



# Inequalities for the eigenvectors associated to extremal eigenvalues in rank one perturbations of symmetric matrices

Jacques Bénasséni, Alain Mom

## ► To cite this version:

Jacques Bénasséni, Alain Mom. Inequalities for the eigenvectors associated to extremal eigenvalues in rank one perturbations of symmetric matrices. *Linear Algebra and its Applications*, 2019, 570, pp.123-137. 10.1016/j.laa.2019.01.021 . hal-01902562

**HAL Id: hal-01902562**

**<https://hal.science/hal-01902562>**

Submitted on 23 Oct 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Inequalities for the eigenvectors associated to extremal eigenvalues in rank one perturbations of symmetric matrices

Jacques Bénasséni and Alain Mom <sup>\*†</sup>

Univ Rennes, CNRS, IRMAR - UMR 6625  
F-35000 Rennes, France

## Abstract

This paper considers the eigenvectors involved in rank one perturbations of symmetric matrices. A doubly stochastic matrix based on the inner products between initial and perturbed eigenvectors is introduced to derive several relations for the latter ones. A majorization theorem used together with this doubly stochastic matrix provides the main results of the paper, which deal with the eigenvectors associated to the largest and smallest non-zero eigenvalues. Further developments are also suggested for infinitesimal perturbations using convergent power expansions of both eigenvalues and eigenvectors.

---

<sup>\*</sup>Email addresses : corresponding author [jacques.benasseni@univ-rennes2.fr](mailto:jacques.benasseni@univ-rennes2.fr),  
[alain.mom@univ-rennes2.fr](mailto:alain.mom@univ-rennes2.fr)

<sup>†</sup>

**Keywords.** doubly stochastic matrix; eigenvalues and eigenvectors; perturbation of rank one ; symmetric matrix.

**AMS subject classifications.** 15A18, 15A42

**Abbreviated title:** Eigenvectors in rank one perturbation

## 1 Introduction

Rank one perturbations are of major interest from both applied and theoretical standpoints. When considering symmetric or hermitian matrices, some central results for eigenvalues are provided by Golub [8], Thompson [15], Bunch, Nielsen and Sorensen [4] or Arbenz and Golub [1]. More recent works focusing on bounds for the perturbed eigenvalues include papers of Ipsen and Nadler [10], Bénasséni [3], Cheng, Luo and Li [5], Cheng, Song, Yang and Si [6] among others. It is also worth noting that Ding and Zhou [7] consider rank-one perturbed matrices of special structure. Finally, Mehl, Mehrmann, Ran and Rodman [12] develop a comprehensive study of the perturbation theory for structured matrices under structured rank one perturbations while Ran and Wojtylak [14] consider curves associated to eigenvalues in the framework of rank one perturbations. However, despite this amount of results for eigenvalues, less work has been devoted to the behavior of eigenvectors under perturbations of restricted rank.

We consider the perturbation of a symmetric matrix  $\mathbf{A}$  of order  $p$  to

$$\mathbf{B} = \mathbf{A} + \tau \mathbf{c} \mathbf{c}^t \quad (1)$$

where  $\tau$  is a real positive constant and  $\mathbf{c}$  is a unit two norm column vector in  $\mathbb{R}^p$  with  $\mathbf{c}^t = (c_1, \dots, c_p)$ . For  $i = 1, \dots, p$ , the eigenvectors  $\mathbf{u}_i$  of  $\mathbf{A}$  associated to the eigenvalues ranked in decreasing order:  $\lambda_1(\mathbf{A}) \geq \lambda_2(\mathbf{A}) \geq \dots \lambda_p(\mathbf{A}) \geq 0$  are assumed to have a unit two norm. The same assumption is made for the eigenvectors  $\mathbf{v}_i$  of  $\mathbf{B}$  associated to the eigenvalues  $\lambda_1(\mathbf{B}) \geq \lambda_2(\mathbf{B}) \geq \dots \lambda_p(\mathbf{B}) \geq 0$ .

This paper considers the inner products  $\mathbf{c}^t \mathbf{u}_i$ ,  $\mathbf{c}^t \mathbf{v}_i$  and  $\mathbf{u}_i^t \mathbf{v}_i$  for  $i = 1, \dots, p$ . Since the vectors  $\mathbf{c}$ ,  $\mathbf{u}_i$  and  $\mathbf{v}_i$  are assumed to have a unit norm, these three inner products are involved in the fundamental cosine formula of spherical geometry. However, the goal of this paper is to provide new relations for these inner products and to derive properties specific to the eigenvectors corresponding to the largest and smallest non-zero eigenvalues. The paper is organized as follows. Section 2 is devoted to preliminary results and provides relations involving the inner products  $\mathbf{c}^t \mathbf{u}_i$ ,  $\mathbf{c}^t \mathbf{v}_i$  and  $\mathbf{u}_i^t \mathbf{v}_i$  for  $i = 1, \dots, p$ . In particular it is shown that the matrix of order  $p$  with elements  $(\mathbf{v}_i^t \mathbf{u}_j)^2$  is a doubly stochastic matrix. Focusing more specifically on the eigenvectors corresponding to the largest and smallest non-zero eigenvalues, Section 3 provides inequalities for the inner products  $\mathbf{c}^t \mathbf{v}_i$  using a majorization theorem of Hardy, Littelwood and Pólya. Finally, in Section 4, additional inequalities are proposed and refined developments are suggested for infinitesimal perturbations.

## 2 Preliminary results

As first lemma we use a result due to Golub [9, Corollary 8.1-5 p.270].

**Lemma 1** *There exist  $p$  real numbers  $q_i$  satisfying  $q_i \geq 0$  and  $\sum_{i=1}^p q_i = 1$  such that:  $\lambda_i(\mathbf{B}) = \lambda_i(\mathbf{A}) + \tau q_i$  for  $i = 1, \dots, p$ .*

It should be noted that  $q_i = (\lambda_i(\mathbf{B}) - \lambda_i(\mathbf{A}))/\tau$  is the percentage of increase of the trace accounted for by the  $i$ th eigenvalue when  $\mathbf{A}$  is modified to  $\mathbf{B}$ . The next proposition provides a new formulation of  $q_i$  which will be useful since  $q_1$  and  $q_p$  play a central role in several results of the paper.

