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Digital Two-dimensional Bijective Reflection and Associated Rotation

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Abstract. In this paper, a new bijective reflection algorithm in two dimensions is proposed along with an associated rotation. The reflection line is defined by an arbitrary Euclidean point and a straight line passing through this point. The reflection line is digitized and the 2D space is paved by digital perpendicular (to the reflection line) straight lines. For each perpendicular line, digital points are reflected by central symmetry with respect to the reflection line. Two consecutive digital reflections are combined to define a digital bijective rotation about arbitrary center (i.e. bijective digital rigid motion).

Key words: Digital Reflection; Bijective Digital Rotation; Digital Rotation; Bijective digital rigid motion

1 Introduction

Reflection transformation is a one of the most basic linear transforms [1] with many potential applications. In particular, they are key to defining n -dimensional rotations [2] and can be used in applications where reflections may play a key role such as lightwave simulations [3–5]. There is however surprisingly few works that deal with the problem such a transform in the digital world. The goal of this paper is to propose an algorithm that performs a digital bijective reflection on a two-dimensional image and to use this to propose a digital bijective rotation algorithm that works for an arbitrary rotation center (i.e. bijective digital rigid motion. See [6] for a first algorithm and a theoretical discussion on the subject of digital rigid motions). While the reflection transform in the digital world has not been studied much, digital rotations and more precisely bijective digital rotations has been a point of interest of the digital community for some years now [7]. The characterization of a discretized rotation is stated in [8] which is the composition of an Euclidean rotation with rounding operations. Using Gaussian integers the bijective discrete rotation is characterized in [9]. In [10], an incremental approach to discretized rotations is explained. In [11], bijective rigid motion in 2D cartesian

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18 grid is stated. Other mentionable work on characterization of bijective rotations
 19 in 3D space is [12]. The bijective rotation is also studied on hexagonal grid in
 20 discrete space [13].

21 A continuous transform applied to a digital image is not, in general, bijective,
 22 nor injective or surjective for that matter, even for an apparently very simple
 23 transform such as the reflection transform (see Figure 1). Our idea is to digitize
 24 the Euclidean reflection straight line as naive digital lines (i.e. 8-connected digital
 25 lines) and to partition the two-dimensional space into naive digital lines that
 26 are perpendicular to the reflection line. A central symmetry is performed on
 27 the points of each *Perpendicular Digital Straight Line* (PDSL) according to the
 28 *Digital Reflection Straight Line* (DRSL). This is always possible because we can
 29 compute the exact pixel to pixel correspondance on each side of the DRSL for
 30 such naive digital lines. The main problem comes from the fact that the digital
 31 reflection line and a given digital perpendicular line may or may not intersect
 32 and so two different cases have to be considered.

33 The organization of the paper is as follows: in Sec. 2, we present the pre-
 34 liminaries. Section 3 presents the mathematical foundation of bijective digital
 35 reflection. The bijective digital rotation about an arbitrary center can be deter-
 36 mined by applying digital reflection twice (see Sec. 4). An error criteria based on
 37 the distance between the continuous and the digital rotated points are presented
 38 in Sec. 4.2. The concluding remarks are presented in Sec. 5.

39 2 Preliminaries

40 Let $\{i, j\}$ denote the canonical basis of the 2-dimensional Euclidean vector space.
 41 Let \mathbb{Z}^2 be the subset of \mathbb{R}^2 that consists of all the integer coordinate points.
 42 A *digital (resp. Euclidean) point* is an element of \mathbb{Z}^2 (resp. \mathbb{R}^2). Two digital
 43 points $p(p_x, p_y)$ and $q(q_x, q_y)$ are said to be 8-connected if $|p_x - q_x| \leq 1$ and
 44 $|p_y - q_y| \leq 1$.

45 For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the biggest integer smaller or equal to x and $\lceil x \rceil$ is the
 46 smallest integer greater or equal to x .

47 A *naive digital straight line* is defined as all the digital points verifying
 48 $-\frac{\max(|a|, |b|)}{2} \leq ax - by < \frac{\max(|a|, |b|)}{2}$, where $\frac{a}{b}$ represents the slope of the *Digital*
 49 *Straight Line* (DSL) and $\omega = \max(|a|, |b|)$ the arithmetical thickness [14]. A
 50 naive DSL is 8-connected such that if you remove any point of the line then it
 51 is not 8-connected anymore [14]. There are no simple points.

52 A *reflection transformation* $\mathcal{R}_{\theta, (x_o, y_o)} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ reflects (or flips) a continu-
 53 ous point, like in a mirror, on a continuous straight line called the reflection line.
 54 The reflection line is defined as the line passing through a point of coordinates
 55 $(x_o, y_o) \in \mathbb{R}^2$ and the vector director $v = (\sin \theta, \cos \theta)$. The corresponding digital
 56 reflection is denoted $R_{\theta, (x_o, y_o)} : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$. A continuous rotation $\mathcal{Rot}_{\theta, (x_o, y_o)}$ of
 57 center (x_o, y_o) and angle θ can be defined as the composition of two continuous
 58 reflections $\mathcal{R}_{\alpha, (x_o, y_o)}$ and $\mathcal{R}_{\alpha + \frac{\theta}{2}, (x_o, y_o)}$.

