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ON SPECIAL L -VALUES OF t -MODULES

BRUNO ANGLÈS, TUAN NGO DAC, AND FLORIC TAVARES RIBEIRO

ABSTRACT. We show that Taelman’s conjecture on special L -values of Anderson t -modules holds for a large class of t -modules. This class contains all mixed A -finite and uniformizable t -modules whose Hodge-Pink weights are at least 1. As a consequence, we deduce various log-algebraicity identities for tensor powers of the Carlitz module, generalizing the work of Anderson-Thakur.

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INTRODUCTION

Let \mathbb{F}_q be a finite field having q elements, q being a power of a prime number p . Let $A = \mathbb{F}_q[\theta]$ with θ an indeterminate over \mathbb{F}_q , $K = \mathbb{F}_q(\theta)$, $K_\infty = \mathbb{F}_q(\frac{1}{\theta})$ and let v_∞ be the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$. We denote by \mathbb{C}_∞ the completion of a fixed algebraic closure of K_∞ . The unique valuation of \mathbb{C}_∞ which extends v_∞ will still be denoted by v_∞ . For $d \in \mathbb{N}$, $A_{+,d}$ denotes the set of monic elements in A of degree d .

In 1930’s, Carlitz [11] introduced the Carlitz zeta values given by

$$\zeta_A(n) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \in K_\infty, \quad n \in \mathbb{Z}.$$

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In 1979, Goss [18] introduced a new type of L -functions in the arithmetic of function fields over finite fields and showed that Carlitz zeta values can be realized as special values of such L -functions (see [19], Chapter 8). The zeta values $\zeta_A(n)$ are related to the so-called Carlitz module which is defined as the \mathbb{F}_q -algebra homomorphism $C : A \rightarrow \mathbb{C}_\infty\{\tau\}$ given by $C_\theta = \theta + \tau$. Here $\mathbb{C}_\infty\{\tau\}$ is the non-commutative ring with the rules $\tau a = a^q \tau$ for $a \in \mathbb{C}_\infty$. The Carlitz exponential $\exp_C \in \mathbb{C}_\infty\{\{\tau\}\}$ is defined by $\exp_C = \sum_{j \geq 0} \frac{\tau^j}{D_j}$ where $D_0 = 1$ and for $j \geq 1$, $D_j = (\theta^{q^j} - \theta)D_{j-1}^q$. It satisfies the functional equation

$$\exp_C(\theta z) = C_\theta(\exp_C(z)), \quad \text{for all } z \in \mathbb{C}_\infty.$$

The Carlitz logarithm is the inverse series $\log_C \in \mathbb{C}_\infty\{\{\tau\}\}$ of the Carlitz exponential. Further, one can define the special L -values $L(C/A, n) \in K_\infty$ for any integer $n \in \mathbb{Z}$ and show that $L(C/A, n) = \zeta_A(n)$.

The following remarkable identity was discovered by Carlitz:

$$\exp_C(\zeta_A(1)) = 1$$

or equivalently

$$\zeta_A(1) = \log_C(1).$$

The first equality is an example of *log-algebraicity identities* for C while the second one is *the class formula* for C .

In 1970's, Drinfeld [15, 16] made a breakthrough by introducing the notion of Drinfeld modules which generalizes the Carlitz module. Several years later, Anderson [1] developed the theory of t -modules (or t -motives) which extends that of Drinfeld modules to higher dimensions. An important example of such objects is the n -th tensor power $C^{\otimes n}$ of the Carlitz module for any $n \in \mathbb{N}$ (see [4]).

Recently, Taelman [24, 25] defined fundamental objects attached to a Drinfeld module: the unit module and the class module. He proved a class formula which states that the special value of the Goss L -function attached to a Drinfeld module at $s = 1$ is the product of a regulator term arising from the unit module and an algebraic term arising from the class module. It is a vast generalization of the above-mentioned class formula for the Carlitz module. Taelman's class formula has been extended to t -modules by Fang [17] and to t -modules with variables by Demeslay [13, 14].

However, we should mention that very few log-algebraicity results are known. Inspired by examples of Carlitz [11] and Thakur [27], Anderson [2, 3] proved further log-algebraicity identities for C (and signed-normalized rank 1 Drinfeld modules in general). His fundamental results are extended and revisited by various works ([4, 5, 6, 7, 9, 12] and [28], Chapter 8). In a work in progress, Papanikolas [22] suggests to extend Anderson's method to develop a theory of log-algebraicity on tensor powers $C^{\otimes n}$ of the Carlitz module.

The present paper grew from an attempt to understand the possible connection between the class formula and the log-algebraicity identities for t -modules. It turned out that for A -finite and uniformizable t -modules, it has been already formulated as a conjecture by Taelman [23]. By Taelman's class formula [25], this conjecture holds for Drinfeld modules. In this paper, we succeed in proving Taelman's conjecture for a large class of t -modules. This class contains all mixed A -finite and uniformizable

t -modules whose Hodge-Pink weights are at least 1. As an application, we obtain various log-algebraicity results for tensor powers of the Carlitz module, generalizing the work of Anderson-Thakur [4] and rediscovering that of Papanikolas [22].

Let us give now more precise statements of our results.

Let F be a finite extension of K and let O_F be the integral closure of A in F . We set $F_\infty := F \otimes_K K_\infty$. Let E be a t -module of dimension d defined over O_F which consists of an \mathbb{F}_q -algebra homomorphism $E : A \rightarrow M_{d \times d}(O_F)\{\tau\}$ such that for all $a \in A$, if we write

$$E_a = \partial_E(a) + E_{a,1}\tau + \dots,$$

then we have $(\partial_E(a) - aI_d)^d = 0$.

We set $\text{Lie}(E)(F_\infty) := F_\infty^d$, $W_E(F_\infty) := \text{Lie}(E)(F_\infty)/(\partial_E(\theta) - \theta I_d) \text{Lie}(E)(F_\infty)$ and write w for the projection $\text{Lie}(E)(F_\infty) \rightarrow W_E(F_\infty)$. The following statement is a slightly modified version of a conjecture formulated by Taelman in 2009 ([23], Conjecture 1).

Conjecture A (Taelman's conjecture). *With the above notation, suppose that E is A -finite and uniformizable. Then there exist an element $a \in A \setminus \{0\}$ and a sub- A -module $Z \subset \text{Lie}(E)(F_\infty)$ of rank $r := \dim_{K_\infty} W_E(F_\infty)$ such that*

- 1) $\exp_E(Z) \subset \text{Lie}(E)(O_F)$,
- 2) $\bigwedge_A^r w(Z) = a \cdot L(\phi/O_F) \cdot \bigwedge_A^r W_E(O_F)$.

For Drinfeld modules, Conjecture A follows from the class formula of Taelman [25]. More generally, it is true for t -modules satisfying $\partial_E(\theta) = \theta I_d$ by the class formula for t -modules proved by Fang [17]. Further, when $F = K$ and E is a tensor power of the Carlitz module, Taelman's conjecture holds by the pioneer work of Anderson and Thakur [4]. However, Conjecture A is not always true (see Proposition A.2).

In this paper, we prove that Conjecture A holds for a large class of t -modules (see Theorem 4.4, Corollaries 4.5 and 4.6).

Theorem B (Theorem 4.4). *Let E/O_F be an A -finite t -module such that $\sigma N \subset (t - \theta)N$ where N is its associated dual t -motive. Then Conjecture A holds for E/O_F .*

Surprisingly, in the above theorem, we do not require the uniformizable assumption for t -modules. For a mixed A -finite and uniformizable t -module E , one can define the Hodge-Pink structure and the Hodge-Pink weights associated to E (by [20], Definition 5.32) and one can show that the condition $\sigma N \subset (t - \theta)N$ is equivalent to the condition that every Hodge-Pink weight of E is at least 1.

The proof of Theorem B is based on two key ingredients: the dictionary between t -modules and dual t -motives ([1, 10, 20]) and the notion of Stark units introduced in our previous papers ([5, 9] and Section 3.2). As an immediate application, Theorem B implies:

Theorem C (Corollaries 4.5 and 4.6).

- 1) *Let E/O_F be an A -finite t -module. Then Conjecture A holds for $E \otimes C/O_F$.*

2) Let E/O_F be a mixed A -finite and uniformizable t -module whose Hodge-Pink weights are at least 1. Then Conjecture A holds for E .

Finally, we deal with tensor powers of the Carlitz module $C^{\otimes n}$ ($n \in \mathbb{N}^*$). As a special case of Theorem B, Taelman's conjecture is true for these t -modules over any finite extension F of K (see Theorem 5.2). Then we apply our techniques to deduce various log-algebraicity results for $C^{\otimes n}$ (see Theorem 5.3 and Proposition 5.5). These results extend the work of Anderson-Thakur [4] and rediscover that of Papanikolas [22] (see Section 5.3).

Theorem D (Theorem 5.3). *Let $n \geq 1$ be an integer and let $C^{\otimes n}/A$ be the n -th tensor power of the Carlitz module. Then we have:*

1) *There exists a nonzero vector $x_n \in M_{n \times 1}(A)$ such that*

$$\exp_{C^{\otimes n}} \left(\left(\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{\partial_{C^{\otimes n}}(a)^n} \right) x_n \right) \in M_{n \times 1}(A).$$

2) *There exists a nonzero vector $X_n \in M_{n \times 1}(A[z])$ such that*

$$\exp_{\widetilde{C^{\otimes n}}} \left(\left(\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{\partial_{C^{\otimes n}}(a)^n} z^d \right) X_n \right) \in M_{n \times 1}(A[z]).$$

We also prove an equivariant log-algebraicity theorem for tensor powers of the Carlitz module over cyclotomic field extensions.

Theorem E (Theorem 5.7). *Let $n \geq 1$ be an integer and let $a \in A$ be a monic polynomial which is square-free. We denote by F the a -th cyclotomic extension of K whose Galois group is Δ_a and by O_F the integral closure of A in L .*

Then there exists a free $A[\Delta_a]$ -module of rank one $M \subset M_{n \times 1}(O_F)$ of the form $M = A[\Delta_a]L(n, \Delta_a)X_n$ for some vector $X_n \in M_{n \times 1}(O_F)$ such that

1) $\exp_{C^{\otimes n}}(L(n, \Delta_a)X_n) \in M_{n \times 1}(O_F)$.

2) If $\iota : M_{n \times 1}(O_F) \rightarrow O_F$ denotes the projection on the last coordinate, then

$$\iota(M) = b(\Delta_a)L(n, \Delta_a)O_F$$

for some $b(\Delta_a) \in A[\Delta_a] \cap K[\Delta_a]^\times$.

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1. BACKGROUND

We recall the notion of Anderson t -modules and dual t -motives. We refer the reader to excellent papers [1, 10, 20] for more details.

