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HIGHLY TRANSITIVE ACTIONS OF GROUPS ACTING ON TREES

PIERRE FIMA, SOYOUNG MOON, AND YVES STALDER

Abstract. We show that a group acting on a non-trivial tree with finite edge stabilizers and icc vertex stabilizers admits a faithful and highly transitive action on an infinite countable set. This result is actually true for infinite vertex stabilizers and some more general, finite or infinite, edge stabilizers that we call highly core-free. We study the notion of highly core-free subgroups and give some examples. In the case of a free product amalgamated over a highly core-free subgroup and an HNN extension with a highly core-free base group we obtain a genericity result for faithful and highly transitive actions.

An action of a countable group Γ on an infinite countable set X is called *highly transitive* if it is n-transitive for all $n \geq 1$. It is easy to see that an action $\Gamma \curvearrowright X$ is highly transitive if and only if the image of Γ in the Polish group $S(X)$ of bijections of X is dense.

An obvious example of a highly transitive and faithful action is given by the action of the countable group of finitely supported permutations of X. This group is not finitely generated and far from being free (it is amenable).

The first explicit construction of a highly transitive and faithful action of the free group $\mathbf{F}_n$, for $2 \leq n \leq \infty$, was published in [McD76] by T.P. McDonough. Then, J. D. Dixon [Di89] showed that most (in a topological sense) finitely generated highly transitive subgroups of $S(X)$ are free.

A.M.W. Glass and S.H. McCleary [GM90] have constructed faithful and highly transitive action of a free product $\Gamma_1 * \Gamma_2$ of non-trivial countable (or finite) groups with Γ_2 having an element of infinite order. They also observed that $\mathbf{Z}_2 * \mathbf{Z}_2$ does not have a faithful and 2-transitive action and they asked for which non-trivial groups Γ_i, $i = 1, 2$, does $\Gamma_1 * \Gamma_2$ have a faithful and highly transitive action.

S.V. Gunhouse [Gu91] completely answered the question by constructing, for any non-trivial countable (or finite) groups Γ_i, $i = 1, 2$, with one of the Γ_i of size at least 3, a faithful and highly transitive action of $\Gamma_1 * \Gamma_2$. A similar result was obtained independently by K.K. Hickin [Hi90]. Recently, the last two authors [MS12] proved a genericity result for faithful and highly transitive actions of free products.

Using the techniques of Hickin, Gunhouse also characterized in his PhD (unpublished result) the non-trivial amalgamated free products with amalgamation over an Artinian group[1], that admit a faithful and highly transitive action. He proved that if $\Gamma = \Gamma_1 \underset{\Sigma}{*} \Gamma_2$ and Σ is an Artinian group, properly included in Γ_1 and Γ_2, then Γ admits a faithful and highly transitive action

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[1]that is, any decreasing chain of its distinct subgroups terminates after a finite number; e.g. finite groups.

if and only if the only subgroup of Σ which is normal in both Γ_1, Γ_2 is the trivial group. By the results of [Co08], this last condition is equivalent, when Σ is finite and has index at least three in one of the Γ_i, to say that Γ is icc.

Other examples of groups admitting a faithful and highly transitive action were given recently: surface groups (D. Kitroser [Ki09]), $\mathrm{Out}(\mathbf{F}_n)$, for $n \geq 4$ (S. Garion and Y. Glasner [GG10]), and non-elementary hyperbolic groups with trivial finite radical (V. V. Chaynikov, [Ch12, Section IV.4]).

In this paper we are interested in groups acting without inversion on a tree and admitting a faithful and highly transitive action.

Observe that, by [MS12, Proposition 1.4], if Γ admits a highly transitive and faithful action then Γ is icc.

We call a tree *non-trivial* if it has at least two edges, e and its inverse edge $\overline{e}$. For general groups acting on trees we obtain the following result.

Theorem A. *Let Γ be a group acting without inversion on a non-trivial tree. If the vertex stabilizers are icc and the edge stabilizers are finite then Γ admits a faithful and highly transitive action.*

We obtain actually a more general statement (Theorem 4.1) including some infinite edge stabilizers that we call highly core-free subgroups (Definition 1.1).

To prove Theorem A we use standard arguments from Bass-Serre theory to reduce the case to either an amalgamated free product or an HNN-extension. For these two cases we prove a genericity result for faithful and highly transitive actions (Theorem 2.2 and Theorem 3.2). We recover, as a particular case of our result on amalgamated free products, the result of D. Kitroser about surface groups (Example 5.1).

The paper is organized as follows. Section 1 is a preliminary section in which we introduce and study the notion of highly core-free subgroups that we use in the paper. In section 2 we study the case of HNN-extensions and the case of amalgamated free products is treated in section 3. We prove Theorem A in section 4. Finally, we give some examples in section 5.

1. Highly core-free subgroups

Let Σ be a subgroup of a group H. For a non-empty subset $S \subset H$, the *normal core of* Σ *with respect to* S is defined by $\mathrm{Core}_S(\Sigma) = \cap_{h \in S} h^{-1}\Sigma h$. Recall that Σ is called a *core-free subgroup* if $\mathrm{Core}_H(\Sigma) = \{1\}$. We need a stronger condition than core-freeness namely, we ask that, for every covering of H with non-empty sets, there exists at least one set in the covering for which the associated normal core of Σ is trivial. Asking for this property to hold also for coverings up to finitely many Σ-classes leads to the following definition.

Definition 1.1. A subgroup $\Sigma < H$ is called *highly core-free* if, for every finite subset $F \subset H$, for any $n \geq 1$, for any non-empty subsets $S_1, \ldots, S_n \subset H$ such that $H - \Sigma F \subset \cup_{k=1}^{n} S_k$ there exists $1 \leq k \leq n$ such that $\mathrm{Core}_{S_k}(\Sigma) = \{1\}$.

It is clear that a highly core-free subgroup is core-free. Also, if H is finite then every non-trivial subgroup is not highly core-free (however there exists finite groups with non-trivial core-free subgroups, for example the permutation group $S(n)$ in $S(n+1)$ is core-free). More

generally, a finite index subgroup is never highly core-free. Indeed, if Σ is a non-trivial finite index subgroup of H and F is a finite subset such that $H = \Sigma F$, take $n = 1$ and $S_1 = \{1\}$ so that we have $\mathrm{Core}_{S_1}(\Sigma) = \Sigma \neq \{1\}$. This shows that the highly core-free condition is interesting only for subgroups of infinite index. Examples of highly core-free subgroups will be given in Example 1.11.

Recall that a subgroup $\Sigma < H$ is core-free if and only if the action $H \curvearrowright H/\Sigma$ on the left cosets is faithful (this argument also shows that a finite index subgroup of an infinite group cannot even be core-free). The highly core-free condition will be equivalent to a stronger condition than the faithfulness of the action on the left cosets. We introduce this notion in the following definition.