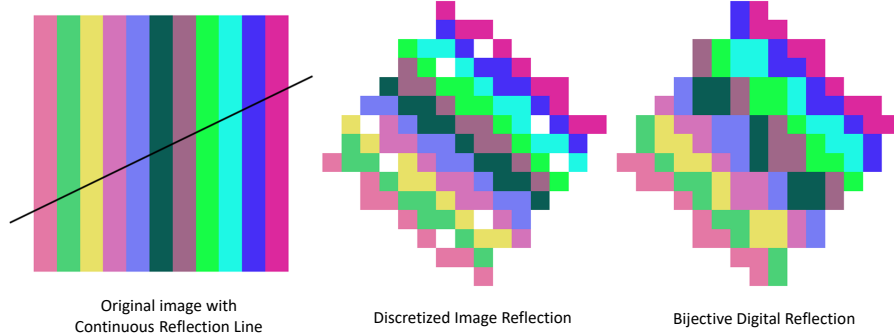


Fig. 1. Reflection Transform applied to an image. On the left, the original image. In the center, the continuous reflection transform composed with a rounding function. On the right, our proposed bijective digital reflection

59 **3 Digital Reflection**

60 **3.1 Principle**

61 Let us consider the continuous reflection transform, $\mathcal{R}_{\theta,(x_o,y_o)}$. It is not difficult
 62 to note that if we simply compose the continuous reflection with a digitization
 63 transform (such as for instance $\mathcal{D} : (x, y) \mapsto (\lfloor x + 0.5 \rfloor, \lfloor y + 0.5 \rfloor)$) then we have
 64 a transform that is, in general, neither injective, surjective and of course not
 65 bijective (see the middle image in Figure 1).

66 To avoid this problem and create a digital bijective reflection transform, a
 67 completely digital framework is proposed here based on the following digital
 68 primitives:

- 69 – A naive DSL called *digital reflection straight line* (DRSL);
- 70 – and a partition of the digital space with naive DSLs that are perpendicular to
 71 the digital reflection straight line. These digital lines are called *Perpendicular*
 72 *Digital Straight Lines* (PDSL).

73 The idea of the reflection method is the following: let us consider a point p
 74 that belongs to a given digital perpendicular straight line P_k (k is a parameter
 75 identifying the PDSL in the partition). The point p may belong to the digital
 76 reflection line (in case there is an intersection) or lie on either side of that line.
 77 Let the digital reflection of p be determined in the following way (see Figure 2):

- 78 – let us first suppose that p lies in the intersection between the PDSL P_k
 79 and the DRSL. Such an intersection does not always exist between the two
 80 digital lines but when it does (in this case point p) then the image of p by
 81 the digital reflection is p ;

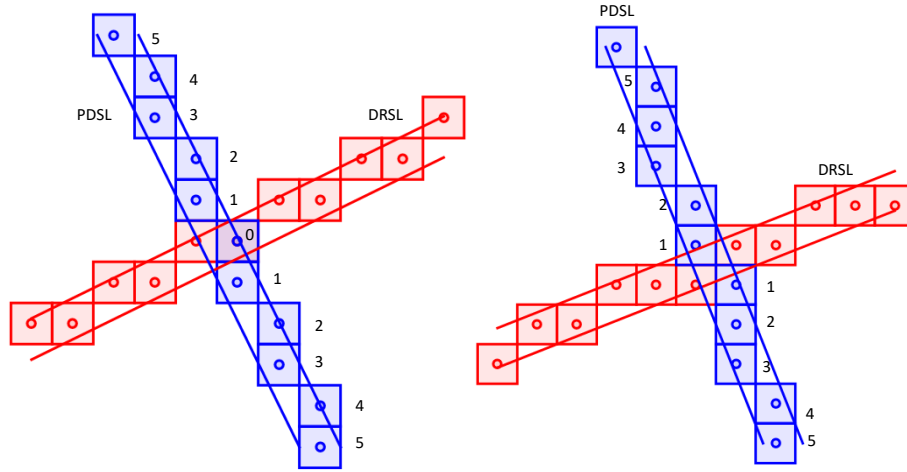


Fig. 2. The image of each point is the mirror image in the perpendicular line according to the digital reflection straight line. The image of a point numbered i is the opposite point numbered i . The point numbered 0 is its own image.

82 – Let us now suppose that p does not belong to the intersection between the
 83 PDSL and the DRSL (whether such an intersection exists or not). The point
 84 p is then located on one side of the DRSL: the digital space is split by the
 85 DRSL into three regions, the DRSL and two regions on each side of it. All
 86 the digital points of P_k on each side of the DRSL can formally be ordered
 87 according to the distance to the DRSL with as first point the closest one to
 88 the DRSL. As we will see, there actually is no real ordering needed. Let us
 89 suppose that p is the n^{th} point of the ordered list on its side and a digital
 90 point q is the n^{th} point of the list on the other side of the DRSL. The digital
 91 reflection of p will then be the point q and vice-versa.

