.4,*

$$\|x_ix_jNL_{kg}(t,\cdot)\|_{L^2} + \|x_ix_jNL_w(t,\cdot)\|_{L^2} \leq C(A+B)B\varepsilon^2t^{1+\frac{\delta+\delta_2}{2}}. \tag{B.1.31}$$

Proof. Straightforward after (1.1.12c), (B.1.27b) and the following inequality

$$\begin{aligned}\|x_ix_jNL_{kg}(t,\cdot)\|_{L^2} + \|x_ix_jNL_w(t,\cdot)\|_{L^2} \\ \lesssim \sum_{|\mu|=0}^{1} \Big\|x_ix_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\Big\|_{L^\infty} \left(\|u_\pm(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1}\right).\end{aligned}$$

□

Lemma B.1.10. *There exists a constant $C>0$ such that, for any $i,j,k=1,2$, every $t\in[1,T]$, B.1.4,*

$$\sum_{|\mu|=0}^{1} \Big\|x_ix_jx_k\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\Big\|_{L^2} \leq CB\varepsilon t^{3+\frac{\delta_2}{2}}. \tag{B.1.32}$$

Proof. Using equality (B.1.9b) we derive that

$$\begin{aligned}\|x_ix_jx_kv_-(t,\cdot)\|_{L^2} \lesssim \sum_{\mu_1,\mu_2=0}^{1} \Big(\|x_i^{\mu_1}x_j^{\mu_2}(Z_kv)_-(t,\cdot)\|_{L^2} + t\|x_i^{\mu_1}x_j^{\mu_2}v_-(t,\cdot)\|_{L^2}\Big) \\ + \sum_{\mu_1,\mu_2,|\mu|=0}^{1} \Big\|x_i^{\mu_1}x_j^{\mu_2}x_k\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_-(t,\cdot)\Big\|_{L^2} \left(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1u_\pm(t,\cdot)\|_{H^{2,\infty}}\right),\end{aligned}$$

so the result of the statement is a straight consequence of (B.1.10a), (B.1.17), (B.1.27a), (B.1.28), a-priori estimates, and the fact that ε_0 is smaller than $(2A)^{-1}$. □

B.2 First range of estimates

The aim of this section is to show that, if a-priori estimates (1.1.11) are satisfied for every $t \in [1,T]$, for some fixed $T > 1$, then in the same interval the semi-classical Sobolev norms of the semi-classical functions $\widetilde{u}, \widetilde{v}$ introduced in (3.2.2) grow in time at a moderate rate t^β, for some small $\beta > 0$. More precisely, in lemma B.2.1 we prove that this is the case for the $H^s_h(\mathbb{R}^2)$ norm of $\widetilde{u}$, $\widetilde{u}^{\Sigma_j,k}$ (see definition (3.2.41)) for any $s \leq n-15$, and for the $L^2(\mathbb{R}^2)$ norm of those functions when operators Ω_h and $\mathcal{M}$, introduced in (1.2.40) and (1.2.49) respectively, are acting on them and frequencies are less or equal than $h^{-\sigma}$, for some small $\sigma > 0$. Lemma B.2.14 shows that this moderate growth is also enjoyed by the $H^s_h(\mathbb{R}^2)$ norm of $\widetilde{v}$, again for $s \leq n = 15$, and by the $L^2(\mathbb{R}^2)$ norm of $\mathcal{L}\widetilde{v}$ (see (1.2.68)) when restricted to frequencies $|\xi| \lesssim h^{-\sigma}$. The proof of this latter lemma will require some intermediate results, among which lemma B.2.8 that provides us with a first non-sharp estimate of the $L^\infty(\mathbb{R}^2)$ norm of Klein-Gordon functions $v_\pm$ when one Klainerman vector field is acting on them (and again frequencies are localized for $|\xi| \lesssim t^\sigma$). This estimate will successively improved to the sharpest one (B.4.50) in lemma B.4.14 of section B.4.

As said at the beginning of this chapter, we prove the below results under the hypothesis that a-priori estimates (1.1.11) are satisfied in some fixed $[1,T]$, with $\varepsilon_0 < (2A+B)^{-1}$. We remind here that, if $\chi \in C^\infty_0(\mathbb{R}^2)$ and $\sigma > 0$, $\chi(t^{-\sigma}D_x)$ is a bounded operator from H^s to L^2 with norm $O(t^{\sigma s})$, and on L^∞ uniformly in time.

Lemma B.2.1. *Let $\widetilde{u}, \widetilde{u}^{\Sigma_j,k}$ be defined, respectively, in* (3.2.2) *and* (3.2.41)*, and $s \leq n-15$. There exists a constant $C > 0$ such that, for any $\theta_0, \chi \in C^\infty_0(\mathbb{R}^2)$ and every $t \in [1,T]$,*

$$\|\widetilde{u}(t,\cdot)\|_{H^s_h} + \|\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{H^s_h} \leq CB\varepsilon t^{\frac{\delta}{2}+\kappa}, \tag{B.2.1a}$$

$$\|\Omega_h \widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_2}{2}+\kappa}, \tag{B.2.1b}$$

$$\sum_{|\mu|=1} \Big(\|\mathrm{Op}^w_h(\chi(h^\sigma\xi))\mathcal{M}^\mu \widetilde{u}(t,\cdot)\|_{L^2} + \|\mathcal{M}^\mu \widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} \Big) \leq C(A+B)\varepsilon t^{\frac{\delta_2}{2}+\kappa}, \tag{B.2.1c}$$

$$\sum_{|\mu|=1} \|\theta_0(x)\Omega_h \mathcal{M}^\mu \widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_1}{2}+\kappa}, \tag{B.2.1d}$$

with $\kappa = \sigma\rho$ if $\rho \geq 0$, 0 otherwise.

Proof. We warn the reader that, throughout the proof, C and β will denote positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$. We will also use the following concise notation

$$\phi^j_k(\xi) := \Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi),$$

reminding that

$$\Big\| \mathrm{Op}^w_h(\phi^j_k(\xi)) \Big\|_{\mathcal{L}(L^2)} = O(h^{-\kappa}), \tag{B.2.2}$$

with $\kappa = \sigma\rho$ if $\rho \geq 0$, 0 otherwise.

Inequality (B.2.1a) is straightforward after (B.2.2), definitions (3.2.2) and (3.1.15), inequality (3.1.20a), and a-priori estimate (1.1.11b). By commutating $\mathrm{Op}^w_h(\phi^j_k(\xi))$ with $\mathcal{M}$ (the commutator with Ω_h being zero if φ, χ_0 are supposed to be radial) and using (B.2.2) we observe that there is some $\chi \in C^\infty_0(\mathbb{R}^2)$ such that

$$\|\Omega_h \widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} \lesssim h^{-\kappa} \|\mathrm{Op}^w_h(\chi_0(h^\sigma\xi))\Omega_h \widetilde{u}(t,\cdot)\|_{L^2},$$

$$\|\mathcal{M}\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2} \lesssim h^{-\kappa} \sum_{|\nu|=0}^{1} \|\mathrm{Op}^w_h(\chi(h^\sigma\xi))\mathcal{M}^\nu \widetilde{u}(t,\cdot)\|_{L^2},$$

$$\|\theta_0(x)\Omega_h\mathcal{M}\widetilde{u}^{\Sigma_j,k}(t,\cdot)\|_{L^2}\lesssim\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}+h^{-\kappa}\sum_{\mu=0}^{1}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\Omega_h^\mu\widetilde{u}(t,\cdot)\|_{L^2}.$$

Therefore, as $h=t^{-1}$, in order to prove (B.2.1b)-(B.2.1d) it is enough to show that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$,

$$\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\Omega_h\widetilde{u}(t,\cdot)\|_{L^2}\leq CB\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.3a}$$

$$\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}\leq C(A+B)\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.3b}$$

$$\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}\leq CB\varepsilon t^{\frac{\delta_1}{2}}. \tag{B.2.3c}$$

Estimate (B.2.3a) follows from definitions (3.2.2) and (3.1.15), inequality (A.37a) with $u=v=v_\pm$, and a-priori estimates (1.1.11), as

$$\begin{aligned}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\Omega_h\widetilde{u}(t,\cdot)\|_{L^2}&\lesssim\|(\Omega u)_-(t,\cdot)\|_{L^2}+\|\chi(t^{-\sigma}D_x)\Omega(u^{NF}-u_-)(t,\cdot)\|_{L^2}\\&\lesssim\|\Omega U(t,\cdot)\|_{L^2}+t^\beta\left(\|V(t,\cdot)\|_{L^2}+\|\Omega V(t,\cdot)\|_{L^2}\right)\|V(t,\cdot)\|_{H^{17,\infty}}\\&\leq C(1+A\varepsilon t^{-1+\beta})E_3^2(t;W)^{\frac12}\leq CB\varepsilon t^{\frac{\delta_2}{2}}.\end{aligned}$$

From equality (3.2.9a) and definition (3.1.15) of u^{NF} we deduce that

$$\begin{aligned}&\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{M}_n\widetilde{u}(t,\cdot)\|_{L^2}\lesssim\|Z_nU(t,\cdot)\|_{L^2}+\|\chi(t^{-\sigma}D_x)Z_n(u^{NF}-u_-)(t,\cdot)\|_{L^2}\\&+\|\widetilde{u}(t,\cdot)\|_{L^2}+\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))[t(tx_j)[q_w+c_w](t,tx)]\|_{L^2(dx)}+\|\chi(t^{-\sigma}D_x)(x_nr_w^{NF})(t,\cdot)\|_{L^2},\end{aligned} \tag{B.2.4}$$

with q_w, c_w and r_w^{NF} given by (3.1.17), (3.1.18) and (3.1.19) respectively. We first notice that, after inequality (A.37b) with $u=v=v_\pm$, (B.1.6a) with $s=0$, a-priori estimates, and the fact that $A\varepsilon_0\leq1$,

$$\begin{aligned}&\|\chi(t^{-\sigma}D_x)Z_n(u^{NF}-u_-)(t,\cdot)\|_{L^2}\\&\qquad\lesssim t^\beta\left(\|D_tV(t,\cdot)\|_{L^2}\|V(t,\cdot)\|_{H^{13}}+\|V(t,\cdot)\|_{H^{15,\infty}}\|Z_nV(t,\cdot)\|_{L^2}\right)\leq CB\varepsilon t^{\beta+\delta}.\end{aligned} \tag{B.2.5}$$

Let us also observe that from (3.1.17), (3.1.18) we have that

$$\begin{aligned}q_w(t,x)+c_w(t,x)&=\frac12\Im\left[\overline{v_-}\,D_1v_--\overline{\frac{D_x}{\langle D_x\rangle}v_-}\cdot\frac{D_xD_1}{\langle D_x\rangle}v_-\right](t,x)\\&=\frac{h^2}{2}\Im\left[\overline{\widetilde{V}}\,\mathrm{Op}_h^w(\xi_1)\widetilde{V}-\overline{\mathrm{Op}_h^w\Big(\frac{\xi_1}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot\mathrm{Op}_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)\widetilde{V}\right]\Big(t,\frac{x}{t}\Big),\end{aligned} \tag{B.2.6}$$

where $\widetilde{V}(t,x):=tv_-(t,tx)$ is such that, for every $s,\rho\geq0$,

$$\|\widetilde{V}(t,\cdot)\|_{H_h^s}=\|v_-(t,\cdot)\|_{H^s},\quad\|\widetilde{V}(t,\cdot)\|_{H_h^{\rho,\infty}}=t\|v_-(t,\cdot)\|_{H^{\rho,\infty}}.$$

Moreover, by (3.2.8) with $w=v_-$ and $f=NL_{kg}$

$$\begin{aligned}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H_h^1}&\lesssim\|Z_jv_-(t,\cdot)\|_{L^2}+\|v_-(t,\cdot)\|_{L^2}\\&\quad+\left(\|x_jv_\pm(t,\cdot)\|_{L^\infty}+\left\|x_j\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{L^\infty}\right)\|U(t,\cdot)\|_{H^1}.\end{aligned} \tag{B.2.7}$$

Using (B.2.6) along with the definition of $\mathcal{L}_j$ in (1.2.68) we derive that

$$\begin{aligned}t(tx_j)[q_w+c_w](t,tx)&=\frac12\Im\left[\overline{\widetilde{V}}\mathrm{Op}_h^w(\xi_1)(h\mathcal{L}_j\widetilde{V})+\overline{\widetilde{V}}\mathrm{Op}_h^w\Big(\frac{\xi_1\xi_j}{\langle\xi\rangle}\Big)\widetilde{V}+\overline{\widetilde{V}}[x_j,\mathrm{Op}_h^w(\xi_1)]\widetilde{V}\right.\\&\quad-\overline{\mathrm{Op}_h^w\Big(\frac{\xi}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot\mathrm{Op}_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)(h\mathcal{L}_j\widetilde{V})-\overline{\mathrm{Op}_h^w\Big(\frac{\xi}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot\mathrm{Op}_h^w\Big(\frac{\xi\xi_1\xi_j}{\langle\xi\rangle^2}\Big)\widetilde{V}\\&\quad\left.-\overline{\mathrm{Op}_h^w\Big(\frac{\xi}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot\Big[x_j,\mathrm{Op}_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)\Big]\widetilde{V}\right](t,x),\end{aligned} \tag{B.2.8}$$

so after estimates (1.1.11) and (B.1.10b)

$$\begin{aligned}\|t(tx_j)[q_w+c_w](t,t\cdot)\|_{L^2(dx)} &\lesssim \left[\|\widetilde{V}(t,\cdot)\|_{H^1_h}+h\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1_h}\right]\|\widetilde{V}(t,\cdot)\|_{H^{1,\infty}_h}\\ &\leq CA(A+B)\varepsilon^2 t^{\frac{\delta}{2}}.\end{aligned}\tag{B.2.9}$$

Moreover, from (3.1.19), the fact that $x_j e^{ix\cdot\xi}=D_{\xi_j}e^{ix\cdot\xi}$, integration by parts, and inequalities (A.26a) with $\rho=2$ (after the first part of lemma A.8), (A.36a), we get that

(B.2.10)

$$\begin{aligned}&\|\chi(t^{-\sigma}D_x)(x_n r_w^{NF})(t,\cdot)\|_{L^2}\\ &\quad\lesssim t^\beta\left[\|x_n v_-(t,\cdot)\|_{L^\infty}\|NL_{kg}(t,\cdot)\|_{H^{15}}+\|V(t,\cdot)\|_{H^{15}}\|x_n NL_{kg}(t,\cdot)\|_{L^\infty}\right.\\ &\quad\left.+\|NL_{kg}(t,\cdot)\|_{L^2}\left(\|V(t,\cdot)\|_{H^{13}}+\|V(t,\cdot)\|_{H^{13,\infty}}\right)+\|V(t,\cdot)\|_{H^{13}}\|NL_{kg}(t,\cdot)\|_{L^\infty}\right]\\ &\quad\leq CB\varepsilon t^{\frac{\delta_2}{2}},\end{aligned}$$

where last estimate follows from (B.1.10b), (B.1.15a), inequalities (B.1.4a), (B.1.4b), (B.1.4c) with $s=15$, and a-priori estimates (1.1.11). Consequently, from (B.2.4), (B.2.5), (B.2.9), (B.2.10), (B.2.1a) and a-priori estimate (1.1.11d) with $k=2$, we obtain (B.2.3b).

Let us now apply $\theta_0\big(\frac{x}{t}\big)\phi_k^j(D_x)\Omega$ to both sides of (3.2.9a) to deduce that

(B.2.11)

$$\begin{aligned}&\left\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h\mathcal{M}_n\widetilde{u}(t,\cdot)\right\|_{L^2}\lesssim\|\Omega Z_n U(t,\cdot)\|_{L^2}\\ &+\left\|\theta_0\Big(\frac{x}{t}\Big)\phi_k^j(D_x)\Omega Z_n(u^{NF}-u_-)(t,\cdot)\right\|_{L^2}+\sum_{\mu=0}^{1}\|\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\Omega_h^\mu\widetilde{u}(t,\cdot)\|_{L^2}\\ &+\left\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h[t(tx_j)(q_w+c_w)(t,tx)]\right\|_{L^2(dx)}+\left\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h[t(tx_n)r_w^{NF}](t,tx)]\right\|_{L^2(dx)}.\end{aligned}$$

In order to estimate the second addend in the above right hand side we first commute Z_n to Ω, reminding that

$$[\Omega,Z_1]=-Z_2\quad\text{and}\quad[\Omega,Z_2]=Z_1,$$

and use that

$$\theta_0\Big(\frac{x}{t}\Big)\phi_k^j(D_x)Z_j=\Big[t\theta_0^j\Big(\frac{x}{t}\Big)\phi_k^j(D_x)+\theta_0\Big(\frac{x}{t}\Big)[\phi_k^j(D_x),x_j]\Big]\partial_t+t\theta_0\Big(\frac{x}{t}\Big)\phi_k^j(D_x)\partial_j,$$

with $\theta_0^j(z):=\theta_0(z)z_j$. Observe that commutator $[\phi_k^j(D_x),x_j]$ is bounded on L^2 with norm $O(t)$, and that its symbol is still supported for moderate frequencies $|\xi|\lesssim t^{-\sigma}$. Therefore, for some new $\chi\in C_0^\infty(\mathbb{R}^2)$ we have that

$$\begin{aligned}\left\|\theta_0\Big(\frac{x}{t}\Big)\phi_k^j(D_x)\Omega Z_n(u^{NF}-u_-)(t,\cdot)\right\|_{L^2}&\lesssim t\left\|\chi(t^{-\sigma}D_x)\partial_{t,x}(u^{NF}-u_-)(t,\cdot)\right\|_{L^2}\\ &\quad+t\left\|\chi(t^{-\sigma}D_x)\partial_{t,x}\Omega(u^{NF}-u_-)(t,\cdot)\right\|_{L^2},\end{aligned}$$

so using (A.26a) with $\rho=2$ (because of first part of lemma A.8) and (A.37a), both considered with $u=\partial_{t,x}v_\pm, v=v_\pm$, and $u=v_\pm, v=\partial_{t,x}v_\pm$, we obtain that the above right hand side is estimated by

$$t^{1+\beta}\left[\left(\|\partial_{t,x}V(t,\cdot)\|_{L^2}+\|\Omega\partial_{t,x}V(t,\cdot)\|_{L^2}\right)\|V(t,\cdot)\|_{H^{17,\infty}}+\left(\|V(t,\cdot)\|_{L^2}+\|\Omega V(t,\cdot)\|_{L^2}\right)\|\partial_{t,x}V(t,\cdot)\|_{H^{17,\infty}}\right]$$

From (B.1.6a) and (B.1.6b) with $s=0$, (B.1.6c) and a-priori estimates, we hence deduce that

$$\left\|\theta_0\Big(\frac{x}{t}\Big)\phi_k^j(D_x)\Omega Z_n(u^{NF}-u_-)(t,\cdot)\right\|_{L^2}\leq CB\varepsilon t^{\beta+\frac{\delta_2}{2}}.\tag{B.2.12}$$

As concerns, instead, the estimate of the fourth L^2 norm in the right hand side of (B.2.11), we observe that from equality (B.2.8), Leibniz rule and (B.2.2)

$$\text{(B.2.13)}\quad \left\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h[t(tx_j)[q_w+c_w](t,tx)]\right\|_{L^2} \lesssim \sum_{\mu=0}^{1} h^{-\kappa}\|\widetilde{V}(t,\cdot)\|_{H_h^{2,\infty}}\|\Omega_h^\mu \widetilde{V}(t,\cdot)\|_{H_h^1}$$
$$+\sum_{\mu=0}^{1} h^{1-\kappa}\|\widetilde{V}(t,\cdot)\|_{H_h^{1,\infty}}\|\Omega_h^\mu \mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1} + h^{1-\kappa}\|\Omega_h\widetilde{V}(t,\cdot)\|_{L^\infty}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H_h^1},$$

with $\kappa=\sigma\rho$ if $\rho\geq 0$, 0 otherwise. Using the semi-classical Sobolev injection, (B.2.7) and the fact that $\|\Omega_h\widetilde{V}(t,\cdot)\|_{H_h^s} = \|\Omega v_-(t,\cdot)\|_{H^s}$ for any $s\geq 0$, together with (B.1.10b) and a-priori estimates, we see that

$$\text{(B.2.14)}\qquad h\|\Omega_h\widetilde{V}(t,\cdot)\|_{L^\infty}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H_h^1} \lesssim \|\Omega\widetilde{V}(t,\cdot)\|_{H_h^2}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H_h^1} \leq CB\varepsilon t^{\frac{3\delta_2}{2}}.$$

Also, from (3.2.8) with $w=v_-$ and $f=NL_{kg}$

$$\|\Omega_h\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{L^2} \lesssim \|\Omega Z_j v_-(t,\cdot)\|_{L^2} + \sum_{\mu=0}^{1}\|\Omega^\mu v_-(t,\cdot)\|_{L^2} + \|\Omega\left(x_j NL_{kg}\right)(t,\cdot)\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{\frac{1}{2}+\frac{\delta_2}{2}},$$

where last inequality is obtained using (1.1.11c), (1.1.11d) and estimates (B.1.15a), (B.1.26a). Therefore

$$h\|\widetilde{V}(t,\cdot)\|_{H_h^{1,\infty}}\|\Omega_h\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{L^2} \leq CAB(A+B)\varepsilon^3 t^{-\frac{1}{2}+\frac{\delta_2}{2}},$$

which combined with (B.2.13), (B.2.14) and a-priori estimates gives that

$$\text{(B.2.15)}\qquad \left\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h[t(tx_j)[q_w+c_w](t,tx)]\right\|_{L^2} \leq CB\varepsilon t^{\frac{3\delta_2}{2}}.$$

We estimate the latter L^2 norm in (B.2.11) recalling definition (3.1.19) of r_w^{NF}, commutating Ω and x_n, and using that

$$\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))x_n = \theta_0^n(x)\mathrm{Op}_h^w(\phi_k^j(\xi)) + \theta_0(x)[\mathrm{Op}_h^w(\phi_k^j(\xi)),x_n],$$

where

$$[\mathrm{Op}_h^w(\phi_k^j(\xi)),x_n] = -ih\mathrm{Op}_h^w(\partial_n\phi_k^j(\xi))$$

is uniformly bounded on L^2. After (3.1.22a), (3.1.22c) with $\theta\ll 1$ small, and a-priori estimates (1.1.11) we derive that, for some $\chi\in C_0^\infty(\mathbb{R}^2)$,

$$\left\|\theta_0(x)\mathrm{Op}_h^w(\phi_k^j(\xi))\Omega_h[t(tx_n)r_w^{NF}](t,tx)\right\|_{L^2(dx)} \lesssim \sum_{\mu=0}^{1} t\|\chi(t^{-\sigma}D_x)\Omega^\mu r_w^{NF}(t,\cdot)\|_{L^2} \leq CB\varepsilon.$$

Combining (B.2.11), (B.2.12), (B.2.15) and above estimate together with (1.1.11d), (B.2.1a), (B.2.3a), and assuming $3\delta_2\leq\delta_1$, we finally obtain (B.2.3c) and the conclusion of the proof. □

In the following lemma we explain how we estimate the L^2 or the L^∞ norm of products supported for moderate frequencies $|\xi|\lesssim t^\sigma$, when we have a control on high Sobolev norms of, at least, all factors but one. This type of estimate will be frequently used in most of the results that follow.

Lemma B.2.2. *Let $n \in \mathbb{N}$, $n \geq 2$, and $w_1, \dots, w_n$ such that $w_1 \in L^2(\mathbb{R}^2)$, $w_2, \dots, w_n \in L^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$, for some large positive s. Let also $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\sigma > 0$. There exists some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$, equal to 1 on the support of χ, such that for $L = L^2$ or $L = L^\infty$*

$$\big\|\chi(t^{-\sigma}D_x)\,[w_1 \dots w_n]\big\|_L \lesssim \left\|\big[\chi_1(t^{-\sigma}D_x)w_1\big]\prod_{j=2}^n \chi(t^{-\sigma}D_x)w_j\right\|_{L(dx)} + t^{-N(s)}\|w_1\|_{L^2(dx)}\sum_{j=2}^n\prod_{k\neq j}\|w_k\|_{L^\infty}\|w_j\|_{H^s(dx)},$$

with $N(s)$ as large as we want as long as $s > 0$ is large.

Proof. The idea of the proof is to decompose each factor w_j, for $j = 2, \dots, n$ into

$$\chi(t^{-\sigma}D_x)w_j + (1-\chi)(t^{-\sigma}D_x)w_j, \tag{B.2.16}$$

and to estimate the L^2 norm of product

$$\chi(t^{-\sigma}D_x)\left[w_1\prod_{\substack{k=2,\dots,n\\ k\neq j}}\widetilde{w}_k\left[(1-\chi)(t^{-\sigma}D_x)w_j\right]\right], \tag{B.2.17}$$

where $\widetilde{w}_k$ is either w_k or $\chi(t^{-\sigma}D_x)w_k$, with the L^2 norm of w_1 times the L^∞ norm of all remaining factors, reminding that $\chi(t^{-\sigma}D_x)$ is uniformly bounded on L^∞ and that by Sobolev injection and (B.1.2),

$$\big\|(1-\chi)(t^{-\sigma}D_x)w_j\big\|_{L^\infty(dx)} \lesssim t^{-N(s)}\|w_j\|_{H^s(dx)}, \tag{B.2.18}$$

with $N(s)$ as large as we want as long as $s > 0$ is large. The L^∞ norm of (B.2.17) is estimated in the same way, using firstly the $L^2 - L^\infty$ continuity of operator $\chi(t^{-\sigma}D_x)$ acting on the entire product.

The end of the statement follows from the observation that, if $\mathrm{supp}\chi \subset B_C(0)$ for some $C > 0$, then

$$\mathrm{supp}\hat{w}_1 \subset \{\xi : |\xi| \geq C_1 > nC\} \quad \Rightarrow \quad \chi(t^{-\sigma}D_x)\Big[w_1\prod_{j=2}^n\chi(t^{-\sigma}D_x)w_j\Big] \equiv 0. \tag{B.2.19}$$

□

Remark B.2.3. Property (B.2.19) is more general, meaning that if $\chi, \chi_j \in C_0^\infty(\mathbb{R}^2)$ with $\mathrm{supp}\chi \subset B_C(0)$, $\mathrm{supp}\chi_j \subset B_{C_j}(0)$ for some $C, C_j > 0$, for every $j = 2, \dots, n$, then

$$\mathrm{supp}\hat{w}_1 \subset \Big\{\xi : |\xi| \geq C_1 > C + \sum_{j=2}^n C_j\Big\} \quad \Rightarrow \quad \chi(t^{-\sigma}D_x)\Big[w_1\prod_{j=2}^n\chi_j(t^{-\sigma}D_x)w_j\Big] \equiv 0.$$

We have seen at the beginning of section B.1, and already used in the previous lemma's proof, that, if $w \in H^s(\mathbb{R}^2)$ for some large $s > 0$, the L^2 norm (resp. L^∞ norm) of this function when restricted to large frequencies $|\xi| \gtrsim t^\sigma$ decays fast in time as $t^{-\sigma s}$ (resp. $t^{-\sigma(s-1)-1}$ after the semi-classical Sobolev injection). The aim of the following lemma is to show that, even if we don't have a control on the $H^s(\mathbb{R}^2)$ norm of $(\Gamma u)_\pm$, $(\Gamma v)_\pm$, for $\Gamma \in \{\Omega, Z_m, m = 1, 2\}$ and s larger than 2, the L^2 norm (resp. L^∞ norm) of products as in (B.2.21) still have a good decay in time.

Lemma B.2.4. *Let $w \in \{u, v\}$ and for any $\Gamma \in \{\Omega, Z_m, m = 1, 2\}$*

$$(\Gamma w)_\pm = \begin{cases} (D_t \pm |D_x|)(\Gamma u), & \text{if } w = u, \\ (D_t \pm \langle D_x \rangle)(\Gamma v), & \text{if } w = v. \end{cases}$$

Let also $n \in \mathbb{N}^*$, $w_1, \dots, w_n$ be such that $w_1, xw_1 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $w_j \in L^\infty(\mathbb{R}^2)$ for $j = 2, \dots, n$, $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$, and $a(D_x) = D_x^\alpha \big(\frac{D_x}{\langle D_x \rangle}\big)^\beta \big(\frac{D_x}{|D_x|}\big)^\gamma$ for any $\alpha, \beta, \gamma \in \mathbb{N}^2$ with $|\alpha|, |\beta|, |\gamma| \le 1$. Then for $L = L^2$ or $L = L^\infty$ we have that

(B.2.20a)

$$\|a(D_x)(\Omega w)_\pm w_1 \dots w_n\|_{L(dx)} \lesssim \Big\| \big[\chi(t^{-\sigma} D_x) a(D_x)(\Omega w)_\pm\big] \prod_{j=1}^n w_j \Big\|_{L(dx)} + t^{-N(s)} \|w_\pm(t, \cdot)\|_{H^s} \Big(\sum_{|\mu|=0}^1 \|x^\mu w_1\|_{L(dx)} \Big) \prod_{j=2}^n \|w_j\|_{L^\infty(dx)}$$

and, for $m = 1, 2$,

$$\|a(D_x)(Z_m w)_\pm w_1 \dots w_n\|_{L(dx)} \lesssim \Big\| \big[\chi(t^{-\sigma} D_x) a(D_x)(Z_m w)_\pm\big] \prod_{j=1}^n w_j \Big\|_{L(dx)} + t^{-N(s)} \left(\|w_\pm(t, \cdot)\|_{H^s} + \|D_t w_\pm(t, \cdot)\|_{H^s} \right) \Big(\sum_{\mu=0}^1 \|x_m^\mu w_1\|_{L(dx)} + t \|w_1\|_{L(dx)} \Big) \prod_{j=2}^n \|w_j\|_{L^\infty(dx)}, \tag{B.2.20b}$$

with $N(s)$ as large as we want as long as $s > 0$ is large.