**Proposition 2** *For  $i = 1, \dots, p$  we have:*

$$q_i = \frac{(\mathbf{c}^t \mathbf{u}_i)(\mathbf{c}^t \mathbf{v}_i)}{\mathbf{u}_i^t \mathbf{v}_i} \quad (2)$$

if  $\mathbf{u}_i^t \mathbf{v}_i \neq 0$

*Proof:* The proof of the result is straightforward multiplying the left-hand side of (1) by  $\mathbf{u}_i^t$  and its right-hand side by  $\mathbf{v}_i$  and using  $\mathbf{A}\mathbf{u}_i = \lambda_i(\mathbf{A})\mathbf{u}_i$  and  $\mathbf{B}\mathbf{v}_i = \lambda_i(\mathbf{B})\mathbf{v}_i$ .

**Proposition 3** *Let  $\mathbf{M} = (m_{ij})$  denote the  $p \times p$  matrix whose entries are defined by:  $m_{ij} = (\mathbf{v}_i^t \mathbf{u}_j)^2$ . Then  $\mathbf{M}$  is a doubly stochastic matrix.*

*Proof:* For  $i = 1, \dots, p$  we have:

$$\sum_{j=1}^p m_{ij} = \sum_{j=1}^p (\mathbf{v}_i^t \mathbf{u}_j)^2 = \sum_{j=1}^p (\mathbf{v}_i^t \mathbf{u}_j)(\mathbf{u}_j^t \mathbf{v}_i) = \mathbf{v}_i^t \left( \sum_{j=1}^p \mathbf{u}_j \mathbf{u}_j^t \right) \mathbf{v}_i = \mathbf{v}_i^t \mathbf{v}_i = 1$$

since  $\sum_{j=1}^p \mathbf{u}_j \mathbf{u}_j^t$  is the identity matrix and the eigenvectors  $\mathbf{v}_i$  are normalized for the two norm. Therefore all the rows of  $\mathbf{M}$  sum to one. Similarly we get

$\sum_{i=1}^p m_{ij} = 1$  for  $j = 1, \dots, p$  so that all the columns of  $\mathbf{M}$  also sum to one. Since all the entries of  $\mathbf{M}$  are nonnegative, this matrix is doubly stochastic.

**Theorem 4** *Let  $\mathbf{l}_\mathbf{A}$  and  $\mathbf{l}_\mathbf{B}$  denote the two column vectors of  $\mathbb{R}^p$  defined by  $\mathbf{l}_\mathbf{A}^t = (\lambda_1(\mathbf{A}), \lambda_2(\mathbf{A}), \dots, \lambda_p(\mathbf{A}))$  and  $\mathbf{l}_\mathbf{B}^t = (\lambda_1(\mathbf{B}), \lambda_2(\mathbf{B}), \dots, \lambda_p(\mathbf{B}))$ . Denote also by  $\mathbf{h}$  the column vector defined by  $\mathbf{h}^t = ((\mathbf{c}^t \mathbf{v}_1)^2, \dots, (\mathbf{c}^t \mathbf{v}_p)^2)$ . Then:*

$$\mathbf{M} \mathbf{l}_\mathbf{A} = \mathbf{l}_\mathbf{B} - \tau \mathbf{h} \quad (3)$$

*Proof:* Using the definition of  $\mathbf{M}$ , the  $i$ th row of  $\mathbf{M} \mathbf{l}_\mathbf{A}$  may be written as:

$$\begin{aligned} [\mathbf{M} \mathbf{l}_\mathbf{A}]_i &= \sum_{j=1}^p \lambda_j(\mathbf{A}) m_{ij} = \sum_{j=1}^p \lambda_j(\mathbf{A}) (\mathbf{v}_i^t \mathbf{u}_j)^2 = \sum_{j=1}^p \lambda_j(\mathbf{A}) (\mathbf{v}_i^t \mathbf{u}_j) (\mathbf{u}_j^t \mathbf{v}_i) \\ &= \mathbf{v}_i^t \left[ \sum_{j=1}^p \lambda_j(\mathbf{A}) \mathbf{u}_j \mathbf{u}_j^t \right] \mathbf{v}_i \\ &= \mathbf{v}_i^t \mathbf{A} \mathbf{v}_i. \end{aligned}$$

From (1) we then have:

$$[\mathbf{M} \mathbf{l}_\mathbf{A}]_i = \mathbf{v}_i^t (\mathbf{B} - \tau \mathbf{c} \mathbf{c}^t) \mathbf{v}_i = \lambda_i(\mathbf{B}) - \tau (\mathbf{c}^t \mathbf{v}_i)^2$$

since  $\mathbf{B} \mathbf{v}_i = \lambda_i(\mathbf{B}) \mathbf{v}_i$  and  $\mathbf{v}_i^t \mathbf{v}_i = 1$ . This proves (3).

**Corollary 5** *For  $i = 1, \dots, p$ , we have  $\lambda_i(\mathbf{B}) - \tau (\mathbf{c}^t \mathbf{v}_i)^2 \geq 0$ .*

*Proof:* The result is obvious from (3) since all the entries in  $\mathbf{M}$  and  $\mathbf{l}_\mathbf{A}$  are non negative.