Definition 1.2. An action $H \curvearrowright X$ is called *highly faithful* if, for any $n \geq 1$, for any non-empty subsets $T_1, \ldots, T_n \subset X$ such that $X - \cup_{k=1}^n T_k$ is finite, there exists $1 \leq k \leq n$ satisfying the following property:

$$\text{if } h \in H \text{ is such that } hx = x \text{ for all } x \in T_k \text{ then } h = 1.$$

Lemma 1.3. *Let $\Sigma < H$ be a subgroup of a group H. The following are equivalent.*

(1) *Σ is highly core-free.*
(2) *The action $H \curvearrowright \Sigma\backslash H$ on right cosets is highly faithful.*
(3) *The action $H \curvearrowright H/\Sigma$ on left cosets is highly faithful.*

Proof. As a preliminary, assume $S_1, \ldots, S_n$ are subsets of H and $T_1, \ldots, T_n$ are the images of $S_1, \ldots, S_n$ in $\Sigma\backslash H$. Then, for any k, the following are equivalent:

- $\forall h \in H$ $(\forall x \in T_k,\ h \cdot x = x \Longrightarrow h = 1)$;
- $\forall h \in H$ $(\forall g \in S_k,\ \Sigma g h^{-1} = \Sigma g \Longrightarrow h = 1)$;
- $\forall h \in H$ $(\forall g \in S_k,\ g^{-1}\Sigma g = (g^{-1}\Sigma g)h \Longrightarrow h = 1)$;
- $\mathrm{Core}_{S_k}(\Sigma) = \{1\}$.

$2 \Longrightarrow 1$. Take F and $S_1, \ldots, S_n$ as in Definition 1.1, and $T_1, \ldots, T_n$ as before. Then $(\Sigma\backslash H) - \cup_{k=1}^n T_k$ is finite. Take k as in Definition 1.2 and we get $\mathrm{Core}_{S_k}(\Sigma) = \{1\}$.

$1 \Longrightarrow 2$. Take $T_1, \ldots T_n$ as in Definition 1.2 and set S_k to be the preimage of T_k in H. Then $H - \Sigma F \subset \cup_{k=1}^n S_k$ for some finite set F. Take k as in Definition 1.1 and we get $\forall h \in H$ $(\forall x \in T_k,\ hx = x \Longrightarrow h = 1)$.

$2 \Longleftrightarrow 3$. Follows from the fact that the H-sets H/Σ and $\Sigma\backslash H$ are isomorphic. □

Example 1.4. The highly core-free condition is stricly stronger than core-freeness, even for infinite groups. Indeed, let $H = S_\infty$ be the group of finitely supported permutations of $\mathbf{N}$ and $\Sigma = \mathrm{Stab}_{S_\infty}(0) < H$ be the stabilizer of $0 \in \mathbf{N}$. Since $S_\infty \curvearrowright \mathbf{N}$ is transitive and faithful, Σ is core-free. However, Σ is not highly core-free since the action $S_\infty \curvearrowright \mathbf{N}$ is not highly faithful: it suffices to write $\mathbf{N} = \{0, 1\} \cup \{k \in \mathbf{N} : k \geq 2\}$ and to use directly the definition.

We are grateful to the referee for having suggested the following example.

Example 1.5. Let $G = \mathrm{PSL}_d(K)$ and $X = \mathbf{P}^1(K^d)$, where $d \geq 2$ and K is a countable field. The natural action $G \curvearrowright X$ is highly faithful. Indeed, assume that $T_1, \ldots, T_n$ are non-empty subsets of X such that $X - \cup_{k=1}^n T_i$ is finite. There exist some index k and elements $\bar{u}_1, \ldots, \bar{u}_{d+1} \in T_k$, that we represent by vectors $u_1, \ldots, u_{d+1} \in K^n$, such that any strict subfamily of $(u_1, \ldots, u_n)$ is linearly independant in K^n. Let us now suppose that some $g \in G$,

represented by a matrix $A \in \mathrm{SL}_d(K)$, satisfies $gx = x$ for all $x \in T_k$. Then one has $g\bar{u}_j = \bar{u}_j$ for all j, hence $Au_j = \lambda_j u_j$ for some eigenvalues $\lambda_1, \dots, \lambda_n$. If there are several eigenvalues, our linear independance condition implies that the sum of eigenspaces dimensions of A is at least $d+1$, which is impossible. Consequently, A is a scalar matrix, that is, g is trivial in G. This proves that $G \curvearrowright X$ is highly faithful. Consequently, by Lemma 1.3, the stabilizer in G of any point $x \in X$ is a highly core-free subgroup.

Given a set X and an integer $k \geq 2$, let $X^{(k)}$ be the *complement of the large diagonal in* X^k:

$$X^{(k)} = \{(x_1, \dots, x_k) \in X^k \,:\, x_i \neq x_j \text{ for all } i \neq j\}.$$

We give some more caracterisations of core-freeness in the next lemma.

Lemma 1.6. *Let Σ be a non-trivial subgroup of an infinite group H. The following are equivalent.*

(1) *Σ is highly core-free.*
(2) *$\forall n \geq 1$, $\forall x_1, \dots, x_n \in H - \{1\}$, the set*
$$H_{\bar{x}} = \{\Sigma h \in \Sigma\backslash H \,:\, \Sigma h x_i \neq \Sigma h \text{ for all } i\} \quad \text{is an infinite subset of } \Sigma\backslash H.$$
(3) *$\forall k \geq 2$, $\forall \bar{x} = (x_1, \dots, x_k) \in H^{(k)}$, the set*
$$G_{\bar{x}} = \{\Sigma h \in \Sigma\backslash H \,:\, \Sigma h x_i \neq \Sigma h x_j,\ \forall i \neq j\} \quad \text{is an infinite subset of } \Sigma\backslash H.$$
(4) *For every free action $H \curvearrowright X$, $\forall k \geq 2$ and $\forall \bar{x} = (x_1, \dots, x_k) \in X^{(k)}$,*
$$E_{\bar{x}} = \{\Sigma h \in \Sigma\backslash H \,:\, \Sigma h x_i \cap \Sigma h x_j = \emptyset,\ \forall i \neq j\} \quad \text{is an infinite subset of } \Sigma\backslash H.$$
(5) *For every free action $H \curvearrowright X$, $\forall k \geq 2$, $\forall \bar{x} = (x_1, \dots, x_k) \in X^{(k)}$ and all $F \subset X$ finite,*
$$E_{F,\bar{x}} = \{h \in H \,:\, hx_i \notin \Sigma F \text{ for all } i \text{ and } \Sigma h x_i \cap \Sigma h x_j = \emptyset,\ \forall i \neq j\} \quad \text{is non-empty.}$$

Proof. $1 \implies 2$. Fix $n \geq 1$ and $x_1, \dots, x_n \in H - \{1\}$. For $1 \leq k \leq n$, set $T_k = \{\Sigma h \in \Sigma\backslash H \,:\, \Sigma h x_k = \Sigma h\}$. We have then

$$\Sigma\backslash H = H_{\bar{x}} \cup \bigcup_{k=1}^{n} T_k\,.$$

Assume by contradiction that $H_{\bar{x}}$ is finite. Since $H \curvearrowright \Sigma\backslash H$ is highly faithful (see Lemma 1.3), there exists k such that $\forall x \in H$, $(\forall \Sigma h \in T_k,\ \Sigma h x^{-1} = \Sigma h \implies x = 1)$. But we also have $\forall \Sigma h \in T_k$, $\Sigma h x_k = \Sigma h$ just by definition of T_k. Hence, we get $x_k^{-1} = 1$, which is impossible since $x_k \in H - \{1\}$. This proves that $H_{\bar{x}}$ is infinite, as we wished.

