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ASYMPTOTIC EFFICIENCY IN THE AUTOREGRESSIVE PROCESS DRIVEN BY A STATIONARY GAUSSIAN NOISE

MARIUS SOLTANE

ABSTRACT. The first purpose of this article is to obtain *a.s.* asymptotic properties of the maximum likelihood estimator in the autoregressive process driven by a stationary Gaussian noise. The second purpose is to show the local asymptotic normality property of the likelihoods ratio in order to get a notion of asymptotic efficiency and to build an asymptotically uniformly invariant most powerful procedure for testing the significance of the autoregressive parameter.

1. INTRODUCTION.

Classical autoregressive processes driven by strong white noise were introduced by Box-Jenkins and studied as early in [4]. Now models using autoregressive processes with dependent perturbations are widely used in various fields, especially in econometrics and finance. The asymptotic behavior of the least square estimator (LSE) is generally degraded for this type of process and no consistent for the autoregressive parameter (see [2] for an illustration of this fact where the author consider an AR(1) process driven by an AR(1) noise). A more general study with an AR(p) process driven by an AR(1) noise was realized in [17] and some asymptotic properties of the maximum likelihood estimator (MLE) in the model presented later was studied in [5]. In this study no attention is paid to the obtention of *a.s.* properties of the estimation and no rate of convergence is obtained. We also address the questions of the asymptotic efficiency for the MLE and the optimality of the test of significance of the parameter driving the autoregressive dynamics.

We consider in this paper the stochastic process (X_n) indexed on \mathbb{N} and satisfying for all $n \in \mathbb{N}$,

$$(1.1) \quad X_n = \sum_{i=1}^p \theta_i X_{n-i} + \xi_n.$$

In (1.1) the nuisance process (ξ_n) is a stationary centered Gaussian process and we assume that $X_{-p} = \dots = X_{-1} = 0$.

To obtain an explicit formula for the MLE, a transformation of the model is carried out, in order to obtain an independent noise. Typically, the arguments used to obtain the asymptotic properties of the estimators in this type of processes call for results on martingales. To apply these results, ergodicity arguments are invoked but we will see later that this can not be verified, strictly speaking. We will therefore present in section 5 a new method to apply the

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standard results for martingales and to obtain the desired properties. The second section is devoted to the presentation of the model in particular, we recall the well-know results related to this AR process. The third section contains the presentation of the results established on the MLE. In particular, we are getting *a.s.* convergence for the filtered process, which furthermore makes it possible to obtain quadratic strong law for the MLE and its the strong consistency. In the same section, we also get the LAN property which will allow us to build an optimal test. For the sake of clarity, a part containing the auxiliary results precedes the part containing the proofs of the mains results

2. PRELIMINARIES.

2.1. Model and assumptions.

In the rest of the article $\|x\|$ refers to the euclidian norm of a vector x and Id_p is the identity matrix of size $p \times p$. When M is a matrix, $\|M\|$ is the usual matrix norm induced by the Euclidian norm. Finally, A^* is the transpose of A .

We use (1.1) in order to write the model in a vectorial form. Let

$$A_0 = \begin{pmatrix} \theta_1 & \theta_2 & \dots & \dots & \theta_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \text{ and } Y_n = (X_n, X_{n-1}, \dots, X_{n-p+1})^*.$$

Then, for all n ,

$$(2.1) \quad Y_n = A_0 Y_{n-1} + b \xi_n,$$

where $b = (1, 0_{1 \times (p-1)})^*$. In all that follows, we retain the following hypotheses :

- (H_1) $\rho(A_0) < 1$ where ρ refers to the spectral radius of the matrix A_0 . The Parametric space is therefore $\Theta = \{\theta \in \mathbb{R}^p | \rho(A_0) < 1\}$.
- (H_2) the covariance function r of the nuisance process satisfies $r(n) = O(\frac{c}{n^\alpha})$ when $n \rightarrow \infty$. In this relation, $\alpha > 0$ and c is a positive constant.
- (H_3) Let (β_n) be the PACF of (ξ_n) , we suppose that $\beta_n^2 = O(\frac{1}{n^\alpha})$ for some $\alpha > 1$.

