

Energy efficient communication with neuro-inspired detector

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Abstract—This paper focuses on the theoretical performance of a neuro-inspired digital communication, where the input signal is below the activation threshold of the neuron, making the noise crucial for the detection. We first derive the error probability of a neuro-inspired detector build upon a single Integrate-and-Fire (IF) neuron and then extend this result to a IF-based detector build upon multiple neurons. In this case, we propose to optimize the number of neurons in parallel, leading to an error probability of below 10^{-4} for a large noise range, proving the strength of such a IF-based detector.

I. INTRODUCTION

With the introduction of various communicating devices, Internet of Things (IoT) becomes more and more concrete. *Ad hoc* sensor networks designed, for instance for smart cities, smart buildings or 5G, create on one hand an increasing demand for transmission rates but on the other hand suffer from their energy consumption. Indeed, one major source of energy loss is the radio transceiver. Staying in an idle mode, while waiting for incoming data, costs a large amount of energy to a node. In order to achieve energy efficient communications, two main solutions have been proposed in the literature: either receivers wake-up in a periodic fashion, or they are waken-up when receiving a specific signal, as for instance the node ID in an *ad hoc* network, which is called wake-up radio. The latter seems to offer a promising decrease of energy waste. Unfortunately, most of these devices are build upon micro-controllers that consume a large amount of energy, decreasing thus the energy efficiency of the wake-up receivers.

To overcome thus energy consumption issue due to the use of standard micro-controllers, many research are turning to neuro-science: our brain is indeed a very good example of a complex system where many excitable cells, called neurons, communicate with each other through the transmission of bioelectrical signals, called spikes, in order to perform complex tasks such as detection or classification, with a very low energy consumption. Inspired by our brain, neuromorphic systems are designed to have a very low energy consumption: a recently developed CMOS based neuron [1], [2] was proven to consume only few femto-Joule which is several order of magnitude lower than what standard processors would need to execute similar operations. This neuron is based on a Morris-Lecar model [3] but various models of spiking neurons can be found in the literature, from the simplest one, Integrate-and-

Fire (IF) [4], to more complex ones such as Morris-Lecar. Thus, if the wake-up receivers were designed with neuromorphic circuits, one could achieve high energy efficiency for IoT and *ad hoc* networks.

One natural question arising is whether a neuro-inspired detection is reliable enough in the context of *ad hoc* networks, where one needs to detect a weak signal surrounded by noise.

One major advantage of a neuro-based detector, besides its very low energy consumption, is its nonlinear behavior. These non linearities are a key ingredient of stochastic resonance (SR), a paradoxical phenomenon, where the addition of noise enhances the overall performance of a nonlinear system. From a neuroscience point of view, SR is ubiquitous and crucial in facilitating the information processing. The idea of adding noise in order to improve the performance of a system is rather counter-intuitive but some example can be found even in the communication field with dithers in lattice coding [5], with noise aided LDPC decoders [6], or even the effect of SR in locally optimal detectors [7]. Since in *ad hoc* sensor network, many sensors communicate over the same resources, there will be a large amount of surrounding noise when considering a given communication. Since the neuro-inspired detector is non linear, it can definitely takes advantage of this surrounding noise.

In this paper, we propose to study the theoretical performance of a neuro-based digital communication, where the signal detection is performed using an IF neuron. Our main contributions are the following:

- We derive the theoretical performance of a digital communication setup using a single IF detector.
- We show that stochastic resonance occurs with the single IF detector.
- We derive the theoretical performance of a digital communication setup using multiple IF detectors in parallel.

The remaining of the paper is organized as follows: Section II presents some mathematical tools and distributions. The IF neuron and the digital communication setup are presented in Section III and the error probability of a single neuron-based detection is derived in Section IV. Section V presents an extension to multiple neurons in parallel in order to achieve an enough reliable detection for digital communication and Section VI concludes the paper.

II. NOTATIONS AND DEFINITIONS

A. Notations

The following notations will be used throughout the paper.