$2 \implies 3$. Let $k \geq 2$ and $\bar{x} = (x_1, \dots, x_k) \in H^{(k)}$. Consider the collection of non-trivial elements $y_{ij} = x_i x_j^{-1} \in H$ for $i \neq j$. It is easy to see that $H_{\bar{y}} \subset G_{\bar{x}}$.

$3 \implies 4$. Since $H \curvearrowright X$ is free, we may assume that $X = H \times I$, where $I \subset \mathbf{N}$ and the action is given by $h(g, n) = (hg, n)$ for $h \in H$ and $(g, n) \in X$. Let $k \geq 2$ and $\bar{x} = (x_1, \dots, x_k) \in X^{(k)}$ and write $x_i = (g_i, n_i)$. Let m be the size of the set $\{g_i \,:\, 1 \leq i \leq k\}$. If $m \geq 2$ let $\bar{y} = (y_1, \dots y_m) \in H^{(m)}$ such that $\{g_i \,:\, 1 \leq i \leq k\} = \{y_1, \dots, y_m\}$; it is then easy to see that $G_{\bar{y}} \subset E_{\bar{x}}$. If $m = 1$, all the g_i's are equal to $y \in H$. Since the x_i's are pairwise distinct, the n_i's must be pairwise distinct. Hence, $E_{\bar{x}} = \Sigma\backslash H \supset G_{\bar{y}}$ for any $y = (y_1, y_2) \in H^{(2)}$. In both cases, $E_{\bar{x}}$ contains some $G_{\bar{y}}$, and using (3), we conclude that $E_{\bar{x}}$ is infinite.

$4 \implies 3$. Applying 4 to the free action $H \curvearrowright H$ we conclude that 3 holds.

$3 \implies 2$. Let $n \geq 1$, $x_1, \ldots, x_n \in H - \{1\}$. To show that $H_{\bar{x}}$ is infinite, we may suppose that the x_i's are pairwise distinct. Define $y_i = x_i$ for $1 \leq i \leq n$. Define $k = n+1$ and $y_k = 1$. Then $\bar{y} = (y_1, \ldots, y_k) \in H^{(k)}$ and it is easy to see that $G_{\bar{y}} \subset H_{\bar{x}}$.

$2 \implies 1$. Assume (1) does not hold, that is, that the action $H \curvearrowright \Sigma\backslash H$ is not highly faithful. This means that there exist non-empty subsets $T_1, \ldots, T_n \subset X$ such that $X - \cup_{k=1}^n T_k$ is finite, and for all $1 \leq k \leq n$ there exists $x_k \in H$ such that:

$$x_k \neq 1 \text{ and } \Sigma h x_k = x_k \text{ for all } \Sigma h \in T_k.$$

Set now $H_{\bar{x}} = \{\Sigma h \in \Sigma\backslash H \,:\, \Sigma h x_i \neq \Sigma h \text{ for all } 1 \leq i \leq n\}$. We get immediately $T_k \cap H_{\bar{x}} = \emptyset$ for all k, which implies that $H_{\bar{x}}$ is finite. Hence (2) does not hold.

$4 \Leftrightarrow 5$. Let $k \geq 2$, $x_1, \ldots x_k \in X^{(k)}$ and $F \subset X$ a finite subset. Observe that,

$$\Sigma E_{F,\overline{x}} = E_{\overline{x}} - \bigcup_{i=1}^{k} F_i \quad \text{where} \quad F_i = \{\Sigma h \in \Sigma\backslash H \,:\, h x_i \in \Sigma F\}.$$

Since $H \curvearrowright X$ is free and F is finite, the set F_i is finite for all $1 \leq i \leq k$. Suppose condition (4) holds. Then Σ must have infinite index in H and, since $E_{\overline{x}}$ is infinite and $\bigcup_{i=1}^k F_i$ is finite, $\Sigma E_{F,\overline{x}}$ is an infinite subset of $\Sigma\backslash H$. In particular, $E_{F,\overline{x}}$ is non-empty.
Suppose condition (4) does not hold and let $(x_1, \ldots, x_k) \in X^{(k)}$ and $Y \subset H$ a finite subset such that $E_{\overline{x}} = \Sigma Y$. Define $F = \{y x_i \,:\, y \in Y,\ 1 \leq i \leq k\}$. We have (we do not need the freeness assumption here) $\Sigma E_{F,\overline{x}} \subset E_{\overline{x}} \setminus \Sigma Y = \emptyset$. □

Motivated by the preceding lemma, we give the following definition.

Definition 1.7. Let $H \curvearrowright X$ be an action of a group H on an infinite set X and $\Sigma < H$ be a subgroup.

- We say that Σ *is highly core-free with respect to the action* $H \curvearrowright X$ if, $\forall k \geq 2$, $\forall \bar{x} = (x_1, \ldots, x_k) \in X^{(k)}$, the subset
$$E_{\bar{x}} = \{\Sigma h \in \Sigma\backslash H \,:\, \Sigma h x_i \cap \Sigma h x_j = \emptyset,\ \forall i \neq j\} \quad \text{is infinite.}$$
- We say that Σ *is strongly highly core-free with respect to the action* $H \curvearrowright X$ if, $\forall k \geq 2$, $\forall \bar{x} = (x_1, \ldots, x_k) \in X^{(k)}$, $\forall F \subset X$ a finite subset, the subset
$$E_{F,\bar{x}} = \{h \in H \,:\, h x_i \notin \Sigma F \text{ for all } i \text{ and } \Sigma h x_i \cap \Sigma h x_j = \emptyset,\ \forall i \neq j\} \quad \text{is non-empty.}$$

The two notions are not equivalent: actions with fixed points are possible in the first case and not in the second one. For free actions they are equivalent (see the proof of Lemma 1.6). In general, we have the following result.

Proposition 1.8. *Let $H \curvearrowright X$ be an action of a group H on an infinite set X and $\Sigma < H$ be a subgroup. If Σ is strongly highly core-free with respect to $H \curvearrowright X$ then it is also highly core-free with respect to $H \curvearrowright X$. The converse is true when, for all $x \in X$, the map*

$$\varphi_x \,:\, \Sigma\backslash H \to \Sigma\backslash X; \quad \Sigma h \mapsto \Sigma h x$$

has finite fibers.