Proof. Let us remind definition (1.1.6) of Klainerman vector fields Ω, Z_m, for $m = 1, 2$, and decompose factor $a(D_x)(\Gamma w)_\pm$ in frequencies by means of operator $\chi(t^{-\sigma} D_x)$. When dealing with product

$$\big[(1 - \chi)(t^{-\sigma} D_x) a(D_x)(\Gamma w)_\pm\big] w_1 \cdots w_n \tag{B.2.21}$$

the idea is to discharge on w_1 factors x and/or t defining Γ, after a previous commutation between $D_t \pm |D_x|$ if $w = u$ (resp. $D_t \pm \langle D_x \rangle$ if $w = v$) and Γ, and between $(1 - \chi)(t^{-\sigma} D_x) a(D_x)$ and the mentioned factors x, t. For instance, if $w = u$ and $\Gamma = Z_1$

$$\begin{aligned} \big[(1 - \chi)(t^{-\sigma} D_x) a(D_x)(Z_1 u)_\pm\big] w_1 &= \big[(1 - \chi)(t^{-\sigma} D_x) a(D_x)(\partial_t u)_\pm\big](x_1 w_1) \\ &+ \big[(1 - \chi)(t^{-\sigma} D_x) a(D_x)(\partial_1 u)_\pm\big](t w_1) + \Big[(1 - \chi)(t^{-\sigma} D_x) a(D_x) \frac{D_1}{|D_x|} u_\pm\Big] w_1 \\ &+ \Big[\big[(1 - \chi)(t^{-\sigma} D_x) a(D_x), x_1\big] D_t u_\pm\Big] w_1, \end{aligned} \tag{B.2.22}$$

from which we deduce, using the Sobolev injection together with (B.1.2), that

$$\big\| \big[(1 - \chi)(t^{-\sigma} D_x) a(D_x)(Z_1 u)_\pm\big] w_1 \big\|_L \lesssim t^{-N(s)} \left(\|u_\pm(t, \cdot)\|_{H^s} + \|D_t u(t, \cdot)\|_{H^s} \right) \Big(\sum_{\mu=0}^1 \|x_1^\mu w_1\|_L + t \|w_1\|_L \Big),$$

with $N(s)$ large as long as s is large. Analogous inequalities can be obtained for $\Gamma = \Omega, Z_2$ and/or $w = v$. This concludes the proof of the statement since the L norm of (B.2.21) is bounded by the L norm of $\big[(1 - \chi)(t^{-\sigma} D_x) a(D_x)(Z_1 u)_\pm\big] w_1$ times the L^∞ norm of the remaining factors. □

Corollary B.2.5. *If the hypothesis of lemma B.2.4 are satisfied and in addition $w_1, \dots, w_n \in H^s(\mathbb{R}^2)$, we have that*

$$\begin{aligned}\|a(D_x)(\Omega w)_\pm w_1 \cdots w_n\|_L \lesssim{}& \Big\| \big[\chi(t^{-\sigma}D_x)a(D_x)(\Omega w)_\pm\big] \prod_{j=1}^n \chi(t^{-\sigma}D_x)w_j \Big\|_L \\ &+ t^{-N(s)}\|w_\pm(t,\cdot)\|_{H^s(dx)}\Big(\sum_{|\mu|=0}^1 \|x^\mu w_1\|_L\Big)\prod_{j=2}^n \|w_j\|_{L^\infty} \\ &+ t^{-N(s)}\|(\Omega w)_\pm(t,\cdot)\|_{L^2}\sum_{j=1}^n \prod_{k\neq j}\|w_k\|_{L^\infty}\|w_j\|_{H^s}\end{aligned} \tag{B.2.23a}$$

and, for $m = 1, 2$,

$$\begin{aligned}\|a(D_x)(Z_m w)_\pm w_1 \cdots w_n\|_L \lesssim{}& \Big\| \big[\chi(t^{-\sigma}D_x)a(D_x)(Z_m w)_\pm\big] \prod_{j=1}^n \chi(t^{-\sigma}D_x)w_j \Big\|_L \\ &+ t^{-N(s)}\left(\|w_\pm(t,\cdot)\|_{H^s} + \|D_t w_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{\mu=0}^1 \|x_m^\mu w_1\|_L + t\|w_1\|_L\Big)\prod_{j=2}^n \|w_j\|_{L^\infty} \\ &+ t^{-N(s)}\|(Z_m w)_\pm(t,\cdot)\|_{L^2}\sum_{j=1}^n \prod_{k\neq j}\|w_k\|_{L^\infty}\|w_j\|_{H^s},\end{aligned} \tag{B.2.23b}$$

with $N(s)$ as large as we want as long as $s > 0$ is large. Moreover, there exists $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that, for any fixed $j_0 \in \{1, \dots, n\}$,

$$\begin{aligned}&\big\|\chi(t^{-\sigma}D_x)\big[a(D_x)(\Omega w)_\pm w_1 \cdots w_n\big]\big\|_L \\ &\lesssim \Big\| \big[\chi(t^{-\sigma}D_x)a(D_x)(\Omega w)_\pm\big]\big[\chi_1(t^{-\sigma}D_x)w_{j_0}\big] \prod_{\substack{j=1,\dots,n\\ j\neq j_0}} \chi(t^{-\sigma}D_x)w_j \Big\|_L \\ &+ t^{-N(s)}\|w_\pm(t,\cdot)\|_{H^s}\Big(\sum_{|\mu|=0}^1 \|x^\mu w_1\|_L\Big)\prod_{j=2}^n \|w_j\|_{L^\infty} \\ &+ t^{-N(s)}\|(\Omega w)_\pm(t,\cdot)\|_{L^2}\sum_{\substack{j=1,\dots,n\\ j\neq j_0}} \prod_{k\neq j}\|w_k\|_{L^\infty}\|w_j\|_{H^s}\end{aligned} \tag{B.2.24a}$$

and, for $m = 1, 2$,

$$\begin{aligned}&\big\|\chi(t^{-\sigma}D_x)\big[a(D_x)(Z_m w)_\pm w_1 \cdots w_n\big]\big\|_L \\ &\lesssim \Big\| \big[\chi(t^{-\sigma}D_x)a(D_x)(Z_m w)_\pm\big]\big[\chi_1(t^{-\sigma}D_x)w_{j_0}\big] \prod_{\substack{j=1,\dots,n\\ j\neq j_0}} \chi(t^{-\sigma}D_x)w_j \Big\|_L \\ &+ t^{-N(s)}\left(\|w_\pm(t,\cdot)\|_{H^s} + \|D_t w_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{\mu=0}^1 \|x_m^\mu w_1\|_L + t\|w_1\|_L\Big)\prod_{j=2}^n \|w_j\|_{L^\infty} \\ &+ t^{-N(s)}\|(Z_m w)_\pm(t,\cdot)\|_{L^2}\sum_{\substack{j=1,\dots,n\\ j\neq j_0}} \prod_{k\neq j}\|w_k\|_{L^\infty}\|w_j\|_{H^s}.\end{aligned} \tag{B.2.24b}$$

Proof. The inequalities of the statement mainly follows from (B.2.20). In fact, by decomposing each factor w_j appearing in the first norm in the right hand sides of (B.2.20) as in (B.2.16), and then using the following inequality, for $\Gamma \in \{\Omega, Z_m, m = 1, 2\}$ and $\widetilde{w}_k$ either equal to w_k or to $\chi(t^{-\sigma}D_x)w_k$,

$$\left\| [\chi(t^{-\sigma}D_x)a(D_x)(\Gamma w)_\pm] \prod_{\substack{k=1,\dots,n \\ k\neq j}} \widetilde{w}_k \left[(1-\chi)(t^{-\sigma}D_x)w_j\right] \right\|_L \lesssim t^{-N(s)} \|(\Gamma w)_\pm(t,\cdot)\|_{L^2} \prod_{\substack{k=1,\dots,n \\ k\neq j}} \|w_k\|_{L^\infty} \|w_j\|_{H^s},$$

with $N(s)$ as large as we want as long as $s > 0$, which is obtained from (B.1.2) together with the $L^2 - L^\infty$ and $L^\infty - L^\infty$ continuity of operator $\chi(t^{-\sigma}D_x)$, we obtain (B.2.23).

On the other hand, if the product in the left hand side of (B.2.20) is localized in frequencies by means of operator $\chi(t^{-\sigma}D_x)$, so it is for the product in the first norm of the same inequalities. Inequalities (B.2.24) are then derived by bounding these L norms by means lemma B.2.2, where the role of w_1 is here played by w_{j_0}, for some fixed $j_0 \in \{1, \dots, n\}$. □

The following two lemmas are stated and proved in view of lemma B.2.8, in which we recover a first non-sharp estimate on the L^∞ norm of the Klein-Gordon component when one Klainerman vector field is acting on it and its frequencies are less or equal than t^σ, for some small $\sigma > 0$. This estimate will be successively refined in lemma B.4.14.

Lemma B.2.6. *Let $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, and $w = w(t,x)$ such that, if $\widetilde{w}(t,x) := tw(t,tx)$, $\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}(t,\cdot) \in L^2(\mathbb{R}^2)$ for any $|\mu| \leq 1$. Then*

$$\left\|\chi(t^{-\sigma}D_x)w(t,\cdot)\right\|_{L^\infty} \lesssim t^{-1+\beta} \sum_{|\mu|=0}^{1} \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}(t,\cdot)\right\|_{L^2}, \tag{B.2.25}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. Since

$$\chi(t^{-\sigma}D_x)w(t,y) = t^{-1}\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{w}(t,x)|_{x=\frac{y}{t}},$$

the goal is to prove that

$$\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{w}(t,\cdot)\right\|_{L^\infty} \lesssim h^{-\beta} \sum_{|\mu|=0}^{1} \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}(t,\cdot)\right\|_{L^2}, \tag{B.2.26}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$. So let $w^\chi := \mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{w}$ and take $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ equal to 1 on the support of χ, so that

$$\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{w} = \mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\widetilde{w}^\chi.$$

For a $\gamma \in C_0^\infty(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin and with sufficiently small support, we consider the following decomposition

$$\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi + \mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi$$

and immediately observe that, from inequality (3.2.17b),

$$\left\|\mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi(t,\cdot)\right\|_{L^\infty}\lesssim h^{-\beta}\sum_{|\mu|=0}^{1}\|\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}^\chi(t,\cdot)\|_{L^2}.$$

After lemma 1.2.38 there exists a family of smooth cut-off functions $\theta_h(x)$ such that equality (1.2.67) holds. Then, if we also consider a new cut-off function χ_2 equal to 1 on the support of χ_1 and a small $\sigma_1>\sigma$, by symbolic calculus and remark 1.2.22 we derive that for any $N\in\mathbb{N}$

$$\begin{aligned}\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi&=\theta_h(x)\mathrm{Op}_h^w(\chi_2(h^\sigma\xi))\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\\&\quad+\mathrm{Op}_h^w(r_\infty(x,\xi))\widetilde{w}^\chi+\theta_h(x)\mathrm{Op}_h^w(r_\infty^1(x,\xi))\widetilde{w}^\chi\end{aligned}$$

with $r_\infty, r_\infty^1\in h^N S_{\frac12,\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. It is enough to take $N=1$ to have, by proposition 1.2.37, that

$$\|\mathrm{Op}_h^w(r_\infty)\widetilde{w}^\chi(t,\cdot)\|_{L^\infty}+\|\theta_h(x)\mathrm{Op}_h^w(r_\infty^1)\widetilde{w}^\chi(t,\cdot)\|_{L^\infty}\le h^{-\beta}\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}.$$

As function $\phi(x):=\sqrt{1-|x|^2}$ is well defined on the support of θ_h we are allowed to to write the following:

$$\begin{aligned}&\left\|\theta_h(x)\mathrm{Op}_h^w(\chi_2(h^{\sigma_1}\xi))\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi(t,\cdot)\right\|_{L^\infty}\\&=\left\|e^{\frac{i}{h}\phi}\theta_h(x)\mathrm{Op}_h^w(\chi_2(h^{\sigma_1}\xi))\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi(t,\cdot)\right\|_{L^\infty}\\&\lesssim\left\|\mathrm{Op}_h^w(\chi_2(h^{\sigma_1}\xi))\left[e^{\frac{i}{h}\phi}\theta_h(x)\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\right]\right\|_{L^\infty}+\|\mathrm{Op}_h^w(r_\infty)\widetilde{w}^\chi(t,\cdot)\|_{L^\infty},\end{aligned}$$

for a new $r_\infty\in h^N S_{\frac12,\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1}\big)$. This latter r_∞ comes out from the commutation between $e^{\frac{i}{h}\phi}\theta_h(x)$ and $\mathrm{Op}_h^w(\chi_2(h^{\sigma_1}\xi))$, whose symbol is computed using (1.2.18) until a large enough order M. We notice that we gain a factor $h^{|\alpha|(\sigma_1-\sigma)}$ at each order of the mentioned asymptotic development as $\sigma_1>\sigma$. Moreover, those terms write in terms of the derivatives of χ_2 and hence vanish on the support of χ_1. By proposition 1.2.21 and remark 1.2.22 we then deduce that the composition of the mentioned commutator with $\mathrm{Op}_h^w\big(\gamma(\frac{x-p'(\xi)}{\sqrt{h}})\chi_1(h^\sigma\xi)\big)$ is an operator of symbol r_∞, with N as large as we want.

Using the classical Sobolev injection, symbolic calculus and lemma 3.2.16 we find that

$$\begin{aligned}&\left\|\mathrm{Op}_h^w(\chi_2(h^{\sigma_1}\xi))\left[e^{\frac{i}{h}\phi}\theta_h(x)\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\right]\right\|_{L^\infty}\\&\lesssim|\log h|\left[\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}+\sum_{j=1}^{2}\left\|D_j\left[e^{\frac{i}{h}\phi}\theta_h(x)\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\right]\right\|_{L^2}\right]\\&\lesssim|\log h|\left[\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}+\sum_{j=1}^{2}h^{-1}\left\|\mathrm{Op}_h^w\big((\xi_j+d_j\phi(x))\theta_h(x)\big)\mathrm{Op}_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\right\|_{L^2}\right]\\&\lesssim|\log h|\left[\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}+h^{-\beta}\sum_{|\mu|=0}^{1}\|\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}^\chi(t,\cdot)\|_{L^2}\right].\end{aligned}$$

Finally, commutating $\mathcal{L}$ with $\mathrm{Op}_h^w(\chi(h^\sigma\xi))$ defining $\widetilde{w}^\chi$, and reminding that $\chi_1\equiv1$ on the support of χ, we obtain

$$\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{w}^\chi(t,\cdot)\|_{L^\infty}\lesssim h^{-\beta}\sum_{|\mu|=0}^{1}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}(t,\cdot)\|_{L^2},$$

for every $t \in [1,T]$, and hence (B.2.26). □

Lemma B.2.7. *Let I be a multi-index of length j, with $j = 1, 2$, and*

(B.2.27)
$$v^{I,NF}(t,x) := (\Gamma^I v)_-(t,x) - \frac{i}{4(2\pi)^2} \sum_{j_1,j_2 \in \{+,-\}} \int e^{ix\cdot\xi} B^1_{(j_1,j_2,+)}(\xi,\eta) \widehat{v^I_{j_1}}(\xi-\eta) \hat{u}_{j_2}(\eta) d\xi d\eta,$$

with $B^1_{(j_1,j_2,+)}$ given by (2.2.42) with $j_3 = +$ and $k = 1$. Then there exists a constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, and every $t \in [1,T]$,

$$\left\| \chi(t^{-\sigma} D_x) \left(v^{I,NF} - (\Gamma^I v)_- \right)(t,\cdot) \right\|_{L^\infty} \leq \frac{1}{2} \left\| \chi(t^{-\sigma} D_x)(\Gamma^I v)_-(t,\cdot) \right\|_{L^\infty} + CB\varepsilon t^{-1}. \tag{B.2.28}$$

Moreover, for every $m = 1, 2$, $t \in [1,T]$,

$$\left\| \chi(t^{-\sigma} D_x) Z_m \left(v^{I,NF} - (\Gamma^I v)_- \right)(t,\cdot) \right\|_{L^2} \leq C(A+B) B\varepsilon^2 t^{2\sigma + \frac{\delta_{3-j}+\delta_2}{2}}. \tag{B.2.29}$$

Proof. We first notice that, after definition (B.2.27) and inequalities (A.15), (1.1.10), we have the following explicit expression:

(B.2.30)
$$v^{I,NF} - (\Gamma^I v)_- = -\frac{i}{2} \left[(D_t \Gamma^I v)(D_1 u) - (D_1 \Gamma^I v)(D_t u) + D_1[(\Gamma^I v) D_t u] - \langle D_x \rangle [(\Gamma^I v) D_1 u] \right].$$

From the above equality together with lemma B.2.2 with $L = L^\infty$ and $w_1 = (\Gamma^I v)_\pm$, and (1.1.5), (1.1.10), we deduce that there exists some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$, equal to 1 on the support of χ, and $s > 0$ sufficiently large such that

(B.2.31)
$$\begin{aligned}
&\left\| \chi(t^{-\sigma} D_x)(v^{I,NF} - v^I_-)(t,\cdot) \right\|_{L^\infty} \\
&\lesssim t^\sigma \sum_{\mu=0}^{1} \left\| [\chi_1(t^{-\sigma} D_x)(\Gamma^I v)_\pm(t,\cdot)][\chi(t^{-\sigma} D_x) \mathrm{R}_1^\mu u_\pm](t,\cdot) \right\|_{L^\infty} t^{-2} \|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2} \|u_\pm(t,\cdot)\|_{H^s} \\
&\lesssim t^\sigma \sum_{\mu=0}^{1} \left\| [\chi_1(t^{-\sigma} D_x)(\Gamma^I v)_\pm(t,\cdot)][\chi(t^{-\sigma} D_x) \mathrm{R}_1^\mu u_\pm](t,\cdot) \right\|_{L^\infty} + B^2 \varepsilon^2 t^{-3/2},
\end{aligned}$$

where the latter inequality follows from after a-priori energy estimates (1.1.11c), (1.1.11d). Our aim is to truncate factor $(\Gamma^I v)_\pm$ in the above right hand side rather with the same operator $\chi(t^{-\sigma} D_x)$ appearing on the left hand side. We hence proceed by picking some $\kappa \geq 1$ and decomposing $\chi(t^{-\sigma} D_x) \mathrm{R}_1^\mu u_\pm$ as

$$\chi(t^{-\sigma} D_x) \mathrm{R}_1^\mu u_\pm = \chi(t^\kappa D_x) \mathrm{R}_1^\mu u_\pm + (1-\chi)(t^\kappa D_x) \chi(t^{-\sigma} D_x) \mathrm{R}_1^\mu u_\pm, \tag{B.2.32}$$

noticing that, as $\chi(t^\kappa \xi)$ is supported for very small frequencies $|\xi| \lesssim t^{-\kappa}$, by Sobolev injection we have that

$$\|\chi(t^\kappa D_x) \mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty} \lesssim t^{-\kappa} \|u_\pm(t,\cdot)\|_{L^2}.$$

Consequently, using the $L^2 - L^\infty$ continuity of $\chi_1(t^{-\sigma} D_x)$ along with a-priori estimates (1.1.11c), (1.1.11d), we have that for any for $\mu = 0, 1$

$$\begin{aligned}
\left\| [\chi_1(t^{-\sigma} D_x)(\Gamma^I v)_\pm(t,\cdot)] \chi(t^\kappa D_x) \mathrm{R}_1^\mu u_\pm(t,\cdot) \right\|_{L^\infty} &\lesssim t^{\sigma-\kappa} \|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2} \|u_\pm(t,\cdot)\|_{L^2} \\
&\leq CB\varepsilon t^{-\kappa+\sigma+\frac{\delta_{3-j}+\delta}{2}}.
\end{aligned}$$

Choosing $\kappa = 1 + \sigma + \frac{\delta+\delta_1}{2}$ we deduce from (B.2.32) and the above inequality that

$$\begin{aligned}(\text{B.2.33})\quad &\big\|[\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)]\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty}\\ &\lesssim \big\|[\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)](1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty} + CB\varepsilon t^{-1}.\end{aligned}$$

We then decompose $(\Gamma^I v)_\pm$ in frequencies using the wished operator $\chi(t^{-\sigma}D_x)$. In order to estimate the L^∞ norm of

$$[(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm](1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm$$

we first commute Γ^I to operator $D_t \pm \langle D_x\rangle$ (see (2.1.15b)) and successively look at it as a linear combination of derivations of the form $x^\alpha t^a \partial_x^\alpha \partial_t^b$, with $1 \le |\alpha| + a \le 2$, $1 \le |\beta| + b \le 2$. By commutating x^α to $(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)$, multiplying it against the wave factor, and successively combining the classical Sobolev injection with inequality (B.1.2), we find that

$$\begin{aligned}(\text{B.2.34})\quad &\big\|[(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)](1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty}\\ &\lesssim t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s} + \|D_t^2 v_\pm(t,\cdot)\|_{H^s}\right)\\ &\qquad\qquad \times \sum_{\substack{1\le|\alpha|+a\le 2\\ |\mu|=0,1}} \big\|x^\alpha t^a (1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm\big\|_{L^\infty}.\end{aligned}$$

Using system (2.1.2) with $|I| = 0$, (B.1.4c) with $s = 1$, (B.1.6a) and a-priori estimates, it is straightforward to check that

$$(\text{B.2.35})\qquad \|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s} + \|D_t^2 v_\pm(t,\cdot)\|_{H^s} \le CB\varepsilon t^{\frac{\delta}{2}}.$$

Also,

$$(\text{B.2.36})\qquad t^a \big\|(1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm\big\|_{L^\infty} \lesssim t^{a+\sigma}\|u_\pm(t,\cdot)\|_{L^2} \le CB\varepsilon t^{a+\sigma+\frac{\delta}{2}},$$

and for $|\alpha| \in \{1,2\}$ we have that

$$(\text{B.2.37})\qquad \big\|x^\alpha(1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm\big\|_{L^\infty} \le CB\varepsilon t^{|\alpha|+|\alpha|\kappa+\frac{\delta}{2}}.$$

In fact, when $|\alpha| = 1$ this can be proved by commutating x^α with $(1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)$, using that

$$[x_n, (1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)] = -it^{\kappa}(\partial_n\chi)(t^{\kappa}D_x) + it^{-\sigma}(\partial_n\chi)(t^{-\sigma}D_x), \quad n = 1,2,$$

is bounded from L^2 to L^∞ uniformly in t, and together with estimates (1.1.11d), (B.1.16a), and the following inequality

$$\begin{aligned}&\big\|(1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\big[x^\alpha \mathrm{R}_1^\mu u_\pm\big](t,\cdot)\big\|_{L^\infty}\\ &\qquad\lesssim t^{\kappa}\left[\sum_{|\mu|=1}\|Z^\mu u_\pm(t,\cdot)\|_{L^2} + t\|u_\pm(t,\cdot)\|_{H^1} + \|xNL_w(t,\cdot)\|_{L^2}\right],\end{aligned}$$

which is obtained by writing

$$\begin{aligned}&(1-\chi)(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)x_n\mathrm{R}_1^\mu\\ &= t^{\kappa}\widetilde{\chi}_1(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)x_n|D_x|\mathrm{R}_1^\mu + t^{\kappa}\widetilde{\chi}_1(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)[|D_x|,x]\mathrm{R}_1^\mu\\ &= t^{\kappa}\widetilde{\chi}_1(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu\left[x_n|D_x| - tD_n + \frac{1}{2i}\frac{D_n}{|D_x|}\right]\\ &+ t^{\kappa}\widetilde{\chi}_1(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu\left[tD_n - \frac{1}{2i}\frac{D_n}{|D_x|}\right] + \delta_{\mu=1}it^{\kappa}\widetilde{\chi}_1(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)Op(|\xi|\partial_n(\xi_1|\xi|^{-1}))\\ &- it^{\kappa}\widetilde{\chi}_1(t^{\kappa}D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_n\mathrm{R}_1^\mu\end{aligned}$$

with $\widetilde{\chi}(\xi) := (1-\chi)(\xi)|\xi|^{-1}$ and using relation (3.2.12a) with $w = u_\pm$. The proof for $|\alpha| = 2$ is analogous. It is based on the commutation of x^α with $(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)$ (the commutator is here a $L^2 - L^\infty$ bounded operator with norm $O(t^\kappa)$), on the fact that we can rewrite $(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)x^\alpha \mathrm{R}_1^\mu$ making appear $(x|D_x| - tD_x + \frac{1}{2i}\frac{D_x}{|D_x|})^\alpha$ by considering $\widetilde{\chi}_2(\xi) := (1-\chi)(\xi)|\xi|^{-2}$ instead of previous $\widetilde{\chi}_1$, and use relation (3.2.12a). Doing so we derive the following inequality

$$\begin{aligned}&\big\|(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\big[x^\alpha \mathrm{R}_1^\mu u_\pm\big](t,\cdot)\big\|_{L^\infty}\\ &\quad\lesssim t^{2\kappa}\left[\sum_{|\mu|=2}\|Z^\mu u_\pm(t,\cdot)\|_{L^2} + \sum_{|\mu|\leq 1} t^{2-|\mu|}\|Z^\mu u_\pm(t,\cdot)\|_{H^1} + \sum_{|\mu|=1}^{2}\|x^\mu NL_w(t,\cdot)\|_{L^2}\right] \leq CB\varepsilon t^{2+2\kappa+\frac{\delta}{2}},\end{aligned}$$

last estimate following from a-priori estimates, (B.1.16a) and (B.1.31). Summing up (B.2.35), (B.2.36), (B.2.37), together with the previous choice of κ and the fact that in (B.2.34) $N(s) \geq 6$ if $s > 0$ is sufficiently large, we deduce that

(B.2.38)
$$\big\|[(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)](1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty} \leq CB\varepsilon t^{-\frac{3}{2}}.$$

Therefore, from (B.2.31), (B.2.33),(B.2.38), and the uniform continuity on L^∞ of $\chi(t^{-\sigma}D_x)$, we find that

$$\big\|\chi(t^{-\sigma}D_x)\left(v^{I,NF} - v_-^I\right)(t,\cdot)\big\|_{L^\infty} \lesssim \sum_{\mu=0}^{1} t^\sigma \|\chi(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)\|_{L^\infty}\|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty} + CB\varepsilon t^{-1},$$

and as σ is small and $\varepsilon_0 < (2A)^{-1}$, from (1.1.11a) we obtain (B.2.28).

In order to prove (B.2.29) we apply Z_m to equality (B.2.30) and apply the Leibniz rule. As

$$[Z_m, D_t] = -D_m, \quad [Z_m, D_1] = -\delta_{m1}D_t, \quad [Z_m, \langle D_x\rangle] = -D_m\langle D_x\rangle^{-1}D_t, \tag{B.2.39}$$

with δ_{m1} the Kronecker delta, we find that

(B.2.40)
$$\begin{aligned}&2i\chi(t^{-\sigma}D_x)Z_m(v^{I,NF} - v_-^I)\\ &= \chi(t^{-\sigma}D_x)\Big[(D_tZ_m\Gamma^I v)(D_1u) - (D_1Z_m\Gamma^I v)(D_tu) + D_1[(Z_m\Gamma^I v)(D_tu)] - \langle D_x\rangle[(Z_m\Gamma^I v)(D_1u)]\\ &\quad + (D_t\Gamma^I v)(D_1Z_mu) - (D_1\Gamma^I v)(D_tZ_mu) + D_1[(\Gamma^I v)(D_tZ_mu)] - \langle D_x\rangle[(\Gamma^I v)(D_1Z_mu)]\\ &\quad - (D_m\Gamma^I v)(D_1u) + \delta_{m1}(D_t\Gamma^I v)(D_tu) - \delta_{m1}D_t[(\Gamma^I v)(D_tu)] + \frac{D_m}{\langle D_x\rangle}D_t[(\Gamma^I v)(D_1u)]\\ &\quad - \delta_{m1}(D_t\Gamma^I v)(D_tu) + (D_1\Gamma^I v)(D_mu) - \delta_{m1}D_1[(\Gamma^I v)(D_tu)] + \delta_{m1}\langle D_x\rangle[(\Gamma^I v)(D_tu)]\Big].\end{aligned}$$

The L^2 norm of all products in the above second, fourth and fifth line, i.e. those in which Z_m is not acting on the wave component u, is estimated by

$$\begin{aligned}&\sum_{\mu=0}^{1} t^\sigma\left(\|(Z_m\Gamma^I v)_\pm(t,\cdot)\|_{L^2} + \|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2}\right)\left(\|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty} + \|D_tu_\pm(t,\cdot)\|_{L^\infty}\right)\\ &\qquad\qquad\leq CAB\varepsilon^2 t^{-\frac{1}{2}+\frac{\delta_0}{2}+\sigma},\end{aligned} \tag{B.2.41}$$

after inequality (B.1.5b) with $s = 0$ and a-priori estimates. The L^2 norm of products appearing in the second line are, instead, estimated by using (1.1.10) and (B.2.24b) with $L = L^2$, $\Gamma w = Z_m u$,

$s > 0$ sufficiently large so that $N(s) \geq 2$. It is hence bounded by

$$
\begin{aligned}
&t^{\sigma} \left\| \chi(t^{-\sigma} D_x)(\Gamma^I v)_{\pm}(t,\cdot) \right\|_{L^\infty} \|(Z_m u)_{\pm}(t,\cdot)\|_{L^2} \\
&+ t^{-2} \Big(\sum_{|\mu|=0}^{1} \|x^\mu (\Gamma^I v)_{\pm}(t,\cdot)\|_{L^2} + t \|(\Gamma^I v)_{\pm}(t,\cdot)\|_{L^2} \Big) \left(\|u_{\pm}(t,\cdot)\|_{H^s} + \|D_t u_{\pm}(t,\cdot)\|_{H^s} \right) \\
&\leq C B^2 \varepsilon^2 t^{2\sigma + \frac{\delta_{3-j}+\delta_2}{2}},
\end{aligned}
$$

where the latter estimate is obtained using the fact that $\chi(t^{-\sigma} D_x)$ is a bounded operator from L^2 to L^∞ with norm $O(t^\sigma)$, together with (B.1.5a), (B.1.17) and a-priori estimates. That concludes, together with (B.2.41), the proof of (B.2.29) and of the statement. □

Lemma B.2.8. *There exists a constant $C > 0$ such that, for any $\rho \in \mathbb{N}$, $\chi \in C_0^\infty(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin, $\sigma > 0$ small, and every $t \in [1, T]$,*

$$
\sum_{|I|=1} \|\chi(t^{-\sigma} D_x) V^I(t,\cdot)\|_{H^{\rho,\infty}} \leq C B \varepsilon t^{-1+\beta+\frac{\delta_1}{2}}, \tag{B.2.42}
$$

with $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$.

Proof. Since $\chi(t^{-\sigma} D_x)$ is a bounded operator from L^∞ to $H^{\rho,\infty}$ with norm $O(t^{\sigma\rho})$, for any $\rho \in \mathbb{N}$, it is enough to prove that the L^∞ norm of $\chi(t^{-\sigma} D_x) V^I(t,\cdot)$ is bounded by the right hand side of (B.2.42). Moreover, as this latter inequality is automatically satisfied when Γ is a spatial derivative after a-priori estimate (1.1.11b) and the fact that operator $\chi(t^{-\sigma} D_x)$ is uniformly bounded on L^∞, for the rest of the proof we will assume that $\Gamma \in \{\Omega, Z_j, j = 1, 2\}$ is a Klainerman vector field. We also warn the reader that, throughout the proof, C and β will denote some positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$.

