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We study the dynamics of Fermi-Pasta-Ulam (FPU) one-dimensional models with both harmonic and anharmonic power-law long-range interactions. We show that the dynamics is described in the continuum limit by a generalized Boussinesq fractional differential equation, whose derivation is performed in full detail. We also discuss a version of the model where couplings are alternating in sign.

Keywords: *FPU models, long-range interactions, Generalized Fractional Boussinesq Equations (GFBE), fractional calculus.*

I. INTRODUCTION

It is nowadays well recognized that the study of the Fermi-Pasta-Ulam (FPU) model, first performed six decades ago, leads to deep insights into the behavior of discrete nonlinear systems and into fundamental problems of statistical mechanics [1, 2]. The FPU model describes the dynamics of an oscillator chain with nearest neighbor nonlinear couplings among masses, exhibiting at low energy mode recurrence [3]. This remarkable phenomenon was historically tackled by considering the continuum limit, pioneered in the classical paper by N. Zabusky and M. Kruskal [4]. In that limit one finds a Boussinesq equation [5] and after a change of variables a Korteweg-de Vries (KdV) equation [4] which gives an accurate description of waves and localized solutions of the FPU model. Performing in a controlled way the continuum limit for generalized FPU models is of paramount importance, since it allows one to construct solitonic solutions and to determine the low-energy properties of these systems.

Our goal here is to derive the effective equations describing the dynamics of FPU models characterized by both harmonic and anharmonic Long-Range Interactions (LRI). LRI have been intensively studied in the last decades for a variety of physical systems [6] and considerable interest has been devoted to the study of power-law LRI. Pioneering work by F. Dyson [7] has revealed that the one-dimensional Ising ferromagnet with power-law couplings among the spins displays a non trivial phase transition for values of the power exponent s in the range $1 < s \leq 2$.

Coupled oscillators with power-law LRI were also studied [8–16]. The attention was mainly focused on long-range interactions in DNA [8, 10], the existence of standing localized solutions like breathers [9], deriving the continuum counterpart of the discrete long-range models which implies the use of fractional derivatives [11–13], weakly chaotic citeChristodoulidi2014 and thermalization [15] properties caused by the long-range character of the interactions and the existence of solitons in a long-range extension of the quartic FPU chain [16].

We will show in the following that fractional differential equations describe the continuum limit of the FPU model with both harmonic and anharmonic power-law LRI. Fractional calculus has gained considerable interest and importance as an extension of differential equations with integer order derivatives. These techniques are used for the investigation of various problems in physics, engineering, life sciences and economy [17–21]. The use of fractional derivatives may lead to an elegant and more compact way of treating dynamical systems with non-local interactions and/or couplings. Typical examples are physical phenomena with long memory [22, 23] and random displacement with space jumps of arbitrary lengths [24].

Fractional derivatives are also used in condensed matter physics. We can mention the recent studies of classical spin systems [25, 26] and of fermionic quantum chains [27, 28] with long-range couplings, which can be described by effective field theories with a dispersion relation associated in real space to a kinetic term with fractional derivatives.

In this paper, we derive the fractional partial differential equations which describe the FPU model with LRI in the continuum limit. We consider power-law LRI couplings in both the harmonic and anharmonic terms. We derive the Generalized Fractional Boussinesq Equation (GFBE) for the displacement field of the long-range FPU model.

The paper is organized as follows. In Section II we discuss the derivation of the GFBE for the α–FPU model when both the harmonic and the anharmonic terms have long-range couplings. The situation in which the couplings alternate in sign is also presented in section III. In Section IV, we repeat all the derivations for the β–FPU model. We discuss our results and perspectives in Section V. In Appendix A we briefly review the basic definitions of fractional calculus. In Appendixes B-D some derivations of lengthy formulas needed for continuum approximation of long-range FPU models are presented. Appendix E is devoted to the case in which cubic and quartic terms are both present.

II. THE α–FPU MODEL WITH POWER-LAW LONG-RANGE INTERACTIONS

We consider in this Section a Hamiltonian where both the couplings in the harmonic and the anharmonic terms of the α–FPU model have a power-law behavior with different exponents s_1 and s_2, respectively

$$H\left(u_n, \dot{u}_n\right) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \dot{u}_n^2 + \frac{\chi}{2} \sum_{\substack{n,m=-\infty \\ m<n}}^{+\infty} \frac{\left[u_n - u_m\right]^2}{\left|a\left(n-m\right)\right|^{s_1}} + \frac{\gamma}{3} \sum_{\substack{n,m=-\infty \\ m<n}}^{+\infty} \frac{\left[u_n - u_m\right]^3}{\left|a\left(n-m\right)\right|^{s_2}}, \tag{1}$$

where χ and γ are positive constants giving the strength of the quadratic and cubic potential, n and m stand for the indices of the lattice sites and a is the lattice spacing (from now on we put for simplicity $a \equiv 1$, except where it is necessary for the treatment).

We choose the power-law decay exponents in the range $1 < s_1, s_2 < 3$. For smaller values the Hamiltonian diverges, while when the powers tend to infinity, we of course get back the conventional α–FPU model. When both powers are finite and above 3 the system becomes effectively short-range. When s_1 (s_2) is in the range $(1,3)$ and s_2 (s_1) is above 3, then the effective continuum equation displays long-range behaviour in the anharmonic (harmonic) terms. In the following the latter cases will be separately considered. The power $s_1 = 2$ (with $s_2 \to \infty$) has been considered for crack front propagation along disordered planes between solid blocks [29, 30] and contact lines of liquid spreading on solid surfaces [31].

From Eq. (1) we get the following equations of motion:

$$\ddot{u}_n + \chi \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} + \gamma \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^2}{|a(n-m)|^{s_2}} f_{n,m} = 0 \tag{2}$$

where

$$f_{n,m} \equiv \begin{cases} +1, & m < n \\ -1, & m > n \end{cases} .$$

To understand the qualitative property of the wave equation, we study the dispersion relation. We consider Eq. (2) in which we ignore the anharmonic term, thus, it admits the "plane wave" reading

$$u_n(t) = u_0 e^{i(\omega t - k x_n)} \tag{3}$$

where $x_n = na$. The corresponding linear dispersion relation is

$$\omega^2(k) = 2\chi \sum_{l=1}^{\infty} \frac{1 - \cos(kl)}{l^{s_1}}. \tag{4}$$

Eq. (4) is plotted in Fig. (1) for three specific values of s_1. It is straightforward to observe that the dispersion relation diverges for $s_1 < 1$.