92 Figure 2 shows the two possible cases: on the left, the perpendicular digital
 93 line has an intersection point (the point marked as point 0) with the digital
 94 reflection line, while on the right there is no intersection point. The points marked
 95 by the number i (from 1 to 5) are images of each other. The image of the
 96 point marked 0 is itself. The idea is to create a digital transformation that is
 97 bijective and easily reversible. Let us now, in the next section, present the various
 98 definitions and mathematical details to make this work out.

99 3.2 Mathematical Details

100 The mathematical details of the works is stated here. Let us consider the con-
 101 tinuous reflection \mathcal{L} line defined by the continuous point $(x_o, y_o) \in \mathbb{R}^2$ and the

102 direction vector $v = (a, b) = (\sin \theta, \cos \theta)$. The analytical equation of the reflec-
 103 tion line is given by: $\mathcal{L} = \{(x, y) \in \mathbb{R}^2 : a(x - x_o) - b(y - y_o) = 0\}$.

The *digital reflection straight line (DRSL)* the digitization (as naive DSL) of \mathcal{L} defined as all the digital points verifying:

$$DRSL : -\frac{\max(|a|, |b|)}{2} \leq a(x - x_o) - b(y - y_o) < \frac{\max(|a|, |b|)}{2}$$

The *perpendicular digital straight lines (PDSL)* P_k are naive DSLs defined as all the digital points (x, y) verifying :

$$P_k : \frac{(2k - 1) \max(|a|, |b|)}{2} \leq b(x - x_o) + a(y - y_o) < \frac{(2k + 1) \max(|a|, |b|)}{2}$$

104 It is easy to see that all the PDSLs are perpendicular to the DRSL and that
 105 the set of PDSLs, as a set of naive digital lines, partitions the two dimensional
 106 digital space.

107 Let us now suppose for what follows, w.l.o.g, that we have $-\pi/4 \leq \theta \leq \pi/4$,
 108 then $0 \leq |a| \leq b$ and thus $\max(|a|, |b|) = b$. The DRSL is then defined by
 109 $-\frac{b}{2} \leq a(x - x_o) - b(y - y_o) < \frac{b}{2}$ and the PDSLs are then defined by $P_k :$
 110 $\frac{(2k-1)b}{2} \leq b(x - x_o) + a(y - y_o) < \frac{(2k+1)b}{2}$.

111 The method supposes that we are able to formally order the points of a given
 112 perpendicular line that are located on either side of the reflection line. This is
 113 always possible because the PDSLs are naive lines and for an angle verifying
 114 $-\pi/4 \leq \theta \leq \pi/4$, there is one and only one point per integer ordinate y (For
 115 angles $\pi/4 \leq \theta \leq 3\pi/4$, we will have, symmetrically, one and only one point per
 116 integer abscissa x in a given PDSL).

More precisely, $\frac{(2k-1)b}{2} \leq b(x - x_o) + a(y - y_o) < \frac{(2k+1)b}{2}$ means that $\frac{2k-1}{2} +$
 $x_o - \frac{a}{b}(y - y_o) \leq x < \frac{2k+1}{2} + x_o - \frac{a}{b}(y - y_o)$. Since $\frac{2k+1}{2} - \frac{2k-1}{2} = 1$, there is one
 and only one value x . For a given ordinate y , the abscissa x in the PDSL P_k is
 given by the following function:

$$\mathcal{X}(y) = \left\lceil \frac{(2k - 1)}{2} + x_o - (a/b)(y - y_o) \right\rceil$$

Let us note that for a given digital point $p(x, y)$, it is easy to determine
 the PDSL P_k it belongs to: $\frac{(2k-1)b}{2} \leq b(x - x_o) + a(y - y_o) < \frac{(2k+1)b}{2}$ leads to
 $2k \leq 2 \frac{b(x-x_o) + a(y-y_o)}{b} + 1 < 2k + 2$ and thus:

$$k = \left\lceil (x - x_o) + \frac{a}{b}(y - y_o) + \frac{1}{2} \right\rceil$$

117 The next question that arises is the determination and the localization of
 118 the potential intersection point between the digital reflection line and a given
 119 digital perpendicular line. As already mentioned, there may be 0 or 1 digital
 120 intersection points between the DRSL and a given PDSL since both are naive
 121 digital lines. We propose here a simple criterion to determine the existence of