Definition 1.1. Let $L \subset \mathbb{C}_\infty$ be a field containing K .

1) An Anderson t -module E/L of dimension $d \geq 1$ defined over L (a t -module for short) consists of an \mathbb{F}_q -algebra homomorphism $E : A \rightarrow M_{d \times d}(L)\{\tau\}$ such that for all $a \in A$, if we write

$$E_a = E_{a,0} + E_{a,1}\tau + \dots, \quad E_{a,i} \in M_{d \times d}(L),$$

we will set $\partial_E(a) := E_{a,0}$ and require

$$(\partial_E(a) - aI_d)^d = 0_d.$$

2) A Drinfeld module is a t -module of dimension 1.

3) Let E and E' be two t -modules over L of dimension d and d' respectively. Then a morphism $f : E \rightarrow E'$ over L is a morphism $f : L^d \rightarrow L^{d'}$ over L commuting with the actions of A .

Let E/L be a t -module of dimension d . Note that $M_{d \times 1}(L)$ is equipped with two structures of A -module: the first one is induced by E and we denote by $E(L)$ the corresponding A -module, the second one is induced by $\partial_E : A \rightarrow M_{d \times d}(L)$, $a \mapsto E_{a,0}$, and we denote by $\text{Lie}(E)(L)$ the corresponding A -module.

Until the end of this section, we suppose further that L is perfect and for $c \in L$, we set $c^{(-1)} = c^{1/q}$. We introduce the non-commutative ring $L[t, \sigma]$ with

$$tc = ct, \quad t\sigma = \sigma t, \quad \sigma c = c^{(-1)}\sigma, \quad c \in L.$$

Definition 1.2.

1) A dual t -motive $N(L)$ over L is a left $L[t, \sigma]$ -module which is free and finitely generated over $L\{\sigma\}$ such that there exists an integer $d \in \mathbb{N}$ with $(t - \theta)^d(N(L)/\sigma N(L)) = 0$.

2) Morphisms of dual t -motives are morphisms of left $L[t, \sigma]$ -modules.

There is an explicit correspondence between t -modules and dual t -motives as follows. Let E/L be a t -module of dimension d defined over L . We define the map $*$: $L\{\tau\} \rightarrow L\{\sigma\}$ by $(\sum a_i \tau^i)^* := \sum \sigma^i a_i = \sum a_i^{(-i)} \sigma^i$. For any matrix $B \in M_{d \times d}(L\{\sigma\})$, we put $B^* \in M_{d \times d}(L\{\tau\})$ given by $B_{ij}^* := (B_{ji})^*$. We set $N(L) := M_{1 \times d}(L\{\sigma\})$. It is equipped with an action of $\mathbb{F}_q[t]$ given by

$$a(t) \cdot h := hE_a^* = h \left(\sum_{i \geq 0} \sigma^i E_{a,i}^* \right), \quad h \in N(L), a \in A.$$

Then $N(L)$ is a dual t -motive and the functor $E \mapsto N(L)$ is covariant. Anderson proved that it is in fact an equivalence of categories between the category of dual t -motives and that of t -modules ([10], Theorem 4.4.1).

Let $\delta_1 : N(L) \rightarrow E(L)$ be the homomorphism of A -modules defined by

$$\delta_1(h) := (\delta_1(a_1), \dots, \delta_1(a_d))^t$$

where $h = (a_1, \dots, a_d) \in N(L)$ and for $b = \sum_j b_j \sigma^j = \sum_j \sigma^j b_j^{(j)} \in L\{\sigma\}$, we set $\delta_1(b) = \sum_j b_j^{(j)}$. This map induces a commutative diagram

$$\begin{array}{ccc} N(L)/(\sigma - 1)N(L) & \xrightarrow{\delta_1} & E(L) \\ a(t) \downarrow & & \downarrow E_a \\ N(L)/(\sigma - 1)N(L) & \xrightarrow{\delta_1} & E(L). \end{array}$$

We set

$$N_0(L) := M_{1 \times d}(L) \subset N(L).$$

Observe that in general, $N_0(L)$ is not stable under the action of $\mathbb{F}_q[t]$.

Now we define another canonical map of A -modules

$$\delta_0 : N(L) \longrightarrow \text{Lie}(E)(L).$$

For any $h \in N(L)$, we put $h = (a_1, \dots, a_d)$ with $a_i \in L\{\sigma\}$. Then we define $\delta_0(h) := (\delta_0(a_1), \dots, \delta_0(a_d))^t$ where for $b = \sum_j b_j \sigma^j \in L\{\sigma\}$, we set $\delta_0(b) = b_0$. The map δ_0 induces an isomorphism of A -modules

$$\delta_0 : N(L)/\sigma N(L) \xrightarrow{\sim} \text{Lie}(E)(L).$$

We observe that δ_0 induces an isomorphism of L -vector spaces

$$\psi : N_0(L) \xrightarrow{\sim} \text{Lie}(E)(L).$$

Thus we get an induced structure of A -module on $N_0(L)$ defined by

$$\partial_E(a) \cdot h_0^t := \psi^{-1}(\partial_E(a) \cdot \psi(h_0)), \quad a \in A, h_0 \in N_0(L).$$

In particular, $N_0(\widehat{LK_\infty})$ is a K_∞ -vector space equipped with an action of A via ∂_E .

2. INVERSE OF THE FROBENIUS

We keep the notation of Section 1. Recall that $L \subset \mathbb{C}_\infty$ is an extension field of K which is perfect and E/L is a t -module whose corresponding dual t -motive is denoted by $N(L)$.

We set $P_0(t) = 1$ and for $j \geq 1$, we choose $P_j(t) \in K[t]$ such that

$$P_j(t)(t - \theta^{q^j})^d \equiv 1 \pmod{(t - \theta)^d K[t]}.$$

Recall that $(t - \theta)^d N(L) \subset \sigma N(L)$. For $j \in \mathbb{N}$, we define the j -th inverse of the Frobenius $\varphi_j : N(L) \rightarrow N_0(L)$ as follows. Let $h \in N(L)$, then there exists a unique $y \in N(L)$ such that

$$\prod_{k=0}^{j-1} (t - \theta^{q^{-k}})^d h = \sigma^j y.$$

We set

$$\varphi_j(h) := \psi^{-1}(\delta_0(P_0(t)P_1(t) \cdots P_j(t)y)).$$

One easily verifies that φ_j is well-defined.

Lemma 2.1. *We have the following properties:*

- i) $\varphi_j(h_1 + \lambda h_2) = \varphi_j(h_1) + \lambda^{q^j} \varphi_j(h_2)$ for all $h_1, h_2 \in N(L)$ and $\lambda \in L$,
- ii) $\varphi_j(t \cdot h) = \partial_E(\theta) \cdot \varphi_j(h)$ for all $h \in N(L)$,
- iii) $\varphi_j(\sigma^k h) = \varphi_{j-k}(h)$ for all $j, k \in \mathbb{N}$ and $h \in N(L)$ where we put $\varphi_n = 0$ if $n < 0$.

Proof. Left to the reader. □

We also recall that there exist two series $\exp_E, \log_E \in I_d + \tau M_{d \times d}(L)\{\{\tau\}\}$ such that

$$\exp_E \partial_E(a) = E_a \exp_E, \quad \text{for all } a \in A,$$

$$\log_E E_a = \partial_E(a) \log_E, \quad \text{for all } a \in A,$$

$$\exp_E \log_E = \log_E \exp_E = I_d.$$

Furthermore, \exp_E converges on $\text{Lie}(E)(\mathbb{C}_\infty)$ and therefore induces a homomorphism of A -modules $\exp_E : \text{Lie}(E)(\mathbb{C}_\infty) \rightarrow E(\mathbb{C}_\infty)$.

Proposition 2.2. *We write $\log_E = \sum_{k \geq 0} L_k \tau^k$ with $L_k \in M_{d \times d}(L)$. Then for all $k \geq 0$, we have the following identity*

$$\varphi_k(h) = \tau^k(h) L_k^*, \quad \text{for all } h \in N_0(L)$$

where $\tau^k(a_1, \dots, a_d) := (a_1^{q^k}, \dots, a_d^{q^k})$ for $a_1, \dots, a_d \in L$.

Proof. Let $k \in \mathbb{N}$ be an integer. By Lemma 2.1, Part i), there exists $Q_k \in M_{d \times d}(L)$ such that for $h \in N_0(L)$, we have

$$\varphi_k(h) = \tau^k(h) Q_k.$$

In particular, $Q_0 = I_d$. Let $h \in N_0(L)$, then by Lemma 2.1, Part ii), we have

$$\varphi_k(t \cdot h) = \tau^k(h) Q_k (\partial_E(\theta))^*.$$

Recall that $t \cdot h := h E_\theta^* = h \left(\sum_{i \geq 0} \sigma^i E_{\theta, i}^* \right)$, then Lemma 2.1, Part iii) implies that

$$\varphi_k(t \cdot h) = \tau^k(h) \sum_{i \geq 0} \tau^{k-i}(E_{\theta, i}^*) Q_{k-i}.$$

It follows that for all $k \geq 0$, we have

$$Q_k (\partial_E(\theta))^* = \sum_{i \geq 0} \tau^{k-i}(E_{t, i}^*) Q_{k-i}.$$

Recall that \log_E is the unique series in $I_d + \tau M_{d \times d}(L)\{\{\tau\}\}$ such that $\log_E E_\theta = \partial_E(\theta) \log_E$. We deduce that $Q_k = L_k^*$ for all $k \geq 0$. □

Remark 2.3. Similarly, we can calculate the coefficients of the exponential series \exp_E using the t -motive attached to E (see [20], Chapter 5).

3. STATEMENT OF TAE LMAN'S CONJECTURE

Let F be a finite extension of K and let O_F be the integral closure of A in F . We set $F_\infty := F \otimes_K K_\infty$. Let E be a t -module defined over O_F , i.e. for all $a \in A$, we have $E_a \in M_{d \times d}(O_F)\{\tau\}$.

Recall that for an O_F -algebra B , $E(B)$ denotes the A -module B^d equipped with the action of A via E and the tangent space $\text{Lie}(E)(B)$ denotes the A -module B^d equipped with the action of A via the map $\partial_E : A \rightarrow M_{d \times d}(O_F)$. By [17], Lemma 1.7, the latter map can be uniquely extended to a map $\partial_E : K_\infty \rightarrow M_{d \times d}(F_\infty)$ given by

$$\partial_E \left(\sum_{i \geq m} a_i \theta^{-i} \right) = \sum_{i \geq m} a_i \partial_E(\theta)^{-i}.$$

3.1. The class formula à la Taelman.

Following Taelman [24], we define the class module by

$$H(E/O_F) := \frac{E(F_\infty)}{\exp_E(\text{Lie}(E)(F_\infty)) + E(O_F)}.$$

This is an A -module which is finitely generated and of torsion (see [24] for Drinfeld modules and [17] for t -modules). We denote by $[H(E/O_F)]_A \in A$ the monic generator of the Fitting ideal of $H(E/O_F)$. More generally, for any finitely generated and torsion A -module M , we denote by $[M]_A \in A$ the monic generator of the Fitting ideal of M .