Finally in the next section we shall use a theorem of majorization due to Hardy, Littelwood and Pólya which requires to introduce the following definition:

**Definition 6** Given a  $p$ -tuple  $\mathbf{x}$  of nonnegative real numbers with  $\mathbf{x}^t = (x_1, x_2, \dots, x_p)$  we denote by  $\mathbf{x}^*$  the  $p$ -tuple with the same entries as  $\mathbf{x}$  but rearranged in non-increasing order. In other words,  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_p^*)^t$  with  $x_1^* \geq x_2^* \geq \dots \geq x_p^*$ . The  $p$ -tuple  $\mathbf{x}$  is said to be majorized by the  $p$ -tuple  $\mathbf{y}$  with the notation  $\mathbf{x} \prec \mathbf{y}$  if :

$$\sum_{j=1}^k x_j^* \leq \sum_{j=1}^k y_j^*$$

for  $k = 1, \dots, p-1$  and

$$\sum_{j=1}^p x_j^* = \sum_{j=1}^p y_j^*.$$

As a simple consequence of the previous definition we have the following corollary whose proof is obvious.

**Corollary 7** Consider two  $p$ -tuples of nonnegative real numbers  $\mathbf{x}$  and  $\mathbf{y}$  such that  $\mathbf{x} \prec \mathbf{y}$ . For any  $p \times p$  permutation matrix  $\mathbf{P}$ , define  $\mathbf{z} = \mathbf{P}\mathbf{x}$ . Letting  $\mathbf{z}^t = (z_1, z_2, \dots, z_p)$  and  $\mathbf{y}^t = (y_1, y_2, \dots, y_p)$  we then have:

$$\sum_{j=1}^k z_j \leq \sum_{j=1}^k y_j^*$$

for  $k = 1, \dots, p-1$  and

$$\sum_{j=1}^p z_j = \sum_{j=1}^p y_j^*.$$

Note that this corollary holds when considering in particular the identity matrix for  $\mathbf{P}$ .

**Theorem 8** For two  $p$ -tuples of nonnegative real numbers  $\mathbf{x}$  and  $\mathbf{y}$ , we have  $\mathbf{x} \prec \mathbf{y}$  if and only if there exists a doubly stochastic matrix  $\mathbf{S}$  such that  $\mathbf{x} = \mathbf{S}\mathbf{y}$ .

This well known result due to Hardy, Littelwood and Pólya can be found in Minc [13, Theorem 2.2 p.110] for example . A comprehensive reference on inequalities in relation with the theory of majorization is also provided by Marshall, Olkin and Arnold [11].

### 3 Main inequalities

**Theorem 9** *Let  $r$  denote the rank of  $\mathbf{A}$ . Then we have the following inequality:*

$$(\mathbf{c}^t \mathbf{v}_1)^2 \geq \frac{\lambda_1(\mathbf{B}) - \lambda_1(\mathbf{A})}{\tau} = q_1 \quad (4)$$

Furthermore, if  $\mathbf{c} \in \text{range}(A)$ , we have:

$$(\mathbf{c}^t \mathbf{v}_r)^2 \leq \frac{\lambda_r(\mathbf{B}) - \lambda_r(\mathbf{A})}{\tau} = q_r \quad (5)$$

and if  $\mathbf{c} \notin \text{range}(A)$ , we have:

$$(\mathbf{c}^t \mathbf{v}_{r+1})^2 \leq \frac{\lambda_{r+1}(\mathbf{B}) - \lambda_{r+1}(\mathbf{A})}{\tau} = \frac{\lambda_{r+1}(\mathbf{B})}{\tau} \quad (6)$$

*Proof:* From Proposition 3 we know that the matrix  $\mathbf{M}$  involved in Equation (3) is doubly stochastic. Therefore we can derive from this proposition that:

$$\mathbf{l}_{\mathbf{B}} - \tau \mathbf{h} \prec \mathbf{l}_{\mathbf{A}} \quad (7)$$

by the Theorem 8 of Hardy, Littelwood and Pólya. Noting that  $\mathbf{l}_{\mathbf{A}} = \mathbf{l}_{\mathbf{A}}^*$  and using Corollary 7 because the ranking order of the entries  $\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2$  is unknown , we then have:

$$\lambda_1(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_1)^2 \leq \lambda_1(\mathbf{A})$$



which proves (4).

Now assume that  $\mathbf{c} \in \text{range}(\mathbf{A})$  so that both  $\mathbf{A}$  and  $\mathbf{B}$  are of rank  $r$ . Then for  $i \geq r + 1$  we have  $\lambda_i(\mathbf{B}) = \lambda_i(\mathbf{A}) = 0$  and from Corollary 5:

$$\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2 = -\tau(\mathbf{c}^t \mathbf{v}_i)^2 \geq 0$$

so that  $(\mathbf{c}^t \mathbf{v}_i) = 0$ . Then:

$$\sum_{i=1}^r (\mathbf{c}^t \mathbf{v}_i)^2 = \sum_{i=1}^p (\mathbf{c}^t \mathbf{v}_i)^2 = \mathbf{c}^t \left( \sum_{i=1}^p \mathbf{v}_i \mathbf{v}_i^t \right) \mathbf{c} = \mathbf{c}^t \mathbf{c} = 1$$

and

$$\sum_{i=1}^r (\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2) = \text{Tr}(\mathbf{B}) - \tau = \text{Tr}(\mathbf{B} - \tau \mathbf{c} \mathbf{c}^t) = \text{Tr}(\mathbf{A}) = \sum_{i=1}^r \lambda_i(\mathbf{A})$$

From (7) used together with Corollary 7, we have:

$$\sum_{i=1}^{r-1} (\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2) \leq \sum_{i=1}^{r-1} \lambda_i(\mathbf{A})$$

and

$$\sum_{i=1}^r (\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2) = \sum_{i=1}^r \lambda_i(\mathbf{A})$$

so that:

$$\lambda_r(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_r)^2 \geq \lambda_r(\mathbf{A})$$

which proves (5).