- $\mathbb{1}_{[0,T]}(t)$ denotes the indicator function over the interval $[0, T]$: $\forall t \in [0, T], \mathbb{1}_{[0,T]}(t) = 1$.
- $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2 . $\Phi(t)$ denotes the cumulative distribution function (cdf) of a standard normal distribution $\mathcal{N}(0, 1)$.
- $\text{erfc}(x)$ is the complementary error function, defined as $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) dt$.
- $L(\mu, c)$ denotes the Levy distribution, with location parameter μ and scale parameter c . Its cdf is denoted by $F_L(x)$ and is given as $F_L(x) = \text{erfc}\left(\sqrt{\frac{c}{2(x-\mu)}}\right)$.
- $IG(\mu, \lambda)$ denotes the Inverse Gaussian distribution with mean μ and shape parameter c . Its cdf is denoted by $F_I(x)$ and is given as $F_I(x) = \Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right)$.

B. Brownian motion and first hitting time

As we will show later on in the paper, the neuron output can be expressed as a Brownian motion. In this subsection, we will briefly introduce Brownian motion and the notion of first hitting time, that will be needed to derive the theoretical performance of our IF-based detector.

1) *Standard Brownian motion*: Let Z_s follow a standard normal distribution $Z_s \sim \mathcal{N}(0, 1)$ and denote by W_t its integral over a the time interval $[0, t]$, i.e. $W_t = \int_0^t Z_s(\tau) d\tau$. W_t is a standard Brownian motion and is such that $W_t \sim \mathcal{N}(0, t)$.

2) *Brownian motion with drift* [8]: Let $Y(t)$ denote a Brownian motion with drift μ . $Y(t)$ is given as

$$Y(t) = \mu t + \sigma W_t. \quad (1)$$

3) *First hitting time*: Since events are usually triggered when the particle first reaches a given threshold, characterizing the first hitting time is of important matter. Let T_a denote the needed time for the signal to reach for the first time the value $a > 0$.

Lemma 1 [9] *The first hitting time T_a of a Brownian motion $Y(t)$ as given in (1) with a non-zero drift $\mu \neq 0$ starting in position y_0 at time $t = 0$ follows the Inverse Gaussian distribution: $T_a \sim IG\left(\frac{a-y_0}{\mu}, \frac{(a-y_0)^2}{\sigma^2}\right)$.*

Remark 1 *Note that if the drift μ equals zero, then the inverse Gaussian distribution reduces to a Levy distribution. Thus, the first hitting time T_a of a Brownian motion $Y(t)$ as given in (1) with a zero drift $\mu = 0$ starting in position y_0 at time $t = 0$ follows the Levy distribution: $T_a \sim L\left(0, \frac{(a-y_0)^2}{\sigma^2}\right)$.*

III. DIGITAL COMMUNICATION WITH NEURO-BASED DETECTION

A. Integrate-and-Fire neuron

Integrate-and-Fire (IF) neurons are among the easiest models of spiking neurons, where the state of the neuron is characterized only by its membrane potential.

The electrical model of an IF neuron consists in a capacitor C in parallel of a resistor R that is driven by an input current. The neuron output is a function of the voltage. Throughout this paper, we focus on the perfect integrator [4], where the resistor R is negligible, i.e. the membrane potential's decay over time is neglected. Such a case can be achieved in practice when the duration of a symbol is much larger compared to the time constants of the electronic circuit. The membrane potential V of the perfect IF neuron is thus given as the integral of the injected current I_{inj} as $V(t) = \frac{1}{C}V_{rest} + \frac{1}{C} \int_0^t I_{inj}(\tau) d\tau$, where V_{rest} is the membrane potential without excitation.