The last assumption is slightly stronger than $(\beta_n^2) \in \ell^1(\mathbb{N})$ which holds in this study, but it will be required in our technical proofs. Let f_ξ the spectral density of the process (ξ_n) , not thats (H_2) can be rewritten (see [5]) in term on condition of the spectral density f_ξ as

$$(2.2) \quad \int_{-\pi}^{\pi} |\log f_\xi(\lambda)| d\lambda < \infty.$$

2.2. Model Transformation.

In this section, we present a linear transformation in order to obtain a Markov process driven by independent noise. Let $\sigma_1 \varepsilon_1 = \xi_1$ and for all $n \geq 2$,

$$(2.3) \quad \sigma_n \varepsilon_n = \xi_n - \mathbb{E}(\xi_n | \xi_1, \dots, \xi_{n-1}),$$

where (ε_n) are *i.i.d.* and $\varepsilon_n \sim \mathcal{N}(0, 1)$. By the Theorem of Normal Correlation (Theorem 13.1 in [16]) we have,

$$(2.4) \quad \sigma_n \varepsilon_n = \sum_{i=1}^n k(n, i) \xi_i,$$

where $(k(n, i)_{\{1 \leq i \leq n, n \in \mathbb{N}^*\}})$ is a deterministic kernel and (σ_n^2) is the variance of innovations. Let

$$(2.5) \quad \beta_{n-1} = -k(n, 1).$$

By the Durbin-Levinson algorithm (see [12]), the following relations are true and make it possible to calculate the coefficients intervening in (2.4).

$$(2.6) \quad \sigma_n^2 = \prod_{i=1}^{n-1} (1 - \beta_i^2), \quad n \geq 2, \quad \sigma_1 = 1,$$

$$(2.7) \quad \sum_{i=1}^n k(n, i) r(i) = \beta_n \sigma_n^2, \quad k(n, n) = 1,$$

$$(2.8) \quad k(n+1, n+1-i) = k(n, n-i) - \beta_n k(n, i), \quad 1 \leq i \leq n-1.$$

Now, let also

$$(2.9) \quad Z_n = \sum_{i=1}^n k(n, i) Y_i,$$

and,

$$(2.10) \quad \zeta_n = \left(\begin{array}{c} Z_n \\ \sum_{k=1}^{n-1} \beta_k Z_k \end{array} \right).$$

The initial estimation problem of θ is replaced by the estimation of the unknown parameter θ from the observations $\zeta = (\zeta_n, n \geq 1)$. It was shown in [6] that (ζ_n) is a $2p$ -dimensional Markov process. More precisely, for all $n \in \mathbb{N}^*$,

$$(2.11) \quad \zeta_n = \tilde{A}_{n-1} \zeta_{n-1} + \ell \sigma_n \varepsilon_n,$$

where

$$\tilde{A}_n = \left(\begin{array}{cc} A_0 & \beta_n A_0 \\ \beta_n Id_p & Id_p \end{array} \right), \quad \ell = (1, 0_{1 \times (2p-1)})^* \text{ and } \zeta_0 = 0_{2p \times 1}.$$

Therefore, the log-likelihood function is given by

$$(2.12) \quad \log \mathcal{L}(\theta, X^{(n)}) = -\frac{1}{2} \sum_{i=1}^n \left(\frac{\ell^*(\zeta_i - \tilde{A}_{i-1} \zeta_{i-1})}{\sigma_i} \right)^2 - \frac{n}{2} \log 2\pi - \frac{1}{2} \sum_{i=1}^n \sigma_i^2$$

where $\theta = (\theta_1, \dots, \theta_p)$ and $X^{(n)} = (X_0, X_1, \dots, X_n)$.