Finally if  $\mathbf{c} \notin \text{range}(\mathbf{A})$ , the matrix  $\mathbf{B}$  is of rank  $r + 1$  so that  $\lambda_{r+1}(\mathbf{B}) > 0$  and  $\lambda_i(\mathbf{B}) = 0$  for  $i \geq r + 2$  while we have always  $\lambda_i(\mathbf{A}) = 0$  for  $i \geq r + 1$ . Therefore from Corollary 5 we have for  $i \geq r + 2$ :

$$\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2 = -\tau(\mathbf{c}^t \mathbf{v}_i)^2 \geq 0$$

so that  $(\mathbf{c}^t \mathbf{v}_i) = 0$  and

$$\sum_{i=1}^{r+1} (\mathbf{c}^t \mathbf{v}_i)^2 = \sum_{i=1}^p (\mathbf{c}^t \mathbf{v}_i)^2 = \mathbf{c}^t \left( \sum_{i=1}^p \mathbf{v}_i \mathbf{v}_i^t \right) \mathbf{c} = \mathbf{c}^t \mathbf{c} = 1.$$

Then

$$\sum_{i=1}^{r+1} (\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2) = \text{Tr}(\mathbf{B}) - \tau = \text{Tr}(\mathbf{B} - \tau \mathbf{c} \mathbf{c}^t) = \text{Tr}(\mathbf{A}) = \sum_{i=1}^r \lambda_i(\mathbf{A}).$$

From (7) used again together with Corollary 7, we have:

$$\sum_{i=1}^r (\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2) \leq \sum_{i=1}^r \lambda_i(\mathbf{A})$$

and

$$\sum_{i=1}^{r+1} (\lambda_i(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_i)^2) = \sum_{i=1}^{r+1} \lambda_i(\mathbf{A})$$

so that:

$$\lambda_{r+1}(\mathbf{B}) - \tau(\mathbf{c}^t \mathbf{v}_{r+1})^2 \geq \lambda_{r+1}(\mathbf{A})$$

which proves (6) since  $\lambda_{r+1}(\mathbf{A}) = 0$ .

The following corollary provides two additional inequalities.

**Corollary 10** *With the notation of the previous theorem we have:*

$$\frac{|\mathbf{c}^t \mathbf{u}_1|}{|\mathbf{u}_1^t \mathbf{v}_1|} \leq |\mathbf{c}^t \mathbf{v}_1| \quad (8)$$

and

$$(\mathbf{c}^t \mathbf{v}_1)^2 \geq (\mathbf{c}^t \mathbf{u}_1)^2 \quad (9)$$

and

$$(\mathbf{u}_1^t \mathbf{v}_1)^2 \geq (\mathbf{c}^t \mathbf{u}_1)^2 \quad (10)$$

*Proof:* First, since

$$0 \leq q_1 = \frac{(\mathbf{c}^t \mathbf{u}_1)(\mathbf{c}^t \mathbf{v}_1)}{\mathbf{u}_1^t \mathbf{v}_1} \leq 1,$$

we have

$$q_1 = \frac{|\mathbf{c}^t \mathbf{u}_1| |\mathbf{c}^t \mathbf{v}_1|}{|\mathbf{u}_1^t \mathbf{v}_1|}.$$

From (4) in Theorem 9 we then get

$$(\mathbf{c}^t \mathbf{v}_1)^2 \geq \frac{|\mathbf{c}^t \mathbf{u}_1| |\mathbf{c}^t \mathbf{v}_1|}{|\mathbf{u}_1^t \mathbf{v}_1|}.$$

Thus,

$$\frac{|\mathbf{c}^t \mathbf{u}_1|}{|\mathbf{u}_1^t \mathbf{v}_1|} \leq |\mathbf{c}^t \mathbf{v}_1|.$$

or equivalently

$$(\mathbf{c}^t \mathbf{v}_1)^2 \geq \frac{(\mathbf{c}^t \mathbf{u}_1)^2}{(\mathbf{u}_1^t \mathbf{v}_1)^2} \quad (11)$$

Inequality (9) follows since  $(\mathbf{u}_1^t \mathbf{v}_1)^2 \leq 1$ .

Finally, noting that in (11) we have  $(\mathbf{c}^t \mathbf{v}_1)^2 \leq 1$ , we get  $(\mathbf{u}_1^t \mathbf{v}_1)^2 \geq (\mathbf{c}^t \mathbf{u}_1)^2$ .

## 4 Further inequalities

### 4.1 First inequalities

In practice, the eigenvalues of the perturbed matrix  $\mathbf{B}$  are unknown in contrast to those of the initial one  $\mathbf{A}$ . Therefore, it is interesting to obtain inequalities similar to those of Theorem 9 in the previous section but involving only the eigenvalues of  $\mathbf{A}$ . This can be achieved by using results of Ipsen

and Nadler [10] or inequalities of Bénasséni [2]. A lower bound to  $(\mathbf{c}^t \mathbf{v}_1)^2$  based only on the eigenvalues of the initial matrix  $\mathbf{A}$  and the vector  $\mathbf{c}$  can be obtained by introducing the lower bound to  $\lambda_1(\mathbf{B})$  given in the Theorem 2.4 of Ipsen and Nadler or the results in Section 3 of Bénasséni. In the same way, other inequalities in these two papers can be used to obtain upper bounds to  $\mathbf{c}^t \mathbf{v}_r$  depending only  $\mathbf{A}$  and the vector  $\mathbf{c}$ . For the sake of brevity, we only illustrate this approach in the case  $r = p$  with the upper bound for  $\mathbf{c}^t \mathbf{v}_p$ . First we use the upper bound  $\lambda_{\min}(\mathbf{U}_+)$  to  $\lambda_p(\mathbf{B})$  defined in Theorem 2.1 and Corollary 2.2 of Ipsen and Nadler [10]. Letting  $\text{gap}_p = \lambda_{p-1}(\mathbf{A}) - \lambda_p(\mathbf{A})$  and  $\text{norm}^2 = (\mathbf{c}^t \mathbf{u}_{p-1})^2 + (\mathbf{c}^t \mathbf{u}_p)^2$  we get:

$$\lambda_p(\mathbf{B}) \leq \lambda_p(\mathbf{A}) + \frac{\text{gap}_p + \tau \text{norm}^2 - \sqrt{(\text{gap}_p + \tau \text{norm}^2)^2 - 4\tau \text{gap}_p (\mathbf{c}_p)^2}}{2} \quad (12)$$