For small wavenumbers $k \to 0$, $\omega(k) \propto |k|$ for $s_1 > 3$ and the phase and group velocities are constant. If instead $1 < s_1 < 3$, $\omega(k) \propto |k|^{(s_1-1)/2}$ and the phase and group velocities are given by $v_{ph} \propto |k|^{(s_1-3)/2}$ and $v_g \propto |k|^{(s_1-3)/2}(s_1-1)/2 = v_{ph}(s_1-1)/2$. They both diverge in the limit $k \to 0$.

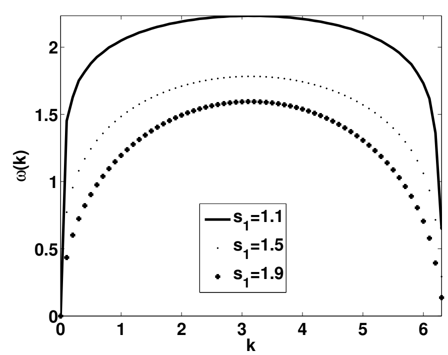

FIG. 1. Dispersion relation, Eq. (4), for $s_1 = 1.1$ (solid), $s_1 = 1.5$ (thin dots), and $s_1 = 1.9$ (thick dots).

Now we are ready for deriving the fractional partial differential equation which describes Hamiltonian (1) in the long wavelength limit. We start by defining the Fourier transform of $u_n(t)$ as

$$u_n(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \, e^{ikn} \hat{u}(k,t) \tag{5}$$

where k is the wave number, and

$$\hat{u}\left(k,t\right)=\sum_{n=-\infty}^{+\infty}e^{-ikn}u_{n}\left(t\right). \tag{6}$$

Basic properties and definition of fractional calculus are given for convenience in Appendix A and the details of deriving of continuum limits of terms in Eq. (2) are given in Appendix B. Finally, we get

$$\sum_{\substack{m=-\infty\\m\neq n}}^{+\infty}\frac{u_{n}-u_{m}}{\left|a\left(n-m\right)\right|^{s_{1}}}\simeq\frac{\pi}{\Gamma\left(s\right)\sin\frac{s_{1}-1}{2}\pi}\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\;e^{ipx}\left|p\right|^{s_{1}-1}\hat{u}\left(p,t\right), \tag{7}$$

$$\sum_{\substack{m=-\infty\\m\neq n}}^{+\infty}\frac{\left(u_{n}-u_{m}\right)^{2}}{\left|a\left(n-m\right)\right|^{s_{2}}}f_{n,m}\approx\frac{u\left(x,t\right)}{2\pi}\int\limits_{-\infty}^{+\infty}dp\;e^{ipx}\frac{2\left(2-2^{s_{2}-1}\right)\pi}{\Gamma\left(s_{2}\right)\sin\left(s_{2}\pi\right)}\Big(\left(-ip\right)^{s_{2}-1}-\left(ip\right)^{s_{2}-1}\Big)\tilde{u}\left(p,t\right). \tag{8}$$

In this limit, we redefine respectively Eq. (5) and Eq. (6) as

$$\hat{u}\left(p,t\right)=\int\limits_{-\infty}^{+\infty}dx\,e^{-ipx}\,u\left(x,t\right),\qquad\qquad u\left(x,t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\,e^{ipx}\,u\left(p,t\right). \tag{9}$$

We now introduce the Fourier transformation of fractional differentiation of order α via the relations

$$D_{x^{+}}^{\alpha}u\left(x,t\right)\equiv\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\left(x,t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\left(+ip\right)^{\alpha}e^{ipx}\,\hat{u}\left(p,t\right) \tag{10}$$

and

$$D_{x^{-}}^{\alpha}u\left(x,t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\left(-ip\right)^{\alpha}e^{ipx}\,\hat{u}\left(p,t\right). \tag{11}$$

The Fourier transform involving the absolute value of momentum $|p|^{\alpha}$ is expressed by a Riesz derivative in real space as

$$-\frac{\partial^{\alpha}}{\partial\left|x\right|^{\alpha}}u\left(x,t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\left|p\right|^{\alpha}\hat{u}\left(p,t\right)e^{ipx}, \tag{12}$$

and using Eqs. (10)-(11) one gets

$$\frac{\partial^{\alpha}}{\partial\left|x\right|^{\alpha}}u\left(x,t\right)=-\frac{1}{2\cos\frac{\alpha\pi}{2}}[D_{x^{+}}^{\alpha}+D_{x^{-}}^{\alpha}]u\left(x,t\right). \tag{13}$$

Combining respectively Eqs. (7) and (8) with Eqs. (10), (11) and (13), one gets

$$\sum_{\substack{m=-\infty\\m\neq n}}^{+\infty}\frac{u_{n}-u_{m}}{\left|a\left(n-m\right)\right|^{s_{1}}}\approx-\frac{\pi}{\Gamma\left(s_{1}\right)\sin\frac{s_{1}-1}{2}\pi}\frac{\partial^{s_{1}-1}}{\partial\left|x\right|^{s_{1}-1}}u\left(x,t\right), \tag{14}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{\left(u_n-u_m\right)^2}{\left|a\left(n-m\right)\right|^{s_2}} f_{n,m} \approx \frac{2\left(2-2^{s_2-1}\right)\pi}{\Gamma\left(s_2\right)\sin\left(s_2\pi\right)} u\left(x,t\right)\left[D_{x^-}^{s_2-1}-D_{x^+}^{s_2-1}\right]u\left(x,t\right). \tag{15}$$

Finally, Eqs. (14), (11) and (2) combine into the following Generalized Fractional Boussinesq Equation (GFBE):

$$\frac{\partial^2 u\left(x,t\right)}{\partial t^2} - g_{s_1-1}\frac{\partial^{s_1-1}u\left(x,t\right)}{\partial\left|x\right|^{s_1-1}} - h_{s_2-1}u\left(x,t\right)\left[D_{x^-}^{s_2-1}-D_{x^+}^{s_2-1}\right]u\left(x,t\right)=0, \tag{16}$$

where the constants g_{s_1-1} and h_{s_2-1} are given by

$$g_{s_1-1}=\frac{\chi\pi}{\Gamma\left(s_1\right)\sin\left(\frac{s_1-1}{2}\pi\right)}, \quad h_{s_2-1}=\frac{2\left(2-2^{s_2-1}\right)\gamma\pi}{\Gamma\left(s_2\right)\cos\frac{s_2\pi}{2}} \tag{17}$$

where Γ stands for the Euler's gamma function.