Proof. The proof of the Proposition is the same as the proof of of $4 \Leftrightarrow 5$ in Lemma 1.6. The only remark to make is that, for all $(x_1, \ldots x_n) \in X^{(n)}$ and $F \subset X$, we have

$$\Sigma E_{F,\overline{x}} = E_{\overline{x}} - \bigcup_{i=1}^{n} \varphi_{x_i}^{-1}(\Sigma F)$$

and the $\varphi_{x_i}^{-1}(\Sigma F)$'s are finite since φ_x has finite fibers. $\square$

Notice that if Σ is highly core-free with respect to the action $H \curvearrowright X$, then the action $H \curvearrowright X$ has no finite orbits, except maybe fixed points. Notice also that if Σ is highly core-free with respect to the action $H \curvearrowright X$, then the action $\Sigma \curvearrowright X$ has infinitely many orbits.

Remark 1.9. Let H be a group and $\Sigma < H$ a subgroup. If there exists a highly transitive action $H \curvearrowright X$ such that the action $\Sigma \curvearrowright X$ has infinitely many orbits, then Σ is strongly highly core-free with respect to the action $H \curvearrowright X$.

Proof. Let $k \geq 2$, $\bar{x} = (x_1, \ldots, x_k) \in X^{(k)}$ and $F \subset X$ a finite subset. We want to prove that

$$E_{F,\bar{x}} = \{h \in H \,:\, hx_i \notin \Sigma F \text{ for all } i \text{ and } \Sigma hx_i \cap \Sigma hx_j = \emptyset,\ \forall i \neq j\} \neq \emptyset.$$

Let $Y = \Sigma F \cup (\cup_{j=1}^{k} \Sigma x_j)$. Since $\Sigma \curvearrowright X$ has infinitely many orbits there exists an element $(z_1, \ldots, z_k) \in X^{(k)}$ such that $\Sigma z_i \subset Y^c$ and $\Sigma z_i \cap \Sigma z_j = \emptyset$ for all $i \neq j$. By high transitivity of $H \curvearrowright X$ there exists $h \in H$ such that $hx_i = z_i$ for every i. Then $hx_i = z_i \notin \Sigma F$ for all i and $\Sigma hx_i \cap \Sigma hx_j = \Sigma z_i \cap \Sigma z_j = \emptyset$ for all $i \neq j$ so $h \in E_{F,\bar{x}}$. $\square$

Let H be a group. For $g \in H$ denote by $Cl_H(g) = \{hgh^{-1} \,:\, h \in H\}$ the conjugacy class of g and by $C_H(g) = \{h \in H \,:\, gh = hg\}$ the centralizer of g. Recall that the group H is called icc if for every $g \in H - \{1\}$ the set $Cl_H(g)$ is infinite (or, equivalently, the subgroup $C_H(g)$ has infinite index). We call a subgroup $\Sigma < H$ *icc relative to* H if, for every $\sigma \in \Sigma - \{1\}$, the set $Cl_H(\sigma)$ is infinite.

Lemma 1.10. *Assume $\Sigma < H$ be a subgroup such that:*

- *Σ has infinite index;*
- *Σ is icc relative to H;*
- *for any $h \in H - \{1\}$ the set $\Sigma \cap Cl_H(h)$ is finite.*

Then, Σ is highly core-free.

Proof. Take $n \geq 1$ and $x_1, \ldots, x_n \in H - \{1\}$. By Lemma 1.6, it suffices to prove that the set

$$H_{\bar{x}} = \{\Sigma h \in \Sigma \backslash H \,:\, \Sigma hx_i \neq \Sigma h \text{ for all } i\}$$

is infinite. We note that, since Σ has infinite index, $\Sigma \backslash H$ is infinite and the unique orbit of the action $H \curvearrowright \Sigma \backslash H$ is infinite.

We then remark that one has $\Sigma hx_i = \Sigma h \Leftrightarrow hx_ih^{-1} \in \Sigma \Leftrightarrow x_i \in h^{-1}\Sigma h$. Hence, equality $\Sigma hx_i = \Sigma h$ can occur only if $x_i \in Y := \cup_{\sigma \in \Sigma - \{1\}} Cl_H(\sigma)$, so that we may assume that $x_1, \ldots, x_n$ lie in Y. Since Σ is icc relative to H, all orbits of the action $H \curvearrowright Y$ by conjugation are infinite.

Set $F = \big(\cup_{1 \leq i \leq n} Cl_H(x_i)\big) \cap \Sigma = \cup_{1 \leq i \leq n} \big(Cl_H(x_i) \cap \Sigma\big)$; this set is finite by hypothesis. Morover, one has $h\{x_1, \ldots, x_n\}h^{-1} \cap \Sigma = h\{x_1, \ldots, x_n\}h^{-1} \cap F$ for all $h \in H$.

Now, note that all orbits of the action $H \curvearrowright (\Sigma \backslash H \sqcup Y)$ are infinite, and take any $k \geq 1$, and any $\Sigma h_1, \ldots, \Sigma h_k \in \Sigma \backslash H$. Then consider the finite sets $A = \{\Sigma h_1, \ldots, \Sigma h_k\} \sqcup \{x_1, \ldots, x_n\}$ and $B = \{\Sigma\} \sqcup F$. By Neumann's Lemma [Ne76, Lemma 2.3], there exists $h \in H$ such that $h \cdot A \cap B = \emptyset$. This implies:

- $\{\Sigma h_1h^{-1}, \ldots, \Sigma h_kh^{-1}\} \cap \{\Sigma\} = \emptyset$, hence $\Sigma h \notin \{\Sigma h_1, \ldots, \Sigma h_k\}$, and
- $h\{x_1, \ldots, x_n\}h^{-1} \cap F = \emptyset$, hence $h\{x_1, \ldots, x_n\}h^{-1} \cap \Sigma = \emptyset$, hence $\Sigma hx_i \neq \Sigma h$ for all i, hence $h \in H_{\bar{x}}$.

This proves that $H_{\bar{x}}$ is infinite, as desired. $\square$

Example 1.11. The easiest example of groups satisfying the hypothesis of Lemma 1.10 is a finite relatively icc subgroup Σ of H. In particular, a finite subgroup of an icc group. Other examples are given below.

Observe that, if $\Sigma < H$ is malnormal, then $C_H(\sigma) \subset \Sigma$ for all $\sigma \in \Sigma - \{1\}$. Hence, if Σ is malnormal and has infinite index then Σ is relatively icc.

Suppose that $\Sigma < H$ is a malnormal subgroup then, for all $h \in H - \{1\}$ with the property that $Cl_H(h) \cap \Sigma \neq \emptyset$, there exists $\sigma_0 \in \Sigma - \{1\}$ such that $Cl_H(h) \cap \Sigma = Cl_\Sigma(\sigma_0)$. Indeed, let $h \neq 1$ and $\sigma_0 = ghg^{-1} \in \Sigma$ for some $g \in H$. If $\sigma = tht^{-1} \in Cl_H(h) \cap \Sigma$ then $\sigma \neq 1$ and, $\sigma = tg^{-1}\sigma_0 gt^{-1} = tg^{-1}\sigma_0(tg^{-1})^{-1} \in tg^{-1}\Sigma(tg^{-1})^{-1}$. So $\sigma \in \Sigma \cap tg^{-1}\Sigma gt^{-1}$ and thus $tg^{-1} \in \Sigma$ by malnormality of Σ. Hence $\sigma \in Cl_\Sigma(\sigma_0)$ and this proves $Cl_H(h) \cap \Sigma \subset Cl_\Sigma(\sigma_0)$. The other inclusion is obvious.

