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Uncertainty Quantification and Global Sensitivity Analysis for Economic Models*

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Abstract

We present a global sensitivity analysis that quantifies the impact of parameter uncertainty on model outcomes. Specifically, we propose variance-decomposition-based Sobol' indices to establish an importance ranking of parameters and univariate effects to determine the direction of their impact. We employ the state-of-the-art approach of constructing a polynomial chaos expansion of the model, from which Sobol' indices and univariate effects are then obtained analytically, using only a limited number of model evaluations. We apply this analysis to several quantities of interest of a standard real-business-cycle model and compare it to traditional local sensitivity analysis approaches. The results show that local sensitivity analysis can be very misleading, whereas the proposed method accurately and efficiently ranks all parameters according to importance, identifying interactions and nonlinearities.

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1 Introduction

The question whether quantitative results of an economic model are sensitive to specific parameterization assumptions is important, not only for the credibility of a specific study but also for the general advancement of a quantitative approach to economic analysis. Since many studies have implications for policymakers, there is a strong case to be made for structured sensitivity analyses to become an essential part of quantitative studies of economic models. Moreover, a sensitivity analysis can go beyond simple robustness checks and answer more detailed questions such as which parameters—and which interactions between them—are driving the conclusions derived from an economic model. Such an importance ranking informs the researcher on which parts to focus on when calibrating or extending a model, or the policymaker on which parameters need further scrutiny.

The economic literature is well aware of the need for a structured sensitivity analysis for quantitative models.[1] However, with few exceptions, current practice involves a high degree of subjective and somewhat arbitrary judgments. Typically, some parameters are chosen and individually changed to a different value to assess the partial influence on the results. Such "one-at-a-time" approaches tend to be unstructured and, more importantly, suffer from the fact that they are only *local*, i.e., highly dependent on the chosen parameter values. Moreover, they cannot account for possible interactions between parameters and non-linear relationships that are often encountered in economic models.

The present paper proposes methods for *global* sensitivity analysis that overcome the mentioned deficiencies of local approaches. Specifically, we employ Sobol' indices and univariate effects, which—in contrast to the local sensitivity analyses typically used in economics—accurately identify nonlinearities and interactions in the mapping from parameters to model outcomes. Such global methods have been developed in the last decades in the engineering and applied mathematics fields as part of the more general topic of uncertainty quantification.[2] While local sensitivity analyses rely on comparisons of model outcomes at few selected parameter values, global methods quantitatively formalize the uncertainty surrounding parameter values and propagate it through the model to

[1] See Leamer (1985), Kydland (1992), Canova (1995), Hansen and Heckman (1996), among others, who advocate a structured sensitivity analysis. Canova (1994) and Gregory and Smith (1995) propose global sensitivity analysis as a means to partly answer to the statistical weaknesses of calibration.

[2] See, e.g., Sudret (2007) or Borgonovo and Plischke (2016) for an overview.

evaluate the importance of each parameter, as well as interactions between parameters. The methods we propose are easy to deploy because they are non-intrusive, meaning that they treat the economic model as a black box and thus require no changes to existing code. Also, they are implemented in various freely available software toolboxes.

Many global sensitivity analysis methods are described in the literature, characterized by varying degrees of complexity, as well as underlying assumptions.[3] For an in-depth unifying view of most global sensitivity measures, the reader is directed to Borgonovo, Hazen, and Plischke (2016). In this paper we are interested in the class of importance measures, as our aim is that of providing a robust quantitative assessment of the importance of each of the input parameters with respect to the model outcomes. Arguably one of the most widely accepted importance measures in the engineering and applied mathematics communities is variance decomposition, also known as analysis of variance (ANOVA). Based on the functional model decomposition introduced in Sobol' (1993), variance decomposition allows one to compute total Sobol' indices, which represent the fraction of the variance of the outcome that is explained by each parameter. Total Sobol' indices contain a parameter's direct impact as well as its impact due to interactions with other parameters. First-order Sobol' indices isolate the direct impact, while higher-order Sobol' indices identify the contributions due to the joint effects of groups of parameters at a time, thereby exposing interactions in an economic model. As a result, we get a complete, global importance ranking of all parameters and their interactions, which can be helpful for interpreting model mechanics, as well as guiding model calibration and further model development.

A second sensitivity question of central economic interest is in which direction each parameter affects the outcomes, not just locally but globally. We answer this with so-called univariate effects, which display the slope of that relationship for each parameter over its range, averaging over all other parameters. They help a researcher to find regions of high and low sensitivity, and can be interpreted as a robust direction of change under parameter uncertainty. Thus, they can be very useful for economic policy analysis.

The global approach we propose starts by representing the uncertainty about each parameter by a (potentially bounded) probability distribution. This parameter uncertainty is propagated through the economic model by repeated evaluation at randomly drawn

[3] See Iooss and Lemaître (2014) for a comprehensive review.

parameter vectors. The required sampling from the parameter distributions could be done by Monte Carlo simulation. However, due to the slow convergence properties of Monte Carlo simulations, a very large number of draws would be required, in particular if higher-order Sobol' indices and univariate effects are to be estimated. We overcome this problem by approximating the mapping from parameters to quantities of interest with a so-called sparse polynomial chaos expansion. The Sobol' indices and univariate effects are then computed analytically from the coefficients of the polynomial with high accuracy and at no additional cost (see Sudret (2008)).

We exemplify the approach for the canonical real-business-cycle (RBC) model with capital adjustment costs. This model has been widely studied and is well understood.[4] We consider several quantities of interest, i.e., endogenous outcomes, that are frequently found in the traditional RBC literature. In the main part of the paper we focus on the capital-output ratio, because it is often used as a calibration target, and the ratio of the variance of log production in the model over its empirical counterpart. This second variable, which we will refer to as the production variance ratio, has often been employed to assess how much of the observed fluctuations can be explained by the model (see, e.g., Canova (1995) or Eichenbaum (1991)). Results for other quantities are then summarized to show the broad applicability of global sensitivity analysis.

We find that the local sensitivity measures typically employed in economics can be highly misleading. For example, the relative importance of variance and autocorrelation of total factor productivity (TFP) shocks in determining the production variance ratio flips depending on which parameter vectors are considered. Of course, we know that it is the combination of both that drives the unconditional variance of TFP and production, but this interaction cannot be picked up by a local measure where one parameter at a time is changed individually. Another wide-spread local method is scenario analysis, which—as we show—suffers from similar weaknesses. By contrast, the global Sobol' indices we compute establish an unambiguous ranking of autocorrelation and standard deviation and accurately quantify the contribution of the interaction between the two parameters. More generally, our global analysis shows that only few parameters and interactions matter for each quantity of interest. Therefore, when calibrating or estimating such an RBC

[4]For example, Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) use it to compare different solution methods. Den Haan, Judd, and Juillard (2011) do the same for the multi-country extension.

model, a researcher can focus on a small subset of the parameters.

The univariate effects uncover a non-linear relationship between autocorrelation and the variance of production, with the impact becoming stronger the higher the value of the autocorrelation parameter. Thus, the empirically much researched question of the value of the autocorrelation of TFP shocks is of paramount importance for the results of the RBC model. Another interesting nonlinearity is found for the depreciation rate, whose impact on the capital-output ratio is decreasing and convex. This exemplifies the importance of univariate effects for understanding model properties, but also for economic policy analysis if the parameter under consideration is a policy variable, e.g., a tax rate.

In the economic literature, there are only few papers that perform a global sensitivity analysis (GSA). An early example is Harrison and Vinod (1992) who assume distributions over the elasticities of a static, deterministic general equilibrium model of the macroeconomy to study robustness of their simulation results. A specific field where GSA has received a bit more attention is the economics of climate change. Anderson et al. (2014) compute various global sensitivity indices for the DICE model of Nordhaus (2008). Saltelli and D'Hombres (2010) show that the sensitivity analysis of the Stern (2007) report is not robust. In a highly complex model, Cai, Judd, and Lontzek (2015) perform an extensive sensitivity analysis of the social cost of carbon using local methods on a wide range of parameter vectors.

Canova (1994, 1995) proposes a global sensitivity analysis to put the macroeconomic calibration approach on a statistically more rigorous footing. He analyzes the RBC model and puts great effort into specifying the distributions over the parameters, for which he uses existing studies. All of the above papers rely on Monte Carlo simulations and therefore cannot (accurately) compute interactions or univariate effects.

Ratto (2008) employs first-order Sobol' indices to study the influence of structural parameters on reduced form estimation of linear or linearized DSGE models. More recently, Scheidegger and Bilionis (2017) propose Gaussian process machine learning to solve economic models with very high-dimensional state spaces. They show how this framework lends itself naturally to uncertainty quantification. The main advantage of polynomial chaos expansions as proposed in the present paper lies in the very fast convergence rate of the estimation of Sobol' indices of first and higher orders (Sudret

(2008), Le Gratiet et al. (2016)). In addition, the presented methodology is non-intrusive, thus requiring no changes to existing code.

The paper is structured as follows: in Section two, we introduce a general framework for uncertainty quantification, followed by the theory and numerical techniques employed in global sensitivity analysis. Section three first presents the economic model and then the parameterization for the sensitivity analyses. In Section four, we present and discuss results for local sensitivity analyses, and in Section five for our global sensitivity analysis. Section six concludes.

2 Uncertainty quantification framework

2.1 Introduction

Uncertainty quantification aims at identifying the sources of uncertainty or lack of knowledge that can affect parameters of a model and, subsequently, the predictions obtained from this model. In this paper we call the *computational model* a mapping:

$$\boldsymbol{\theta} \in \mathcal{D}_{\boldsymbol{\theta}} \subset \mathbb{R}^M \mapsto \boldsymbol{y} = \mathcal{M}(\boldsymbol{\theta}) \in \mathbb{R}^Q. \tag{1}$$

To simplify the presentation, we assume in this section $Q = 1$, i.e., we consider a scalar *quantity of interest* (QoI) y. Due to uncertainties in the model parameters, the latter are represented by a random vector $\boldsymbol{\Theta}$ of prescribed joint probability density function (PDF) $f_{\boldsymbol{\Theta}}$ defined over a probabilistic space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, where Ω is the space of outcomes, $\mathcal{F}$ is the associated σ-algebra, and $\mathbb{P}$ is the probability measure associated with the PDF $f_{\boldsymbol{\Theta}}$. For instance, without any further information, the various input parameters $\{\Theta_i, i = 1, \ldots, M\}$ may be considered as statistically independent, and be assigned prescribed ranges.[5]

[5]The following derivations however hold whatever the PDF (e.g., Gaussian, Beta, etc.) of these input parameters.

Uncertainty propagation techniques aim at characterizing the statistical properties of the (random) output of the model

$$Y = \mathcal{M}(\boldsymbol{\Theta}), \tag{2}$$

i.e., estimate its statistical moments (mean μ_Y, variance σ_Y^2) or its probability density function f_Y. *Sensitivity analysis* aims at determining which input parameters $\{\Theta_i, i = 1, \ldots, M\}$ (or combination thereof) contribute the most to the uncertainty of the QoI. In particular, methods for *global sensitivity analysis* developed in the sequel aim at apportioning the variance σ_Y^2 to each input parameter Θ_i, pairs (Θ_i, Θ_j), etc., in order to determine those parameters whose uncertainty explain most of the QoI's variance, as well as to detect those whose uncertainty has no impact on the predictions.

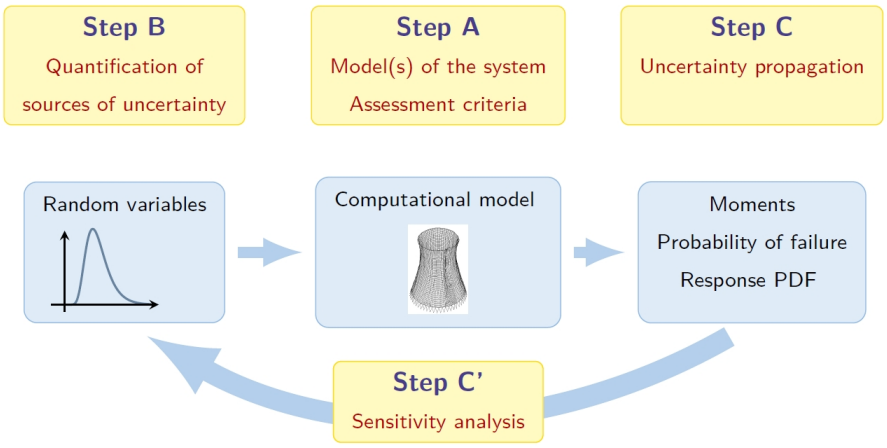


Figure 1: Uncertainty quantification framework

Figure 1 summarizes the different concepts presented above (after Sudret (2007)):

- In Step A, the computational model $\mathcal{M}$ is defined, which requires to identify input parameters and output quantities of interest.
- In Step B, the uncertainty in the input parameters is described by a joint PDF that best represents the available information. In the present case, bounds on the various parameters will be selected based on literature review, see details in Section 3.3. Alternatively, the PDF could result from a previous estimation of the economic model.

- In Step C, uncertainty propagation is carried out so as to analyze the moments and distributions of the QoI, for instance by plotting histograms, or calculating low-probability events.

- Finally, in Step C', sensitivity analysis is carried out to rank the input parameters according to their impact onto the prediction uncertainty, or to identify the direction of change of that impact. The acquired knowledge can be used to focus a more detailed analysis on a subset of the input variables (dimensionality reduction), hence mitigating the computational costs associated with the curse of dimensionality.