III. α–FPU MODEL WITH ALTERNATING MASSES AND INTERACTIONS

The main goal of the paper is to establish that general FPU chains with LRI are mapped in fractional equations of motion and we aim at illustrating it in variety of models. To investigate how general is this mapping, we consider in this Section an α-FPU model with alternating in signs kinetic and interacting terms. Despite the physical realization of both alternating masses and interaction terms is certainly not easily implementable, alternating/varying interactions are rather common and one can think to implement alternating in sign hopping coefficients in ultracold chains using a variation of well-known shaking techniques [32]. The corresponding Hamiltonian is written as

$$H\left(u_n,\dot{u}_n\right)=\frac{1}{2}\sum_{n=-\infty}^{+\infty}\dot{u}_n^2\left(-1\right)^n+\frac{\chi}{2}\sum_{\substack{n,m=-\infty \\ m<n}}^{+\infty}\frac{\left[u_n-u_m\right]^2\left(-1\right)^n}{\left|a\left(n-m\right)\right|^s}+\frac{\gamma}{3}\sum_{\substack{n,m=-\infty \\ m<n}}^{+\infty}\frac{\left[u_n-u_m\right]^3\left(-1\right)^n}{\left|a\left(n-m\right)\right|^s}, \tag{18}$$

where χ and γ are positive constants. Here again the exponent is chosen in the range of $1<s<3$. Applying the Hamiltonian formalism to Eq. (18), one gets the following equation on the lattice

$$\ddot{u}_n+\chi\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty}\frac{u_n-u_m}{\left|a\left(n-m\right)\right|^s}+\gamma\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty}\frac{\left[u_n-u_m\right]^2}{\left|a\left(n-m\right)\right|^s}=0. \tag{19}$$

We can obtain a plane wave solution as

$$u_n\left(t\right)=u_0e^{i\left(\omega t-kx_n\right)} \tag{20}$$

where $x_n=na$. The dispersion relation of the fractional wave equation is given by Eq. (4).

To derive the continuum equation describing the system while using the long wavelength limit for the corresponding lattice-field model, we use the expressions of the Fourier transform and its inverse defined in Eqs. (5)-(6). The third term of Eq. (19) is therefore transformed into (see Appendix C):

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{\left(u_n-u_m\right)^2}{\left|a\left(n-m\right)\right|^s}\approx\frac{\left(2-2^{s-1}\right)\pi}{\Gamma\left(s\right)\sin\left(\frac{s-1}{2}\pi\right)}\frac{u\left(x,t\right)}{2\pi}\int_{-\infty}^{+\infty}dp\left|p\right|^{s-1}e^{ipx}\tilde{u}\left(p,t\right). \tag{21}$$

Using Riesz fractional derivative definitions Eqs. (12) and (13), we obtain

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^s} \approx -\frac{\left(2-2^{s-1}\right)\pi}{\Gamma(s)\sin\left(\frac{s-1}{2}\pi\right)} u(x,t) \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x,t) \tag{22}$$

.

Finally, Eqs. (14), (22) and (19) combine into the following fractional differential equation:

$$\frac{\partial^2}{\partial t^2} u(x,t) - g_{s-1} \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x,t) - k_{s-1} u(x,t) \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x,t) = 0, \tag{23}$$

with

$$k_{s-1} = \frac{\left(2-2^{s-1}\right)\gamma\pi}{\Gamma(s)\sin\frac{s-1}{2}\pi}. \tag{24}$$

Despite the presence of the alternating terms $(-1)^n$ in Eq. (18), one finds again a fractional differential equation, even simpler than Eq. (1). One can also write Eq. (23) as

$$u_{tt} - g_{s-1} D^{\alpha}_{|x|} u - k_{s-1} u D^{\alpha}_{|x|} u = 0 \tag{25}$$

where $\alpha = s-1$, $g_{s-1} \neq 0$ and $k_{s-1} \neq 0$ are non-zero constants.

IV. THE β–FPU MODEL WITH POWER-LAW LONG-RANGE INTERACTIONS

In this section we study the effect of LRI both in the harmonic and anharmonic terms of an extended β–FPU Hamiltonian model. The main difference with the previous Sections is that now the interaction is quartic instead of cubic - for the rest our goal is to parallel the results presented in the previous Sections. The model reads

$$H(u_n,\dot{u}_n) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \dot{u}_n^2 + \frac{\chi}{4} \sum_{\substack{n,m=-\infty \\ m\neq n}}^{+\infty} \frac{[u_n-u_m]^2}{|a(n-m)|^{s_1}} + \frac{\gamma}{8} \sum_{\substack{n,m=-\infty \\ m\neq n}}^{+\infty} \frac{[u_n-u_m]^4}{|a(n-m)|^{s_2}} \tag{26}$$

where χ and γ are positive constants. Hamiltonian (26) is referred to as the extended β–FPU model. Again we choose the parameters describing the order of the fractional space derivative in the range of $(1 < s_1, s_2 < 3)$. The discrete equation of motion reads

$$\ddot{u}_n + \chi \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{u_n-u_m}{|a(n-m)|^{s_1}} + \gamma \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{[u_n-u_m]^3}{|a(n-m)|^{s_2}} = 0 \tag{27}$$

Doing the same analytical calculation illustrated in detail in the previous Section and Appendixes B, C, we can derive the continuum equation describing the macroscopic system within the long-wavelength limit framework. The third term of Eq. (27) can be rewritten as (see for more details Appendix D):

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^3}{|a(n-m)|^{s_2}} \approx \frac{3\left(1-2^{s_2-1}-3^{s-2}\right)\pi}{\Gamma(s_2)\sin\left(\frac{s-1}{2}\pi\right)} \frac{u^2(x,t)}{2\pi} \int_{-\infty}^{+\infty} dp\, |p|^{s_2-1} e^{ipx} \tilde{u}(p,t). \tag{28}$$

Using Riesz fractional derivative definitions, Eqs. (12) and (13), we obtain

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^3}{|a(n-m)|^s} \approx -\frac{3\left(1-2^{s_2-1}-3^{s_2-2}\right)\pi}{\Gamma(s_2)\sin\left(\frac{s_2-1}{2}\pi\right)} u^2(x,t)\,\frac{\partial^{s_2-1}}{\partial|x|^{s_2-1}}u(x,t)\,. \tag{29}$$

Substituting Eqs. (14) and (29) into Eq. (27) it follows that Eq. (27) is reduced into the following nonlinear fractional equation:

$$\frac{\partial^2}{\partial t^2}u(x,t) - g_{s_1-1}\frac{\partial^{s_1-1}}{\partial|x|^{s_1-1}}u(x,t) - j_{s_2-1}u^2(x,t)\,\frac{\partial^{s_2-1}}{\partial|x|^{s_2-1}}u(x,t) = 0, \tag{30}$$

with

$$j_{s_2-1} = 3\gamma\pi\frac{1-2^{s_2-1}-3^{s_2-2}}{\Gamma(s_2)\sin\frac{s_2-1}{2}\pi}, \tag{31}$$

We report in Appendix E the results for the $\alpha+\beta$–FPU model in which cubic and quartic terms appears simultaneously.