We set

$$U(E/O_F) := \{x \in \text{Lie}(E)(F_\infty) : \exp_E(x) \in E(O_F)\}.$$

Then $U(E/O_F)$ is an A -lattice in $\text{Lie}(E)(F_\infty)$, i.e. $U(E/O_F)$ is discrete and co-compact in $\text{Lie}(E)(F_\infty)$ (see [24] for Drinfeld modules and [17] for t -modules). This module is called *the unit module* attached to E/O_F . We denote by $[\text{Lie}(E)(O_F)]_A : U(E/O_F)]_A \in K_\infty$ the co-volume of two lattices $\text{Lie}(E)(O_F)$ and $U(E/O_F)$.

Here is the statement of the class formula à la Taelman:

Theorem 3.1 (The class formula). *The infinite product*

$$L(E/O_F) := \prod_{\mathfrak{p}} \frac{[\text{Lie}(E)(O_F/\mathfrak{p})]_A}{[E(O_F/\mathfrak{p})]_A}$$

where \mathfrak{p} runs through the set of maximal ideals of O_F converges in K_∞ . Further, we have

$$L(E/O_F) = [\text{Lie}(E)(O_F) : U(E/O_F)]_A \cdot [H(E/O_F)]_A.$$

Theorem 3.1 was first proved by Taelman [25] for Drinfeld modules. Shortly after, the class formula was extended to t -modules by Fang [17] and to t -modules with variables by Demeslay [13]. While the proof of Demeslay follows closely the original proof of Taelman, that of Fang is based on the work of V. Lafforgue [21] using shtukas and the theory of Fontaine in equal characteristics.

3.2. Stark units for t -modules.

The module of Stark units attached to a Drinfeld module was introduced in [9] and developed further in [5]. In this section, we extend this notion for t -modules.

Let z be an indeterminate over K_∞ and let $\mathbb{T}_z(K_\infty) := \mathbb{F}_q[z][(\frac{1}{\theta})]$ be the Tate algebra in the variable z with coefficients in K_∞ . We set

$$\mathbb{T}_z(F_\infty) = F_\infty \otimes_{K_\infty} \mathbb{T}_z(K_\infty).$$

We still denote by $\tau : \mathbb{T}_z(F_\infty) \rightarrow \mathbb{T}_z(F_\infty)$ the continuous $\mathbb{F}_q[z]$ -algebra homomorphism such that $\tau(x) = x^q$ for all $x \in F_\infty$.

Let $\tilde{E} : \mathbb{F}_q(z)[\theta] \rightarrow M_{d \times d}(F(z))\{\tau\}$ be the $\mathbb{F}_q(z)$ -algebra homomorphism given by

$$\tilde{E}_\theta = \sum_{i \geq 0} z^i E_{\theta, i} \tau^i.$$

We also denote by $\partial_{\tilde{E}} : \mathbb{F}_q(z)[(\frac{1}{\theta})] \rightarrow M_{d \times d}(F_\infty \otimes_{K_\infty} \mathbb{F}_q(z)[(\frac{1}{\theta})])$ the continuous homomorphism of $\mathbb{F}_q(z)$ -algebras given by

$$\partial_{\tilde{E}} \left(\sum_{i \geq i_0} x_i \theta^{-i} \right) = \sum_{i \geq i_0} x_i \partial_E(\theta)^{-i}, \quad x_i \in \mathbb{F}_q(z), i_0 \in \mathbb{Z}.$$

There exists a unique element $\exp_{\tilde{E}} \in I_d + \tau M_{d \times d}(F(z))\{\tau\}$ such that

$$\exp_{\tilde{E}} \partial_{\tilde{E}}(\theta) = \tilde{E}_\theta \exp_{\tilde{E}}.$$

If $\exp_E = \sum_{i \geq 0} E_i \tau^i$ with $E_i \in M_{d \times d}(F)$, then we have $\exp_{\tilde{E}} = \sum_{i \geq 0} z^i E_i \tau^i$. In particular, $\exp_{\tilde{E}}$ converges on $\text{Lie}_{\tilde{E}}(F_\infty \otimes_{K_\infty} \mathbb{F}_q(z)[(\frac{1}{\theta})])$ and induces a homomorphism of A -modules $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(F_\infty)) \rightarrow \tilde{E}(\mathbb{T}_z(F_\infty))$.

Let $\text{ev} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(F_\infty)) \rightarrow \text{Lie}(E)(F_\infty)$ be the evaluation at $z = 1$. We observe that ev induces a short exact sequence of A -modules:

$$0 \rightarrow (z-1) \text{Lie}_{\tilde{E}}(\mathbb{T}_z(F_\infty)) \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(F_\infty)) \rightarrow \text{Lie}(E)(F_\infty) \rightarrow 0.$$

Definition 3.2. The module of z -units and the module of Stark units attached to E are defined by

$$\begin{aligned} U(\tilde{E}/O_F[z]) &:= \{x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(F_\infty)) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(O_F[z])\}, \\ U_{\text{St}}(E/O_F) &:= \text{ev}(U(\tilde{E}/O_F[z])). \end{aligned}$$

We observe that $U_{\text{St}}(E/O_F) \subset U(E/O_F)$.

Theorem 3.3. *The module of Stark units $U_{\text{St}}(E/O_F)$ is an A -lattice in $\text{Lie}(E)(F_\infty)$ and we have*

$$L(E/O_F) = [\text{Lie}(E)(O_F) : U_{\text{St}}(E/O_F)]_A.$$

Proof. The proof uses similar arguments to those employed in [9], Section 2. We give a detailed proof for the convenience of the reader.

1) We set

$$H := \frac{\tilde{E}(\mathbb{T}_z(F_\infty))}{\tilde{E}(O_F[z]) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(F_\infty)))}.$$

Thus H is an $A[z]$ -module via \tilde{E} . Let u_1, \dots, u_m be an A -basis of O_F where $m := [L : K]$. We set $\mathfrak{M} := \frac{1}{\theta} \mathbb{F}_q[z][[\frac{1}{\theta}]]$ and get a direct sum of $\mathbb{F}_q[z]$ -modules

$$\mathbb{T}_z(F_\infty)^d = O_F[z]^d \oplus (\oplus_{i=1}^m u_i \mathfrak{M}^d).$$

By [1], Proposition 2.1.4, there exists an integer $N \geq 1$ such that $\exp_{\tilde{E}}$ induces an automorphism of $\mathbb{F}_q[z]$ -modules on $\oplus_{i=1}^m u_i \frac{1}{\theta^N} \mathfrak{M}^d$. Thus H is a finitely generated $\mathbb{F}_q[z]$ -module. Since $\exp_{\tilde{E}} \equiv I_d \pmod{z}$, we finally obtain

$$\frac{H}{zH} = \{0\}.$$

Consequently, H is a finitely generated and torsion $\mathbb{F}_q[z]$ -module.

2) Let $\alpha : \mathbb{T}_z(F_\infty)^d \rightarrow H$ be the homomorphism of $\mathbb{F}_q[z]$ -modules given by

$$\alpha(x) \equiv \frac{1}{z-1} (\exp_{\tilde{E}}(x) - \exp_E(x)) \pmod{O_F[z]^d + \exp_{\tilde{E}}(\mathbb{T}_z(F_\infty)^d)}.$$

We observe that

$$\alpha(U(E/O_F)) \subset H[z-1] := \{h \in H, (z-1)h = 0\}$$

and for all $x \in U(E/O_F)$, we have

$$\alpha(\partial_E(\theta)x) \equiv \tilde{E}_\theta(\alpha(x)) \pmod{O_F[z]^d + \exp_{\tilde{E}}(\mathbb{T}_z(F_\infty)^d)}.$$

We deduce that α induces a homomorphism of A -modules still denoted by $\alpha : U(E/O_F) \rightarrow H[z-1]$. Let $x \in U_{\text{St}}(E/O_F)$, then there exist $u, v \in \mathbb{T}_z(F_\infty)^d$ such that

$$\begin{aligned} \exp_{\tilde{E}}(u) &\in O_F[z]^d, \\ x &= u + (z-1)v. \end{aligned}$$

We observe that $\exp_{\tilde{E}}(u) - \exp_E(x) \in (z-1)O_F[z]^d$. It implies $\alpha(x) = 0$. Therefore α induces an A -module homomorphism still denoted by $\alpha : \frac{U(E/O_F)}{U_{\text{St}}(E/O_F)} \rightarrow H[z-1]$.

3) Let $x \in U(E/O_F)$ such that $\exp_{\tilde{E}}(x) \in O_F[z]^d + (z-1)\exp_{\tilde{E}}(\mathbb{T}_z(F_\infty)^d)$. Then we have

$$x \in \{y \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(F_\infty)) \mid \exp_{\tilde{E}}(y) \in \tilde{E}(O_F[z])\} + (z-1)\mathbb{T}_z(F_\infty)^d.$$

Thus we get

$$x = \text{ev}(x) \in U_{\text{St}}(E/O_F).$$

We conclude that the map α is injective. It follows that $\frac{U(E/O_F)}{U_{\text{St}}(E/O_F)}$ is a finite A -module and $U_{\text{St}}(E/O_F)$ is an A -lattice in $\text{Lie}(E)(F_\infty)$.

4) Finally, let $x \in \mathbb{T}_z(F_\infty)^d$ such that

$$(z-1)x = a + \exp_{\tilde{E}}(b), \quad a \in O_F[z]^d, b \in \mathbb{T}_z(F_\infty)^d.$$

Thus $\exp_E(\text{ev}(b)) = -\text{ev}(a) \in O_F$. It follows that $\text{ev}(b) \in U(E/O_F)$.

