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Thomas Chambrion, Laurent Thomann

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THOMAS CHAMBRION AND LAURENT THOMANN

Abstract. In this paper we study the bilinear-control problem for the linear and non-linear Schrödinger equation with harmonic potential. By the means of different examples, we show how space-time smoothing effects (Strichartz estimates, Kato smoothing effect) enjoyed by the linear flow, can help to prove obstructions to controllability.

1. Introduction and results

1.1. **Introduction.** In this paper, for $d \geq 1$, we consider the bi-linear control problem for the quantum harmonic oscillator

$$\begin{cases} i\partial_t \psi + H\psi = u(t)K(x)\psi - \sigma|\psi|^2\psi, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.1}$$

where

$$H = -\Delta + |x|^2 = \sum_{j=1}^{d} \big(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \big)$$

is the harmonic oscillator, $K : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a given potential and where the control u belongs to $L^r_{loc}(\mathbb{R};\mathbb{R})$ for some $r \geq 1$. In the sequel, we will either study the case $\sigma = 0$ and we will refer to this equation as the bi-linear Schrödinger equation, or the case $\sigma = 1$ (respectively $\sigma = -1$) which corresponds to the non-linear Schrödinger equation with a cubic defocusing (respectively focusing) non-linearity. We call the linear operator $\psi \mapsto K\psi$ the *control operator*, while the (possibly non-linear) map $\psi \longmapsto iH\psi + i\sigma|\psi|^2\psi$ is usually called the *drift*.

The controllability question, crucial for applications, amounts to find which solutions ψ of (1.1) can be obtained with a suitable choice of u. For a given source ψ_0, the *attainable set* from ψ_0 with controls in $L^r_{loc}(\mathbb{R};\mathbb{R})$ is the set of ψ_f for which there exist a time $T \geq 0$ and a control u in $L^r([0,T];\mathbb{R})$ such that the solution ψ of (1.1) at time T satisfies $\psi(T,\cdot) = \psi_f(\cdot)$. A system is *controllable* in a given space X if the attainable set from any point of X contains X.

A celebrated result [1, Theorem 3.6] (see also [21] for the case of the Schrödinger equation) states that for bi-linear equations posed in a Banach space with linear drift and bounded control operator, the attainable set (from any source) with $L^r_{loc}(\mathbb{R},\mathbb{R})$ controls, $r > 1$, is contained in a countable union of compact sets. In an infinite dimensional Banach space, a countable union of compact sets is meager in Baire sense. Hence, this result represents a deep topological obstruction to controllability of bi-linear control systems. Notice that this negative result does not prohibit controllability in smaller spaces, endowed with stronger norms, where the control operator is not continuous anymore.

In [10], using energy estimates, we have proved obstructions to controllability for non-linear wave equations with L^1_{loc} controls and bounded control operators. Here using more refined inequalities,

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as the Kato smoothing effect and Strichartz estimates, we are able to tackle the case of the (non-linear) Schrödinger equation involving possibly non-linear drifts and unbounded control operators.

Concerning the study of the well-posedness of Schrödinger equations with potentials, we refer to [11, 17, 8].

For (local) exact controllability results for NLS on a finite length interval we refer to [2, 4, 3]. For both the case of the bi-linear and non-linear Schrödinger equations, to get positive exact controllability results, the main difficulty is the choice of the ambient space. This space has to be chosen such that the equation is well-posed *and* the control operator is not bounded. In [2, 4, 3] the fact that the control operator is not continuous is a consequence that the Schrödinger equation is studied on a *finite length* interval with well chosen boundary conditions. Here instead, we study the equation on $\mathbb{R}^d$ and therefore take advantage of dispersive effects.

For approximate controllability results for the bi-linear Schrödinger equation see [9, 15].

On the other hand, in the particular case $K(x) = x$ (which does not fall in the scope of our analysis), with an explicit change of variable, one can show that the atteignable set is a finite dimensional manifold [16]. Notice that this result also holds for the non-linear equation, see [13].

We refer to the introduction of [4] for more references on control problems and concerning results on the optimal control problem of the non-linear Schrödinger equation, see [12].

For an overview of results concerning the control of (1.1), see [13]. For an overview of controllability results of bi-linear control systems, we refer to [14].

In the sequel, we will need the harmonic Sobolev spaces, in other words, the Sobolev spaces based on the domain of the harmonic oscillator. For $s \geq 0$, $p \geq 1$ we define

$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbb{R}^d) = \big\{ f \in L^p(\mathbb{R}^d),\ H^{s/2} f \in L^p(\mathbb{R}^d) \big\},$$

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^d) = \mathcal{W}^{s,2}.$$

The natural norms are denoted by $\|f\|_{\mathcal{W}^{s,p}}$ and up to equivalence of norms (see *e.g.* [22, Lemma 2.4]), for $1 < p < +\infty$, we have

$$\|f\|_{\mathcal{W}^{s,p}} = \|H^{s/2} f\|_{L^p} \equiv \|(-\Delta)^{s/2} f\|_{L^p} + \|\langle x \rangle^s f\|_{L^p}, \tag{1.2}$$

with the notation $\langle x \rangle = (1 + |x|^2)^{1/2}$.

1.2. **A smoothing property for the bi-linear equation.** Consider the equation

$$\begin{cases} i\partial_t \psi + H\psi = u(t) K(x) \psi, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \psi(0,x) = \psi_0(x) \in \mathcal{H}^k(\mathbb{R}^d), \end{cases} \tag{1.3}$$

in any dimension $d \geq 1$ and regularity $k \geq 0$. Assume that $K \in \mathcal{W}^{k,\infty}(\mathbb{R}^d)$. Then for all integer $k \geq 0$, the control operator

$$\begin{array}{rcl} \mathcal{H}^k(\mathbb{R}^d) & \longrightarrow & \mathcal{H}^k(\mathbb{R}^d) \\ \psi & \longmapsto & K\psi, \end{array} \tag{1.4}$$

is continuous (see (2.11) for the proof), and therefore the general result of Ball-Marsden-Slemrod [1, Theorem 3.6] applies to (1.3). This result shows that, for fixed initial condition $\psi_0 \in \mathcal{H}^k(\mathbb{R}^d)$, the atteignable set of (1.3)

$$\bigcup_{t \in \mathbb{R}} \bigcup_{\substack{u \in L^r_{loc}(\mathbb{R}), \\ r > 1}} \big\{ \psi(t) \big\},$$

is a countable union of compact subsets of $\mathcal{H}^k(\mathbb{R}^d)$.

Our next results (Theorem 1.1 and Corollary 1.2) give a more precise description of the atteignable set of (1.3), under the assumption $u \in L^2_{loc}(\mathbb{R})$.

Theorem 1.1. *Let $d \geq 1$ and $k \geq 0$ be an even integer. Let $u \in L^2_{loc}(\mathbb{R})$ and $K \in \mathcal{W}^{k+1,\infty}(\mathbb{R}^d)$. Let $\psi_0 \in \mathcal{H}^k(\mathbb{R}^d)$, then the equation (1.3) admits a unique global solution $\psi \in \mathcal{C}(\mathbb{R}; \mathcal{H}^k(\mathbb{R}^d))$.*

Moreover for all $\sigma < 1/2$, there exists $\alpha > 0$ such that

$$\psi(t) - e^{itH}\psi_0 \in \mathcal{C}^\alpha\big(\mathbb{R}; \mathcal{H}^{k+\sigma}(\mathbb{R}^d)\big), \tag{1.5}$$

and for all $T > 0$,

$$\|\psi(t) - e^{itH}\psi_0\|_{\mathcal{C}^\alpha([-T,T];\mathcal{H}^{k+\sigma}(\mathbb{R}^d))} \leq C(T, k, \|\psi_0\|_{\mathcal{H}^k(\mathbb{R}^d)}, \|u\|_{L^2([-T,T])}). \tag{1.6}$$

The proof of (1.6) relies on the Kato smoothing effect for the linear Schrödinger equation. It can be stated like this: for all $\sigma < 1/2$ there exists $C > 0$ such that for all $\varphi \in L^2(\mathbb{R}^d)$

$$\Big\| \frac{1}{\langle x \rangle^{\frac{1}{2}}} H^{\frac{\sigma}{2}} e^{itH}\varphi \Big\|_{L^2([-2\pi,2\pi]\times\mathbb{R}^d)} \leq C\|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{1.7}$$

We refer to [19, Théorème 15] for the proof of (1.7). This inequality shows that the solution of the linear Schrödinger flow enjoys a gain of $1/2$ derivative locally in space.

It is likely that the statement of Theorem 1.1 holds for any $k \in \mathbb{N}$, but at the price of more technicalities, therefore we only gave the proof for $k \in 2\mathbb{N}$, which allows to work with differential operators instead of pseudo-differential operators.

The result also holds for perturbations of H, namely, when H is replaced with $H + W$, where W is in the Schwartz class $\mathcal{S}(\mathbb{R}^d; \mathbb{R})$. In the argument one has to replace uK with $uK - W$.

The smoothing property stated in Theorem 1.1 leads to the following obstruction to controllability of equation (1.3).

Corollary 1.2. *Under the assumptions of Theorem 1.1, for all $\sigma < 1/2$, $T > 0$, and $K > 0$, the set*

$$\bigcup_{\substack{t\in[-T,T] \\ \|u\|_{L^2([-T,T];\mathbb{R})}\leq K}} \big\{\psi(t) - e^{itH}\psi_0\big\}$$

is a compact of $\mathcal{H}^{k+\sigma}(\mathbb{R}^d)$.

As a consequence, the set

$$\bigcup_{t\in\mathbb{R}} \bigcup_{u\in L^2_{loc}(\mathbb{R})} \big\{\psi(t) - e^{itH}\psi_0\big\}$$

is a countable union of compact subsets of $\mathcal{H}^{k+\sigma}(\mathbb{R}^d)$.

1.3. **Strichartz estimates and obstructions to the controllability of the non-linear equation.** The Strichartz estimates are crucial tools in the study of the well-posedness of non-linear Schrödinger equation at low regularity. Let us recall them: a couple $(q, r) \in [2, +\infty]^2$ is called admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \text{and} \quad (d, q, r) \neq (2, 2, +\infty),$$

and if one defines

$$X^s_T := \bigcap_{(q,r)\ admissible} L^q\big([-T,T]; \mathcal{W}^{s,r}(\mathbb{R}^d)\big), \tag{1.8}$$

then for all $T > 0$ there exists $C_T > 0$ so that for all $\psi_0 \in \mathcal{H}^s(\mathbb{R}^d)$ we have

$$\|e^{itH}\psi_0\|_{X^s_T} \leq C_T\|\psi_0\|_{\mathcal{H}^s(\mathbb{R}^d)}. \tag{1.9}$$

We will also need the inhomogeneous version of Strichartz: for all $T>0$, there exists $C_T>0$ so that for all admissible couple (q,r) and function $F\in L^{q'}([T,T];\mathcal{W}^{s,r'}(\mathbb{R}^d))$,

$$\Big\| \int_0^t e^{i(t-\tau)H}F(\tau)d\tau\Big\|_{X_T^s} \leq C_T\|F\|_{L^{q'}([-T,T],\mathcal{W}^{s,r'}(\mathbb{R}^d))}, \tag{1.10}$$

where q' and r' are the Hölder conjugate of q and r. We refer to [19] for a proof.