The neuron "fires" when its membrane potential V reaches a threshold γ . In other words, for a time $t = t_\gamma$ such that $V(t_\gamma) = \gamma$, a spike is generated. Since we are not interested in modeling the shape of the spike, we model this spike generation solely by an amplitude change as $V(t_\gamma) = \gamma + V_s$. After a spike generation, the membrane potential is reset to its resting value V_{rest} and the integration process is restarted. Throughout this paper, we assume that the integration process is immediately restarted, such that no refractory period exists and we assume that $V_{rest} = 0$. The obtained results can be easily extended to a case with $V_{rest} \neq 0$ and with a refractory period.

B. Communication setup

The studied detection setup is described in Fig. 1. Let $\{i_{inj}(k)\}_{k \in \mathbb{N}}$ denote the symbol sequence, where $\forall k$,

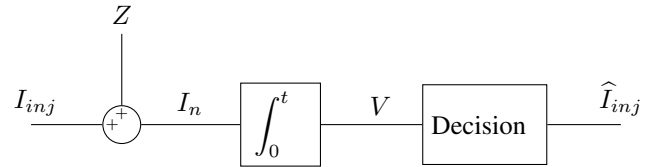


Fig. 1. Studied communication setup

$i_{inj}(k) \in \{0, A\}$ such that $\mathbb{P}(i_{inj}(k) = 0) = \mathbb{P}(i_{inj}(k) = A) = 1/2$.

Let I_n denote the neuron input. I_n is given as a noisy version of I_{inj} as $I_n = I_{inj} + Z$, where Z is assumed to be an additive white Gaussian noise of variance σ_N^2 .

Without loss of generality, we restrict our study on one time interval $[0, T]$. Let $i_{inj} \in \{0, A\}$ denote the information symbol. The membrane potential V of the neuron at time $t \in [0, T]$ is thus given as

$$V(t) = \frac{1}{C} \int_0^t (i_{inj} \mathbb{1}_{[0,T]}(\tau) + Z(\tau)) d\tau. \quad (2)$$

The goal is, based on the generated spike train, to decide whether the information symbol i_{inj} was equal to 0 or A .

Detecting spikes at the decoder is an easy task, since we model a spike by a very large amplitude change. We propose thus to use a hard decision rule with threshold K based on the number of generated spikes N_s over the interval $[0, T]$ as

$$\hat{i}_{inj} = \begin{cases} A & \text{if } N_s \geq K \\ 0 & \text{if } N_s \leq K. \end{cases} \quad (3)$$

In the remaining of the paper, we restrict our study to the case $K = 1$, but it can be easily extended to arbitrary values of K .

An example of the evolution of the output potential $V(t)$ over the interval $[0, T]$ is given in Fig. 2, where $i_{inj} = A$, $A = 1$, $C = 1$ and $\sigma_N^2 = 0$. For this example, $N_s = 3$ spikes are generated over the time interval $[0, T]$.

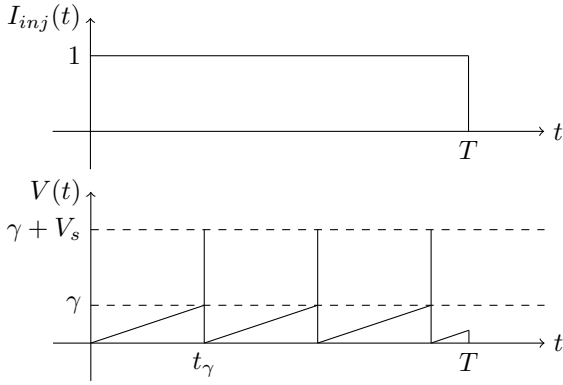


Fig. 2. Example of the evolution of the membrane voltage $V(t)$ in absence of additive noise Z over the interval $[0, T]$, when $i_{inj} = A$, $A = 1$ and $C = 1$. Under this setup, $N_s = 3$ spikes have been generated.