The above procedure can be carried out using Monte Carlo simulation (MCS), which is a standard technique to estimate statistical properties based on random number generation. However, when dealing with computationally expensive models, the well-known slow convergence rate of MCS methods hinders their application in many practical scenarios. An alternative, much more efficient approach is given by spectral representations and, in particular, polynomial chaos expansions.

2.2 Polynomial chaos expansions

Instead of being represented through samples as in Monte Carlo simulation, the model output may be represented as a series expansion in an abstract space of random variables (spectral representation). More specifically, assuming that Y has finite variance, it belongs to the Hilbert space of second-order random variables and may be cast as follows (Soize and Ghanem (2004)):

$$Y = \sum_{j=0}^{\infty} b_j \, Z_j. \tag{3}$$

In the above equation, $\{Z_j\}_{j=0}^{\infty}$ is a numerable set of random variables (which form a basis of the Hilbert space), and $\{b_j\}_{j=0}^{\infty}$ are coefficients to be computed. The latter may be interpreted as the *coordinates* of Y in this basis. In the sequel we focus on *polynomial chaos expansions* (PCE), in which the basis terms $\{Z_j\}_{j=0}^{\infty}$ are multivariate orthonormal polynomials w.r.t. the joint PDF of the input random vector $\boldsymbol{\Theta}$.[6] Note that, while Eq. (3) is exact, approximations are in practice obtained by truncating the series to a finite number of terms.

[6] See, e.g., Ghanem and Spanos (2003) or Ghanem and Spanos (2003).

We relegate the construction of the basis to Appendix B, because it is formally identical to other orthonormal polynomials that are often used in economics, e.g., Chebyshev polynomials.[7] Let us stress, however, that a crucial difference is that, in the case of polynomial chaos expansions, the weight functions are given by $f_{\boldsymbol{\Theta}}$, so that the polynomials efficiently capture the uncertainty in the model parameters. Once such a basis is constructed, and a truncation scheme is selected (typically, the maximum total polynomial degree p, cf. Appendix B, or more sophisticated truncation schemes as in Blatman and Sudret (2010a)), the spectral expansion in Eq. (3) becomes:

$$Y = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} b_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\Theta}), \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_M)$ is a multi-index that identifies the polynomial degree in each input variable θ_i, $\Psi_{\boldsymbol{\alpha}}$ is a multivariate orthogonal polynomial built by tensor product of the underlying univariate polynomials of degree α_i and $\mathcal{A}$ is the selected truncation set.

2.2.1 Computation of coefficients by least-squares

The literature on polynomial chaos expansions proposes many alternative approaches to compute the expansion coefficients denoted by $\{b_{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \mathcal{A}\}$. Even when limiting the scope to so-called *non-intrusive approaches*, which rely upon repeated evaluations of the model $\mathcal{M}$ for selected realizations of the input vector, one can mention projection methods (Le Maître, Knio, Najm, and Ghanem (2001)), sparse grids (Keese and Matthies (2003), Ganapathysubramanian and Zabaras (2007)), stochastic collocation (Xiu and Hesthaven (2005)) and least-square minimization (Berveiller, Sudret, and Lemaire (2006)). In this paper we focus on the latter approach for several reasons. First, thanks to recent advances in the field of compressive sensing (see, e.g., Chen, Donoho, and Saunders (1998), Efron et al. (2004)), sparse regression-based PCE (Blatman and Sudret (2010a)) has become a staple method in applied sciences due its efficiency when a limited computational budget is available, even in high dimensional problems. Second, no dedicated sampling algorithm is required to generate the pool of full model evaluations needed to calculate the PCE coefficients (as opposed, e.g., to sparse-grid-based methods). As a matter of fact, sparse PCE can be seen as a post-processing step of an existing MCS. Finally, this family of

[7] See Judd (1998).

methods allows for some noise in the QoI, which is useful in economic applications and which we discuss further in the context of the real-business-cycle model in Section 3.2.

Considering a truncation set $\mathcal{A} \subset \mathbb{N}^M$, the series expansion in Eq. (4) is cast as the sum of the truncated series and a residual ε

$$Y = \mathcal{M}(\boldsymbol{\Theta}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} b_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\Theta}) + \varepsilon. \tag{5}$$

The least-square minimization approach consists of finding the set of coefficients $\mathfrak{B} = \{b_{\boldsymbol{\alpha}}, \, \boldsymbol{\alpha} \in \mathcal{A}\}$ which minimizes the mean square error $\mathbb{E}\left[\varepsilon^2\right]$. This set is computed at once by solving:

$$\mathfrak{B} = \arg \min_{\boldsymbol{b} \in \mathbb{R}^{\text{card}\mathcal{A}}} \mathbb{E}\left[\left(\mathcal{M}(\boldsymbol{\Theta}) - \sum_{\boldsymbol{\alpha} \in \mathcal{A}} b_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\Theta})\right)^2\right]. \tag{6}$$

In practice one replaces the expectation operator in Eq. (6) by the empirical mean over a sample set:

$$\hat{\mathfrak{B}} = \arg \min_{\boldsymbol{b} \in \mathbb{R}^{\text{card}\mathcal{A}}} \frac{1}{N} \sum_{i=1}^{N} \left(\mathcal{M}(\boldsymbol{\theta}^{(i)}) - \sum_{\boldsymbol{\alpha} \in \mathcal{A}} b_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}^{(i)})\right)^2. \tag{7}$$

In this expression, $\mathcal{X}_{ED} = \left\{\boldsymbol{\theta}^{(i)}, \, i = 1, \ldots, N\right\}$ is a sample set of points called *experimental design* (ED) that is typically chosen so as to cover the input parameter space $\mathcal{D}_{\boldsymbol{\Theta}}$. To solve the least-square minimization problem in Eq. (7), the computational model $\mathcal{M}$ is first run for each point in the ED, and the results are stored in a vector $\mathcal{Y} = \left\{y^{(1)} = \mathcal{M}(\boldsymbol{\theta}^{(1)}), \ldots, y^{(n)} = \mathcal{M}(\boldsymbol{\theta}^{(N)})\right\}^{\mathsf{T}}$. Then one calculates the *information matrix* by evaluating the basis polynomials on each point in the ED:

$$\mathbf{A} = \left\{\mathbf{A}_{ij} \overset{\text{def}}{=} \Psi_j(\boldsymbol{\theta}^{(i)}), \; i = 1, \ldots, N, \quad j = 1, \ldots, \text{card } \mathcal{A}\right\}. \tag{8}$$

The solution of the least-square minimization problem finally reads

$$\hat{\mathfrak{B}} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1} \mathbf{A}^{\mathsf{T}} \, \mathcal{Y}. \tag{9}$$

The ED may be built from Monte Carlo simulation, Latin Hypercube Sampling (LHS, see McKay, Beckman, and Conover (1979)) or quasi-random sequences (Niederreiter (1992)). The size of the ED is of crucial importance for a robust analysis. Typical *oversampling rates* $(N/\text{card } \mathcal{A}) = 2$ to 3 are used in practice (Berveiller, Sudret, and Lemaire (2006)).

2.2.2 Error estimation and sparse PCE

As discussed above, the proper truncation set (e.g., the maximal degree of polynomials to be included in the truncated series) depends on the problem under consideration. In order to assess the accuracy of any truncated series, the generalization error $\mathbb{E}\left[\varepsilon^2\right]$ in Eq. (6) shall be estimated. This could be done using a *validation set* $\mathcal{X}_{val} = \{\boldsymbol{\theta}_k,\, k = 1, \ldots, n_{val}\}$ as follows:

$$err(\mathcal{X}_{val}) \overset{\text{def}}{=} \frac{1}{n_{val}} \sum_{k=1}^{n_{val}} \left(\mathcal{M}(\boldsymbol{\theta}_k) - \sum_{\boldsymbol{\alpha} \in \mathcal{A}} b_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}_k) \right)^2, \tag{10}$$

where the validation points may be sampled by Monte Carlo simulation and where n_{val} is large enough, typically equal to 10^4 or 10^5. Such an estimator is, however, not affordable in the general case since the very principle of constructing PC expansions is to limit the number of runs of the original model $\mathcal{M}$. Reusing the ED $\mathcal{X}_{ED}$ in the above equation is not a viable option due to *overfitting*. Indeed, doing so, the so-called *empirical error*, $err(\mathcal{X}_{ED})$, would strongly underestimate the true error $\mathbb{E}\left[\varepsilon^2\right]$.

A good compromise between accuracy and efficiency is obtained by using the *leave-one-out* error estimator (Blatman and Sudret (2010a), Le Gratiet, Marelli, and Sudret (2016)). The principle is the following: a PC expansion $\mathcal{M}^{\mathsf{PC}\setminus i}$ is constructed using an experimental design $\mathcal{X}_{ED}\setminus\boldsymbol{\theta}^{(i)} \overset{\text{def}}{=} \left\{\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(i-1)}, \boldsymbol{\theta}^{(i+1)}, \ldots, \boldsymbol{\theta}^{(N)}\right\}$, and the error is computed on the point that has been left apart:

$$\Delta_i \overset{\text{def}}{=} \mathcal{M}(\boldsymbol{\theta}^{(i)}) - \mathcal{M}^{\mathsf{PC}\setminus i}(\boldsymbol{\theta}^{(i)}). \tag{11}$$

Then the operation is repeated for $i = 1, \ldots, N$ excluding each point in turn. The leave-one-out error is defined by:

$$err_{LOO} \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \Delta_i^2 = \frac{1}{N} \sum_{i=1}^{N} \left(\mathcal{M}(\boldsymbol{\theta}^{(i)}) - \mathcal{M}^{\mathsf{PC}\setminus i}(\boldsymbol{\theta}^{(i)}) \right)^2 \tag{12}$$

and turns out to be, after basic algebra:

$$err_{LOO} = \sum_{i=1}^{N} \left(\frac{\mathcal{M}(\boldsymbol{\theta}^{(i)}) - \mathcal{M}^{\mathsf{PC}}(\boldsymbol{\theta}^{(i)})}{1 - h_i} \right)^2, \tag{13}$$

where h_i is the i-th diagonal term of matrix $\mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ (matrix $\mathbf{A}$ is defined in Eq. (8))

and $\mathcal{M}^{\mathsf{PC}}(\cdot)$ is now the PC expansion built up at once from the *full* experimental design $\mathcal{X}_{ED}$. The error in Eq. (13) requires neither additional model evaluations nor additional PCEs and can thus be computed at very low cost. As a conclusion, as soon as an experimental design is available, the size of which is sufficiently large compared to the number of unknown PCE coefficients, the latter can be computed from a mere least-square minimization (Eq. (9)) and a very good and cheap error estimator is given by Eq. (13).

This error estimator allows for degree-adaptive PCE construction. To see how, define the *standard* truncation scheme by $\mathcal{A}^{M,p} = \left\{\boldsymbol{\alpha} \in \mathbb{N}^M \ : \ |\boldsymbol{\alpha}| \leq p\right\}$, see Appendix B for details.[8] For a given ED, different $\mathcal{A}^{M,p}$ are tried out by varying the maximal polynomial degree p, and the best expansion according to Eq. (13) is finally retained. Values of $err_{LOO} \leq 10^{-2}$ guarantee a sufficient accuracy in practice for moment- and sensitivity analysis. We corroborate this for the real-business-cycle model in Sections 5.2 and 5.4.3.

When the number of input parameters is large (e.g., $M \geq 10$), the standard truncation set $\mathcal{A}^{M,p}$ may easily contain thousands to even millions of basis elements. Due to the necessity of oversampling (i.e., having $N >$ card $\mathcal{A}^{M,p}$), the basic least-squares approach detailed above may not be practically feasible anymore due to the associated computational costs, especially in the presence of expensive-to-evaluate modern computational codes, which may take hours to execute even on dedicated high-performance-computing hardware. In the last few years, algorithms for deriving *sparse* expansions have been proposed: in these approaches, instead of computing a possibly big set of coefficients the majority of which are eventually close to zero, one searches directly for the non-zero coefficients. Techniques such as compressive sensing (e.g., orthogonal matching pursuit (Pati et al. (1993)) or least-angle regression (Efron et al. (2004))) have proven effective in selecting only a few basis polynomials out of a large candidate basis set, and then compute the associated coefficients (Blatman and Sudret (2011), Doostan and Owhadi (2011)). A detailed description can be found in these publications and the literature therein. In the real-business-cycle model of Section 3.1, degree-adaptive sparse PCE based on least-angle regression (LAR) (see Blatman and Sudret (2011)) is used.

[8] In economics, this truncation scheme is also known as *complete* polynomials, cf. Judd (1998).

2.2.3 Post-processing of PC expansions

As mentioned previously, the truncated PC expansion

$$\hat{Y} = \mathcal{M}^{\mathsf{PC}}(\boldsymbol{\Theta}) = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} \hat{b}_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\Theta}) \tag{14}$$

is a sample-free representation of the model output. It contains all the information about the statistical properties of the random output $Y = \mathcal{M}(\boldsymbol{\Theta})$. Due to the orthogonality of the PC basis, mean and standard deviation of $\hat{Y}$ may be computed directly from the coefficients $\hat{\mathfrak{B}}$ (see details in Le Gratiet, Marelli, and Sudret (2016)):

$$\begin{aligned} \hat{\mu}_Y &\overset{\text{def}}{=} \mathbb{E}\left[\hat{Y}\right] = \mathbb{E}\left[\sum_{\boldsymbol{\alpha}\in\mathcal{A}} \hat{b}_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\Theta})\right] = \hat{b}_{\mathbf{0}}, \\ \hat{\sigma}^2_Y &\overset{\text{def}}{=} \mathrm{Var}\left[\hat{Y}\right] = \mathbb{E}\left[\left(\hat{Y} - \hat{b}_{\mathbf{0}}\right)^2\right] = \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{A} \\ \boldsymbol{\alpha}\neq\mathbf{0}}} \hat{b}^2_{\boldsymbol{\alpha}}. \end{aligned} \tag{15}$$

In other words, the mean and variance of the random response may be obtained by a mere combination of the PCE coefficients once the latter have been computed. This property, together with the close relation to Sobol' indices and univariate effects presented in detail in Section 2.3, significantly distinguish PCE from formally similar polynomial approximation methods (as used, e.g., in Cai and Judd (2010)). Indeed, especially in the context of moment- and sensitivity analysis, the polynomial expansion in Eq. (14) is never used directly to approximate the full model. Rather, it is the coefficients themselves that are directly used to give a fast-converging estimate of the statistics of the underlying model.