1.3.1. *The Schrödinger equation in dimension $d=1$.* To begin with, we consider the bi-linear Schrödinger equation

$$\begin{cases} i\partial_t\psi + H\psi = u(t)K(x)\psi, & (t,x)\in\mathbb{R}\times\mathbb{R},\\ \psi(0,x)=\psi_0(x)\in\mathcal{H}^s(\mathbb{R}), \end{cases} \tag{1.11}$$

where $K\in\mathcal{H}^s(\mathbb{R})$, for some $s\geq 0$. Then we are able to prove

Theorem 1.3. *(i) Let $K\in L^2(\mathbb{R})$, $u\in L^2_{loc}(\mathbb{R})$, and $\psi_0\in L^2(\mathbb{R})$. There exists a unique global solution to equation (1.11) in the class*

$$\psi\in\mathcal{C}\big(\mathbb{R};L^2(\mathbb{R})\big)\cap L^4_{loc}\big(\mathbb{R};L^\infty(\mathbb{R})\big).$$

This solution satisfies

$$\|\psi(t)\|_{L^2(\mathbb{R})}=\|\psi_0\|_{L^2(\mathbb{R})},\quad \forall\, t\in\mathbb{R},$$

and for all $T>0$

$$\|\psi\|_{L^4([-T,T];L^\infty(\mathbb{R}))}\leq C\big(T,\|\psi_0\|_{L^2(\mathbb{R})},\|u\|_{L^2([-T,T])}\big). \tag{1.12}$$

Moreover, the atteignable set

$$\bigcup_{t\in\mathbb{R}}\bigcup_{u\in L^2_{loc}(\mathbb{R})}\big\{\psi(t)\big\}$$

is a countable union of compact subsets of $L^2(\mathbb{R})$.

(ii) More generally, let $s\geq 0$, $K\in\mathcal{H}^s(\mathbb{R})$, $u\in L^2_{loc}(\mathbb{R})$ and $\psi_0\in\mathcal{H}^s(\mathbb{R})$. Then there exists a unique global solution to equation (1.11) in the class

$$\psi\in\mathcal{C}\big(\mathbb{R};\mathcal{H}^s(\mathbb{R})\big)\cap L^4_{loc}\big(\mathbb{R};\mathcal{W}^{s,\infty}(\mathbb{R})\big).$$

This solution satisfies

$$\|\psi(t)\|_{L^2(\mathbb{R})}=\|\psi_0\|_{L^2(\mathbb{R})},\quad \forall\, t\in\mathbb{R},$$

and for all $T>0$

$$\|\psi\|_{L^\infty([-T,T];\mathcal{H}^s(\mathbb{R}))}+\|\psi\|_{L^4([-T,T];\mathcal{W}^{s,\infty}(\mathbb{R}))}\leq C\big(T,\|\psi_0\|_{\mathcal{H}^s(\mathbb{R})},\|u\|_{L^2([-T,T])}\big).$$

Moreover, the atteignable set and the atteignable set

$$\bigcup_{t\in\mathbb{R}}\bigcup_{u\in L^2_{loc}(\mathbb{R})}\big\{\psi(t)\big\}$$

is a countable union of compact subsets of $\mathcal{H}^s(\mathbb{R})$.

This result shows that it is not the continuity of the control operator

$$\begin{array}{rcl} \mathcal{H}^s(\mathbb{R}) & \longrightarrow & \mathcal{H}^s(\mathbb{R})\\ \psi & \longmapsto & K\psi, \end{array} \tag{1.13}$$

which matters to get non-controllability results (since the map (1.13) is not continuous in general for a given $K\in\mathcal{H}^s(\mathbb{R})$ when $0\leq s\leq 1/2$). In the proof, we will crucially use the space-time

Strichartz estimates to control $K\psi$ $\big($by showing that $K\psi \in L^2_{loc}\big(\mathbb{R};\mathcal{H}^s(\mathbb{R})\big)$ when $\psi_0 \in \mathcal{H}^s(\mathbb{R})$ and $K \in \mathcal{H}^s(\mathbb{R})\big)$ and to prove the compactness result.

Notice that for $s > 1/2$, the result of Theorem 1.3 is a direct consequence of [1, Theorem 3.6], because in this case, the map (1.13) is continuous (see the discussion at the beginning of Section 1.2).

The previous approach also holds for the non-linear problem. Namely, consider the cubic equation

$$\begin{cases} i\partial_t\psi + H\psi = u(t)K(x)\psi - \sigma|\psi|^2\psi, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ \psi(0,x) = \psi_0(x) \in \mathcal{H}^s(\mathbb{R}), \end{cases} \tag{1.14}$$

where $\sigma = \pm 1$ and $K \in \mathcal{H}^s(\mathbb{R})$ for some $s \geq 0$. Then we have

Theorem 1.4. *Let $s \geq 0$, $K \in \mathcal{H}^s(\mathbb{R})$, $u \in L^2_{loc}(\mathbb{R})$ and $\psi_0 \in \mathcal{H}^s(\mathbb{R})$. Then there exists a unique global solution to equation* (1.14) *in the class*

$$\psi \in \mathcal{C}\big(\mathbb{R};\mathcal{H}^s(\mathbb{R})\big) \cap L^4_{loc}\big(\mathbb{R};\mathcal{W}^{s,\infty}(\mathbb{R})\big).$$

This solution satisfies

$$\|\psi(t)\|_{L^2(\mathbb{R})} = \|\psi_0\|_{L^2(\mathbb{R})}, \quad \forall\, t \in \mathbb{R},$$

and for all $T > 0$

$$\|\psi\|_{L^\infty([-T,T];\mathcal{H}^s(\mathbb{R}))} + \|\psi\|_{L^4([-T,T];\mathcal{W}^{s,\infty}(\mathbb{R}))} \leq C\big(T, \|\psi_0\|_{\mathcal{H}^s(\mathbb{R})}, \|u\|_{L^2([-T,T])}\big). \tag{1.15}$$

Moreover, the atteignable set

$$\bigcup_{t\in\mathbb{R}} \bigcup_{u\in L^2_{loc}(\mathbb{R})} \big\{\psi(t)\big\}$$

is a countable union of compact subsets of $\mathcal{H}^s(\mathbb{R})$.

This result is relevant in the sense that it shows that the non-linear term does not help to control the equation.

All the results of this section also hold for perturbations of H, namely, when H is replaced with $H+W$, where W is in the Schwartz class $\mathcal{S}(\mathbb{R}^d;\mathbb{R})$. The term $W\psi$ can be treated as a perturbation of the non-linear term.

1.3.2. *The non-linear Schrödinger equation in dimension $d = 3$.* In order to get similar results to Theorem 1.4 in higher dimension, one needs to impose more regularity on the initial condition and more regularity on the potential. This in turn will allow us to consider a larger set of controls, namely $u \in \cup_{r>1} L^r_{loc}(\mathbb{R})$ instead of $u \in L^2_{loc}(\mathbb{R})$, as assumed in Theorem 1.4.

In this paragraph, we fix $d = 3$ and we study the defocusing non-linear problem

$$\begin{cases} i\partial_t\psi + H\psi = u(t)K(x)\psi - |\psi|^2\psi, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ \psi(0,x) = \psi_0(x) \in \mathcal{H}^1(\mathbb{R}^3). \end{cases} \tag{1.16}$$

To begin with, thanks to (1.9) and (1.10) we are able to state a global well-posedness result adapted to our control problem.

Proposition 1.5. *Let $u \in L^1_{loc}(\mathbb{R})$.*

(*i*) *Let $K \in \mathcal{W}^{1,\infty}(\mathbb{R}^3)$. For $\psi_0 \in \mathcal{H}^1(\mathbb{R}^3)$ the equation* (1.16) *admits a unique global solution $\psi \in \mathcal{C}(\mathbb{R};\mathcal{H}^1(\mathbb{R}^3))$. This defines a global flow $\psi(t) = \Phi^u(t)(\psi_0)$.*

(*ii*) *Moreover, this solution ψ satisfies the bound*

$$\|\psi\|_{L^\infty([-T,T];\mathcal{H}^1(\mathbb{R}^3))} \leq C(\|\psi_0\|_{\mathcal{H}^1(\mathbb{R}^3)})(1+\int_{-T}^{T}|u(s)|ds), \tag{1.17}$$

for some $C = C(\|\psi_0\|_{\mathcal{H}^1(\mathbb{R}^3)})$. Furthermore, the following bound holds true

$$\|\psi\|_{L^2([-T,T];\mathcal{W}^{1,6}(\mathbb{R}^3))} \leq C\big(T, \|\psi_0\|_{\mathcal{H}^1(\mathbb{R}^3)}, \int_{-T}^{T}|u(s)|ds\big). \tag{1.18}$$

(*iii*) *Let $k \geq 1$ be an integer and assume that $K \in \mathcal{W}^{k,\infty}(\mathbb{R}^3)$. Then for $\psi_0 \in \mathcal{H}^k(\mathbb{R}^3)$ the equation* (1.16) *admits a unique global solution $\psi \in \mathcal{C}(\mathbb{R};\mathcal{H}^k(\mathbb{R}^3))$ which satisfies the bounds*

$$\|\psi\|_{L^\infty([-T,T];\mathcal{H}^k(\mathbb{R}^3))} \leq C(T,k,\|\psi_0\|_{\mathcal{H}^k(\mathbb{R}^3)}, \int_{-T}^{T}|u(s)|ds), \tag{1.19}$$

and

$$\|\psi\|_{L^2([-T,T];\mathcal{W}^{k,6}(\mathbb{R}^3))} \leq C\big(T, \|\psi_0\|_{\mathcal{H}^k(\mathbb{R}^3)}, \int_{-T}^{T}|u(s)|ds\big). \tag{1.20}$$

The proof relies on a fixed point argument in Strichartz spaces which are well-adapted to control the non-linear term in (1.16). Notice that from (1.18), we deduce that, for almost all $t \in \mathbb{R}$,

$$\psi(t) \in \mathcal{W}^{1,6}(\mathbb{R}^3). \tag{1.21}$$

This is a smoothing effect for the solution, but can not be interpreted as an obstruction to controllability of the equation (1.16), since the set of times such that (1.21) holds true depends on the control u.

We now state our result concerning the lack of controllability of (1.16)

Theorem 1.6. *Let $K \in \mathcal{W}^{1,\infty}(\mathbb{R}^3)$ and $\psi_0 \in \mathcal{H}^1(\mathbb{R}^3)$. Denote by ψ the solution of equation* (1.16) *defined in Proposition 1.5. Then the attainable set*

$$\bigcup_{t\in\mathbb{R}} \bigcup_{\substack{u\in L^r_{loc}(\mathbb{R}),\\ r>1}} \{\psi(t)\}$$

is a countable union of compact subsets of $\mathcal{H}^1(\mathbb{R}^3)$.

We are able to prove similar results in dimensions $d = 1$ and $d = 2$, but we do not detail them, since the proofs are similar. The same result also holds for the bi-linear Schrödinger equation, but it is not relevant to state it here, since it is a direct application of [1, Theorem 3.6] (see the discussion at the beginning of Section 1.2).

Again, the results of this section also hold for perturbations of H, namely, when H is replaced with $H + W$, where W is in the Schwartz class $\mathcal{S}(\mathbb{R}^d;\mathbb{R})$. The term $W\psi$ can be treated as a perturbation of the non-linear term, and the corresponding energy functional is still coercive, which is needed in our argument.