In this paper, we aim to characterize the theoretical performance of such an IF-based detection in the context of energy efficient communication and wake-up radio. As such, the transmitted power is low, and if there are many users, the noise variance can easily be very high, thus our goal is to detect a signal with a very low amplitude compared to the noise. We aim to show that in an extreme case, where the spiking threshold γ of the neuron is such that $\gamma > AT$, i.e. where without channel noise no detection of $I_{inj} = A$ can occur, one can achieve a small error probability by taking advantage of the surrounding noise.

The next Section presents one of our main contribution.

IV. PERFORMANCE ANALYSIS OF AN IF-NEURO BASED DETECTOR

The following Proposition presents the error probability of a digital communication set-up using our proposed IF-based detector.

Proposition 1 *When using the proposed IF-based detector with $K = 1$, threshold γ , over the AWGN channels with noise*

variance σ_N^2 , the error probability $\mathbb{P}_e = \mathbb{P}(\hat{i}_{inj} \neq i_{inj})$ is given as

$$\mathbb{P}_e = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{\gamma^2 C^2}{2T\sigma_N^2}} \right) + \frac{1}{2} - \frac{1}{2} \Phi \left(\frac{TA - \gamma C}{\sqrt{\sigma_N^2 T}} \right) - \frac{1}{2} \exp \left(\frac{2\gamma C A}{\sigma_N^2} \right) \Phi \left(-\frac{TA + \gamma C}{\sqrt{\sigma_N^2 T}} \right).$$

Proof: The error probability can be expressed as

$$\mathbb{P}_e = \frac{1}{2} \mathbb{P}(\hat{i}_{inj} = A | i_{inj} = 0) + \frac{1}{2} \mathbb{P}(\hat{i}_{inj} = 0 | i_{inj} = A). \quad (4)$$

Let us compute the two conditional error probabilities separately.

a) : Let us first assume that $i_{inj} = 0$. The output of the neuron at time t is thus given as $V(t) = \frac{1}{C} \int_0^t Z(\tau) d\tau = \frac{\sigma_N}{C} W_t$. Using Remark 1, the first hitting time T_γ follows a Levy distribution: $T_\gamma \sim L \left(0, \frac{\gamma^2 C^2}{\sigma_N^2} \right)$.

The probability $\mathbb{P}(\hat{i}_{inj} = A | i_{inj} = 0)$ corresponds to have the first hitting time $T_\gamma = t_\gamma$ inside the time interval $[0, T]$, leading to

$$\mathbb{P}(\hat{i}_{inj} = A | i_{inj} = 0) = F_L(T) = \operatorname{erfc} \left(\sqrt{\frac{\gamma^2 C^2}{2T\sigma_N^2}} \right). \quad (5)$$

b) : Let us now assume that $i_{inj} = A$. The output of the neuron at time t is thus given as $V(t) = \frac{1}{C} \int_0^t A + N(\tau) d\tau = \frac{A}{C} t + \sigma_N W_t$. Using Lemma 1, the first hitting time T_γ follows an Inverse Gaussian distribution: $T_\gamma \sim IG \left(\frac{\gamma C}{A}, \frac{\gamma^2 C^2}{\sigma_N^2} \right)$.

The probability $\mathbb{P}(\hat{i}_{inj} = 0 | i_{inj} = A)$ corresponds to have the first hitting time T_γ outside the interval $[0, T]$, leading to

$$\begin{aligned} \mathbb{P}(\hat{i}_{inj} = 0 | i_{inj} = A) &= 1 - F_I(T) \\ &= 1 - \Phi \left(\frac{TA - \gamma C}{\sqrt{\sigma_N^2 T}} \right) - \exp \left(\frac{2\gamma C A}{\sigma_N^2} \right) \Phi \left(-\frac{TA + \gamma C}{\sqrt{\sigma_N^2 T}} \right). \end{aligned} \quad (6)$$