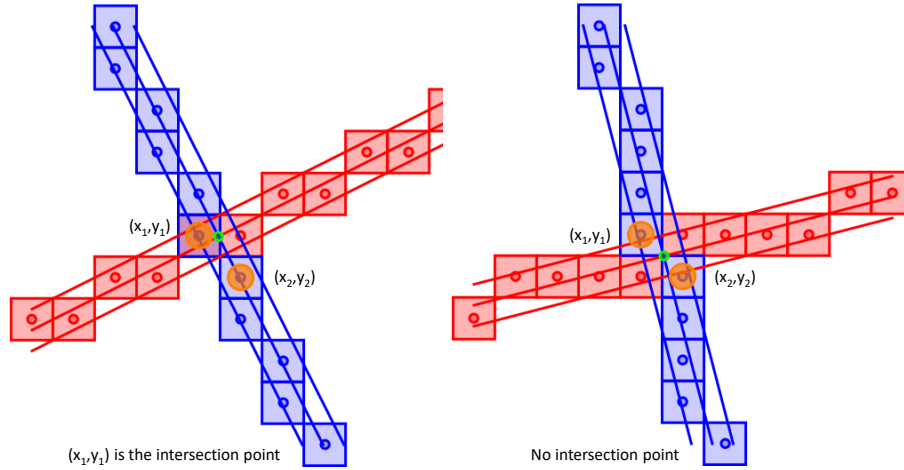


Fig. 3. The green point represents the intersection point between the continuous reflection line and the continuous perpendicular line. The orange disks mark the two candidate points for the digital intersection. On the left (x_1, y_1) is the digital intersection point between the DRSL and the PDSL. On the right there is no digital intersection point.

122 such an intersection point and its localization, if it exists. If the point does not
 123 exist, it will yield the closest point of the PDSL to the DRSL on either side of
 124 the digital reflection line.

125 Firstly, let us note that the DRSL is the digitization of the continuous re-
 126 flection line of equation: $a(x - x_o) - b(y - y_o) = 0$ which lies in the middle of
 127 the strip defining the DRSL. In the same way, the continuous straight line of
 128 equation $b(x - x_o) + a(y - y_o) = kb$ lies in the middle of the strip defining the
 129 perpendicular digital straight line P_k . The intersection point of those two con-
 130 tinuous lines is given by $(kb^2 + x_o, abk + y_o)$ (since $a^2 + b^2 = 1$). It is easy to
 131 understand that the only digital intersection point, if it exists, has an ordinate
 132 value given by either $\lceil abk + y_o \rceil$ or $\lfloor abk + y_o \rfloor$. This is a direct consequence of
 133 the fact that the DRSL is a naive digital line of slope between -1 and 1 and
 134 thus with only one digital point per integer ordinate value. We have therefore a
 135 very simple test to determine the existence and coordinates of the intersection
 136 point (see Figure 3):

- 137 – let us define $(x_1, y_1) = (\mathcal{X}(\lceil abk + y_o \rceil), \lceil abk + y_o \rceil)$
- 138 and $(x_2, y_2) = (\mathcal{X}(\lfloor abk + y_o \rfloor), \lfloor abk + y_o \rfloor)$
- 139 – If (x_1, y_1) belongs to the DRSL then there is an intersection point, (x_1, y_1) in
 140 this case. A central symmetry of the PDSL points can be performed around
 141 the ordinate value y_1 . The digital reflection of a point $p(x_p, y_p) \in P_k$ is
 142 given by $(\mathcal{X}(2y_1 - y_p), 2y_1 - y_p)$. Let us note here that the central symmetry

Algorithm 1: REFLECTION TRANSFORM $R_{\theta, (x_o, y_o)}$

Input : $(x, y) \in \mathbb{Z}^2, (x_o, y_o) \in \mathbb{R}^2, -\pi/4 \leq \theta < \pi/4$
Output: $(x', y') \in \mathbb{Z}^2$
1 $k = \lfloor (x - x_o) + \frac{a}{b}(y - y_o) + \frac{1}{2} \rfloor$
2 Function $\mathcal{X}(y) : \mathbb{Z} \mapsto \mathbb{Z} : \mathcal{X}(y) = \lfloor \frac{(2k-1)}{2} + x_o - (a/b)(y - y_o) \rfloor$
3 $(x_1, y_1) = (\mathcal{X}(\lceil abk + y_o \rceil), \lceil abk + y_o \rceil)$
4 $(x_2, y_2) = (\mathcal{X}(\lfloor abk + y_o \rfloor), \lfloor abk + y_o \rfloor)$
5 If $-b/2 \leq a(x_1 - x_o) - b(y_1 - y_o) < b/2$ Then
6 $(x', y') = (\mathcal{X}(2y_1 - y), 2y_1 - y)$
7 Elself $-b/2 \leq a(x_2 - x_o) - b(y_2 - y - o) < b/2$ Then
8 $(x', y') = (\mathcal{X}(2y_2 - y), 2y_2 - y)$
9 Else $(x', y') = (\mathcal{X}(y_1 + y_2 - y), y_1 + y_2 - y)$
10 return (x', y')

143 around the ordinate y_1 does not mean here that there is a central symmetry
 144 around the point (x_1, y_1) because space partition would not be guaranteed
 145 and thus bijectivity would be lost. The abscissa $\mathcal{X}(2y_1 - y_p)$ is computed so
 146 that the reflected point still belongs to P_k .