Remark 1.7. Let $k \geq 1$ be an integer. As a consequence of Proposition 1.5 (*iii*) we may similarly prove that for $K \in \mathcal{W}^{k,\infty}(\mathbb{R}^3)$ and $\psi_0 \in \mathcal{H}^k(\mathbb{R}^3)$, the attainable set

$$\bigcup_{t\in\mathbb{R}} \bigcup_{\substack{u\in L^r_{loc}(\mathbb{R}),\\ r>1}} \{\psi(t)\}$$

is a countable union of compact subsets of $\mathcal{H}^k(\mathbb{R}^3)$.

Remark 1.8. It is worth noticing that the different results developed in this paper (excepted Corollary 1.2) also hold for the Schrödinger equation, in the case where H is replaced with $\Delta = \sum_{j=1}^{d} \partial_{x_j}^2$, in other words for equations of the form

$$i\partial_t \psi + \Delta\psi = u(t)K(x)\psi + \sigma|\psi|^2\psi. \tag{1.22}$$

In the argument, it is enough to observe that the inequalities (1.7), (1.9) and (1.10) hold true for the operator Δ (instead of H) and the usual Sobolev spaces $H^s(\mathbb{R}^d)$, $W^{s,p}(\mathbb{R}^d)$ (instead of $\mathcal{H}^s(\mathbb{R}^d)$, $\mathcal{W}^{s,p}(\mathbb{R}^d)$). In this setting, the conclusion of Corollary 1.2 is that the atteignable set is meagre in the sense of Baire (the compactness is lost because the embedding $H^{s_2}(\mathbb{R}^d) \subset H^{s_1}(\mathbb{R}^d)$ is not compact, $s_1 < s_2$).

One should be able to adapt the approach developed in [13, Section 2.2] (in particular [13, Lemma 1]) to the equation (1.22). However, the argument of [13, Section 2.2] does not apply to (1.16), because it heavily relies on the space translation invariance of the problem.

1.4. **Notations.** In this paper $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded. For $x \in \mathbb{R}^d$, we write $\langle x\rangle = (1+|x|^2)^{1/2}$. We will sometimes use the notations $L^p_T = L^p([0,T])$ and $L^p_T X = L^p([0,T];X)$ for $T > 0$.

2. Proof of the results concerning the bi-linear equation and the Kato smoothing effect

2.1. Proof of Theorem 1.1.

To begin with, we observe that it is enough to work with non-negative times, by reversibility of the Schrödinger equation. Therefore in the sequel we assume $t \geq 0$.

Local existence: We consider the map

$$\Phi(\psi)(t) = e^{itH}\psi_0 - i\int_0^t u(s)e^{i(t-s)H}(K\psi)ds, \tag{2.1}$$

and we will show that it is a contraction in the space

$$B_{k,T,R} = \big\{\|\psi\|_{L^\infty_T\mathcal{H}^k} \leq R\big\},$$

with $R > 0$ and $T > 0$ to be fixed. From the fact that e^{itH} is unitary in $\mathcal{H}^k$ and thanks to the Leibniz rule we deduce that

$$\begin{aligned}\|\Phi(\psi)(t)\|_{\mathcal{H}^k} &\leq \|\psi_0\|_{\mathcal{H}^k} + \int_0^t |u(s)|\big\|K\psi(s)\big\|_{\mathcal{H}^k}ds \\ &\leq \|\psi_0\|_{\mathcal{H}^k} + c\big\|K\big\|_{\mathcal{W}^{k,\infty}}\int_0^t |u(s)|\big\|\psi(s)\big\|_{\mathcal{H}^k}ds. \end{aligned}\tag{2.2}$$

Therefore we have

$$\|\Phi(\psi)\|_{L^\infty_T\mathcal{H}^k} \leq \|\psi_0\|_{\mathcal{H}^k} + c\Big(\int_0^T |u(s)|ds\Big)\|K\|_{\mathcal{W}^{k,\infty}}\|\psi\|_{L^\infty_T\mathcal{H}^k}.$$

We now choose $R = 2\|\psi_0\|_{\mathcal{H}^k}$ and we fix $T > 0$ such that $c\int_0^T |u(s)|ds \leq \|K\|^{-1}_{\mathcal{W}^{k,\infty}}/2$. As a consequence, Φ maps $B_{k,T,R}$ into itself. With similar estimates we can show that Φ is a contraction in $B_{k,T,R}$, namely

$$\|\Phi(\psi_1) - \Phi(\psi_2)\|_{L^\infty_T\mathcal{H}^k} \leq c\Big(\int_0^T |u(s)|ds\Big)\|K\|_{\mathcal{W}^{k,\infty}}\|\psi_1 - \psi_2\|_{L^\infty_T\mathcal{H}^k}.$$

Global existence: Assume that $T^\star > 0$ is the maximal time of existence of the problem (1.3). From the bound (2.2), with $\Phi(\psi) = \psi$ we deduce

$$\|\psi(t)\|_{\mathcal{H}^k} \leq \|\psi_0\|_{\mathcal{H}^k} + c\big\|K\big\|_{\mathcal{W}^{k,\infty}} \int_0^t |u(s)| \big\|\psi(s)\big\|_{\mathcal{H}^k} ds.$$

Therefore, by the Grönwall lemma, we get that for all $t \leq T^\star$

$$\|\psi(t)\|_{\mathcal{H}^k} \leq \|\psi_0\|_{\mathcal{H}^k} e^{c\|K\|_{\mathcal{W}^{k,\infty}} \int_0^t |u(s)| ds}.$$

The previous bound combined with the local existence theory implies that $T^\star = +\infty$. There exists a unique global solution $\psi \in \mathcal{C}(\mathbb{R}; \mathcal{H}^k(\mathbb{R}^d))$ to (1.3).

Proof of the smoothing effect: In order to prove (1.5), we use the Kato smoothing effect (1.7). Let ψ be the solution to (1.3) and set $\psi_1(t) = \psi(t) - e^{itH}\psi_0$. Then ψ_1 solves

$$\psi_1(t) = -i \int_0^t u(s) e^{i(t-s)H} (K e^{isH} \psi_0) ds - i \int_0^t u(s) e^{i(t-s)H} (K \psi_1(s)) ds.$$

Therefore

$$\begin{aligned} \|\psi_1\|_{L^\infty_T \mathcal{H}^{k+\sigma}} &\leq \|uKe^{itH}\psi_0\|_{L^1_T \mathcal{H}^{k+\sigma}} + \|uK\psi_1\|_{L^1_T \mathcal{H}^{k+\sigma}} \\ &\leq \|u\|_{L^2_T} \|Ke^{itH}\psi_0\|_{L^2_T \mathcal{H}^{k+\sigma}} + \|u\|_{L^2_T} \|K\psi_1\|_{L^2_T \mathcal{H}^{k+\sigma}}. \end{aligned} \tag{2.3}$$

- We write $k = 2j$ with $j \geq 1$. Firstly we show that

$$\|Ke^{itH}\psi_0\|_{L^2_T \mathcal{H}^{2j+\sigma}} \leq C_T \tag{2.4}$$

using the Leibniz rule.

$$\begin{aligned} \|Ke^{itH}\psi_0\|_{\mathcal{H}^{2j+\sigma}} &= \|H^j(Ke^{itH}\psi_0)\|_{\mathcal{H}^\sigma} \\ &\leq C \sum_{\substack{j_1,j_2,j_3 \in \mathbb{N}^d \\ |j_1|+|j_2|+|j_3|=2j \\ |j_3| \neq 2j}} \|x^{j_1} \partial^{j_2} K \partial^{j_3} (e^{itH}\psi_0)\|_{\mathcal{H}^\sigma} + \|KH^j(e^{itH}\psi_0)\|_{\mathcal{H}^\sigma}, \end{aligned} \tag{2.5}$$

where ∂^ℓ stands for derivatives in x of order $|\ell|$. Each term in the sum is bounded by

$$\begin{aligned} \|x^{j_1} \partial^{j_2} K \partial^{j_3} (e^{itH}\psi_0)\|_{\mathcal{H}^\sigma} &\leq \|x^{j_1} \partial^{j_2} K \partial^{j_3} (e^{itH}\psi_0)\|_{\mathcal{H}^1} \\ &\leq \|K\|_{\mathcal{W}^{|j_1|+|j_2|+1,\infty}} \|\psi_0\|_{\mathcal{H}^{j_3+1}} \\ &\leq \|K\|_{\mathcal{W}^{2j+1,\infty}} \|\psi_0\|_{\mathcal{H}^{2j}} \end{aligned}$$

thus

$$\|x^{j_1} \partial^{j_2} K \partial^{j_3} (e^{itH}\psi_0)\|_{L^2_T \mathcal{H}^\sigma} \leq C T^{1/2}. \tag{2.6}$$

To control the contribution of the second term in (2.5), we write

$$\begin{aligned} \|KH^j(e^{itH}\psi_0)\|_{\mathcal{H}^\sigma} &= \|Ke^{itH}(H^j\psi_0)\|_{\mathcal{H}^\sigma} \\ &\leq \big\|[H^{\sigma/2}, K] e^{itH}(H^j\psi_0)\big\|_{L^2} + \|KH^{\sigma/2} e^{itH}(H^j\psi_0)\|_{L^2}. \end{aligned} \tag{2.7}$$

We use the commutator estimate [18, Lemma 18] to get the bound

$$\big\|[H^{\sigma/2}, K] e^{itH}(H^j\psi_0)\big\|_{L^2} \leq C\|\psi_0\|_{\mathcal{H}^{2j}}. \tag{2.8}$$

By the smoothing effect (1.7),

$$\|KH^{\sigma/2} e^{itH}(H^j\psi_0)\|_{L^2_T L^2} \leq C_T, \tag{2.9}$$

hence by (2.7), (2.8) and (2.9)

$$\|KH^j(e^{itH}\psi_0)\|_{L^2_T \mathcal{H}^\sigma} \leq C_T. \tag{2.10}$$

Hence, from (2.6) and (2.10) we deduce (2.4). In the case $k = j = 0$, the estimate (2.4) is deduced from (2.7)–(2.10).

• We now show that $\|K\psi_1\|_{L^2_T\mathcal{H}^{k+\sigma}} \leq CT^{1/2}\|\psi_1\|_{L^\infty_T\mathcal{H}^{k+\sigma}}$. By the fractional Leibniz rule (A.1), we have, for all $p > 2$

$$\|K\psi_1\|_{\mathcal{H}^{k+\sigma}} \leq C\|K\|_{L^\infty}\|\psi_1\|_{\mathcal{H}^{k+\sigma}} + C\|K\|_{\mathcal{W}^{k+\sigma,p}}\|\psi_1\|_{L^q}, \tag{2.11}$$

with $q = 2p/(p-2)$. For $p \gg 2$ large enough (hence $q > 2$ small enough), by the Sobolev inequalities, $\|\psi_1\|_{L^q} \leq C\|\psi_1\|_{\mathcal{H}^{k+\sigma}}$ which controls the first term in (2.11). To treat the second, we claim that $\|K\|_{\mathcal{W}^{k+\sigma,p}} \leq C\|K\|_{\mathcal{W}^{k+1,\infty}}$, for $p \gg 2$ large enough. Actually, we observe that the decay $|K| \leq C\langle x\rangle^{-k-1}$ implies that $K \in L^r(\mathbb{R}^d)$ for $r \gg 2$ large enough. Then one uses the interpolation inequality

$$\|K\|_{\mathcal{W}^{(1-\theta)s,r/\theta}(\mathbb{R}^d)} \leq \|K\|^{1-\theta}_{\mathcal{W}^{s,\infty}(\mathbb{R}^d)}\|K\|^{\theta}_{L^r(\mathbb{R}^d)}, \quad 0 \leq \theta \leq 1,$$

with $s = k+1$, θ such that $(1-\theta)s = k+\sigma$ and $r = \theta p$.