Instead of proving the result of the statement directly on $\chi(t^{-\sigma} D_x) v^I_{\pm}$ we do it for $\chi(t^{-\sigma} D_x) v^{I,NF}$, where $v^{I,NF}$ has been introduced in (B.2.27) and is considered here for $|I| = 1$ and $\Gamma^I = \Gamma$. In fact, by (B.2.28)

$$
\left\| \chi(t^{-\sigma} D_x) v^I_-(t,\cdot) \right\|_{L^\infty} \leq 2 \left\| \chi(t^{-\sigma} D_x) v^{I,NF}(t,\cdot) \right\|_{L^\infty} + C B \varepsilon t^{-1}. \tag{B.2.43}
$$

The advantage of dealing with this new function is related to the fact that it is solution to a half Klein-Gordon equation with a more suitable non-linearity (see (B.2.44)) than the equation satisfied by v^I_-. In fact, it is a computation to show that from definition (B.2.27)

$$
[D_t + \langle D_x \rangle] v^{I,NF}(t,x) = NL^{I,NF}_{kg}, \tag{B.2.44}
$$

where

$$
NL^{I,NF}_{kg} = r^{I,NF}_{kg}(t,x) + Q^{\mathrm{kg}}_0(v_{\pm}, D_1 u^I_{\pm}) + G^{\mathrm{kg}}_1(v_{\pm}, D u_{\pm}), \tag{B.2.45}
$$

$G^{\mathrm{kg}}_1(v_{\pm}, D u_{\pm}) = G_1(v, \partial u)$ with G_1 given by (1.1.16), and

$$
\begin{aligned}
r^{I,NF}_{kg}(t,x) &= -\frac{i}{4(2\pi)^2} \\
&\times \sum_{j_1, j_2 \in \{+,-\}} \int e^{ix\cdot\xi} B^1_{(j_1,j_2,+)}(\xi,\eta) \left[\widehat{\Gamma^I N L_{kg}}(\xi-\eta) \hat{u}_{j_2}(\eta) - \widehat{v^I_{j_1}}(\xi-\eta) \widehat{NL_w}(\eta) \right] d\xi d\eta,
\end{aligned} \tag{B.2.46}
$$

with $B^1_{(j_1,j_2,+)}$ given by (2.2.42) when $j_3 = +$ and $k = 1$. After (B.2.27) and (A.15) it appears that $r^{I,NF}_{kg}$ has the following nice explicit expression

$$r^{I,NF}_{kg} = -\frac{i}{2}\left[(\Gamma^I NL_{kg})D_1 u - (D_1\Gamma^I v)NL_w + D_1[(\Gamma^I v)NL_w]\right]. \tag{B.2.47}$$

Using lemma B.2.6 and relation (3.2.8) with $w = v^{I,NF}$, and reminding that $\|tw(t,t\cdot)\|_{L^2} = \|w(t,\cdot)\|_{L^2}$, we find the following

$$\begin{aligned}\big\|\chi(t^{-\sigma}D_x)v^{I,NF}(t,\cdot)\big\|_{L^\infty} &\\ \lesssim t^{-1+\beta}\sum_{|\mu|=0}^{1}\big\|\chi(t^{-\sigma}D_x)Z^\mu v^{I,NF}(t,\cdot)\big\|_{L^2} &+ \sum_{j=1}^{2} t^{-1+\beta}\Big\|\chi(t^{-\sigma}D_x)\big[x_j NL^{I,NF}_{kg}\big](t,\cdot)\Big\|_{L^2}.\end{aligned} \tag{B.2.48}$$

From equality (B.2.30), along with (1.1.5), (1.1.10), and a-priori estimates (1.1.11a), (1.1.11d), we immediately see that

$$\begin{aligned}\big\|\chi(t^{-\sigma}D_x)(v^{I,NF}-v^I_-)(t,\cdot)\big\|_{L^2} &\lesssim t^\sigma\|v^I_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{L^\infty} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{L^\infty}\right)\\ &\leq CAB\varepsilon^2 t^{-\frac{1}{2}+\frac{\delta_2}{2}+\sigma},\end{aligned} \tag{B.2.49}$$

and as $\sigma, \delta_2 \ll 1$ are small

$$\begin{aligned}\big\|\chi(t^{-\sigma}D_x)v^{I,NF}(t,\cdot)\big\|_{L^2} &\leq \big\|\chi(t^{-\sigma}D_x)v^I_-(t,\cdot)\big\|_{L^2} + \big\|\chi(t^{-\sigma}D_x)(v^{I,NF}-v^I_-)(t,\cdot)\big\|_{L^2}\\ &\leq CB\varepsilon t^{\frac{\delta_2}{2}}.\end{aligned} \tag{B.2.50}$$

Moreover, from (B.2.29) and a-priori estimate (1.1.11d) we have that, for every $m = 1, 2$, $t \in [1, T]$,

$$\big\|\chi(t^{-\sigma}D_x)Z_m v^{I,NF}(t,\cdot)\big\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_1}{2}}. \tag{B.2.51}$$

Finally, from (B.2.47), (1.1.5), (1.1.10), (B.1.10b), (B.1.26a) and a-priori estimates, we derive that

$$\begin{aligned}&\Big\|\chi(t^{-\sigma}D_x)\left[x_j r^{I,NF}_{kg}\right](t,\cdot)\Big\|_{L^2} \lesssim \|x_j\Gamma^I NL_{kg}(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{L^\infty} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{L^\infty}\right)\\ &+\sum_{\mu=0}^{1} t^\sigma\left(\|x_j^\mu v_\pm(t,\cdot)\|_{L^\infty} + \left\|x_j^\mu\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{L^\infty}\right)\|v^I_\pm(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\ &\leq C(A+B)B\varepsilon^2 t^{\frac{\delta_2}{2}},\end{aligned} \tag{B.2.52}$$

while from (B.1.5a) with $s = 0$, (B.1.10b) and a-priori estimates

$$\begin{aligned}&\Big\|\chi(t^{-\sigma}D_x)\left[x_j Q^{\mathrm{kg}}_0\left(v_\pm, D_1 u^I_\pm\right)\right](t,\cdot)\Big\|_{L^2} + \Big\|\chi(t^{-\sigma}D_x)\left[x_j G^{\mathrm{kg}}_1\left(v_\pm, Du_\pm\right)\right](t,\cdot)\Big\|_{L^2}\\ &\lesssim \left(\|x_j v_\pm(t,\cdot)\|_{L^\infty} + \left\|x_j\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{L^\infty}\right)\left(\|u^I_\pm(t,\cdot)\|_{H^1} + \|D_t u_\pm(t,\cdot)\|_{L^2}\right)\\ &\leq C(A+B)B\varepsilon t^{\delta_2}.\end{aligned}$$

Therefore, from (B.2.45) we deduce that

$$\|\chi(t^{-\sigma}D_x)\big[x_j NL^{I,NF}_{kg}\big](t,\cdot)\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{\delta_2}, \tag{B.2.53}$$

so injecting (B.2.50), (B.2.51), (B.2.53) into (B.2.48), and summing it up with (B.2.43), we obtain the result of the statement. □

As done for the Klein-Gordon component in the above lemma, we also derive an estimate for the uniform norm of the wave component when a Klainerman vector field acts on it and its frequencies less or equal than t^σ (see lemma B.2.10). We first need the following result.

Lemma B.2.9. *Let $\Gamma \in \mathcal{Z}$, index J be such that $\Gamma^J = \Gamma$, and $\widetilde{u}^J(t,x) := t(\Gamma u)_-(t,tx)$. There exists a constant $C > 0$ such that, for any $\theta_0, \chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$, and every $t \in [1,T]$,*

$$\|\widetilde{u}^J(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.54a}$$

$$\|\Omega_h \widetilde{u}^J(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_1}{2}}, \tag{B.2.54b}$$

$$\big\|\mathcal{M}\widetilde{u}^J(t,\cdot)\big\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_1}{2}}, \tag{B.2.54c}$$

$$\big\|\theta_0(x)\mathrm{Op}_h^w(\chi(h^\sigma\xi))\Omega_h\mathcal{M}\widetilde{u}^J(t,\cdot)\big\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_0}{2}}. \tag{B.2.54d}$$

Proof. We warn the reader that, throughout the proof, C will denote a positive constant that may change line after line. We also recall that

$$[D_t + \langle D_x\rangle](\Gamma u)_-(t,x) = \Gamma NL_w(t,x).$$

Estimates (B.2.54a) and (B.2.54b) are straightforward after (1.1.11d) and the fact that

$$\|\widetilde{u}^J(t,\cdot)\|_{L^2} = \|(\Gamma u)_-(t,\cdot)\|_{L^2}, \quad \|\Omega_h\widetilde{u}^J(t,\cdot)\|_{L^2} = \|(\Omega\Gamma u)_-(t,\cdot)\|_{L^2}.$$

From (3.2.6) with $w = (\Gamma u)_-$ and $f = \Gamma NL_w$, estimates (1.1.11d), (B.1.26b), along with the fact that $\delta_2 \ll \delta_1$ (e.g. $2\delta_2 \leq \delta_1$), and $(A+B)\varepsilon_0 < 1$, we obtain (B.2.54c). By (3.2.6) we also derive that, for any $n = 1, 2$,

$$\begin{aligned}\big\|\theta_0(x)\mathrm{Op}_h^w(\chi(h^\sigma\xi))\Omega_h\mathcal{M}_n\widetilde{u}^J(t,\cdot)\big\|_{L^2} &\lesssim \|\Omega Z_n(\Gamma u)_-(t,\cdot)\|_{L^2} + \sum_{\mu=0}^{1}\|\Omega^\mu(\Gamma u)_-(t,\cdot)\|_{L^2} \\ &\quad + \Big\|\theta_0\Big(\frac{x}{t}\Big)\chi(t^{-\sigma}D_x)\Omega[x_n\Gamma NL_w](t,\cdot)\Big\|_{L^2}.\end{aligned} \tag{B.2.55}$$

The first two norms in the above right hand side are controlled by $E_3^0(t;W)^{1/2}$ and are hence bounded by $CB\varepsilon t^{\frac{\delta_0}{2}}$. By commutating x_n with $\chi(t^{-\sigma}D_x)\Omega$, and using that $\theta_0\big(\frac{x}{t}\big)x_n = t\theta_0^n\big(\frac{x}{t}\big)$, with $\theta_0^n(z) := \theta_0(z)z_n$, we deduce that

$$\Big\|\theta_0\Big(\frac{x}{t}\Big)\chi(t^{-\sigma}D_x)\Omega[x_n\Gamma NL_w](t,\cdot)\Big\|_{L^2} \lesssim t\sum_{\mu=0}^{1}\|\chi_1(t^{-\sigma}D_x)\Omega^\mu\Gamma NL_w\|_{L^2},$$

for some new $\chi_1 \in C_0^\infty(\mathbb{R}^2)$. On the one hand, using (B.1.20b), (B.1.6a) with $s = 0$ and a-priori estimates we derive that

$$t\|\Gamma NL_w\|_{L^2} \lesssim t\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\left(\|(\Gamma v)_\pm(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1} + \|D_t v_\pm(t,\cdot)\|_{L^2}\right) \lesssim CB\varepsilon t^{\frac{\delta_2}{2}}. \tag{B.2.56}$$

On the other hand, when we compute $\Omega\Gamma NL_w$ we find among the out-coming quadratic terms the following ones

$$Q_0^{\mathrm{w}}((\Omega v)_\pm, D_1(\Gamma v)_\pm) \quad \text{and} \quad Q_0^{\mathrm{w}}((\Gamma v)_\pm, D_1(\Omega v)_\pm),$$

which we estimate in the L^2 norm (when truncated for frequencies less or equal than t^σ) by means of (B.2.24a) with $L = L^2$, $\Gamma w = \Omega v$, and $s > 0$ large enough to have $N(s) \geq 3$. From

(B.1.17), (B.2.42) and a-priori estimates, we obtain that

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)Q_0^{\mathrm{w}}((\Omega v)_\pm, D_1(\Gamma v)_\pm)\right\|_{L^2} + \left\|\chi(t^{-\sigma}D_x)Q_0^{\mathrm{w}}((\Gamma v)_\pm, D_1(\Omega v)_\pm)\right\|_{L^2}\\
&\lesssim t^\sigma \|\chi(t^{-\sigma}D_x)(\Omega v)_\pm(t,\cdot)\|_{L^\infty}\|(\Gamma v)_\pm(t,\cdot)\|_{H^1} + \sum_{|\mu|=0}^{1} t^{-3}\|v_\pm(t,\cdot)\|_{H^s}\|x^\mu(\Gamma v)_\pm(t,\cdot)\|_{H^1}\\
&\leq CB^2\varepsilon^2 t^{-1+\beta+\frac{\delta_1+\delta_2}{2}},
\end{aligned}$$

with $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$. All remaining quadratic contributions to $\Omega\Gamma NL_w$ are estimated with

$$\begin{aligned}
\|(\Omega\Gamma v)_\pm(t,\cdot)\|_{H^1}\|v_\pm(t,\cdot)\|_{H^{2,\infty}} + \|(\Omega v)_\pm(t,\cdot)\|_{L^2}\left(\|v_\pm(t,\cdot)\|_{H^{1,\infty}} + \|D_t v_\pm(t,\cdot)\|_{L^\infty}\right)\\
+ \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(\Omega v)_\pm(t,\cdot)\|_{H^1} + \|D_t(\Omega v)_\pm(t,\cdot)\|_{L^2}\right),
\end{aligned}$$

and are hence bounded by $C(A+B)B\varepsilon^2 t^{-1+\frac{\delta_1}{2}}$ after (B.1.6b), (B.1.6c) and the a-priori estimates. This finally implies that

$$t\left\|\chi(t^{-\sigma}D_x)\Omega\Gamma NL_w(t,\cdot)\right\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta_1+\delta_2}{2}},$$

which, together with (B.2.56) and the fact that $\beta + \frac{\delta_1+\delta_2}{2} \leq \frac{\delta_0}{2}$, as $\delta_2 \ll \delta_1 \ll \delta_0$ and $\beta > 0$ is as small as we want provided that σ is small, gives

$$\left\|\theta_0\Big(\frac{x}{t}\Big)\chi(t^{-\sigma}D_x)\Omega[x_n\Gamma NL_w](t,\cdot)\right\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_0}{2}}.$$

□

Lemma B.2.10. *There exists a constant $C > 0$ such that, for any $\rho \in \mathbb{N}$, $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin, $\sigma > 0$ small, and every $t \in [1,T]$,*

$$\sum_{|J|=1}\sum_{|\mu|=0}^{1} \|\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^J(t,\cdot)\|_{H^{\rho,\infty}} \leq C(A+B)\varepsilon t^{-\frac{1}{2}+\beta+\frac{\delta_1}{2}}, \tag{B.2.57}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Proof. We warn the reader that, throughout the proof, C and β will denote two positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$. Moreover, since $\chi(t^{-\sigma}D_x)$ is a bounded operator from L^∞ to $H^{\rho,\infty}$ with norm $O(t^{\sigma\rho})$, for any $\rho \in \mathbb{N}$, we can reduce to prove that the L^∞ norm of $\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^J(t,\cdot)$ is bounded by the right hand side of (B.2.57). We observe that this estimate is automatically satisfied when J is such that Γ^J is a spatial derivative, as a consequence of a-priori estimate (1.1.11a). We therefore assume that Γ^J is one of the Klainerman vector fields Ω, Z_m, for $m \in \{1,2\}$.

Introducing $\widetilde{u}^J(t,x) := tu_-^J(t,tx)$, passing to the semiclassical setting ($t \mapsto t$, $x \mapsto \frac{x}{t}$, and $h := 1/t$), and reminding that $u_+^J = -\overline{u_-^J}$, inequality (B.2.57) becomes

$$\sum_{|\mu|=0}^{1} \left\|\mathrm{Op}_h^w\Big(\chi(h^\sigma\xi)(\xi|\xi|^{-1})^\mu\Big)\widetilde{u}_-^J(t,\cdot)\right\|_{L^\infty} \leq C(A+B)\varepsilon h^{-\frac{1}{2}-\beta-\frac{\delta_1}{2}}. \tag{B.2.58}$$

We consider a Littlewood-Paley decomposition such that

$$\chi(h^\sigma\xi) = \widetilde{\chi}(h^{-1}\xi) + \sum_k (1-\widetilde{\chi})(h^{-1}\xi)\psi(2^{-k}\xi)\chi(h^\sigma\xi), \tag{B.2.59}$$

for some suitably supported $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^2)$, $\psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, and immediately observe that the above sum is restricted to indices k such that $h \lesssim 2^k \lesssim h^{-\sigma}$. By the classical Sobolev injection, the uniform continuity of $\mathrm{Op}_h^w(\xi|\xi|^{-1})$ on L^2, and a-priori estimate (1.1.11d), we derive that for any $|\mu| \leq 1$, every $t \in [1,T]$,

$$\begin{aligned} \big\|\mathrm{Op}_h^w\big(\widetilde{\chi}(h^{-1}\xi)(\xi|\xi|^{-1})^\mu\big)\widetilde{u}^J(t,\cdot)\big\|_{L^\infty} &= \|\chi(D_x)\mathrm{Op}_h^w((\xi|\xi|^{-1})^\mu)\widetilde{u}^J(t,\cdot)\|_{L^\infty} \\ &\lesssim \|u_-^J(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_2}{2}}. \end{aligned} \tag{B.2.60}$$

If we concisely denote by $\phi_k(\xi)$ the k-th addend in decomposition (B.2.59) and introduce two smooth cut-off functions χ_0, γ, with χ_0 radial and equal to 1 on the support of ϕ_k, γ with sufficiently small support, we can write

$$\begin{aligned} \mathrm{Op}_h^w\left(\phi_k(\xi)(\xi|\xi|^{-1})^\mu\right)\widetilde{u}^J = \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J \\ + \mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J. \end{aligned}$$

On the one hand, after proposition 1.2.30, the fact that $2^k \lesssim h^{-\sigma}$, a-priori estimate (1.1.11d), and the uniform L^2 continuity of $\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))$, we have that for any $|\mu| \leq 1$

$$\begin{aligned} &\Big\|\mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\Big\|_{L^\infty} \\ &\lesssim h^{-\frac{1}{2}-\beta}\left(\|\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{L^2} + \|\theta_0(x)\Omega_h\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{L^2}\right) \\ &\lesssim h^{-\frac{1}{2}-\beta}\left(\|u_-^J(t,\cdot)\|_{L^2} + \|\Omega u_-^J(t,\cdot)\|_{L^2}\right) \leq CB\varepsilon h^{-\frac{1}{2}-\beta-\frac{\delta_1}{2}}. \end{aligned} \tag{B.2.61}$$

On the other hand, using that $(1-\gamma)(z) = \gamma_1^j(z)z_j$, where $\gamma_1^j(z) := (1-\gamma)(z)z_j|z|^{-2}$ is such that $|\partial_z^\alpha\gamma_1^j(z)| \leq \langle z\rangle^{-1-|\alpha|}$, we derive from (1.2.52b), the commutation between $\mathcal{M}$ with $\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))$, and lemma B.2.9, that

$$\begin{aligned} &\Big\|\mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J\Big\|_{L^\infty} \\ &\lesssim h^{-\beta}\sum_{\gamma,|\nu|=0}^{1}\|(\theta_0(x)\Omega_h)^\gamma\mathcal{M}^\nu\mathrm{Op}_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\beta+\frac{\delta_0}{2}}. \end{aligned}$$

Combining this estimate with (B.2.61) we deduce that

$$\|\mathrm{Op}_h^w\left(\phi_k(\xi)(\xi|\xi|^{-1})^\mu\right)\widetilde{u}^J(t,\cdot)\|_{L^\infty} \leq C(A+B)\varepsilon h^{-\frac{1}{2}-\beta-\frac{\delta_1}{2}},$$

for any $|\mu| \leq 1$, and hence (B.2.58) after (B.2.59), (B.2.60), up to a further loss $|\log h|$, as a consequence of the fact that the sum in (B.2.59) is finite and taken over indices k such that $\log h \lesssim k \lesssim \log h^{-1}$. □

Lemma B.2.11. *There exists a positive constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin, $\sigma > 0$, and every $t \in [1,T]$,*

$$\sum_{|\mu|=0}^{1}\Big\|\chi(t^{-\sigma}D_x)\Big[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_\pm(t,\cdot)\Big]\Big\|_{L^\infty} \leq CB\varepsilon t^{\beta+\frac{\delta_1}{2}}, \tag{B.2.62}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. We warn the reader that, throughout the proof, C and β will denote two positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$. As $\Gamma v_+ = -\overline{\Gamma v_-}$, it is enough to prove the statement for Γv_-.

If Γ is a spatial derivative, estimate (B.2.62) is just consequence of the uniform continuity of $\chi(t^{-\sigma}D_x)$ on L^∞ and of (B.1.10b). We then assume that $\Gamma \in \{\Omega, Z_m, m = 1, 2\}$ is a Klainerman vector field. First of all, we observe that by (B.1.9b) with $w = (\Gamma v)_-$ and $f = \Gamma NL_{kg}$, along with the classical Sobolev injection,

(B.2.63)
$$\sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu (\Gamma v)_-(t,\cdot) \Big\|_{L^\infty} \lesssim \|Z_j(\Gamma v)_-(t,\cdot)\|_{H^1} + t\|(\Gamma v)_-(t,\cdot)\|_{H^2} + \sum_{\mu=0}^{1} \|x_j^\mu \Gamma NL_{kg}(t,\cdot)\|_{L^\infty}.$$

From equality (B.1.20a) and lemma B.2.4 with $L = L^\infty$ and $s > 0$ large enough so that $N(s) \geq 3$, together with estimates (1.1.11), (B.1.5a), (B.1.5b), (B.1.5c), (B.1.6a), (B.1.10a), (B.1.11), (B.2.42), and (B.2.57), we get that

(B.2.64)
$$\begin{aligned}
&\|\Gamma NL_{kg}(t,\cdot)\|_{L^\infty} \\
&\lesssim \sum_{\mu=0}^{1} \big(\|\chi(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{H^{1,\infty}} \|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}} \big) + \|v_\pm(t,\cdot)\|_{H^{1,\infty}} \big\|\chi(t^{-\sigma}D_x)(\Gamma u)_\pm(t,\cdot)\big\|_{H^{2,\infty}} \\
&+ \|v_\pm(t,\cdot)\|_{H^{1,\infty}} \times \sum_{|\mu|=0}^{1} \big(\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|D_t \mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{1,\infty}} \big) \\
&+ t^{-3} \big(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s} \big) \Big(\sum_{|\mu|,|\nu|=0}^{1} \Big\| x^\mu D_1 \Big(\frac{D_x}{|D_x|} \Big)^\nu u_\pm(t,\cdot) \Big\|_{L^2} + t\|u_\pm(t,\cdot)\|_{L^2} \Big) \\
&+ t^{-3} \Big(\sum_{|\mu|=0}^{1} \|x^\mu v_\pm(t,\cdot)\|_{L^2} + t\|v_\pm(t,\cdot)\|_{L^2} \Big) \big(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s} \big) \\
&\leq CAB\varepsilon^2 t^{-\frac{3}{2}+\beta+\frac{\delta_1}{2}}.
\end{aligned}$$

Moreover, as

$$\Big\| x_j Q_0^{\mathrm{kg}}\big((\Gamma v)_\pm, D_1 u_\pm\big) \Big\|_{L^\infty} \lesssim \sum_{|\mu|,\nu=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu (\Gamma v)_-(t,\cdot) \Big\|_{L^\infty} \|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}, \tag{B.2.65}$$

(B.2.66)
$$\Big\| x_j G_1^{\mathrm{kg}}\big(v_\pm, Du_\pm\big) \Big\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_\pm(t,\cdot) \Big\|_{L^\infty} \big(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|D_t u_\pm(t,\cdot)\|_{H^{1,\infty}} \big),$$

and by lemma B.2.4 with $L = L^\infty$, $w = u$, and $s > 0$ large enough so that $N(s) \geq 3$,

(B.2.67)
$$\begin{aligned}
\Big\| x_j Q_0^{\mathrm{kg}}(v_\pm, D_1(\Gamma u)_\pm) \Big\|_{L^\infty} &\lesssim \sum_{|\mu|=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu v_\pm(t,\cdot) \Big\|_{L^\infty} \|\chi(t^{-\sigma}D_x)(\Gamma u)_\pm(t,\cdot)\|_{H^{2,\infty}} \\
&+ t^{-3} \sum_{|\mu|,\nu=0}^{1} \Big(\big\| x^\mu x_j^\nu v_\pm(t,\cdot) \big\|_{L^2} + t\|x_j^\nu v_\pm(t,\cdot)\|_{L^2} \Big) \big(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s} \big),
\end{aligned}$$

we derive that

(B.2.68)
$$\|x_j \Gamma NL_{kg}(t,\cdot)\|_{L^\infty} \leq CA\varepsilon t^{-\frac{1}{2}} \sum_{|\mu|,\nu=0}^{1} \Big\| x_j \Big(\frac{D_x}{\langle D_x \rangle} \Big)^\mu (\Gamma v)_-(t,\cdot) \Big\|_{L^\infty} + C(A+B)B\varepsilon^2 t^{-\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}},$$

as follows using (B.1.5a), (B.1.5b) with $s = 1$, (B.1.10a), (B.1.10b),(B.1.27a), (B.2.57) and a-priori estimates. By injecting the above inequality into (B.2.63) and using the fact that $\varepsilon_0 < (2CA)^{-1}$, we initially obtain that

$$\|x_j(\Gamma v)_-(t,\cdot)\|_{L^\infty} + \left\|x_j \frac{D_x}{\langle D_x\rangle}(\Gamma v)_-(t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon t^{1+\frac{\delta_2}{2}}. \tag{B.2.69}$$

If we take any smooth cut-off function χ and use equality (B.1.9b), instead of (B.2.63) we find that

(B.2.70)

$$\sum_{|\mu|=0}^{1} \left\|\chi(t^{-\sigma}D_x)\Big[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}(\Gamma v)_-(t,\cdot)\Big]\right\|_{L^\infty} \lesssim \|Z_j(\Gamma v)_-(t,\cdot)\|_{H^1} + t\|\chi(t^{-\sigma}D_x)(\Gamma v)_-(t,\cdot)\|_{L^\infty} + \sum_{\mu=0}^{1}\left\|\chi(t^{-\sigma}D_x)\big[x_j^\mu \Gamma NL_{kg}(t,\cdot)\big]\right\|_{L^\infty},$$

where now

$$\big\|\chi(t^{-\sigma}D_x)\big[x_j\Gamma NL_{kg}(t,\cdot)\big]\big\|_{L^\infty} \lesssim \|x_j\Gamma NL_{kg}(t,\cdot)\|_{L^\infty} \leq C(A+B)B\varepsilon^2 t^{\frac{1}{2}+\frac{\delta_2}{2}},$$

as follows injecting (B.2.69) into (B.2.68). Therefore, from (B.2.64), lemma B.2.8 and a-priori estimate (1.1.11d) with $k = 2$, we find that

$$\big\|\chi(t^{-\sigma}D_x)\big[x_j(\Gamma v)_-(t,\cdot)\big]\big\|_{L^\infty} + \left\|\chi(t^{-\sigma}D_x)\left[x_j\frac{D_x}{\langle D_x\rangle}(\Gamma v)_-(t,\cdot)\right]\right\|_{L^\infty} \leq CB\varepsilon t^{\frac{1}{2}+\frac{\delta_2}{2}}. \tag{B.2.71}$$

Finally, by means of lemma B.2.2 with $L = L^\infty$, $w_1 = x(\Gamma v)_\pm$, and $s > 0$ such that $N(s) \geq 2$, we derive that for any $\chi \in C_0^\infty(\mathbb{R}^2)$ there is some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)x_jQ_0^{\mathrm{kg}}((\Gamma v)_\pm, D_1u_\pm)\right\|_{L^\infty} &\lesssim \sum_{|\mu|,\nu=0}^{1}\left\|\chi_1(t^{-\sigma}D_x)\left[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}(\Gamma v)_-(t,\cdot)\right]\right\|_{L^\infty}\|\chi(t^{-\sigma}D_x)\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}} \\ &\quad + \sum_{\mu=0}^{1} t^{-2}\left\|x_j^\mu(\Gamma v)_\pm(t,\cdot)\right\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}.\end{aligned}$$

Then, combining such inequality with (B.2.66), (B.2.67), together with (B.1.17), (B.2.71), and all the other inequalities to which we already referred before, from (B.1.20a) we find that

$$\big\|\chi(t^{-\sigma}D_x)\big[x_j\Gamma NL_{kg}(t,\cdot)\big]\big\|_{L^\infty} \leq C(A+B)\varepsilon^2 t^{\delta_2},$$

which injected into (B.2.70) finally implies, together with (1.1.11d) with $k = 2$, lemma B.2.8, and (B.2.64), the wished estimate (B.2.62). □

Making use of lemmas B.2.8 and B.2.11, estimate (B.2.52) can be improved of a factor $t^{-\frac{1}{2}}$. This improvement, that will be useful to derive (B.4.30), is showed in the following lemma.