Finally, replacing (5) and (6) into (4) concludes the proof. \blacksquare

Fig. 3 shows the error probability \mathbb{P}_e as a function of the normalized noise variance σ_N^2/F_e , where F_e is the sampling frequency, when the duration of a symbol is set to $T = 50$ and the amplitude A is set to $A = 1$. We consider normalized noise variance in order to obtain performance closed to one obtained by analog circuits used for the hardware neuron. We used an IF neuron with $\gamma = 51$ and $C = 1$, thus in absence of noise, no signal detection can occur if $I_{inj} = A$. The curve marked with circle corresponds to the simulated error probability when 10^5 symbols i_{inj} have been generated, whereas the curve marked with squares corresponds to the error probability given in Proposition 1. First, one can note that the simulated results fit the theory and second, that there exists a specific value of the noise variance such that the error probability is minimal, proving thus the existence of SR in this IF-based communication scheme. Nevertheless, with a single IF neuron, one can achieve an error probability of 0.24 in the

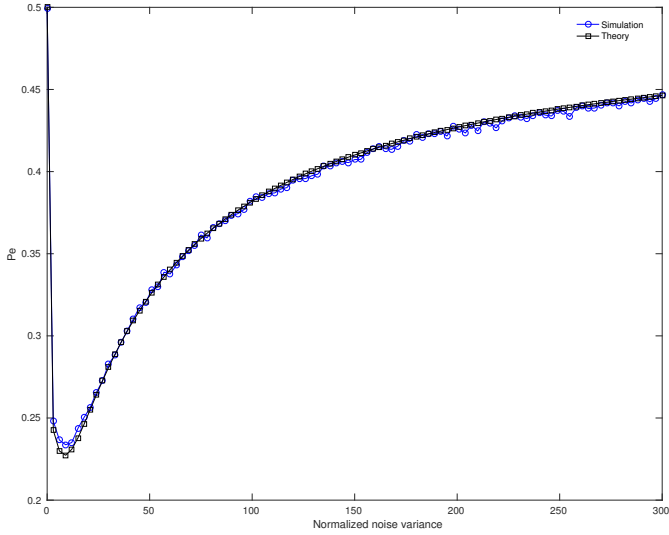


Fig. 3. Error probability \mathbb{P}_e as a function of the normalized noise variance σ_N^2/F_e , $\gamma = 51$, $T = 50$, $K = 1$, $A = 1$, $C = 1$, 10^5 symbols i_{inj}

best case. First note that this error probability is too large to consider an application to digital communication with a single IF-based detector. Moreover, even if this value was smaller, it is achieved only for a specific value of normalized noise variance. Unfortunately, one can not control the noise power of the channel, thus one can not guarantee a low enough error probability with this single IF-based detector.

If instead one is interested in achieving a small error probability for a fixed noise variance when the transmitted power can be large, the best solution is to set the amplitude A as large as possible. Indeed, if A becomes large, then the conditional error probability $\mathbb{P}(\hat{i}_{inj} = 0 | i_{inj} = A)$ tends to 0. Thus the error probability tends to $\mathbb{P}_e \rightarrow \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{\gamma^2 C^2}{2T\sigma_N^2}} \right)$.

In the following Section we propose to use parallel neurons in order to overcome the single IF neuron issues and to achieve, as we will see in the numerical simulations, an error probability of the order of 10^{-12} (for a specific value of the noise variance) and below 10^{-4} for a large range of noise variance, allowing thus the use of our proposed IF-based detector for energy efficient communication.

V. PERFORMANCE ANALYSIS OF N_p IF-NEURO BASED DETECTOR IN PARALLEL

Let assume that we have N_p IF-based decoders with the same input sequence. In the remaining of the paper, we assume that we can have at most $N_{max} = 50$ neurons in parallel. Each detector outputs its own decided symbol, with $K = 1$ and the overall decoder decides whether the input symbol was equal to 0 or A . Again, this decision step is performed using a hard decision rule based on the number N_1 of single IF-based detectors that have decided that the input symbol was equal to A . The output symbol is decoded as follows:

$$\hat{i}_{inj} = \begin{cases} A & \text{if } N_1 \geq K_p \\ 0 & \text{else} \end{cases}$$

This situation is depicted in Fig. 4.