V. CONCLUSIONS AND PERSPECTIVES

Starting from the lattice dynamics of the FPU model with both harmonic and anharmonic long-range power law couplings, we have derived in the continuum limit a Generalized Fractional Boussinesq Equation (GFBE) as it is usually done for short range model [4, 5]. We have performed the analytical derivations by two different methods: i) using Riesz derivative and Hurwitz formulas of fractional calculus ii) performing a direct analysis of a Fourier spectrum in the $k \to 0$ limit. We also dealt with a variant of the model where masses and couplings are alternating in sign getting GFBE in formula (23).

In general, the presence of long-range couplings in the FPU model is reflected into the appearance of nonlocal terms in the continuum equations, the nonlocality is mathematically represented by fractional derivatives in GFBE. When the power of the couplings tends to infinity, the interactions become short-range, nonlocality is removed and fractional derivatives convert to ordinary partial derivatives. In this paper the fractional derivatives are mainly considered in the Riesz sense.

Our systematic formulation of mechanics based on fractional derivatives can be used to develop models of biological systems in which fractional power-law interaction are essential elements of biological phenomena (for instance the anomalous diffusion in cell biology). In the area of physics, fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle spreads at a rate inconsistent with the classical Brownian motion model [24]. In particular, the Riesz fractional derivative includes a left Riemann-Liouville derivative and a right Riemann-Liouville derivative that allow the modeling of flow regime impacts from either side of the domain [22]. Moreover, our formulation may be applicable to a study of condensed matter physics [33].

As future work, one should find the corresponding exact analytical localized solutions (solitonic solutions) and perform a systematic comparison of numerical simulations of the long-range FPU lattice dynamics where the initial condition will be given by the analytical results in the continuum limit. The effect of different boundary conditions should be analyzed in much detail due to its importance for models with long-range interactions. Another challenge for the future would be to find the correct transformation of space and time variable which would allow one to derive from the GFBE the generalized KDV valid for long-range interactions.

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Appendix A: Basic definitions of fractional calculus

In this Appendix we briefly review basic notions of fractional differentiation and integration. Fractional differentiation is an extension of the order of differentiation from an integer to real number. We introduce definitions given by Riemann-Liouville, Caputo and Jumarie (the modified Riemann-Liouville). We also introduce Riesz derivatives, which are the most useful representation applicable to a macroscopic model with long-range interaction [34]. In addition, we present fractional integration, which is the inverse operation of fractional differentiation.

Definition 1. A real function $u(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbf{R}$ if there exists a real number $p > \mu$, such that $u(x) = x^p u_1(x)$, where $u_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $u^{(n)} \in \mathbf{N}$ [17].

Definition 2. The definition of a fractional derivative of order α ($\alpha > 0$) for $u(x)$ with respect to x is formulated in the following two ways:

$$D_{x^+}^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_{-\infty}^{x} dy \; u(y)\, (x-y)^{n-\alpha-1}, \tag{A1}$$

$$D_{x^-}^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int\limits_{x}^{+\infty} dy \; u(y)\, (y-x)^{n-\alpha-1}. \tag{A2}$$

In this formula, the integer n is chosen for a given real number α such that $n-1 \leqslant \alpha < n$, and $\Gamma(n-\alpha)$ denotes Euler's gamma function. When an order α is an integer n, a fractional derivative Eq. (A1) and Eq. (A2) is reduced to standard derivatives of integer order:

$$D_{x^+}^{n} = \left(\frac{d}{dx}\right)^n, \quad D_{x^-}^{n} = \left(-\frac{d}{dx}\right)^n. \tag{A3}$$

Therefore, $D_{x^+}^{n} = (-1)^n D_{x^-}^{n}$, but in general $D_{x^+}^{\alpha} \neq (-1)^\alpha D_{x^-}^{\alpha}$; see Ref. [34] and

$$D_{x^+}^{\alpha} x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \qquad for \qquad x > 0. \tag{A4}$$

The formulations of fractional differentiation [17] include integration of order α for $u(x)$ was defined in the following two ways:

$$I_{x^+}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{x} dy \, u(y)\, (x-y)^{\alpha-1}, \tag{A5}$$

$$I_{x^-}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{x}^{+\infty} dy \, u(y)\, (y-x)^{\alpha-1}. \tag{A6}$$

$$I^0 u(x) = u(x). \tag{A7}$$

Fractional differentiation of order α is the inverse operation of fractional integration of order α:

$$D_{\mu^+}^{\alpha} I_{\mu^+}^{\alpha} u(x) = u(x), \quad D_{\mu^-}^{\alpha} I_{\mu^-}^{\alpha} u(x) = u(x). \tag{A8}$$

If we assume $u(x)$ to be a regular function in $-\infty < x < \infty$, an integer derivative and a fractional derivative are commutable as follows

$$D^{\alpha}_{\mu^+}\left(\partial_\nu u(x)\right)=\partial_\nu\left(D_{\mu^+}u(x)\right),\quad D^{\alpha}_{\mu^-}\left(\partial_\nu u(x)\right)=\partial_\nu\left(D_{\mu^-}u(x)\right). \tag{A9}$$

However, the commutation relation between two fractional derivatives is not satisfied in general:

$$D^{\alpha}_{\mu^+}\left(D^{\beta}_{\mu^-}u\right)\neq D^{\beta}_{\mu^-}\left(D^{\alpha}_{\mu^+}u\right). \tag{A10}$$