By the preceding remarks any infinite index and malnormal subgroup $\Sigma < H$ such that $Cl_\Sigma(\sigma)$ is finite for all $\sigma \in \Sigma$ satisfy the hypothesis of Lemma 1.10. Here are some particular examples.

- A finite malnormal subgroup Σ of an infinite group H.
- $H = \Sigma * G$ such that G and Σ are non-trivial and Σ abelian. More generally, an abelian malnormal subgroup Σ of infinite index of an infinite group H.
- Let K be an infinite (commutative) field. Let $H = K^* \ltimes K$ and $\Sigma = K^* < H$. Then Σ is malnormal in H (see Example 4 in [HW11]).
- Let $H = \langle a, b\rangle \simeq \mathbf{F}_2$ be a free group on 2 generators a and b. The infinite cyclic subgroup Σ generated by any primitive element (e.g. $a^k b a^{-l} b^{-1}$ with $k \neq 0 \neq l \in \mathbf{Z}$) is malnormal (see Example 7.A. in [HW11] and [BMR99]).

2. The case of an HNN-extension

Let H be a countable infinite group, $\Sigma < H$ a subgroup and $\theta : \Sigma \to H$ an injective group homomorphism. Let $\Gamma = \mathrm{HNN}(H, \Sigma, \theta) = \langle H, t \,|\, \theta(\sigma) = t\sigma t^{-1}\, \forall \sigma \in \Sigma\rangle$ and, for $\epsilon \in \{-1, 1\}$,

$$\Sigma_\epsilon = \begin{cases} \Sigma & \text{if} \quad \epsilon = 1, \\ \theta(\Sigma) & \text{if} \quad \epsilon = -1. \end{cases}$$

Let $H \curvearrowright X$ be an action of H on an infinite countable set X. Let $S(X)$ be the Polish group of bijections of X. Although the action is not supposed to be faithful we write, in order to simplify the notations, the same symbol $h \in S(X)$ for the action of $h \in H$ on the set X.

Define

$$Z = \{w \in S(X) \,:\, \theta(\sigma) = w\sigma w^{-1} \ \ \forall \sigma \in \Sigma\}.$$

Then Z is a closed subset of $S(X)$. Suppose moreover that the actions $\Sigma_\epsilon \curvearrowright X$ are free for $\epsilon \in \{-1, 1\}$ and have infinitely many orbits (this last assumption is automatically satisfied if Σ_ϵ is highly core-free with respect to the action $H \curvearrowright X$ for all $\epsilon \in \{-1, 1\}$). With these assumptions, the set Z is non-empty.

For all $w \in Z$ there exists a unique group homomorphism $\pi_w : \Gamma \to S(X)$ such that $\pi_w(t) = w$ and $\pi_w(h) = h$ for all $h \in H$.

Lemma 2.1. *If, for each $\epsilon \in \{-1, 1\}$, Σ_ϵ is strongly highly core-free with respect to the action $H \curvearrowright X$ then the set $O = \{w \in Z : \pi_w$ is highly transitive$\}$ is a dense G_δ in Z.*

Proof. We have $O = \cap_{n\geq 2} \cap_{\bar{x},\bar{y}\in X^{(n)}} O_{\bar{x},\bar{y}}$ where, for $\bar{x} = (x_1, \dots x_n), \bar{y} = (y_1, \dots, y_n) \in X^{(n)}$,

$$O_{\bar{x},\bar{y}} = \{w \in Z \,:\, \exists g \in \Gamma,\ \pi_w(g)x_k = y_k \text{ for all } 1 \leq k \leq n\}.$$

It is easy to see that $O_{\bar{x},\bar{y}}$ is open in Z. Let us show that $O_{\bar{x},\bar{y}}$ is dense in Z. Let $w \in Z$ and $F \subset X$ be a finite subset. We claim that there exists $g, h \in H$ such that $hx_k \notin \Sigma F$, $g^{-1}y_k \notin w(\Sigma F)$ for all $1 \leq k \leq n$ and the sets Σhx_k, $\Sigma w^{-1}g^{-1}y_k$ for $1 \leq k \leq n$ are pairwise disjoint. Indeed, since $\theta(\Sigma)$ is highly core-free with respect to the action $H \curvearrowright X$, the set

$$E_{wF,\bar{y}} = \{h \in H \,:\, hy_i \notin \theta(\Sigma)wF \text{ for all } i \text{ and } \theta(\Sigma)hy_i \cap \theta(\Sigma)hy_j = \emptyset,\ \forall i \neq j\}$$

is not empty. Take $g^{-1} \in E_{wF,\bar{y}}$ and let $F' := F \cup (\cup_{k=1}^n w^{-1}g^{-1}y_k)$. Since Σ is highly core-free with respect to $H \curvearrowright X$, the set $E_{F',\bar{x}} = \{h \in H \,:\, hx_i \notin \Sigma F'\ \forall i \text{ and } \Sigma hx_i \cap \Sigma hx_j = \emptyset\ \forall i \neq j\}$ is not empty and we take $h \in E_{F',\bar{x}}$. It is clear that g, h satisfy the claimed properties.

Define $Y = \sqcup_{k=1}^n \Sigma hx_k \sqcup \Sigma w^{-1}g^{-1}y_k$. Then $F \subset Y^c$ and $w(Y) = \sqcup_{k=1}^n \theta(\Sigma)whx_k \sqcup \theta(\Sigma)g^{-1}y_k$. Since the actions $\Sigma_\epsilon \curvearrowright X$ are free, we can define $\gamma \in S(X)$ by $\gamma|_{Y^c} = w|_{Y^c}$ and,

$$\gamma(\sigma hx_k) = \theta(\sigma)g^{-1}y_k, \quad \gamma(\sigma w^{-1}g^{-1}y_k) = \theta(\sigma)whx_k,\ \ \forall \sigma \in \Sigma,\ \forall 1 \leq k \leq n.$$

By construction $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, $\pi_\gamma(gth)x_k = g\gamma hx_k = gg^{-1}y_k = y_k$ for all $1 \leq k \leq n$. □

Theorem 2.2. *Let H be an infinite countable group, $\Sigma < H$ a subgroup and $\theta : \Sigma \to H$ an injective group homomorphism. Suppose that, for each $\epsilon \in \{-1, 1\}$, Σ_ϵ is highly core-free in H. Then $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$ admits a highly transitive and faithful action on an infinite countable set. Moreover, if H is amenable and Σ is finite, this action can be chosen to be amenable.*