Let $c \in O_F[z]^d$ and $d \in \mathbb{T}_z(F_\infty)^d$ such that $a - \text{ev}(a) = (z-1)c$ and $b - \text{ev}(b) = (z-1)d$. We have

$$(z-1)(x - c - \exp_{\tilde{E}}(d)) = \exp_{\tilde{E}}(\text{ev}(b)) - \exp_E(\text{ev}(b)).$$

In other words,

$$\alpha(\text{ev}(b)) \equiv x \pmod{O_F[z]^d + \exp_{\tilde{E}}(\mathbb{T}_z(F_\infty)^d)}.$$

Therefore we have an isomorphism of A -modules

$$\frac{U(E/O_F)}{U_{\text{St}}(E/O_F)} \simeq H[z-1].$$

We have another isomorphism of A -modules

$$\frac{H}{(z-1)H} \simeq H(E/O_F).$$

Therefore

$$[H(E/O_F)]_A = [H/(z-1)H]_A = [H[z-1]]_A.$$

We conclude

$$[H(E/O_F)]_A = [U(E/O_F)/U_{\text{St}}(E/O_F)]_A.$$

The Theorem follows immediately from this equality and the class formula for t -modules (see for example [17], Theorem 1.7). \square

3.3. Taelman's conjecture.

Definition 3.4. We keep the notation of Sections 3.1 and 3.2. We define

$$W_E := \text{Lie}(E)/(\partial_E(\theta) - \theta I_d) \text{Lie}(E)$$

We denote by w the projection $\text{Lie}(E) \rightarrow W_E$ and set $r := \dim_{K_\infty} W_E(F_\infty)$.

The following conjecture for A -finite and uniformizable t -modules is a slightly modified version of that formulated by Taelman in 2009 ([23], Conjecture 1).

Conjecture 3.5 (Taelman's conjecture). *Let E be an A -finite and uniformizable t -module defined over O_F with $r = \dim_{K_\infty} W_E(F_\infty)$ (see Definition 3.4). Then there exist an element $a \in A \setminus \{0\}$ and a sub- A -module $Z \subset \text{Lie}(E)(F_\infty)$ of rank r such that*

- 1) $\exp_E(Z) \subset \text{Lie}(E)(O_F)$,
- 2) $\bigwedge_A^r w(Z) = a \cdot L(E/O_F) \cdot \bigwedge_A^r W_E(O_F)$.

Remark 3.6.

- 1) We refer the reader to [20], Sections 5.2 and 5.5 (see also [10, 23]) for the definition of A -finite and uniformizable t -modules.
- 2) We should mention that Drinfeld modules are always A -finite and uniformizable. For Drinfeld modules, Conjecture 3.5 follows immediately from the class formula of Taelman [25] (see Theorem 3.1).
- 3) The original form of Taelman's conjecture ([23], Conjecture 1) requires a stronger condition that $a = 1$. If $\partial_E(\theta) = \theta I_d$, then by Theorem 3.3, Taelman's original conjecture is true. However, we provide below a counterexample for this strong form, see Proposition 3.7.

Proposition 3.7. *Let $n = q+1$ and let $C^{\otimes n}$ be the n -th tensor power of the Carlitz module. We denote by $\exp_{C^{\otimes n}}$ the exponential series attached to $C^{\otimes n}$. Then there does not exist $x = (x_1, \dots, x_n)^t \in M_{n \times 1}(K_\infty)$ such that $\exp_{C^{\otimes n}}(x) \in M_{n \times 1}(A)$ and $x_n = \zeta_A(n)$.*

Proof. It is clear that $C^{\otimes n}$ is A -finite and uniformizable. Suppose that there exists $x = (x_1, \dots, x_n)^t \in M_{n \times 1}(K_\infty)$ such that $\exp_{C^{\otimes n}}(x) \in M_{n \times 1}(A)$ and $x_n = \zeta_A(n)$.

Anderson and Thakur ([4], Section 3.8) proved that there exists $z = (z_1, \dots, z_n) \in K_\infty^n$ such that

$$\exp_{C^{\otimes n}}(z) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + C_{\theta^q - \theta}^{\otimes n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in M_{n \times 1}(A)$$

and

$$z_n = (\theta^q - \theta)\zeta_A(n).$$

By a theorem of Yu ([30], Theorem 2.3), it follows that $z = (\theta^q - \theta)x$. Thus

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + C_{\theta^q - \theta}^{\otimes n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \exp_\phi(z) = C_{\theta^q - \theta}^{\otimes n} \exp_{C^{\otimes n}}(x).$$

Since $\exp_\phi(x) \in M_{n \times 1}(A)$, there exists $y = (y_1, \dots, y_n)^t \in M_{n \times 1}(A)$ such that

$$C_\theta^{\otimes n}(y) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We obtain a contradiction since the last equation has no solutions in $M_{n \times 1}(A)$. The proof is finished. \square

4. TAE LMAN'S CONJECTURE FOR A CLASS OF t -MODULES

In this section, we prove Taelman's conjecture for a large class of t -modules which states that *if E/O_F is an A -finite t -module such that $\sigma N \subset (t - \theta)N$ where N is its associated dual t -motive, then Taelman's Conjecture is true for E/O_F* (see Theorem 4.4).

Remark 4.1. Let E be a mixed A -finite and uniformizable t -module (see [20], Sections 5.4 and 5.5). By [20], Definition 5.32, one can associate to E its Hodge-Pink structure and its Hodge-Pink weights. By the exact sequence (5.35) of [20], Section 5.6, the assumption $\sigma N(L) \subset (t - \theta)N(L)$ is equivalent to the condition that every Hodge-Pink weight of E is at least 1.

4.1. Setup.

Let F be a finite extension of K and let O_F be the integral closure of A in F . We set $F_\infty := F \otimes_K K_\infty$. Let E be a t -module defined over O_F , i.e. for all $a \in A$, we have $E_a \in M_{d \times d}(O_F)\{\tau\}$.

We denote by L the perfection of F and by L_∞ the completion of $L \otimes_K K_\infty$. Let $\tau : L_\infty \rightarrow L_\infty$ be the map which sends x to x^q . It is an isomorphism of \mathbb{F}_q -algebras.

Let $\sigma : L_\infty \rightarrow L_\infty$ be the map which maps x to $x^{1/q}$. Finally, **we assume that E is A -finite**, i.e. $N(L)$ is a free $L[t]$ -module of finite rank. We write

$$N(L) = \bigoplus_{i=1}^n L[t]v_i,$$

and we say that E is of rank n , i.e. $\text{rk } E = n$. Recall that

$$N(L) = \bigoplus_{i=1}^d L\{\sigma\}e_i,$$

where (e_1^t, \dots, e_d^t) is the canonical basis of $\text{Lie}(E)(L_\infty)$. We also have

$$t \cdot e_i = e_i \left(\sum_{i=0}^r \sigma^i E_{\theta,i}^* \right),$$

where $E_\theta = \sum_{i=0}^r E_{\theta,i}\tau^i$, $E_{\theta,i} \in M_{d \times d}(O_F)$, and $\partial_E(\theta) = E_{\theta,0}$.

We have set $N_0(L_\infty) = \bigoplus_{i=1}^d L_\infty e_i$. By Section 2, for all $k \geq 0$, we have constructed the maps $\varphi_k : N(L_\infty) \rightarrow N_0(L_\infty)$ verifying the following properties:

- i) $\varphi_j(h_1 + \lambda h_2) = \varphi_j(h_1) + \lambda^q \varphi_j(h_2)$ for all $h_1, h_2 \in N(L_\infty)$ and $\lambda \in L_\infty$,
- ii) $\varphi_j(t \cdot h) = \partial_E(\theta)\varphi_j(h)$ for all $h \in N(L_\infty)$,
- iii) $\varphi_j(\sigma^k h) = \varphi_{j-k}(h)$ for all $j, k \in \mathbb{N}$ and $h \in N(L_\infty)$ where we put $\varphi_n = 0$ if $n < 0$,
- iv) $\varphi_k(h) = \tau^k(h) L_k^*$ for all $k \geq 0$ and $h \in N_0(L_\infty)$ where $\log_E = \sum_{k \geq 0} L_k \tau^k$ with $L_k \in M_{d \times d}(L_\infty)$.

4.2. A stabilization result.

Proposition 4.2. *With the notation of Section 4.1, there exists a sub- K_∞ -vector space V' of $N_0(F_\infty)$ verifying the following properties:*

- 1) *We have $\dim_{K_\infty} V' \leq nm = \text{rk } E \cdot [F : K]$.*
- 2) *For all k sufficiently large, the sub- K_∞ -vector space of $N_0(F_\infty)$ generated by $\varphi_k(N_0(F_\infty))$ is always contained in V' .*

Proof. We choose elements x_1, \dots, x_m such that $F = \bigoplus_{l=1}^m Kx_l$. For $i = 1, \dots, d$, we put

$$e_i = \sum_{j=1}^n f_{i,j}(t)v_j, \quad f_{i,j}(t) \in L[t].$$

Then there exists an integer $k_0 \in \mathbb{N}$ such that for all i, j , we have $f_{i,j}(t)^{(k_0)} \in F[t]$. It follows that if $k \geq k_0$, we get

$$f_{i,j}(t)^{(k)} \in F^{q^{k-k_0}}[t], \quad \text{for all } i, j.$$

Therefore, if $q^{k-k_0} \geq d$, then

$$\varphi_k(e_i) \in \sum_{l=1}^m \sum_{j=1}^n \partial_E(K_\infty) x_l^q x_l^{k-k_0} \varphi_k(v_j).$$

Next for $i = 1, \dots, n$, we put

$$\sigma v_i = \sum_{j=1}^n g_{i,j}(t)v_j, \quad g_{i,j}(t) \in L[t].$$

Then there exists an integer $k_1 \in \mathbb{N}$ such that for all i, j , $g_{i,j}(t)^{(k_1)} \in F[t]$. Therefore, if $q^{k-k_1} \geq d$, then

$$\varphi_{k-1}(v_i) = \varphi_k(\sigma v_i) \in \sum_{l=1}^m \sum_{j=1}^n \partial_E(K_\infty) x_l^{q^{k-k_1}} \varphi_k(v_j).$$

Now observe that in $M_{d \times d}(F_\infty)$, if $q^k \geq d$, we have

$$\sum_{l=1}^m \partial_E(K_\infty) x_l^{q^k} I_d = \partial_E(K_\infty) F^{q^k} I_d.$$

Let s be the least integer such that $s \geq \max\{k_0, k_1\}$ and $q^{s-k_0} \geq d$, $q^{s-k_1} \geq d$. We set $k_2 := \inf\{s - k_0, s - k_1\}$. Then by the previous discussion, for all $k \geq s$, we get

$$\varphi_k(e_i) \in \sum_{l=1}^m \sum_{j=1}^n \partial_E(K_\infty) x_l^{q^{k_2}} \varphi_k(v_j),$$

and

$$\varphi_{k-1}(v_i) = \varphi_k(\sigma v_i) \in \sum_{l=1}^m \sum_{j=1}^n \partial_E(K_\infty) x_l^{q^{k_2}} \varphi_k(v_j).$$

For $k \geq s$, we set

$$V'_k := \sum_{l=1}^m \sum_{j=1}^n \partial_E(K_\infty) x_l^{q^{k_2}} \varphi_k(v_j), \quad V' := \cup_{k \geq s} V'_k,$$

$$V_k := \partial_E(K_\infty) \varphi_k(N_0(F_\infty)) \subset N_0(F_\infty), \quad V := \cup_{k \geq s} V_k.$$

It follows that for all $k \geq s$, we have the inclusions $V_k \subset V'_k$, and $V'_{k-1} \subset V'_k$. Since $\dim_{K_\infty} V'_k \leq nm = \text{rk } E[F : K]$, we deduce that for k sufficiently large, $V' = V'_k$ and

$$\dim_{K_\infty} V' \leq nm.$$

Since $V_k \subset V \subset V'$, the Theorem follows. \square

4.3. An injectivity result.

We will give a stronger version of Proposition 4.2 under the assumption that $\sigma N(L) \subset (t - \theta)N(L)$.