Definition 3. The Caputo fractional derivative D^α of $u(x)$ is defined as [17]

$$D^{\alpha}u(x)=\frac{1}{\Gamma(n-\alpha)}\int\limits_0^x(x-\xi)^{n-\alpha-1}u^{(n)}(\xi)d\xi\quad(\alpha>0), \tag{A11}$$

for $n-1<\alpha\leqslant n$, $n\in\mathbf{N}$, $x>0$, $u\in C^n_{-1}$. The following are two basic properties of the Caputo fractional derivative:
(1) Let $u\in C^n_{-1}$, $n\in\mathbf{N}$. Then $D^\alpha u$, $0\leqslant\alpha\leqslant n$ is well defined and $D^\alpha u\in C_{-1}$.
(2) Let $n-1<\alpha\leqslant n$, $n\in\mathbf{N}$ and $u\in C^n_\lambda$, $\lambda\geqslant -1$.
Then one has

$$(I^\alpha D^\alpha)u(x)=u(x)-\sum_{k=0}^{n-1}u^k(0^+)\frac{x}{k!},\quad(x>0). \tag{A12}$$

In this paper only real and positive α has been considered. Similar to integer-order differentiation, the Caputo fractional differentiation is a linear operation

$$D^\alpha(\lambda f(x)+\mu g(x))=\lambda D^\alpha f(x)+\mu D^\alpha g(x), \tag{A13}$$

where λ, μ are constants.

Definition 4. A generalization of the classical Leibniz rule

$$D^n(fg)=\sum_{k=0}^{\infty}\left(\begin{array}{c}n\\k\end{array}\right)\left(f^{n-k}\right)g^k \tag{A14}$$

from integer n to fractional α contains an infinite series

$$D^\alpha(fg)(x)=\sum_{k=0}^{\infty}\left(\begin{array}{c}\alpha\\k\end{array}\right)\left(D^{\alpha-k}_x f\right)(x)D^k_x\,g, \tag{A15}$$

with $f(\xi)$ continuous in $[0,\,x]$ and $g(\xi)$ having $(n+1)$ continuous derivatives in $[0,\,x]$. The sum is infinite and contains integrals of fractional order (for $k>[\alpha]+1$).

Definition 5. For n to be the smallest integer that exceeds α, the Caputo fractional derivative operator of order $\alpha>0$ with respect to x_μ, is formulated as [17]

$$D^\alpha_{x_\mu+}u(x)=\frac{1}{\Gamma(n-\alpha)}\int\limits_{-\infty}^{x_\mu}dy\frac{\partial^n}{\partial y^n}u(x_1,...,y,x_{\mu+1},...)(x_\mu-y)^{n-\alpha-1} \tag{A16}$$

and

$$D^\alpha_{x_\mu-}u(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int\limits_{x_\mu}^{+\infty}dy\frac{\partial^n}{\partial y^n}u(x_1,...,y,x_{\mu+1},...)(y-x_\mu)^{n-\alpha-1} \tag{A17}$$

if $n-1<\alpha<n$. Of course $D_x^\alpha u(x)=\frac{\partial^n u(x)}{\partial x^n}$ if $\alpha\in\mathbf{N}$. The corresponding Fourier transformation of order α is defined as [34]

$$D_{x^+}^\alpha u\left(x,t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\left(+ip\right)^\alpha\hat{u}\left(p,t\right)e^{ipx} \tag{A18}$$

$$D_{x^-}^\alpha u\left(x,t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\left(-ip\right)^\alpha\hat{u}\left(p,t\right)e^{ipx}, \tag{A19}$$

where

$$\hat{u}\left(p,t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}dp\,u\left(x,t\right)e^{-ipx}. \tag{A20}$$

Riesz derivatives can be defined as follows, except when α is an odd number [35]:

$$\frac{\partial^\alpha}{\partial\left|x\right|^\alpha}u\left(x,t\right)=-\frac{1}{2\cos\frac{\alpha\pi}{2}}\left[D_{x^+}^\alpha+D_{x^-}^\alpha\right]u\left(x,t\right). \tag{A21}$$

The above derivative is singular at $\alpha=1,3,5,...$. Riesz derivatives have symmetry with respect to the transformation $x\to-x$. When $u(x)$ is a regular function in $-\infty<x<\infty$, the two formulations provided by Riemann-Liouville and Caputo are equivalent. We note that the commutation relation between the two fractional derivatives defined by caputo is the same as that of the Riemann-Liouville derivatives.

Definition 6. The relationship between left and right-derivatives in real space reads

$$D_{x^-}^\alpha u\left(x\right)=D_{(-x)^+}^\alpha u\left(x\right) \tag{A22}$$

if $u(x)$ is a real and even function, and

$$D_{x^-}^\alpha u\left(x\right)=-D_{(-x)^+}^\alpha u\left(x\right) \tag{A23}$$

if $u(x)$ is a pure imaginary and odd function. Notice that the above transformation is valid only for $x<0$.

Definition 7. The modified Riemann-Liouville derivative reads

$$D_x^\alpha u\left(x\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dx}\int\limits_0^x\left(x-\xi\right)^{-\alpha}\left(u\left(\xi\right)-u\left(0\right)\right)d\xi, \tag{A24}$$

where u is a continuous (but not necessarily differentiable) function. However, there exists a non-commutative property

$$D^{\alpha+\beta}\neq D^\alpha D^\beta\neq D^\beta D^\alpha. \tag{A25}$$

The Riemann-Liouville fractional derivative has some notable disadvantages in applications such as nonzero of the fractional derivative of constants,

$$D_t^\alpha C=\frac{t^{-\alpha}}{\Gamma(1-\alpha)}C, \tag{A26}$$

which means that the dissipation does not vanish for a system in equilibrium. The Caputo fractional differentiation of a constant results in zero. Therefore, to use the usual initial value problems

$$u(t_0) = t_0, \qquad \left(D_t^k f\right)(t_0) = C_k, \qquad k = 1, ..., n, \tag{A27}$$

one is lead to the application of Caputo fractional derivatives instead of the Riemman-Liouville derivative.

Finally, we should clearly emphasize that many usual properties of the ordinary derivative D^n are not realized for fractional derivative operators D^α. For example, the Leibniz rule, chain rule, semi-group property $\left(D_x^\alpha D_x^\alpha \neq D_x^{2\alpha}\right)$ have strongly complicated analogs for the operators D^α. We refer to [17, 35, 36] for more informations on the mathematical properties of fractional derivatives and integrals.