Incorporating the right hand side of (12) in Inequality (5) of Theorem 9 yields:

$$(\mathbf{c}^t \mathbf{v}_p)^2 \leq \frac{\text{gap}_p + \tau \text{norm}^2 - \sqrt{(\text{gap}_p + \tau \text{norm}^2)^2 - 4\tau \text{gap}_p (\mathbf{c}_p)^2}}{2\tau} \quad (13)$$

Finally, inequalities in Sections 2 and 3 of Bénasséni [2] can also be used. His Section 2 provides for example the following simple inequality:

$$\lambda_p(\mathbf{B}) \leq \lambda_p(\mathbf{A}) + \tau \frac{(\mathbf{c}^t \mathbf{u}_p)^2 + \sqrt{(\mathbf{c}^t \mathbf{u}_p)^4 + 4(\mathbf{c}^t \mathbf{u}_p)^2 \{\|\mathbf{c}\|^2 - (\mathbf{c}^t \mathbf{u}_p)^2\}}}{2} \quad (14)$$

which is sharper than the usual inequality:  $\lambda_r(\mathbf{B}) \leq \lambda_r(\mathbf{A}) + \tau \|\mathbf{c}\|^2$ . Incorporating the right hand side of this inequality in (5) of Theorem 9 we get:

$$(\mathbf{c}^t \mathbf{v}_p)^2 \leq \frac{(\mathbf{c}^t \mathbf{u}_p)^2 + \sqrt{(\mathbf{c}^t \mathbf{u}_p)^4 + 4(\mathbf{c}^t \mathbf{u}_p)^2 \{1 - (\mathbf{c}^t \mathbf{u}_p)^2\}}}{2} \quad (15)$$

since  $\mathbf{c}$  is normalized.

## 4.2 Inequalities for infinitesimal perturbations

When considering infinitesimal perturbations, power expansions of the eigenvalues allow the derivation of refined bounds for  $\mathbf{c}^t \mathbf{v}_1$  and  $\mathbf{c}^t \mathbf{v}_p$  as detailed in the remaining of this section. It should be noted that these infinitesimal perturbations corresponding to small values of  $\tau$  are involved in many fields of applications such as statistics for example where one is concerned by the effects of small changes in data on some parameters to be evaluated.

More specifically, we focus on a specific eigenvalue  $\lambda(\mathbf{A})$  of the initial matrix  $\mathbf{A}$  assumed to be simple and on its corresponding normalized eigenvector  $\mathbf{u}$ . We know from matrix perturbation analysis that, for sufficiently small values of  $\tau$ , there exist an eigenvalue  $\lambda(\mathbf{B})$  of  $\mathbf{B}$  and a corresponding non normalized eigenvector  $\mathbf{v}$  which can be expressed under the following convergent power expansions:

$$\lambda(\mathbf{B}) = \lambda(\mathbf{A}) + \tau\mu_1 + \tau^2\mu_2 + \dots \tau^n\mu_n + O(\tau^{n+1}) \quad (16)$$

and

$$\mathbf{v} = \mathbf{u} + \tau\mathbf{w}_1 + \tau^2\mathbf{w}_2 + \dots \tau^n\mathbf{w}_n + O(\tau^{n+1}) \quad (17)$$

Letting  $\mathbf{C} = \mathbf{c}\mathbf{c}^t$  for notational convenience, a lemma of Wang and Liski [16] provides some general relations for the real numbers  $\mu_j$  and the vectors  $\mathbf{w}_j$

involved in Equations (16) and (17) when considering perturbation (1). This lemma is given below.

**Lemma 11** *Using the notation  $\mathbf{M}^+$  for the Moore-Penrose inverse of any matrix  $\mathbf{M}$  and considering the perturbation  $\mathbf{B} = \mathbf{A} + \tau\mathbf{C}$ , a solution for the parameters  $\mu_j$  and  $\mathbf{w}_j$  involved in Equations (16) and (17) is provided by the following relations:*

$$\mu_1 = \mathbf{u}^t \mathbf{C} \mathbf{u} \quad (18)$$

$$\mathbf{w}_1 = -[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+ \mathbf{C} \mathbf{u} \quad (19)$$

and for  $j = 2, \dots, n$ :

$$\mu_j = \mathbf{u}^t \mathbf{C} \mathbf{w}_{j-1} \quad (20)$$

$$\mathbf{w}_j = -[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+ [\mathbf{C} \mathbf{w}_{j-1} - \sum_{i=1}^{j-1} \mu_i \mathbf{w}_{j-i}] \quad (21)$$

where  $\mathbf{I}_p$  denote the identity matrix of order  $p$ .

The proof of this lemma is provided by Wang and Liski [16] through a usual approach detailed for example in Wilkinson [17, pp.66-70]. We follow below their derivation while introducing some necessary complements for the sake of generality.

*Proof:*

Equating the coefficients of  $\tau^j$  for  $j = 1, \dots, n$  in the relation:

$$\mathbf{B} \mathbf{v} = \lambda(\mathbf{B}) \mathbf{v} \quad (22)$$

provides the following set of equations:

$$\mathbf{A}\mathbf{w}_1 + \mathbf{C}\mathbf{u} = \lambda(\mathbf{A})\mathbf{w}_1 + \mu_1\mathbf{u} \quad (23)$$

$$\mathbf{A}\mathbf{w}_2 + \mathbf{C}\mathbf{w}_1 = \lambda(\mathbf{A})\mathbf{w}_2 + \mu_1\mathbf{w}_1 + \mu_2\mathbf{u} \quad (24)$$

and more generally for  $j = 2, \dots, n$

$$\mathbf{A}\mathbf{w}_j + \mathbf{C}\mathbf{w}_{j-1} = \lambda(\mathbf{A})\mathbf{w}_j + \sum_{i=1}^{j-1} \mu_i\mathbf{w}_{j-i} + \mu_j\mathbf{u} \quad (25)$$

First, multiplying the left-hand side of (23) by  $\mathbf{u}^t$  and using the relations  $\mathbf{A}\mathbf{u} = \lambda(\mathbf{A})\mathbf{u}$  and  $\mathbf{u}^t\mathbf{u} = 1$ , we get (18).