Proposition 4.3. *Suppose further that $\sigma N(L) \subset (t - \theta)N(L)$. Then there exists a sub- K_∞ -vector space W of $N_0(F_\infty)$ verifying the following properties:*

- 1) We have $\dim_{K_\infty} W = nm = \text{rk } E \cdot [F : K]$.
- 2) For all k sufficiently large, the sub- K_∞ -vector space of $N_0(F_\infty)$ generated by $\varphi_k(N_0(F_\infty))$ is always contained in W .
- 3) The projection $w : N_0(F_\infty) \rightarrow \frac{N_0(F_\infty)}{(\partial_E(\theta) - \theta I_d)N_0(F_\infty)}$ induces an isomorphism of K_∞ -vector spaces

$$W \simeq \frac{N_0(F_\infty)}{(\partial_E(\theta) - \theta I_d)N_0(F_\infty)}.$$

Proof. Recall that $\{v_1, \dots, v_n\}$ is an $L[t]$ -basis of the free $L[t]$ -module $N(L)$. Recall also that $w : N_0(F_\infty) \rightarrow \frac{N_0(F_\infty)}{(\partial_E(\theta) - \theta I_d)N_0(F_\infty)}$ and for k sufficiently large,

$$V' = \sum_{l=1}^m \sum_{j=1}^n \partial_E(K_\infty) x_l^{q^{k_2}} \varphi_k(v_j),$$

where (x_1, \dots, x_m) be a K -basis of F .

We claim that $w|_{V'}$ is injective. In fact, for $j = 1, \dots, n$ we write

$$(t - \theta)^d \dots (t - \theta^{q^{1-k}})^d v_j = \sigma^k y_j, \quad y_j \in N(L).$$

Recall that $\ell_k = (\theta - \theta^q) \dots (\theta - \theta^{q^k})$, it follows that

$$\varphi_k(v_j) \equiv \frac{1}{\ell_k^d} y_j \pmod{(t - \theta)N(L_\infty)}.$$

Suppose that there exist elements $\delta_{j,l} \in K_\infty$ such that

$$\sum_{l=1}^m \sum_{j=1}^n \delta_{j,l} x_l^{q^{k_2}} w(y_j) = 0.$$

In other words,

$$\sum_{l=1}^m \sum_{j=1}^n \delta_{j,l} x_l^{q^{k_2}} y_j \in (t - \theta)N(L_\infty).$$

It implies that

$$\sigma^k \left(\sum_{l=1}^m \sum_{j=1}^n \delta_{j,l} x_l^{q^{k_2}} y_j \right) \in (t - \theta^{q^{-k}})N(L_\infty).$$

Hence

$$\sum_{j=1}^n \left(\sum_{l=1}^m \delta_{j,l}^{q^{-k}} x_l^{q^{k_2-k}} \right) (t - \theta)^d \dots (t - \theta^{q^{1-k}})^d v_j \in (t - \theta^{q^{-k}})N(L_\infty).$$

We deduce that for $j = 1, \dots, n$,

$$\sum_{l=1}^m \delta_{j,l} x_l^{q^{k_2}} = 0.$$

If F/K is a separable extension, then $(x_1^{q^{k_2}}, \dots, x_m^{q^{k_2}})$ is still a K_∞ -basis of F_∞ . We get immediately that $\delta_{j,l} = 0$ for all j, l . Thus $\dim_{K_\infty} w(V') = nm$. Since $\dim_{K_\infty} V' \leq nm$, it follows that $w|_{V'}$ is injective and $\dim_{K_\infty} V' = nm$. We set $W := V'$.

Now suppose that F/K is not separable. We have already observed that

$$\sum_{l=1}^m \partial_E(K_\infty) x_l^{q^{k_2}} I_d = \partial_E(K_\infty) F^{q^{k_2}} I_d.$$

We choose k_2 sufficiently large such that every element of $F^{q^{k_2}}$ is separable over K . Thus $F^{\text{sep}} := KF^{q^{k_2}}$ is the maximal separable extension of K contained in F . We rearrange x_1, \dots, x_m such that for some $1 \leq r \leq m$, we have

$$F^{\text{sep}} = \bigoplus_{l=1}^r K x_l^{q^{k_2}}.$$

It follows that for all $k \geq k_2$, $F^{\text{sep}} = \bigoplus_{l=1}^r Kx_l^{q^k}$. Further, for $i = r+1, \dots, m$, we write

$$x_i^{q^{k_2}} = \sum_{l=1}^r a_{i,j} x_l^{q^{k_2}}, \quad a_{i,j} \in K.$$

It implies that for k sufficiently large,

$$\sum_{l=1}^m \partial_E(K_\infty) x_l^{q^k} = \sum_{l=1}^r \partial_E(K_\infty) x_l^{q^k}.$$

Thus for k_2 chosen sufficiently large, we deduce again that $w|_{V'}$ is injective and $\dim_{K_\infty} V' = n[F : K]_{\text{sep}}$. We choose $y_1, \dots, y_h \in F$ such that $F = F^{\text{sep}} \oplus \bigoplus_{l=1}^h Ky_l$ and we set

$$W := V' + \sum_{l=1}^h \sum_{j=1}^n \partial_E(K_\infty) y_l \varphi_k(v_j).$$

By similar arguments, we show that $w|_W$ is injective and

$$\dim_{K_\infty} W = nm.$$

Now we claim that

$$N_0(F_\infty) \cap (t - \theta)N(L_\infty) = (\partial_E(\theta) - \theta I_d)N_0(F_\infty).$$

In fact, for $x \in N_0(F_\infty)$, we have $(t - \theta)x \equiv (\partial_E(\theta) - \theta I_d)x \pmod{\sigma}$. It implies the inclusion

$$(\partial_E(\theta) - \theta I_d)N_0(F_\infty) \subset N_0(F_\infty) \cap (t - \theta)N(L_\infty).$$

On the other hand, let $x \in N_0(F_\infty)$ such that $x = (t - \theta)y$ for some $y \in N(L_\infty)$. We have $y \in \bigoplus_{j=1}^n F_\infty[t]v_j$. We know that $y \equiv y' \pmod{\sigma}$ for some $y' \in N_0(F_\infty)$. It follows that

$$(t - \theta)y \equiv (\partial_E(\theta) - \theta I_d)y' \pmod{\sigma}.$$

Hence

$$x = (\partial_E(\theta) - \theta I_d)y' \in (\partial_E(\theta) - \theta I_d)N_0(F_\infty).$$

From the above discussion, it follows that the natural map of K_∞ -vector spaces $w : N_0(F_\infty) \rightarrow \frac{N_0(F_\infty)}{(\partial_E(\theta) - \theta I_d)N_0(F_\infty)}$ induces an isomorphism of K_∞ -vector spaces

$$W \simeq \frac{N_0(F_\infty)}{(\partial_E(\theta) - \theta I_d)N_0(F_\infty)}.$$

The proof is finished. \square

4.4. Main result.

Recall that the map $\delta_0 : N(L)/\sigma N(L) \rightarrow \text{Lie}(E)(L)$ induces an isomorphism of A -modules $N_0(F_\infty) \simeq \text{Lie}(E)(F_\infty)$ (see Section 1).

Theorem 4.4. *Let E/O_F be an A -finite t -module such that $\sigma N(L) \subset (t - \theta)N(L)$. Then there exists a sub- K_∞ -vector space W of $\text{Lie}(E)(F_\infty)$ such that the following properties hold:*

1) *The natural map of K_∞ -vector spaces $\text{Lie}(E)(F_\infty) \rightarrow \frac{\text{Lie}(E)(F_\infty)}{(\partial_E(\theta) - \theta I_d)\text{Lie}(E)(F_\infty)}$ induces an isomorphism of K_∞ -vector spaces*

$$W \simeq \frac{\text{Lie}(E)(F_\infty)}{(\partial_E(\theta) - \theta I_d)\text{Lie}(E)(F_\infty)}.$$

2) We have the following inclusion

$$U(E/O_F) \subset \text{Lie}(E)(F) + W.$$

Moreover, $U(E/O_F) \cap W$ and $\text{Lie}(E)(O_F) \cap W$ are A -lattices in W and

$$\left[(\text{Lie}(E)(O_F) \cap W) : (U(E/O_F) \cap W) \right]_A = \alpha L(E/O_F)$$

for some $\alpha \in K^\times$.

In particular, Conjecture 3.5 holds for E/O_F .

Proof. We set $\log_E = \sum_{k \geq 0} L_k \tau^k$, $L_k \in M_{d \times d}(F)$. By Proposition 4.3 there exist a sub- K_∞ -vector space W of $\text{Lie}(E)(F_\infty)$ and an integer $k_2 \geq 0$ such that

i) for all $k \geq k_2$, $L_k \tau^k(\text{Lie}(E)(F_\infty)) \subset W$,

ii) the natural map of K_∞ -vector spaces $\text{Lie}(E)(F_\infty) \rightarrow \frac{\text{Lie}(E)(F_\infty)}{(\partial_E(\theta) - \theta I_d) \text{Lie}(E)(F_\infty)}$ induces an isomorphism of K_∞ -vector spaces $W \simeq \frac{\text{Lie}(E)(F_\infty)}{(\partial_E(\theta) - \theta I_d) \text{Lie}(E)(F_\infty)}$.

Let $u \in U_{St}(E/O_F)$. Then there exists a polynomial $\sum_{k=0}^N b_k z^k \in M_{d,1}(O_F[z])$ such that

$$\sum_{i \geq 0} \sum_{k+l=i} z^i L_l \tau^l(b_k) \in M_{d,1}(F_\infty \otimes_{K_\infty} \mathbb{T}_z(K_\infty)),$$

and

$$u = \sum_{i \geq 0} \sum_{k+l=i} z^i L_l \tau^l(b_k) \Big|_{z=1}.$$

For $i \geq N + q^{k_2}$, we have

$$\sum_{k+l=i} L_l \tau^l(b_k) \in W.$$

Since $\frac{U(E/O_F)}{U_{St}(E/O_F)}$ is a finite A -module, it implies that

$$U(E/O_F) \subset \text{Lie}(E)(F) + W.$$

It is clear that $\text{Lie}(E)(O_F) \cap W$ is an A -lattice in W .