From a functional point of view, however, the function $\boldsymbol{\theta} \mapsto \mathcal{M}^{\mathsf{PC}}(\boldsymbol{\theta})$ in Eq. (14) can still be viewed as a *surrogate* of the original model $\mathcal{M}$, i.e., an analytical, easy-to-evaluate function that gives a good approximation of the true model output $\mathcal{M}(\boldsymbol{\theta})$. The quality of the approximation is not ensured pointwise uniformly, but in the mean-square sense, as can be seen from the derivation of the PC coefficients (Eqs. (6)-(9)). Importantly, the construction of PCE using the parameters' input distributions ensures that the stochastic properties of the uncertainty are retained, which is crucial for the GSA we propose in this paper. One can take advantage of this feature to obtain accurate plots of the output distribution, i.e., the PDF of the output random variable $Y = \mathcal{M}(\boldsymbol{\Theta})$. For this purpose,

a large Monte Carlo sample set $\mathcal{X} = \{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n\}$ is drawn from the input distribution $f_{\boldsymbol{\Theta}}$, say $n = 10^6$. Then the surrogate model $\mathcal{M}^{\mathsf{PC}}$ is run onto this sample set in no time. The sample set of PCE outputs $\mathcal{Y}^{\mathsf{PC}} = \left\{\mathcal{M}^{\mathsf{PC}}(\boldsymbol{\theta}_1), \ldots, \mathcal{M}^{\mathsf{PC}}(\boldsymbol{\theta}_n)\right\}$ is then plotted as a histogram, or using kernel density smoothing techniques (Wand and Jones (1995)).

2.3 Sensitivity analysis

2.3.1 Global sensitivity analysis

Global sensitivity analysis aims at quantifying which are the input parameters $\{\Theta_i\}_{i=1}^M$ or combinations thereof that best explain the variability of the quantity of interest $Y = \mathcal{M}(\boldsymbol{\Theta})$ (Saltelli, Chan, and Scott (2000), Saltelli (2008)). This variability being described by the variance $\mathrm{Var}[Y]$, the question reduces to apportioning the latter to each input parameter $\{\Theta_1, \ldots, \Theta_M\}$, pairs (Θ_i, Θ_j), etc. For this purpose, variance decomposition techniques (a.k.a. functional ANOVA) have gained interest since the mid 90's. The Sobol' decomposition (Sobol' (1993)) states that any square integrable function $\mathcal{M}$ with respect to a probability measure associated with a PDF $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \prod_{i=1}^M f_{\Theta_i}(\theta_i)$ (independent components[9]) may be cast as:

$$\mathcal{M}(\boldsymbol{\theta}) = \mathcal{M}_0 + \sum_{i=1}^{M} \mathcal{M}_i(\theta_i) + \sum_{1 \le i < j \le M} \mathcal{M}_{ij}(\theta_i, \theta_j) + \cdots + \mathcal{M}_{12\ldots M}(\boldsymbol{\theta}), \tag{16}$$

that is, as a sum of a constant $\mathcal{M}_0$, univariate functions $\{\mathcal{M}_i(\theta_i)\,,\, 1 \le i \le M\}$, bivariate functions $\{\mathcal{M}_{ij}(\theta_i, \theta_j)\,,\, 1 \le i < j \le M\}$, etc. Using the *set notation* for indices

$$\mathbf{u} \overset{\text{def}}{=} \{i_1, \ldots, i_s\} \subset \{1, \ldots, M\}\,, \tag{17}$$

the Sobol' decomposition in Eq. (16) reads:

$$\mathcal{M}(\boldsymbol{\theta}) = \mathcal{M}_0 + \sum_{\substack{\mathbf{u} \subset \{1, \ldots, M\} \\ \mathbf{u} \neq \emptyset}} \mathcal{M}_{\mathbf{u}}(\boldsymbol{\theta}_{\mathbf{u}}), \tag{18}$$

[9] Extensions of functional ANOVA to the case of dependent random variables exist—see, e.g., Kucherenko, Tarantola, and Annoni (2012), Mara and Tarantola (2012), Caniou and Sudret (2013). However, their interpretation is more complex due to the need to distinguish between contributions due to interaction and to correlation. An important class of global sensitivity measures that do not rely on the independence of the input parameters is that of moment-independent measures, first introduced by Borgonovo (2007).

where $\boldsymbol{\theta}_\mathbf{u}$ is a subvector of $\boldsymbol{\theta}$ which only contains the components that belong to the index set $\mathbf{u}$. It can be proven that the Sobol' decomposition is unique when the orthogonality between summands is required, namely:

$$\mathbb{E}\left[\mathcal{M}_\mathbf{u}(\boldsymbol{\theta}_\mathbf{u})\,\mathcal{M}_\mathbf{v}(\boldsymbol{\theta}_\mathbf{v})\right] = 0 \quad \forall\, \mathbf{u}, \mathbf{v} \subset \{1, \ldots, M\}, \quad \mathbf{u} \neq \mathbf{v}. \tag{19}$$

Orthogonality with the constant term $\mathcal{M}_0$ implies in particular that $\mathbb{E}\left[\mathcal{M}_\mathbf{u}(\boldsymbol{\theta}_\mathbf{u})\right] = 0\ \forall\, \mathbf{u} \subset \{1, \ldots, M\}$. The existence and uniqueness of Eq. (16) together with the orthogonality property in Eq. (19) now allow one to decompose the variance $D \overset{\text{def}}{=} \mathrm{Var}\left[\mathcal{M}(\boldsymbol{\Theta})\right]$ as follows:

$$D = \mathrm{Var}\left[\sum_{\substack{\mathbf{u}\subset\{1,\ldots,M\}\\ \mathbf{u}\neq\emptyset}} \mathcal{M}_\mathbf{u}(\boldsymbol{\Theta}_\mathbf{u})\right] = \sum_{\substack{\mathbf{u}\subset\{1,\ldots,M\}\\ \mathbf{u}\neq\emptyset}} \mathrm{Var}\left[\mathcal{M}_\mathbf{u}(\boldsymbol{\Theta}_\mathbf{u})\right] = \sum_{\substack{\mathbf{u}\subset\{1,\ldots,M\}\\ \mathbf{u}\neq\emptyset}} D_\mathbf{u} \tag{20}$$

where the *partial variances* are defined by:

$$D_\mathbf{u} \overset{\text{def}}{=} \mathrm{Var}\left[\mathcal{M}_\mathbf{u}(\boldsymbol{\Theta}_\mathbf{u})\right] = \mathbb{E}\left[\mathcal{M}_\mathbf{u}^2(\boldsymbol{\Theta}_\mathbf{u})\right]. \tag{21}$$

2.3.2 Sobol' indices

The so-called *Sobol' indices* $S_\mathbf{u}$ are defined as the ratio of the partial variances $D_\mathbf{u}$ to the total variance D. Due to Eq. (20) they obviously sum up to 1. Hence each index is interpreted as the share of variance that is explained by the group of parameters $\boldsymbol{\Theta}_\mathbf{u}$. The *first-order indices* correspond to single input variables, i.e., $\mathbf{u} = \{i\}$:

$$S_i = \frac{D_i}{D} = \frac{\mathrm{Var}\left[\mathcal{M}_i(\Theta_i)\right]}{\mathrm{Var}\left[Y\right]}. \tag{22}$$

The second-order indices ($\mathbf{u} = \{i, j\}$) read:

$$S_{ij} = \frac{D_{ij}}{D} = \frac{\mathrm{Var}\left[\mathcal{M}_{ij}(\Theta_i, \Theta_j)\right]}{\mathrm{Var}\left[Y\right]}, \tag{23}$$

etc. Note that the *total Sobol' index* S_i^T, which quantifies the total impact of a given parameter Θ_i including all interactions with other parameters, may be computed by the

sum of the Sobol' indices of any order that involve Θ_i:

$$S_i^T = \sum_{i \in \mathbf{u}} S_{\mathbf{u}}. \tag{24}$$

Sobol' indices allow for an in-depth analysis of the relative impact of the uncertainties affecting the model predictions. The formulæ above are interpreted as follows:

- *Factor setting*: the total Sobol' index S_i^T indicates the share of the total variance D explained by the input parameter θ_i, alone or in combination with any other parameter(s). If this is negligible (in pratice, if $S_i^T < 1\%$), this means that parameter θ_i could be set to a deterministic value without changing the distribution of the quantity of interest.

- *Screening*: the first-order Sobol' index S_i indicates by what percentage the total variance D would be reduced, should the parameter θ_i be perfectly known and set to a fixed value. It allows to determine which parameter(s) shall be investigated in priority, should one want to decrease the prediction variability.

Classically, Sobol' indices are evaluated by Monte Carlo simulation. Detailed expressions of the estimators of first-order and total indices can be found in Sobol' (1993), Sobol' (2001), Janon et al. (2014). In practice, two sample sets of the input vector $\mathbf{\Theta}$ are used for computing *each* first-order (resp. total) index. Typically $n_S = 10^3$ to 10^4 samples are needed for accuratey estimating each index, leading to a total cost of $(M+1) \cdot n_S$. This high computational cost is affordable when the considered model $\mathcal{M}$ is analytical, or at least very fast to evaluate. Fortunately, the technique of polynomial chaos expansions presented above allows for a straightforward evaluation of Sobol' indices.

2.3.3 PC expansion-based Sobol' indices

Sobol' indices are considered as the most versatile sensitivity measures for general computational models, since they do not rely on any assumption of linearity nor monotonicity of the model $\mathcal{M}$ (Saltelli (2008)). Their estimation by Monte Carlo simulation is, however, computationally demanding, as mentioned above. A number of recent approaches has been proposed to reduce the computational burden associated to their estimation, mostly

based on recent developments of stochastic collocation techniques (Ma and Zabaras (2010), Yang et al. (2012)). In this paper, we choose to follow the approach proposed by Sudret (2008), where the Sobol' indices are derived by directly post-processing the coefficients of the PCE. When combined with their sparse-regression-based calculation, this approach has been extensively shown to be computationally very efficient (see, e.g., Blatman and Sudret (2010b), Deman et al. (2016)). Indeed, the Sobol' decomposition (Eq. (16)) of a truncated PC expansion $\mathcal{M}^{\mathsf{PC}}(\boldsymbol{\theta}) = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} \hat{b}_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta})$ can be derived *analytically*, as shown below.

For any subset of variables $\mathbf{u} = \{i_1, \ldots, i_s\} \subset \{1, \ldots, M\}$ let us define the set of multivariate polynomials $\Psi_{\boldsymbol{\alpha}}$ which depend *only* on $\mathbf{u}$ by

$$\mathcal{A}_{\mathbf{u}} = \{\boldsymbol{\alpha} \in \mathcal{A} \,:\, \alpha_k \neq 0 \;\text{ if and only if }\; k \in \mathbf{u}\}\,. \tag{25}$$

One can observe that the $\mathcal{A}_{\mathbf{u}}$'s form a partition of $\mathcal{A}$ since

$$\bigcup_{\mathbf{u}\subset\{1,\ldots,M\}} \mathcal{A}_{\mathbf{u}} = \mathcal{A}. \tag{26}$$

Thus a truncated PC expansion such as in Eq. (14) may be rewritten as follows by simple reordering of the terms:

$$\mathcal{M}^{\mathsf{PC}}(\boldsymbol{\theta}) = b_0 + \sum_{\substack{\mathbf{u}\subset\{1,\ldots,M\}\\ \mathbf{u}\neq\emptyset}} \mathcal{M}^{\mathsf{PC}}_{\mathbf{u}}(\boldsymbol{\theta}_{\mathbf{u}}) \tag{27}$$

where:

$$\mathcal{M}^{\mathsf{PC}}_{\mathbf{u}}(\boldsymbol{\theta}_{\mathbf{u}}) \overset{\text{def}}{=} \sum_{\boldsymbol{\alpha}\in\mathcal{A}_{\mathbf{u}}} b_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}). \tag{28}$$

Consequently, due to the orthogonality of the PC basis, the partial variance $D_{\mathbf{u}}$ in Eq. (21) reduces to:

$$D_{\mathbf{u}} = \mathrm{Var}\left[\mathcal{M}^{\mathsf{PC}}_{\mathbf{u}}(\boldsymbol{\Theta}_{\mathbf{u}})\right] = \sum_{\boldsymbol{\alpha}\in\mathcal{A}_{\mathbf{u}}} b^2_{\boldsymbol{\alpha}}, \tag{29}$$

i.e., again, a mere sum of squares of selected coefficients. The Sobol' indices $S_{\mathbf{u}}$ can then be computed by dividing the above results by the total variance (Eq. (15)). In other words, from a given PC expansion, the Sobol' indices *of any order* may be obtained by a mere combination of the squares of the coefficients. As an illustration, the first-order PC-based

Sobol' indices read:

$$S_i^{\mathsf{PC}} = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_i} b_{\boldsymbol{\alpha}}^2 / D \qquad \mathcal{A}_i = \{\boldsymbol{\alpha} \in \mathcal{A} \,:\, \alpha_i > 0\,,\, \alpha_{j \neq i} = 0\}\,, \tag{30}$$

whereas the total PC-based Sobol' indices are:

$$S_i^{T,\mathsf{PC}} = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_i^T} b_{\boldsymbol{\alpha}}^2 / D \qquad \mathcal{A}_i^T = \{\boldsymbol{\alpha} \in \mathcal{A} \,:\, \alpha_i > 0\}\,. \tag{31}$$

As a conclusion, polynomial chaos expansions not only provide a surrogate model for a possibly computationally expensive model as those used nowadays in economics, but also yield at no cost the full set of sensitivity indices that are useful for a better understanding of the single and joint effects of input parameters on quantities of interest.