Now in order to get (19), rewrite (23) as :

$$[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]\mathbf{w}_1 = -(\mathbf{C} - \mu_1\mathbf{I}_p)\mathbf{u}. \quad (26)$$

The following two remarks are necessary for the derivation.

First, using (18) we get  $\mathbf{u}^t(\mathbf{C} - \mu_1\mathbf{I}_p)\mathbf{u} = 0$  so that  $(\mathbf{C} - \mu_1\mathbf{I}_p)\mathbf{u}$  is in the column space of  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]$ . Thus Equation (26) is consistent and all its solutions can be expressed as:

$$\mathbf{w}_1 = -[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^{-}(\mathbf{C} - \mu_1\mathbf{I}_p)\mathbf{u} \quad (27)$$

where  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^{-}$  denotes any generalized inverse of  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]$ .

Second, if we have two solutions  $\mathbf{w}_1$  and  $\mathbf{w}'_1$  of (26), these solutions satisfy:  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p](\mathbf{w}_1 - \mathbf{w}'_1) = 0$ . Therefore  $\mathbf{w}'_1 - \mathbf{w}_1$  is in the null space of  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]$  and we have :

$$\mathbf{w}'_1 = \mathbf{w}_1 + \alpha\mathbf{u} \quad (28)$$

for some coefficient  $\alpha \in \mathbb{R}$  since  $\lambda(\mathbf{A})$  is simple. Note that one particular solution of (26) is given by:

$$\mathbf{w}_1 = -[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+(\mathbf{C} - \mu_1\mathbf{I}_p)\mathbf{u} \quad (29)$$

using the Moore-Penrose inverse:

$$[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+ = \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{\mathbf{u}_i \mathbf{u}_i^t}{\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})}.$$

Then, all the solutions of (26) are of the form:

$$\mathbf{w}_1 = -[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+(\mathbf{C} - \mu_1\mathbf{I}_p)\mathbf{u} + \alpha\mathbf{u} \quad (30)$$

or equivalently

$$\mathbf{w}_1 = -[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+\mathbf{C}\mathbf{u} + \alpha\mathbf{u} \quad (31)$$

since  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+\mathbf{u} = 0$ . The choice of (19) as particular solution is justified by attractive properties which are detailed at the end of the proof.

Note that  $\mathbf{u}^t \mathbf{w}_1 = 0$  since  $\mathbf{u}$  is in the null space of  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+$ .

Now in order to derive (21) for  $j = 2, \dots, n$  we have simply to rewrite (25) as:

$$[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]\mathbf{w}_j = -\mathbf{C}\mathbf{w}_{j-1} + \sum_{i=1}^{j-1} \mu_i \mathbf{w}_{j-i} + \mu_j \mathbf{u}. \quad (32)$$

Premultiplying this last equation by  $\mathbf{u}^t$  we get:

$$\mathbf{u}^t [-\mathbf{C}\mathbf{w}_{j-1} + \sum_{i=1}^{j-1} \mu_i \mathbf{w}_{j-i} + \mu_j \mathbf{u}] = 0 \quad (33)$$

Therefore the right hand side of (32) is in the column space of  $[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]$  and Equation (32) is consistent. Similarly to  $\mathbf{w}_1$  we can choose (21) as a



particular solution.

Finally, in order to derive the parameters  $\mu_j$  involved in the expansion of  $\lambda(\mathbf{A})$  in (16), we have simply to note from (21) that:

$$\mathbf{u}^t \mathbf{w}_j = 0 \quad (34)$$

Then, developing (33) and using (34) we get (20).

Now, some justifications of the choice of the Moore-Penrose inverse for  $\mathbf{w}_1$  in (19) are given below.

First, among all the solutions provided by (27), the use of the Moore-Penrose inverse is the only one which preserves the orthogonality between  $\mathbf{w}_1$  and  $\mathbf{u}$ . This is the same between  $\mathbf{w}_j$  and  $\mathbf{u}$  for  $j = 1, \dots, n$ . This orthogonality is crucial for the recurrence process and motivates the choice of (19) and (21) as solutions.

Second, it should be noted from (28) that:

$$\|\mathbf{w}'_1\|^2 = \|\mathbf{w}_1\|^2 + \alpha^2$$

since  $\mathbf{u}^t \mathbf{w}_1 = 0$ . Then, focusing on the first order approximation  $\mathbf{v}^{(1)} = \mathbf{u} + \tau \mathbf{w}'_1$  of  $\mathbf{v}$  we have:

$$\begin{aligned} \|\mathbf{u} + \tau \mathbf{w}'_1\|^2 &= \|\mathbf{u}\|^2 + \tau^2 \|\mathbf{w}_1\|^2 + \tau^2 \alpha^2 + 2\tau\alpha \\ &= \|\mathbf{u}\|^2 + 2\tau\alpha + \tau^2(\|\mathbf{w}_1\|^2 + \alpha^2) \\ &= 1 + 2\tau\alpha + \tau^2 \|\mathbf{w}'_1\|^2. \end{aligned}$$

Hence, we see that the first order approximation of the norm of  $\mathbf{v}^{(1)}$  is given

by:  $1+2\tau\alpha$  which is equal to one if and only if  $\alpha = 0$  or equivalently  $\mathbf{w}'_1 = \mathbf{w}_1$ . Having an approximation of  $\mathbf{v}$  whose norm only differs from that of the initial vector  $\mathbf{u}$  by terms of degree 2 or greater allows a better estimation of the direction of  $\mathbf{v}$  by  $\mathbf{v}^{(1)}$ .

It should be noted that the previous theorem holds for any symmetric perturbation matrix  $\mathbf{C}$ . However, in the framework of rank one perturbation, the following proposition provides developed formulations involving the perturbation vector  $\mathbf{c}$ .