Finally, we have

$$\dim_{K_\infty} \frac{\text{Lie}(E)(F_\infty) + W}{W} = (d - n)[F : K],$$

where $n[F : K] = \dim_{K_\infty} \frac{\text{Lie}(E)(F_\infty)}{(\partial_E(\theta) - \theta I_d) \text{Lie}(E)(F_\infty)}$. Therefore $U(E/O_F) \cap W$ is an A -lattice in W . The last assertions follow immediately from the class formula (Theorem 3.1). \square

As an immediate consequence, we obtain the following result:

Corollary 4.5. *Let E/O_F be an A -finite t -module. Then Conjecture 3.5 is true for the tensor product $E \otimes C$.*

Proof. We denote by N_C the dual t -motive attached to C . It is isomorphic to $L[t]$ where σ acts as follows: for $h = \sum_{k=0}^n \alpha_k t^k \in L[t]$,

$$\sigma h = \sigma \left(\sum_{k=0}^n \alpha_k t^k \right) = (t - \theta) \sum_{k=0}^n \alpha_k \frac{1}{t} t^k.$$

Let $N(L)$ be the dual t -motive associated to E . Then the dual t -motive $N_{E \otimes C}$ associated to $E \otimes C$ is defined to be the tensor product $N(L) \otimes_{L[t]} N_C$ as an $L[t]$ -module on which σ acts diagonally. Note that $E \otimes C$ is also A -finite because E is A -finite. Further, we see that $\sigma(N_{E \otimes C}) \subset (t - \theta)N_{E \otimes C}$. Therefore we can apply Theorem 4.4 to $E \otimes C$ to obtain the Corollary. \square

By Remark 4.1, the above Theorem implies:

Corollary 4.6. *Let E/O_F be a mixed A -finite and uniformizable t -module whose Hodge-Pink weights are at least 1. Then Conjecture 3.5 holds for E .*

5. A DETAILED EXAMPLE: TENSOR POWERS OF THE CARLITZ MODULE

In this section, we study in details the case of tensor powers of the Carlitz module over any finite extension of K . As an application, we obtain various log-algebraicity theorems on tensor powers of the Carlitz module.

5.1. Taelman's conjecture for tensor powers of the Carlitz module.

Let F be a finite extension of K and O_F be the integral closure of A in F . Let $x_1, \dots, x_m \in F$ such that $F = \bigoplus_{j=1}^m Kx_j$. We fix a K -embedding of F in \mathbb{C}_∞ . Let $n \geq 1$ be an integer. We define the n -th tensor power of the Carlitz module $C^{\otimes n} : A \rightarrow M_{n \times n}(\mathbb{C}_\infty)\{\tau\}$ by

$$C_\theta^{\otimes n} = \begin{pmatrix} \theta & 1 & \cdots & 0 \\ & \theta & \ddots & \vdots \\ & & \theta & 1 \\ & & & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \tau.$$

The map $\partial_{C^{\otimes n}} : A \rightarrow M_{n \times n}(\mathbb{C}_\infty)$ is given by

$$\partial_{C^{\otimes n}}(\theta) = \begin{pmatrix} \theta & 1 & \cdots & 0 \\ & \theta & \ddots & \vdots \\ & & \theta & 1 \\ & & & \theta \end{pmatrix}.$$

We observe that $C^{\otimes n}$ is defined over O_F .

The corresponding dual t -motive of $C^{\otimes n}$ is $N = \mathbb{C}_\infty[t]$ whose action of σ is given by $\sigma \cdot h := (t - \theta)^h h^{(-1)}$ for all $h \in N$. In other words, if we write $h = \sum_{i \geq 0} a_i t^i$ with $a_i \in \mathbb{C}_\infty$, then

$$\sigma \left(\sum_{i \geq 0} a_i t^i \right) = (t - \theta)^n \left(\sum_{i \geq 0} a_i^{1/q} t^i \right).$$

We observe that $\{(t - \theta)^i\}_{0 \leq i \leq n-1}$ is a basis of N as a $\mathbb{C}_\infty\{\sigma\}$ -module

$$N = \sum_{i=0}^{n-1} \mathbb{C}_\infty\{\sigma\}(t - \theta)^i.$$

We set

$$N_0 := \sum_{i=0}^{n-1} \mathbb{C}_\infty (t - \theta)^i.$$

Recall that the canonical map of A -modules $\delta_0 : N \rightarrow \text{Lie}(C^{\otimes n})(\mathbb{C}_\infty)$ which maps $\sum_{i=0}^{n-1} (\sum_{j \geq 0} \sigma^j a_{i,j})(t - \theta)^i$ to $(a_{d-1,0}, \dots, a_{1,0})^t$ induces an isomorphism of A -modules $N/\sigma N \simeq \text{Lie}(C^{\otimes n})(\mathbb{C}_\infty)$.

Let v be the place of F above ∞ that corresponds to the K -embedding of F into \mathbb{C}_∞ , then $F_v = FK_\infty \subset \mathbb{C}_\infty$. Furthermore, we have

$$N_0(F_v) = \sum_{i=0}^{n-1} \sum_{j=1}^m K_\infty x_j (t - \theta)^i.$$

Then as a consequence of the proof of Proposition 4.2, we get:

Proposition 5.1. *We set $\log_{C^{\otimes n}} = \sum_{k \geq 0} L_k \tau^k$ with $L_k \in M_{d \times d}(K)$. Let k_0 be the smallest integer such that $q^{k_0} \geq n$. For $k \in \mathbb{N}$, we denote by $V_k(F_\infty) \subset \text{Lie}(C^{\otimes n})(F_\infty)$ the K_∞ -vector space (via $\partial_{C^{\otimes n}}$) generated by $L_k \tau^k(x)$ where x runs through the set $\text{Lie}(C^{\otimes n})(F_\infty)$.*

Then for $k \geq k_0$, we have

$$V_k(F_\infty) \subset V_{k_0}(F_\infty) =: W(F_\infty).$$

By Theorem 4.4, Taelman's conjecture is true for $C^{\otimes n}/O_F$:

Theorem 5.2. *With the above notation, we have*

$$U(C^{\otimes n}/F) \subset \text{Lie}(C^{\otimes n})(F) + W(F_\infty).$$

Furthermore, $U(C^{\otimes n}/O_F) \cap W(F_\infty)$ and $\text{Lie}(C^{\otimes n})(O_F) \cap W(F_\infty)$ are A -lattices in $W(F_\infty)$, and

$$\frac{[(\text{Lie}(C^{\otimes n})(O_F) \cap W(F_\infty)) : (U(C^{\otimes n}/O_F) \cap W(F_\infty))]_A}{L(C^{\otimes n}/O_F)} \in K^\times.$$

In particular, Conjecture 3.5 holds for $C^{\otimes n}/O_F$.

5.2. Log-algebraicity for tensor powers of the Carlitz module.

In this section, we apply our techniques to obtain log-algebraicity identities for $C^{\otimes n}/A$.

We take $F = K$ and consider $C^{\otimes n}/A$ as a t -module defined over A . The L -value of $C^{\otimes n}/A$ at 1 is known to be equal to the Carlitz zeta value at n given by

$$L(C^{\otimes n}/A) = \zeta_A(n) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \in K_\infty.$$

We also set

$$\zeta_A(n, z) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} z^d \in \mathbb{T}_z(K_\infty).$$

Recall that k_0 be the smallest integer such that $q^{k_0} \geq n$. We define $u_n := (y_{n-1}, \dots, y_0)^t \in M_{n \times 1}(K)$ by

$$(5.1) \quad u_n := \varphi_{k_0}(1) = \frac{1}{((t - \theta^q) \dots (t - \theta^{q^{k_0}}))^n} \equiv \sum_{i=0}^{n-1} y_i (t - \theta)^i \pmod{(t - \theta)^n}.$$

We deduce that

$$W(K_\infty) = \partial_{C^{\otimes n}}(K_\infty)u_n.$$

We are ready to state a log-algebraicity theorem for $C^{\otimes n}/A$.

Theorem 5.3.

1) *There exists $b \in A \setminus \{0\}$ such that*

$$\partial_{C^{\otimes n}}(b \zeta_A(n)) \cdot u_n \in U_{\text{St}}(C^{\otimes n}/A).$$

In particular,

$$\exp_{C^{\otimes n}} \left(\left(\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{\partial_{C^{\otimes n}}(a)^n} \right) \cdot \partial_{C^{\otimes n}}(b) u_n \right) \in M_{n \times 1}(A).$$

2) *There exists $b(z) \in A[z] \setminus \{0\}$ such that*

$$\partial_{\widetilde{C^{\otimes n}}}(b(z) \zeta_A(n, z)) \cdot u_n \in U(\widetilde{C^{\otimes n}}/A[z])$$

where $U(\widetilde{C^{\otimes n}}/A[z])$ is defined in Definition 3.2. In particular, we have

$$\exp_{\widetilde{C^{\otimes n}}}(\partial_{\widetilde{C^{\otimes n}}}(\zeta_A(n, z)) \cdot \partial_{\widetilde{C^{\otimes n}}}(b(z))u_n) \in M_{n \times 1}(A[z]).$$

Proof.

1) Theorem 5.2 implies that

$$U_{\text{St}}(C^{\otimes n}/A) \subset \text{Lie}(C^{\otimes n})(K) + W(K_\infty) = \text{Lie}(C^{\otimes n})(K) + \partial_{C^{\otimes n}}(K_\infty)u_n.$$

We observe that $\dim_{K_\infty} W(K_\infty) = 1$. Combining with the class formula for $C^{\otimes n}/A$ (Theorem 3.1), we deduce that there exists $b \in A \setminus \{0\}$ such that

$$\partial_{C^{\otimes n}}(b L(C^{\otimes n}/A))u_n = \partial_{C^{\otimes n}}(b \zeta_A(n))u_n \in U_{\text{St}}(C^{\otimes n}/A).$$

Thus we get Part 1.

2) By similar arguments, Part 2 follows from the class formula for $C^{\otimes n}/A$ with an extra variable proved by Demeslay (see [14]). \square

5.3. Relations with the works of Anderson-Thakur and Papanikolas.

We keep the notation of Section 5.2. Using Anderson's method ([2, 3]), Papanikolas ([22], Theorem 7.3.3) obtained an explicit log-algebraicity theorem for tensor powers of the Carlitz module which generalizes the fundamental theorem of Anderson and Thakur ([4], Theorem 3.8.3). In this section, we will present another proof of Papanikolas' theorem as a direct consequence of Theorem 5.3.