147 – Else if (x_2, y_2) belongs to the the DRSL then there is an intersection point,
 148 (x_2, y_2) . A central symmetry of the PDSL points can be performed around
 149 the ordinate value y_2 . The digital reflection of a point $p(x_p, y_p) \in P_k$ is given
 150 by $(\mathcal{X}(2y_2 - y_p), 2y_2 - y_p)$.
 151 – Otherwise there is no intersection point and (x_1, y_1) and (x_2, y_2) are the first
 152 points of the PDSL on each side of the Reflection line. The central symmetry
 153 can be performed around the ordinate value $\frac{y_1 + y_2}{2}$. The digital reflection of
 154 a point $p(x_p, y_p) \in P_k$ is given by $(\mathcal{X}(y_1 + y_2 - y_p), y_1 + y_2 - y_p)$.

155 The important point here is that, since there is only one point per ordinate
 156 y , a central symmetry on the ordinate leads directly to the reflection point.
 157 Algorithm 1 presents the digital reflection transform method.

158 3.3 Bijectivity of the Digital Reflection Transform

159 Algorithm 1 provides a digital reflection method for a digital point. It is not
 160 very difficult to see that this defines a bijective digital reflection transform. Let
 161 us briefly summarize the arguments for this:

162 – The computation of the value k in line 1 of algorithm 1 yields the same value
 163 for any point of a given PDSL (and only for those).
 164 – Line 6, 8 and 9 ensure that the image of a digital point of a given PDSL P_k
 165 is a digital point belonging to P_k . Indeed the ordinate values $2y_1 - y, 2y_2 - y$
 166 and $y_1 + y_2 - y$ are integers if y_1, y_2 and y are integers and $(\mathcal{X}(y), y)$ is, by
 167 construction, a point of P_k if y is an integer.
 168 – The reflection of the reflection of a digital point $(x, y) \in P_k$ is the digital point
 169 (x, y) : Let us consider $(x, y) \mapsto (\mathcal{X}(y'), y') \mapsto (\mathcal{X}(y''), y'')$ where $y' = 2y_1 - y,$

170 $y' = 2y_2 - y$ or $y' = y_1 + y_2 - y$. Let us note first that $(x, y) = (\mathcal{X}(y), y)$. For all
 171 three cases, we have respectively $y'' = 2y_1 - y' = 2y - 1 - (2y_1 - y) = y$, $y'' =$
 172 $2y_2 - y' = 2y_2 - (2y_2 - y) = y$ and $y'' = y_1 + y_2 - y' = y_1 + y_2 - (y_1 + y_2 - y) = y$
 173 which proves the point.

174 4 Reflection Based Rotation

As mentioned in the preliminaries, a continuous rotation transform $\mathcal{R}ot_{\theta, (x_o, y_o)}$ of center (x_o, y_o) and angle θ can be defined as the composition of two reflections based on two reflection lines passing through (x_o, y_o) with an angle $\theta/2$ between the two lines. In the same way, a digital rotation:

$$Rot_{\theta, (x_o, y_o)}(x, y) = \left(R_{\alpha + \frac{\theta}{2}, (x_o, y_o)} \circ R_{\alpha, (x_o, y_o)} \right) (x, y)$$

175 Let us note right away that, since the digital reflection transform is bijective,
 176 the digital rotation based on the reflection transforms will be bijective as well.
 177 Furthermore, the inverse transform is easily defined. This last point may seem
 178 obvious but it is not because a digital transform is bijective that the inverse
 179 transform is easily computed.

180 Compared to previous digital rotations methods [7, 15, 13, 10], there is an
 181 extra parameter that comes into play: the angle α .

182 4.1 Rotation Evaluation Criteria

183 In order to evaluate the “quality” of such a digital rotation, let us present some
 184 simple error measures [7]. Each grid point has one and only one image through
 185 a bijective digital rotation but that does not mean that the digital rotation is
 186 a good approximation of the continuous one. To measure how “wrong” we are
 187 by choosing the digital rotation over the continuous one, we are considering two
 188 distance criteria ([7]). Let us denote $\mathcal{R}ot_{\theta, (x_o, y_o)}(p)$ the continuous rotation of
 189 center (x_o, y_o) and angle θ of a grid point $p \in \mathbb{Z}^2$ and $Rot_{\theta, (x_o, y_o)}(p)$ the digital
 190 rotation of center (x_o, y_o) of a grid point p . /par The *Maximum Distance quality*
 191 *criteria (MD)* consists in computing $\max_{p \in \mathbb{Z}^2} (d(\mathcal{R}ot_{\theta, (x_o, y_o)}(p), Rot_{\theta, (x_o, y_o)}(p)))$.
 192 The *average distance quality criteria (AD)* consists in computing
 193 $avg_{p \in \mathbb{Z}^2} (d(\mathcal{R}ot_{\theta, (x_o, y_o)}(p), Rot_{\theta, (x_o, y_o)}(p)))$, where $avg_{p \in \mathbb{Z}^2}$ is the average dis-
 194 tance over the grid.