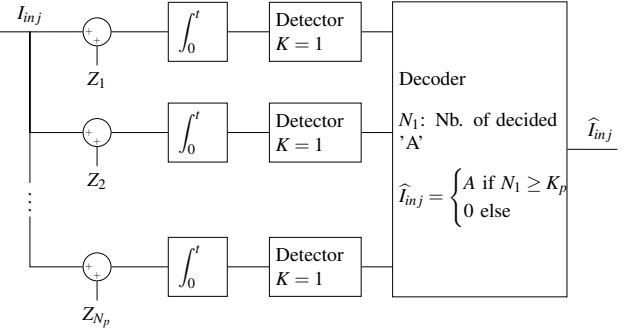


Fig. 4. N_p neuro-based detectors with $K = 1$. The overall decoder decides $I_{inj} = A$ if at least K_p of the N_p decoders have decided $\hat{I}_{inj} = A$

To ease the readability, we denote by $\mathbb{P}(j|k)$, $j, k \in \{0, A\}$ the conditional probability $\mathbb{P}(j|k) = \mathbb{P}(\hat{i}_{inj} = j | i_{inj} = k)$.

Proposition 2 When using the proposed IF-based detector with $K = 1$ and threshold γ over N_p parallel AWGN channels with noise variance σ_N^2 , for a given K_p , the error probability $\mathbb{P}_{e, N_p}^{K_p} = \mathbb{P}(\hat{i}_{inj} \neq i_{inj})$ is given as

$$\mathbb{P}_{e, N_p}^{K_p} = \sum_{k=K_p}^{N_p} \binom{N_p}{k} \frac{\mathbb{P}(A|0)^k \mathbb{P}(0|0)^{N_p-k}}{2} + \sum_{k=N_p-K_p+1}^{N_p} \binom{N_p}{k} \frac{\mathbb{P}(0|A)^k \mathbb{P}(A|A)^{N_p-k}}{2},$$

where $\mathbb{P}(A|0)$, respectively $\mathbb{P}(0|A)$, is given in (5), respectively in (6).

Proof: The error probability is given as

$$\mathbb{P}_{e, N_p}^{K_p} = \frac{1}{2} \mathbb{P}(\text{at least } K_p \text{ decided } \hat{i}_{inj} = A | i_{inj} = 0) + \frac{1}{2} \mathbb{P}(\text{at most } K_p - 1 \text{ decided } \hat{i}_{inj} = A | i_{inj} = A)$$

In the following, we propose a way to determine the optimal number N_p^* of neuron in parallel and the optimal threshold K_p^* for a given noise variance. Since we assume that we can at most have access to $N_{max} = 50$ IF-neurons in parallel, N_p^* and K_p^* are chosen as

$$N_p^*, K_p^* = \arg \min_{N_p, K_p} \mathbb{P}_{e, N_p}^{K_p} \quad \text{subject to } \begin{aligned} N_p &\leq N_{max}, \\ K_p &\leq N_{max} \end{aligned} \quad (7)$$

The obtained results are summarized in Table I and the corresponding error probability curves are given in Fig. 5 as a function of the noise variance. Table II presents the optimal number to put in parallel if one sets K_p to $K_p = 2$. Indeed, varying the decoding threshold without increasing the energy

$-10 \log(\sigma_N^2/F_e)$	8	0	-5	-10	-15	-20	-25	-30
K_p^*	2	2	4	11	21	30	37	42
N_p^*	50	50	50	50	50	50	50	50

TABLE I

OPTIMAL NUMBER OF PARALLEL NEURONS N_p^* AND K_p^* OBTAINED BY EXHAUSTIVE SEARCH.