2.3.4 Univariate effects

While Sobol' indices provide quantitative insight on the importance of a parameter, they do not include information about the direction in which it affects the quantities of interest. Which parameters have an overall positive, which a negative relationship? Is the relationship of the input parameter to the model outcome linear or non-linear? In which regions of the parameter range is the sensitivity the largest? These questions can be answered with univariate effects, originally introduced by Younes et al. (2013). Univariate effects can be defined as the conditional expectation of a quantity of interest as a function of a single parameter, where expectations are taken over all other parameters:

$$\mathcal{M}_i^{(1)}(\theta_i) = \mathbb{E}\left[\mathcal{M}(\boldsymbol{\Theta} | \Theta_i = \theta_i)\right]. \tag{32}$$

They can thus be interpreted as an average or robust relationship between an input parameter and the quantity of interest. In the case of PCE models, univariate effects have an analytical closed form that is closely related to the first-order Sobol' decomposition in Eq. (30) (Deman et al. (2016)):

$$\mathcal{M}_i^{(1)}(\theta_i) = b_{\mathbf{0}} + \sum_{\boldsymbol{\alpha} \in \mathcal{A}_i} b_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\theta_i), \qquad \mathcal{A}_i = \{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_i > 0, \alpha_{i \neq j} = 0\}\,. \tag{33}$$

All the techniques described will be applied to the economic model of Section 3.1.

2.4 Relevance for structural estimation

Sobol' indices and univariate effects can play a major role in structural estimation. The identification of a parameter (or a set thereof) through structural estimation is only possible if the corresponding total Sobol' index is significant. A negligible total Sobol' index means that the QoI under scrutiny is not affected by the value of the underlying random variable. Thus, additional empirical evidence on such QoI cannot constrain the parameter's value.

On the other hand, while a large total Sobol' index means that the paremeter affects the QoI, it is possible that its effect is highly non-linear, hence leading to multiple local maxima in the corresponding likelihood function. In other words, a significant total Sobol' index is a necessary but not sufficient condition for the identifiability of a parameter in structural estimation. Univariate effects can provide an effective tool to identify this particular scenario.

The PCE-based approach presented here could be particularly useful as a pre-step in structural estimation because of its efficiency and accuracy in identifying irrelevant parameters. Showing this in an economic application would, in our view, be very interesting, but is outside the scope of the present paper, so that we leave it for future research.

3 Economic model and parameterization

The presented tools for uncertainty quantification are generically applicable to any economic model. To illustrate their use, we apply them to a canonical real-business-cycle (RBC) model with capital adjustment costs, because this model has often been used as a test bench for introducing new numerical methods, see for example Den Haan, Judd, and Juillard (2011), Brumm and Scheidegger (2017), or Winschel and Kraetzig (2010).

In the subsection on the RBC model, we also define the QoIs for which we perform uncertainty quantification. In the subsection on the parameterization we also include parameter bounds, which are essential to the local and global sensitivity analyses later on.

3.1 Real-business-cycle model and quantities of interest

We first summarize the standard RBC model before defining the quantities of interest. The allocation problem is described by the dynamic optimization

$$\max_{\{c_t,l_t,i_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t^{\chi}(1-l_t)^{1-\chi})^{1-\frac{1}{\tau}}}{(1-\frac{1}{\tau})}. \tag{34}$$

The objective function is a discounted sum of utilities of consumption c_t and leisure $1-l_t$ in each period, where β is the discount factor, τ is the intertemporal elasticity of substitution (IES), and χ is the leisure share parameter in utility. The decision variables are consumption c_t, labor l_t, and investment i_t. The aggregate resource constraint is

$$q_t = c_t + i_t + \frac{\phi}{2} k_t \left(\frac{i_t}{k_t} - \delta\right)^2, \tag{35}$$

where q_t denotes the quantity of produced goods, k_t the capital stock, and δ the depreciation rate of capital. Production q_t can be used for consumption and investment, the latter being subject to a convex adjustment cost. These investment adjustment costs are modeled like in Den Haan, Judd, and Juillard (2011), with ϕ governing the size of the costs. The production technology

$$q_t = \exp(z_t) k_t^{\alpha} l_t^{1-\alpha} \tag{36}$$

depends on productivity z_t, capital k_t, labor l_t and the capital share α. The capital transition and stochastic productivity processes are given by

$$k_{t+1} = i_t + (1-\delta)k_t, \tag{37}$$

$$z_{t+1} = \rho z_t + e_{t+1}, \tag{38}$$

where ρ is the autocorrelation coefficient of the productivity process with independent, identically and normally distributed shocks $e_{t+1} \sim \mathcal{N}(0, \sigma)$.

Using the notation of the uncertainty quantification framework introduced above, we have a parameter vector containing eight parameters,

$$\boldsymbol{\theta} = \{\beta, \tau, \chi, \alpha, \delta, \rho, \phi, \sigma\}. \tag{39}$$

From an uncertainty quantification perspective, the parameter vector is the input to the RBC model, which itself can be treated as a black box, $\mathcal{M}(\boldsymbol{\theta})$.[10] In this model, all parameters are continuous, but for the GSA methods we propose it is no problem to have discrete parameters, e.g., the number of countries, as an input.

Turning to the quantities of interest, let us first repeat that the basic RBC model was chosen for illustrating the methodology, not because it features particularly insightful QoIs. For most of the paper, we consider just two quantities of interest, $\boldsymbol{y} = \{y_1, y_2\} = \mathcal{M}(\boldsymbol{\theta})$, to keep the exposition clear. The first QoI is the average capital-output ratio,

$$y_1 = \mathbb{E}\left[\frac{k_t}{q_t}\right], \tag{40}$$

which is often used as a calibration target, e.g., Kydland and Prescott (1982) or Cooley and Prescott (1995). Our sensitivity analysis can help identify which parameters are most relevant for calibrating it (and which are not), and—by looking at univariate effects—in which regions of the parameter space the capital-output ratio is particularly sensitive.[11]

The second QoI is the ratio of the variance of log production in the model over its empirical counterpart. This second variable, which we will refer to as the production variance ratio, is frequently the quantity of interest in standard RBC models, where it is used to assess how much of the observed fluctuations can be explained by the model (see, e.g., Eichenbaum (1991), King and Rebelo (1999), or Canova (1995)). Denoting by $\hat{\sigma}_q^2$ the empirical variance of log production in the data, the variance ratio is

$$y_2 = \frac{1}{\hat{\sigma}_q^2}\mathrm{Var}\left[\log(q_t)\right]. \tag{41}$$

We set $\hat{\sigma}_q^2 = 3.28$, a standard value in the literature (see, e.g., King and Rebelo (1999)). In Section 5.6 we discuss and present results for other quantities of interest.

[10] This is different from the literature that considers robust decision making under model uncertainty, cf. our discussion in the concluding Section 6. Under that approach, the agent takes the parameter uncertainty into account such that the optimization problem in Eq. (34) needs to be modified accordingly.

[11] Alternatively, one can first calibrate the model to match the capital-output ratio and then study how sensitive the results are. This is relevant, since the empirical values found in the literature range from 2.5 to 4 and higher (cf., e.g., McGrattan and Prescott (2017)), which deliver different parameter values.

3.2 Accuracy of model solution for global sensitivity analysis

The Sobol' decomposition in Eq. (16) is exact when there is no error in the QoIs. Nonetheless, the methods we propose are robust to some numerical error in the evaluation of the model for the following important reasons. First, as discussed in Section 2.2.1, we compute the PCE by least-squares regression, which allows for small noise in the data.[12] Second, Sobol' indices and univariate effects are based on conditional expectations, see Equations (22) and (32). As long as numerical error in the model evaluation is unbiased and small relative to the variance of the QoIs, the error is integrated out and the results of the GSA are unaffected.[13] Third, for Sobol' indices and univariate effects to be used as a sensitivity measure, it is not necessary that their values be known exactly. For example, it does not matter whether the total Sobol' index of the capital share, α, is 0.83 or 0.84, because small differences do not affect the interpretation in a sensitivity context. The interpretation would rather be that, e.g., the variance of the QoI explained by α is "between 80 and 85 percent", and α is "clearly more important than" or "approximately of equal importance as" another parameter.

Thus, while the canonical RBC model presented above can be solved easily at high accuracy, our methods are well suited to more expensive models that are harder to evaluate. For the reasons mentioned in the previous paragraph, it is not necessary to solve expensive models at very high accuracy. On top of that, we employ a degree-adaptive, sparse PCE, as explained in Section 2.2.2, meaning that a small number of model evaluations is often sufficient. We discuss this further in Section 5.4.3, where we study the convergence of Sobol' indices with respect to the experimental design.

We solve the RBC model with global non-linear methods (Cai and Judd (2010)) using the generic toolbox of Miranda and Fackler (2002). Specifically, we use a time iteration algorithm to solve the Euler equation for optimal choices. We employ Chebyshev collocation with a polynomial of degree 11 (Judd (1992)). To compute our QoIs, the model is simulated for 5000 periods, discarding the first 1000. This is sufficient for an accurate

[12] In addition, if the model cannot be solved to the desired accuracy at a given parameter vector, e.g., due to non-convergence for numerical reasons, the vector can simply be discarded from the experimental design.

[13] If the error is large relative to the variance of the QoIs, or if it is biased, then the model results are likely to be unreliable independently of sensitivity analysis.

and stable solution of the model.[14] Relative Euler equation errors, evaluated along the simulation path for all parameter vectors and reported in $\log_{10}$-scale, are in the range of -5.0 to -6.2 for the average error and in the range of -3.0 to -4.1 for the maximum error. This is in line with other studies (e.g., Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006)).

To illustrate the robustness of the global sensitivity measures, we reduced the degree of the Chebyshev polynomial to three. Average Euler errors increase markedly to a range of $[-2.3, -4.2]$ in $\log_{10}$-scale. While small changes in the Sobol' indices and univariate effects are observable, the results in terms of importance ranking, interactions, and direction of change do not change at all.

3.3 Parametrization for sensitivity analysis

Generally, the parameter vector, $\boldsymbol{\theta}$, can be determined either by estimation or by calibration of the model. Since we do not want to distract from the paper's contribution, we will simply parametrize the model by setting $\boldsymbol{\theta}$ to values that are common in the literature. However, it is important to note that the methods for uncertainty quantification and global sensitivity analysis proposed in this paper are just as applicable and relevant when the parameters are first calibrated or estimated.

Our baseline parameterization, $\boldsymbol{\theta^0}$, closely follows Cooley and Prescott (1995), whose values are considered standard in the literature. They are displayed in the second column of Table 1. Since Cooley and Prescott (1995) do not have adjustment costs, we take the value for ϕ from Juillard and Villemot (2011). As is typical in the RBC literature, the values are for quarterly data.

The lower and upper bounds for each parameter, $\underline{\theta}_i$ and $\overline{\theta}_i$, are set symmetrically around each baseline value. In the context of sensitivity analysis, the bounds should be chosen to represent values at the upper and the lower end of what most economists would still find reasonable. For example, in an RBC model, a value for the capital share of $\alpha = 0.9$ is theoretically possible, but wouldn't be considered plausible and is thus not included in our range. The restriction that the bounds be symmetric around the mean

[14] Increasing the order of the polynomial to 15 and the number of simulation periods to 10 000 does not affect our results at all.

is not necessary, but facilitates the discussion of the local sensitivity analysis. It does, however restrict the ranges that we can consider, for example for the discount factor β, since $\beta < 1$ is also required. The bounds are displayed in the third and fourth column of Table 1. We base them on Canova (1994), who performs an extensive literature review of the parameter values of this RBC model.[15]

Table 1: Parameter values

Parameter	Baseline, θ_i^0	Lower bound, $\underline{\theta}_i$	Upper bound, $\overline{\theta}_i$
Discount factor, β	0.98	0.97	0.99
IES, τ	0.6	0.2	1
Leisure share in utility, χ	0.3	0.2	0.4
Capital share, α	0.35	0.2	0.5
Depreciation rate, δ	0.02	0.01	0.03
Capital adjustment cost, ϕ	0.5	0.00	1.00
Autocorrelation of TFP, ρ	0.95	0.92	0.98
Standard deviation of TFP, σ	0.007	0.005	0.009

The three values θ_i^0, $\underline{\theta}_i$, and $\overline{\theta}_i$ for each parameter are used in the local sensitivity analysis in the next section. The global sensitivity analysis also uses them, but additionally specifies a distribution over each parameter, cf. Section 5.1.

4 Local sensitivity analysis

This section presents two local sensitivity measures that are often encountered in quantitative economic work, namely one-at-time finite differences and scenario analysis.[16] Thereby, we can compare them to the global sensitivity measures that are presented in Section 5. Generally, local measures are intuitive and easy to implement, but suffer from three important drawbacks. First, they are valid only locally at the chosen evaluation points and may differ substantially for other, even close-by points. Second, they typically rely on a linear approximation of the slope, so that non-linearities are not accounted for. And third, they either do not capture interactions between the parameters, or if they do, they

[15] Canova (1994) considers log utility and therefore does not have bounds for the IES. Instead, we cover the same range of the IES as Juillard and Villemot (2011).