2.3. Construction of the MLE and reminders of known properties.

Using (2.12), it follows that the MLE is given by

$$(2.13) \quad \hat{\theta}_n = \left(\sum_{i=1}^n \frac{a_{i-1}^* \zeta_{i-1} \zeta_{i-1}^* a_{i-1}}{\sigma_i^2} \right)^{-1} \left(\sum_{i=1}^n \frac{a_{i-1}^* \zeta_{i-1} \ell^* \zeta_i}{\sigma_i^2} \right),$$

where $a_n = (Id_p, \beta_n Id_p)^*$. The matrix $I(\theta)$ is the unique solution of the Lyapunov equation given by

$$(2.14) \quad I(\theta) = A_0^* I(\theta) A_0 + b b^*,$$

and we have the following properties (see Theorem 1 in [5]) :

$$(2.15) \quad \hat{\theta}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}_\theta^{(n)}} \theta,$$

$$(2.16) \quad \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, I(\theta)^{-1})$$

under $\mathbb{P}_\theta^{(n)}$.

3. MAINS RESULTS.

3.1. Almost sure properties of the MLE.

The results of this part is the strong consistency of the MLE, the quadratic strong law for the MLE and a law of the iterated logarithm. All the results presented in this section are valid under $(H_1), (H_2), (H_3)$.

Theorem 3.1. *The MLE is strongly consistent, i.e.*

$$(3.1) \quad \hat{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta.$$

Proof. See Section 6.1. □

Theorem 3.2. *We have the following quadratic strong law for the MLE,*

$$(3.2) \quad \frac{1}{\log n} \sum_{k=1}^n (\hat{\theta}_k - \theta)(\hat{\theta}_k - \theta)^* \xrightarrow[n \rightarrow \infty]{a.s.} I(\theta)^{-1}.$$

The limit above is the same as the asymptotic covariance matrix in (2.16) and $I(\theta)$ is defined in (2.14).

Proof. See Section 6.2. □

To conclude this section, we give the LLI of the MLE and hence the convergence rate of the MLE.

Proposition 3.1. *We have the following properties for all $v \in \mathbb{R}^p$,*

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(\frac{n}{2 \log \log n} \right)^{\frac{1}{2}} v^* (\hat{\theta}_n - \theta) &= - \liminf_{n \rightarrow \infty} \left(\frac{n}{2 \log \log n} \right)^{\frac{1}{2}} v^* (\hat{\theta}_n - \theta) \\ &= (v^* I(\theta)^{-1} v)^{\frac{1}{2}} \text{ a.s.} \end{aligned}$$

Consequently,

$$(3.3) \quad \left\| \hat{\theta}_n - \theta \right\|^2 = O \left(\frac{\log \log n}{n} \right) \text{ a.s.}$$

Proof. See Section 6.3. □

3.2. Local asymptotic normality property and application.

The LAN (local asymptotic normality) property is an important notion under which we can define a notion of asymptotic efficiency for estimators (see [14]). Before stating the results, we remind for the reader's convenience some properties and definitions under LAN statistical experiments. The LAN property for stationary Gaussian process was obtained in [9] with conditions on the spectral density. We present here direct computation based on the particular autoregressive structure in order to obtain the LAN property.

Definition 3.1. *We will say that a family of measures $\mathbb{P}_\theta^{(n)}$ is LAN in $\theta_0 \in M \subset \mathbb{R}^d$ if the following conditions are satisfied about the likelihood ratio,*

$$(3.4) \quad L_n(u) = \frac{d\mathbb{P}_{\theta_0 + \phi_n(\theta_0)u}^{(n)}}{d\mathbb{P}_{\theta_0}^{(n)}},$$

$$(3.5) \quad L_n(u) = \exp \left(\langle u, Z_n(\theta_0) \rangle - \frac{1}{2} \langle u, J(\theta_0)u \rangle + R_n(\theta_0, u) \right),$$

where

$$(3.6) \quad Z_n(\theta_0) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, J(\theta_0)),$$

and,

$$(3.7) \quad R_n(\theta_0, u) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} 0$$

under $\mathbb{P}_{\theta_0}^{(n)}$.