Lemma B.2.12. *Let I be a multi-index of length 1 and $r_{kg}^{I,NF}$ be given by* (B.2.46)*. There exists a constant $C > 0$ such that, for any $\rho \in \mathbb{N}$, $\chi \in C_0^\infty(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin, $\sigma > 0$ small, $j = 1, 2$, and every $t \in [1, T]$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_jr_{kg}^{I,NF}\right](t,\cdot)\right\|_{L^2} \leq C(A+B)AB\varepsilon^3 t^{-\frac{1}{2}+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.2.72}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. Let us remind the explicit expression (B.2.47) of $r_{kg}^{I,NF}$ and consider the cubic term $x_j\Gamma^I NL_{kg}(D_1u)$. Reminding (1.1.5) and applying lemma B.2.2 with $L = L^2$ and $s > 0$ sufficiently large so that $N(s) \geq 2$, together with (B.1.26a) and a-priori estimates, we derive that there is some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that

(B.2.73)
$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}(D_1u)\right](t,\cdot)\right\|_{L^2}\\
&\lesssim \left\|\chi_1(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^2}\|\mathrm{R}_1u_\pm(t,\cdot)\|_{L^\infty} + t^{-2}\|x_j\Gamma^I NL_{kg}(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&\leq CA\varepsilon t^{-\frac{1}{2}}\left\|\chi_1(t^{-\sigma}D_x)\left[x_jNL_{kg}^I\right](t,\cdot)\right\|_{L^2} + C(A+B)B\varepsilon^2t^{-1}.
\end{aligned}$$

Then, recalling (B.1.20a) and using again lemma B.2.2 with $L = L^2$, $w_1 = (\Gamma v)_\pm$, and s as before, in order to estimate the contribution coming from the first quadratic term in the right hand side of (B.1.20a), we find that there is a new $\chi_2 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned}
&\left\|\chi_1(t^{-\sigma}D_x)\left[x_jNL_{kg}^I\right](t,\cdot)\right\|_{L^2}\\
&\lesssim \sum_{|\mu|=0}^1\left\|\chi_2(t^{-\sigma}D_x)\Big[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_\pm\Big](t,\cdot)\right\|_{L^\infty}\|u_\pm(t,\cdot)\|_{H^1} + t^{-2}\|x_j(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&+\sum_{|\mu|=0}^1\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1} + \|u_\pm(t,\cdot)\|_{H^1} + \|D_tu_\pm(t,\cdot)\|_{L^2}\right)\\
&\leq C(A+B)B\varepsilon^2t^{\beta+\frac{\delta+\delta_1}{2}},
\end{aligned}$$

where the latter estimate is obtained from (B.1.5a) with $s = 0$, (B.1.10b), (B.1.17) with $k = 1$, (B.2.62) and a-priori estimates. This implies, combined with (B.2.73), that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_jNL_{kg}^I(D_1u)\right](t,\cdot)\right\|_{L^2} \leq C(A+B)AB\varepsilon^3t^{-\frac{1}{2}+\beta+\frac{\delta+\delta_1}{2}},$$

and from (B.2.47), (B.1.10b) and a-priori estimates,

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_jr_{kg}^{I,NF}\right](t,\cdot)\right\|_{L^2} \lesssim \left\|\chi(t^{-\sigma}D_x)\left[x_jNL_{kg}^I(D_1u)\right](t,\cdot)\right\|_{L^2}\\
&+\sum_{\mu=0}^1 t^\sigma\left(\|x_j^\mu v_\pm(t,\cdot)\|_{L^\infty} + \left\|x_j^\mu\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{L^\infty}\right)\|v_\pm^I(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&\leq C(A+B)AB\varepsilon^3t^{-\frac{1}{2}+\beta+\frac{\delta+\delta_1}{2}},
\end{aligned}$$

which concludes the proof of the statement. □

Lemma B.2.13. *Let I be a multi-index of length 2. There exists a constant $C > 0$ such that, for every $j = 1, 2$, $t \in [1, T]$,*

$$\left\|x_j\Gamma^I NL_{kg}(t,\cdot)\right\|_{L^2} \leq C(A+B)B\varepsilon^2t^{\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}}, \tag{B.2.74}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. We remind the reader about (B.1.23). Instead of using (B.1.24), which was obtained by Sobolev injection, we apply lemma B.2.4 with $L = L^2$, $\Gamma w = \Gamma^{I_2}u$, $s > 0$ sufficiently large so that $N(s) \geq 3$, and exploit the fact that we have an estimate of the $H^{\rho,\infty}$ norm of $D_1u^{I_2}$ when

truncated for frequencies less or equal than t^σ (see lemma B.2.10). Therefore, for $(I_1, I_2) \in \mathfrak{I}(I)$ such $|I_1| = |I_2| = 1$ we obtain that

$$\begin{aligned}
&\left\| x_j Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, D_1 u_\pm^{I_2}\right)(t,\cdot)\right\|_{L^2} \lesssim \sum_{\mu=0}^{1} \left\| x_j^\mu v_\pm^{I_1}(t,\cdot)\right\|_{L^2} \left\| \chi(t^{-\sigma} D_x) u_\pm^{I_2}(t,\cdot)\right\|_{H^{2,\infty}} \\
&+ t^{-3}\left(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\right)\left[\sum_{|\mu|=0}^{2} \|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2} + \sum_{|\mu|=0}^{1} t\|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2}\right] \\
&\leq C(A+B)B\varepsilon^2 t^{\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}},
\end{aligned}$$

last estimate following from lemma B.2.10 together with (B.1.5a), (B.1.17) with $k = 1$, (B.1.28), a-priori estimates, and the fact that $\delta_1, \delta_2 \ll 1$ are small. Consequently, from the following inequality

$$\begin{aligned}
\|x_j \Gamma^I NL_{kg}(t,\cdot)\|_{L^2} &\lesssim \sum_{\mu=0}^{1} \|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}} \sum_{\substack{|J|\leq 2 \\ \mu=0,1}} \|x_j^\mu (\Gamma^J v)_-(t,\cdot)\|_{L^2} \\
&+ \sum_{|\mu|=0}^{1} \left\| x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty} \Big[\|u_\pm^I(t,\cdot)\|_{H^1} + \sum_{|J|<2} \big(\|u_\pm^J(t,\cdot)\|_{H^1} + \|D_t u_\pm^J(t,\cdot)\|_{L^2}\big)\Big] \\
&+ \sum_{|I_1|=|I_2|=1} \left\| x_j Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, D_1 u_\pm^{I_2}\right)(t,\cdot)\right\|_{L^2},
\end{aligned}$$

together with (B.1.10b), (B.1.5a) with $s = 0$, (B.1.7), and (B.1.17) with $k = 1$, we finally derive (B.2.74). □

Lemma B.2.14. *Let us fix $s \in \mathbb{N}$. There exists a constant $C > 0$ such that, if we assume that a-priori estimates (1.1.11) are satisfied in some interval $[1, T]$, for a fixed $T > 1$, with $n \geq s + 2$, then we have, for any $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\sigma > 0$ small,*

$$\|\widetilde{v}(t,\cdot)\|_{H_h^s} \leq CB\varepsilon t^{\frac{\delta}{2}}, \tag{B.2.75a}$$

$$\sum_{|\mu|=1} \|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.75b}$$

for every $t \in [1, T]$.

Proof. We warn the reader that, throughout the proof, C and β will denote two positive constants that may change line after line, with $\beta > 0$ is small as long as σ is small.

It is straightforward to check that the H_h^s norm of $\widetilde{v}$ is bounded by energy $E_n(t; W)^{\frac{1}{2}}$ whenever $n \geq s + 2$, after definitions (3.2.2), (3.1.3), inequality (3.1.7a), and a-priori estimates (1.1.11a), (1.1.11b).

In order to prove (B.2.75b) we first use relation (3.2.9b) and definition (3.1.3) to derive that

$$\begin{aligned}
\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m \widetilde{v}(t,\cdot)\|_{L^2} &\lesssim \|Z_m V(t,\cdot)\|_{L^2} + \|\chi(t^{-\sigma} D_x) Z_m(v^{NF} - v_-)(t,\cdot)\|_{L^2} \\
&+ \|\widetilde{v}(t,\cdot)\|_{L^2} + \|\chi(t^{-\sigma} D_x)[x_m r_{kg}^{NF}](t,\cdot)\|_{L^2},
\end{aligned} \tag{B.2.76}$$

with r_{kg}^{NF} given by (3.1.5). Using (1.1.5) we can rewrite (3.1.10) and (3.1.11) similarly to (B.2.30), (B.2.47), as:

$$v^{NF} - v_- = -\frac{i}{2}\left[(D_t v)(D_1 u) - (D_1 v)(D_t u) + D_1[v D_t u] - \langle D_x\rangle[v D_1 u]\right] \tag{B.2.77}$$

and

$$r_{kg}^{NF} = -\frac{i}{2}\left[NL_{kg}D_1u - (D_1v)NL_w + D_1(vNL_w)\right]. \tag{B.2.78}$$

From (B.2.78) and (1.1.5), together with estimates (1.1.11) and (B.1.10b),

(B.2.79)

$$\begin{aligned}\|\chi(t^{-\sigma}D_x)(x_m r_{kg}^{NF})(t,\cdot)\|_{L^2} &\lesssim t^{\sigma}\left(\|x_n v_-(t,\cdot)\|_{L^\infty} + \left\|x_n\frac{D_x}{\langle D_x\rangle}v_-(t,\cdot)\right\|_{L^\infty}\right)\\ &\times\Big[\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\|U(t,\cdot)\|_{L^2} + \|V(t,\cdot)\|_{H^{2,\infty}}\|V(t,\cdot)\|_{L^2}\Big]\\ &+ \|V(t,\cdot)\|_{H^{1,\infty}}^2\|V(t,\cdot)\|_{H^1} \leq C(A+B)AB\varepsilon^3 t^{-\frac{1}{2}+\sigma+\frac{(\delta+\delta_2)}{2}}.\end{aligned}$$

Similarly to (B.2.40),

(B.2.80)

$$\begin{aligned}&2i\chi(t^{-\sigma}D_x)Z_m(v^{NF}-v_-)\\ &= \chi(t^{-\sigma}D_x)\Big[(D_tZ_mv)(D_1u) - (D_1Z_mv)(D_tu) + D_1[(Z_mv)(D_tu)] - \langle D_x\rangle[(Z_mv)(D_1u)]\\ &\quad + (D_tv)(D_1Z_mu) - (D_1v)(D_tZ_mu) + D_1[v(D_tZ_mu)] - \langle D_x\rangle[v(D_1Z_mu)]\\ &\quad - (D_mv)(D_1u) + \delta_{m1}(D_tv)(D_tu) - \delta_{m1}D_t[v(D_tu)] + \frac{D_m}{\langle D_x\rangle}D_t[v(D_1u)]\\ &\quad - \delta_{m1}(D_tv)(D_tu) + (D_1v)(D_mu) - \delta_{m1}D_1[v(D_tu)] + \delta_{m1}\langle D_x\rangle[v(D_tu)]\Big].\end{aligned}$$

We bound the L^2 norm of all products in the first line of the above equality by means of lemma B.2.2, and all the others by the L^∞ norm of the Klein-Gordon factor times the L^2 norm of the wave one. In this way we get that, for some new $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ and $s > 0$ sufficiently large, we derive that

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)Z_m(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\\ &\quad\lesssim t^{\sigma}\left\|\chi_1(t^{-\sigma}D_x)(Z_mv)_\pm(t,\cdot)\right\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2} + t^{-1}\|(Z_mv)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\ &\qquad + t^{\sigma}\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(Z_mu)_\pm(t,\cdot)\|_{L^2} + \|u_\pm(t,\cdot)\|_{L^2} + \|D_tu_\pm(t,\cdot)\|_{L^2}\right).\end{aligned}$$

Consequently, using estimates (1.1.11), (B.1.5a) with $s = 0$, and (B.2.42), we obtain that

$$\|\chi(t^{-\sigma}D_x)Z_m(v^{NF}-v_-)(t,\cdot)\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.2.81}$$

which plugged into (B.2.76), along with (B.2.79), (B.2.75a) and (1.1.11d), gives (B.2.75b). □

B.3 Last range of estimates

The aim of this section is to show that a-priori estimates (1.1.11) also infer a moderate growth in time of the $L^2(\mathbb{R}^2)$ norm of $\mathcal{L}^\mu\widetilde{v}$, for $|\mu| = 2$, when this function is restricted to frequencies less or equal than $h^{-\sigma}$, for $\sigma > 0$ small. This is proved in lemma B.3.7. Lemmas from B.3.1 to B.3.6 are intermediate technical results.

Lemma B.3.1. *Let us consider v^{NF} introduced in* (3.1.3) *and $v^{I,NF}$ as in* (B.2.27) *with $|I| = 1$ and $\Gamma^I = Z_n$, for $n \in \{1,2\}$. There exists a constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, and every $t \in [1,T]$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[(Z_nv)_- - v^{I,NF}\right](t,\cdot)\right\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.1}$$

(B.3.2) $$\big\|\chi(t^{-\sigma}D_x)\big[x_m Z_n(v^{NF}-v_-)(t,\cdot)\big]\big\|_{L^2}+\big\|\chi(t^{-\sigma}D_x)\big[x_m\big((Z_nv)_- - v^{I,NF}\big)\big](t,\cdot)\big\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta_1+\delta_2}{2}}.$$

The same estimates hold true when Z_n is replaced with Ω.

Proof. By comparing equality (B.2.30), with $|I|=1$ and $\Gamma^I=Z_n$, with (B.2.80) we see that $\chi(t^{-\sigma}D_x)(v^{I,NF}-(Z_nv)_-)$ corresponds to the first line in the right hand side of (B.2.80). Therefore, inequality (B.3.1) is automatically satisfied after (B.2.81), which was obtained by estimating the right hand side of (B.2.80) term by term. In order to prove (B.3.2), let us consider equality (B.2.80) but with $\chi(t^{-\sigma}D_x)$ replaced with $\chi(t^{-\sigma}D_x)x_m$. The L^2 norm of each product in the second to fourth line is bounded by

$$t^\sigma\sum_{\mu,\nu=0}^{1}\Big\|x_m^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\Big\|_{L^\infty}\left(\|(Z_mu)_\pm(t,\cdot)\|_{L^2}+\|u_\pm(t,\cdot)\|_{L^2}+\|D_tu_\pm(t,\cdot)\|_{L^2}\right),$$

and then by the right hand side of (B.3.2) after (1.1.11), (B.1.5a) with $s=0$, and (B.1.10b). Using lemma B.2.2 with $L=L^2$ and $s>0$ large enough to have $N(s)\geq 2$, we obtain that the L^2 norm of products in the first line of (the modified) (B.2.80) is bounded by

$$\sum_{\mu,\nu=0}^{1}\Big\|\chi_1(t^{-\sigma}D_x)\Big[x_m^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu(Z_mv)_\pm(t,\cdot)\Big]\Big\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2} + \sum_{\mu=0}^{1}t^{-N(s)}\|x_m^\mu(Z_mv)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s},$$

for some smooth cut-off χ_1, and hence by the right hand side of (B.3.2) after (1.1.11), (B.1.17) and (B.2.62) with $\Gamma=Z_m$. This concludes the proof of (B.3.2).

When Z_n is replaced with Ω, instead of referring to (B.2.80) one uses that

$$\begin{aligned}2i\Omega\left(v^{NF}-v_-\right) &= (D_t\Omega v)(D_1u)-(D_1\Omega v)(D_tu)+D_1[(\Omega v)(D_tu)]-\langle D_x\rangle[(\Omega v)(D_1u)]\\ &\quad+(D_tv)(D_1\Omega u)-(D_1v)(D_t\Omega u)+D_1[v(D_t\Omega u)]-\langle D_x\rangle[v(D_1\Omega u)]\\ &\quad-(D_tv)(D_2u)+(D_2v)(D_tu)-D_2[v(D_tu)]+\langle D_x\rangle[v(D_2u)]\end{aligned}$$

and applies the same argument as above to recover the wished estimates. □

Lemma B.3.2. *Let v^{NF} be defined as in* (3.1.3). *There exists a constant $C>0$ such that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, $m=1,2$,*

(B.3.3a) $$\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))[tZ_nv^{NF}(t,tx)]\big\|_{L^2(dx)}\leq CB\varepsilon t^{\frac{\delta_2}{2}},$$

(B.3.3b) $$\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m[tZ_nv^{NF}(t,tx)]\big\|_{L^2(dx)}\leq CB\varepsilon t^{\frac{\delta_1}{2}},$$

for every $t\in[1,T]$.

Proof. Let us write Z_nv^{NF} as follows:

(B.3.4) $$Z_nv^{NF}=Z_n(v^{NF}-v_-)+\big[(Z_nv)_- - v^{I,NF}\big]+v^{I,NF}+\frac{D_n}{\langle D_x\rangle}v^{NF}+\frac{D_n}{\langle D_x\rangle}(v_- - v^{NF}),$$

with $v^{I,NF}$ given by (B.2.27) with $|I| = 1$ and $\Gamma^I = Z_n$. From the fact that $\|tw(t,t\cdot)\|_{L^2} = \|w(t,\cdot)\|_{L^2}$ and estimates (1.1.11), (3.1.8a), (B.2.50),(B.2.81), (B.3.1), along with the following one

$$\begin{aligned}&\|\chi(t^{-\sigma}D_x)D_n\langle D_x\rangle^{-1}v^{NF}(t,\cdot)\|_{L^2}\\&\quad\le \|\chi(t^{-\sigma}D_x)D_n\langle D_x\rangle^{-1}v_-(t,\cdot)\|_{L^2} + \|\chi(t^{-\sigma}D_x)D_n\langle D_x\rangle^{-1}(v_- - v^{NF})(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta}{2}},\end{aligned}$$

we immediately obtain (B.3.3a).

From (B.3.4) we also derive that

(B.3.5)
$$\begin{aligned}&\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m[tZ_nv^{NF}(t,tx)]\big\|_{L^2(dx)} \lesssim \big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tZ_n(v^{NF}-v_-)(t,tx)\big]\big\|_{L^2(dx)}\\&+\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[t\big((Z_nv)_- - v^{I,NF}\big)(t,tx)\big]\big\|_{L^2(dx)} + \big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tv^{I,NF}(t,tx)\big]\big\|_{L^2(dx)}\\&+\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}v^{NF}(t,tx)\big]\big\|_{L^2(dx)}\\&+\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}(v_- - v^{NF})(t,tx)\big]\big\|_{L^2(dx)}.\end{aligned}$$

By relation (3.2.8) with $w = v^{I,NF}$ and estimates (B.2.50), (B.2.51), (B.2.53), it follows that

$$\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tv^{I,NF}(t,tx)\big]\big\|_{L^2} \le CB\varepsilon t^{\frac{\delta_1}{2}},$$

while from (3.2.2) and (B.2.75b) we have that

$$\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}v^{NF}(t,tx)\big]\big\|_{L^2} \le CB\varepsilon t^{\frac{\delta_2}{2}}.$$

The remaining L^2 norms in the right hand side of (B.3.5) are estimated reminding definition (1.2.68) of $\mathcal{L}_m$ and using the fact that

(B.3.6) $$\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m[tw(t,tx)]\|_{L^2(dx)} \lesssim \|\chi(t^{-\sigma}D_x)[x_mw(t,\cdot)]\|_{L^2} + t\|\chi(t^{-\sigma}D_x)w(t,\cdot)\|_{L^2}.$$

Therefore, by (B.2.81) and lemma B.3.1 we derive that

$$\begin{aligned}&\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tZ_n(v^{NF}-v_-)(t,tx)\big]\big\|_{L^2(dx)}\\&\qquad+\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[t\big((Z_nv)_- - v^{I,NF}\big)(t,tx)\big]\big\|_{L^2(dx)} \le C(A+B)B\varepsilon^2t^{\beta+\frac{\delta+\delta_1}{2}},\end{aligned}$$

while from (3.1.8a), a-priori estimates, together with the following inequality

$$\begin{aligned}\|\chi(t^{-\sigma}D_x)[x_m(v_- - v^{NF})](t,\cdot)\|_{L^2} &\lesssim \sum_{\mu,\nu=0}^{1} t^\sigma\Big\|x_m^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_-(t,\cdot)\Big\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2}\\&\le C(A+B)B\varepsilon^2t^{\sigma+\frac{\delta+\delta_2}{2}},\end{aligned}$$

which follows by (B.2.77), (1.1.5), (B.1.10b), (1.1.11b), (1.1.11c), we derive

$$\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}(v_- - v^{NF})(t,tx)\big]\big\|_{L^2} \le C(A+B)B\varepsilon^2t^{\sigma+\frac{(\delta+\delta_2)}{2}}.$$

□

In the following lemma we are going to prove that the product of the semiclassical wave function $\widetilde{u}$ with the Klein-Gordon one $\widetilde{v}$ enjoys a better L^2 (resp. L^∞) estimate than the one roughly obtained by taking the L^2 (resp. L^∞) norm of the former times the L^∞ norm of the latter. Estimates

$$\|\widetilde{v}\widetilde{u}(t,\cdot)\|_{L^2} \lesssim \|\widetilde{v}(t,\cdot)\|_{L^\infty}\|\widetilde{u}(t,\cdot)\|_{L^2} \leq CAB\varepsilon^2 h^{-\frac{\delta}{2}},$$
$$\|\widetilde{v}\widetilde{u}(t,\cdot)\|_{L^\infty} \lesssim \|\widetilde{v}(t,\cdot)\|_{L^\infty}\|\widetilde{u}(t,\cdot)\|_{L^\infty} \leq CA^2\varepsilon^2 h^{-\frac{1}{2}-\frac{\delta}{2}},$$

which follows from (B.2.1a), (B.3.8), (B.3.9), can be in fact improved of a factor $h^{1/2}$ (see (B.3.7)). This comes from the fact that the main contribution to $\widetilde{u}$ is localized around manifold Λ_w introduced in (3.2.43), whereas $\widetilde{v}$ concentrates around Λ_{kg} defined in (1.2.66), and these two manifolds are disjoint.

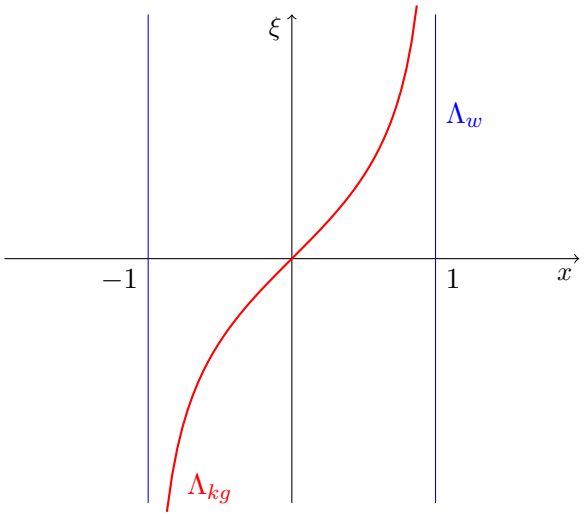

Figure B.1: Manifolds Λ_{kg} and Λ_w.

Lemma B.3.3. *Let $h = t^{-1}$, $\widetilde{u}, \widetilde{v}$ be defined in (3.2.2), $a_0(\xi) \in S_{0,0}(1)$, and $b_1(\xi) = \xi_j$ or $b_1(\xi) = \xi_j\xi_k|\xi|^{-1}$, with $j,k \in \{1,2\}$. There exists a constant $C > 0$ such that, for any $\chi, \chi_1 \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$, and every $t \in [1,T]$, we have that*

(B.3.7a) $$\big\|[\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\big\|_{L^2} \leq C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}},$$

(B.3.7b) $$\big\|[\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\big\|_{L^\infty} \leq C(A+B)B\varepsilon^2 h^{-\beta-\frac{\delta+\delta_1}{2}},$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. Before entering in the details of the proof, we warn the reader that C and β denote two positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$. Also, we will denote by $R(t,x)$ any contribution, in what follows, that satisfies inequalities (B.3.7), and by χ_2 a smooth cut-off function, identically equal to 1 on the support of χ_1, so that

$$\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\widetilde{u} = \mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\mathrm{Op}_h^w(\chi_2(h^\sigma\xi))\widetilde{u},$$

assuming that at any time $\widetilde{u}$ can be replaced with $\mathrm{Op}_h^w(\chi_2(h^\sigma\xi))\widetilde{u}$. Finally, it is useful to remind that from (3.2.2), (3.1.15), (3.1.20b), (3.1.20c), and a-priori estimates,

(B.3.8) $$\|\widetilde{u}(t,\cdot)\|_{H_h^{\rho+1,\infty}} + \sum_{|\mu|=1} \|\mathrm{Op}_h^w\big((\xi|\xi|^{-1})^\mu\big)\widetilde{u}(t,\cdot)\|_{H_h^{\rho+1,\infty}} \leq CA\varepsilon h^{-\frac{1}{2}},$$

while by (3.1.3), (3.1.7b) (with $\theta \ll 1$ small enough) and a-priori estimates,

$$\|\widetilde{v}(t,\cdot)\|_{H_h^{\rho,\infty}} \leq CA\varepsilon, \tag{B.3.9}$$

for every $t \in [1,T]$.

First of all, we take $\gamma \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin and with sufficiently small support, and define

$$\widetilde{v}_{\Lambda_{kg}}(t,x) := \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{v}(t,x),$$
$$\widetilde{v}_{\Lambda_{kg}^c}(t,x) := \mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{v}(t,x),$$

with $p(\xi) := \langle\xi\rangle$, so that

$$\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v} = \widetilde{v}_{\Lambda_{kg}} + \widetilde{v}_{\Lambda_{kg}^c}. \tag{B.3.10}$$

The following estimates hold:

$$\|\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\|_{L^\infty} \leq CA\varepsilon h^{-\beta}, \tag{B.3.11a}$$

$$\|\widetilde{v}_{\Lambda_{kg}^c}(t,\cdot)\|_{L^\infty} \leq CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}. \tag{B.3.11b}$$

The former one is a straight consequence of proposition 1.2.39 with $p = +\infty$ and (B.3.9). On the other hand, if we write

$$(1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi) = \sum_{j=1}^2 \gamma_1^j\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big(\frac{x_j-p_j'(\xi)}{\sqrt{h}}\Big)$$

with $\gamma_1^j(z) := (1-\gamma)(z)z_j|z|^{-2}$ such that $|\partial_z^\alpha\gamma_1^j(z)| \lesssim \langle z\rangle^{-1-|\alpha|}$, and use (1.2.69) with $c(x,\xi) = \chi(h^\sigma\xi)a_0(\xi)$, we obtain that

(B.3.12)

$$\begin{aligned}
\|\widetilde{v}_{\Lambda_{kg}^c}(t,\cdot)\|_{L^\infty} \lesssim{}& \sum_{j=1}^2 \sqrt{h}\,\Big\|\mathrm{Op}_h^w\Big(\gamma_1^j\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\mathcal{L}_j\widetilde{v}(t,\cdot)\Big\|_{L^\infty} \\
&+ \sum_{j=1}^2 \sqrt{h}\,\Big\|\mathrm{Op}_h^w\Big(\gamma_1^j\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\partial_j\big(\chi(h^\sigma\xi)a_0(\xi)\big)\Big)\widetilde{v}(t,\cdot)\Big\|_{L^\infty} \\
&+ \sum_{j=1}^2\sum_{|\alpha|=2} \sqrt{h}\,\Big\|\mathrm{Op}_h^w\Big((\partial^\alpha\gamma_1^j)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)(\partial_\xi^\alpha p')(\xi)\Big)\widetilde{v}(t,\cdot)\Big\|_{L^\infty} \\
&+ \|\mathrm{Op}_h^w(r(x,\xi))\widetilde{v}(t,\cdot)\|_{L^\infty},
\end{aligned}$$

with $r \in h^{1-\beta}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. Since γ_1^j vanishes in a neighbourhood of the origin, we derive from inequality (3.2.17b), equation (3.1.4) and relation (3.2.8) with $w = v^{NF}$, lemmas B.2.14, B.3.2, and estimate (B.2.79), that the first sum in the above right hand side is bounded by the right hand side of (B.3.7b). The same is true for the above second and third sums after (3.2.17b) and lemma B.2.14, and for the above latter L^∞ norm because of proposition 1.2.37 and estimate (B.2.75a).

After decomposition (B.3.10) and estimates (B.2.1a), (B.3.8), and (B.3.11b), we see that

$$\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}\,\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} = \widetilde{v}_{\Lambda_{kg}}\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} + R(t,x).$$

For some suitably supported $\chi_0 \in C_0^\infty(\mathbb{R}^2)$, $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, we also consider the following decomposition

$$\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u} = \mathrm{Op}_h^w(\chi_0(h^{-1}\xi)b_1(\xi))\widetilde{u} + \sum_k \mathrm{Op}_h^w\big((1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_1(h^\sigma\xi)b_1(\xi)\big)\widetilde{u},$$

and observe that, from proposition 1.2.36 and the classical Sobolev injection,

$$\big\|\mathrm{Op}_h^w(\chi_0(h^{-1}\xi)b_1(\xi))\widetilde{u}(t,\cdot)\big\|_{L^2} + \big\|\mathrm{Op}_h^w(\chi_0(h^{-1}\xi)b_1(\xi))\widetilde{u}(t,\cdot)\big\|_{L^\infty} \lesssim h\|\widetilde{u}(t,\cdot)\|_{L^2}.$$

Combining the above decomposition and estimate with (B.3.11a) and (B.2.1a) we derive that

$$\widetilde{v}_{\Lambda_{kg}}\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} = \sum_k \widetilde{v}_{\Lambda_{kg}}\mathrm{Op}_h^w(\phi_k(\xi)b_1(\xi))\widetilde{u} + R(t,x), \tag{B.3.13}$$

where $\phi_k(\xi) := (1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi(h^\sigma\xi)$. We can further decompose $\mathrm{Op}_h^w(\phi_k(\xi)b_1(\xi))\widetilde{u}$ by defining

$$\widetilde{u}^k_{\Lambda_w}(t,x) := \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big)\widetilde{u}(t,x),$$
$$\widetilde{u}^k_{\Lambda_w^c}(t,x) := \mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big)\widetilde{u}(t,x),$$

and observe that

$$\Big\|\widetilde{u}^k_{\Lambda_w^c}(t,\cdot)\Big\|_{L^2} \lesssim h^{\frac{1}{2}-\beta}\left[\|\widetilde{u}(t,\cdot)\|_{L^2} + \sum_{\mu,|\nu|=0}^{1}\|(\theta_0(x)\Omega_h)^\mu\mathcal{M}^\nu\mathrm{Op}_h^w(\chi_2(h^\sigma\xi))\widetilde{u}(t,\cdot)\|_{L^2}\right]$$
$$\leq CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}},$$

and

$$\Big\|\widetilde{u}^k_{\Lambda_w^c}(t,\cdot)\Big\|_{L^\infty} \lesssim h^{-\beta}\left[\|\widetilde{u}(t,\cdot)\|_{L^2} + \sum_{\mu,|\nu|=0}^{1}\|(\theta_0(x)\Omega_h)^\mu\mathcal{M}^\nu\mathrm{Op}_h^w(\chi_2(h^\sigma\xi))\widetilde{u}(t,\cdot)\|_{L^2}\right]$$
$$\leq CB\varepsilon h^{-\beta-\frac{\delta_1}{2}},$$

as follows by using the following equality

$$(1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi) = \sum_{j=1}^{2}\gamma_1^j\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi|-\xi_j}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi),$$

with $\gamma_1^j(z) := (1-\gamma)(z)z_j|z|^{-2}$, together with (1.2.52) with $a \equiv 1$, $p = 1$, and lemma B.2.1. Then, as the sum over k in the right hand side of (B.3.13) is actually restricted to indices k such that $h \lesssim 2^k \lesssim h^{-\sigma}$, the above estimates and (B.3.11a) imply that

$$\sum_k \widetilde{v}_{\Lambda_{kg}}\mathrm{Op}_h^w(\phi_k(\xi)b_1(\xi))\widetilde{u} = \sum_k \widetilde{v}_{\Lambda_{kg}}\widetilde{u}^k_{\Lambda_w} + R(t,x).$$

Moreover, using lemma 1.2.38, symbolic calculus and remark 1.2.22, each $\widetilde{v}_{\Lambda_{kg}}\widetilde{u}^k_{\Lambda_w}$ in the above right hand side can be replaced with

$$\frac{\theta_h(x)}{|x|^2-1}\widetilde{v}_{\Lambda_{kg}}\ (|x|^2-1)\widetilde{u}^k_{\Lambda_w}$$

up to a new remainder $R(t,x)$. Since $|\theta_h(x)(|x|^2-1)^{-1}| \lesssim h^{-2\sigma}$ on the support of $\theta_h(x)$, from proposition 1.2.36 and estimates (B.2.75a), (B.3.11a), we get that

$$\Big\|\theta_h(x)\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\Big\|_{L^2} \leq CB\varepsilon h^{-\frac{\delta}{2}-\beta}\|\theta_h(x)(|x|^2-1)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\|_{L^\infty}, \tag{B.3.14a}$$

$$\Big\|\theta_h(x)\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\Big\|_{L^\infty} \leq CA\varepsilon h^{-\beta}\|\theta_h(x)(|x|^2-1)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\|_{L^\infty}. \tag{B.3.14b}$$

Then the end of the proof relies on the fact that $\theta_h(x)(|x|^2-1)\widetilde{u}^k_{\Lambda_w}$ can be expressed in terms of $h\mathcal{M}\widetilde{u}$. In fact, for a fixed $N\in\mathbb{N}$ and up to some negligible multiplicative constants, we have from proposition 1.2.21 that

(B.3.15)

$$\begin{aligned}\big[\theta_h(x)(|x|^2-1)\big]\sharp\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big] &= \gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\theta_h(x)(|x|^2-1)\\ &+ h\left\{\theta_h(x)(|x|^2-1),\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\right\}\\ &+\sum_{|\alpha|=2}^{N-1}h^{|\alpha|}\partial_x^\alpha\big[\theta_h(x)(|x|^2-1)\big]\,\partial_\xi^\alpha\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big]+r_N(x,\xi),\end{aligned}$$

with

$$\begin{aligned}r_N(x,\xi) = \frac{h^N}{(\pi h)^4}\sum_{|\alpha|=N}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\int_0^1\partial_x^\alpha[\theta_h(x)(|x|^2-1)]|_{(x+tz)}(1-t)^{N-1}dt\\ \times\partial_\xi^\alpha\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big]\,|_{(x+y,\xi+\eta)}dydzd\eta d\zeta.\end{aligned} \tag{B.3.16}$$