Appendix B

We observe that

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \sum_{\substack{m=-\infty \\ m-n\neq 0}}^{+\infty} \frac{1}{|a(n-m)|^{s_1}} e^{ikn}\hat{u}(k,t) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \sum_{\substack{m=-\infty \\ m-n\neq 0}}^{+\infty} \frac{1}{|a(n-m)|^{s_1}} e^{ikm}\hat{u}(k,t), \tag{B1}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} = \sum_{\substack{m=-\infty \\ m-n\neq 0}}^{0} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} - \sum_{\substack{m=1 \\ m-n\neq 0}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}}. \tag{B2}$$

Assuming $n' = m - n$, one gets

$$\sum_{\substack{m=-\infty \\ m-n\neq 0}}^{0} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} = \frac{u_n}{2\pi}\int_{-\infty}^{+\infty} dk \left(\sum_{n'=1}^{+\infty} \frac{1-2e^{-ikn'}+e^{-i\left(k+k'\right)n'}}{|an'|^{s_2}}\right) e^{ikn}\tilde{u}(k,t), \tag{B3}$$

$$\sum_{\substack{m=1 \\ m-n\neq 0}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} = \frac{u_n}{2\pi}\int_{-\infty}^{+\infty} dk \left(\sum_{n'=1}^{+\infty} \frac{1-2e^{ikn'}+e^{i\left(k+k'\right)n'}}{|an'|^{s_2}}\right) e^{ikn}\tilde{u}(k,t), \tag{B4}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \left(\sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} \frac{1}{|an'|^{s_1}}\right) e^{ikn}\hat{u}(k,t) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \left(\sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} \frac{e^{ikn'}}{|an'|^{s_1}}\right) e^{ikn}\hat{u}(k,t), \tag{B5}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} = \frac{u_n}{2\pi}\int_{-\infty}^{+\infty} dk \sum_{n'=1}^{+\infty} \left(\frac{\left(1-2e^{-ikn'}+e^{-i\left(k+k'\right)n'}\right) - \left(1-2e^{ikn'}+e^{i\left(k+k'\right)n'}\right)}{|an'|^{s_2}}\right) e^{ikn}\tilde{u}(k,t), \tag{B6}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} = \frac{u_n}{2\pi}\int_{-\infty}^{+\infty} dk \sum_{n'=1}^{+\infty} \left(2\left(\frac{e^{ikn'}-e^{-ikn'}}{|an'|^{s_2}}\right) - \left(\frac{e^{i\left(k+k'\right)n'}-e^{-i\left(k+k'\right)n'}}{|an'|^{s_2}}\right)\right) e^{ikn}\tilde{u}(k,t). \tag{B7}$$

We can rewrite the Eqs. (B5) and (B7), respectively as

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\ e^{ikn}\ \hat{u}(k,t) \left[\tilde{J}_1(0) - \tilde{J}_1(k)\right], \tag{B8}$$

and

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} = \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk\ e^{ikn}\ \tilde{u}(k,t) \left[2\tilde{J}_2(k) - \tilde{J}_2(k+k')\right]. \tag{B9}$$

In Eqs. (B8)-(B9) we used the following notation:

$$\tilde{J}_1(k) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{e^{ikn}}{|an|^{s_1}} \quad \text{and} \quad \tilde{J}_2(k) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{e^{ikn}}{|an|^{s_2}}. \tag{B10}$$

One has $\sum\limits_{n=1}^{\infty} \frac{e^{+ikn}}{|n|^s} = L_{i,s}\left(e^{+ikn}\right)$, where $L_{i,s}$ is the polylogarithmic function such that

$$L_{i,s}(e^{\mu}) = \Gamma(1-s)(-\mu)^{s-1} + \sum_{n=0}^{\infty} \frac{\zeta(s-n)}{n!} (\mu)^n. \tag{B11}$$

Using the Hurwitz formula [37], one has that

$$\tilde{J}_1(k) = \sum_{n=1}^{+\infty} \frac{e^{-ikn} + e^{ikn}}{|an|^{s_1}} = a^{-s_1} \left[\Gamma(1-s_1)\left((-ik)^{s_1-1} + (ik)^{s_1-1}\right) + \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!} \left((-ik)^{2n} + (ik)^{2n}\right)\right]. \tag{B12}$$

Since

$$(ik)^{\alpha} + (-ik)^{\alpha} = 2|k|^{\alpha} \cos\left(\frac{\alpha\pi}{2}\right), \tag{B13}$$

then

$$(ik)^{2n} + (-ik)^{2n} = 2(-1)^n |k|^{2n}. \tag{B14}$$

Therefore

$$\tilde{J}_1(k) = 2a^{-s_1} \left[\Gamma(1-s_1)\cos\left(\frac{s_1-1}{2}\pi\right)|k|^{s_1-1} + \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!}(-1)^n |k|^{2n}\right]. \tag{B15}$$

Using the relation

$$\Gamma(1-s)\cos\left(\frac{s-1}{2}\pi\right) = -\frac{\pi}{2\Gamma(s)\sin\left(\frac{s-1}{2}\pi\right)},$$

one gets

$$\tilde{J}_1(k) = 2a^{-s_1} \left[-\frac{\pi}{2\Gamma(s_1)\sin\left(\frac{s_1-1}{2}\pi\right)}|k|^{s_1-1} + \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!}(-1)^n |k|^{2n}\right], \tag{B16}$$

and

$$\tilde{J}_1(0) - \tilde{J}_1(k) = \frac{\pi a^{-s_1}}{\Gamma(s_1)\sin\left(\frac{s_1-1}{2}\pi\right)} |k|^{s_1-1} - 2a^{-s_1} \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!} (-1)^n |k|^{2n} + 2a^{-s_1}\zeta(s_1). \tag{B17}$$

It is easy to observe that, $\tilde{J}_1(0) - \tilde{J}_1(k) \sim \omega^2(k)$. In the same way, ie. combining the polylogaritmic function and the Hurwitz formula [37], we obtain

$$\begin{aligned} & 2\tilde{J}_2(k) - \tilde{J}_2(k+k') = \\ & = 2a^{-s_2}\left(2-2^{s_2-1}\right)\Gamma(1-s_2)\left[(-ik)^{s_2-1} - (ik)^{s_2-1}\right] + 2a^{-s_2} \sum_{n=0}^{+\infty} \frac{\left(2-2^{s_2-1}\right)\zeta(s_2-n)}{n!}\left[(-ik)^n - (ik)^n\right]. \end{aligned} \tag{B18}$$

In Eqs. (B17) and (B18), ζ is the Riemann zeta function. In the continuum limit ($k \to 0$) the long-wavelength modes are singled out and the leading term of Eqs. (B8) and (B9) are derived from the first term of the right-hand side of Eqs. (B17) and (B18), respectively.