As a conclusion, with (2.3) we infer

$$\|\psi_1\|_{L^\infty_T\mathcal{H}^{k+\sigma}} \leq C_T + CT^{1/2}\|u\|_{L^2_T}\|\psi_1\|_{L^\infty_T\mathcal{H}^{k+\sigma}},$$

which implies, for $T > 0$ small enough and which only depends on u and K, that $\|\psi_1\|_{L^\infty_T\mathcal{H}^{k+\sigma}} \leq 2C_T$. We are able to iterate this argument to obtain that

$$\psi_1 \in \mathcal{C}\big(\mathbb{R}; \mathcal{H}^{k+\sigma}(\mathbb{R}^d)\big), \tag{2.12}$$

with the bound

$$\|\psi_1\|_{L^\infty_T\mathcal{H}^{k+\sigma}} \leq C(T, k, \|\psi_0\|_{\mathcal{H}^k}, \|u\|_{L^2_T}). \tag{2.13}$$

Notice that the previous estimate implies

$$\psi_1 \in L^2\big([-T,T]; \mathcal{H}^{k+\sigma}(\mathbb{R}^d)\big). \tag{2.14}$$

Let us now show that for all $T > 0$, $\partial_t\psi \in L^2_T\mathcal{H}^{k-2}$, which in turn will imply that

$$\partial_t\psi_1 \in L^2\big([-T,T]; \mathcal{H}^{k-2}(\mathbb{R}^d)\big). \tag{2.15}$$

From the equation (1.3), we get for all $-T \leq t \leq T$

$$\|\partial_t\psi(t)\|_{\mathcal{H}^{k-2}} \leq \|\psi(t)\|_{\mathcal{H}^k} + |u(t)|\|K\|_{\mathcal{W}^{k+1,\infty}}\|\psi(t)\|_{\mathcal{H}^k},$$

thus

$$\|\partial_t\psi\|_{L^2_T\mathcal{H}^{k-2}} \leq CT^{1/2}\|\psi\|_{L^\infty_T\mathcal{H}^k} + \|u\|_{L^2_T}\|K\|_{\mathcal{W}^{k+1,\infty}}\|\psi\|_{L^\infty_T\mathcal{H}^k}, \tag{2.16}$$

hence the result.

By the interpolation Lemma A.4 in the appendix, applied to (2.14) and (2.15), there exist $\alpha > 0$ and $\kappa > 0$ such that

$$\psi_1 \in \mathcal{C}^\alpha\big([-T,T]; \mathcal{H}^{k+\sigma-\kappa}(\mathbb{R}^d)\big). \tag{2.17}$$

Finally we interpolate (2.12) and (2.17), and thus, for all $\sigma' < \sigma$, there exists $\alpha' > 0$ such that $\psi_1 \in \mathcal{C}^{\alpha'}\big([-T,T]; \mathcal{H}^{k+\sigma'}(\mathbb{R}^d)\big)$. The bound (1.6) follows from (2.13), (2.16), by the interpolation argument.

2.2. **Proof of Corollary 1.2.** Fix $\psi_0 \in \mathcal{H}^k(\mathbb{R}^d)$. Let $(u_n)_{n\geq 1}$ be such that $\|u_n\|_{L^2_T} \leq K$ and consider ψ_n the solution of (1.3) associated to u_n, and let $(t_n)_{n\geq 1} \subset [-T,T]$. Set $\Psi_n(t_n) = \psi_n(t_n) - e^{it_nH}\psi_0$. Up to a subsequence, we can assume that $t_n \to t$ for some $t \in [-T,T]$. Let $\sigma < \sigma' < 1/2$, then by (1.6),

$$\|\Psi_n\|_{\mathcal{C}^\alpha_T \mathcal{H}^{k+\sigma'}} \leq C.$$

By the compact embedding $\mathcal{C}^\alpha\big([-T,T];\mathcal{H}^{k+\sigma'}(\mathbb{R}^d)\big) \subset \mathcal{C}\big([-T,T];\mathcal{H}^{k+\sigma}(\mathbb{R}^d)\big)$, there exists $\Psi \in \mathcal{C}\big([-T,T];\mathcal{H}^{k+\sigma}(\mathbb{R}^d)\big)$ such that $\Psi_n \to \Psi$. Next

$$\begin{aligned}\|\Psi_n(t_n) - \Psi(t)\|_{\mathcal{H}^{k+\sigma}} &\leq \|\Psi_n(t_n) - \Psi(t_n)\|_{\mathcal{H}^{k+\sigma}} + \|\Psi(t_n) - \Psi(t)\|_{\mathcal{H}^{k+\sigma}} \\ &\leq \sup_{\tau\in[-T,T]} \|\Psi_n(\tau) - \Psi(\tau)\|_{\mathcal{H}^{k+\sigma}} + \|\Psi(t_n) - \Psi(t)\|_{\mathcal{H}^{k+\sigma}}.\end{aligned}$$

The first term in the previous line tends to 0 since $\Psi_n \to \Psi$, and the second as well, since $\Psi \in \mathcal{C}\big([-T,T];\mathcal{H}^{k+\sigma}(\mathbb{R}^d)\big)$.

Remark 2.1. A similar approach can be developed for the Klein-Gordon equation (even in the non-linear case)

$$\begin{cases}\partial_t^2\psi - \Delta\psi + m\psi = u(t)B(x)\psi - \psi^3, \quad (t,x) \in \mathbb{R}\times\mathcal{M}, \\ \psi(0,.) = \psi_0 \in H^1(\mathcal{M}), \\ \partial_t\psi(0,.) = \psi_1 \in L^2(\mathcal{M}),\end{cases} \tag{2.18}$$

where $\mathcal{M}$ is a boundaryless compact manifold of dimension 1 or 2, with $m \geq 0$ and where the potential B is assumed to be regular enough. Actually, the mild solution to (2.18) reads

$$\psi(t) = S_0(t)\psi_0 + S_1(t)\psi_1 + \int_0^t S_1(t-s)\big(u(s)B(x)\psi(s) - \psi^3(s)\big)ds$$

where

$$S_0(t) = \cos(t\sqrt{-\Delta+m}) \quad \text{and} \quad S_1(t) = \frac{\sin(t\sqrt{-\Delta+m})}{\sqrt{-\Delta+m}}.$$

In this context, the smoothing is realised by the gain of derivative induced by S_1. For non-controllability results for (2.18), we refer to [10, Section 3].

3. The Schrödinger equation in dimension $d = 1$

We prove Theorem 1.3 and Theorem 1.4 at the same time, namely we consider the equation

$$\begin{cases}i\partial_t\psi + H\psi = u(t)K(x)\psi - \sigma|\psi|^2\psi, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \\ \psi(0,x) = \psi_0(x) \in \mathcal{H}^s(\mathbb{R}),\end{cases} \tag{3.1}$$

with $\sigma = 0$ or $\sigma = 1$.

Local existence: Let $\psi_0 \in \mathcal{H}^s(\mathbb{R})$. We consider the map

$$\Phi(\psi)(t) = e^{itH}\psi_0 - i\int_0^t u(\tau)e^{i(t-\tau)H}(K\psi)d\tau + i\sigma\int_0^t e^{i(t-\tau)H}(|\psi|^2\psi)d\tau,$$

and we will show that it is a contraction in some Banach space. By a usual interpolation argument, one can prove that the Strichartz norm in X^s_T can be defined by $\|\psi\|_{X^s_T} = \|\psi\|_{L^\infty_T\mathcal{H}^s} + \|\psi\|_{L^4_T\mathcal{W}^{s,\infty}}$ (see (1.8)), and we define the space

$$B_{s,T,R} = \big\{\|\psi\|_{X^s_T} \leq R\big\},$$

with $R > 0$ and $T > 0$ to be fixed.

By the Strichartz estimates (1.9) and (1.10) we get

$$\begin{aligned}\|\Phi(\psi)\|_{X_T^s} &\leq c\|\psi_0\|_{\mathcal{H}^s} + c\int_0^T \big\||\psi|^2\psi\big\|_{\mathcal{H}^s} ds + c\int_0^T |u(\tau)|\big\|K\psi\big\|_{\mathcal{H}^s} d\tau \\ &\leq c\|\psi_0\|_{\mathcal{H}^s} + c\big\||\psi|^2\psi\big\|_{L_T^1\mathcal{H}^s} + c\|u\|_{L_T^2}\big\|K\psi\big\|_{L_T^2\mathcal{H}^s}.\end{aligned}$$

• By the generalised Leibniz rule (A.1),

$$\big\||\psi|^2\psi\big\|_{\mathcal{H}^s} \leq c\|\psi\|_{L^\infty}^2\|\psi\|_{\mathcal{H}^s},$$

thus by Hölder

$$\big\||\psi|^2\psi\big\|_{L_T^1\mathcal{H}^s} \leq c\|\psi\|_{L_T^2L^\infty}^2\|\psi\|_{L_T^\infty\mathcal{H}^s} \leq cT^{1/2}\|\psi\|_{L_T^4L^\infty}^2\|\psi\|_{X_T^s}. \tag{3.2}$$

Since $\|\psi\|_{L_T^4L^\infty} \leq \|\psi\|_{X_T^s} \leq R$, we get $\big\||\psi|^2\psi\big\|_{L_T^1\mathcal{H}^s} \leq cT^{1/2}R^3$.

• Let us now prove that there exists $\kappa > 0$ such that

$$\|K\psi\|_{L_T^2\mathcal{H}^s} \leq cT^\kappa\|K\|_{\mathcal{H}^s}\|\psi\|_{X_T^s}. \tag{3.3}$$

In the case $s = 0$ we simply write $\|K\psi\|_{L_T^2L^2} \leq \|K\|_{L^2}\|\psi\|_{L_T^2L^\infty} \leq T^{1/2}\|K\|_{L^2}\|\psi\|_{X_T^0}$. Assume now $s > 0$. By (A.1),

$$\|K\psi\|_{\mathcal{H}^s} \leq c\|K\|_{\mathcal{H}^s}\|\psi\|_{L^\infty} + c\|K\|_{L^p}\|\psi\|_{\mathcal{W}^{s,q}}$$

for all $2 < p, q < \infty$ such that $1/p + 1/q = 1/2$. Then, by Sobolev, if $p > 2$ is small enough, $\|K\|_{L^p} \leq c\|K\|_{\mathcal{H}^s}$. Finally using that $\|\psi\|_{L_T^2\mathcal{W}^{s,q}} \leq c\|\psi\|_{X_T^s}$, we obtain (3.3).

Putting the previous estimates together we have

$$\|\Phi(\psi)\|_{X_T^s} \leq c\|\psi_0\|_{\mathcal{H}^s} + cT^{1/2}R^3 + cT^\kappa\|u\|_{L_T^2}\|K\|_{\mathcal{H}^s}R.$$

We now choose $R = 2c\|\psi_0\|_{\mathcal{H}^s}$. Then for $T > 0$ small enough, Φ maps $B_{s,T,R}$ into itself. With similar estimates we can show that Φ is a contraction in $B_{s,T,R}$, namely

$$\|\Phi(\psi_1) - \Phi(\psi_2)\|_{X_T^s} \leq \big[cT^{1/2}R^2 + cT^\kappa\|u\|_{L_T^2}\|K\|_{\mathcal{H}^s}\big]\|\psi_1 - \psi_2\|_{X_T^s}.$$

As a conclusion there exists a unique fixed point to Φ, which is a local solution to (3.1).