As

$$|x|^2-1 = x\cdot x-\frac{\xi\cdot\xi}{|\xi|^2} = (x|\xi|-\xi)\cdot\frac{x}{|\xi|}+(x|\xi|-\xi)\cdot\frac{\xi}{|\xi|^2},$$

the first term in the right hand side of (B.3.15) appears to be linear combination of products of the form $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\phi_k(\xi)a(x)b_0(\xi)(x_j|\xi|-\xi_j)$, for some smooth compactly supported function $a(x)$, and $b_0(\xi)$ such that $|\partial^\alpha b_0(\xi)|\lesssim|\xi|^{-|\alpha|}$. From (1.2.52b) and lemma B.2.1, we hence deduce that

$$\Big\|\mathrm{Op}^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\theta_h(x)(|x|^2-1)\Big)\widetilde{u}(t,\cdot)\Big\|_{L^\infty}\leq CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}. \tag{B.3.17a}$$

An explicit computation shows that

$$\begin{aligned}h\left\{\theta_h(x)(|x|^2-1),\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\right\}\\ =\sum_i h^{\frac{1}{2}+\sigma}\partial_{x_i}[\theta_h(x)(|x|^2-1)]\sum_j(\partial_j\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(x_j\frac{\xi_i}{|\xi|}-\delta_{ij}\Big)\phi_k(\xi)b_1(\xi)\\ +h\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\partial_x[\theta_h(x)(|x|^2-1)]\partial_\xi[\phi_k(\xi)b_1(\xi)],\end{aligned}$$

with $\delta_{ij}=1$ being the Kronecker delta. One the one hand, since the first contribution to the above right hand side is still supported for $|x|<1-ch^{2\sigma}$, we can multiply and divide it by $|x|^2-1$ so that it writes as linear combination of terms of the form $h^{\frac{1}{2}-\sigma}\gamma_1\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\phi_k(\xi)a(x)b_0(\xi)(x_j|\xi|-\xi_j)$, for a new $\gamma_1\in C_0^\infty(\mathbb{R}^2)$, and some new $a(x),b_0(\xi)$ with the same properties as the ones we

considered before. On the other hand, as $\partial_\xi[\phi_k(\xi)b_1(\xi)]$ is uniformly bounded and supported for frequencies of size 2^k, the second term in the above right hand side writes as linear combination of products of the form $h\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\phi_k^1(\xi)a(x)b_0(\xi)$, for some new $\phi_k^1 \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$. Therefore, inequality (1.2.52b), proposition 1.2.30, and lemma B.2.1, give that

$$\Big\|h\mathrm{Op}_h^w\Big(\Big\{\theta_h(x)(|x|^2-1),\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big\}\Big)\widetilde{u}(t,\cdot)\Big\|_{L^\infty} \leq CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}. \tag{B.3.18}$$

As concerns $|\alpha|$-order terms, for each fixed $2\leq|\alpha|\leq N-1$, we find using (1.2.25) that they are given by

$$\begin{aligned} h^{|\alpha|}\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\partial_x^\alpha[\theta_h(x)(|x|^2-1)]\partial_\xi^\alpha(\phi_k(\xi)b_1(\xi)) \\ + \sum_{\substack{|\beta_1|+|\beta_2|=|\alpha|\\ |\beta_1|\geq 1}}\sum_{j=1}^{|\beta_1|}h^{|\alpha|-j(\frac{1}{2}-\sigma)}\gamma_j\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\theta}_j(x)b_{j-|\beta_1|}(\xi)\partial_\xi^{\beta_2}(\phi_k(\xi)b_1(\xi)), \end{aligned}$$

for some $\gamma_j,\widetilde{\theta}_j\in C_0^\infty(\mathbb{R}^2)$. Since $|\alpha|\geq 2$ and $|\partial_\xi^\mu(\phi_k(\xi)b_1(\xi))|\lesssim 2^{-k(|\mu|-1)}$, for any $\mu\in\mathbb{N}^2$, by proposition 1.2.30 and lemma B.2.1 we obtain that the action of their quantization on $\widetilde{u}$ is estimated in the uniform norm by

$$\begin{aligned} &\Big[h^{|\alpha|-\frac{1}{2}-\beta}2^{-k(|\alpha|-1)}+\sum_{1\leq j\leq|\alpha|}h^{|\alpha|-j(\frac{1}{2}-\sigma)}2^{k(j+1-|\alpha|)}h^{-\frac{1}{2}-\beta}\Big] \\ &\qquad\times\Big[\|\widetilde{u}(t,\cdot)\|_{L^2}+\sum_{\mu,|\nu|=0}^{1}\|(\theta\Omega_h)^\mu\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)\widetilde{u}(t,\cdot)\|_{L^2}\Big]\leq CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}. \end{aligned} \tag{B.3.19}$$

Finally, by integrating in $dyd\zeta$ and using (1.2.24) in (B.3.16) we find that $r_N(x,\xi)$ can be written as

$$\begin{aligned} \sum_{j\leq N}h^{N-j(\frac{1}{2}-\sigma)}\frac{1}{(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int_0^1\theta_N(x+tz)(1-t)^{N-1}dt \\ \times\gamma_j\Big(\frac{x|\xi+\eta|-(\xi+\eta)}{h^{1/2-\sigma}}\Big)\phi_k^j(\xi+\eta)b_{j+1-N}(\xi+\eta)dzd\eta, \end{aligned}$$

for some new smooth compactly supported $\theta_N,\gamma_j,\phi_k^j$.From the last part of proposition 1.2.31 then follows that the quantization of the above integral is a bounded operator from L^2 to L^∞, with norm controlled by

$$\sum_{\substack{j\leq N\\ i\leq 6}}h^{N-j(\frac{1}{2}-\sigma)}2^{k(1+j-N)}(h^{-\frac{1}{2}+\sigma}2^k)^i(h^{-1}2^k)\lesssim h$$

if N is sufficiently large (e.g. $N\geq 10$), and consequently that

$$\|\mathrm{Op}_h^w(r_N(x,\xi))\widetilde{u}(t,\cdot)\|_{L^\infty}\lesssim h\|\widetilde{u}^k(t,\cdot)\|_{L^2}\leq CB\varepsilon h^{1-\frac{\delta}{2}}. \tag{B.3.20}$$

Finally, summing up the above estimates with formulas from (B.3.15) to (B.3.19) we obtain that

$$\|\theta_h(x)(|x|^2-1)\widetilde{u}_{\Lambda_w}(t,\cdot)\|_{L^\infty}\lesssim CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}},$$

which injected in (B.3.14) gives that $\theta_h(x)\widetilde{v}_{\Lambda_{kg}}\widetilde{u}_{\Lambda_w}^k$ is a remainder $R(t,x)$. That concludes the proof of the statement. □

A similar result to the one proved in lemma B.3.3 holds true when $\widetilde{u}$ in the left hand side of (B.3.7) is replaced with

$$\widetilde{u}^J(t,x) := t(\Gamma u)_-(t,tx) \tag{B.3.21}$$

with $\Gamma \in \{\Omega, Z_m, m=1,2\}$ being a Klainerman vector field, as briefly shown in the following:

Lemma B.3.4. *Let $h=t^{-1}$, $\widetilde{v}$ be defined in* (3.2.2), *$\widetilde{u}^J$ as in* (B.3.21), *$a_0(\xi) \in S_{0,0}(1)$, and $b_1(\xi)=\xi_j$ or $b_1(\xi)=\xi_j\xi_k|\xi|^{-1}$, with $j,k\in\{1,2\}$. There exists a constant $C>0$ such that, for any $\chi,\chi_1 \in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$, and every $t\in[1,T]$, we have that*

$$\big\|[\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}^J(t,\cdot)]\big\|_{L^2} \leq C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta'}, \tag{B.3.22a}$$

$$\big\|[\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}^J(t,\cdot)]\big\|_{L^\infty} \leq C(A+B)B\varepsilon^2 h^{-\beta'}, \tag{B.3.22b}$$

with $\beta'>0$ small, $\beta\to 0$ as $\sigma,\delta_0\to 0$.

Proof. The proof of this result is analogous to that of lemma B.3.3 except that, instead of referring to (B.3.8), we should use that

$$\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{H_h^{\rho+1,\infty}} + \sum_{|\mu|=1}\|\mathrm{Op}_h^w\big(\chi(h^\sigma\xi))(\xi|\xi|^{-1})^\mu\big)\widetilde{u}^J(t,\cdot)\|_{H_j^{\rho+1,\infty}} \leq CA\varepsilon h^{-\frac{1}{2}-\beta-\frac{\delta_1}{2}}, \tag{B.3.23}$$

which is the semiclassical translation of (B.2.57), and to lemma B.2.9 instead of lemma B.2.1. □

Lemma B.3.5. *Let $a_0(\xi)\in S_{0,0}(1)$, $b_1(\xi)\in\{\xi_j,\xi_j\xi_k|\xi|^{-1},|\xi|,j,k=1,2\}$, $b_0(\xi)\in\{1,\xi_j|\xi|^{-1},j=1,2\}$. There exists a constant $C>0$ such that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, and every $t\in[1,T]$,*

$$\big\|\chi(t^{-\sigma}D_x)\big[[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-\big](t,\cdot)\big\|_{L^\infty} \leq C(A+B)AB\varepsilon^3 t^{-\frac{5}{2}+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.24}$$

with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$. Consequently

$$\big\|\chi(t^{-\sigma}D_x)r_{kg}^{NF}(t,\cdot)\big\|_{L^\infty} \leq C(A+B)AB\varepsilon^3 t^{-\frac{5}{2}+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.25}$$

where r_{kg}^{NF} is given by (B.2.78).

Proof. We warn the reader that we denote by C and β two positive constants that may change line after line during this proof, with $\beta\to 0$ as $\sigma\to 0$. Moreover, we are going to denote generically by $R(t,x)$ each term satisfying

$$\|R(t,\cdot)\|_{L^\infty} \leq C(A+B)AB\varepsilon^3 t^{-\frac{5}{2}+\beta+\frac{\delta+\delta_1}{2}}.$$

From lemma B.2.2 with $L=L^\infty$ and $s>0$ large enough to have $N(s)\geq 3$, and a-priori estimates (1.1.11), we can reduce ourselves to estimate the L^∞ norm of the product in the left hand side of (B.3.24) when all its factors are supported for moderate frequencies less or equal than t^σ, up to remainders $R(t,x)$. Moreover, since

$$\big\|\chi(t^{-\sigma}D_x)a_0(D_x)[v^{NF}-v_-](t,\cdot)\big\|_{L^\infty} \leq CA^2\varepsilon^2 t^{-\frac{3}{2}+\sigma} \tag{B.3.26a}$$

and

$$\big\|\chi(t^{-\sigma}D_x)b_1(D_x)[u^{NF}-u_-](t,\cdot)\big\|_{L^\infty} \leq CA^2\varepsilon^2 t^{-2+\beta}, \tag{B.3.26b}$$

as follows by (B.2.77) and (3.1.15), (A.26b) with $\rho = 2$ (as consequence of lemma A.8), together with a-priori estimates, we can also suppose v_- (resp. u_-) be replaced with v^{NF} (resp. u^{NF}), up to some new $R(t,x)$. This reduces us to prove that

$$\big\|[\chi(t^{-\sigma}D_x)a_0(D_x)v^{NF}][\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}][\chi(t^{-\sigma}D_x)b_0(D_x)u_-](t,\cdot)\big\|_{L^\infty} \\ \leq C(A+B)AB\varepsilon^3 t^{-\frac{5}{2}+\beta+\frac{\delta+\delta_1}{2}},$$

or rather, reminding (1.1.11a), to show that

$$\big\|[\chi(t^{-\sigma}D_x)a_0(D_x)v^{NF}][\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}](t,\cdot)\big\|_{L^\infty} \leq C(A+B)B\varepsilon^2 t^{-2+\beta+\frac{\delta+\delta_1}{2}}.$$

But after writing the above product in the semi-classical setting and reminding definition (3.2.2), one can immediately check that this estimate is satisfied thanks to (B.3.7b), which concludes the proof of (B.3.24).

The last part of the statement follows from (3.1.11), the fact that

$$\left\|\chi(t^{-\sigma}D_x)\left[-\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\,NL_w + D_1\big[\langle D_x\rangle^{-1}(v_+ - v_-)\,NL_w\right](t,\cdot)\right\|_{L^\infty} \leq CA^3\varepsilon^3 t^{-3+\sigma}$$

for every $t \in [1,T]$, which is consequence of (B.1.3b) and a-priori estimate (1.1.11b), and from the observation that the remaining contributions to r_{kg}^{NF} are products of the form

$$[a_0(D_x)v_-][b_1(D_x)u_-]\mathrm{R}_1 u_-,$$

with $a_0(\xi)$ equal to 1 or to $\xi_j\langle\xi\rangle^{-1}$, and $b_1(\xi)$ equal to ξ_1 or to $\xi_j\xi_1|\xi|^{-1}$, for $j = 1,2$. □

Lemma B.3.6. *Under the same assumptions as in lemma B.3.5,*

(B.3.27a)
$$\big\|\chi(t^{-\sigma}D_x)\big[x_n[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-\big](t,\cdot)\big\|_{L^2(dx)} \leq C(A+B)^2B\varepsilon^3 t^{-1+\beta+\frac{\delta_1}{2}},$$

(B.3.27b)
$$\big\|\chi(t^{-\sigma}D_x)\big[x_mx_n[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-\big](t,\cdot)\big\|_{L^2(dx)} \leq C(A+B)^2B\varepsilon^3 t^{\beta+\frac{\delta_1}{2}},$$

for every $t \in [1,T]$, $m,n = 1,2$, with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$. Moreover,

$$\big\|\chi(t^{-\sigma}D_x)\big[x_nr_{kg}^{NF}(t,\cdot)\big]\big\|_{L^2(dx)} \leq C(A+B)^2B\varepsilon^3 t^{-1+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.28a}$$

$$\big\|\chi(t^{-\sigma}D_x)\big[x_mx_nr_{kg}^{NF}(t,\cdot)\big]\big\|_{L^2(dx)} \leq C(A+B)^2B\varepsilon^3 t^{\beta+\frac{\delta+\delta_1}{2}}. \tag{B.3.28b}$$

Proof. We warn the reader that we will denote by C and β two positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$. We also denote by $R(t,x)$ any contribution verifying

$$\big\|\chi(t^{-\sigma}D_x)\big[x_nR(t,\cdot)\big]\big\|_{L^2(dx)} \leq C(A+B)^2B\varepsilon^3 t^{-1+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.29a}$$

$$\big\|\chi(t^{-\sigma}D_x)\big[x_mx_nR(t,\cdot)\big]\big\|_{L^2(dx)} \leq C(A+B)^2B\varepsilon^3 t^{\beta+\frac{\delta+\delta_1}{2}}. \tag{B.3.29b}$$

Let us first notice that, after (B.1.3b), (B.1.10a), (B.1.27a) and a-priori estimates, we have that

$$\left\|\chi(t^{-\sigma}D_x)\left[-x_n\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\,NL_w + x_nD_1\big[\langle D_x\rangle^{-1}(v_+ - v_-)\,NL_w\right](t,\cdot)\right\|_{L^2} \\ \lesssim t^\sigma\sum_{\mu=0}^{1}\|x_n^\mu v_\pm(t,\cdot)\|_{L^2}\,\|NL_w(t,\cdot)\|_{L^\infty} \leq CA^2B\varepsilon^3 t^{-1+\sigma+\frac{\delta}{2}}$$

and

$$\Big\|\chi(t^{-\sigma}D_x)\left[-x_mx_n\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\,NL_w+x_mx_nD_1\big[\langle D_x\rangle^{-1}(v_+-v_-)\,NL_w\right](t,\cdot)\Big\|_{L^2}$$
$$\lesssim t^\sigma\sum_{\mu_1,\mu_2=0}^{1}\|x_m^{\mu_1}x_n^{\mu_2}v_\pm(t,\cdot)\|_{L^2}\,\|NL_w(t,\cdot)\|_{L^\infty}\leq C(A+B)AB\varepsilon^3t^{\sigma+\frac{\delta}{2}}.$$

Therefore, since from (3.1.11) and (B.1.1b) the remaining contributions to r_{kg}^{NF} are of the form

$$[a_0(D_x)v_-][b_1(D_x)u_-]\mathrm{R}_1u_-$$

with $a_0(\xi)$ equal to 1 or to $\xi_j\langle\xi\rangle^{-1}$, and $b_1(\xi)$ equal to ξ_1 or to $\xi_j\xi_1|\xi|^{-1}$, for $j=1,2$, estimates (B.3.28) will follow from (B.3.27). Our aim is hence to prove that the above product is a remainder $R(t,x)$.

Applying lemma B.2.2 with $L=L^2$, $w_1=x_na_0(D_x)v_-$ (resp. $w_1=x_mx_na_0(D_x)v_-$), $s>0$ sufficiently large so that $N(s)>2$, and using estimates (B.1.10a) (resp. (B.1.27a)), (1.1.11a), (1.1.11c), we can suppose all above factors truncated for moderate frequencies less or equal than t^σ, up to remainders $R(t,x)$. Let us also observe that, from (B.1.10b), (B.3.26b) and (1.1.11c),

$$\Big\|\chi(t^{-\sigma}D_x)\Big[[\chi_1(t^{-\sigma}D_x)[x_na_0(D_x)v_-]][\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)][\chi(t^{-\sigma}D_x)b_0(D_x)u_-]\Big](t,\cdot)(t,\cdot)\Big\|_{L^2}$$
$$\lesssim\sum_{|\mu|=0}^{1}\Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty}\|\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2}$$
$$\leq C(A+B)A^2B\varepsilon^4t^{-2+\beta+\frac{\delta+\delta_2}{2}},$$

and that, using additionally estimate (B.1.27b),

$$\Big\|\chi(t^{-\sigma}D_x)\Big[[\chi_1(t^{-\sigma}D_x)[x_mx_na_0(D_x)v_-][\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)][\chi(t^{-\sigma}D_x)b_0(D_x)u_-]\Big](t,\cdot)(t,\cdot)\Big\|_{L^2}$$
$$\lesssim\sum_{|\mu|,|\nu|=0}^{1}\Big\|x_mx_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty}\|\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)\|_{L^2}\|\mathrm{R}^\nu u_\pm(t,\cdot)\|_{L^\infty}$$
$$\leq C(A+B)A^2B\varepsilon^4t^{-1+\beta+\frac{\delta+\delta_2}{2}}.$$

This means that we can actually replace u_- by u^{NF} up to some new $R(t,x)$. Furthermore, we can also substitute $\chi_1(t^{-\sigma}D_x)[x_m^kx_na_0(D_x)v_-]$ with $\chi(t^{-\sigma}D_x)[x_m^kx_na_0(D_x)v_-^{NF}]$, for any $k\in\{0,1\}$, up to a new remainder $R(t,x)$ in consequence of a-priori estimate (1.1.11a), the fact that

$$\|u^{NF}(t,\cdot)\|_{L^2}\leq CB\varepsilon t^{\frac{\delta}{2}},\tag{B.3.30}$$

(see (B.2.1a) in semi-classical coordinates), and the following inequalities

$$\big\|\chi_1(t^{-\sigma}D_x)\left[x_na_0(D_x)(v^{NF}-v_-)\right](t,\cdot)\big\|_{L^\infty}$$
$$\lesssim\sum_{\mu,\nu,\kappa=0}^{1}t^\sigma\Big\|x_n^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\Big\|_{L^\infty}\|\mathrm{R}_1^\kappa u_\pm(t,\cdot)\|_{L^\infty}\leq C(A+B)A\varepsilon^2t^{-\frac12+\sigma+\frac{\delta_2}{2}}\tag{B.3.31a}$$

and

$$\big\|\chi_1(t^{-\sigma}D_x)\left[x_mx_na_0(D_x)(v^{NF}-v_-)\right](t,\cdot)\big\|_{L^\infty}$$
$$\lesssim\sum_{\mu_1,\mu_2,\nu,\kappa=0}^{1}\Big\|x_m^{\mu_1}x_n^{\mu_2}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\Big\|_{L^\infty}\|\mathrm{R}_1^\kappa u_\pm(t,\cdot)\|_{L^\infty}\leq C(A+B)A\varepsilon^2t^{\frac12+\frac{\delta_2}{2}},\tag{B.3.31b}$$

derived from (3.1.10), (B.1.10b), (B.1.27b), (1.1.11a) and (1.1.11b). This reduces us to prove that, for $k=0,1$,

$$\Big\| \big[\chi_1(t^{-\sigma}D_x)[x_m^k x_n a_0(D_x) v_-^{NF}]\big]\big[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}\big]\big[\chi(t^{-\sigma}D_x)b_0(D_x)u_-\big](t,\cdot)\Big\|_{L^2(dx)} \leq C(A+B)^2 B\varepsilon^3 t^{-1+k+\beta+\frac{\delta_1}{2}},$$

or rather, after (1.1.11a), that

$$\big\| \big[\chi_1(t^{-\sigma}D_x)[x_m^k x_n a_0(D_x) v_-^{NF}]\big]\,[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}](t,\cdot)\big\|_{L^2(dx)} \leq C(A+B)B\varepsilon^2 t^{-\frac{1}{2}+k+\beta+\frac{\delta+\delta_1}{2}}.$$

Passing to the semi-classical setting, this corresponds to prove that

(B.3.32)
$$\sum_{k=0}^{1}\Big\| \big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_m^k x_n \mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\,[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}](t,\cdot)\Big\|_{L^2(dx)} \leq C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}}.$$

First of all let us notice that, from the commutation of x_n with $\mathrm{Op}_h^w(a_0(\xi))$ and definition (1.2.68) of $\mathcal{L}_n$,

$$x_n \mathrm{Op}_h^w(a_0(\xi))\widetilde{v} = h\mathrm{Op}_h^w(a_0(\xi))\mathcal{L}_n\widetilde{v} + \mathrm{Op}_h^w\Big(a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v} - \frac{h}{2i}\mathrm{Op}_h^w\big(\partial_{\xi_n}a_0(\xi)\big)\widetilde{v}, \tag{B.3.33}$$

while from the commutation of x_m with $\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))$, definition (1.2.49) of $\mathcal{M}_m$, and symbolic calculus,

(B.3.34)
$$\begin{aligned} x_m \mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} &= h\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi)|\xi|^{-1})\mathcal{M}_m\widetilde{u} - \frac{h}{2i}\mathrm{Op}_h^w\big(\partial_{\xi_m}(\chi(h^\sigma\xi)b_1(\xi)|\xi|^{-1})|\xi|\big)\widetilde{u} \\ &\quad + \mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\widetilde{u} - \frac{h}{2i}\mathrm{Op}_h^w\big(\partial_{\xi_m}(\chi(h^\sigma\xi)b_1(\xi))\big)\widetilde{u}. \end{aligned}$$

On the one hand, using equality (B.3.33), lemma B.2.14, and estimates (B.3.7a), (B.3.8), we deduce that

$$\begin{aligned} &\Big\| [\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)[x_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\,[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}](t,\cdot)\Big\|_{L^2} \\ &\leq \Big\| \Big[\mathrm{Op}_h^w\Big(\chi_1(h^\sigma\xi)a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v}(t,\cdot)\Big][\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\Big\|_{L^2} + CAB\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta_2}{2}} \\ &\leq C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}}. \end{aligned} \tag{B.3.35}$$

On the other hand, when we deal with the L^2 norm in the left hand side of (B.3.32) corresponding to $k=1$ we first commute x_m with $\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))$ and see, using symbolic calculus, that

$$\begin{aligned} &\Big\| [\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_m x_n \mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\,[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}](t,\cdot)\Big\|_{L^2(dx)} \\ &\leq \Big\| [h^\sigma \mathrm{Op}_h^w((\partial\chi_1)(h^\sigma\xi))[x_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\,[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}](t,\cdot)\Big\|_{L^2(dx)} \\ &+ \Big\| [\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\,[x_m\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}](t,\cdot)\Big\|_{L^2(dx)}. \end{aligned}$$

The first norm in the above right hand side satisfies an inequality analogous to (B.3.35). In order to derive an estimate for the latter one, we first use equality (B.3.33) and observe the following: from the semi-classical Sobolev injection and estimates (B.2.1c), (B.2.75b), we have that

$$\begin{aligned} &\big\|h^2\big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\mathcal{L}_n\widetilde{v}\big]\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\mathcal{M}_m\widetilde{u}\big](t,\cdot)\big\|_{L^2} \\ &\lesssim h\,\|\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\mathcal{L}_n\widetilde{v}(t,\cdot)\|_{L^2}\,\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\mathcal{M}_m\widetilde{u}(t,\cdot)\big\|_{L^2} \\ &\leq C(A+B)B\varepsilon^2 h^{1-\delta_2-\beta}; \end{aligned} \tag{B.3.36}$$

a similar chain of inequalities as in (2.2.80), together with (3.1.20a), (3.1.20b) and (1.1.11), gives that for any $\theta \in]0,1[$

$$\begin{aligned}(\text{B.3.37}) \quad \big\|\mathrm{Op}_h^w(b_1(\xi)\xi_m|\xi|^{-1})\widetilde{u}(t,\cdot)\big\|_{L^\infty} &= t\,\big\|b_1(D_x)D_m|D_x|^{-1}u^{NF}(t,\cdot)\big\|_{L^\infty} \\ &\lesssim t\|u^{NF}(t,\cdot)\|_{H^{3,\infty}}^{1-\theta}\|u^{NF}(t,\cdot)\|_{H^2} \leq CA^{1-\theta}B^\theta\varepsilon t^{\frac{1}{2}+\frac{(1+\delta)}{2}\theta}.\end{aligned}$$

Therefore, from equality (B.3.34) and estimates (B.2.75b), (B.3.8), (B.3.37), (B.3.36), we find that

(B.3.38)
$$h\,\big\|\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\mathcal{L}_n\widetilde{v}\,\big[x_m\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\big\|_{L^2} \leq C(A+B)B\varepsilon^2h^{\frac{1}{2}-\frac{\delta_2}{2}-\frac{(1+\delta)\theta}{2}}.$$

Moreover, using again (B.3.34) along with(B.2.1a), (B.2.1c), (B.3.7a) and (B.3.9),

$$\begin{aligned}&\Big\|\Big[\mathrm{Op}_h^w\Big(\chi_1(h^\sigma\xi)a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v}+h\mathrm{Op}_h^w\big(\chi_1(h^\sigma\xi)\partial_{\xi_n}a_0(\xi)\big)\widetilde{v}\Big]\,\big[x_m\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2(dx)}\\ &\leq\Big\|\Big[\mathrm{Op}_h^w\Big(\chi_1(h^\sigma\xi)a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v}\Big]\Big[\mathrm{Op}_h^w\Big(\chi(h^\sigma\xi)b_1(\xi)\frac{\xi_m}{|\xi|}\Big)\widetilde{u}\Big](t,\cdot)\Big\|_{L^2(dx)}+C(A+B)B\varepsilon^2h^{1-\beta-\frac{\delta_2}{2}}\\ &\leq C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}}.\end{aligned}$$

Choosing $\theta \ll 1$ small enough, this concludes that

(B.3.39)
$$\Big\|\,[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_mx_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\,[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}](t,\cdot)\Big\|_{L^2(dx)} \leq C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}}$$

and, together with (B.3.35), the proof of (B.3.32). □

We can finally prove the following:

Lemma B.3.7. *There exists a constant $C>0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, and every $t\in[1,T]$,*

$$(\text{B.3.40}) \qquad \sum_{|\mu|=2}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\beta+\frac{\delta+\delta_1}{2}},$$

with $\beta>0$ small, $\beta\to0$ as $\sigma\to0$.

Proof. From relation (3.2.9b) and the commutation between $\mathcal{L}_m$ and $\mathrm{Op}_h^w(\langle\xi\rangle)$ we deduce that

(B.3.41)
$$\begin{aligned}\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\mathcal{L}_n\widetilde{v}(t,\cdot)\|_{H_h^1} &\lesssim \sum_{\mu=0}^{1}\Big[\,\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m^\mu\big[tZ_nv^{NF}(t,tx)\big]\big\|_{L^2(dx)}\\ &+\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m^\mu\mathrm{Op}_h^w\Big(\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v}(t,\cdot)\Big]\Big\|_{L^2(dx)}+\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m^\mu\big[t(tx_n)r_{kg}^{NF}(t,tx)\big]\big\|_{L^2(dx)}\Big],\end{aligned}$$

so the result of the statement follows from lemmas B.2.14, B.3.2, and inequalities (B.3.6), (B.3.28). □

B.4 The sharp decay estimate of the Klein-Gordon solution with a Klainerman vector field

This last section is devoted to prove that, for any admissible vector field Γ, the $L^\infty(\mathbb{R}^2)$ norm of functions $(\Gamma v)_\pm$, when restricted to moderate frequencies less or equal than t^σ, for some small $\sigma > 0$, decays in time at the same sharp rate t^{-1} of the two-dimensional linear Klein-Gordon solution. This result is proved in lemma B.4.14 under the hypothesis that a-priori estimates (1.1.11) are satisfied in some fixed interval $[1, T]$, with $\varepsilon_0 < (2A + B)^{-1}$ and $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1$ sufficiently small, and is fundamental when proving lemmas 2.1.2 and 2.1.3. All the other lemmas of this section are to be meant as preparatory intermediate results.

Lemma B.4.1. *With the convention that $D = D_1$ whenever $|I_1|+|I_2| = 2$, $D \in \{D_j, D_t, j = 1, 2\}$ otherwise, there exists a positive constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, $n = 1, 2$, and every $t \in [1, T]$,*

$$\sum_{\substack{|I_1|+|I_2|\leq 2\\ |I_1|<2}} \Big\|\chi(t^{-\sigma}D_x)\big[x_n Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2})\big](t,\cdot)\Big\|_{L^2(dx)} \leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_2}{2}}, \tag{B.4.1}$$

with $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$.

Proof. We estimate the L^2 norms in the left hand side of (B.4.1) separately.