Proof. Consider the free action $\Gamma \curvearrowright X$ with $X = \Gamma \times \mathbf{N}$ given by $g(x, n) = (gx, n)$ for $g \in \Gamma$ and $(x, n) \in X$. Since for each $\epsilon \in \{-1, 1\}$, Σ_ϵ is highly core-free in H, it is also strongly highly core-free with respect to $H \curvearrowright X$. Hence, we can apply Lemma 2.1 to the action $H \curvearrowright X$ and it suffices to show that the set $O = \{w \in Z \,:\, \pi_w \text{ is faithful}\}$ is a dense G_δ in Z. Writing $O = \cap_{g\in\Gamma-\{1\}} O_g$, where the set $O_g = \{w \in Z \,:\, \pi_w(g) \neq \mathrm{id}\}$ is obviously open, it suffices to show that O_g is dense. We are going to prove directly that O itself is dense. Write $X = \cup^{\uparrow} X_n$ where $X_n = \{(x, k) \in X \,:\, k \leq n\}$ is infinite and globally invariant under Γ. Let $w \in Z$ and $F \subset X$ a finite subset. Let $N \in \mathbf{N}$ large enough such that $\Sigma F \cup w(\Sigma F) \subset X_N$. Since Σ_ϵ has infinite index in Γ, the set $X_N - \Sigma F$ (resp. $X_N - w(\Sigma F)$) has infinitely many Σ-orbits (resp. $\theta(\Sigma)$-orbits) and is globally invariant under Σ (resp. $\theta(\Sigma)$). Hence, there exists a bijection $\gamma_0 : X_N - \Sigma F \to X_N - w(\Sigma F)$ satisfying $\gamma_0\sigma = \theta(\sigma)\gamma_0$ for all $\sigma \in \Sigma$. Define $\gamma \in S(X)$ by $\gamma|_{\Sigma F} = w|_{\Sigma F}$, $\gamma|_{X_N - \Sigma F} = \gamma_0$ and $\gamma|_{X_N^c} = t|_{X_N^c}$. By construction, $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, since $\pi_\gamma(g)(x, n) = (gx, n)$ for all $n > N$ and since $\Gamma \curvearrowright X$ is faithful, it follows that π_γ is faithful.

If H is amenable and Σ is finite, let $(D_n)_{n\geq 1}$ be a Følner sequence in H such that $|D_n| \to \infty$. Then $C_n = D_n \times \{0\}$ is a Følner sequence for $H \curvearrowright X$ such that $|C_n| \to \infty$. We may apply [Fi12, Lemma 3.2] to conclude. □

3. The case of an amalgamated free product

Let Γ_1, Γ_2 be two infinite countable groups and Σ a common subgroup of Γ_1, Γ_2. Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be the amalgamated free product. Suppose that, for $i = 1, 2$, there is an action $\Gamma_i \curvearrowright X$ on an infinite countable set X such that the two actions of Σ on X are free.

Observe that if the two actions of Σ on X have infinitely many orbits (which is automatic when Σ is highly core-free with respect to the action $\Gamma_i \curvearrowright X$ for $i = 1, 2$) then we may suppose, up to conjugating the action of Γ_2 by an element in $S(X)$, that the two actions of Σ on X coincide. Hence we do suppose that these two actions actually coincide.

As before, although the actions $\Gamma_i \curvearrowright X$ are not supposed to be faithful, we use the same symbol $g \in S(X)$ to denote the action of the element $g \in \Gamma_i$ on the set X for $i = 1, 2$. Define

$$Z = \{w \in S(X) \,:\, w\sigma = \sigma w \ \forall \sigma \in \Sigma\}.$$

It is easy to check that Z is a closed subgroup of $S(X)$.

For all $w \in Z$ there exists a unique group homomorphism $\pi_w : \Gamma \to S(X)$ such that $\pi_w(g) = g$ and $\pi_w(h) = w^{-1}hw$ for all $g \in \Gamma_1$, $h \in \Gamma_2$.

Lemma 3.1. *If, for $i = 1, 2$, Σ is strongly highly core-free with respect to the action $\Gamma_i \curvearrowright X$, then the set $O = \{w \in Z \,:\, \pi_w$ is highly transitive$\}$ is a dense G_δ in Z.*

Proof. Write $O = \cap_{n\geq 2} \cap_{\bar{x},\bar{y}\in X^{(n)}} O_{\bar{x},\bar{y}}$ where, for $\bar{x} = (x_1, \dots x_n), \bar{y} = (y_1, \dots, y_n) \in X^{(n)}$,

$$O_{\bar{x},\bar{y}} = \{w \in Z \,:\, \exists g \in \Gamma,\ \pi_w(g)x_k = y_k \text{ for all } 1 \leq k \leq n\}.$$

Since $O_{\bar{x},\bar{y}}$ is obviously open in Z, it suffices to show that it is dense in Z. Let $w \in Z$ and $F \subset X$ be a finite subset. We first prove the following claim.

Claim. There exists $g_1, g_2 \in \Gamma_1$, $h \in \Gamma_2$ and $z_1, \dots, z_n \in X$ such that the sets $\Sigma g_1 x_k$, $\Sigma g_2^{-1} y_k$, $\Sigma w^{-1} z_k$, $\Sigma w^{-1} h z_k$ and F are pairwise disjoint for $1 \leq k \leq n$.

Proof of the claim. Since Σ is highly core-free with respect to the action $\Gamma_1 \curvearrowright X$, the set

$$E_{F,\bar{x}} = \{h \in \Gamma_1 \,:\, hx_i \notin \Sigma F \text{ for all } i \text{ and } \Sigma hx_i \cap \Sigma hx_j = \emptyset,\ \forall i \neq j\}$$

is not empty. Take $g_1 \in E_{F,\bar{x}}$. Let $F' := F \cup \{g_1 x_i \,:\, 1 \leq i \leq n\}$. Since the set

$$E_{F',\bar{y}} = \{h \in \Gamma_1 \,:\, hy_i \notin \Sigma F' \text{ for all } i \text{ and } \Sigma hy_i \cap \Sigma hy_j = \emptyset,\ \forall i \neq j\}$$

is not empty, we can take $g_2^{-1} \in E_{F',\bar{y}}$. Let

$$Y = w\big(\Sigma F \cup (\cup_{i=1}^n \Sigma g_1 x_i) \cup (\cup_{i=1}^n \Sigma g_2^{-1} y_i)\big).$$

Since $\Sigma \curvearrowright X$ has infinitely many orbits, we can find n distinct Σ-classes $\Sigma z_1, \dots, \Sigma z_n$ in the complement of Y. Let $F'' := \cup_{i=1}^n \Sigma z_i \cup Y$. Since Σ is highly core-free with respect to the action $\Gamma_2 \curvearrowright X$, the set

$$E_{F'',\bar{z}} = \{h \in \Gamma_2 \,:\, hz_i \notin \Sigma F'' \text{ for all } i \text{ and } \Sigma hz_i \cap \Sigma hz_j = \emptyset,\ \forall i \neq j\}$$

is not empty. So there is $h \in E_{F'',\bar{z}}$ and this proves the claim.