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} \simeq \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \left(\frac{\pi a^{-s_1}}{\Gamma(s_1)\sin\left(\frac{s_1-1}{2}\pi\right)} |k|^{s_1-1} \right) e^{ikn}\ \hat{u}(k,t). \tag{B19}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} \approx \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk \left\{ 2a^{-s_2}\Gamma(1-s_2)\left(2-2^{s_2-1}\right)\left((-ik)^{s_2-1} - (ik)^{s_2-1}\right)\right\} e^{ikn}\tilde{u}(k,t). \tag{B20}$$

In continuum limit one puts $u_n(t) = u(x = na, t)$, obtaining

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} \simeq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{a}\, e^{ik\frac{x}{a}} \left(\frac{\pi}{\Gamma(s_1)\sin\left(\frac{s_1-1}{2}\pi\right)} \right) \left|\frac{k}{a}\right|^{s_1-1}\ \hat{u}(k,t). \tag{B21}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} \approx \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{a} \left\{ \frac{2\left(2-2^{s_2-1}\right)\pi}{\Gamma(s_2)\sin(s_2\pi)} \left(\left(-i\frac{k}{a}\right)^{s_2-1} - \left(i\frac{k}{a}\right)^{s_2-1}\right)\right\} e^{ikn}\tilde{u}(k,t). \tag{B22}$$

Setting $\frac{k}{a} = p$, it follows

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} \simeq \frac{\pi}{\Gamma(s)\sin\frac{s_1-1}{2}\pi} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp\ e^{ipx}\, |p|^{s_1-1}\ \hat{u}(p,t), \tag{B23}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} \approx \frac{u(x,t)}{2\pi} \int_{-\infty}^{+\infty} dp\ e^{ipx} \frac{2\left(2-2^{s_2-1}\right)\pi}{\Gamma(s_2)\sin(s_2\pi)} \left((-ip)^{s_2-1} - (ip)^{s_2-1}\right)\tilde{u}(p,t). \tag{B24}$$

Appendix C

One has

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^s} = \sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{u_n^2-2u_nu_m-u_m^2}{|a(n-m)|^s}, \tag{C1}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^s} = \frac{u_n}{2\pi}\int_{-\infty}^{+\infty} dk \sum_{\substack{n'=-\infty \\ n'\neq 0}}^{n'=+\infty} \left(\frac{1}{|an'|^s} - 2\frac{e^{ikn'}}{|an'|^s} + \frac{e^{i\left(k+k'\right)n'}}{|an'|^s}\right) e^{ikn}\tilde{u}(k,t), \tag{C2}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^s} = \frac{u_n}{2\pi}\int_{-\infty}^{+\infty} dk\ e^{ikn}\ \tilde{u}(k,t)\left[\tilde{J}(0) - 2\tilde{J}(k) + \tilde{J}(k+k')\right], \tag{C3}$$

where

$$\tilde{J}(k) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{e^{ikn}}{|an|^s}. \tag{C4}$$

Using again the Hurwitz formula [37], $\tilde{J}(0) - 2\tilde{J}(k) + \tilde{J}(k+k')$ is rewritten as

$$\begin{aligned}&\tilde{J}(0) - 2\tilde{J}(k) + \tilde{J}(k+k') = \\ &= \frac{\pi a^{-s}}{\Gamma(s)\sin\left(\frac{s-1}{2}\pi\right)}\left(2|k|^{s-1} - |k+k'|^{s-1}\right) + 2a^{-s}\sum_{n=0}^{n=+\infty}\frac{\zeta(s-2n)}{(2n)!}(-1)^n\left(2|k|^{2n} - |k+k'|^{2n}\right) + 2a^{-s}\zeta(s).\end{aligned} \tag{C5}$$

In the continuum limit ($k \to 0$), the leading term of Eq. (C3) is derived from the first term of the right-hand side of Eq. (C5). In continuum limit one puts $u_n(t) = u(x = na, t)$, obtaining

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^s} \approx \frac{u_n}{2\pi}\int_{-\infty}^{+\infty} dk\ e^{i\frac{k}{a}x}\left\{\frac{\pi a^{-s}}{\Gamma(s)\sin\left(\frac{s-1}{2}\pi\right)}\left(2|k|^{s-1} - |k+k'|^{s-1}\right)\right\}\tilde{u}(k,t). \tag{C6}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^s} \approx \frac{u(x,t)}{2\pi}\int_{-\infty}^{+\infty} \frac{dk}{a} e^{i\frac{k}{a}x}\left\{\frac{\pi a^{-1}}{\Gamma(s)\sin\left(\frac{s-1}{2}\pi\right)}\left(2\left|\frac{k}{a}\right|^{s-1} - \left|\frac{k+k'}{a}\right|^{s-1}\right)\right\}\tilde{u}(k,t), \tag{C7}$$

ie.

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty} \frac{(u_n-u_m)^2}{|a(n-m)|^s} \approx \frac{u(x,t)}{2\pi}\int_{-\infty}^{+\infty} \frac{dk}{a}\ e^{i\frac{k}{a}x}\left(\frac{\left(2-2^{s-1}\right)\pi}{\Gamma(s)\sin\left(\frac{s-1}{2}\pi\right)}\right)\left|\frac{k}{a}\right|^{s-1}\ \tilde{u}(k,t), \tag{C8}$$