The sequence $(\phi_n(\theta_0))$ satisfied

$$(3.8) \quad \phi_n(\theta_0) \xrightarrow[n \rightarrow \infty]{} 0.$$

In this definition $u \in K \subset \mathbb{R}^d$, $\phi_n(\theta_0)$ are non-singular matrix rate and $J(\theta_0)$ is a non singular $d \times d$ matrix.

Theorem 3.3. Suppose that the family of measures $\{\mathbb{P}_\theta^{(n)}, \theta \in M \subset \mathbb{R}^d\}$ is LAN in θ_0 . Then for any $\delta > 0$,

$$(3.9) \quad \liminf_{n \rightarrow \infty} \sup_{\|\phi_n(\theta_0)^{-1}(\hat{\theta}_n - \theta)\| \leq \delta} \mathbb{E}_{\theta_0}^{(n)} \left(f \left(\phi_n(\theta_0)^{-1}(\hat{\theta}_n - \theta) \right) \right) \geq \int_{\mathbb{R}^m} f \left(J(\theta_0)^{-\frac{1}{2}} x \right) \Phi_d(x) dx,$$

for any estimator $\hat{\theta}_n$ and for any cost function f such that f is continuous, symmetric, quasi-convex and $f(z) \exp(-\frac{\|z\|^2}{2}) \rightarrow 0$ when $\|z\| \rightarrow \infty$. Here Φ_d is the density of the standard d -dimensional Gaussian distribution.

For a proof of the last result see Theorem 12.1, chapter 2 in [13]. We can now give the LAN property in the model that interests us.

Theorem 3.4. With the notation of Definition 3.1 and denoting $\phi_n(\theta_0) = \frac{1}{\sqrt{n}} Id_p$ we have

$$(3.10) \quad \log \left(\frac{\mathcal{L}(\theta_0 + \phi_n(\theta_0)u, X^{(n)})}{\mathcal{L}(\theta_0, X^{(n)})} \right) = \langle u, \frac{M_n}{\sqrt{n}} \rangle - \frac{1}{2} \langle u, I(\theta_0)u \rangle + R_n(\theta_0, u),$$

where $(\frac{M_n}{\sqrt{n}})$ satisfied condition (3.6) under $\mathbb{P}_{\theta_0}^{(n)}$ with $J(\theta_0) = I(\theta_0)$ and $R_n(\theta_0, u)$ satisfied condition (3.7) under $\mathbb{P}_{\theta_0}^{(n)}$. In this Theorem, $u \in B(0; R)$ for any $R > 0$.

Proof. See Section 6.4. □

We are now in position to give a result concerning the asymptotic efficiency of the MLE.

Proposition 3.2. Under (H_1) and (H_2) , the MLE is asymptotically efficient, more precicely the lower-bound given by the Theorem 3.3 is reached for the MLE.

Proof. See Section 6.5. □

We will now focus on the optimality of the multidimensional hypotheses test in the autoregressive setting. Always for reader's convenience we recall notions and results on the tests which were introduced in [8]. Suppose that the family of measures $\mathbb{P}_\theta^{(n)}$ is LAN in θ_0 . We would like to build an optimal procedure to test $\theta = \theta_0$ against $\theta \neq \theta_0$.