$-10 \log(\sigma_N^2/F_e)$	8	0	-5	-10	-15	-20	-25	-30
K_p^*	2	2	2	2	2	2	2	2
N_p^*	50	50	27	9	4	3	2	2

TABLE II

OPTIMAL NUMBER OF PARALLEL NEURONS N_p^* OBTAINED BY EXHAUSTIVE SEARCH WHEN $K_p = 2$.

consumption or the size of the circuit for each variance if the entire circuit is build upon CMOS technology is a hard task. Thus, these two set-ups allows the compare the gap between optimality over both (N_p, K_p) and only over N_p for a fixed K_p .

Fig. 5 compares the achievable error probabilities in five cases: three cases where $K_p = 2$, the single neuron case and one case with optimal N_p^* and K_p^* .

For the case $K = 2$, we consider 2 cases: either we optimize the number of neurons in parallel N_p under the constraint $K_p = 2$ or we set this number of neuron in parallel to $N_p = 16$, which is optimal for a normalized variance of 0.5.

From Fig. 5, one can see that putting neurons in parallel improves significantly the error probability for a large range of normalized variance compared to the single neuron case. Nevertheless, one can also note that for a large value of the normalized noise variance, having an unappropriated number in parallel can decrease the performance compared to the single neuron case (For instance, for a normalized noise variance from 13 to 25, having 16 neurons in parallel yields an error probability larger than the one obtained in the single neuron case).

One can also note that there is a large gap between the two curves where one optimize the number of neuron in parallel for a fixed K_p or when one optimize both values.

We can also note that the error probability is below 10^{-4} for a very large range of normalized noise variance. As such, a detector based on this very low-consuming parallel IF-based architecture can be used in the context of energy efficient communication and in wake-up radio scenario.

VI. CONCLUSION

In this paper we characterized the theoretical performance of a neuro-based digital communication. We showed that with only one neuron, the error probability is too high to consider a digital communication setup so we proposed to use parallel neurons. With this new neuro-based architecture, we showed that one can achieve a very low error probability, less than 10^{-4} , for a large range of noise variance. Moreover, our detection scheme applies to below-threshold signal detection where we took advantage of the surrounding noise thanks to the non-linear behavior of our decoder. Nevertheless, this large performance improvement requires having N_p i.i.d. white

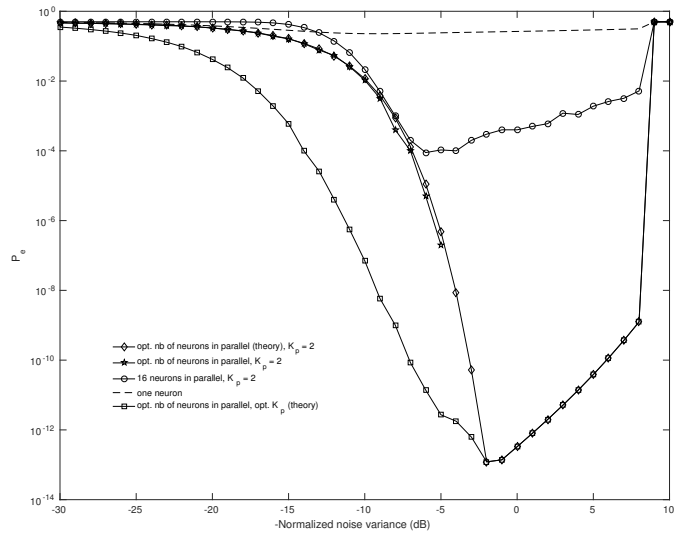


Fig. 5. Error probability $\mathbb{P}_{e, N_p}^{K_p}$ as a function of the normalized SNR - comparison between the single neuron case, the parallel neurons case with $K_p = 2$ and optimal N_p and the parallel neurons case with optimal N_p and K_p when $C = 1$, $\gamma = 51$, $A = 1$ and $T = 50$

noise sources of same variance and N_p identical neurons, which is in practice very hard to achieve. To overcome this issue, one could transform our parallel decoder into a sequential one requiring only one neuron and one noise source by turning the encoding into a repetition, and allowing a latency equal to the number of needed neurons in parallel.

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