195 The idea is to measure what error is made by using the digital rotation in-
 196 stead of the continuous one in terms of distance to the optimal position (to the
 197 continuously rotated point). Let us finish by noting that for the best known dig-
 198 ital bijective rotations [15, 9, 11] and for the angles where the digitized rotations
 199 are bijective (except for the trivial $k\pi/2$ angles), we obtain the best maximum
 200 MD values of $\frac{\sqrt{2}}{2} \approx 0.7$ and average distance AD of ≈ 0.3 . Note also that this is
 201 not obtained for all angles.

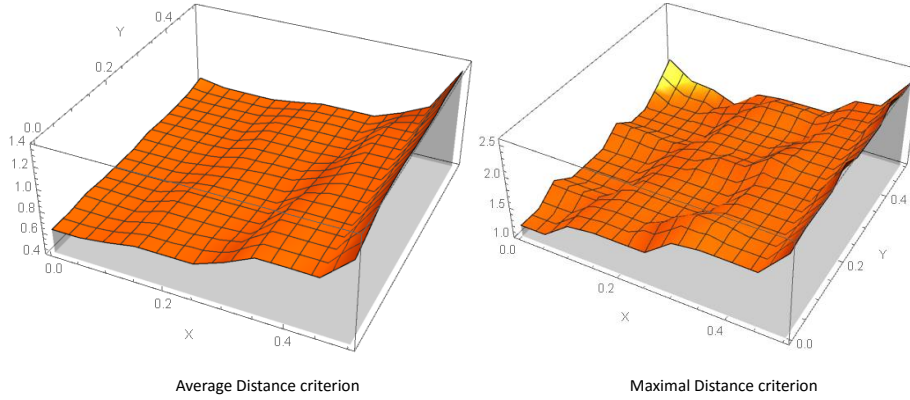


Fig. 4. Rotations of center (x, y) , angle $\pi/8$ and $\alpha = 0$

202 **4.2 Evaluation Analysis**

203 It is difficult to give a completely detailed rotation evaluation here due to lack
 204 of space as there are four different parameters that can influence the outcome of
 205 the evaluation criteria: the center coordinates (x_o, y_o) and the angles θ and α .

206 At first, we wanted to have an idea on the effect of the rotation center on
 207 the rotation error measure with the angle $\alpha = 0$ and $\theta = \pi/8$. This means
 208 that the first reflection corresponds to a reflection with the continuous reflection
 209 line $y = y_o$, for $-0.5 \leq y_o < 0.5$ and the digital reflection $y = 0$. In Figure 4 we
 210 represented the influence of moving the center on the interval $[0, 0.5]^2$. As can be
 211 seen, the error measures are almost not affected by the shift on the ordinate but
 212 greatly by a shift on the abscissa of the rotation center. This can be explained
 213 by the fact that the shift on the perpendicular lines has a direct influence on
 214 the lateral error that adds up to the global error. Let us note that the surfaces
 215 are similar looking for other couple of angles (α, θ) even though the amplitude
 216 of the error measure may be different.

217 Next, we wanted to have an idea on the effect of the angle α on the rotation
 218 error measures. So, for each rotation angle $0 \leq \theta \leq \pi/4$, the average error
 219 measures for 500 rotations with randomly chosen center and respectively angle
 220 $\alpha = 0$ and a randomly chosen angle $0 \leq \alpha \leq \pi/4$. The result can be seen in figure
 221 5. It can be seen that the angle α has only a minor influence on the average error
 222 but the maximal error is typically significantly increased for an angle $\alpha > 0$. It
 223 is interesting to note that this is not always so clear: the digital rotation of angle
 224 $\pi/6$ and center $(0, 0)$ has an average / maximum error of $(0.6367, 1.3972)$ for
 225 $\alpha = 0$ and $(0.5726, 1.4186)$ for $\alpha = \pi/6$. What can be noticed as well is that the
 226 error ratio between $\alpha = 0$ and random chosen α is relatively stable over all the
 227 rotation angles. The error measure in general are higher than those obtained for