Proof of the bound (1.15) *for* $s = 0$*:* Before we turn to the proof of the global existence, we prove this particular case of (1.15). Assume that one can solve (3.1) on $[0, T^\star)$, and let $T < T^\star$. Clearly, $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$ for all $0 \leq t \leq T$. Let $0 \leq t_0 < T$ and $\delta > 0$ such that $t_0 + \delta \leq T$. We have for all $0 \leq t \leq \delta$

$$\psi(t + t_0) = e^{itH}\psi(t_0) + i\int_{t_0}^{t_0+t} e^{i(t_0+t-\tau)H}(|\psi|^2\psi)d\tau - i\sigma\int_{t_0}^{t_0+t} u(\tau)e^{i(t_0+t-\tau)H}(K\psi)d\tau,$$

which implies, by the Strichartz estimates (1.9) and (1.10)

$$\begin{aligned}\|\psi\|_{L^4([t_0,t_0+\delta];L^\infty)} &\leq c\|\psi\|_{L_T^\infty L^2} + c\|\psi\|_{L_T^\infty L^2}^2\|\psi\|_{L^{4/3}([t_0,t_0+\delta];L^\infty)} + c\|K\|_{L^2}\|\psi\|_{L_T^\infty L^2}\|u\|_{L^{4/3}([t_0,t_0+\delta])} \\ &\leq c\|\psi\|_{L_T^\infty L^2} + c\delta^{2/3}\|\psi\|_{L_T^\infty L^2}^2\|\psi\|_{L^4([t_0,t_0+\delta];L^\infty)} + cT^{1/4}\|K\|_{L^2}\|\psi\|_{L_T^\infty L^2}\|u\|_{L_T^2} \\ &\leq c\|\psi_0\|_{L^2} + c\delta^{2/3}\|\psi_0\|_{L^2}^2\|\psi\|_{L^4([t_0,t_0+\delta];L^\infty)} + cT^{1/4}\|K\|_{L^2}\|\psi_0\|_{L^2}\|u\|_{L_T^2}.\end{aligned}$$

We pick $\delta = \delta(T) > 0$ such that $c\delta^{2/3}\|\psi_0\|_{L^2}^2 = 1/2$, thus the previous estimate gives

$$\|\psi\|_{L^4([t_0,t_0+\delta];L^\infty)} \leq 2c\|\psi_0\|_{L^2}(1 + \|u\|_{L_T^2}).$$

We write this estimate for $t_0 = 0, \delta, \ldots, j\delta$ with $j \in \mathbb{N}$ such that $j\delta < T < (j+1)\delta$. We sum up and we obtain

$$\|\psi\|_{L_T^4L^\infty} \leq C\big(T, \|\psi_0\|_{L^2}, \|u\|_{L_T^2}\big). \tag{3.4}$$

Global existence: Thanks to (3.2) and (3.4), the time of existence given in the local theory only depends on $\|\psi_0\|_{L^2}$ and $\|u\|_{L^2_T}$, thus the local argument can be iterated. As a conclusion, the problem (3.1) is globally well-posed and one has the bound

$$\|\psi\|_{L^\infty([-T,T];\mathcal{H}^s(\mathbb{R}))} + \|\psi\|_{L^4([-T,T];\mathcal{W}^{s,\infty}(\mathbb{R}))} \leq C\big(T, \|\psi_0\|_{\mathcal{H}^s(\mathbb{R})}, \|u\|_{L^2([-T,T])}\big).$$

The compactness argument: Let $u_n \rightharpoonup u$ weakly in $L^2([0,T];\mathbb{R})$. Notice in particular that $\|u_n\|_{L^2_T} \leq C(T)$ for some $C(T) > 0$. We have

$$\psi(t) = e^{itH}\psi_0 - i\int_0^t u(\tau)e^{i(t-\tau)H}(K\psi(\tau))d\tau + i\sigma\int_0^t e^{i(t-\tau)H}(|\psi|^2\psi)d\tau,$$

and

$$\psi_n(t) = e^{itH}\psi_0 - i\int_0^t u_n(\tau)e^{i(t-\tau)H}(K\psi_n(\tau))d\tau + i\sigma\int_0^t e^{i(t-\tau)H}(|\psi_n|^2\psi_n)d\tau.$$

We set $z_n = \psi - \psi_n$, then z_n satisfies

$$z_n = \mathcal{L}(\psi,\psi_n) + \mathcal{N}(\psi,\psi_n), \tag{3.5}$$

with

$$\mathcal{L}(\psi,\psi_n) = -i\int_0^t \big(u(\tau) - u_n(\tau)\big)e^{i(t-\tau)H}(K\psi)d\tau - i\int_0^t u_n(\tau)e^{i(t-\tau)H}\big(K(\psi-\psi_n)\big)d\tau$$

and

$$\mathcal{N}(\psi,\psi_n) = i\sigma\int_0^t e^{i(t-\tau)H}\big((\psi-\psi_n)(\psi+\psi_n)\overline{\psi}\big)d\tau + i\sigma\int_0^t e^{i(t-\tau)H}\big((\overline{\psi}-\overline{\psi_n})\psi_n^2\big)d\tau.$$

Let us prove that $z_n \longrightarrow 0$ in $L^\infty([0,T];\mathcal{H}^s(\mathbb{R}))$. To begin with, we state an analogous result to [1, Lemma 3.7].

Lemma 3.1. *Denote by*

$$\epsilon_n = \Big\| \int_0^t \big(u_n(\tau) - u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau\Big\|_{L^\infty_T\mathcal{H}^s(\mathbb{R})}.$$

Then $\epsilon_n \longrightarrow 0$, when $n \longrightarrow +\infty$.

Proof. We proceed by contradiction. Assume that there exists $\epsilon > 0$, a subsequence of u_n (still denoted by u_n) and a sequence $t_n \longrightarrow t \in [0,T]$ such that

$$\Big\| \int_0^{t_n} \big(u_n(\tau) - u(\tau)\big)e^{i(t_n-\tau)H}(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})} \geq \epsilon. \tag{3.6}$$

Up to a subsequence, we can assume that for all $n \geq 1$, $t_n \leq t$ or $t_n \geq t$. We only consider the first case, since the second is similar. By the Minkowski inequality and the unitarity of $e^{i\tau H}$

$$\begin{aligned}\Big\| \int_0^{t_n} \big(u_n(\tau) - u(\tau)\big)\big(e^{i(t_n-\tau)H} - e^{i(t-\tau)H}\big)(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})} \leq \\ \leq \int_0^{t_n} \big|u_n(\tau) - u(\tau)\big|\Big\|\big(e^{i(t_n-\tau)H} - e^{i(t-\tau)H}\big)(K\psi(\tau))\Big\|_{\mathcal{H}^s(\mathbb{R})}d\tau \\ = \int_0^{t_n} \big|u_n(\tau) - u(\tau)\big|\Big\|\big(e^{it_nH} - e^{itH}\big)(K\psi(\tau))\Big\|_{\mathcal{H}^s(\mathbb{R})}d\tau.\end{aligned}$$

Then by Cauchy-Schwarz

$$\Big\| \int_0^{t_n} \big(u_n(\tau)-u(\tau)\big)\big(e^{i(t_n-\tau)H}-e^{i(t-\tau)H}\big)(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})} \leq \\ \leq \big\|u_n-u\big\|_{L^2_T}\Big\|\big(e^{it_nH}-e^{itH}\big)(K\psi(\tau))\Big\|_{L^2_{\tau\in[0,T]}\mathcal{H}^s(\mathbb{R})}.$$

Now, using (1.15), observe that

$$\|K\psi\|_{L^2_T\mathcal{H}^s(\mathbb{R})} \leq \|K\|_{\mathcal{H}^s(\mathbb{R})}\|\psi\|_{L^2_T\mathcal{W}^{s,\infty}(\mathbb{R})} < \infty. \tag{3.7}$$

Hence Lemma 3.2 below (with $d=1$ and $q=2$) applies to conclude, with the previous lines, that

$$\Big\| \int_0^{t_n} \big(u_n(\tau)-u(\tau)\big)\big(e^{i(t_n-\tau)H}-e^{i(t-\tau)H}\big)(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})} \longrightarrow 0, \tag{3.8}$$

when $n \longrightarrow +\infty$.

By the Minkowski inequality, the unitarity of $e^{i\tau H}$ and the Hölder inequality

$$\begin{aligned}\Big\| \int_{t_n}^{t} \big(u_n(\tau)-u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})} &\leq \int_{t_n}^{t} \big|u_n(\tau)-u(\tau)\big|\big\|K\psi(\tau)\big\|_{\mathcal{H}^s(\mathbb{R})}d\tau \\ &\leq \|u_n-u\|_{L^{4/3}_{\tau\in[t_n,t]}}\|K\psi\|_{L^4_T\mathcal{H}^s(\mathbb{R})} \\ &\leq |t-t_n|^{1/4}\|u_n-u\|_{L^2_T}\|K\|_{\mathcal{H}^s(\mathbb{R})}\|\psi\|_{L^4_T\mathcal{W}^{s,\infty}(\mathbb{R})},\end{aligned} \tag{3.9}$$

where we used that $\|\psi\|_{L^4_T\mathcal{W}^{s,\infty}(\mathbb{R})} < \infty$ by (1.15). Then, the term (3.9) tends to 0. We combine this with (3.8) to deduce

$$\Big\| \int_0^{t_n} \big(u_n(\tau)-u(\tau)\big)e^{i(t_n-\tau)H}(K\psi(\tau))d\tau - \int_0^{t} \big(u_n(\tau)-u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})} \longrightarrow 0. \tag{3.10}$$

Let us now prove that $\displaystyle\int_0^t \big(u_n(\tau)-u(\tau)\big)e^{i(t-\tau)H}(K\psi(\tau))d\tau \longrightarrow 0$ in $\mathcal{H}^s(\mathbb{R})$, to reach a contradiction with (3.6). We set $v(\tau)=e^{i(t-\tau)H}(K\psi(\tau))$. Then by (3.7), $v\in L^2([0,T];\mathcal{H}^s(\mathbb{R}))$. We expand v on a Hilbertian basis $(h_k)_{k\geq 0}$ of $L^2(\mathbb{R})$ (the Hermite functions for instance),

$$v(\tau,x)=\sum_{k=0}^{+\infty}\alpha_k(\tau)h_k(x),$$

so that we have $\|v(\tau,\cdot)\|^2_{\mathcal{H}^s}=\displaystyle\sum_{k=0}^{+\infty}(2k+1)^s|\alpha_k(\tau)|^2$.

Then, there exists $M>0$ large enough such that the function $g(\tau,x)=\sum_{k=0}^{M}\alpha_k(\tau)h_k(x)$ satisfies $\|v-g\|_{L^2([0,T];\mathcal{H}^s(\mathbb{R}))}\leq \epsilon/(4\rho)$ where $\rho=\sup_{n\geq 0}\|u_n-u\|_{L^2_T}$.