Appendix D

We start by observing

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{\left(u_n-u_m\right)^3}{\left|a\left(n-m\right)\right|^s}=\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{u_n^3-3u_n^2u_m+3u_nu_m^2-u_m^3}{\left|a\left(n-m\right)\right|^s}, \tag{D1}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{\left(u_n-u_m\right)^3}{\left|a\left(n-m\right)\right|^{s_2}}=\frac{u_n^2}{2\pi}\int_{-\infty}^{+\infty}dk\sum_{\substack{n'=-\infty \\ n'\neq 0}}^{n'=+\infty}\left(\frac{1}{\left|an'\right|^{s_2}}-3\frac{e^{ikn'}}{\left|an'\right|^{s_2}}+3\frac{e^{i\left(k+k'\right)n'}}{\left|an'\right|^{s_2}}-\frac{e^{i\left(k+k'+k''\right)n'}}{\left|an'\right|^{s_2}}\right)e^{ikn}\tilde{u}\left(k,t\right), \tag{D2}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{\left(u_n-u_m\right)^3}{\left|a\left(n-m\right)\right|^{s_2}}=\frac{u_n^2}{2\pi}\int_{-\infty}^{+\infty}dk\ e^{ikn}\ \tilde{u}\left(k,t\right)\left[\tilde{J}_2\left(0\right)-3\tilde{J}_2\left(k\right)+3\tilde{J}_2\left(k+k'\right)-\tilde{J}_2\left(k+k'+k''\right)\right]. \tag{D3}$$

Once again combining the polylogarithmic function and the Hurwitz formula, we obtain

$$\begin{aligned}\tilde{J}_2\left(0\right)-3\tilde{J}_2\left(k\right)+3\tilde{J}_2\left(k+k'\right)-\tilde{J}_2\left(k+k'+k''\right)=\frac{\pi a^{-s_2}}{\Gamma\left(s_2\right)\sin\left(\frac{s_2-1}{2}\pi\right)}\left(3\left|k\right|^{s_2-1}-3\left|k+k'\right|^{s_2-1}-\left|k+k'+k''\right|^{s_2-1}\right)\\ +2a^{-s_2}\sum_{n=0}^{n=+\infty}\frac{\zeta\left(s_2-2n\right)}{\left(2n\right)!}\left(-1\right)^n\left(3\left|k\right|^{2n}+3\left|k+k'\right|^{2n}+\left|k+k'+k''\right|^{2n}\right)+2a^{-s_2}\zeta\left(s_2\right),\end{aligned} \tag{D4}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{\left(u_n-u_m\right)^3}{\left|a\left(n-m\right)\right|^{s_2}}\approx\frac{u_n^2}{2\pi}\int_{-\infty}^{+\infty}dk\ e^{i\frac{k}{a}x}\left\{\frac{\pi a^{-s}}{\Gamma\left(s\right)\sin\left(\frac{s-1}{2}\pi\right)}\left(3\left|k\right|^{s-1}-3\left|k+k'\right|^{s-1}-\left|k+k'+k''\right|^{s-1}\right)\right\}\tilde{u}\left(k,t\right). \tag{D5}$$

In continuum limit one puts $u_n(t)=u(x=na,t)$, obtaining

$$\begin{aligned}\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{\left(u_n-u_m\right)^3}{\left|a\left(n-m\right)\right|^{s_2}}\approx\\ \approx\frac{u^2\left(x,t\right)}{2\pi}\int_{-\infty}^{+\infty}dk\ e^{i\frac{k}{a}x}\left\{\frac{\pi a^{-1}}{\Gamma\left(s_2\right)\sin\left(\frac{s_2-1}{2}\pi\right)}\left(3\left|\frac{k}{a}\right|^{s_2-1}-3\left|\frac{k+k'}{a}\right|^{s_2-1}-\left|\frac{k+k'+k''}{a}\right|^{s_2-1}\right)\right\}\tilde{u}\left(k,t\right),\end{aligned} \tag{D6}$$

$$\sum_{\substack{m=-\infty \\ m\neq n}}^{m=+\infty}\frac{\left(u_n-u_m\right)^3}{\left|a\left(n-m\right)\right|^{s_2}}\approx\frac{u^2\left(x,t\right)}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{a}\ e^{i\frac{k}{a}x}\left(\frac{\left(3-3.2^{s_2-1}-3^{s_2-1}\right)\pi}{\Gamma\left(s_2\right)\sin\left(\frac{s_2-1}{2}\pi\right)}\right)\left|\frac{k}{a}\right|^{s_2-1}\tilde{u}\left(k,t\right), \tag{D7}$$

Appendix E: The (α+β)–FPU model with power-law long-range interactions

For the sake of completeness, we report in this Appendix results for the α+β–FPU model in which cubic and quartic terms are both present. The Hamiltonian of the model reads

$$H\left(u_n,\dot{u}_n\right)=\frac{1}{2}\sum_{n=-\infty}^{+\infty}\dot{u}_n^2+\frac{\chi}{4}\sum_{\substack{n,m=-\infty\\ m\neq n}}^{+\infty}\frac{\left[u_n-u_m\right]^2}{\left|a\left(n-m\right)\right|^s}+\frac{\gamma}{3}\sum_{\substack{n,m=-\infty\\ m<n}}^{+\infty}\frac{\left[u_n-u_m\right]^3}{\left|a\left(n-m\right)\right|^s}+\frac{\lambda}{8}\sum_{\substack{n,m=-\infty\\ m\neq n}}^{+\infty}\frac{\left[u_n-u_m\right]^4}{\left|a\left(n-m\right)\right|^s} \tag{E1}$$

One gets

$$\ddot{u}_n+\chi\sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty}\frac{u_n-u_m}{\left|a\left(n-m\right)\right|^s}+\gamma\sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty}\frac{\left[u_n-u_m\right]^2}{\left|a\left(n-m\right)\right|^s}f_{n,m}+\lambda\sum_{\substack{m=-\infty\\ m\neq n}}^{+\infty}\frac{\left[u_n-u_m\right]^3}{\left|a\left(n-m\right)\right|^s}=0 \tag{E2}$$

where again

$$f_{n,m}=\begin{cases}+1 & ,m<n\\ -1 & ,m>n\end{cases}$$

In the continuum limit for a lattice field model, introducing Eqs. (14), (15) and (29) into Eq. (E2) we get

$$\frac{\partial^2}{\partial t^2}u\left(x,t\right)-g_{s-1}\frac{\partial^{s-1}}{\partial\left|x\right|^{s-1}}u\left(x,t\right)-h_{s-1}u\left(x,t\right)\left[D_{x^-}^{s-1}-D_{x^+}^{s-1}\right]u\left(x,t\right)-r_{s-1}u^2\left(x,t\right)\frac{\partial^{s-1}}{\partial\left|x\right|^{s-1}}u\left(x,t\right)=0, \tag{E3}$$

where the constants g_{s-1} and k_{s-1} are given by Eq. (17) and Eq. (24), respectively (when $s_2=s_1=s$). $r_{s-1}=j_{s-1}\left(\gamma=\lambda\right)$ and j_{s-1} is given by Eq. (31).