• When $|I_1| = 0$, $|I_2| = 2$, we derive from (B.1.10b) and (1.1.11d) that

$$\begin{aligned}\Big\|\chi(t^{-\sigma}D_x)\big[x_n Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm^{I_2})\big](t,\cdot)\Big\|_{L^2(dx)} &\lesssim \sum_{|\mu|=0}^{1} \Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty} \|u_\pm^{I_2}(t,\cdot)\|_{H^1}\\ &\leq C(A+B)B\varepsilon t^{\frac{\delta_1+\delta_2}{2}};\end{aligned}$$

• When $|I_1| = |I_2| = 1$ and $\Gamma^{I_2} \in \{\Omega, Z_m, m = 1, 2\}$ is a Klainerman vector field we use inequalities (B.2.24) with $L = L^2$, $w_{j_0} = x_n(D_x\langle D_x\rangle^{-1})^\mu v_\pm^{I_1}$ with $|\mu| = 0, 1$, and s large enough so that $N(s) \geq 2$, to derive that

$$\begin{aligned}&\Big\|\chi(t^{-\sigma}D_x)\Big[x_n Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2})\Big](t,\cdot)\Big\|_{L^2(dx)}\\ &\lesssim \sum_{|\mu|=0}^{1} \Big\|\chi_1(t^{-\sigma}D_x)\Big[x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm^{I_1}\Big](t,\cdot)\Big\|_{L^\infty} \|u_\pm^{I_2}(t,\cdot)\|_{H^1}\\ &+ \sum_{\substack{|\mu|=0,1,2\\ |\nu|=0,1}} t^{-2}\Big(\|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2} + t\|x^\nu v_\pm^{I_1}(t,\cdot)\|_{L^2}\Big)\big(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\big)\\ &\leq CB^2\varepsilon^2 t^{\frac{\delta_1+\delta_2}{2}},\end{aligned}$$

where last estimate is deduced using (B.1.5a), (B.1.17), (B.1.28), (B.2.62) and (1.1.11d);

• When $|I_1| = |I_2| = 1$ and Γ^{I_2} is a spatial derivative we use lemma B.2.2 with $L = L^2$, $w_1 = x_n(D_x\langle D_x\rangle^{-1})^\mu v_\pm^{I_1}$ with $|\mu| = 0, 1$, s large enough so that $N(s) \geq 1$, and again estimates

(B.1.17), (B.2.62) and (1.1.11d). We obtain that

$$\begin{aligned}
&\Big\|\chi(t^{-\sigma}D_x)\Big[x_nQ_0^{\mathrm{kg}}(v_\pm^{I_1},D_1u_\pm^{I_2})\Big](t,\cdot)\Big\|_{L^2(dx)}\\
&\lesssim\sum_{|\mu|=0}^{1}\Big\|\chi_1(t^{-\sigma}D_x)\Big[x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm^{I_1}\Big](t,\cdot)\Big\|_{L^\infty}\|u_\pm(t,\cdot)\|_{H^2}\\
&+\sum_{|\mu|=0}^{1}t^{-1}\|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\leq CB^2\varepsilon^2t^{\frac{\delta+\delta_1}{2}};
\end{aligned}$$

• When $|I_1|+|I_2|\leq 1$, by the assumption derivative D can be equal to D_x or to D_t. Then
- If $|I_1|=0$, after (B.1.5a), (B.1.7), (B.1.10b) and (1.1.11)

$$\begin{aligned}
&\Big\|\chi(t^{-\sigma}D_x)\big[x_nQ_0^{\mathrm{kg}}(v_\pm,Du_\pm^{I_2})\big](t,\cdot)\Big\|_{L^2(dx)}\\
&\lesssim\sum_{|\mu|=0}^{1}\Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\Big\|_{L^\infty}\Big(\|u_\pm^{I_2}(t,\cdot)\|_{H^1}+\|D_tu_\pm^{I_2}(t,\cdot)\|_{L^2}\Big)\leq C(A+B)B\varepsilon^2t^{\delta_2};
\end{aligned}$$

- If $|I_1|=1$, $|I_2|=0$, using lemma B.2.2 as done above, together with (B.1.5a), (B.1.10), (B.1.17), (B.2.62) and a-priori estimates, we derive that

$$\begin{aligned}
&\Big\|\chi(t^{-\sigma}D_x)\big[x_nQ_0^{\mathrm{kg}}(v_\pm^{I_1},Du_\pm)\big](t,\cdot)\Big\|_{L^2(dx)}\\
&\lesssim\sum_{|\mu|=0}^{1}\Big\|\chi_1(t^{-\sigma}D_x)\Big[x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm^{I_1}(t,\cdot)\Big]\Big\|_{L^\infty}(\|u_\pm(t,\cdot)\|_{H^1}+\|D_tu_\pm(t,\cdot)\|_{L^2})\\
&+\sum_{|\mu|=0}^{1}t^{-1}\Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm^{I_1}(t,\cdot)\Big\|_{L^2}(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s})\\
&\leq CB^2\varepsilon^2t^{\beta+\frac{\delta+\delta_1}{2}}.
\end{aligned}\tag{B.4.2}$$

□

Lemma B.4.2. *There exists a positive constant $C>0$ such that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, $\rho\in\mathbb{N}$, and every $t\in[1,T]$,*

$$\sum_{|I|=2}\big\|\chi(t^{-\sigma}D_x)V^I(t,\cdot)\big\|_{H^{\rho,\infty}}\leq CB\varepsilon t^{-1+\beta+\frac{\delta_0}{2}},\tag{B.4.3}$$

with $\beta>0$ small such that $\beta\to 0$ as $\sigma\to 0$.

Proof. Estimate (B.4.3) is evidently satisfied when Γ^I contains at least one spatial derivative thanks to lemma B.2.8. We then focus on the case when Γ^I is the product of two Klainerman vector fields. As $v_+^I=-\overline{v_-^I}$, we prove the statement for $\chi(t^{-\sigma}D_x)v_-^I$. Moreover, from the $L^\infty-H^{\rho,\infty}$ continuity of $\chi(t^{-\sigma}D_x)$ with norm $O(t^{\sigma\rho})$, we can assume the $H^{\rho,\infty}$ norm in (B.4.3) replaced with the L^∞ one, up to a loss $t^{\sigma\rho}$.

As done in lemma B.2.8, instead of proving the statement directly on $\chi(t^{-\sigma}D_x)v_-^I$ we do it for $\chi(t^{-\sigma}D_x)v^{I,NF}$, with $v^{I,NF}$ introduced in (B.2.27) and considered here with $|I|=2$. This

is justified by inequality (B.2.43). From definition (B.2.27) of $v^{I,NF}$, equation (B.1.18a) and equality (B.1.23) one can check that

$$[D_t + \langle D_x\rangle]v^{I,NF} = NL_{kg}^{I,NF}$$
$$\text{where } NL_{kg}^{I,NF} = r_{kg}^{I,NF}(t,x) + \sum_{\substack{(I_1,I_2)\in\mathcal{J}(I)\\ |I_1|<2}} c_{I_1,I_2} Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2}), \tag{B.4.4}$$

with $r_{kg}^{I,NF}$ given by the same integral expression as in (B.2.46) but with $|I| = 2$ (and hence having the explicit expression (B.2.47)), and $c_{I_1,I_2} \in \{-1,0,1\}$, $c_{I_1,I_2} = 1$ when $|I_1| + |I_2| = 2$ (in which case derivative D corresponds to D_1). It is straightforward to show that inequalities (B.2.48), (B.2.49), (B.2.50) hold even when $|I| = 2$, up to replacing δ_2 with δ_1. Therefore, using those latter ones together with

$$\sum_{j=1}^{2} \left\|\chi(t^{-\sigma}D_x) Z_j v^{I,NF}(t,\cdot)\right\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_0}{2}},$$

which is consequence of (1.1.11d) with $k = 0$ and of (B.2.29) with $j = 2$, we derive that

$$\left\|\chi(t^{-\sigma}D_x) v^{I,NF}(t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon t^{\frac{\delta_0}{2}} + \sum_{j=1}^{2} Ct^{-1+\beta} \left\|\chi(t^{-\sigma}D_x)\left[x_j NL_{kg}^{I,NF}\right](t,\cdot)\right\|_{L^2(dx)}.$$

The only thing we need to show in order to prove the statement is hence that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j NL_{kg}^{I,NF}\right](t,\cdot)\right\|_{L^2(dx)} \leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta_1+\delta_2}{2}}. \tag{B.4.5}$$

But from (B.4.4) and (B.2.47) with $|I| = 2$ we have that

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_j NL_{kg}^{I,NF}\right](t,\cdot)\right\|_{L^2(dx)} \lesssim \|x_j NL_{kg}^{I}(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{L^\infty} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{L^\infty}\right)\\
&+ \sum_{\mu=0}^{1} t^\sigma \left(\|x_j^\mu v_\pm(t,\cdot)\|_{L^\infty} + \left\|x_j^\mu \frac{D_x}{\langle D_x\rangle} v_\pm(t,\cdot)\right\|_{L^\infty}\right)\|v_\pm^I(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+ \sum_{\substack{(I_1,I_2)\in\mathcal{J}(I)\\ |I_1|<2}} \left\|\chi(t^{-\sigma}D_x)\left[x_j Q_0^{\mathrm{kg}}(v_\pm^{I_1}, Du_\pm^{I_2})\right](t,\cdot)\right\|_{L^2}
\end{aligned} \tag{B.4.6}$$

so (B.4.5) follows from a-priori estimates, (B.1.10b), (B.2.74) and (B.4.1). As $\delta_2 \ll \delta_1 \ll \delta_0$, that concludes that

$$\left\|\chi(t^{-\sigma}D_x) v^{I,NF}(t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon t^{-1+\beta+\frac{\delta_0}{2}}. \tag{B.4.7}$$

□

Lemma B.4.3. *There exists a positive constant $C > 0$ such that, for any multi-index I of length 2, any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, $j = 1,2$, and every $t \in [1,T]$*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j(\Gamma^I v)_\pm\right](t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon t^{\beta+\frac{\delta_0}{2}}, \tag{B.4.8}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. If Γ^I contains at least one spatial derivative (B.4.8) is satisfied after (1.1.11b), (B.1.10b) and (B.2.62). Let us then assume that Γ^I is product of two Klainerman vector fields.

From equation (B.1.18a), equality (B.1.9b) with $w = (\Gamma^I v)_-$, the $L^2 - L^\infty$ continuity of operator $\chi(t^{-\sigma}D_x)\langle D_x\rangle^{-1}$ with norm $O(t^\sigma)$, and the L^∞ continuity of $\chi(t^{-\sigma}D_x)D_x\langle D_x\rangle^{-1}$ with norm $O(t^\sigma)$, we derive that

(B.4.9)
$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)\left[x_j(\Gamma^I v)_-\right](t,\cdot)\right\|_{L^\infty(dx)} \lesssim t^\sigma \|Z_j(\Gamma^I v)_-(t,\cdot)\|_{L^2} + t\left\|\chi(t^{-\sigma}D_x)(\Gamma^I v)_-(t,\cdot)\right\|_{L^\infty} \\ + t^\sigma\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty}.\end{aligned}$$

Reminding (B.1.23) and applying lemma B.2.2 with $L = L^\infty$ and $w_1 = (D_x\langle D_x\rangle^{-1})^\mu v_\pm^I$, for $|\mu| = 0, 1$, to the contribution coming from the first quadratic term in the right hand side of (B.1.23), we find that there is some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that

(B.4.10)
$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty} &\lesssim \sum_{\mu,\nu=0}^{1}\left\|\chi_1(t^{-\sigma}D_x)\left[x_j^\mu(\Gamma^I v)_\pm\right](t,\cdot)\right\|_{L^\infty}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}} \\ &+ t^{-N(s)}\sum_{\mu=0}^{1}\left\|x_j^\mu(\Gamma^I v)_\pm(t,\cdot)\right\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s} \\ &+ \sum_{\substack{(I_1,I_2)\in\mathcal{J}(I)\\ |I_1|<2}}\left\|\chi(t^{-\sigma}D_x)\left[x_jQ_0^{\mathrm{kg}}\left(v_\pm^{I_1},Du_\pm^{I_2}\right)\right](t,\cdot)\right\|_{L^\infty}.\end{aligned}$$

Therefore, picking $s > 0$ large so that $N(s) > 1$ and using the $L^2 - L^\infty$ continuity of $\chi_1(t^{-\sigma}D_x)$ with norm $O(t^\sigma)$, together with the estimates (1.1.11), (B.1.17) with $k = 2$, along with (B.4.1), we find at first that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty} \leq CAB\varepsilon^2 t^{\frac{1}{2}+\sigma+\frac{\delta_1}{2}}.$$

Injecting the above estimate, together with (1.1.11d) and (B.4.3), into (B.4.9) we derive that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j(\Gamma^I v)_-\right](t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon t^{\frac{1}{2}+\sigma+\frac{\delta_1}{2}}.$$

The above inequality holds for any $\chi \in C_0^\infty(\mathbb{R}^2)$, so injecting it into (B.4.10) and using again a-priori estimates, (B.1.17), (B.4.1), together with the fact that $\beta + (\delta+\delta_2)/2 \leq \delta_1/2$ as β is as small as we want as long as σ is small and $\delta, \delta_2 \ll \delta_1$, we find the following enhanced estimate

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty} \leq C(A+B)B\varepsilon^2 t^{\sigma+\frac{\delta_1}{2}}.$$

Consequently, summing up this latter one with (1.1.11d) and (B.4.3), we end up with (B.4.8). □

Lemma B.4.4. *Let $\Gamma \in \mathcal{Z}$ be an admissible vector field. There exists a positive constant C such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, $i, j = 1, 2$, and every $t \in [1, T]$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_ix_j(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty(dx)} \leq CB\varepsilon t^{1+\beta+\frac{\delta_1}{2}}, \tag{B.4.11}$$

with $\beta > 0$ such that $\beta \to 0$ as $\sigma \to 0$.

Proof. Since $(\Gamma v)_+ = -\overline{(\Gamma v)_+}$ we reduce to prove that inequality (B.4.11) holds true for $(\Gamma v)_-$. Moreover, we only focus on the case where $\Gamma \in \{\Omega, Z_m, m = 1, 2\}$ is a Klainerman vector field, as (B.4.11) with Γ being a spatial derivative is simply a consequence of (B.1.27b).

We remind that $(\Gamma v)_-$ is solution to non-linear Klein-Gordon equation (B.1.18a) with $\Gamma^I = \Gamma$, and that the non-linearity ΓNL_{kg} is given by (B.1.20a). Hence, multiplying x_i to relation (B.1.9b) with $w = (\Gamma v)_-$ and making some commutations we find that

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_i x_j(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty(dx)}\\
&\lesssim \sum_{\mu=0}^{1}\Big[\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu Z_j(\Gamma v)_-\right](t,\cdot)\right\|_{L^\infty(dx)} + t\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu(\Gamma v)_-\right](t,\cdot)\right\|_{L^\infty(dx)}\Big]\\
&+ \sum_{\mu=0}^{1}\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu x_j\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^\infty(dx)}.
\end{aligned} \tag{B.4.12}$$

At first, we estimate the latter contribution in the above right hand side using that $\chi(t^{-\sigma}D_x)$ is a continuous $L^2 - L^\infty$ operator with norm $O(t^\sigma)$ together with estimates (1.1.11), (B.1.5a) with $s = 0$, (B.1.10b), (B.1.17), (B.1.27b), (B.1.28):

(B.4.13)

$$\begin{aligned}
&\sum_{\mu=0}^{1}\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu x_j\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^\infty(dx)} \lesssim \sum_{\mu=0}^{1} t^\sigma\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu x_j\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^2(dx)}\\
&\lesssim \sum_{\mu_1,\mu_2,\nu=0}^{1} t^\sigma\|x_i^{\mu_1}x_j^{\mu_2}(\Gamma v)_\pm(t,\cdot)\|_{L^2(dx)}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+ \sum_{\mu,|\nu|=0}^{1} t^\sigma\left\|x_i^\mu x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty(dx)}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1} + \|u_\pm(t,\cdot)\|_{H^1} + \|D_t u_\pm(t,\cdot)\|_{L^2}\right)\\
&\leq C(A+B)B\varepsilon^2 t^{\frac{3}{2}+\sigma+\frac{\delta_2}{2}}.
\end{aligned}$$

Injecting this estimate, along with (B.2.42), (B.2.62), (B.4.3) and (B.4.8), into (B.4.12) we deduce that for any smooth cut-off function χ

$$\left\|\chi(t^{-\sigma}D_x)\left[x_i x_j(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty(dx)} \leq CB\varepsilon t^{\frac{3}{2}+\sigma+\frac{\delta_2}{2}}. \tag{B.4.14}$$

Now, if we change the approach of bounding the $L^\infty(dx)$ norm of $x_i^\mu x_j Q_0^{\mathrm{kg}}((\Gamma v)_-, D_1 u_\pm)$, which is one of the contributions to $x_i^\mu x_j\Gamma NL_{kg}$ after (B.1.20a), and make use of lemma B.2.2 with $L = L^\infty$ instead of (B.4.13), we see that

$$\begin{aligned}
&\sum_{\mu=0}^{1}\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu x_j\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^\infty(dx)}\\
&\lesssim \sum_{\mu_1,\mu_2,\nu=0}^{1}\left\|\chi_1(t^{-\sigma}D_x)\left[x_i^{\mu_1}x_j^{\mu_2}(\Gamma v)_\pm\right](t,\cdot)\right\|_{L^\infty(dx)}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+ \sum_{\mu_1,\mu_2=0}^{1} t^{-N(s)}\|x_i^{\mu_1}x_j^{\mu_2}(\Gamma v)_\pm(t,\cdot)\|_{L^2(dx)}\|u_\pm(t,\cdot)\|_{H^s}\\
&+ \sum_{\mu,|\nu|=0}^{1} t^\sigma\left\|x_i^\mu x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty(dx)}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1} + \|u_\pm(t,\cdot)\|_{H^1} + \|D_t u_\pm(t,\cdot)\|_{L^2}\right).
\end{aligned}$$

Then, choosing $s > 0$ sufficiently large so that $N(s) \geq 3$ and using again (1.1.11), (B.1.10b), (B.1.17) with $k = 1$, (B.1.27b), (B.1.28), (B.2.62), together with (B.4.14), we obtain that

$$\sum_{\mu=0}^{1} \left\| \chi(t^{-\sigma} D_x) \left[x_i^\mu x_j \Gamma N L_{kg} \right](t,\cdot) \right\|_{L^\infty(dx)} \leq C(A+B) B \varepsilon^2 t^{1+\sigma+\frac{\delta_2}{2}},$$

which enhances (B.4.13) of a factor $t^{1/2}$. Combining the above estimate with (B.2.42), (B.2.62), (B.4.3) and (B.4.8), we finally end up with the result of the statement. □

Lemma B.4.5. *Let $\Gamma \in \{\Omega, Z_m, m = 1,2\}$ be a Klainerman vector field, $v^{I,NF}$ the function defined in* (B.2.27) *with $|I| = 1$ and $\Gamma^I = \Gamma$, and $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ the multiplier introduced in* (2.2.42) *(resp. in* (2.2.53)*) for any $k = 1,2$ (resp. $k = 3$), any $j_i \in \{+,-\}$ for $i = 1,2,3$. Let us define*

(B.4.15)

$$\begin{aligned} V_\Gamma^{NF}(t,x) := v^{I,NF}(t,x) &- \frac{i}{4(2\pi)^2} \sum_{j_1,j_2 \in \{+,-\}} \int e^{ix\cdot\xi} B^1_{(j_1,j_2,+)}(\xi,\eta) \hat{v}_{j_1}(\xi-\eta) \widehat{(\Gamma u)_{j_2}}(\eta) d\xi d\eta \\ &+ \delta_\Omega \frac{i}{4(2\pi)^2} \sum_{j_1,j_2 \in \{+,-\}} \int e^{ix\cdot\xi} B^2_{(j_1,j_2,+)}(\xi,\eta) \hat{v}_{j_1}(\xi-\eta) \hat{u}_{j_2}(\eta) d\xi d\eta \\ &+ \delta_{Z_1} \frac{i}{4(2\pi)^2} \sum_{j_1,j_2 \in \{+,-\}} \int e^{ix\cdot\xi} B^3_{(j_1,j_2,+)}(\xi,\eta) \hat{v}_{j_1}(\xi-\eta) \hat{u}_{j_2}(\eta) d\xi d\eta, \end{aligned}$$

where δ_Ω (resp. δ_{Z_1}) is equal to 1 if $\Gamma = \Omega$ (resp. if $\Gamma = Z_1$), 0 otherwise. There exists a constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$, and every $t \in [1,T]$,

$$\left\| \chi(t^{-\sigma} D_x) \big(V_\Gamma^{NF} - (\Gamma v)_- \big)(t,\cdot) \right\|_{L^\infty} \leq C(A+B) A \varepsilon^2 t^{-\frac{5}{4}}. \tag{B.4.16}$$

Moreover, for every $m = 1,2$ and $t \in [1,T]$

$$\left\| |\chi(t^{-\sigma} D_x) Z_m \big(V_\Gamma^{NF} - (\Gamma v)_- \big)(t,\cdot) \right\|_{L^2} \leq C(A+B) B \varepsilon^2 t^{3\sigma+\delta_2}. \tag{B.4.17}$$

Proof. From definition (B.4.15) of V_Γ^{NF} and equalities (A.15), (A.16), we find that

$$\begin{aligned} V_\Gamma^{NF} - (\Gamma v)_- = v^{I,NF} &- (\Gamma v)_- \\ &- \frac{i}{2} \left[(D_t v)(D_1 \Gamma u) - (D_1 v)(D_t \Gamma u) + D_1[v(D_t \Gamma u)] - \langle D_x \rangle [v(D_1 \Gamma u)] \right] \\ &+ \delta_\Omega \frac{i}{2} \left[(D_t v)(D_2 u) - (D_2 v)(D_t u) + D_2[v D_t u] - \langle D_x \rangle [v D_2 u] \right] \\ &+ \delta_{Z_1} \frac{i}{2} \left[(D_t v)(D_t u) + v(|D_x|^2 u) - \langle D_x \rangle [v(D_t u)] \right], \end{aligned} \tag{B.4.18}$$

where $v^{I,NF} - (\Gamma v)_-$ has the explicit expression (B.2.30). We use (1.1.5), (1.1.10) and lemma B.2.2 with $L = L^\infty$, $w_1 = \mathrm{R}_1^\mu u_\pm$ (resp. $w_1 = \mathrm{R}_1^\mu (\Gamma u)_\pm$) for $\mu = 0,1$, and $s > 0$ large enough to have $N(s) \geq 2$, in order to estimate the L^∞ norm of products appearing in (B.2.30) (resp. in the second line in the above right hand side). For some new $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ we have that

$$\begin{aligned} \left\| \chi(t^{-\sigma} D_x) \big(V_\Gamma^{NF} - (\Gamma v)_- \big)(t,\cdot) \right\|_{L^\infty} &\lesssim \left\| \chi(t^{-\sigma} D_x) \big(v^{I,NF} - (\Gamma v)_- \big)(t,\cdot) \right\|_{L^\infty} \\ &+ \sum_{|\mu|=0}^{1} t^\sigma \|v_\pm(t,\cdot)\|_{H^{1,\infty}} \left\| \chi_1(t^{-\sigma} D_x) \mathrm{R}_1^\mu (\Gamma u)_\pm(t,\cdot) \right\|_{L^\infty} + t^{-2} \|v_\pm(t,\cdot)\|_{H^s} \|(\Gamma u)_\pm(t,\cdot)\|_{L^2} \\ &+ \sum_{|\mu|=0}^{1} t^\sigma \|v_\pm(t,\cdot)\|_{H^{1,\infty}} \|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}, \end{aligned}$$

with

$$\big\|\chi(t^{-\sigma}D_x)\big(v^{I,NF}-(\Gamma v)_-\big)(t,\cdot)\big\|_{L^\infty}\lesssim\sum_{\mu=0}^{1}t^\sigma\,\big\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\big\|_{L^\infty}\,\|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty}\\+t^{-2}\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}.$$

Estimate (B.4.16) follows then from (1.1.11), (B.2.42) and (B.2.57).

In order to derive (B.4.17) we apply Z_m to (B.4.18) and use the Leibniz rule, reminding formulas (B.2.39). Among the quadratic terms coming out from the action of Z_m on the second line in (B.4.18) we see appear products where Z_m is acting on v and Γ on u. We estimate the L^2 norm of those ones, when truncated by operator $\chi(t^{-\sigma}D_x)$, using inequalities (B.2.24) with $L=L^2$, $w=u$, $w_{j_0}=(D_x\langle D_x\rangle^{-1})^\mu Z_m v$ for $|\mu|=0,1$, and $s>0$ large enough to have $N(s)>1$. We bound instead the L^2 norm of all other remaining products with the L^∞ norm of factor that does not contain any vector field times the L^2 norm of the remaining one. Hence

(B.4.19)

$$\begin{aligned}&\big\|\chi(t^{-\sigma}D_x)Z_m\big(V_\Gamma^{NF}-(\Gamma v)_-\big)(t,\cdot)\big\|_{L^2}\lesssim\big\|\chi(t^{-\sigma}D_x)Z_m\big(v^{I,NF}-(\Gamma v)_-\big)(t,\cdot)\big\|_{L^2}\\&+t^\sigma\,\big\|\chi_1(t^{-\sigma}D_x)(Z_m v)_\pm(t,\cdot)\big\|_{L^\infty}\,\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}\\&+t^{-N(s)}\Big(\sum_{|\mu|=0}^{1}\|x^\mu(Z_m v)_\pm(t,\cdot)\|_{L^2}+t\|(Z_m v)_\pm(t,\cdot)\|_{L^2}\Big)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right)\\&+t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(Z_m\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|D_t(\Gamma u)_\pm(t,\cdot)\|_{L^2}\right)\\&+\sum_{|\mu|=0}^{1}t^\sigma\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}\\&+t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(Z_m u)_\pm(t,\cdot)\|_{H^1}+\|u_\pm(t,\cdot)\|_{H^1}+\|D_t u_\pm(t,\cdot)\|_{L^2}\right),\end{aligned}$$

and estimate (B.4.17) is obtained from (1.1.11), (B.1.5a), (B.1.7), (B.1.17), (B.2.29) with $j=1$, and (B.2.42). □

Lemma B.4.6. *Let $\Gamma\in\{\Omega,Z_m,m=1,2\}$ be a Klainerman vector field, V_Γ^{NF} the function defined in* (B.4.15) *and*

$$\widetilde{V}^\Gamma(t,x):=tV_\Gamma^{NF}(t,tx).\tag{B.4.20}$$

There exists a positive constant $C>0$ such that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, and every $t\in[1,T]$,

$$\Big\|\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^2}\le CB\varepsilon t^{\frac{\delta_2}{2}},\tag{B.4.21a}$$

$$\sum_{|\mu|=1}\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^2}\le CB\varepsilon t^{\frac{\delta_1}{2}}.\tag{B.4.21b}$$

Proof. Let us recall equalities (B.2.30) with $\Gamma^I=\Gamma$ and (B.4.18). From a-priori estimates we immediately derive that, for every $t\in[1,T]$,

$$\|[V_\Gamma^{NF}-(\Gamma v)_-](t,\cdot)\|_{L^2}\le CAB\varepsilon t^{-\frac12+\frac{\delta_2}{2}+\sigma},$$

and consequently that

$$\|\widetilde{V}^\Gamma(t,\cdot)\|_{L^2}=\|V_\Gamma^{NF}(t,\cdot)\|_{L^2}\le CB\varepsilon t^{\frac{\delta_2}{2}}.\tag{B.4.22}$$

Using definition (B.4.15) one can check that V_Γ^{NF} is solution to

$$[D_t + \langle D_x\rangle]V_\Gamma^{NF}(t,x) = NL_\Gamma^{kg,c}(t,x) - \delta_{Z_1} Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\big) \tag{B.4.23}$$

with

$$\begin{aligned}
NL_\Gamma^{kg,c}(t,x) &= r_{kg}^{I,NF}(t,x) \\
&\quad - \frac{i}{4(2\pi)^2}\int e^{ix\cdot\xi} B^1_{(j_1,j_2,+)}(\xi,\eta)\left[\widehat{NL_{kg}}(\xi-\eta)\widehat{(\Gamma u)_{j_2}}(\eta) - \hat v_{j_1}(\xi-\eta)\widehat{\Gamma NL_w}(\eta)\right]d\xi d\eta \\
&\quad + \delta_\Omega \frac{i}{4(2\pi)^2}\int e^{ix\cdot\xi} B^2_{(j_1,j_2,+)}(\xi,\eta)\left[\widehat{NL_{kg}}(\xi-\eta)\hat u_{j_2}(\eta) - \hat v_{j_1}(\xi-\eta)\widehat{NL_w}(\eta)\right]d\xi d\eta \\
&\quad + \delta_{Z_1} \frac{i}{4(2\pi)^2}\int e^{ix\cdot\xi} B^3_{(j_1,j_2,+)}(\xi,\eta)\left[\widehat{NL_{kg}}(\xi-\eta)\hat u_{j_2}(\eta) - \hat v_{j_1}(\xi-\eta)\widehat{NL_w}(\eta)\right]d\xi d\eta,
\end{aligned} \tag{B.4.24}$$

and $r_{kg}^{I,NF}$ given by (B.2.46) (or, explicitly, by (B.2.47)) with $|I| = 1$. Superscript c in $NL_\Gamma^{kg,c}$ stands for *cubic* and wants to stress out the fact that, passing from function $(\Gamma v)_-$ to V_Γ^{NF}, we have replaced all quadratic terms in the right hand side of (B.1.18a) (when $|I| = 1$ and $\Gamma^I = \Gamma$) with cubic ones. Hence, from relation (3.2.8) with $w = V_\Gamma^{NF}$ and equation (B.4.23) we get that

$$\begin{aligned}
&\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\widetilde V^\Gamma(t,\cdot)\right\|_{H^1} \lesssim \|\chi(t^{-\sigma}D_x)Z_m V_\Gamma^{NF}(t,\cdot)\|_{L^2} + \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi)\xi_m\langle\xi\rangle^{-1})\widetilde V^\Gamma(t,\cdot)\right\|_{L^2} \\
&+ \left\|\chi(t^{-\sigma}D_x)\left[x_m NL_\Gamma^{kg,c}\right](t,\cdot)\right\|_{L^2(dx)} + \delta_{Z_1}\left\|\chi(t^{-\sigma}D_x)\left[x_m Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\big)\right](t,\cdot)\right\|_{L^2(dx)}
\end{aligned} \tag{B.4.25}$$

After (1.1.11d) with $k = 1$,(B.4.17), and the fact that σ can be chosen sufficiently small so that $3\sigma + \delta_2 \le \delta_1/2$, as $\delta_2 \ll \delta_1$, it is straightforward to see that

$$\|\chi(t^{-\sigma}D_x)Z_m V_\Gamma^{NF}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta_1}{2}}. \tag{B.4.26}$$

Moreover, from (B.1.3a), (B.1.10b) and a-priori estimates,

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_m Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\big)\right](t,\cdot)\right\|_{L^2(dx)} \\
&\qquad \lesssim \sum_{|\mu|=0}^{1}\left\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty(dx)}\|NL_w(t,\cdot)\|_{L^2} \le C(A+B)AB\varepsilon^3 t^{-1+\frac{\delta+\delta_2}{2}}.
\end{aligned} \tag{B.4.27}$$

Using instead equalities (A.15) and (A.16) we derive the following explicit expression for $NL_\Gamma^{kg,c}$:

$$\begin{aligned}
NL_\Gamma^{kg,c}(t,x) &= r_{kg}^{I,NF}(t,x) - \frac{i}{2}\left[NL_{kg}(D_1\Gamma u) - (D_1 v)\Gamma NL_w + D_1[v\Gamma NL_w]\right] \\
&\quad + \delta_\Omega\frac{i}{2}\left[NL_{kg}(D_2 u) - (D_2 v)NL_w + D_2[vNL_w]\right] \\
&\quad + \delta_{Z_1}\left[NL_{kg}(D_t u) + (D_t v)NL_w - \langle D_x\rangle[vNL_w]\right].
\end{aligned} \tag{B.4.28}$$

Hence, reminding estimates (1.1.11), (B.1.3a), (B.1.5a) with $s = 0$, (B.1.10b), (B.2.72), and equality (B.1.20b) from which follows that

$$\|\Gamma NL_w(t,\cdot)\|_{L^2} \lesssim \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|v_\pm^I(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1} + \|D_t v_\pm(t,\cdot)\|_{L^2}\right), \tag{B.4.29}$$

we find that

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_j NL_\Gamma^{kg,c}\right](t,\cdot)\right\|_{L^2(dx)} \lesssim \left\|\chi(t^{-\sigma}D_x)\left[x_j r_{kg}^{I,NF}(t,\cdot)\right](t,\cdot)\right\|_{L^2(dx)} \\
&+\sum_{|\mu|,\nu=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\right\|_{L^\infty(dx)}\|\mathrm{R}^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|u_\pm(t,\cdot)\|_{L^2}\right) \\
&+\sum_{k,|\mu|=0}^{1}\left\|x_j^k\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\right\|_{L^\infty(dx)}\left(\|\Gamma NL_w(t,\cdot)\|_{L^2}+\|NL_w(t,\cdot)\|_{L^2}\right) \\
&\leq C(A+B)AB\varepsilon^2 t^{-\frac{1}{2}+\beta+\frac{\delta+\delta_1}{2}}.
\end{aligned}
\tag{B.4.30}
$$

By injecting the above estimate, together with (B.4.21a), (B.4.26), (B.4.27), into (B.4.25) we finally deduce (B.4.21b) and conclude the proof of the statement. □

Lemma B.4.7. *Let $\Gamma \in \{\Omega, Z_m, m=1,2\}$ be a Klainerman vector field and I_1, I_2 two multi-indices such that $\Gamma^{I_1}=\Gamma$ and $\Gamma^{I_2}=Z_m\Gamma$, with $m\in\{1,2\}$. Let also $v^{I,NF}$ be the function defined in* (B.2.27) *for a generic multi-index I of length equal to 1 or 2. There exists a constant $C>0$ such that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, $m,n=1,2$, every $t\in[1,T]$,*

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}+\left\|\chi(t^{-\sigma}D_x)\left(v^{I_2,NF}-(Z_m\Gamma v)_-\right)(t,\cdot)\right\|_{L^2} \\
&\qquad\leq C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1+\delta_2}{2}}
\end{aligned}
\tag{B.4.31a}
$$

and

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}+\left\|\chi(t^{-\sigma}D_x)\left[x_n\left(v^{I_2,NF}-(Z_m\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2} \\
&\qquad\leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_1+\delta_2}{2}},
\end{aligned}
\tag{B.4.31b}
$$

with $\beta>0$ small such that $\beta\to 0$ as $\sigma\to 0$. Moreover, if V_Γ^{NF} is the function defined in (B.4.15)*, then for every $t\in[1,T]$*

$$
\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(V_\Gamma^{NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\leq C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1+\delta_2}{2}},
\tag{B.4.32a}
$$

$$
\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(V_\Gamma^{NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_1+\delta_2}{2}}.
\tag{B.4.32b}
$$

Proof. We warn the reader that throughout the proof we denote by C and β two positive constants that may change line after line, with $\beta\to 0$ as $\sigma\to 0$.