Recall that (see Definition 3.2)

$$U(\widetilde{C^{\otimes n}}/A[z]) = \{x \in \text{Lie}_{\widetilde{C^{\otimes n}}}(\mathbb{T}_z(K_\infty)) \mid \exp_{\widetilde{C^{\otimes n}}}(x) \in \widetilde{C^{\otimes n}}(A[z])\}.$$

Lemma 5.4. *Let $x, y \in U(\widetilde{C^{\otimes n}}/A[z])$. Suppose that $\iota(x) = \iota(y)$ where ι is the projection on the last coordinate. Then $x = y$.*

Proof. We set $u := x - y \in U(\widetilde{C^{\otimes n}}/A[z])$. Then $\iota(u) = 0$. We have to show that $u = 0$.

We recall that $\text{ev} : \text{Lie}_{\widetilde{C^{\otimes n}}}(\mathbb{T}_z(K_\infty)) \rightarrow \text{Lie}(C^{\otimes n})(K_\infty)$ is the evaluation at $z = 1$. Since $u \in U(\widetilde{C^{\otimes n}}/A[z])$, it follows that $\text{ev}(u) \in U(C^{\otimes n}/A)$. Since $\iota(\text{ev}(u)) = 0$, by [30], Theorem 2.3, we deduce that $\text{ev}(u) = 0$. Thus $u = (z - 1)v$, and

$$(z - 1) \exp_{\widetilde{C^{\otimes n}}}(v) \in \widetilde{C^{\otimes n}}(A[z]).$$

Therefore

$$v \in U(\widetilde{C^{\otimes n}}/A[z]).$$

Furthermore $\iota(v) = 0$. We can then continue with the same arguments applied to v . Since $z - 1$ is irreducible in $\mathbb{T}_z(K_\infty)$, we deduce that $u = 0$. \square

Following Anderson and Thakur ([4], Section 3.8), we set

$$\gamma_0(t) = 1; \quad \gamma_j(t) = \prod_{l=1}^j (\theta^{t^l} - t^{t^l}), \quad j \geq 1.$$

Then the Anderson-Thakur polynomials $\alpha_n(t) \in A[t]$ ($n \in \mathbb{N}$) is defined by the generating series

$$\sum_{n \geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n = x \left(1 - \sum_{j \geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j} \right)^{-1}$$

where $\Gamma_n \in A$ is the factorial of n introduced by Carlitz (see [4], Section 3 for details). For $n \geq 1$, we put

$$\alpha_n(t) = \sum_{j=0}^m g_j t^j, \quad g_j \in A,$$

and we set

$$\mathfrak{z}_n(z) := \sum_{j=0}^m C_{g_j}^{\otimes n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta^j z \end{pmatrix} \in \widetilde{C^{\otimes n}}(A[z]).$$

Anderson and Thakur proved a fundamental theorem ([4], Theorem 3.8.3) which states that there exists $\mathfrak{z}_n(z) \in U(\widetilde{C^{\otimes n}}/A[z])$ such that $\iota(\mathfrak{z}_n(z)) = \Gamma_n \zeta_A(n, z)$ and

$$\exp_{\widetilde{C^{\otimes n}}}(\mathfrak{z}_n(z)) = \mathfrak{z}_n(z).$$

As an immediate consequence of Theorem 5.2 and Lemma 5.5, we get the following Proposition which is one of the main theorems of [22] (see [22], Theorem 7.3.3).

Proposition 5.5. *We have*

$$\mathfrak{z}_n(z) = \partial_{\widetilde{C^{\otimes n}}} \left(\zeta_A(n, z) \frac{\Gamma_n}{\iota(u_n)} \right) \cdot u_n.$$

5.4. Tensor powers of the Carlitz module over cyclotomic extensions.

For every $\zeta \in \overline{\mathbb{F}}_q$, we denote by $P \in A_+$ the prime such that $P(\zeta) = 0$. Then the \mathbb{F}_q -algebra homomorphism $\chi_\zeta : A \rightarrow \overline{\mathbb{F}}_q, b(\theta) \mapsto b(\zeta)$ induces a group isomorphism

$$\chi_\zeta : (A/PA)^\times \simeq \mathbb{F}_{q^{\deg P}}^\times.$$

Let $a \in A$ be a monic polynomial which is square-free. We set $\lambda_a := \exp_C(\tilde{\pi}/a) \in \mathbb{C}_\infty$ and denote by $F := K_a = K(\lambda_a)$ the a th cyclotomic extension of K . In this section, we consider $C^{\otimes n}$ as a t -module over O_F .

We recall that K_a/K is a finite abelian extension unramified outside A and ∞ whose Galois group $\Delta_a = \text{Gal}(K_a/K)$ is isomorphic to $(A/aA)^\times$. This isomorphism is given as follows: if $b \in A$ is prime to A , there exists a unique element $\sigma_b \in \Delta_a$ such that

$$\sigma_b(\lambda_a) = C_b(\lambda_a).$$

We set $\widehat{\Delta}_a = \text{Hom}(\Delta_a, \overline{\mathbb{F}}_q) \simeq \Delta_a$. For every character $\chi \in \widehat{\Delta}_a \setminus \{1\}$, there exist unique elements $\zeta_1, \dots, \zeta_l \in \overline{\mathbb{F}}_q$, and unique integers $n_1, \dots, n_l \in \{1, \dots, q-1\}$ such that for any $b \in A$ prime to A , we have

$$\chi(\sigma_b) = \prod_{k=1}^l \chi_{\zeta_k}(b)^{n_k}.$$

For $1 \leq k \leq l$, we denote by P_k the prime of A such that $P_k(\zeta_k) = 0$. By [26], Section 9.8, the Gauss-Thakur sum attached to the character χ is defined by

$$g(\chi) = \prod_{k=1}^l \left(\sum_{\substack{b \in A \setminus P_k A \\ \deg b < \deg P_k}} \chi_{\zeta_k}(b)^{-1} C_b(\lambda_{P_k}) \right)^{n_k}.$$

Finally, we set $g(1) = 1$.

We set

$$\eta_a := \sum_{\chi \in \widehat{\Delta}_a} g(\chi) \in O_F,$$

then we have $O_F = A[\Delta_a]\eta_a$.

We put $\mathbb{F} = \mathbb{F}_q(\chi(\Delta_a), \chi \in \widehat{\Delta}_a)$ and denote by $\tau : F_\infty \otimes_{\mathbb{F}_q} \mathbb{F} \rightarrow F_\infty \otimes_{\mathbb{F}_q} \mathbb{F}$ the \mathbb{F} -algebra homomorphism such that for all $x \in F_\infty$, $\tau(x) = x^q$. Then for $\chi \in \widehat{\Delta}_a$, we write $\chi = \prod_{k=1}^l \chi_{\zeta_k}^{n_k}$ with $\zeta_1, \dots, \zeta_l \in \mathbb{F}$ and $n_1, \dots, n_l \in \{1, \dots, q-1\}$ and we have

$$\tau(g(\chi)) = \prod_{k=1}^l (\zeta_k - \theta)^{n_k} g(\chi),$$

$$\delta(g(\chi)) = \chi(\delta)g(\chi), \quad \delta \in \Delta_a \simeq \text{Gal}(L(\mathbb{F})/K(\mathbb{F})).$$

The L -values attached to the character χ are defined by

$$L(n, \chi) = \sum_{d \geq 0} \sum_{a \in A_{d,+}} \frac{\prod_{k=1}^l (\chi_{\zeta_k}(a))^{n_k}}{a^n} \in K_\infty \otimes_{\mathbb{F}_q} \mathbb{F}, \quad n \geq 1,$$

where $\chi = \prod_{k=1}^l \chi_{\zeta_k}^{n_k}$.

We define $e_\chi = \frac{1}{|\Delta_a|} \sum_{\delta \in \Delta_a} \chi(\delta)^{-1} \delta \in \mathbb{F}[\Delta_a]$. The equivariant L -values are given by

$$L(n, \Delta_a) = \sum_{\chi \in \widehat{\Delta}_a} L(n, \chi) e_\chi \in K_\infty[\Delta_a]^\times, \quad n \geq 1.$$

Recall that $\iota : \text{Lie}(C^{\otimes n})(F_\infty) \rightarrow F_\infty$ be the projection on the last coordinate.

We recall the equivariant class formula for $C^{\otimes n}/O_F$.

Theorem 5.6. *Let $n \geq 1$ be an integer and recall that $L = K(\lambda_a)$. We have the following equality in $K_\infty[\Delta_a]$:*

$$L(n, \Delta_a) = [\text{Lie}(C^{\otimes n})(O_F) : U_{\text{St}}(C^{\otimes n}/O_F)]_{A[\Delta_a]}.$$

Proof. Using similar methods to those introduced in [8], one can prove (see [14] and [13], Theorem 4.15):

$$L(n, \Delta_a) = [\text{Lie}(C^{\otimes n})(O_F) : U(C^{\otimes n}/O_F)]_{A[\Delta_a]} \cdot \text{Fitt}_{A[\Delta_a]} H(C^{\otimes n}/O_F).$$

By similar arguments to the proof of Theorem 3.3, we have

$$\text{Fitt}_{A[\Delta_a]} H(C^{\otimes n}/O_F) = \text{Fitt}_{A[\Delta_a]} \frac{U(C^{\otimes n}/O_F)}{U_{\text{St}}(C^{\otimes n}/O_F)}.$$

The Theorem follows immediately. \square

We are ready to prove an equivariant generalization of Theorem 5.3.

Theorem 5.7. *Let $n \geq 1$ be an integer. Then there exists a free $A[\Delta_a]$ -module of rank one $M \subset U_{\text{St}}(C^{\otimes n}/O_F)$ such that $M = A[\Delta_a]L(n, \Delta_a)X_n$ for some vector $X_n \in M_{n \times 1}(O_F)$.*

In particular, $\exp_{C^{\otimes n}}(L(n, \Delta_a)X_n) \in M_{n \times 1}(O_F)$.