We have

$$\int_0^t \big(u_n(\tau)-u(\tau)\big)g(\tau)d\tau=\sum_{k=0}^{M}h_k\int_0^t\big(u_n(\tau)-u(\tau)\big)\alpha_k(\tau)d\tau,$$

thus

$$\Big\| \int_0^t \big(u_n(\tau)-u(\tau)\big)g(\tau)d\tau\Big\|^2_{\mathcal{H}^s(\mathbb{R})}=\sum_{k=0}^{M}(2k+1)^s\Big|\int_0^t\big(u_n(\tau)-u(\tau)\big)\alpha_k(\tau)d\tau\Big|^2\longrightarrow 0,$$

by the weak convergence of (u_n). Finally, for n large enough,

$$\begin{aligned}\Big\| \int_0^t \big(u_n(\tau)-u(\tau)\big)v(\tau)d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})} &\leq \frac{\epsilon}{4\rho}\big\|u_n-u\big\|_{L^2_T} + \Big\| \int_0^t \big(u_n(\tau)-u(\tau)\big)g(\tau)d\tau\Big\|_{\mathcal{H}^s(\mathbb{R})}\\ &\leq \frac{\epsilon}{2},\end{aligned}$$

which together with (3.6) and (3.10) gives the contradiction. □

We state a convergence result (slightly more general than what we need here)

Lemma 3.2. *Let $d\geq 1$, $2\leq q<\infty$ and $s\geq 0$. Assume that $F\in L^q([0,T];\mathcal{H}^s(\mathbb{R}^d))$ and $t_n\longrightarrow t$. Then, when $n\longrightarrow+\infty$,*

$$\big\|\big(e^{it_nH}-e^{itH}\big)F(\tau,x)\big\|_{L^q_{\tau\in[0,T]}\mathcal{H}^s(\mathbb{R}^d)}\longrightarrow 0.$$

Proof. By unitarity of $e^{i\tau H}$, we can assume that $t=0$. Then, up to replacing F by $H^{s/2}F$, it is enough to prove the result for $s=0$. We expand F on the Hilbertian basis $(h_k)_{k\geq 0}$ of $L^2(\mathbb{R}^d)$ given by the Hermite functions: $F(\tau,x)=\sum_{k=0}^{+\infty}\alpha_k(\tau)h_k(x)$. Thus

$$\|F\|^q_{L^q_TL^2(\mathbb{R}^d)}=\int_0^T\Big(\sum_{k=0}^{+\infty}|\alpha_k(\tau)|^2\Big)^{q/2}d\tau<\infty. \tag{3.11}$$

We can write

$$e^{it_nH}F(\tau,x)=\sum_{k=0}^{+\infty}\alpha_k(\tau)e^{i(2k+1)t_n}h_k(x),$$

which gives

$$\big\|\big(e^{it_nH}-1\big)F(\tau,x)\big\|^2_{L^2(\mathbb{R}^d)}=\sum_{k=0}^{+\infty}|e^{i(2k+1)t_n}-1|^2|\alpha_k(\tau)|^2,$$

and we conclude with the Lebesgue convergence theorem thanks to the bound

$$\big\|\big(e^{it_nH}-1\big)F(\tau,x)\big\|_{L^2(\mathbb{R}^d)}\leq 2\Big(\sum_{k=0}^{+\infty}|\alpha_k(\tau)|^2\Big)^{1/2}\in L^q([0,T]),$$

by (3.11). □

By Lemma A.2

$$\begin{aligned}\|\mathcal{N}(\psi,\psi_n)(t)\|_{\mathcal{H}^s(\mathbb{R})} &\leq \int_0^t\|(\psi-\psi_n)(\psi+\psi_n)\overline{\psi}\|_{\mathcal{H}^s(\mathbb{R})}d\tau+\int_0^t\|(\overline{\psi}-\overline{\psi_n})\psi_n^2\|_{\mathcal{H}^s(\mathbb{R})}d\tau\\ &\leq c\int_0^t\|z_n\|_{\mathcal{H}^s(\mathbb{R})}\big(\|\psi\|^2_{\mathcal{H}^s(\mathbb{R})\cap\mathcal{W}^{s,\infty}(\mathbb{R})}+\|\psi_n\|^2_{\mathcal{H}^s(\mathbb{R})\cap\mathcal{W}^{s,\infty}(\mathbb{R})}\big)d\tau. \qquad (3.12)\end{aligned}$$

To simplify the exposition, we write $\mathcal{Y}^s(\mathbb{R})=\mathcal{H}^s(\mathbb{R})\cap\mathcal{W}^{s,\infty}(\mathbb{R})$ in the next lines. Thus, by (3.5), (3.12), and the inhomogeneous Strichartz estimate (1.10) (with q and r to be fixed later), for all $0\leq t\leq T$

$$\|z_n(t)\|_{\mathcal{H}^s(\mathbb{R})}\leq\epsilon_n+c\big\|u_nKz_n\big\|_{L^{q'}_t\mathcal{W}^{s,r'}(\mathbb{R})}+c\int_0^t\|z_n\|_{\mathcal{H}^s(\mathbb{R})}\big(\|\psi\|^2_{\mathcal{Y}^s(\mathbb{R})}+\|\psi_n\|^2_{\mathcal{Y}^s(\mathbb{R})}\big)d\tau.$$

Then by Grönwall, for all $0\leq t\leq T$ and (1.15)

$$\begin{aligned}\|z_n(t)\|_{\mathcal{H}^s(\mathbb{R})} &\leq \big(\epsilon_n+c\big\|u_nKz_n\big\|_{L^{q'}_t\mathcal{W}^{s,r'}(\mathbb{R})}\big)e^{c\int_0^t\left(\|\psi\|^2_{\mathcal{Y}^s(\mathbb{R})}+\|\psi_n\|^2_{\mathcal{Y}^s(\mathbb{R})}\right)d\tau}\\ &\leq C_1(T)\big(\epsilon_n+c\big\|u_nKz_n\big\|_{L^{q'}_t\mathcal{W}^{s,r'}(\mathbb{R})}\big). \qquad (3.13)\end{aligned}$$

Now we claim that

$$\big\|Kz_n\big\|_{\mathcal{W}^{s,r'}(\mathbb{R})} \leq c\|K\|_{\mathcal{H}^s(\mathbb{R})}\|z_n\|_{\mathcal{H}^s(\mathbb{R})}, \tag{3.14}$$

if r is large enough.

If $s=0$ we choose $r=\infty$ and we clearly have $\big\|Kz_n\big\|_{L^1(\mathbb{R})} \leq \|K\|_{L^2(\mathbb{R})}\|z_n\|_{L^2(\mathbb{R})}$.

If $s>0$, by (A.1) we have

$$\big\|Kz_n\big\|_{\mathcal{W}^{s,r'}(\mathbb{R})} \leq c\|K\|_{\mathcal{H}^s(\mathbb{R})}\|z_n\|_{L^q(\mathbb{R})} + c\|z_n\|_{\mathcal{H}^s(\mathbb{R})}\|K\|_{L^q(\mathbb{R})},$$

with $q>2$ such that $1/2+1/q=1/r'$. Now, if $r<\infty$ is large enough, then $q>2$ is close to 2, and by Sobolev $\|K\|_{L^q(\mathbb{R})} \leq c\|K\|_{\mathcal{H}^s(\mathbb{R})}$ and $\|z_n\|_{L^q(\mathbb{R})} \leq c\|z_n\|_{\mathcal{H}^s(\mathbb{R})}$, hence (3.14).

We come back to (3.13) and by (3.14) we get

$$\|z_n(t)\|_{\mathcal{H}^s(\mathbb{R})} \leq C_1(T)\Big(\epsilon_n + \|K\|_{\mathcal{H}^s(\mathbb{R})}\Big(\int_0^t |u_n(\tau)|^{q'}\|z_n(\tau)\|^{q'}_{\mathcal{H}^s(\mathbb{R})}d\tau\Big)^{1/q'}\Big),$$

for some $1<q'<2$. Then there exists $C_2(T)>0$ such that

$$\|z_n(t)\|^{q'}_{\mathcal{H}^s(\mathbb{R})} \leq C_2(T)\Big(\epsilon_n^{q'} + \|K\|^{q'}_{\mathcal{H}^s(\mathbb{R})}\int_0^t |u_n(\tau)|^{q'}\|z_n(\tau)\|^{q'}_{\mathcal{H}^s(\mathbb{R})}d\tau\Big),$$

and by the Grönwall lemma we get, for all $0\leq t\leq T$

$$\|z_n(t)\|_{\mathcal{H}^s(\mathbb{R})} \leq C_3(T)\epsilon_n e^{C_3(T)\|K\|^{q'}_{\mathcal{H}^s(\mathbb{R})}\int_0^t |u_n(\tau)|^{q'}d\tau}$$

which in turn implies

$$\|z_n\|_{L^\infty_T\mathcal{H}^s(\mathbb{R})} \leq C_3(T)\epsilon_n e^{C_3(T)\|K\|^{q'}_{\mathcal{H}^s(\mathbb{R})}\int_0^T |u_n(\tau)|^{q'}d\tau} \leq C_4(T)\epsilon_n,$$

and this latter term tends to 0, which concludes the proof.

4. The non-linear Schrödinger equation in dimension $d=3$

4.1. **Proof of Proposition 1.5.** We first prove (i).

Local existence: We consider the map

$$\Phi(\psi)(t) = e^{itH}\psi_0 + i\int_0^t e^{i(t-s)H}(|\psi|^2\psi)ds - i\int_0^t u(s)e^{i(t-s)H}(K\psi)ds, \tag{4.1}$$

and we will show that it is a contraction in some Banach space. Namely, we define the Strichartz norm $\|\psi\|_{X^1_T} = \|\psi\|_{L^\infty_T\mathcal{H}^1} + \|\psi\|_{L^2_T\mathcal{W}^{1,6}}$ and the space

$$B_{T,R} = \big\{\|\psi\|_{X^1_T} \leq R\big\},$$

with $R>0$ and $T>0$ to be fixed.