[16] Elasticities are another local sensitivity measure, which is conceptually related to OAT analysis.

cannot isolate them. Because of these important drawbacks, this section only presents the methods without giving economic interpretation. The economic interpretation is instead given in the section on the global sensitivity measures.

4.1 One-at-a-time finite differences

One of the most common sensitivity analyses in numerical economics consists of changing a single parameter value, while keeping all others fixed, and reporting the change in the quantity of interest. Often this is interpreted as a robustness check. When performed for all parameters in turn, this procedure is known in the uncertainty quantification literature as one-at-a-time (OAT) finite differences (e.g., Borgonovo and Plischke (2016)).

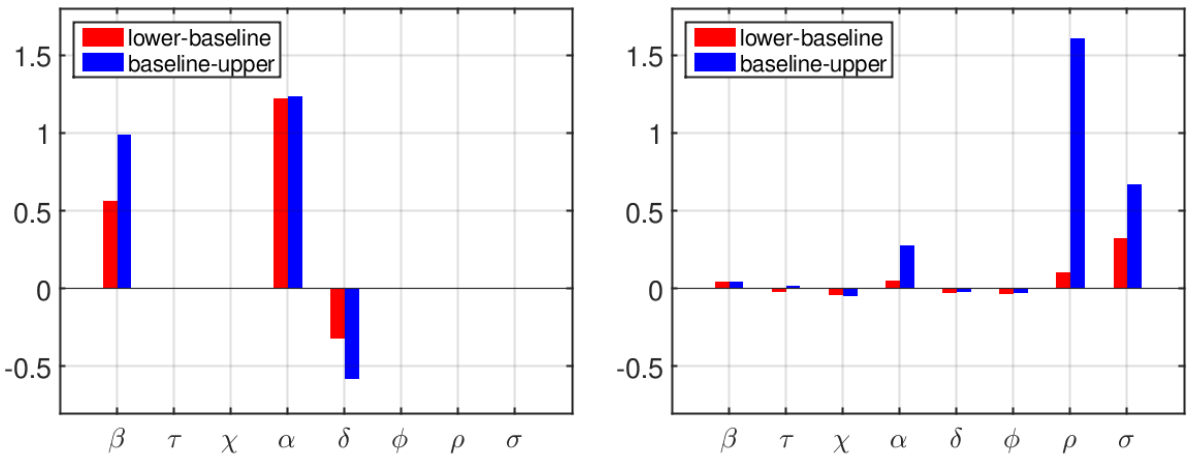
Since OAT is a local measure, we compute it for two different points. Specifically, we first change one parameter at a time from its lower bound to its baseline value given in Table 1, while keeping all other parameters fixed at their lower bound values. The corresponding change in each QoI is:

$$OAT_i^1 = \mathcal{M}(\underline{\boldsymbol{\theta}}_{\sim \boldsymbol{i}}, \theta_i^0) - \mathcal{M}(\underline{\boldsymbol{\theta}}), \tag{42}$$

where $\underline{\boldsymbol{\theta}}_{\sim \boldsymbol{i}}$ means that all parameters but i take their lower bound values. We then do the same starting at the baseline values in Table 1 and changing one parameter at a time to its upper bound, thereby keeping the same direction and size of change as in Eq. (42):

$$OAT_i^2 = \mathcal{M}(\boldsymbol{\theta}^{\mathbf{0}}_{\sim \boldsymbol{i}}, \overline{\theta}_i) - \mathcal{M}(\boldsymbol{\theta}^0). \tag{43}$$

Figure 2 plots the values of OAT_i^1 in red and OAT_i^2 in blue. Turning first to average capital-output ratio in the left-hand panel, we observe that only three of the eight parameters have a non-zero impact. When looking at the first evaluation point, OAT_i^1, represented in red, it is clear that capital share, α, is the most important parameter, followed by discount factor, β, and depreciation rate δ. However, when we move to the second evaluation point, OAT_i^2, represented in blue, α and β are much more similar in their impact, because the importance of β has increased whereas the effect of α remains unchanged. The reason is that for β, nonlinearities, interactions, or both are at work,



(a) Average capital-output ratio

(b) Production variance ratio

Figure 2: One-at-a-time (OAT) sensitivity indices, showing the impact of changing a single parameter while keeping all others fixed (cf. Eqs. (42) and (43)). Red bars represent parameter changes from lower bound to baseline value, blue bars from baseline value to upper bound, cf. Table 1.

whereas capital share, α, seems to have a linear impact on the average capital-output ratio.

Turning to the production variance ratio in the right-hand panel, we similarly observe that only three parameters matter. However, the set of parameters impacting this QoI is quite different. By comparing the two evaluation points, OAT_i^1 in red and OAT_i^2 in blue, we see clearly the main weakness of local analysis: not only does the impact of autocorrelation, ρ, change dramatically between OAT^1 and OAT^2, but also the importance ranking of ρ and σ is reversed. The reasons are, again, nonlinearities and interactions, which cannot be identified or accounted for with the OAT sensitivity measure. Joint variation of multiple parameters is known as scenario analysis, discussed next.

4.2 Scenario analysis

Scenario analysis is another very common form of sensitivity analysis in economics.[17] In a scenario, typically several parameter values are changed simultaneously to reflect some change in the economic environment.[18] This is intuitively appealing and allows for more

[17] See, e.g., Stern (2007).

[18] More generally, a scenario can be defined as a set of assumptions. As long as the assumptions can be nested using a real-valued parameter, they can be analyzed with the tools discussed in this paper.

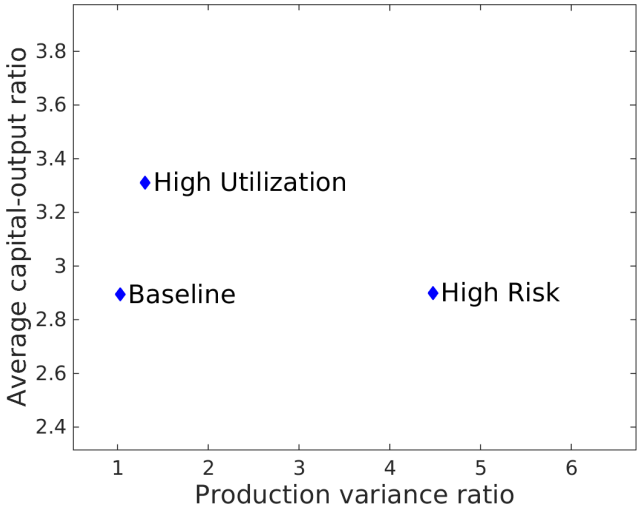


Figure 3: Scenario analysis, each scenario representing a set of parameter values as explained in the enumerated list in the main text.

complex parameter changes than the one-at-a-time finite differences of the previous section. Thereby, scenario analysis is able to capture interactions between parameters, but it is not straightforward to isolate the effect of such interactions.

We consider three scenarios based on the parameter values in Table 1. Due to the simplicity of the canonical RBC model, the scenarios are not meant to capture a particularly plausible economic environment, but rather to exemplify the approach. One drawback that becomes apparent is the high level of discretion typically involved in choosing scenarios and corresponding parameter values, which is due to the local nature of this sensitivity measure. The scenarios are:

1. Scenario "Baseline": all parameters take their baseline values.

2. Scenario "High risk and risk aversion": σ, and ρ are at their upper bounds and τ is at its lower bound, so that risk aversion $\frac{1}{\tau}$ is high. All other parameters are at their baseline values.

3. Scenario "High capital utilization and frictions": α, δ, and ϕ are at their upper bounds, all other parameters are at their baseline values.

Figure 3 plots the three scenarios in a graph with the two quantities of interest, average capital-output ratio and production variance ratio, on the axes. Therefore, we can compare the scenarios and evaluate the impact of joint parameter changes on the two quantities of

interest. However, scenario analysis does not allow us to tell which parameter or which interaction between parameters is important in each case.

For example, the "Baseline" scenario has the lowest level of both QoIs. The "High risk and risk aversion" scenario has a much higher variance ratio, but we cannot say whether this is mostly due to the increase in autocorrelation, ρ, or standard deviation, σ, of TFP shocks. From the results in the previous section, it could be either, cf. Figure 2. One solution would be to combine the scenario analysis with OAT finite differences to tease out individual parameter effects and interactions, known as scenario decomposition or generalized Tornado diagrams (see, e.g., Borgonovo (2010) and Borgonovo, Castaings, and Tarantola (2011)). However, such Tornado diagrams are rarely encountered in economic studies and are outside the scope of this paper, since they also suffer from the fact that they are local and linear.

5 Global sensitivity analysis

In this section, we present the results of the global sensitivity analysis for the canonical RBC model. In contrast to the local sensitivity measures of the previous section, the global measures do not depend on a specific evaluation point. In addition, they fully capture the non-linearity in the mapping from parameters to quantities of interest and allow us to analyze interactions between parameters. All calculations in this section are performed with UQLab©, an actively maintained Matlab© toolbox for uncertainty quantification (Marelli and Sudret (2014, 2017)).[19]

5.1 Parameter distributions

As explained in Section 2, we need to specify a distribution that represents the uncertainty about the value of each parameter. This is a crucial step in GSA, as the sensitivity results depend on it, and should thus be done carefully. Methodologically, there is no limitation on what distributions are allowed.[20] This is determined by the research question and data

[19]The toolbox can be freely downloaded from `www.uqlab.com`.

[20]It is, for example, possible to use discrete distributions with mass points.

availability. In our case, the research question is how sensitive outcomes are with respect to the parameters, and there is only little data on parameter values of RBC models.

The most widely used approach to determine the most suited distributions to represent the lack of knowledge of the specific value of a parameter in the absence of empirical data is given by the principle of maximum entropy (Jaynes (1982)), commonly used to define prior information in Bayesian analysis. The rationale behind this principle is that if only a set of constraints is available regarding the value of an unknown variable—e.g., its maximum and minimum bounds, its moments, or its sign—, the distribution that maximizes the information entropy (a measure of the variability of a random variable) while respecting the constraints should be used to represent its statistical uncertainty. Common examples of such distributions for continuous variables include the uniform distribution when the minimum and maximum values are known, the Gaussian distribution when the first two moments are known, or the exponential distribution when the variable is strictly positive but unbounded.

In our particular case, we have identified the available information on the parameters of the RBC model as the minimum and maximum bounds on the admissible parameters based on the work of Canova (1994), who derives "a least informative (Bayesian) density" for each parameter of the RBC model using a comprehensive literature review. Since we assume that the only knowledge we have are the bounds of each parameter, applying the maximum entropy principle results in the choice of uniform distributions to represent our lack of knowledge on the actual parameter values. Of course, the available information for other economic models may not be the same, hence requiring different distributions to parametrize the corresponding uncertainty. Since often data on parameter values may not be available, the use of literature review, expert opinion, and "commonly used values" is perfectly acceptable. In such circumstances, the uncertainty about a parameter value is usually represented using a continuous distribution rather than point mass for discrete values, unless the parameter is known to have a discrete nature. It should also be noted that the maximum entropy distributions can be updated through Bayesian analysis in the case empirical data were to become available. A uniform distribution is also adequate for our research question, where we want to understand the model sensitivities over plausible ranges of parameters. We do not—and for lack of data could not—ask what empirical

parameter distributions would mean for the distribution of the quantities of interest. Therefore, we assume that all parameters of our model are independently and uniformly distributed with support given by the lower and upper bounds in Table 1.[21]

For this approach, it does not matter whether the values for the baseline parameter vector, $\boldsymbol{\theta^0}$, are taken from the literature or are calibrated to some empirical targets. If the model was calibrated, we could proceed in the same fashion by specifying distributions and performing our global sensitivity analysis. If, instead, a structural model was estimated with data, then one could additionally use the marginal distributions of the parameter estimates as the parameter distributions.

5.2 Polynomial chaos expansion of the economic model

We first demonstrate the convergence of the polynomial chaos expansion of the real-business-cycle model, since the following GSA is based on it. Due to the computational cost of the RBC model, a *maximum* computational budget of 500 model evaluations was available, which allowed us to run both GSA and additional convergence analysis. We therefore generated a set of nested experimental designs of increasing size $N_{ED} = \{50, 60, ..., 500\}$ using nested Latin Hypercube Sampling (Blatman and Sudret (2011)). For each set, a sparse PCE was calculated based on least-angle-regression with an adaptive degree selection in the range $3 \leq p \leq 20$, cf. Section 2.2.2. The resulting set of PCEs was then compared based on their leave-one-out error estimator calculated with Eq. (13).

The resulting convergence plot for each of the two QoIs is displayed in Figure 4. The error for the average capital-output ratio drops very quickly, while the production variance ratio converges at a slower rate. Based on the guidelines given in Le Gratiet, Marelli, and Sudret (2016), a leave-one-out error $err_{LOO} \leq 0.05$ was deemed sufficient for the purposes of first and total Sobol' indices. Therefore a computational budget of $N = 150$ model evaluations was selected for all the subsequent analyses, unless explicitly specified. As shown below, this budget suffices to achieve a very high accuracy of Sobol' indices. However, for more sophisticated economic models, a more realistic computational budget may be in the order of 10^{1-2}. This is typically sufficient for GSA for the reasons discussed

[21] As mentioned in Footnote 9, assuming dependence between parameter distributions makes it difficult to disentangle the effects of parameters.