Proof. Recall that k_0 be the smallest integer such that $q^{k_0} \geq n$. By Theorem 5.2, we have

$$U_{\text{St}}(C^{\otimes n}/O_F) \subset \text{Lie}(C^{\otimes n})(L) + K_\infty[\Delta_a]L_{k_0} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tau^{k_0}(\eta_a) \end{pmatrix}.$$

Let $\chi \in \widehat{\Delta}_a$ be a character. We still denote by τ the continuous homomorphism of \mathbb{F} -vector spaces $\tau \otimes \mathbb{I}_{\mathbb{F}} : \text{Lie}(C^{\otimes n})(F_\infty) \otimes_{\mathbb{F}_q} \mathbb{F} \rightarrow \text{Lie}(C^{\otimes n})(F_\infty) \otimes_{\mathbb{F}_q} \mathbb{F}$. It follows that

$$e_\chi(U_{\text{St}}(C^{\otimes n}/O_F) \otimes_{\mathbb{F}_q} \mathbb{F}) \subset \text{Lie}(C^{\otimes n})(g(\chi)(K \otimes_{\mathbb{F}_q} \mathbb{F})) + (K_\infty \otimes_{\mathbb{F}_q} \mathbb{F}) L_{k_0} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tau^{k_0}(g(\chi)) \end{pmatrix}.$$

Theorem 5.6 implies

$$L(n, \chi) = [\text{Lie}(C^{\otimes n})(O_F \otimes_{\mathbb{F}_q} \mathbb{F}) : e_\chi(U_{\text{St}}(C^{\otimes n}/O_F) \otimes_{\mathbb{F}_q} \mathbb{F})]_{A \otimes_{\mathbb{F}_q} \mathbb{F}}.$$

We conclude that there exists $X_n(\chi) \in e_\chi(\text{Lie}(C^{\otimes n})(O_F) \otimes_{\mathbb{F}_q} \mathbb{F})$ such that

$$1) Z_n(\chi) := L(n, \chi)X_n(\chi) \in e_\chi(U_{\text{St}}(C^{\otimes n}/O_F) \otimes_{\mathbb{F}_q} \mathbb{F}),$$

2) $\iota(X_n(\chi)) = b(\chi)g(\chi)$ for some $b(\chi) \in A \otimes_{\mathbb{F}_q} \mathbb{F} \setminus \{0\}$ where we recall that ι is the projection on the last coordinate.

Further, we can choose elements $X_n(\chi)$ such that $X_n(\chi^\sigma) = \sigma(X_n(\chi))$ for any $\sigma \in \Delta_a$. We set

$$X_n = \sum_{\chi \in \widehat{\Delta}_a} X_n(\chi) \in \text{Lie}(C^{\otimes n})(O_F)$$

and

$$Z_n = \sum_{\chi \in \widehat{\Delta}_a} Z_n(\chi) \in U_{\text{St}}(C^{\otimes n}/O_F).$$

It follows that

1) $Z_n = L(n, \Delta_a)X_n$ and $M = A[\Delta_a]Z_n \subset U_{\text{St}}(C^{\otimes n}/O_F)$ is a free $A[\Delta_a]$ -module of rank one,

2) $\iota(M) = b(\Delta_a)L(n, \Delta_a)O_F$ where $b(\Delta_a) = \sum_{\chi \in \widehat{\Delta}_a} b(\chi)e_\chi \in A[\Delta_a] \cap K[\Delta_a]^\times$.

The proof is finished. \square

As an immediate corollary, if we project on the last coordinate, we obtain an equivariant generalization of Anderson-Thakur's theorem ([4], Theorem 3.8.3):

Corollary 5.8. *Let $n \geq 1$ be an integer. Then there exists a free $A[\Delta_a]$ -module of rank one $M \subset U_{\text{St}}(C^{\otimes n}/O_F)$ such that*

$$\iota(M) = b(\Delta_a)L(n, \Delta_a)$$

for some $b(\Delta_a) \in A[\Delta_a] \cap K[\Delta_a]^\times$.

APPENDIX A. A COUNTEREXAMPLE TO TAE LMAN'S CONJECTURE

In general, Taelman's conjecture is not always true. In this section, we will present a simple counterexample which is a non-trivial extension of the Carlitz module by itself, i.e. we have an exact sequence of t -modules $0 \rightarrow C \rightarrow E \rightarrow C \rightarrow 0$ that does not split. We should mention that this counterexample was independently constructed by Taelman ¹.

Let F/K be a finite extension and let O_F be the integral closure of A in F . Let $E : A \rightarrow M_{2 \times 2}(O_F)\{\tau\}$ be the t -module given by

$$E_\theta = \begin{pmatrix} \theta & \alpha \\ 0 & \theta \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tau, \quad \alpha \in O_F \setminus \{0\}.$$

Thus $\partial_E(\theta) = \begin{pmatrix} \theta & \alpha \\ 0 & \theta \end{pmatrix}$.

Let $\iota : \text{Lie}(E)(F_\infty) \rightarrow F_\infty$ be the projection on the second coordinate, then ι induces an exact sequence of K_∞ -vector spaces:

$$0 \rightarrow (\partial_E(\theta) - \theta I_2) \text{Lie}(E)(F_\infty) \rightarrow \text{Lie}(E)(F_\infty) \rightarrow F_\infty \rightarrow 0.$$

¹L. Taelman, private communication, Sep. 2018.

A.1. The dual t -motive attached to E .

We write down explicitly the dual t -motive attached N over \mathbb{C}_∞ to E . Then $N = \mathbb{C}_\infty\{\sigma\}e_1 \oplus \mathbb{C}_\infty\{\sigma\}e_2$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ where t acts via right multiplication by the matrix $\begin{pmatrix} \theta + \sigma & 0 \\ \alpha & \theta + \sigma \end{pmatrix}$. It follows that

$$(t - \theta)e_1 = \sigma e_1,$$

$$(t - \theta)e_2 = \alpha e_1 + \sigma e_2.$$

Thus we obtain $N = \mathbb{C}_\infty[t]e_1 \oplus \mathbb{C}_\infty[t]e_2$, $N_0 = \mathbb{C}_\infty e_1 \oplus \mathbb{C}_\infty e_2$. As a consequence, the t -module E is A -finite. We observe that $\sigma e_2 \notin (t - \theta)N$ since $\alpha \neq 0$.

Further, one sees that E is pure of weight 1 and the Hodge-Pink weights of E are 0 and 2. In fact, the calculations are done with the dual t -motive N and follow the same lines as those given in [20], Example 4.32.

A.2. Computation of $L(E/O_F)$.

Let \mathfrak{P} be a maximal ideal of O_F and let $P \in A$ be the monic generator of the maximal ideal $\mathfrak{P} \cap A$ of A . We put $f_{\mathfrak{P}} = [O_F/\mathfrak{P} : A/P]$. The projection on the second coordinate induces two exact sequences:

$$0 \rightarrow \text{Lie}(C)\left(\frac{O_F}{\mathfrak{P}}\right) \rightarrow \text{Lie}(E)\left(\frac{O_F}{\mathfrak{P}}\right) \rightarrow \text{Lie}(C)\left(\frac{O_F}{\mathfrak{P}}\right) \rightarrow 0,$$

$$0 \rightarrow C\left(\frac{O_F}{\mathfrak{P}}\right) \rightarrow E\left(\frac{O_F}{\mathfrak{P}}\right) \rightarrow C\left(\frac{O_F}{\mathfrak{P}}\right) \rightarrow 0.$$

Thus we obtain

$$[\text{Lie}(E)\left(\frac{O_F}{\mathfrak{P}}\right)]_A = P^2 f_{\mathfrak{P}},$$

$$[E\left(\frac{O_F}{\mathfrak{P}}\right)]_A = (P^{f_{\mathfrak{P}}} - 1)^2.$$

It follows that

$$L(E/O_F) = \zeta_{O_F}(1)^2 \in K_\infty.$$

A.3. Computation of units and Taelman's conjecture.

We put $\exp_E = \sum_{i \geq 0} E_i \tau^i$, $E_i \in M_{2 \times 2}(F)$ and $E_0 = I_2$. Recall that $\exp_C = \sum_{j \geq 0} \frac{1}{D_j} \tau^j$ where $D_0 = 1$ and for $j \geq 1$, $D_j = (\theta^{q^j} - \theta)D_{j-1}^q$. From the functional equation $\exp_E \partial_E(\theta) = E_\theta \exp_E$, it follows that for $i \geq 1$, we have

$$E_i \partial_E(\theta)^{(i)} - \partial_E(\theta) E_i = E_{i-1}^{(1)}.$$

We deduce that

$$E_i = \begin{pmatrix} \frac{1}{D_i} & e_i \\ 0 & \frac{1}{D_i} \end{pmatrix} \quad \text{for some } e_i \in F.$$

Thus

$$\exp_E = \begin{pmatrix} \exp_C & G_\alpha \\ 0 & \exp_C \end{pmatrix},$$

where $G_\alpha \in F\{\{\tau\}\}$ converges on F_∞ . Thus \exp_E is surjective on $M_{2 \times 1}(\mathbb{C}_\infty)$. Therefore, the t -module E is uniformizable.

Remark A.1. As the referee pointed out, the fact that the t -module E is uniformizable can follow directly from [20]. We are grateful to the referee to share the ideas with us and present below his proof.

In fact, we denote by N_C the dual t -motive attached to C . Then we have a short exact sequence of dual t -motives $0 \rightarrow N_C \rightarrow N \rightarrow N_C \rightarrow 0$. By [20], Lemma 4.20, N is uniformizable since N_C is uniformizable. Thus the t -module E is also uniformizable by [20], Theorem 5.28.

Note also that, if $\begin{pmatrix} x \\ y \end{pmatrix} \in U(E/O_F)$, then we have $\exp_C(y) \in O_F$. Recall that the map of K_∞ -vector spaces $w : \text{Lie}(E)(F_\infty) \rightarrow \frac{\text{Lie}(E)(F_\infty)}{(\partial_E(\theta) - \theta I_2) \text{Lie}(E)(F_\infty)}$ can be identified with the projection on the last coordinate ι . In particular, $w|_{U(E/O_F)}$ is not injective.

For simplicity, we identify w and ι . Suppose that Taelman's conjecture (Conjecture 3.5) holds for E/O_F . It means that there exists $Z \subset U(E/O_F)$ of A -rank $[F : K]$ such that in F_∞ , we have

$$[O_F : \iota(Z)]_A = \beta L(E/O_F), \quad \beta \in K^\times.$$

However, we have seen that $\iota(Z) \subset U(C/O_F)$. It implies that

$$[U(C/O_F) : \iota(Z)]_A \in A \setminus \{0\}.$$

By Taelman's class formula for C/O_F (Theorem 3.1), we obtain

$$[O_F : U(C/O_F)]_A = \beta' L(C/O_F), \quad \beta' \in K^\times.$$

Therefore,

$$[O_F : \iota(Z)] = \beta'' L(C/O_F), \quad \beta'' \in K^\times.$$

Since $L(C/O_F) = \zeta_{O_F}(1)$ and $L(E/O_F) = \zeta_{O_F}(1)^2$, we deduce

$$\zeta_{O_F}(1) \in K^\times,$$

which is conjecturally a contradiction.

For $F = K$, it is known that $\zeta_A(1)$ is transcendental over K (see [29, 30]). We have proved:

Proposition A.2. *Let $\alpha \in A \setminus \{0\}$. Let $E : A \rightarrow M_{2 \times 2}(A)\{\tau\}$ be the t -module defined over A defined by*

$$E_\theta = \begin{pmatrix} \theta & \alpha \\ 0 & \theta \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tau.$$

Then Conjecture 3.5 is false for E/A .

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