By the Strichartz estimates (1.9), (1.10) and the Leibniz rule

$$\begin{aligned}\|\Phi(\psi)\|_{X^1_T} &\leq c\|\psi_0\|_{\mathcal{H}^1} + c\int_0^T \big\||\psi|^2\psi\big\|_{\mathcal{H}^1}ds + c\int_0^T |u(s)|\big\|K\psi\big\|_{\mathcal{H}^1}ds \\ &\leq c\|\psi_0\|_{\mathcal{H}^1} + c\big\|\psi\big\|_{L^\infty_T\mathcal{H}^1}\|\psi\|^2_{L^2_TL^\infty} + c\Big(\int_0^T |u(s)|ds\Big)\big\|K\big\|_{\mathcal{W}^{1,\infty}}\big\|\psi\big\|_{L^\infty_T\mathcal{H}^1}. \end{aligned}\tag{4.2}$$

We now show that there exists $\kappa>0$ such that $\|\psi\|^2_{L^2_TL^\infty} \leq T^\kappa\|\psi\|^2_T$. Let $0<\epsilon<1/2$, then the couple $(q_\epsilon, r_\epsilon) = (\frac{4}{1+2\epsilon}, \frac{3}{1-\epsilon})$ is admissible and by Sobolev $\|\psi\|_{L^\infty} \leq C\|\psi\|_{\mathcal{W}^{1,r_\epsilon}}$. Then by the Hölder inequality,

$$\|\psi\|^2_{L^2_TL^\infty} \leq T^\kappa\|\psi\|^2_{L^{q_\epsilon}_T\mathcal{W}^{1,r_\epsilon}} \leq cT^\kappa\|\psi\|^2_{X^1_T},$$

for some $\kappa > 0$. Thus

$$\|\Phi(\psi)\|_{X_T^1} \leq c\|\psi_0\|_{\mathcal{H}^1} + cT^\kappa R^3 + cR\Big(\int_0^T |u(s)|ds\Big)\big\|K\big\|_{\mathcal{W}^{1,\infty}}.$$

We now choose $R = 4c\|\psi_0\|_{\mathcal{H}^1}$. Then we fix $T_1 = c_1 R^{-2/\kappa}$ with $c_1 > 0$ small enough such that $cT_1^{1/2}R^2 \leq 1/4$ and we fix $T_2 > 0$ such that $c\int_0^{T_2}|u(s)|ds \leq \big\|K\big\|_{\mathcal{W}^{1,\infty}}^{-1}/4$. Therefore, for $T = \min(T_1, T_2)$, Φ maps $B_{T,R}$ into itself. With similar estimates we can show that Φ is a contraction in $B_{T,R}$, namely

$$\|\Phi(\psi_1) - \Phi(\psi_2)\|_{X_T^1} \leq \big[cT^\kappa R^2 + c\Big(\int_0^T |u(s)|ds\Big)\big\|K\big\|_{\mathcal{W}^{1,\infty}}\big]\|\psi_1 - \psi_2\|_{X_T^1}.$$

Energy bound: We define

$$E(t) = \int_{\mathbb{R}^3}\big(\overline{\psi}H\psi + |\psi|^2 + \frac{1}{2}|\psi|^4\big)dx = \int_{\mathbb{R}^3}\big(|\nabla\psi|^2 + |x|^2|\psi|^2 + |\psi|^2 + \frac{1}{2}|\psi|^4\big)dx.$$

Then, using that $\partial_t\overline{\psi} = -i(H\overline{\psi} + |\psi|^2\overline{\psi}) + iu(t)K(x)\overline{\psi}$, we get

$$\begin{aligned} E'(t) &= 2\mathfrak{Re}\int_{\mathbb{R}^3}\partial_t\overline{\psi}\big(\psi + H\psi + |\psi|^2\psi\big)dx \\ &= -2u(t)\mathfrak{Im}\int_{\mathbb{R}^3}K\overline{\psi}H\psi dx \\ &= 2u(t)\mathfrak{Im}\int_{\mathbb{R}^3}\overline{\psi}\nabla K\cdot\nabla\psi dx. \end{aligned}$$

Now we use the assumption $\nabla K \in L^\infty(\mathbb{R}^3)$ to get

$$E'(t) \leq C|u(t)|\|\psi\|_{L^2}\|\nabla\psi\|_{L^2} \leq C|u(t)|\|\psi_0\|_{L^2}E^{1/2}(t),$$

which, by integration, implies

$$E(t) \leq \big(E^{1/2}(0) + 2C\|\psi_0\|_{L^2}\int_0^t |u(s)|ds\big)^2. \tag{4.3}$$

Notice that thanks to the Sobolev inequality, $\|\psi\|_{L^4(\mathbb{R}^3)} \leq C\|\psi\|_{\mathcal{H}^1(\mathbb{R}^3)}$, therefore $E(0) \leq C(\|\psi_0\|_{\mathcal{H}^1(\mathbb{R}^3)})$.

Global existence: Assume that one can solve (1.16) on $[0, T^\star)$. By (4.3), there is a time $T_1^\star > 0$ such that $c(T_1^\star)^\kappa(R^\star)^2 \leq 1/4$ with $R^\star = 4c\|\psi\|_{L^\infty_{T^\star}\mathcal{H}^1}$. Then we fix $T_2^\star > 0$ with

$$c\Big(\int_{T^\star - \frac{T_2^\star}{2}}^{T^\star + \frac{T_2^\star}{2}} |u(s)|ds\Big)\big\|K\big\|_{\mathcal{W}^{1,\infty}} \leq 1/4.$$

As a consequence, with the arguments of the local theory step, we are able to solve the equation (1.16), with an initial condition at $t = T^\star - \min(T_1^\star, T_2^\star)/2$, on the time interval $[T^\star - \min(T_1^\star, T_2^\star)/2, T^\star + \min(T_1^\star, T_2^\star)/2]$. This shows that the maximal solution is global in time.

Proof of (ii): Let $0 \leq \tau < T$ and $\delta > 0$ such that $\tau + \delta \leq T$. By the Gagliardo-Nirenberg and Sobolev inequalities on $\mathbb{R}^3$,

$$\|\psi\|_{L^\infty} \leq C\|\psi\|_{L^6}^{\frac{1}{2}}\|\psi\|_{\mathcal{W}^{1,6}}^{\frac{1}{2}} \leq C\|\psi\|_{\mathcal{H}^1}^{\frac{1}{2}}\|\psi\|_{\mathcal{W}^{1,6}}^{\frac{1}{2}},$$

then by the Hölder inequality

$$\|\psi\|_{L^2([\tau,\tau+\delta];L^\infty)}^2 \leq C\delta^{1/2}\|\psi\|_{L^\infty_T\mathcal{H}^1}\|\psi\|_{L^2([\tau,\tau+\delta];\mathcal{W}^{1,6})}. \tag{4.4}$$

We have for all $0 \leq t \leq \delta$

$$\psi(t+\tau) = e^{itH}\psi(\tau) + i\int_\tau^{\tau+t} e^{i(\tau+t-s)H}(|\psi|^2\psi)ds - i\int_\tau^{\tau+t} u(s)e^{i(\tau+t-s)H}(K\psi)ds,$$

which implies, using the same arguments as in (4.2), that

$$\begin{aligned}\|\psi\|_{L^2([\tau,\tau+\delta];\mathcal{W}^{1,6})} \leq \\ &\leq c\|\psi\|_{L^\infty_T\mathcal{H}^1} + c\big\|\psi\big\|_{L^\infty_T\mathcal{H}^1}\|\psi\|^2_{L^2([\tau,\tau+\delta];L^\infty)} + c\|K\|_{\mathcal{W}^{1,\infty}}\big\|\psi\big\|_{L^\infty_T\mathcal{H}^1}\Big(\int_\tau^{\tau+\delta}|u(s)|ds\Big)\\ &\leq c\|\psi\|_{L^\infty_T\mathcal{H}^1} + c\delta^{1/2}\big\|\psi\big\|^2_{L^\infty_T\mathcal{H}^1}\|\psi\|_{L^2([\tau,\tau+\delta];\mathcal{W}^{1,6})} + c\|\psi\|_{L^\infty_T\mathcal{H}^1}\int_0^T|u(s)|ds,\end{aligned}$$

where we used (4.4). We pick $\delta = \delta(T) > 0$ such that $c\delta^{1/2}\big\|\psi\big\|^2_{L^\infty_T\mathcal{H}^1} = \frac{1}{2}$, thus the previous estimate gives

$$\|\psi\|_{L^2([\tau,\tau+\delta];\mathcal{W}^{1,6})} \leq 2c\|\psi\|_{L^\infty_T\mathcal{H}^1}(1+\int_0^T|u(s)|ds). \tag{4.5}$$

We write this estimate for $\tau = 0, \delta, \dots, j\delta$ with $j \in \mathbb{N}$ such that $j\delta < T < (j+1)\delta$. We sum up and combine with (1.17), which gives

$$\|\psi\|_{L^2_T\mathcal{W}^{1,6}} \leq C\big(T, \|\psi_0\|_{\mathcal{H}^1}, \int_0^T|u(s)|ds\big),$$

which in turn implies (1.18), thanks to (4.6).

Proof of (iii): Let $k \geq 1$, and let $\psi_0 \in \mathcal{H}^k(\mathbb{R}^3)$ and $K \in \mathcal{W}^{k,\infty}(\mathbb{R}^3)$. Local existence in this case is proven as in the case $k = 1$, thanks to a fixed point argument using the Strichartz norms $\|\psi\|_{X^k_T} = \|\psi\|_{L^\infty_T\mathcal{H}^k} + \|\psi\|_{L^2_T\mathcal{W}^{k,6}}$. The globalisation part is obtained as previously, since the local time of existence only depends on the energy norm and on u.

Let us check the bound (1.19). Let $T > 0$. Since ψ is a fixed point in (4.1), we get for all $t \leq T$

$$\begin{aligned}\|\psi(t)\|_{\mathcal{H}^k} &\leq c\|\psi_0\|_{\mathcal{H}^k} + c\int_0^t\big\||\psi|^2\psi\big\|_{\mathcal{H}^k}ds + c\int_0^t|u(s)|\big\|K\psi\big\|_{\mathcal{H}^k}ds\\ &\leq c\|\psi_0\|_{\mathcal{H}^k} + c\int_0^t\big(\big\|\psi(s)\big\|^2_{L^\infty} + |u(s)|\|K\|_{\mathcal{W}^{k,\infty}}\big)\|\psi(s)\|_{\mathcal{H}^k}ds,\end{aligned}$$

where in the previous line we used the Moser estimate (A.2) to bound the non-linear term. Therefore, by the Grönwall lemma, we get

$$\begin{aligned}\|\psi(t)\|_{\mathcal{H}^k} &\leq c\|\psi_0\|_{\mathcal{H}^k}e^{C\int_0^t\left(\|\psi(s)\|^2_{L^\infty}+|u(s)|\right)ds}\\ &\leq C\big(\|\psi_0\|_{\mathcal{H}^k}, \|\psi\|_{L^2_TL^\infty}, \int_0^T|u(s)|ds\big).\end{aligned} \tag{4.6}$$

By Sobolev, from (1.18) we deduce

$$\|\psi\|_{L^2([0,T];L^\infty(\mathbb{R}^3))} \leq C\|\psi\|_{L^2([0,T];\mathcal{W}^{1,6}(\mathbb{R}^3))} \leq C\big(T, \|\psi_0\|_{\mathcal{H}^1}, \int_0^T|u(s)|ds\big),$$

which in turn, by (4.6), implies (1.19).

The estimate (1.20) can be obtained with similar arguments as for the special case $k = 1$. We do not write the details.