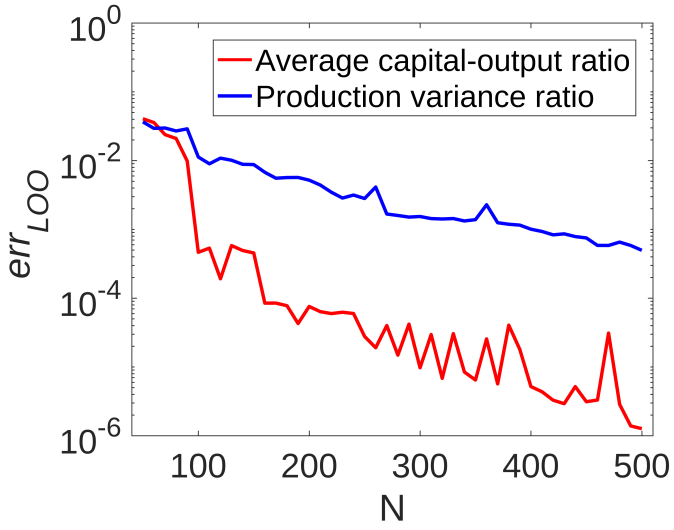


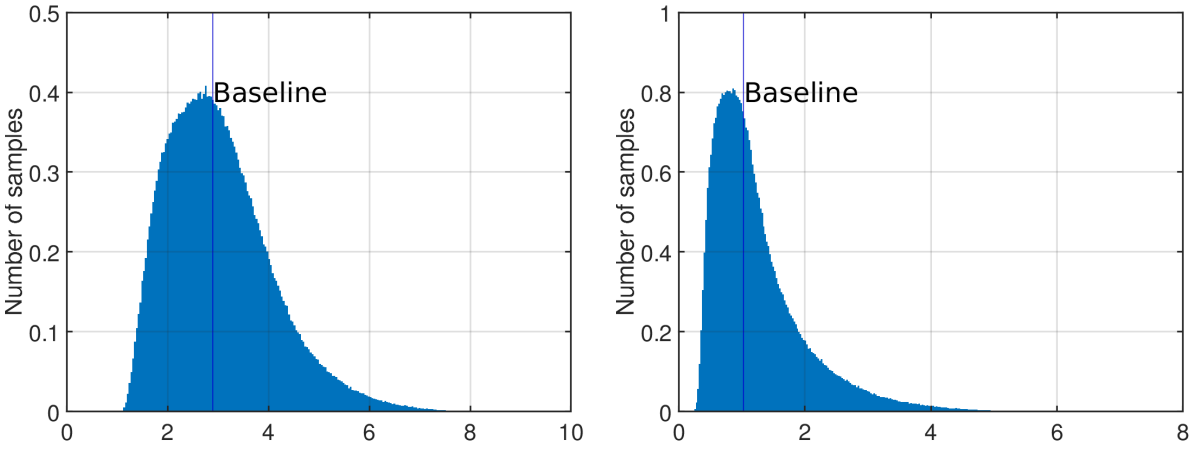
Figure 4: Convergence of the PCE leave-one-out error estimator (Eq. (13)) as a function of the size of the experimental design for both QoIs.

in Sections 3.2 and 5.4.3.

5.3 Histograms of the quantities of interest

To see how the the parameter uncertainty propagates through the model, first consider the resulting histograms of the two quantities of interest. To get the histograms, we evaluate the surrogate model on a Monte Carlo sample of size one million. Such a large number of evaluations would be prohibitively expensive for the RBC model. For smaller sample sizes, the histograms of the surrogate and the original model are virtually identical.

Figure 5 displays the histograms of average capital-output ratio (left) and production variance ratio (right). Both distributions have a notable dispersion and are right-skewed. The values corresponding to the baseline, $\boldsymbol{\theta^0}$, are indicated by vertical lines. For these two QoIs, the baseline values are close to the respective modes of the distribution, but that need not generally be the case. The left-hand figure shows that typical target values for the capital-output ratio in $[2.5, 3.5]$ can be achieved with many different parameter combinations, whereas the more extreme value of 6 considered by McGrattan and Prescott (2017), while still being in the range considered here, substantially limits the possible parameter values. The right-hand figure is very similar to figure 5 in Canova (1994), who, however, had only 1 000 Monte Carlo-based model evaluations, too few to perform a global sensitivity analysis, as we do next.



(a) Average capital-output ratio

(b) Production variance ratio

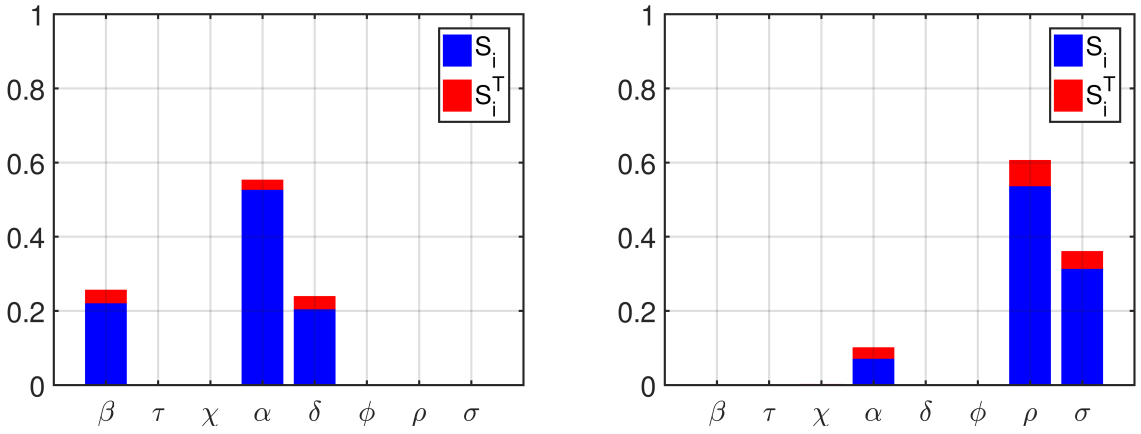
Figure 5: Histograms of the QoIs, calculated by drawing a Monte Carlo sample of size 1 000 000 from the parameter distributions and evaluating each parameter vector with the PCE surrogate of the RBC model. The straight line represents the value if the model is evaluated at the baseline parameter values, cf. Table 1.

5.4 Sobol' indices

5.4.1 Total and first-order Sobol' indices

As described in Section 2.3.2, Sobol' indices are an important tool to establish an importance ranking of input parameters and their interactions. Total Sobol' indices (Eq. (24)) represent the fraction of the variance of a quantity of interest that is explained by the variability of each input parameter, including non-linearities and interactions. Therefore, small total indices are indicative of unimportant variables. First-order indices (Eq. (22)) instead only account for the direct contribution of each parameter, including non-linearities but excluding interactions terms. It is common to compare the two sets of indices to identify the importance of interactions between input variables: if total and first-order indices are very similar, the model is mostly additive (no interactions), otherwise interactions play an important role. Figure 6 shows this comparison between total (in red) and first-order (in blue) indices for both QoIs.

The left-hand panel displays the indices for the average capital-output ratio. The first thing to notice is that only three of the eight parameters have an influence. The other Sobol' indices are zero, meaning that we can fix them to any value without influencing the capital-output ratio. To some extent, this is surprising, because in a general equilibrium



(a) Average capital-output ratio

(b) Production variance ratio

Figure 6: Total Sobol' indices, S_i^T, in red and first-order Sobol' indices, S_i, in blue. The total Sobol' indices are larger by construction, as they include all interactions. Based on $N = 150$ evaluations of the RBC model.

of a dynamic, stochastic economy with rational expectations, as considered here, all parameters can, in principle, affect all endogenous outcomes. Due to the complexity of such models, obtaining the insight that only few parameters matter is generally difficult. Moreover, local analysis would not help, as any results are valid only locally. We can, of course, derive the deterministic steady state of the RBC model, where is easy to see that τ, χ, and ϕ drop out of the equations (and by definition of deterministic steady state, ρ and σ play no role). But that this transfers to the stochastic economy is not immediately clear. Turning to the three parameters with non-zero indices, we first observe that they are of substantial (not just minor) importance. The capital share, α is clearly the most important, indicating that most effort should go into that parameter when calibrating the RBC model. Since estimates of the capital share depend a lot on data quality, period considered, and the measure of capital (or labor) income employed, a global sensitivity analysis should be an integral part of quantitative studies using a neoclassical production function.[22] Typically, when calibrating the RBC model, α is fixed to a value taken from other studies. Given that α is so fixed, we see that the discount factor β and the depreciation rate δ are equally important in determining the capital-output ratio.

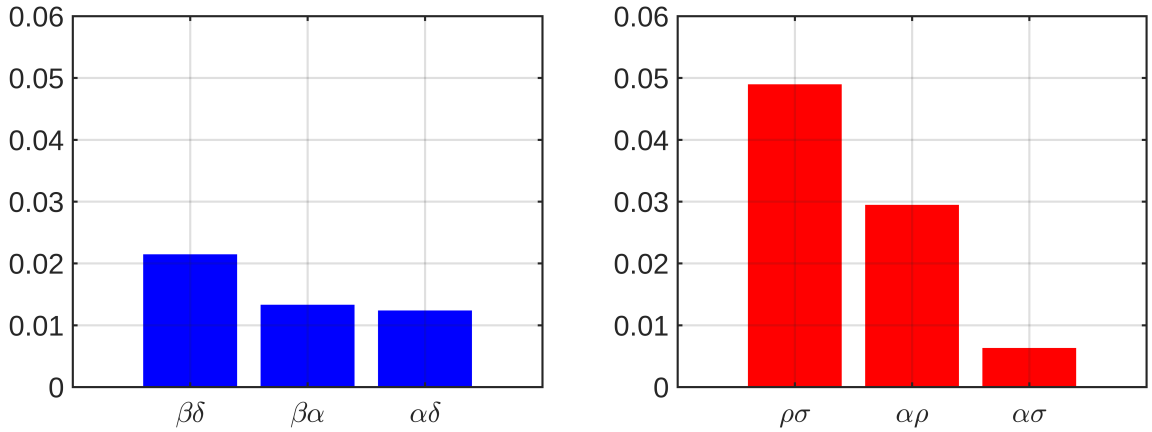
The production variance ratio in the right-hand panel displays a similar pattern in

[22]For example, estimates of the capital share depend on whether nonfarm proprietors' income or intellectual property rights are included in the measure of capital. Also, the capital share has increased over time, cf. Karabarbounis and Neiman (2014) or Elsby, Hobijn, and Şahin (2013).

that again only three parameters are non-zero. Note, however, that the set of parameters governing capital-output ratio and variance ratio is quite different. While it is not surprising that autocorrelation, ρ, and standard deviation, σ, of TFP shocks are the most important parameters for the variance of production, it is to some extent unexpected that no other parameters play a role (α being the only other non-zero parameter, but unimportant compared to ρ and σ). As argued above, in a rational expectations general equilibrium of a dynamic economy, preference parameters could, in principle, have an impact on this QoI. In addition, for such a variance-related QoI, it would be hard to derive insights by resorting to a deterministic steady state. The conclusion we can draw from the figure is that preferences and technology do not matter for determining one of the most relevant quantities of interest of the early RBC literature—it is only the shock process that matters. Thus, the assumption of log-utility found in many papers (e.g., Cooley and Prescott (1995)) is innocuous, at least when the variance ratio is the focus. Last, recall that in the local OAT analysis of Section 4.1, the results on which parameter was more important—ρ or σ—were ambiguous and depended on the evaluation point. There is no such ambiguity here: over the intervals specified, ρ is more important. As there is substantial disagreement in the literature on the empirical value of ρ, GSA should be performed.[23]

For all non-zero parameters, the total Sobol' index in red and the first-order Sobol' index in blue behave very similarly. The total index is larger by construction, as it contains all interactions, whereas the blue bar represents the direct impact of each parameter (excluding interactions). The difference represents the interactions of each parameter. While non-negligible, the interactions are, overall, surprisingly small, given that this is a general equilibrium model, where all parameters could potentially interact. Such interactions are of substantial interest economically, since they help explain the model's mechanics, which often cannot be derived analytically. The figure therefore confirms the inadequacy of OAT measures since they cannot take interactions into account. To see which of any two parameter combinations interact, we next study second-order Sobol' indices.

[23]Estimates of the stochastic process of TFP depend, for example, on how aggregate time series are detrended, cf. Canova (1998).



(a) Average capital-output ratio

(b) Production variance ratio

Figure 7: Second-order Sobol' indices, representing the importance of interactions between any two parameters. Only the three largest are shown for each of the two quantities of interest. Based on $N = 250$ evaluations of the RBC model.

5.4.2 Second-order Sobol' indices

From an economic perspective, interactions between parameters are particularly interesting. However, accurate computation of higher-order indices requires an experimental design of bigger size. To estimate second-order indices, we choose—based on the convergence shown in Figure 4—a sample size of $N = 250$, corresponding to $err_{LOO} \approx 2 \cdot 10^{-3}$ for the production variance ratio.

The three second-order Sobol' indices corresponding to the three parameters that had non-zero total Sobol' indices are displayed in Figure 7 for each of the two QoIs. Generally, they are not large, but, as we show in the next section, they are estimated at very high precision, meaning that we can confidently interpret them.

For average capital-output ratio, the strongest interaction turns out be between those two parameters that, individually, had less of an impact, namely β and δ. The interaction means that, for example, increasing the discount factor will directly increase the capital output ratio (first-order Sobol' index), but this effect is dampened due to depreciation. In other words, the smaller δ, the larger the effect of β. While this interaction can also be seen in the analytic solution for the deterministic steady state, it is not obvious from the deterministic solution that this is the most important interaction, and what that means for the stochastic economy. Another interesting point is that the indices $S_{\beta\alpha}$ and $S_{\alpha\delta}$ are of essentially the same magnitude. It is due to the fact that these two interactions offset

each other that the local OAT for α does not change between evaluation points in Fig. 2a.