End of the proof of the lemma. Define $Y' = \bigsqcup_{k=1}^n \Sigma g_1 x_k \sqcup \Sigma g_2^{-1} y_k \sqcup \Sigma w^{-1} z_k \sqcup \Sigma w^{-1} h z_k$. One has $F \subset Y'^c$ and $w(Y') = \bigsqcup_{k=1}^n \Sigma w g_1 x_k \sqcup \Sigma w g_2^{-1} y_k \sqcup \Sigma z_k \sqcup \Sigma h z_k$. Define $\gamma \in S(X)$ by $\gamma|_{Y'^c} = w|_{Y'^c}$ and, for all $\sigma \in \Sigma$ and all $1 \leq k \leq n$,

$$\gamma(\sigma g_1 x_k) = \sigma z_k,\ \ \gamma(\sigma g_2^{-1} y_k) = \sigma h z_k,\ \ \gamma(\sigma w^{-1} z_k) = \sigma w g_1 x_k,\ \ \gamma(\sigma w^{-1} h z_k) = \sigma w g_2^{-1} y_k.$$

By construction, $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, with $g = g_2 h g_1 \in \Gamma$, one has, for all $1 \leq k \leq n$,

$$\pi_\gamma(g)x_k = g_2\gamma^{-1}h\gamma g_1 x_k = g_2\gamma^{-1}hz_k = g_2 g_2^{-1} y_k = y_k.$$

□

Theorem 3.2. *Suppose that Γ_1, Γ_2 are infinite countable groups with a common subgroup Σ. If Σ is highly core-free in both Γ_1 and Γ_2, then $\Gamma = \Gamma_1 \underset{\Sigma}{*} \Gamma_2$ admits a faithful and highly transitive action on an infinite countable set. If moreover Γ_1, Γ_2 are amenable and Σ is finite then the action can be chosen to be amenable.*

Proof. Consider the free action $\Gamma \curvearrowright X$ with $X = \Gamma \times \mathbf{N}$ given by $g(x,n) = (gx, n)$ for $g \in \Gamma$ and $(x,n) \in X$. Since Σ is highly core-free in Γ_i it is also strongly highly core-free with respect to $\Gamma_i \curvearrowright X$ for $i = 1, 2$. Hence, we can apply Lemma 3.1 to the actions $\Gamma_i \curvearrowright X$, $i = 1, 2$, and it suffices to show that the set $O = \{w \in Z : \pi_w \text{ is faithful}\}$ is a dense G_δ in Z. Writing $O = \cap_{g \in \Gamma - \{1\}} O_g$, where $O_g = \{w \in Z : \pi_w(g) \neq \mathrm{id}\}$ is obviously open, it suffices to show that O_g is dense. Let us prove that O itself is dense. Write $X = \cup^{\uparrow} X_n$ where $X_n = \{(x,k) \in X : k \leq n\}$ is infinite and globally invariant under Γ. Let $w \in Z$ and $F \subset X$ a finite subset. Let $N \in \mathbf{N}$ large enough such that $\Sigma F \cup w(\Sigma F) \subset X_N$. Since Σ has infinite index in Γ, the sets $X_N - \Sigma F$ and $X_N - w(\Sigma F)$ have infinitely many Σ-orbits and are globally invariant under Σ. Hence, there exists a bijection $\gamma_0 : X_N - \Sigma F \to X_N - w(\Sigma F)$ satisfying $\gamma_0 \sigma = \sigma \gamma_0$ for all $\sigma \in \Sigma$. Define $\gamma \in S(X)$ by $\gamma|_{\Sigma F} = w|_{\Sigma F}$, $\gamma|_{X_N - \Sigma F} = \gamma_0$ and $\gamma|_{X_N^c} = \mathrm{id}|_{X_N^c}$. By construction, $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, since $\pi_\gamma(g)(x,n) = (gx, n)$ for all $n > N$ and since $\Gamma \curvearrowright X$ is faithful, it follows that π_γ is faithful.

If Γ_i is amenable for $i = 1, 2$ then there exists Følner sequences $(C'_n)_{n \geq 1}$ in Γ_1 and $(D'_n)_{n \geq 1}$ in Γ_2 such that $|C'_n| = |D'_n| \to \infty$. Since Σ is finite, we may apply [Fi12, Lemma 4.2] to the Følner sequences $C_n = C'_n \times \{0\}$ and $D_n = D'_n \times \{0\}$ for $\Gamma_1 \curvearrowright X$ and $\Gamma_2 \curvearrowright X$ respectively to conclude the proof. □

4. Groups acting on trees

In this section we prove Theorem A.

Let Γ be a group acting without inversion on a non-trivial tree. By [Se77], the quotient graph $\mathcal{G}$ can be equipped with the structure of a graph of groups $(\mathcal{G}, \{\Gamma_p\}_{p \in \mathrm{V}(\mathcal{G})}, \{\Sigma_e\}_{e \in \mathrm{E}(\mathcal{G})})$ where each $\Sigma_e = \Sigma_{\overline{e}}$ is isomorphic to an edge stabilizer and each Γ_p is isomorphic to a vertex stabilizer and such that Γ is isomorphic to the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of this graph of groups i.e., given a fixed maximal subtree $\mathcal{T} \subset \mathcal{G}$, the group Γ is generated by the groups Γ_p for $p \in \mathrm{V}(\mathcal{G})$ and the edges $e \in \mathrm{E}(\mathcal{G})$ with the relations

$$\overline{e} = e^{-1}, \quad s_e(x) = e r_e(x) e^{-1} \ , \forall x \in \Sigma_e \quad \text{and} \quad e = 1 \ \ \forall e \in \mathrm{E}(\mathcal{T}),$$

where $s_e : \Sigma_e \to \Gamma_{s(e)}$ and $r_e = s_{\overline{e}} : \Sigma_e \to \Gamma_{r(e)}$ are respectively the source and range group monomomorphisms. Using Lemma 1.10 one sees that Theorem A is a straightforward corollary of the following result.

Theorem 4.1. *If Γ_p is infinite for all $p \in \mathrm{V}(\mathcal{G})$ and $s_e(\Sigma_e)$ is highly core-free in $\Gamma_{s(e)}$ for all $e \in \mathrm{E}(\mathcal{G})$ then Γ admits a faithful and highly transitive action on an infinite countable set.*

Proof. Let e_0 be one edge of $\mathcal{G}$ and $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by removing the edges e_0 and $\overline{e_0}$.