4.2. **Proof of Theorem 1.6.** We adopt the strategy of Ball-Marsden-Slemrod [1] combined with some non-linear estimates. Let $u_n \rightharpoonup u$ weakly in $L^1([0,T];\mathbb{R})$ and fix $L \geq 0$ such that $\int_0^T |u_n(s)|ds \leq L$, $\int_0^T |u(s)|ds \leq L$. By definition of ψ we have

$$\psi(t) = e^{itH}\psi_0 - i\int_0^t u(s)e^{i(t-s)H}(K\psi)ds + i\int_0^t e^{i(t-s)H}(|\psi|^2\psi)ds,$$

and we define ψ_n

$$\psi_n(t) = e^{itH}\psi_0 - i\int_0^t u_n(s)e^{i(t-s)H}(K\psi_n)ds + i\int_0^t e^{i(t-s)H}(|\psi_n|^2\psi_n)ds.$$

Let us prove that $\|\psi - \psi_n\|_{L^\infty_T\mathcal{H}^1} \longrightarrow 0$. Set $z_n = \psi - \psi_n$, then z_n satisfies

$$z_n = \mathcal{L}(\psi,\psi_n) + \mathcal{N}(\psi,\psi_n)$$

with

$$\mathcal{L}(\psi,\psi_n) = -i\int_0^t \big(u(s)-u_n(s)\big)e^{i(t-s)H}(K\psi)ds - i\int_0^t u_n(s)e^{i(t-s)H}\big(K(\psi-\psi_n)\big)ds$$

and

$$\mathcal{N}(\psi,\psi_n) = -i\int_0^t e^{i(t-s)H}\big((\psi-\psi_n)(\psi+\psi_n)\overline{\psi}\big)ds - i\int_0^t e^{i(t-s)H}\big((\overline{\psi}-\overline{\psi_n})\psi_n^2\big)ds.$$

Since $K \in \mathcal{W}^{1,\infty}(\mathbb{R}^3)$, the map $\psi \longmapsto K\psi$ is continuous from $\mathcal{H}^1(\mathbb{R}^3)$ to $\mathcal{H}^1(\mathbb{R}^3)$ and [1, Lemma 3.7] applies. Thus, when $n \longrightarrow +\infty$

$$\epsilon_n := \sup_{t\in[0,T]} \Big\| \int_0^t \big(u(s)-u_n(s)\big)e^{i(t-s)H}(K\psi)ds\Big\|_{\mathcal{H}^1(\mathbb{R}^3)} \longrightarrow 0.$$

By Lemma A.3,

$$\begin{aligned}\|\mathcal{N}(\psi,\psi_n)(t)\|_{\mathcal{H}^1(\mathbb{R}^3)} &\leq \int_0^t \|(\psi-\psi_n)(\psi+\psi_n)\overline{\psi}\|_{\mathcal{H}^1(\mathbb{R}^3)}ds + \int_0^t \|(\overline{\psi}-\overline{\psi_n})\psi_n^2\|_{\mathcal{H}^1(\mathbb{R}^3)}ds \\ &\leq \int_0^t \|z_n\|_{\mathcal{H}^1(\mathbb{R}^3)}\big(\|\psi\|^2_{\mathcal{W}^{1,6}} + \|\psi_n\|^2_{\mathcal{W}^{1,6}}\big)ds.\end{aligned}$$

Therefore

$$\|z_n(t)\|_{\mathcal{H}^1(\mathbb{R}^3)} \leq \epsilon_n + C\int_0^t |u_n(s)|\|z_n(s)\|_{\mathcal{H}^1(\mathbb{R}^3)}ds + C\int_0^t \|z_n(s)\|_{\mathcal{H}^1(\mathbb{R}^3)}\big(\|\psi\|^2_{\mathcal{W}^{1,6}} + \|\psi_n\|^2_{\mathcal{W}^{1,6}}\big)(s)ds,$$

and by Grönwall

$$\|z_n(t)\|_{\mathcal{H}^1(\mathbb{R}^3)} \leq \epsilon_n \exp\big(C\int_0^t |u_n(s)|ds + C\|\psi\|^2_{L^2_t\mathcal{W}^{1,6}} + C\|\psi_n\|^2_{L^2_t\mathcal{W}^{1,6}}\big).$$

Finally, by (1.18),

$$\|z_n\|_{L^\infty_T\mathcal{H}^1} \leq \epsilon_n C\big(T, \|\psi_0\|_{\mathcal{H}^1}, L\big),$$

which implies the result.

Appendix A. Some Sobolev estimates

In this section we gather some useful estimates in Sobolev spaces. To begin with, we have the following generalised Leibniz rule

Lemma A.1. *Let $s \geq 0$, then the following estimates hold*

$$\|f\, g\|_{\mathcal{W}^{s,q}} \leq C\|f\|_{L^{q_1}} \|g\|_{\mathcal{W}^{s,q_1'}} + C\|g\|_{L^{q_2}} \|f\|_{\mathcal{W}^{s,q_2'}}, \tag{A.1}$$

with $1 < q < \infty$, $1 < q_1,\, q_2 \leq \infty$ and $1 \leq q_1',\, q_2' < \infty$ so that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_1'} = \frac{1}{q_2} + \frac{1}{q_2'}.$$

For the proof with the usual Sobolev spaces, we refer to [20, Proposition 1.1, p. 105]. The result in our context follows by using (1.2). Observe that in this result we must have $q_1',\, q_2' < \infty$ and $q \neq 1, \infty$ which induces some technicalities in this paper.

A particular case of the previous inequality is the Moser estimate

$$\|fg\|_{\mathcal{H}^k} \leq C\big(\|f\|_{L^\infty}\|g\|_{\mathcal{H}^k} + \|g\|_{L^\infty}\|f\|_{\mathcal{H}^k}\big). \tag{A.2}$$

The following lemma will be useful

Lemma A.2. *Let $s \geq 0$. There exists $c > 0$ such that for all $\varphi \in \mathcal{H}^s(\mathbb{R})$, $\chi_1 \in \mathcal{H}^s(\mathbb{R}) \cap \mathcal{W}^{s,\infty}(\mathbb{R})$ and $\chi_2 \in \mathcal{H}^s(\mathbb{R}) \cap \mathcal{W}^{s,\infty}(\mathbb{R})$*

$$\|\varphi\chi_1\chi_2\|_{\mathcal{H}^s(\mathbb{R})} \leq c\|\varphi\|_{\mathcal{H}^s(\mathbb{R})}\|\chi_1\|_{\mathcal{H}^s(\mathbb{R})\cap\mathcal{W}^{s,\infty}(\mathbb{R})}\|\chi_2\|_{\mathcal{H}^s(\mathbb{R})\cap\mathcal{W}^{s,\infty}(\mathbb{R})}.$$

Proof. The case $s = 0$ is directly obtained by writing $\|\varphi\chi_1\chi_2\|_{L^2(\mathbb{R})} \leq c\|\varphi\|_{L^2(\mathbb{R})}\|\chi_1\|_{L^\infty(\mathbb{R})}\|\chi_2\|_{L^\infty(\mathbb{R})}$.

Now we assume that $s > 0$. By (A.1) we have

$$\|\varphi\chi_1\chi_2\|_{\mathcal{H}^s(\mathbb{R})} \leq c\|\varphi\|_{\mathcal{H}^s(\mathbb{R})}\|\chi_1\chi_2\|_{L^\infty(\mathbb{R})} + c\|\varphi\|_{L^p(\mathbb{R})}\|\chi_1\chi_2\|_{\mathcal{W}^{s,q}(\mathbb{R})}$$

for all $2 < p, q < \infty$ such that $1/p + 1/q = 1/2$. Then, by Sobolev, if $p > 2$ is small enough, $\|\varphi\|_{L^p} \leq c\|\varphi\|_{\mathcal{H}^s}$. Next, by (A.1) again,

$$\|\chi_1\chi_2\|_{\mathcal{W}^{s,q}(\mathbb{R})} \leq c\|\chi_1\|_{L^{q_1}(\mathbb{R})}\|\chi_2\|_{\mathcal{W}^{s,q_1'}(\mathbb{R})} + c\|\chi_2\|_{L^{q_1}(\mathbb{R})}\|\chi_1\|_{\mathcal{W}^{s,q_1'}(\mathbb{R})},$$

with $1/q_1 + 1/q_1' = 1/q$. We are able to conclude by observing that

$$\|\chi\|_{L^{q_1}(\mathbb{R})}, \|\chi\|_{\mathcal{W}^{s,q_1'}(\mathbb{R})} \leq \|\chi\|_{\mathcal{H}^s(\mathbb{R})} + \|\chi\|_{\mathcal{W}^{s,\infty}(\mathbb{R})} = \|\chi\|_{\mathcal{H}^s(\mathbb{R})\cap\mathcal{W}^{s,\infty}(\mathbb{R})}.$$

□

In the same spirit we state the following result

Lemma A.3. *There exists $c > 0$ such that for all $\varphi \in \mathcal{H}^1(\mathbb{R}^3)$, $\chi_1 \in \mathcal{W}^{1,6}(\mathbb{R}^3)$ and $\chi_2 \in \mathcal{W}^{1,6}(\mathbb{R}^3)$*

$$\|\varphi\chi_1\chi_2\|_{\mathcal{H}^1(\mathbb{R}^3)} \leq c\|\varphi\|_{\mathcal{H}^1(\mathbb{R}^3)}\|\chi_1\|_{\mathcal{W}^{1,6}(\mathbb{R}^3)}\|\chi_2\|_{\mathcal{W}^{1,6}(\mathbb{R}^3)}.$$

Proof. From the Leibniz rule and Hölder we deduce that

$$\begin{aligned}\|\varphi\chi_1\chi_2\|_{\mathcal{H}^1(\mathbb{R}^3)} &\leq \|\chi_1\chi_2\nabla\varphi\|_{L^2(\mathbb{R}^3)} + \|\varphi\chi_1\nabla\chi_2\|_{L^2(\mathbb{R}^3)} + \|\varphi\chi_2\nabla\chi_1\|_{L^2(\mathbb{R}^3)} + \|\langle x\rangle\varphi\chi_1\chi_2\|_{L^2(\mathbb{R}^3)} \\ &\leq \|\chi_1\|_{L^\infty}\|\chi_2\|_{L^\infty}\big(\|\nabla\varphi\|_{L^2} + \|\langle x\rangle\varphi\|_{L^2}\big) + \|\varphi\|_{L^6}\big(\|\chi_1\|_{L^6}\|\nabla\chi_2\|_{L^6} + \|\chi_2\|_{L^6}\|\nabla\chi_1\|_{L^6}\big).\end{aligned}$$

Then by Sobolev, $\|\chi\|_{L^\infty(\mathbb{R}^3)} \leq C\|\chi\|_{\mathcal{W}^{1,6}(\mathbb{R}^3)}$ and $\|\varphi\|_{L^6(\mathbb{R}^3)} \leq C\|\varphi\|_{\mathcal{H}^1(\mathbb{R}^3)}$, which allows to conclude. □

We recall the following interpolation lemma taken from [7, Lemma 3.3].

Lemma A.4. *Let $T > 0$ and $p \in [1, +\infty]$. Let $-\infty < \sigma_2 \leq \sigma_1 < +\infty$ and assume that $\psi \in L^p\big([-T,T];\mathcal{H}^{\sigma_1}\big)$ and $\partial_t\psi \in L^p\big([-T,T];\mathcal{H}^{\sigma_2}\big)$. Then for all $\epsilon > \sigma_1/p - \sigma_2/p$, $\psi \in L^\infty\big([-T,T];\mathcal{H}^{\sigma_1-\epsilon}\big)$ and*

$$\|\psi\|_{L^\infty_T\mathcal{H}^{\sigma_1-\epsilon}} \leq C\|\psi\|^{1-1/p}_{L^p_T\mathcal{H}^{\sigma_1}}\|\psi\|^{1/p}_{W^{1,p}_T\mathcal{H}^{\sigma_2}}.$$

Moreover, there exists $\eta > 0$ and $\theta \in [0,1]$ so that for all $t_1, t_2 \in [-T,T]$

$$\|\psi(t_1) - \psi(t_2)\|_{\mathcal{H}^{\sigma_1-2\epsilon}} \leq C|t_1 - t_2|^\eta\|\psi\|^{1-\theta}_{L^p_T\mathcal{H}^{\sigma_1}}\|\psi\|^{\theta}_{W^{1,p}_T\mathcal{H}^{\sigma_2}}.$$

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Thomas Chambrion, Université de Lorraine, CNRS, INRIA, IECL, F-54000 Nancy, France
E-mail address: thomas.chambrion@univ-lorraine.fr

Laurent Thomann, Université de Lorraine, CNRS, IECL, F-54000 Nancy, France
E-mail address: laurent.thomann@univ-lorraine.fr