**Proposition 12** *Consider the perturbation  $\mathbf{B} = \mathbf{A} + \tau\mathbf{c}\mathbf{c}^t$  defined in Equation (1). Then the parameters  $\mu_j$  and  $\mathbf{w}_j$  involved in the power expansions (16) and (17) can be expressed as follows for  $j = 1, 2$ :*

$$\mu_1 = (\mathbf{c}^t \mathbf{u})^2 \quad (35)$$

$$\mathbf{w}_1 = -(\mathbf{c}^t \mathbf{u}) \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{(\mathbf{c}^t \mathbf{u}_i)}{\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})} \mathbf{u}_i \quad (36)$$

$$\mu_2 = -(\mathbf{c}^t \mathbf{u})^2 \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})} \quad (37)$$

$$\begin{aligned} \mathbf{w}_2 = (\mathbf{c}^t \mathbf{u}) \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \sum_{\substack{k=1 \\ \lambda_k(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{(\mathbf{c}^t \mathbf{u}_i)(\mathbf{c}^t \mathbf{u}_k)^2}{[\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})][\lambda_k(\mathbf{A}) - \lambda(\mathbf{A})]} \mathbf{u}_i \\ - (\mathbf{c}^t \mathbf{u})^3 \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{(\mathbf{c}^t \mathbf{u}_i)}{[\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})]^2} \mathbf{u}_i \end{aligned} \quad (38)$$

Finally, we have also:

$$\begin{aligned} \mu_3 = (\mathbf{c}^t \mathbf{u})^2 & \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \sum_{\substack{k=1 \\ \lambda_k(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2 (\mathbf{c}^t \mathbf{u}_k)^2}{[\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})][\lambda_k(\mathbf{A}) - \lambda(\mathbf{A})]} \\ & - (\mathbf{c}^t \mathbf{u})^4 \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{[\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})]^2} \quad (39) \end{aligned}$$

*Proof:*

The derivation of the formula is straightforward by simply substituting  $\mathbf{c}\mathbf{c}^t$  for  $\mathbf{C}$  in Theorem 11 and using the formulation of

$$[\mathbf{A} - \lambda(\mathbf{A})\mathbf{I}_p]^+ = \sum_{\substack{i=1 \\ \lambda_i(\mathbf{A}) \neq \lambda(\mathbf{A})}}^p \frac{\mathbf{u}_i \mathbf{u}_i^t}{\lambda_i(\mathbf{A}) - \lambda(\mathbf{A})}$$

already given.

Now we use the previous expansions of the eigenvalues in power series to provides refined bounds derived from Theorem 9.

**Proposition 13** *Assuming that  $\lambda_1(\mathbf{A})$  is simple and that  $\tau$  is sufficiently small in order to express  $\lambda_1(\mathbf{B})$  under the following convergent power series:*

$$\lambda_1(\mathbf{B}) = \lambda_1(\mathbf{A}) + \tau\mu_1 + \tau^2\mu_2 + \dots \tau^n\mu_n + O(\tau^{n+1})$$

*we have the following inequality.*

$$\begin{aligned}
(\mathbf{c}^t \mathbf{v}_1)^2 &\geq (\mathbf{c}^t \mathbf{u}_1)^2 + \tau (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})} \\
&\quad + \tau^2 \left\{ (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \sum_{k=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2 (\mathbf{c}^t \mathbf{u}_k)^2}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})][\lambda_1(\mathbf{A}) - \lambda_k(\mathbf{A})]} \right. \\
&\quad \left. - (\mathbf{c}^t \mathbf{u}_1)^4 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})]^2} \right\} + O(\tau^3) \quad (40)
\end{aligned}$$

*Proof:* This result simply follows from (4) used together with Proposition 12 taking  $\mathbf{u} = \mathbf{u}_1$ .

As simple consequence of this proposition we have the following two corollaries.

**Corollary 14** *Under assumptions in the previous proposition we have also the following inequality:*

$$\begin{aligned}
(\mathbf{c}^t \mathbf{v}_1)^2 &\geq (\mathbf{c}^t \mathbf{u}_1)^2 + \tau (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})} \\
&\quad - \tau^2 (\mathbf{c}^t \mathbf{u}_1)^4 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})]^2} \quad (41)
\end{aligned}$$

*Proof:* The result is obvious simply rewriting (40) as

$$\begin{aligned}
(\mathbf{c}^t \mathbf{v}_1)^2 &\geq (\mathbf{c}^t \mathbf{u}_1)^2 + \tau (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})} \\
&\quad - \tau^2 (\mathbf{c}^t \mathbf{u}_1)^4 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})]^2} \\
&\quad + \tau^2 (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \sum_{k=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2 (\mathbf{c}^t \mathbf{u}_k)^2}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})][\lambda_1(\mathbf{A}) - \lambda_k(\mathbf{A})]} + O(\tau^3) \quad (42)
\end{aligned}$$

and neglecting in the last line the second order term (which is non negative) as well as higher order terms.

**Corollary 15** *Under assumptions of Proposition 13 we have the following inequality:*

$$\begin{aligned}
(\mathbf{c}^t \mathbf{v}_1)^2 &\geq (\mathbf{c}^t \mathbf{u}_1)^2 + \tau (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})} \\
&\quad + \tau^2 \{ (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \sum_{k=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2 (\mathbf{c}^t \mathbf{u}_k)^2}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})][\lambda_1(\mathbf{A}) - \lambda_k(\mathbf{A})]} \} \quad (43)
\end{aligned}$$

if  $(\mathbf{c}^t \mathbf{u}_i)^2 \geq (\mathbf{c}^t \mathbf{u}_1)^2$  for  $i = 2, \dots, p$ .