Case 1: $\mathcal{G}'$ is connected. It follows from Bass-Serre theory that $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$ where H is fundamental group of our graph of groups restricted to $\mathcal{G}'$, $\Sigma = r_{e_0}(\Sigma_{e_0}) < H$ is a subgroup and $\theta : \Sigma \to H$ is given by $\theta = s_{e_0} \circ r_{e_0}^{-1}$. By hypothesis H is infinite and, since $\Sigma < \Gamma_{r(e_0)}$ (resp. $\theta(\Sigma) < \Gamma_{s(e_0)}$) is a highly core-free subgroup, $\Sigma < H$ (resp. $\theta(\Sigma) < H$) is also a highly

core-free subgroup. Thus we may apply Theorem 2.2 to conclude that Γ admits a faithful and highly transitive action.

Case 2: $\mathcal{G}'$ is not connected. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the two connected components of $\mathcal{G}'$ such that $s(e_0) \in \mathrm{V}(\mathcal{G}_1)$ and $r(e_0) \in \mathrm{V}(\mathcal{G}_2)$. Bass-Serre theory implies that $\Gamma = \Gamma_1 *_{\Sigma_{e_0}} \Gamma_2$, where Γ_i is the fundamental group of our graph of groups restricted to $\mathcal{G}_i$, $i = 1, 2$, and Σ_{e_0} is viewed as a highly core-free subgroup of Γ_1 via the map s_{e_0} and as a highly core-free subgroup of Γ_2 via the map r_{e_0} since $s_{e_0}(\Sigma_{e_0})$ is highly core-free in $\Gamma_{s(e_0)}$ and $r_{e_0}(\Sigma_{e_0})$ is highly core-free in $\Gamma_{r(e_0)}$ by hypothesis. Since Γ_1 and Γ_2 are infinite, we may apply Theorem 3.2 to conclude that Γ admits a faithful and highly transitive action. □

5. Examples

Kitroser [Ki09] proved that surface groups admit a faithful and highly transitive action. We recover this result from Theorem 3.2:

Example 5.1. The fundamental group $\pi_1(\Sigma_g)$ of a closed, orientable surface of genus $g > 1$ admits a faithful and highly transitive action.

Indeed, as seen in Example 1.11, the subgroup $\Sigma = \langle [a_1, b_1] \rangle$ generated by the commutator $[a_1, b_1] = a_1 b_1 a_1^{-1} b_1^{-1}$ is highly core-free in $\Gamma_1 = \langle a_1, b_1 \rangle \simeq \mathbf{F}_2$. So Theorem 3.2 implies that the group $\pi_1(\Sigma_2) = \langle a_1, b_1 \rangle *_{\langle c \rangle} \langle a_2, b_2 \rangle$ where $c = [a_1, b_1] = [a_2, b_2]$, admits a faithful and highly transitive action. For $g > 1$, the group $\pi_1(\Sigma_g)$ injects into $\pi_1(\Sigma_2)$ as a subgroup of finite index. Thus by Corollary 1.5 in [MS12], the group $\pi_1(\Sigma_g)$ admits a faithful and highly transitive action as well.

Chaynikov [Ch12, Section IV.4] proved that non-elementary hyperbolic groups with trivial finite radical admit a faithful and highly transitive action. (In fact, this action has more properties.) Of course, we do not recover this result, since our Theorem 4.1 is not applicable to one-ended hyperbolic groups. On the other hand our techniques allow to treat some non-hyperbolic groups, for example $\mathbf{Z}^2 * \mathbf{Z}^2$. Other examples are as follows: let us denote by $\mathbf{Z}[i]$ the ring of Gaussian integers (which is isomorphic to $\mathbf{Z}^2$ as a group) and by $\mathbf{Z}[i]^*$ the group of its invertible elements (which is cyclic of order 4).

Example 5.2. The following groups admit a faithful and highly transitive action:

(1) the group $(\mathbf{Z}[i]^* \ltimes \mathbf{Z}[i]) *_{\mathbf{Z}[i]^*} (\mathbf{Z}[i]^* \ltimes \mathbf{Z}[i])$, and
(2) the group $\mathrm{HNN}(\mathbf{Z}[i]^* \ltimes \mathbf{Z}[i], \mathbf{Z}[i]^*, \theta)$, where θ maps $\mathbf{Z}[i]^*$ onto one of its conjugates.

Indeed, $\mathbf{Z}[i]^*$ is a malnormal subgroup in $\mathbf{Z}[i]^* \ltimes \mathbf{Z}[i]$ and so are all its conjugates (see e.g. [HW11, Propositions 1 and 2]). Since they are moreover finite, with infinite index, these subgroups satisfy the hypotheses of Lemma 1.10, so that they are highly core-free. It then suffices to apply Theorem 4.1.

References

[BMR99] G. Baumslag, A. Myasnikov, and V. Remeslennikov, Malnormality is decidable in free groups, *Internat. J. Algebra Comput.* **9**, (1999), 687- 692.

[Ch12] V. V. Chaynikov, Properties of hyperbolic groups: free normal subgroups, quasiconvex subgroups and actions of maximal growth, Ph.D. Thesis, Vanderbilt University, (2012), available at http://etd.library.vanderbilt.edu/available/etd-06212012-172048/unrestricted/CHAYNIKOV.pdf

[Co08] Y. de Cornulier, Infinite conjugacy classes in groups acting on trees, *Groups Geom. Dyn.* **3**, (2009), 267–277.

[Di89] J. D. Dixon, Most finitely generated permutation groups are free, *Bull. London Math. Soc.* **22**, (1990), 222–226.

[Fi12] P. Fima, Amenable, transitive and faithful actions of groups acting on trees, *arXiv:1202.6467*, to appear in *Annales de l'Institut Fourier*.

[GG10] S. Garion and Y. Glasner, Highly transitive actions of $\mathrm{Out}(\mathbf{F}_n)$, *arXiv:1008.0563*, to appear in *Groups, Geometry and Dynamics*.

[GM90] A. M. W. Glass and S. H. McCleary, Highly transitive representations of free groups and free products, *Bull. Austral. Math. Soc.* **43**, (1991), 19–36.

[Gu91] S. V. Gunhouse, Highly transitive representations of free products on the natural numbers, *Arch. Math.* **58**, (1992), 435–443.

[HW11] P. de la Harpe, C. Weber, Appendix by D. Osin, Malnormal subgroups and Frobenius groups: basics and examples, *arXiv:1104.3065*.

[Hi90] K. K. Hickin, Highly transitive Jordan representations of free products, *J. London Math. Soc.* **46**, (1992), 81–91.

[Ki09] D. Kitroser, Highly transitive actions of surface groups, *Proc. Amer. Math. Soc.* **140**, (2012), 3365–3375.

[McD76] T. P. McDonough, A permutation representation of a free group, *Quart. J. Math. Oxford.* **28**, (1977), 353–356.

[MS12] S. Moon and Y. Stalder, Highly transitive actions of free products, *Algebr. Geom. Topol.* **13**, (2013), 589–607.

[Ne76] P. Neumann, The structure of finitary permutation groups, Arch. Math. (Basel) 27 (1976), no. 1, 3–17.

[Se77] J.-P. Serre, Arbres, amalgames, SL_2, *Astérisque* **46**, Société Mathmatique de France, (1977).

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