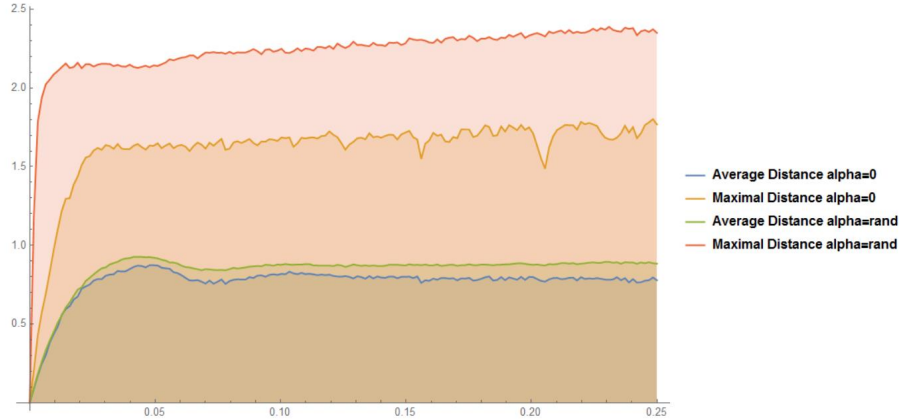


Fig. 5. For each rotation angle $0 \leq \theta \leq \pi/4$, the average error measures of 500 Rotations with random center and respectively $\alpha = 0$ and random α (with $0 \leq \alpha \leq \pi/4$)

228 the shear based method [7] that is also defined for all angles and centers. The
 229 reflection based rotation is however way easier to implement.

230 5 Conclusion

231 A novel bijective digital reflection transform in two dimensions has been pro-
 232 posed. The reflection line is defined by an arbitrary Euclidean point and a
 233 straight line passing through this point. A rotation can be defined by the com-
 234 position of two reflections, and so an associated new bijective digital rotation
 235 transform has been proposed. This new bijective digital rotation is defined for all
 236 angles and for all rotation centers, defining a rigid motion transform (see figure
 237 6 that illustrates the rotation of an image at various angles and centers). In av-
 238 erage the distance between a continuously and a digitally rotated point is about
 239 0.8 (for a pixel of side 1). This reflection based rotation seems well suited for
 240 extensions to higher dimensions which is not easily done with previous methods.
 241 This is the main perspective for the future: exploring digital reflection based
 242 bijective rotations in three and higher dimensions.

243 References

- 244 1. R. Goodman, Alice through looking glass after looking glass: The mathematics of
 245 mirrors and kaleidoscopes, The American Mathematical Monthly 111 (4) (2004)

- 246 281–298.
 247 URL <http://www.jstor.org/stable/4145238>
- 248 2. A. Richard, L. Fuchs, G. Largeteau-Skapin, E. Andres, Decomposition of nd-
 249 rotations: Classification, properties and algorithm, *Graphical Models* 73 (6) (2011)
 250 346–353.
 - 251 3. P. S. Heckbert, P. Hanrahan, Beam tracing polygonal objects, *ACM SIGGRAPH*
 252 *Computer Graphics* 18 (3) (1984) 119–127.
 - 253 4. R. Zrou, P. Y. Chatelier, F. Feschet, R. Malgouyres, Parallelization of a discrete
 254 radiosity method, in: *Euro-Par 2006, Parallel Processing, 12th International Euro-
 255 Par Conference, Dresden, Germany, August 28 - September 1, 2006, Proceedings,*
 256 2006, pp. 740–750.
 - 257 5. F. Mora, G. Ruillet, E. Andres, R. Vauzelle, Pedagogic discrete visualization of
 258 electromagnetic waves, *Eurographics poster session, EUROGRAPHICS, Granada,*
 259 *Spain (January 2003).*
 - 260 6. K. Pluta, P. Romon, Y. Kenmochi, N. Passat, Bijective digitized rigid motions on
 261 subsets of the plane, *Journal of Mathematical Imaging and Vision* 59 (1) (2017)
 262 84–105.
 - 263 7. E. Andres, The quasi-shear rotation, in: *Discrete Geometry for Computer Imagery:*
 264 *6th International Workshop, DGCI'96, Lyon, France, November 13-15, 1996, Pro-*
 265 *ceedings, Vol. 1176, Springer Science & Business Media, 1996, p. 307.*
 - 266 8. B. Nouvel, E. Rémila, Characterization of bijective discretized rotations, in: *Pro-*
 267 *ceedings of the 10th International Conference on Combinatorial Image Analysis,*
 268 *IWCIA'04, Springer-Verlag, Berlin, Heidelberg, 2004, pp. 248–259.*
 - 269 9. T. Roussillon, D. Coeurjolly, Characterization of bijective discretized rotations by
 270 gaussian integers, *Tech. rep., LIRIS UMR CNRS 5205 (January 2016).*
 - 271 10. B. Nouvel, E. Rémila, Incremental and transitive discrete rotations, in: *Proceedings*
 272 *of the 11th International Conference on Combinatorial Image Analysis, IWCIA'06,*
 273 *Springer-Verlag, Berlin, Heidelberg, 2006, pp. 199–213.*
 - 274 11. K. Pluta, P. Romon, Y. Kenmochi, N. Passat, Bijective rigid motions of the 2d
 275 cartesian grid, in: *Proceedings of the 19th IAPR International Conference on Dis-*
 276 *crete Geometry for Computer Imagery - Volume 9647, DGCI 2016, Springer-Verlag,*
 277 *Berlin, Heidelberg, 2016, pp. 359–371.*
 - 278 12. K. Pluta, P. Romon, Y. Kenmochi, N. Passat, Bijectivity certification of 3d dig-
 279 itized rotations, in: *Proceedings of the 6th International workshop on Computer*
 280 *Topology on Image Context, CTIC 2016, 2016, pp. 30–41.*
 - 281 13. K. Pluta, T. Roussillon, D. Coeurjolly, P. Romon, Y. Kenmochi, V. Ostromoukhov,
 282 Characterization of bijective digitized rotations on the hexagonal grid, *Tech. rep.,*
 283 *HAL, submitted to Journal of Mathematical Imaging and Vision. (June 2017).*
 284 URL <https://hal.archives-ouvertes.fr/hal-01540772>
 - 285 14. J.-P. Reveillès, *Calcul en nombres entiers et algorithmique*, Ph.D. thesis, Université
 286 Louis Pasteur, Strasbourg, France (1991).
 - 287 15. M.-A. Jacob, E. Andres, On discrete rotations, in: *Discrete Geometry for Computer*
 288 *Imagery, 1995, p. 161.*