We refer to equality (B.2.40) with $I=I_1$ and bound the L^2 norm of each product in the first, third and fifth line of its right hand side by means of lemma B.2.2 with $L=L^2$. The L^2 norm of the remaining products in the second line of the mentioned equality is instead estimated using inequalities (B.2.24) with $L=L^2$ and $w_{j_0}=(D_x\langle D_x\rangle^{-1})^\mu(\Gamma^{I_1}v)_\pm$. In this way we obtain that there is some $\chi_1\in C_0^\infty(\mathbb{R}^2)$ such that

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\lesssim t^\sigma\|\chi_1(t^{-\sigma}D_x)(Z_m\Gamma v)_\pm(t,\cdot)\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2} \\
&+t^{-N(s)}\|(Z_m\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}+t^\sigma\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{L^\infty}\|(Z_m u)_\pm(t,\cdot)\|_{L^2} \\
&+t^{-N(s)}\left(\sum_{\mu=0}^{1}\|x_m^\mu(\Gamma v)_\pm(t,\cdot)\|_{L^2}+t\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right) \\
&+t^\sigma\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{L^\infty}\left(\|u_\pm(t,\cdot)\|_{L^2}+\|D_t u_\pm(t,\cdot)\|_{L^2}\right) \\
&+t^{-N(s)}\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right).
\end{aligned}
$$

Choosing $s>0$ large so that $N(s)>1$ and using estimates (1.1.11), (B.1.5a), (B.1.17), together with lemmas B.2.8 and B.4.2, we hence find that

$$\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\leq C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1+\delta_2}{2}}.$$

Analogously,

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\lesssim t^{\sigma}\|\chi_1(t^{-\sigma}D_x)\left[x_n(Z_m\Gamma v)_\pm\right](t,\cdot)\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2}\\
&+t^{-N(s)}\|x_n(Z_m\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}+t^{\sigma}\|\chi_1(t^{-\sigma}D_x)\left[x_n(\Gamma v)_\pm\right](t,\cdot)\|_{L^\infty}\|(Z_mu)_\pm(t,\cdot)\|_{L^2}\\
&+t^{-N(s)}\left(\sum_{\mu=0}^{1}\|x_m^\mu x_n(\Gamma v)_\pm(t,\cdot)\|_{L^2}+t\|x_n(\Gamma v)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s}\right)\\
&+t^{\sigma}\|\chi_1(t^{-\sigma}D_x)\left[x_n(\Gamma v)_\pm\right](t,\cdot)\|_{L^\infty}\left(\|u_\pm(t,\cdot)\|_{L^2}+\|D_tu_\pm(t,\cdot)\|_{L^2}\right)\\
&+t^{-N(s)}\|x_n(\Gamma v)_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s}\right),
\end{aligned}$$

so from (1.1.11), (B.1.5a), (B.1.17), (B.1.28), (B.2.62) and (B.4.8), we derive that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_1+\delta_2}{2}}.$$

Inequalities (B.4.31) follows then just by the observation that, after the hypothesis on multi-indices I_1,I_2 and the comparison between (B.2.30) with $I=I_2$ and (B.2.40) with $I=I_1$, $2i\chi(t^{-\sigma}D_x)\left(v^{I_2,NF}-(Z_m\Gamma v)_-\right)$ corresponds to the first line in the right hand side of (B.2.40).

In order to derive estimate (B.4.32a) we apply Z_m to both sides of equality (B.4.18), use (B.4.31), formulas (B.2.39), and successively proceed as follows: products in which Z_m acts on v and Γ on u, that arise from the action of Z_m on the second line of (B.4.18), are estimated using inequalities (B.2.24) with $L=L^2$ and $w=u$; products in which Z_m is acting on v and there are no Klainerman vector fields acting on u are estimated applying lemma B.2.2 with $L=L^2$; the L^2 norm of the remaining ones are controlled by the L^∞ norm of the Klein-Gordon factor times the L^2 norm of the wave one. In this way we get that

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(V_\Gamma^{NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\lesssim\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\\
&+t^{\sigma}\left\|\chi_1(t^{-\sigma}D_x)(Z_mv)_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|u_\pm(t,\cdot)\|_{L^2}\right)\\
&+t^{-N(s)}\left(\sum_{|\mu|=0}^{1}\|x^\mu(Z_mv)_\pm(t,\cdot)\|_{L^2}+t\|(Z_mv)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s}\right)\\
&+\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(Z_m\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|D_t(\Gamma u)_\pm(t,\cdot)\|_{L^2}\right.\\
&\qquad\left.+\|u_\pm(t,\cdot)\|_{L^2}+\|D_tu_\pm(t,\cdot)\|_{L^2}\right).
\end{aligned}$$

Choosing $s>0$ large so that $N(s)>2$ and using (1.1.11) (B.1.5a), (B.1.7), (B.1.17) with $k=1$, (B.2.42), (B.4.31a), we hence recover (B.4.32a). An analogous procedure leads us to the following inequality

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(V_\Gamma^{NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\lesssim\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\\
&+t^{\sigma}\left\|\chi_1(t^{-\sigma}D_x)\left[x_n(Z_mv)_\pm\right](t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|u_\pm(t,\cdot)\|_{L^2}\right)\\
&+t^{-N(s)}\left(\sum_{|\mu|=0}^{1}\|x^\mu x_n(Z_mv)_\pm(t,\cdot)\|_{L^2}+t\|x_n(Z_mv)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s}\right)\\
&+\sum_{|\mu|=0}^{1}\left\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(Z_m\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|D_t(\Gamma u)_\pm(t,\cdot)\|_{L^2}\right.\\
&\qquad\left.+\|u_\pm(t,\cdot)\|_{L^2}+\|D_tu_\pm(t,\cdot)\|_{L^2}\right),
\end{aligned}$$

and estimate (B.4.32b) is obtained by choosing $s > 0$ large so that $N(s) > 1$ and using (B.1.5a), (B.1.7), (B.1.10b), (B.1.17) with $k = 1$, (B.1.28), (B.2.62), (B.4.31b) and a-priori estimates. □

Lemma B.4.8. *Let $\Gamma \in \{\Omega, Z_m, m = 1, 2\}$ be a Klainerman vector field and V_Γ^{NF} be the function defined in* (B.4.15). *There exists a constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, $m, n = 1, 2$, and every $t \in [1, T]$,*

$$\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[tZ_nV_\Gamma^{NF}(t,tx)\right]\right\|_{L^2(dx)} \leq CB\varepsilon t^{\frac{\delta_0}{2}}. \tag{B.4.33}$$

Proof. We warn the reader that, throughout the proof, we denote by C and β two positive constants that may change line after line, with $\beta \to 0$ as $\sigma \to 0$.

Let $v^{I,NF}$ be the function defined in (B.2.27) for a generic multi-index I of length 1 or 2, and I_1, I_2 two multi-indices such that $\Gamma^{I_1} = \Gamma$, $\Gamma^{I_2} = Z_n\Gamma$. Using (2.1.15b) we rewrite $Z_nV_\Gamma^{NF}$ as follows:

$$Z_nV_\Gamma^{NF} = Z_n\left(V_\Gamma^{NF} - (\Gamma v)_-\right) + \left[(Z_n\Gamma v)_- - v^{I_2,NF}\right] + v^{I_2,NF} + \frac{D_n}{\langle D_x\rangle}v^{I_1,NF} + \frac{D_n}{\langle D_x\rangle}\left[(\Gamma v)_- - v^{I_1,NF}\right]$$

so that

$$\begin{aligned}
&\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[tZ_nV_\Gamma^{NF}(t,tx)\right]\right\|_{L^2(dx)}\\
&\lesssim \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[tZ_n\left(V_\Gamma^{NF} - (\Gamma v)_-\right)(t,tx)\right]\right\|_{L^2(dx)}\\
&+ \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\left[(Z_n\Gamma v)_- - v^{I_2,NF}\right](t,tx)\right]\right\|_{L^2(dx)}\\
&+ \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[tv^{I_2,NF}(t,tx)\right]\right\|_{L^2(dx)} + \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\frac{D_n}{\langle D_x\rangle}v^{I_1,NF}(t,tx)\right]\right\|_{L^2(dx)}\\
&+ \left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\frac{D_n}{\langle D_x\rangle}\left[(\Gamma v)_- - v^{I_1,NF}\right](t,tx)\right]\right\|_{L^2(dx)}.
\end{aligned} \tag{B.4.34}$$

Since $v^{I_2,NF}$ satisfies (B.4.4) with $I = I_2$, we derive from relation (3.2.8) with $w = v^{I_2,NF}$ that

$$\begin{aligned}
\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[tv^{I_2,NF}(t,tx)\right]\right\|_{L^2(dx)} &\lesssim \left\|\chi(t^{-\sigma}D_x)Z_m(\Gamma^{I_2}v)_-(t,\cdot)\right\|_{L^2}\\
&+ \left\|\chi(t^{-\sigma}D_x)Z_m\left[v^{I_2,NF} - (\Gamma^{I_2}v)_-\right](t,\cdot)\right\|_{L^2} + \left\|\chi(t^{-\sigma}D_x)v^{I_2,NF}(t,\cdot)\right\|_{L^2}\\
&+ \left\|\chi(t^{-\sigma}D_x)\left[x_mNL_{kg}^{I_2,NF}\right](t,\cdot)\right\|_{L^2}.
\end{aligned}$$

A-priori estimate (1.1.11d) with $k = 0$, (B.2.29) with $I = I_2$, (B.4.5), (B.4.7), the fact that $\delta \ll \delta_2 \ll \delta_1 \ll \delta_0$ and that β is small as long as σ is small, imply

$$\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[tv^{I_2,NF}(t,tx)\right]\right\|_{L^2(dx)} \leq CB\varepsilon t^{\frac{\delta_0}{2}}.$$

Analogously, commutating $\mathcal{L}_m$ with $\mathrm{Op}_h^w(\xi_n\langle\xi\rangle^{-1})$, using (3.2.8) with $w = v^{I_1,NF}$ and the fact that $v^{I_1,NF}$ is solution to (B.2.44) with non-linear term given by (B.2.45), together with inequalities (B.2.50), (B.2.51), (B.2.53), we derive that

$$\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\frac{D_n}{\langle D_x\rangle}v^{I_1,NF}(t,tx)\right]\right\|_{L^2(dx)} \leq CB\varepsilon t^{\frac{\delta_1}{2}}.$$

Finally, the remaining norms in the right hand side of (B.4.34) are estimated by the right hand side of (B.4.33) after (B.3.6) and lemma B.4.7. □

Lemmas B.2.8, B.4.6 and B.4.8 allow us to prove an analogous result to that of lemma B.3.3, where $\widetilde{v}$ is replaced with $\widetilde{V}^\Gamma$ introduced in (B.4.20).

Lemma B.4.9. *Let $h = t^{-1}$, $\widetilde{u}, \widetilde{V}^\Gamma$ be respectively defined in* (3.2.2) *and*(B.4.20)*, $a_0(\xi) \in S_{0,0}(1)$, and $b_1(\xi) = \xi_j$ or $b_1(\xi) = \xi_j\xi_k|\xi|^{-1}$, with $j,k \in \{1,2\}$. There exists a constant $C > 0$ such that, for any $\chi, \chi_1 \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$, and every $t \in [1,T]$, we have that*

$$\left\|[\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{V}^\Gamma(t,\cdot)][\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\right\|_{L^2} \leq C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta'}, \tag{B.4.35a}$$

$$\left\|[\mathrm{Op}_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{V}^\Gamma(t,\cdot)][\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\right\|_{L^\infty} \leq C(A+B)B\varepsilon^2 h^{-\beta'}, \tag{B.4.35b}$$

with $\beta' > 0$ small, $\beta \to 0$ as $\sigma, \delta_0 \to 0$.

Proof. The proof of this result has the same structure as that of lemma B.3.3. Only few differences occur due to to the fact that we are replacing $\widetilde{v}$ with $\widetilde{V}^\Gamma$. We limit here to indicate them.

Instead of referring to estimate (B.3.9) we use the fact that, after (B.2.42) in classical coordinates, there exists a constant $C > 0$ such that for any $\rho \in \mathbb{N}$

$$\left\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot)\right\|_{H^{\rho,\infty}} \leq CB\varepsilon h^{-\beta-\frac{\delta_1}{2}}, \tag{B.4.36}$$

with $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$. We successively decompose $\widetilde{V}^\Gamma$ into $\widetilde{V}^\Gamma_{\Lambda_{kg}} + \widetilde{V}^\Gamma_{\Lambda^c_{kg}}$, with

$$\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x) := \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{V}^\Gamma(t,x),$$
$$\widetilde{V}^\Gamma_{\Lambda^c_{kg}}(t,x) := \mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{V}^\Gamma(t,x).$$

On the one hand, from the fact that above operators are supported for frequencies $|\xi| \lesssim h^\sigma$, together with proposition 1.2.39 with $p = +\infty$ and (B.4.36), we have that

$$\left\|\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon h^{-\beta-\frac{\delta_1}{2}}.$$

On the other hand, combining the analogous of (B.3.12) with lemma B.4.6 (instead of B.2.14), estimates (B.4.26), (B.4.33) (instead of lemma B.3.2) and (B.4.30) (instead of (B.2.79)),

$$\left\|\widetilde{V}^\Gamma_{\Lambda^c_{kg}}(t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}.$$

Lemma B.4.10. *Let $\Gamma \in \{\Omega, Z_m, m = 1,2\}$ be a Klainerman vector field and V_Γ^{NF} be the function defined in* (B.4.15)*. There exists a constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, $m,n = 1,2$, and every $t \in [1,T]$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_m\left(V_\Gamma^{NF} - (\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty} \leq C(A+B)^2\varepsilon^2 t^{-\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}}, \tag{B.4.37a}$$

$$\left\|\chi(t^{-\sigma}D_x)\left[x_nx_m\left(V_\Gamma^{NF} - (\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty} \leq C(A+B)^2\varepsilon^2 t^{\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}}, \tag{B.4.37b}$$

with $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$.

Proof. We remind the reader about explicit expression (B.4.18) of the difference $V_\Gamma^{NF} - (\Gamma v)_-$, and (B.2.30), here considered with $|I| = 1$ such that $\Gamma^I = \Gamma$.

We first use equalities (1.1.5), (1.1.10), and, after some commutations, multiply x_m (together with x_n when proving (B.4.37b)) against each Klein-Gordon factor. Successively, we estimate the contribution coming from $v^{I,NF} - (\Gamma v)_-$ using lemma B.2.2 with $L = L^\infty$, and all products coming from the second line of (B.4.18) by means of inequalities (B.2.24) with $L = L^\infty$, $w = u$ and $w_{j_0} = (D_x\langle D_x\rangle^{-1})^\mu Z_m v$ for $|\mu| = 0, 1$. On the one hand, we obtain that

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_m\left(V_\Gamma^{NF} - (\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty}\\
&\lesssim \sum_{\mu=0}^{1} t^\sigma \left\|\chi_1(t^{-\sigma}D_x)\left[x_m(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty} \|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty} + t^{-N(s)}\|x_m(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&+ \sum_{|\mu|=0}^{1} t^\sigma \left\|x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty} \|\chi_1(t^{-\sigma}D_x)(\Gamma u)_-(t,\cdot)\|_{L^\infty}\\
&+ \sum_{|\mu|=0}^{2} t^{-N(s)}\|x^\mu v_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\right)\\
&+ \sum_{|\mu|,|\nu|=0}^{1} t^\sigma \left\|x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty} \|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}
\end{aligned}
$$

and estimate (B.4.37a) follows choosing $s > 0$ large enough to have $N(s) \geq 2$ and using (1.1.11), (B.1.5a), (B.1.10), (B.1.17) with $k = 1$, (B.2.62), (B.2.57). On the other hand,

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_n x_m\left(V_\Gamma^{NF} - (\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty}\\
&\lesssim \sum_{\mu=0}^{1} t^\sigma \left\|\chi_1(t^{-\sigma}D_x)\left[x_n x_m(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty} \|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty}\\
&+ t^{-N(s)}\|x_n x_m(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&+ \sum_{|\mu|=0}^{1} t^\sigma \left\|x_n x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty} \|\chi_1(t^{-\sigma}D_x)(\Gamma u)_-(t,\cdot)\|_{L^\infty}\\
&+ \sum_{|\mu|=0}^{3} t^{-N(s)}\|x^\mu v_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\right)\\
&+ \sum_{|\mu|,|\nu|=0}^{1} t^\sigma \left\|x_n x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty} \|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}
\end{aligned}
$$

so picking the same s as before and using (B.1.5a), (B.1.10a), (B.1.27), (B.1.28), (B.1.32), (B.2.57) and (B.4.11), together with a-priori estimates, we derive (B.4.37b). □

Lemma B.4.11. *Let $\Gamma \in \{\Omega, Z_m, m = 1, 2\}$ be a Klainerman vector field and $NL_\Gamma^{kg,c}$ be given by* (B.4.28). *There exists a constant $C > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small, $m, n = 1, 2$, and every $t \in [1, T]$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_n NL_\Gamma^{kg,c}\right](t,\cdot)\right\|_{L^2} \leq C(A+B)^2 B\varepsilon^3 t^{-1+\beta'}, \tag{B.4.38a}$$

$$\left\|\chi(t^{-\sigma}D_x)\left[x_m x_n NL_\Gamma^{kg,c}\right](t,\cdot)\right\|_{L^2} \leq C(A+B)^2 B\varepsilon^3 t^{\beta'}, \tag{B.4.38b}$$

with $\beta' > 0$ small such that $\beta' \to 0$ as $\sigma, \delta_0 \to 0$. Moreover, in the same time interval

$$\left\|\chi(t^{-\sigma}D_x)NL_\Gamma^{kg,c}(t,\cdot)\right\|_{L^\infty} \leq C(A+B)^2B\varepsilon^3t^{-\frac{5}{2}+\beta'}. \tag{B.4.39}$$

Proof. We warn the reader that we will denote by C, β, β' some positive constants that may change line after line, with $\beta \to 0$ (resp. $\beta' \to 0$) as $\sigma \to 0$ (resp. as $\sigma, \delta_0 \to 0$). For a seek of compactness we also denote by $R(t,x)$ any contribution verifying

$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)\left[x_nR(t,\cdot)\right]\right\|_{L^2} &\leq C(A+B)^2B\varepsilon^3t^{-1+\beta'},\\ \left\|\chi(t^{-\sigma}D_x)\left[x_mx_nR(t,\cdot)\right]\right\|_{L^2} &\leq C(A+B)^2B\varepsilon^3t^{\beta'},\end{aligned} \tag{B.4.40}$$

together with

$$\left\|\chi(t^{-\sigma}D_x)R(t,\cdot)\right\|_{L^\infty} \leq C(A+B)^2B\varepsilon^3t^{-\frac{5}{2}+\beta'}. \tag{B.4.41}$$

Let us introduce NL_v^{cub} as follows

$$\begin{aligned}NL_v^{cub} :=& -\frac{i}{2}\left[-(D_1\Gamma v)NL_w + D_1\left[(\Gamma v)NL_w\right]\right]\\ &-\frac{i}{2}\left[-(D_1v)\Gamma NL_w + D_1[v\Gamma NL_w]\right] + \frac{i}{2}\delta_\Omega\left[-(D_2v)NL_w + D_2[vNL_w]\right]\\ &+\delta_{Z_1}\left[(D_tv)NL_w^I - \langle D_x\rangle[vNL_w]\right],\end{aligned} \tag{B.4.42}$$

so that from (B.4.24)

$$NL_\Gamma^{kg,c} = \frac{i}{2}\left[(\Gamma NL_{kg})(D_1u) + NL_{kg}(D_1\Gamma u)\right] + \delta_\Omega\frac{i}{2}NL_{kg}(D_2u) + \delta_{Z_1}NL_{kg}(D_tu) + NL_v^{cub}, \tag{B.4.43}$$

with δ_Ω (resp. δ_{Z_1}) equal to 1 when $\Gamma = \Omega$ (resp. $\Gamma = Z_1$), 0 otherwise. After (1.1.5), (1.1.10), and estimates (1.1.11), (B.1.3a), (B.1.6a) with $s = 0$, (B.1.10b), (B.4.29), NL_v^{cub} verifies the following:

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)\left[x_nNL_v^{cub}\right](t,\cdot)\right\|_{L^2}\\ &\lesssim \sum_{\mu,|\nu|=0}^{1} t^\sigma\left\|x_n^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty}\left[\|\Gamma NL_w(t,\cdot)\|_{L^2} + \|NL_w(t,\cdot)\|_{L^2} + \|v_\pm(t,\cdot)\|_{H^{2,\infty}}\|(\Gamma^Iv)_\pm(t,\cdot)\|_{L^2}\right]\\ &\leq C(A+B)AB\varepsilon^3t^{-1+\sigma+\delta_2}.\end{aligned}$$

From the mentioned inequalities and the additional (B.1.27b), it also satisfies

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)\left[x_mx_nNL_v^{cub}\right](t,\cdot)\right\|_{L^2}\\ &\lesssim \sum_{\mu_1,\mu_2,|\nu|=0}^{1} t^\sigma\left\|x_m^{\mu_1}x_n^{\mu_2}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty}\left[\|\Gamma NL_w(t,\cdot)\|_{L^2} + \|NL_w(t,\cdot)\|_{L^2}\right.\\ &\qquad\qquad\left.+\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\|(\Gamma^Iv)_\pm(t,\cdot)\|_{L^2}\right]\\ &\leq C(A+B)AB\varepsilon^3t^{\sigma+\delta_2}.\end{aligned}$$

Moreover, applying twice lemma B.2.2 with $L = L^\infty$ and $s > 0$ large enough to have $N(s) \geq 2$, the first time to estimate products involving Γv and NL_w in (B.4.42), the second one to estimate

the first two quadratic contributions to ΓNL_w (see (B.1.20b)), we derive that there are two smooth cut-off functions χ_1, χ_2 such that

$$\begin{aligned}&\big\|\chi(t^{-\sigma}D_x)NL_v^{cub}(t,\cdot)\big\|_{L^\infty} \lesssim t^\sigma \big\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\big\|_{L^\infty}\|NL_w(t,\cdot)\|_{L^\infty}\\&+t^{-2}\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|NL_w(t,\cdot)\|_{H^s}+t^\sigma\|\chi_1(t^{-\sigma}D_x)\Gamma NL_w(t,\cdot)\|_{L^\infty}\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\\&+t^{-2}\|NL_w^I(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^s}+t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\|NL_w(t,\cdot)\|_{L^\infty}\end{aligned}$$

and

$$\begin{aligned}&\|\chi_1(t^{-\sigma}D_x)\Gamma NL_w(t,\cdot)\|_{L^\infty}\lesssim\|\chi_2(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{H^{2,\infty}}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\&+t^{-2}\|(\Gamma v)_\pm(t,\cdot)\|_{H^1}\|v_\pm(t,\cdot)\|_{H^s}+\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\big(\|v_\pm(t,\cdot)\|_{H^{2,\infty}}+\|D_tv_\pm(t,\cdot)\|_{H^{1,\infty}}\big).\end{aligned}$$

From a-priori estimates, (B.1.3b), (B.1.3c), (B.1.5a) with $s=0$, (B.1.6b) with $s=1$ and $\theta\ll 1$ small, (B.2.42), (B.4.29), we then recover

$$\big\|\chi(t^{-\sigma}D_x)NL_v^{cub}(t,\cdot)\big\|_{L^\infty}\le CA^2B\varepsilon^3t^{-3+\beta'}.$$

Those inequalities make NL_v^{cub} a contribution of the form $R(t,x)$, so from (B.4.43) we are left to prove that the same is true for $\Gamma NL_{kg}(D_1u)$, $NL_{kg}(D_1\Gamma u)$, $NL_{kg}(D_2u)$ and $NL_{kg}(D_tu)$.

We immediately observe, from (B.1.1b) and (1.1.5), that the cubic contributions to $NL_{kg}(D_2u)$ and $NL_{kg}(D_tu)$ are of the form

$$[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-, \tag{B.4.44}$$

with $a_0(\xi)\in\{1,\xi_j\langle\xi\rangle^{-1},j=1,2\}$, $b_1(\xi)\in\{\xi_1,\xi_j\xi_1|\xi|^{-1},j=1,2\}$, $b_0(\xi)\in\{1,\xi_2|\xi|^{-1}\}$. Therefore, lemmas B.3.5, B.3.6 imply that $NL_{kg}(D_2u)$ and $NL_{kg}(D_tu)$ are remainders $R(t,x)$. Furthermore, from (B.1.20a), (1.1.16) and the equation satisfied by $u_\pm$ in (2.1.2) with $|I|=0$,

$$\begin{aligned}\Gamma NL_{kg}=Q_0^{\rm kg}((\Gamma v)_\pm,D_1u_\pm)+Q_0^{\rm kg}(v_\pm,D_1(\Gamma u)_\pm)\\-\delta_\Omega Q_0^{\rm kg}(v_\pm,D_2u_\pm)-\delta_{Z_1}\Big[Q_0^{\rm kg}(v_\pm,|D_x|u_\pm)+Q_0^{\rm kg}\big(v_\pm,Q_0^{\rm w}(v_\pm,D_1v_\pm)\big)\Big],\end{aligned}$$

with δ_Ω (resp. δ_{Z_1}) equal to 1 if $\Gamma=\Omega$ (resp. $\Gamma=Z_1$), 0 otherwise. Estimates (1.1.11a) and (B.4.27) imply that

$$\Big\|\chi(t^{-\sigma}D_x)\Big[x_nQ_0^{\rm kg}\big(v_\pm,Q_0^{\rm w}(v_\pm,D_1v_\pm)\big)(D_1u)\Big](t,\cdot)\Big\|_{L^2(dx)}\le C(A+B)A^2B\varepsilon^4t^{-\frac32+\frac{\delta+\delta_2}{2}},$$

while after (1.1.11a), (B.1.3a), (B.1.27b),

$$\begin{aligned}&\Big\|\chi(t^{-\sigma}D_x)\Big[x_mx_nQ_0^{\rm kg}\big(v_\pm,Q_0^{\rm w}(v_\pm,D_1v_\pm)\big)(D_1u)\Big](t,\cdot)\Big\|_{L^2(dx)}\\&\lesssim\sum_{|\mu|=0}^1\Big\|x_mx_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty(dx)}\|NL_w(t,\cdot)\|_{L^2(dx)}\|{\rm R}_1u_\pm(t,\cdot)\|_{L^\infty}\\&\le C(A+B)A^2B\varepsilon t^{-\frac12+\frac{\delta+\delta_2}{2}}\end{aligned}$$

Also, for any $\theta\in]0,1[$,

$$\begin{aligned}&\Big\|\chi(t^{-\sigma}D_x)\Big[Q_0^{\rm kg}\big(v_\pm,Q_0^{\rm w}(v_\pm,D_1v_\pm)\big)(D_1u)\Big](t,\cdot)\Big\|_{L^\infty(dx)}\\&\lesssim\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\|NL_w(t,\cdot)\|_{H^{1,\infty}}\|{\rm R}_1u_\pm(t,\cdot)\|_{L^\infty}\le CA^{4-\theta}B^\theta\varepsilon^4t^{-\frac72+\theta(1+\frac\delta2)},\end{aligned}$$

as follows from (B.1.3d) with $s=1$ and a-priori estimates. Thus $Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\big)(D_1 u)$ is a remainder $R(t,x)$. The same holds true for

$$\Big[-\delta_\Omega Q_0^{\mathrm{kg}}(v_\pm, D_2 u_\pm) - \delta_{Z_1} Q_0^{\mathrm{kg}}(v_\pm, |D_x| u_\pm)\Big]\,(D_1 u)$$

thanks to lemmas B.3.5 and B.3.6, since the above term is linear combination of products of the form

$$[a_0(D_x)v_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1 u_-,$$

with the same $a_0(\xi)$ as before and $b_1(\xi) \in \{\xi_2, \xi_2\xi_j|\xi|^{-1}, |\xi|, j=1,2\}$, as one can check using (2.1.1) and (1.1.5).