Turning to the production variance ratio we find that the interaction between autocorrelation ρ and standard deviation σ is very important, which is not surprising as these two parameters determine the unconditional variance of TFP and thus of production. That both parameters interact with α is due to the standard modeling of the TFP shocks being multiplicative to production technology, which in turn is determined mainly by α. The result that $S_{\alpha\rho}$ is much larger than $S_{\alpha\sigma}$ reflects the corresponding finding for first-order Sobol' indices, namely that ρ is more important than σ.

Finally, recall that local OAT analysis cannot identify such interactions. The ambiguous results it provided regarding the relative importance of ρ versus σ were in part due to this deficiency, as these two parameters turn out to interact strongly. Since the economic literature typically uses local sensitivity measures, there are nearly no studies that identify interactions.[24] The fast, accurate, and non-intrusive identification of parameter interactions is an important advantage of the global methods we propose.

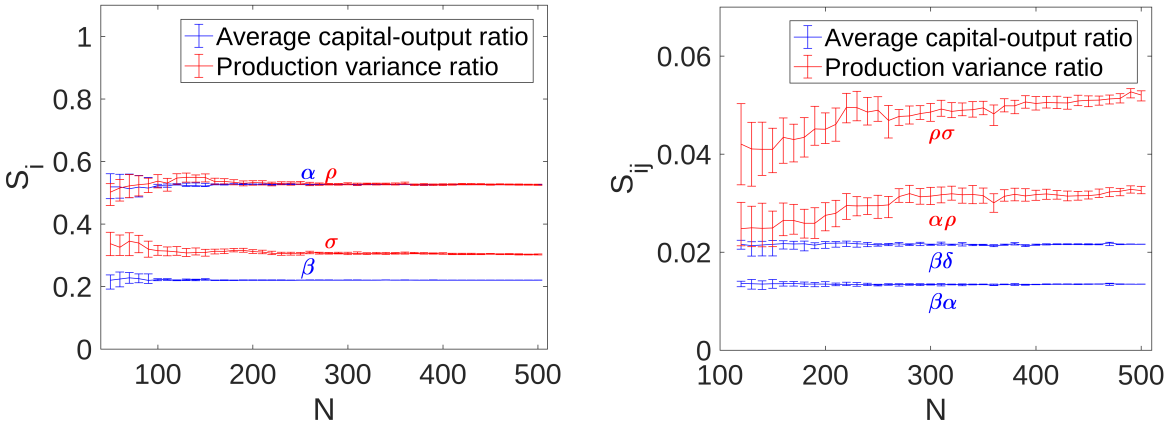
5.4.3 Convergence of Sobol' indices

As discussed in Section 5.2, the main rationale for choosing a minimum experimental size is to achieve a sufficiently low generalization error for the PCE. However, it is worth giving further insight on the convergence behavior of the Sobol' indices, as these are of central interest and tend to converge very fast.[25] Therefore, in this section we look at the estimates of the Sobol' indices as a function of the experimental design size.

The convergence study consists of estimating total, first-order, and second-order indices for a set of increasingly larger experimental designs with a maximum of of $N = 500$. The experimental designs are constructed as described in Section 5.2. Confidence bounds for each index estimate are calculated as the 95% empirical inter-quantile ranges by means of $N_B = 100$ bootstrap replications of the underlying PCE coefficients. The main results are reported in Figure 8 for the two largest first-order Sobol' indices (left panel) and the two largest second-order indices (right panel). Total Sobol' indices are not shown because their convergence behavior is essentially identical to that of their first-order counterparts. From

[24] One exception is Anderson, Borgonovo, Galeotti, and Roson (2014) who compute second-order Sobol' indices for a subset of the parameters of the climate change model of Nordhaus.

[25] See also our discussion relating to the robustness of Sobol' indices in Section 3.2.



(a) First-order Sobol' indices

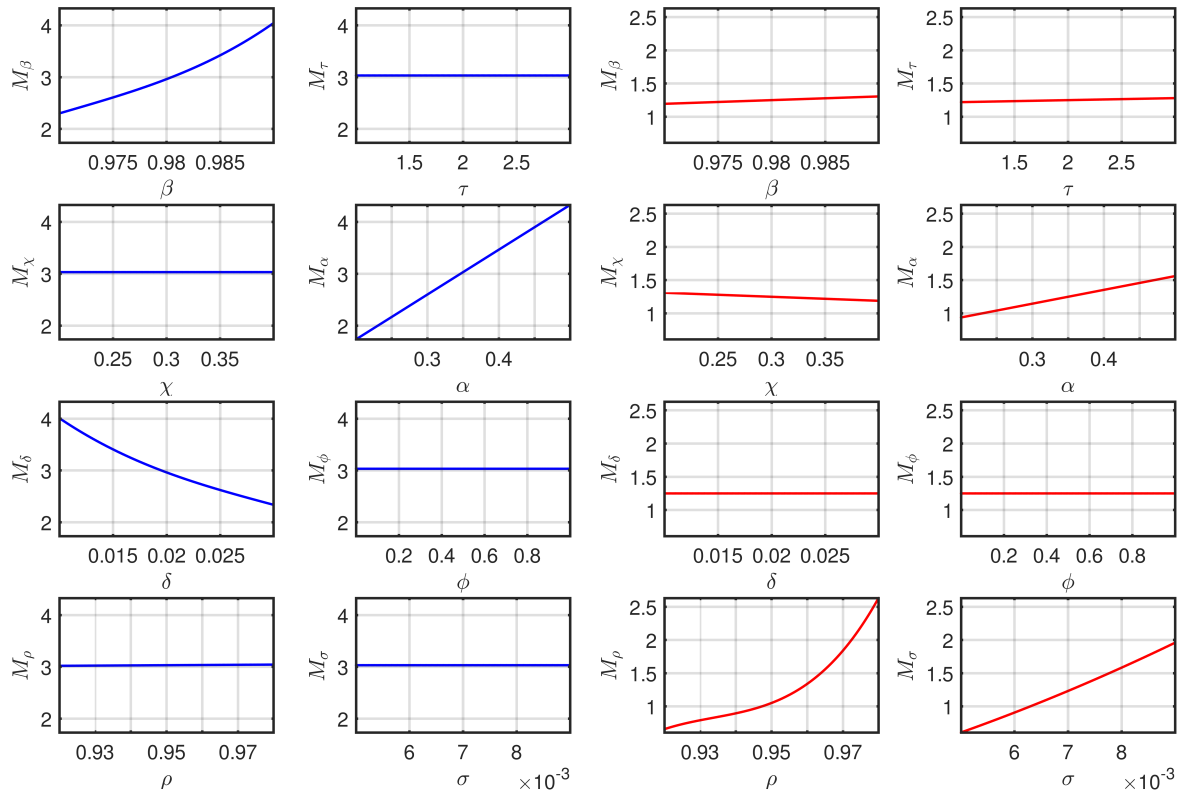
(b) Second-order Sobol' indices

Figure 8: Convergence of estimates of first- and second-order Sobol' indices. The left panel shows the two largest non-zero first-order indices for each QoI. The right panel shows the two largest second-order indices for each QoI. Error bounds are calculated by bootstrap resampling of the PCE coefficients.

the convergence behavior in Figure 8 it is clear that estimators of first-order (and total) indices converge already with as few as $N = 120$ model evaluations, whereas second-order indices require approximately $N = 220$ model evaluations.

Therefore, the empirical error bounds of $err_{LOO} \leq 5 \cdot 10^{-2}$ for total and first-order indices as well as $err_{LOO} \leq 5 \cdot 10^{-3}$ for second-order indices are appropriate. For more expensive models, it is possible to adopt a greedy strategy by gradually enriching the experimental design until the target err_{LOO} for the desired analysis is reached.

In realistic scenarios with highly complex models, the available computational budget can be much lower than the 500 model evaluations we could afford for this model. However, the combination of sparse PCE and Sobol' indices is widely regarded as one of the most computationally effective tools available to perform global sensitivity analysis and can often be performed with affordable experimental designs. This efficiency is largely due to the effectiveness of sparse PCE, whose convergence is very fast globally, a crucial property for the unbiased estimation of Sobol' indices. Further considerations on convergence, even for models that show a relatively poor point-wise convergence of the PCE surrogates, are discussed in Le Gratiet, Marelli, and Sudret (2016). Recent applications with complex, high dimensional and/or highly computationally expensive models can be found, e.g., in Deman et al. (2016), Le Gratiet et al. (2016), Chiaramello et al. (2017).



(a) Average capital-output ratio

(b) Production variance ratio

Figure 9: Univariate effects, which are conditional expectation functions, showing the direction in which a parameter impacts each quantity of interest. Because the mean effect is included, the y-axis represents the expected value of the quantity of interest.

5.5 Univariate effects

As described in Section 2.3.4, univariate effects are conditional expectations functions of a parameter that provide a robust magnitude and sign of the parameter's impact on the QoIs. The univariate effects for the eight input parameters are shown in Figure 9 for average capital-ouput ratio (left) and production variance ratio (right), respectively. The mean effect is included in each, so that the y-axis directly shows the corresponding value of the quantity of interest, cf. Eq. (32). For example, as β ranges from 0.97 to 0.99, expected average capital-output ratio increases from approximately 2.2 to 4, where the expectation is taken over all other parameters.

For capital-output ratio, the parameters which had a total Sobol' index of zero accordingly have univariate effects that are flat. The univariate effect for the discount rate β is increasing and convex. The convexity reflects, of course, the exponential discounting

in the agent's objective. The impact of capital share α is very close to linear over the whole range. This is because—even though α enters as an exponent in the production function—the quantity of interest here is the *ratio* of capital over output, both of which are affected similarly by α. Finally, the univariate effect of the depreciation rate δ is falling and convex. The result that β and δ are nonlinear, while α is close to linear, corresponds to the findings of the local OAT analysis of Section 4.1. However, we now get a much clearer picture as to the exact shape of (non-)linearities over the respective ranges. Together with the insights on interactions that we get from the second-order Sobol' indices, we can only now fully understand and interpret the results of the OAT analysis.

For the production variance ratio, shown in the right panel, we again observe that parameters with first-order Sobol' indices close to zero display univariate effects that are zero or negligibly small, while those corresponding to higher indices (α, ρ, and σ) display sizeable effects. Unsurprisingly, both ρ and σ display a positive slope, but the univariate effect of σ on the variance ratio is close to linear while that of ρ is strongly convex. Therefore, when setting ρ to a high value, e.g., to 0.975, it is much more important to be careful and put effort in obtaining an accurate estimate than it is for low values. Empirically, the discussion centers a lot around values approaching a unit root, which, in light of our results, indeed is an important question when studying how much of the business cycle a given model can explain.

All univariate effects of the RBC model turn out to be monotone in the ranges we specified. Of course, non-monotonic behavior is possible and would be highly relevant for calibrating a model. For example, a parameter could first have a negative impact on a quantity of interest and, after crossing some threshold value, a positive impact. Identifying and interpreting such relationships should typically be of substantial economic relevance. As a point in case, an interesting application of univariate effects would be the study of a policy reform, where the reform is governed by a real-valued parameter, say a tax rate. If the quantity of interest is, for example, social welfare, then the univariate effect can be interpreted as the robust impact of the tax rate on welfare, which often will be non-monotonic and have local or global maxima.[26] Finding such global optima that are

[26] As discussed in our concluding Section 6, robustness in our context refers to the sensitivity of the model's quantities of interest under parameter uncertainty. In particular, this is complementary to the robust decision making under model uncertainty as studied, e.g., in Hansen and Sargent (2007).

robust to parameter uncertainty should make policy recommendations stemming from economic models more credible to policy makers.

5.6 Other quantities of interest

We have focused on two quantities of interest, average capital-output ratio and production variance ratio, to explain the methodology of GSA. In Appendix A, we extend the analysis to four more QoIs that are frequently found in the traditional RBC literature: the variance ratio of log consumption, the correlation of consumption and output, the variance of consumption growth, and the correlation of investment with output. Our aim is to showcase how the methodology applies broadly, picks up the individually different characteristics of the QoIs, and presents them in an easy to convey manner. It is important to note that extending the analysis with more QoIs can be done at very small cost as long as either the QoIs have already been calculated or—in the case of stochastic models like our RBC model—the model simulations have been saved. In that case, there is no need to create a new experimental design and run the model.

We find that there is substantial variation as to which parameters are important for each QoI. One result worth highlighting is that the utility leisure share, χ, and the capital adjustment costs, ϕ, do not impact any of the QoIs we studied. As mentioned previously, these two parameters drop out in the steady state equation of the capital-output ratio, but that this transfers to the stochastic economy and all other QoIs, for which analytical solutions do not exist, is not obvious. One conclusion is that the functional specification of adjustment costs, which is frequently used in the literature, cf. Den Haan, Judd, and Juillard (2011), is not well suited for studying these QoIs. Depending on the research question, it should be modified or extended.

Other QoIs that would be of economic interest and for which GSA may yield important insights are, e.g., welfare, inequality, or bankruptcies. Such questions are, of course, not in the scope of the canonical RBC model, which is used here to explain the methodology.