*Proof:* It should be noted that in Inequality (40) of Proposition 13 the first order term in  $\tau$  is nonnegative. However writing the second order term as

$$\begin{aligned}
(\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \sum_{\substack{k=2 \\ k \neq i}}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2 (\mathbf{c}^t \mathbf{u}_k)^2}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})][\lambda_1(\mathbf{A}) - \lambda_k(\mathbf{A})]} \\
+ (\mathbf{c}^t \mathbf{u}_1)^2 \sum_{i=2}^p \frac{(\mathbf{c}^t \mathbf{u}_i)^2 [(\mathbf{c}^t \mathbf{u}_i)^2 - (\mathbf{c}^t \mathbf{u}_1)^2]}{[\lambda_1(\mathbf{A}) - \lambda_i(\mathbf{A})]^2} \quad (44)
\end{aligned}$$

we see that this term is also non negative under the condition:  $(\mathbf{c}^t \mathbf{u}_i)^2 \geq (\mathbf{c}^t \mathbf{u}_1)^2$  for  $i = 2, \dots, p$  so that it can be neglected as well as higher order terms.

Finally, the last proposition is similar to Proposition 13 when considering the smallest eigenvalue.

**Proposition 16** *Assuming that  $\lambda_p(\mathbf{A}) > 0$  is simple and that  $\tau$  is sufficiently small in order to express  $\lambda_p(\mathbf{B})$  under the following convergent power*

series:

$$\lambda_p(\mathbf{B}) = \lambda_p(\mathbf{A}) + \tau\mu_1 + \tau^2\mu_2 + \dots \tau^n\mu_n + O(\tau^{n+1})$$

we have the following inequality.

$$\begin{aligned} (\mathbf{c}^t \mathbf{v}_p)^2 &\leq (\mathbf{c}^t \mathbf{u}_p)^2 - \tau (\mathbf{c}^t \mathbf{u}_p)^2 \sum_{i=1}^{p-1} \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{\lambda_i(\mathbf{A}) - \lambda_p(\mathbf{A})} \\ &\quad + \tau^2 \left\{ (\mathbf{c}^t \mathbf{u}_p)^2 \sum_{i=1}^{p-1} \sum_{k=1}^{p-1} \frac{(\mathbf{c}^t \mathbf{u}_i)^2 (\mathbf{c}^t \mathbf{u}_k)^2}{[\lambda_i(\mathbf{A}) - \lambda_p(\mathbf{A})][\lambda_k(\mathbf{A}) - \lambda_p(\mathbf{A})]} \right. \\ &\quad \left. - (\mathbf{c}^t \mathbf{u}_p)^4 \sum_{i=1}^{p-1} \frac{(\mathbf{c}^t \mathbf{u}_i)^2}{[\lambda_i(\mathbf{A}) - \lambda_p(\mathbf{A})]^2} \right\} + O(\tau^3) \quad (45) \end{aligned}$$

*Proof:* Since  $\lambda_p(\mathbf{A}) > 0$  we know that  $\mathbf{A}$  is of rank  $r = p$  and we have  $\mathbf{c} \in \text{range}(\mathbf{A})$ . Therefore the inequality can be simply derived using Proposition 12 with  $\mathbf{u} = \mathbf{u}_p$  together with (5).

Corollaries similar to the two previous ones could be also derived for  $(\mathbf{c}^t \mathbf{v}_p)^2$  but are omitted for the sake of brevity.

## References

- [1] P. Arbenz and G.H. Golub. On the spectral decomposition of hermitian matrices modified by low rank perturbations with applications. *SIAM Journal on Matrix Analysis and Applications*, 9:40–58, 1988.
- [2] J. Bénasséni. A note on bounds to the variation of eigenvalues in symmetric matrix perturbation of rank one. *Linear and Multilinear Algebra*, 27:111–116, 1990.

- [3] J. Bénasséni. Lower bounds for the largest eigenvalue of a symmetric matrix under perturbations of rank one. *Linear and Multilinear Algebra*, 59:565–569, 2011.
- [4] J.R. Bunch, C.P. Nielsen, and D.C. Sorensen. Rank-one modification of the symmetric eigenproblem. *Numerische Mathematik*, 31:31–48, 1978.
- [5] G. Cheng, X. Luo, and L. Li. The bounds of the smallest and largest eigenvalues for rank-one modification of the hermitian eigenvalue problem. *Applied Mathematics Letters*, 25:1191–1196, 2012.
- [6] G. Cheng, Z. Song, J. Yang, and J. Si. The bounds of the eigenvalues for rank-one modification of hermitian matrix. *Numerical Linear Algebra with Applications*, 21:98–107, 2014.
- [7] J. Ding and A. Zhou. Eigenvalues of rank-one updated matrices with some applications. *Applied Mathematics Letters*, 20:1223–1226, 2007.
- [8] G.H. Golub. Some modified matrix eigenvalue problems. *SIAM Review*, 15:318–334, 1973.
- [9] G.H. Golub and C.F. Van Loan. *Matrix computations*. The Johns Hopkins University Press, 1983.
- [10] I.C.F. Ipsen and B. Nadler. Refined perturbation bounds for eigenvalues of hermitian and non-hermitian matrices. *SIAM Journal on Matrix Analysis and Applications*, 31:40–53, 2009.
- [11] A. W. Marshall, I. Olkin, and B. Arnold. *Inequalities: Theory of Majorization and Its Application*. Springer, 2nd Edition, 2011.

- [12] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations. *Linear Algebra and its Applications*, 435:687–716, 2011.
- [13] H. Minc. *Nonnegative matrices*. Wiley, 1988.
- [14] A.C.M. Ran and M. Wojtylak. Eigenvalues of rank one perturbations of unstructured matrices. *Linear Algebra and its Applications*, 437:589–600, 2012.
- [15] R.C. Thomson. The behaviour of eigenvalues and singular values under perturbations of restricted rank. *Linear Algebra and its Applications*, 13:69–78, 1976.
- [16] S.-G. Wang and E.P. Liski. Effects of observations on the eigensystem of a sample covariance matrix. *Journal of Statistical Planning and Inference*, 36:215–226, 1993.
- [17] J.H. Wilkinson. *The algebraic eigenvalue problem*. Oxford University Press, 1988.