Summing up, the very contributions for which we have to prove estimates (B.4.40) and (B.4.41) are the following:

$$[a_0(D_x)(\Gamma v)_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1 u_- \tag{B.4.45a}$$

$$[a_0(D_x)v_-]\,[b_1(D_x)(\Gamma u)_-]\,\mathrm{R}_1 u_-, \tag{B.4.45b}$$

which are the remaining types of products in $(\Gamma NL_{kg})(D_1 u)$, and

$$[a_0(D_x)v_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1(\Gamma u)_-, \tag{B.4.45c}$$

which are the products appearing in $NL_{kg}(D_1\Gamma u)$, with a_0 being the same as above and $b_1(\xi)$ equal to ξ_1 or to $\xi_j\xi_1|\xi|^{-1}$, with $j=1,2$. All the manipulations we are going to make in what follows are aimed at showing that these estimates follow from lemmas B.3.3, B.3.4 and B.4.9.

Firstly, we can assume that all factors in (B.4.45) are truncated for moderate frequencies less or equal than t^σ, up to $R(t,x)$ contributions. As regards (B.4.45a), this comes out from the application of lemma B.2.2. In fact, taking $L=L^2$, $w_1 = x_m^k x_n a_0(D_x)(\Gamma v)_-$for $k\in\{0,1\}$, $s>0$ large enough to have $N(s)>2$, and using a-priori estimates and (B.1.17), (B.1.28), we find that there is some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that, for $k=0,1$,

$$\begin{aligned}
&\Big\|\chi(t^{-\sigma}D_x)\big[x_m^k x_n[a_0(D_x)(\Gamma v)_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1 u_-\big]\Big\|_{L^2(dx)}\\
&\lesssim \Big\|\Big[\chi_1(t^{-\sigma}D_x)[x_m^k x_n a_0(D_x)(\Gamma v)_-]\Big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1 u_-\big]\Big\|_{L^2(dx)}\\
&+ t^{-2}\sum_{\mu_1,\mu_2,|\nu|=0}^{1} \|x_m^{\delta_{k1}\mu_1} x_n^{\mu_2}(\Gamma v)_-(t,\cdot)\|_{L^2(dx)}\|\mathrm{R}^\nu u_-(t,\cdot)\|_{H^{2,\infty}}\|u_-(t,\cdot)\|_{H^s}\\
&\lesssim \Big\|\Big[\chi_1(t^{-\sigma}D_x)[x_m^k x_n a_0(D_x)(\Gamma v)_-]\Big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1 u_-\big]\Big\|_{L^2(dx)}\\
&+ CAB^2\varepsilon^3 t^{-\frac{3}{2}(1-k)+\frac{\delta+\delta_2}{2}},
\end{aligned}$$

where δ_{k1} is the Kronecker delta. Taking instead $L=L^\infty$, from a-priori estimates we derive that

$$\begin{aligned}
&\big\|\chi(t^{-\sigma}D_x)\big[[a_0(D_x)(\Gamma v)_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1 u_-\big]\big\|_{L^\infty(dx)}\\
&\lesssim \big\|\big[\chi_1(t^{-\sigma}D_x)[a_0(D_x)(\Gamma v)_-]\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1 u_-\big]\big\|_{L^\infty(dx)}\\
&+ t^{-2}\sum_{|\mu|=0}^{1} \|(\Gamma v)_-(t,\cdot)\|_{L^2}\|\mathrm{R}^\mu u_-(t,\cdot)\|_{H^{2,\infty}}\|u_-(t,\cdot)\|_{H^s}\\
&\lesssim \big\|\big[\chi_1(t^{-\sigma}D_x)[a_0(D_x)(\Gamma v)_-]\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1 u_-\big]\big\|_{L^\infty(dx)}\\
&+ CAB^2\varepsilon^3 t^{-\frac{5}{2}+\frac{\delta+\delta_2}{2}}.
\end{aligned}$$

As concerns instead products (B.4.45b) and (B.4.45c), this follows applying inequalities (B.2.24) with $w = u$, $w_{j_0} = x_m^k x_n a_0(D_x) v_-$ for $k = 0, 1$, and $s > 0$ such that $N(s) \geq$. In fact, for $L = L^2$ we use estimates (1.1.11), (B.1.10), (B.1.27), together with (B.1.32), to derive that for $k \in \{0, 1\}$

$$
\begin{aligned}
&\left\| \chi(t^{-\sigma} D_x) \left[x_m^k x_n [a_0(D_x) v_-] \, [b_1(D_x)(\Gamma u)_-] \, \mathrm{R}_1 u_- \right] \right\|_{L^2(dx)} \\
&\lesssim \left\| \left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n a_0(D_x) v_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x)(\Gamma u)_- \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 u_- \right\|_{L^2(dx)} \\
&+ t^{-3} \sum_{|\mu_1|, \mu_2, \mu_3 = 0}^{1} \left(\left\| x^{\mu_1} x_m^{\delta_{k1} \mu_2} x_n^{\mu_3} v_-(t, \cdot) \right\|_{L^2(dx)} + t \| x_n^{\mu_2} v_-(t, \cdot) \|_{L^2} \right) \| u_\pm(t, \cdot) \|_{H^s} \| \mathrm{R}_1 u_-(t, \cdot) \|_{L^\infty} \\
&+ t^{-3} \left\| x_m^k x_n a_0(D_x) v_-(t, \cdot) \right\|_{L^\infty(dx)} \| (\Gamma u)_-(t, \cdot) \|_{L^2} \| u_-(t, \cdot) \|_{H^s} \\
&\lesssim \left\| \left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n a_0(D_x) v_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x)(\Gamma u)_- \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 u_- \right\|_{L^2(dx)} \\
&+ C A B^2 \varepsilon^2 t^{-(1-k) + \frac{\delta + \delta_2}{2}}.
\end{aligned}
$$

Using instead (B.2.24) with $L = L^\infty$ along with (1.1.11) and (B.1.10b),

$$
\begin{aligned}
&\left\| \chi(t^{-\sigma} D_x) \left[[a_0(D_x) v_-] \, [b_1(D_x)(\Gamma u)_-] \, \mathrm{R}_1 u_- \right] \right\|_{L^\infty} \\
&\lesssim \left\| \left[\chi_1(t^{-\sigma} D_x) a_0(D_x) v_- \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x)(\Gamma u)_- \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 u_- \right\|_{L^\infty} + C A B^2 \varepsilon^2 t^{-\frac{5}{2} + \frac{\delta + \delta_2}{2}}.
\end{aligned}
$$

Secondly, we can assume that in (B.4.45a) (resp. in (B.4.45c)) $b_1(D_x) u_-$ is replaced with $b_1(D_x) u^{NF}$ (with u^{NF} introduced in (3.1.15)). This is justified up to some $R(t, x)$ terms that satisfy (B.4.40) as consequence of (1.1.11a), (B.1.17), (B.1.28) (resp. (B.1.10a), (B.1.27a)), (B.3.26b), and also (B.4.41) because of (1.1.11a), (B.2.42) (resp. (1.1.11b)) and (B.3.26b). Hence we are led to estimate the L^2 norm of

$$
\left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n^l a_0(D_x)(\Gamma v)_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x) u^{NF} \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 u_- \tag{B.4.46a}
$$

$$
\left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n^l a_0(D_x) v_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x)(\Gamma u)_- \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 u_- \tag{B.4.46b}
$$

$$
\left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n^l a_0(D_x) v_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x) u^{NF} \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 (\Gamma u)_- \tag{B.4.46c}
$$

for $k = 0, 1$, $l = 1$, and the L^∞ norm of above products when $k = l = 0$.

Thirdly, we can think of $a_0(D_x)(\Gamma v)_-$ in (B.4.46a) and of $a_0(D_x) v_-$ in (B.4.46b), (B.4.46c) as replaced with $a_0(D_x) V_\Gamma^{NF}$ and $a_0(D_x) v^{NF}$ respectively, where V_Γ^{NF} has been introduced in (B.4.15) and v^{NF} in (3.1.3). For (B.4.46a) (resp. (B.4.46c)) this substitution is justified up to some $R(t, x)$ terms that satisfy (B.4.40) and (B.4.41), the former because of a-priori estimate (1.1.11a), (B.3.30) and (B.4.37) (resp. (B.2.57), (B.3.30) and (B.3.31)), the latter after (1.1.11a), (B.4.16) (resp. (B.3.26a), (B.2.57)) and the classical translation of the semi-classical (B.3.8)

$$
\| u^{NF}(t, \cdot) \|_{H^{\rho, \infty}} + \| \mathrm{R} u^{NF}(t, \cdot) \|_{H^{\rho, \infty}} \leq C B \varepsilon t^{-\frac{1}{2}}.
$$

Therefore, in order to conclude the proof we must prove that, for some $\chi, \chi_1 \in C_0^\infty(\mathbb{R}^2)$ and $k \in \{0, 1\}$,

$$
\begin{aligned}
&\left\| \left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n a_0(D_x)(\Gamma v)_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x) u^{NF} \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 u_- \right\|_{L^2(dx)} \\
&+ \left\| \left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n a_0(D_x) v_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x)(\Gamma u)_- \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 u_- \right\|_{L^2(dx)} \\
&+ \left\| \left[\chi_1(t^{-\sigma} D_x) [x_m^k x_n^l a_0(D_x) v_-] \right] \, \left[\chi(t^{-\sigma} D_x) b_1(D_x) u^{NF} \right] \chi(t^{-\sigma} D_x) \mathrm{R}_1 (\Gamma u)_- \right\|_{L^2(dx)} \\
&\leq C (A + B)^2 B \varepsilon^3 t^{-1 + k + \beta'}
\end{aligned}
$$

and

$$\begin{aligned}
&\big\|\big[\chi_1(t^{-\sigma}D_x)a_0(D_x)(\Gamma v)_-\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}\big]\,\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big\|_{L^\infty(dx)}\\
&+\big\|\big[\chi_1(t^{-\sigma}D_x)a_0(D_x)v_-\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)(\Gamma u)_-\big]\,\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big\|_{L^\infty(dx)}\\
&+\big\|\big[\chi_1(t^{-\sigma}D_x)a_0(D_x)v_-\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}\big]\,\chi(t^{-\sigma}D_x)\mathrm{R}_1(\Gamma u)_-\big\|_{L^\infty(dx)}\\
&\leq C(A+B)^2B\varepsilon^3t^{-\frac{5}{2}+\beta'}.
\end{aligned}$$

Actually, using (1.1.11a), (B.2.57), and passing to the semi-classical framework and unknowns with $\widetilde{V}^\Gamma$ introduced in (B.4.20), $\widetilde{u},\widetilde{v}$ in (3.2.2), and $\widetilde{u}^I(t,x)=t^{-1}(\Gamma u)_-(t,t^{-1}x)$, above inequalities will follow respectively from

$$\begin{aligned}
&\sum_{k=0}^{1}\Big[\,\Big\|\Big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_m^kx_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{V}^\Gamma]\Big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2(dx)}\\
&\qquad+\Big\|\Big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_m^kx_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\Big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^I\big](t,\cdot)\Big\|_{L^2(dx)}\\
&\qquad+\Big\|\Big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_m^kx_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\Big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2(dx)}\Big]\\
&\qquad\leq C(A+B)B\varepsilon^3h^{-\frac{1}{2}-\beta'}
\end{aligned}\tag{B.4.47}$$

and

$$\begin{aligned}
&\Big\|\big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\widetilde{V}^\Gamma\big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^\infty(dx)}\\
&+\big\|\big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\widetilde{v}\big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^I\big](t,\cdot)\big\|_{L^\infty(dx)}\\
&+\big\|\big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\widetilde{v}\big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\big\|_{L^\infty(dx)}\leq C(A+B)B\varepsilon^3h^{-\beta'}.
\end{aligned}\tag{B.4.48}$$

We immediately obtain from inequalities (B.3.35) and (B.3.39) that

$$\sum_{k=0}^{1}\big\|\big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_m^kx_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}]\big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\big\|_{L^2(dx)}\leq C(A+B)B\varepsilon^3h^{-\frac{1}{2}-\beta'}.$$

Moreover, one can check that

$$\begin{aligned}
&\Big\|\Big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{V}^\Gamma]\Big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2(dx)}\\
&\qquad\leq\Big\|\Big[\mathrm{Op}_h^w\Big(\chi_1(h^\sigma\xi)a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{V}^\Gamma\Big][\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}](t,\cdot)\Big\|_{L^2}+CAB\varepsilon^2h^{\frac{1}{2}-\beta'},
\end{aligned}$$

$$\begin{aligned}
&\Big\|\Big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))[x_mx_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{V}^\Gamma]\Big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2(dx)}\\
&\lesssim\Big\|\Big[\mathrm{Op}_h^w\Big(\chi_1(h^\sigma\xi)a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{V}^\Gamma\Big]\Big[\mathrm{Op}_h^w\Big(\chi(h^\sigma\xi)b_1(\xi)\frac{\xi_m}{|\xi|}\Big)\widetilde{u}\Big](t,\cdot)\Big\|_{L^2(dx)}+C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta'},
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\big[x_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}\big]\big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^I\big](t,\cdot)\big\|_{L^2(dx}\\
&\qquad\leq\Big\|\Big[\mathrm{Op}_h^w\Big(\chi_1(h^\sigma\xi)a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v}\Big][\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^I](t,\cdot)\Big\|_{L^2}+CAB\varepsilon^2h^{\frac{1}{2}-\beta'},
\end{aligned}$$

$$\begin{aligned}&\big\|\big[\mathrm{Op}_h^w(\chi_1(h^\sigma\xi))\big[x_mx_n\mathrm{Op}_h^w(a_0(\xi))\widetilde{v}\big]\big]\,\big[\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^J\big](t,\cdot)\big\|_{L^2}\\&\lesssim\Big\|\Big[\mathrm{Op}_h^w\Big(\chi_1(h^\sigma\xi)a_0(\xi)\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v}\Big]\Big[\mathrm{Op}_h^w\Big(\chi(h^\sigma\xi)b_1(\xi)\frac{\xi_m}{|\xi|}\Big)\widetilde{u}^I\Big](t,\cdot)\Big\|_{L^2(dx)}+C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta'}.\end{aligned}$$

This can be done using a similar argument to the one that led us to (B.3.35) and (B.3.39), up to replacing $\widetilde{v}$ with $\widetilde{V}^\Gamma$ in (B.3.33), referring to lemma B.4.6 instead of B.2.14, and to estimate (B.4.36) instead of (B.3.9), in order to derive the former two inequalities; up to replacing $\widetilde{u}$ with $\widetilde{u}^I$ in (B.3.34), using lemma B.2.9 instead of (B.2.1a), (B.2.1c), estimate (B.3.23) instead of (B.3.8), and the fact that for any $\theta\in]0,1[$

$$\begin{aligned}\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\widetilde{u}^I(t,\cdot)\big\|_{L^\infty}&=t\,\big\|\chi(t^{-\sigma}D_x)b_1(D_x)D_m|D_x|^{-1}(\Gamma u)_-(t,\cdot)\big\|_{L^\infty}\\&\lesssim t\|\chi(t^{-\sigma}D_x)(\Gamma u)_-(t,\cdot)\|_{H^{3,\infty}}^{1-\theta}\|(\Gamma u)_-(t,\cdot)\|_{H^2}^{\theta}\leq C(A+B)^{1-\theta}B^\theta\varepsilon t^{\frac{1}{2}+\beta+\frac{\delta_1}{2}+\frac{(1+\delta_1+\delta_2)}{2}\theta},\end{aligned}$$

which is the analogous of (B.3.37) (last estimate deduced using (B.2.57) and (1.1.11d) with $k=1$), to demonstrate the latter two ones. Therefore, above inequalities and (B.3.22a), (B.4.35a) imply (B.4.47). Finally, (B.4.48) is consequence of (B.3.7b), (B.3.22b) and (B.4.35b). That concludes the proof of the statement. □

Lemma B.4.12. *Let $NL_\Gamma^{kg,c}$ be given by* (B.4.28). *There exists a constant $C>0$ such that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, $m,n=1,2$, and every $t\in[1,T]$,*

$$\begin{aligned}\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\Big[t(tx_n)NL_\Gamma^{kg,c}(t,tx)\Big]\Big\|_{L^2(dx)}&\leq C(A+B)^2B\varepsilon^3t^{\beta'},\\\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\Big[t(tx_n)Q_0^{\mathrm{kg}}\big(v_\pm,Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm)\big)(t,tx)\Big]\Big\|_{L^2(dx)}&\leq C(A+B)AB\varepsilon^3t^{\frac{\delta+\delta_2}{2}},\end{aligned}$$

with $\beta'>0$ such that $\beta'\to0$ as $\sigma,\delta_0\to0$.

Proof. Straightforward after (B.3.6), lemma B.4.11, estimate (B.4.27) and the following inequality

$$\begin{aligned}&\Big\|\chi(t^{-\sigma}D_x)\Big[x_mx_nQ_0^{\mathrm{kg}}\big(v_\pm,Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm)\big)\Big](t,\cdot)\Big\|_{L^2}\\&\qquad\lesssim\sum_{|\mu|=0}^1\Big\|x_mx_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty}\|NL_w(t,\cdot)\|_{L^2}\leq C(A+B)AB\varepsilon^3t^{\frac{\delta+\delta_2}{2}},\end{aligned}$$

deduced from (1.1.11), (B.1.3a) and (B.1.27b). □

Lemma B.4.13. *Let $\widetilde{V}^\Gamma$ be the function defined in* (B.4.20). *There exists some positive constant C such that, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, and every $t\in[1,T]$,*

$$\sum_{|\mu|=2}\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi)\mathcal{L}^\mu\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^2}\leq CB\varepsilon t^{\beta'},\tag{B.4.49}$$

with $\beta'>0$ small, $\beta'\to0$ as $\sigma,\delta_0\to0$.

Proof. First of all we remind that V_Γ^{NF} is solution to (B.4.23). From relation (3.2.9b) and the commutation between $\mathcal{L}_m$ and $\mathrm{Op}_h^w(\langle\xi\rangle)$ we deduce that, for any $m,n=1,2$,

$$\begin{aligned}\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\mathcal{L}_n\widetilde{V}^\Gamma(t,\cdot)\Big\|_{H_h^1}&\lesssim\sum_{\mu=0}^1\Big[\,\big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m^\mu\big[tZ_nV_\Gamma^{NF}(t,tx)\big]\big\|_{L^2(dx)}\\&+\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m^\mu\mathrm{Op}_h^w\Big(\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{V}^\Gamma(t,\cdot)\Big]\Big\|_{L^2}+\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m^\mu\big[t(tx_n)NL_\Gamma^{kg,c}(t,tx)\big]\Big\|_{L^2(dx)}\\&+\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}_m^\mu\Big[t(tx_n)Q_0^{\mathrm{kg}}\big(v_\pm,Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm)\big)(t,tx)\Big]\Big\|_{L^2(dx)}\Big].\end{aligned}$$

The result of the statement follows then from (B.4.26), (B.4.27), (B.4.30), (B.4.33), and lemmas B.4.6, B.4.12. □

Lemma B.4.14. *There exists a constant $C > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin, $\sigma > 0$ small, and every $t \in [1, T]$,*

$$\sum_{|I|=1} \|\chi(t^{-\sigma} D_x) V^I(t,\cdot)\|_{L^\infty} \leq CB\varepsilon t^{-1}. \tag{B.4.50}$$

Proof. As this estimate is evidently satisfied when I is such that Γ^I is a spatial derivative after a-priori estimate (1.1.11b), we focus on proving the statement for $\Gamma^I \in \{\Omega, Z_m, m = 1, 2\}$ being a Klainerman vector field. For simplicity, we refer to Γ^I simply by Γ.

Instead of proving the result of the statement directly on $(\Gamma v)_\pm$ we show that

$$\big\|V_\Gamma^{NF}(t,\cdot)\big\|_{L^\infty} \leq CB\varepsilon t^{-1}, \tag{B.4.51}$$

where V_Γ^{NF} has been introduced in (B.4.15). After (B.4.16), the above inequality evidently implies the statement. The main idea to derive the sharp decay estimate in (B.4.51) is to use the same argument that, in subsection 3.2.1, led us to the propagation of a-priori estimate (1.1.11b), i.e. to move to the semi-classical setting and deduce an ODE from equation (B.4.23) satisfied by V_Γ^{NF}. The most important feature that will provide us with (B.4.51) is that the uniform norm of all involved non-linear terms is integrable in time. Before going into the details, we also remind the reader our choice to denote by C, β and β' some positive constants that may change line after line, with $\beta \to 0$ (resp. $\beta' \to 0$) as $\sigma \to 0$ (resp. as $\sigma, \delta_0 \to 0$).

Let us consider $\widetilde{V}^\Gamma(t,x) := tV_\Gamma^{NF}(t,tx)$, operator Γ^{kg} as follows

$$\Gamma^{kg} := \mathrm{Op}_h^w\Big(\gamma\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big),$$

with $\gamma, \chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that $\gamma \equiv 1$ close to the origin, $\chi_1 \equiv 1$ on the support of χ, $p(\xi) := \langle\xi\rangle$, and

$$\begin{aligned}
\widetilde{V}_{\Lambda_{kg}}^\Gamma(t,x) &:= \Gamma^{kg}\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,x),\\
\widetilde{V}_{\Lambda_{kg}^c}^\Gamma(t,x) &:= \mathrm{Op}_h^w\Big((1-\gamma)\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,x),
\end{aligned}$$

so that

$$\mathrm{Op}_h^w(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot) = \widetilde{V}_{\Lambda_{kg}}^\Gamma + \widetilde{V}_{\Lambda_{kg}^c}^\Gamma.$$

It immediately follows from inequality (3.2.18b) and lemmas B.4.6, B.4.13, that

$$\Big\|\widetilde{V}_{\Lambda_{kg}^c}^\Gamma(t,\cdot)\Big\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{2} h^{\frac{1}{2}-\beta}\Big\|\mathrm{Op}_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^2} \leq CB\varepsilon t^{-\frac{1}{2}+\beta'}. \tag{B.4.52}$$

On the other hand, as V_Γ^{NF} is solution to (B.4.23) an explicit computation shows that $\widetilde{V}^\Gamma$ satisfies the following semi-classical pseudo-differential equation:

$$\big[D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle)\big]\,\widetilde{V}^\Gamma(t,x) = h^{-1}NL_\Gamma^{kg,c}(t,tx) - \delta_{Z_1}h^{-1}Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\big)(t,tx),$$

with $NL_\Gamma^{kg,c}$ given explicitly by (B.4.28). Applying successively operators $\mathrm{Op}_h^w(\chi(h^\sigma\xi))$ and Γ^{kg} to the above equation we find, from symbolic calculus and the first part of lemma 3.2.5, that

$\widetilde{V}^{\Gamma}_{\Lambda_{kg}}$ satisfies

$$
\begin{aligned}
(\text{B.4.53}) \quad & [D_t - \mathrm{Op}^w_h(x\cdot\xi - \langle\xi\rangle)]\,\widetilde{V}^{\Gamma}_{\Lambda_{kg}}(t,x) = h^{-1}\Gamma^{kg}\mathrm{Op}^w_h(\chi(h^\sigma\xi))\left[NL^{kg,c}_{\Gamma}(t,tx)\right] \\
& - \delta_{Z_1}h^{-1}\Gamma^{kg}\mathrm{Op}^w_h(\chi(h^\sigma\xi))\big[Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)\big)(t,tx)\big] - \mathrm{Op}^w_h(b(x,\xi))\mathrm{Op}^w_h(\chi(h^\sigma\xi))\widetilde{V}^{\Gamma}(t,x) \\
& \qquad + i\sigma h^{1+\sigma}\Gamma^{kg}\mathrm{Op}^w_h\big((\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\big)\widetilde{V}^{\Gamma},
\end{aligned}
$$

with symbol b given by (3.2.27). Since γ's derivatives vanish in a neighbourhood of the origin and $\partial\chi_1 \equiv 0$ on the support of χ, from symbolic calculus of lemma 1.2.24 and remark 1.2.22, 1.2.42, together with inequalities (3.2.17b), (3.2.18b), that

$$
\begin{aligned}
&\left\|\mathrm{Op}^w_h(b(x,\xi))\mathrm{Op}^w_h(\chi(h^\sigma\xi))\widetilde{V}^{\Gamma}(t,\cdot)\right\|_{L^\infty} \\
&\qquad \lesssim h^{\frac{3}{2}-\beta}\sum_{|\mu|=0}^{2}\|\mathrm{Op}^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{V}^{\Gamma}(t,\cdot)\|_{L^2} + h^2\left\|\widetilde{V}^{\Gamma}(t,\cdot)\right\|_{L^2} \leq CB\varepsilon t^{-\frac{3}{2}+\beta'},
\end{aligned}
$$

where last estimate is obtained using lemmas B.4.6, B.4.13. Moreover, reminding lemma 1.2.38 and using symbolic calculus we see that, for any $N \in \mathbb{N}$ as large as we want,

$$
\begin{aligned}
(\text{B.4.54}) \quad & h^{1+\sigma}\left\|\Gamma^{kg}\mathrm{Op}^w_h((\partial\chi(h^\sigma\xi)\cdot(h^\sigma\xi))\widetilde{V}^{\Gamma}(t,\cdot)\right\|_{L^\infty} \\
& \qquad \leq h^{1+\sigma}\left\|\Gamma^{kg}\theta_h(x)\mathrm{Op}^w_h((\partial\chi(h^\sigma\xi)\cdot(h^\sigma\xi))\widetilde{V}^{\Gamma}(t,\cdot)\right\|_{L^\infty} + h^N\|\widetilde{V}^{\Gamma}(t,\cdot)\|_{L^2},
\end{aligned}
$$

where $\theta_h(x)$ is a smooth cut-off function supported in closed ball $\overline{B_{1-ch^{2\sigma}}(0)}$, with $c>0$ small. Denoting $(\partial\chi)(\xi)\cdot\xi$ concisely by $\widetilde{\chi}(\xi)$, we observe from proposition 1.2.39 with $p=+\infty$, together with the uniform continuity on L^∞ of operator $\widetilde{\chi}(t^{-\sigma}D_x)$, the definition of $\widetilde{V}^{\Gamma}$ in terms of V^{NF}_{Γ}, and (B.4.16), that

$$
\begin{aligned}
h^{1+\sigma}\left\|\Gamma^{kg}\theta_h(x)\mathrm{Op}^w_h(\widetilde{\chi}(h^\sigma\xi))\widetilde{V}^{\Gamma}(t,\cdot)\right\|_{L^\infty} &\lesssim h^{1-\beta}\left\|\theta_h(x)\mathrm{Op}^w_h(\widetilde{\chi}(h^\sigma\xi))\widetilde{V}^{\Gamma}(t,\cdot)\right\|_{L^\infty} \\
&\leq t^\beta\left\|\theta_h\Big(\frac{\cdot}{t}\Big)\widetilde{\chi}(t^{-\sigma}D_x)(\Gamma v)_-(t,\cdot)\right\|_{L^\infty} + C(A+B)B\varepsilon^2t^{-\frac{5}{4}+\beta}.
\end{aligned}
$$

Using the fact that, for $\theta^j_h(z) := \theta_h(z)z_j$,

$$
\theta_h\Big(\frac{x}{t}\Big)(\Omega v)_- = t\Big[\theta^1_h\Big(\frac{x}{t}\Big)\partial_2v_- - \theta^2_h\Big(\frac{x}{t}\Big)\partial_1v_-\Big]
$$

and

$$
\theta_h\Big(\frac{x}{t}\Big)(Z_mv)_- = t\Big[\theta^m_h\Big(\frac{x}{t}\Big)\partial_tv_- + \theta_h\Big(\frac{x}{t}\Big)\partial_mv_-\Big] + \theta_h\Big(\frac{x}{t}\Big)\frac{D_m}{\langle D_x\rangle}v_-, \quad m=1,2,
$$

and making some commutations, we can express $(\Gamma v)_-$ in terms of v_- and its derivatives up to a loss in t. Thus, from the classical Sobolev injection combined inequality (B.1.2), we obtain that

$$
\begin{aligned}
t^{-\beta}\left\|\widetilde{\chi}(t^{-\sigma}D_x)\theta_h\Big(\frac{\cdot}{t}\Big)(\Gamma v)_-(t,\cdot)\right\|_{L^\infty} &\lesssim t^{-N(s)+1+\beta}\left(\|D_tv_\pm(t,\cdot)\|_{H^s} + \|v_\pm(t,\cdot)\|_{H^s}\right) \\
&\leq CB\varepsilon t^{-\frac{3}{2}},
\end{aligned}
$$

last estimate following by taking $s>0$ large enough to have $N(s)\geq 3$ and using a-priori estimates along with (B.1.6a) with $s=0$. From (B.4.21a) and (B.4.54) we hence derive that

$$
h^{1+\sigma}\left\|\Gamma^{kg}\mathrm{Op}^w_h((\partial\chi(h^\sigma\xi)\cdot(h^\sigma\xi))\widetilde{V}^{\Gamma}(t,\cdot)\right\|_{L^\infty} \leq CB\varepsilon t^{-\frac{3}{2}},
$$

so the last two terms in the right hand side of equation (B.4.53) are remainders $R(t,x)$ such that

$$\|R(t,\cdot)\|_{L^\infty} \leq CB\varepsilon t^{-\frac{5}{4}}, \tag{B.4.55}$$

for every $t \in [1,T]$.

After proposition 1.2.39 with $p = +\infty$, estimate (B.4.39), and the fact that for any $\theta \in]0,1[$,

$$\left\| Q_0^{\mathrm{kg}}\left(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)\right)(t,\cdot)\right\|_{L^\infty(dx)} \leq CA^{3-\theta}B^\theta\varepsilon^3 t^{-3+\theta(1+\frac{\delta}{2})},$$

as follows by (B.1.3c) with $s=1$ and a-priori estimates, we deduce (up to taking $\theta \ll 1$ small in the above inequality) that also the first two non-linear terms in the right hand side of (B.4.53) satisfy (B.4.55) and can be included into $R(t,x)$. Therefore, $\widetilde{V}^\Gamma_{\Lambda_{kg}}$ satisfies

$$\left[D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle)\right]\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x) = R(t,x),$$

and using (3.2.21) along with inequality (3.2.23b), together with lemmas B.4.6, B.4.13, we deduce that, for the same family of cut-off functions θ_h introduced above, $\widetilde{V}^\Gamma_{\Lambda_{kg}}$ is solution to the following ODE:

$$D_t\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x) = -\theta_h(x)\phi(x)\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x) + R(t,x), \tag{B.4.56}$$

with $\phi(x) = \sqrt{1-|x|^2}$. Since the inhomogeneous term $R(t,x)$ decays, in the uniform norm, at a rate which is integrable in time, we get that

$$\|\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,\cdot)\|_{L^\infty} \lesssim \|\widetilde{V}^\Gamma_{\Lambda_{kg}}(1,\cdot)\|_{L^\infty} + CB\varepsilon \leq CB\varepsilon,$$

which summed up with (B.4.52) implies (B.4.51), and hence the conclusion of the proof. □

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