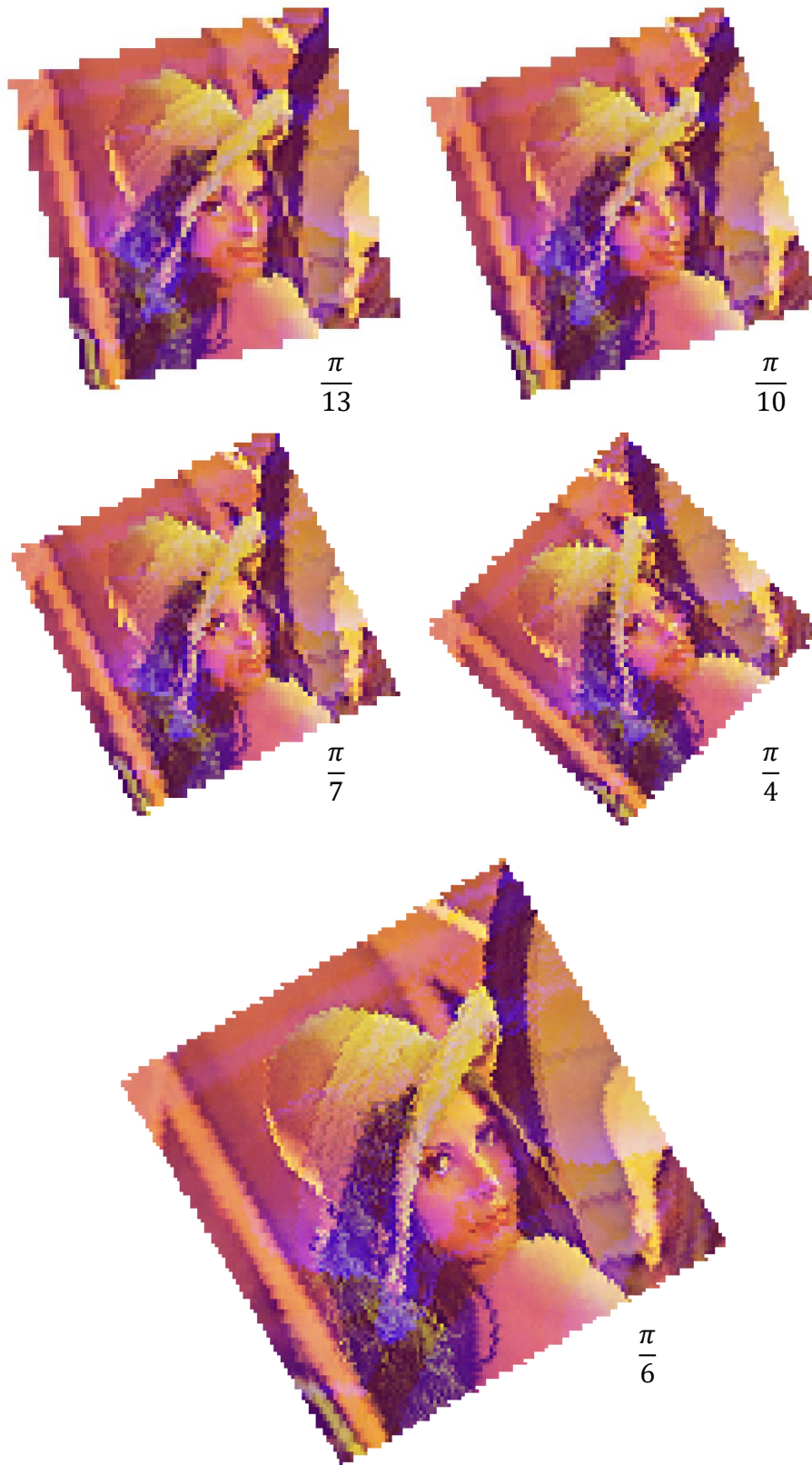


Fig. 6. Rotation of the Lena image by various angles and randomly chosen center.