6 Conclusion

This paper introduces Sobol' indices and univariate effects as tools for global sensitivity analysis (GSA) and uncertainty quantification in economics and shows how to accurately compute them with a limited computational budget using polynomial chaos expansions. We apply this methodology to the canonical real-business-cycle model with capital adjustment costs and compare it to traditional local methods, such as one-at-a-time parameter changes and scenario analyses. The comparison shows that the traditional local sensitivity analysis can be misleading as it depends on the chosen evaluation points. Only the global analysis captures non-linearities and identifies interactions, both of which are central to economic models. For univariate effects, which are conditional expectation functions of each parameter, it is worth emphasizing the economic significance for public policy evaluation. For example, the policy parameter could be a tax rate and the quantity of interest could be social welfare. Then the univariate effect could be interpreted as the robust impact of the tax on welfare under parameter uncertainty.[27]

With respect to robust economic policy analysis, a related strand of the economic literature has studied the impact of model uncertainty, where the decision maker cannot assign probabilities to different, competing economic models because of fundamental, irreducible uncertainty. In that strand of literature, non-Bayesian approaches to decision making are used, such as min-max (e.g., Hansen and Sargent (2007)) or min-max regret (e.g., Brock et al. (2003), Brock et al. (2007)), yielding optimal policies that perform sufficiently well under all—and, in particular, adverse—model specifications.[28] The sensitivity analysis we propose is complementary, as it can be applied to a min-max (regret) model in order to understand the importance of the parameters that are not part of the fundamental model uncertainty, for example the parameters in the decision maker's utility function. Combining model uncertainty and parameter uncertainty is an interesting undertaking that we leave for future research.

While we parameterized the model with standard values from the literature to focus on the sensitivity analysis, the methods we propose are equally well suited if the model is calibrated to empirical targets or even structurally estimated. For a structurally estimated

[27] A nice example is the ongoing work of Gersbach, Liu, and Tischhauser (2018), who apply the methods we propose to study robustness of forward guidance for monetary policy.

[28] An insightful study comparing different approaches is Cai and Sanstad (2016).

model, one can use the moments of the parameter estimates to specify the distributions needed for uncertainty quantification, or even the posterior marginal distributions (obtained, e.g., through kernel density estimation). For a calibrated model, uncertainty quantification puts the calibration procedure on a more rigorous statistical footing, as argued by Eichenbaum (1991), Gregory and Smith (1995), and in particular Canova (1994, 1995).

There are several freely available toolboxes that implement GSA and uncertainty quantification. The analysis in this paper is done with the Matlab© toolbox UQLab, which is non-intrusive, i.e., it treats the model as a black box. Therefore, no changes have to be made to an existing model solution code and the proposed methods can readily be deployed. In addition, the methods in this paper are straight-forward to parallelize by distributing the experimental design points.

For academic research, the insights offered by GSA can help economists understand mechanisms and interactions in complex models and inform them as to where to direct efforts, e.g., when extending the model or calibrating it. For policy-oriented work, a GSA is crucial for assessing the plausibility and credibility of policy recommendations.

Appendix

A Other quantities of interest

In this section we present results for the quantities of interest discussed in Section 5.6. These are the variance ratio of log consumption, $y_3 = \frac{1}{\hat{\sigma}_c^2}\mathrm{Var}\left[\log(c_t)\right]$, the correlation of consumption and output, $y_4 = corr(c_t, q_t)$, the variance of consumption growth, $y_5 = var(\frac{c_{t+1}}{c_t} - 1)$, and the correlation of investment with output, $y_6 = corr(i_t, q_t)$.

Fig. 10 displays the total Sobol' indices for all four QoIs. The Sobol' indices for consumption variance are, as expected, very similar to those of production variance. The variance of consumption growth, by contrast, depends mostly on the standard deviation of TFP shocks, σ, and only very little on the autocorrelation, ρ. Turning to the correlations of consumption and investment with output, it is well known that the standard RBC model has difficulty matching the data (cf. King and Rebelo (1999)). The Sobol' indices

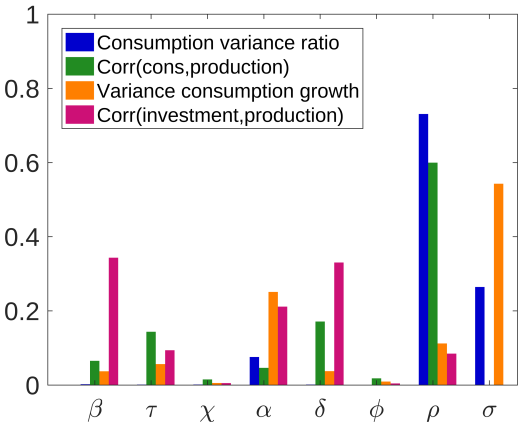


Figure 10: Total Sobol' indices for other QoIs.

show that, while both are influenced by the same set of parameters, the importance of the parameters differs strongly. This knowledge can help to calibrate the model.

The corresponding univariate effects are displayed together in Fig. 11. When the univariate effects for different QoIs are plotted together, it is common to scale them by mean and variance, since otherwise some effects may be hard to see due to scaling issues. This is different of how we represented the univariate effects in Section 5.5, but the advantage here is that the direction of change can easily be compared. The information on the importance of each parameter is given by the Sobol' indices of the previous figure. For example, Fig. 11 shows a positive impact of ϕ on three of the QoIs, but the corresponding total Sobol' indices are essentially zero, meaning that these univariate effects can be ignored. On the other hand, the capital share α has non-zero Sobol' indices for all QoIs, and we see that its univariate effects have positive slope for the consumption variance ratio and the correlation of investment with production, while having a negative impact on the variance of consumption growth and the correlation of consumption with production. Also, one can see that the impact is close to linear for some parameters and nonlinear for others.

B Polynomial basis

In this appendix we provide details on how to construct a polynomial basis in Section 2.2. While this can also be found in, e.g., Judd (1998), we restate it here as the context is very different, and because the construction is crucial to the PCE.

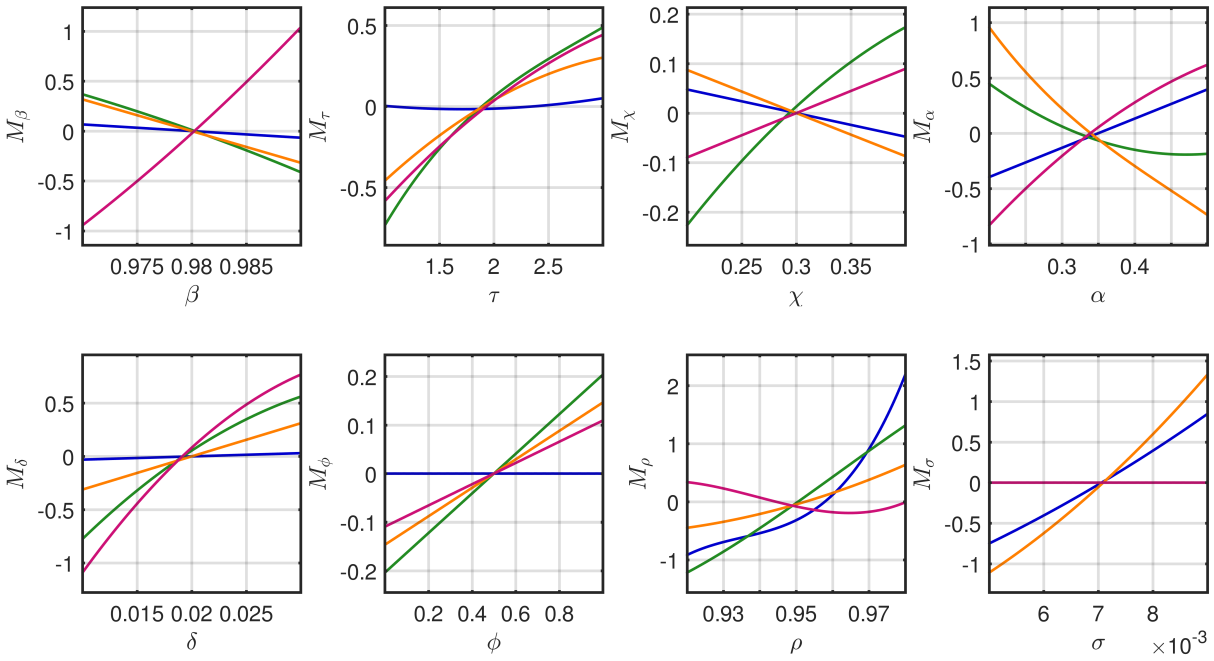


Figure 11: Univariate effects

A suitable basis for Eq. (3) is given by orthonormal polynomials with respect to a weight function that corresponds to the PDF of the input random variables. In the sequel we assume that the input variables are statistically *independent*, so that the joint PDF is the product of the M marginal distributions: $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \prod_{i=1}^{M} f_{\theta_i}(\theta_i)$, where f_{θ_i}'s are the marginal distributions of each variable $\{\theta_i,\, i = 1, \ldots, M\}$ defined on $\mathcal{D}_{\Theta_i}$. For each variable Θ_i and any two functions $\phi_1, \phi_2 : \theta \in \mathcal{D}_{\Theta_i} \mapsto \mathbb{R}$, we define the functional inner product by the following integral (provided it exists):

$$\langle \phi_1, \phi_2 \rangle_i \stackrel{\text{def}}{=} \mathbb{E}\left[\phi_1(\Theta_i)\, \phi_2(\Theta_i)\right] = \int_{\mathcal{D}_{\Theta_i}} \phi_1(\theta)\, \phi_2(\theta)\, f_{\Theta_i}(\theta)\, d\theta. \tag{44}$$

where $\mathbb{E}\left[\cdot\right]$ is the expectation operator. Using the above notation, classical algebra allows one to build a family of *orthogonal polynomials* $\{P_k^{(i)},\, k \in \mathbb{N}\}$ satisfying

$$\left\langle P_j^{(i)}, P_k^{(i)} \right\rangle_i \stackrel{\text{def}}{=} \mathbb{E}\left[P_j^{(i)}(\Theta_i)\, P_k^{(i)}(\Theta_i)\right] = a_j^{(i)}\, \delta_{jk}, \tag{45}$$

see, e.g., Abramowitz and Stegun (1970). In the above equation subscript k denotes the degree of the polynomial $P_k^{(i)}$, δ_{jk} is the Kronecker symbol equal to 1 when $j = k$ and 0 otherwise and $a_j^{(i)} \stackrel{\text{def}}{=} \| P_j^{(i)} \|_i^2 = \left\langle P_j^{(i)}, P_j^{(i)} \right\rangle_i$ corresponds to the squared norm of $P_j^{(i)}$. For standard distributions, the associated families of orthogonal polynomials are well-known.

For instance, if $\Theta_i \sim \mathcal{U}(-1,1)$ has a uniform distribution over $[-1,1]$, the resulting family is that of the so-called *Legendre polynomials* (Xiu and Karniadakis (2002)). The obtained polynomials may be normalized as follows:

$$\psi_j^{(i)} = P_j^{(i)} / \sqrt{a_j^{(i)}} \quad i = 1, \ldots, d, \quad j \in \mathbb{N}. \tag{46}$$

From the sets of univariate orthonormal polynomials one can now build *multivariate* orthonormal polynomials by *tensor product*. For this purpose, let us define the multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^M$, which are ordered lists of natural integers $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_M)$, $\alpha_i \in \mathbb{N}$. One can associate a multivariate polynomial $\Psi_{\boldsymbol{\alpha}}$ to any multi-index $\boldsymbol{\alpha}$ by

$$\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \prod_{i=1}^{M} \psi_{\alpha_i}^{(i)}(\theta_i), \tag{47}$$

where the univariate polynomials $\left\{\psi_k^{(i)},\, k \in \mathbb{N}\right\}$ are defined in Eq. (46). Due to Eq. (45) and the above tensor product construction, the multivariate polynomials in the input vector $\boldsymbol{\Theta}$ are also orthonormal, i.e.,

$$\mathbb{E}\left[\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\Theta})\, \Psi_{\beta}(\boldsymbol{\Theta})\right] \stackrel{\text{def}}{=} \int_{\mathcal{D}_{\boldsymbol{\Theta}}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \Psi_{\beta}(\boldsymbol{\theta})\, f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta} = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} \qquad \forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^M, \tag{48}$$

where $\delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ is the Kronecker symbol which is equal to 1 if $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and zero otherwise. With this notation, it can be proven that the set of all multivariate polynomials in the input random vector $\boldsymbol{\Theta}$ forms a basis of the Hilbert space in which $Y = \mathcal{M}(\boldsymbol{\Theta})$ is represented (Soize and Ghanem (2004)):

$$Y = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^M} b_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\Theta}). \tag{49}$$

The representation of the random response in Eq. (49) is exact when the infinite series is considered. However, in practice, only a finite number of terms can be computed. For this purpose a *truncation scheme* $\mathcal{A}$ has to be selected. Since the basis consists of multivariate polynomials, it is natural to consider all the polynomials up to a given maximum degree. Let us define the *total degree* of a multivariate polynomial $\Psi_{\boldsymbol{\alpha}}$ by $|\boldsymbol{\alpha}| \stackrel{\text{def}}{=} \sum_{i=1}^{M} \alpha_i$. The *standard truncation scheme* consists in selecting all polynomials such that the total degree $|\boldsymbol{\alpha}|$ is smaller than or equal to a given p. The maximal polynomial degree p may typically be equal to $3-10$ in practical applications. Note that the cardinality

of the truncation set $\mathcal{A}^{M,p} = \left\{\boldsymbol{\alpha} \in \mathbb{N}^M \ : \ |\boldsymbol{\alpha}| \leq p\right\}$ increases polynomially with M and p, since card $\mathcal{A}^{M,p} = \binom{M+p}{p} = \frac{(M+p)!}{M!\,p!}$. Thus the number of coefficients to be computed increases dramatically when M is large, say $M > 10$. This complexity is referred to as the *curse of dimensionality*. This issue is however solved satisfactorily using specific algorithms to compute sparse PCE, which we discuss in Section 2.2.2.

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