Definition 3.2. A test ϕ_n^1 is said AUMP(α) (asymptotically uniformly most powerful of level α) if

$$(3.11) \quad \limsup_{n \rightarrow \infty} \mathbb{E}_{\theta_0}^{(n)}(\phi_n^1) \leq \alpha,$$

and for any other test ϕ_n^2 of asymptotic level α ,

$$(3.12) \quad \limsup_{n \rightarrow \infty} \mathbb{E}_{\theta_0 + \phi_n(\theta_0)u}^{(n)}(\phi_n^2) \leq \liminf_{n \rightarrow \infty} \mathbb{E}_{\theta_0 + \phi_n(\theta_0)u}^{(n)}(\phi_n^1).$$

Remark 3.1. We give a lemma set in [8] in order to formalize the next definition. We formulate this lemma in our context, i.e. without the parameters of nuisance since in our case, it is possible to compute them (via the Durbin-Levinson algorithm).

Lemma 3.1. With the notation of Definition 3.1, for every test ϕ_n^1 and every subsequence n' , we can find a subsequence n'' of n' and a test ϕ from \mathbb{R}^p to $[0; 1]$ such that for every $u \in K$,

$$(3.13) \quad \lim_{n'' \rightarrow \infty} \mathbb{E}_{\theta_0 + \phi_{n''}(\theta_0)u}^{(n'')}(\phi_{n''}^1) = \int_{\mathbb{R}^d} \phi(x) \Phi_d(x - J(\theta_0)u) dx,$$

where Φ_d is defined as in Theorem 3.3.

We will now introduce an invariance principle by rotation who is involved in the next Definition.

Definition 3.3. A test ϕ_n^1 is AUMPI(α) if the condition of the Definition 3.2 are satisfied and for all subsequence n' the corresponding test ϕ (obtained via Lemma 3.1) satisfied $\phi(Ru) = \phi(u)$ for any rotation from \mathbb{R}^d to \mathbb{R}^d .

To finish this section we give an AUMPI test to test the significance of the autoregressive parameter.

Theorem 3.5. The test

$$(3.14) \quad \tilde{\phi}_n = \mathbb{1} \left\{ 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_n, X^{(n)})}{\mathcal{L}(\theta, X^{(n)})} \right) \geq C_\alpha \right\}$$

is AUMPI(α) to test $\theta = \theta_0$ against $\theta \neq \theta_0$ where C_α is the α -quantile of χ_p^2 .

Proof. See Section 6.6 □

4. CONCLUSION.

We have seen through this study that the classical properties on the stable autoregressive processes concerning the MLE are preserved despite the harmful effects of the filter which leads to the lack of ergodicity. On the other hand, the results obtained in [5] are sufficient to deduce the LAN property which leads to the asymptotic efficiency for the estimation of the autoregressive parameter and to the construction of an asymptotically optimal procedure to test the significance of the same parameter. It would be interesting in the future to extend this study without any assumption on convergence rate of the PACF of the nuisance process. It would be just as interesting to build a procedure to detect a change in the autoregressive dynamic. They would be needed for that convergence rate in (5.6) and (5.11) in order to apply the method proposed in [10].

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5. AUXILIARY RESULTS.

This section is devoted to the numerous technical lemmas and proposition that we will use for the proof of the results of section 3. Before starting the proofs, we give technical results established in [5]. We can write

$$(5.1) \quad \widehat{\theta}_n - \theta = (\langle M \rangle_n)^{-1} M_n,$$

where

$$\langle M \rangle_n = \sum_{i=1}^n \frac{a_{i-1}^* \zeta_{i-1} \zeta_{i-1}^* a_{i-1}}{\sigma_i^2} \quad \text{and} \quad M_n = \sum_{i=1}^n \frac{a_{i-1}^* \zeta_{i-1} \varepsilon_i}{\sigma_i}.$$

Let $\mathcal{F}_n = \mathcal{F}(X_0, X_1, \dots, X_n)$ be the σ -algebra generated by the values of the process up to time n . It follows that (M_n) is a \mathcal{F}_n -martingale, and that $(\langle M \rangle_n)$ is its bracket process.

We also have the following properties :

$$(5.2) \quad \sum_{i=1}^{\infty} \beta_i^2 < \infty,$$

$$(5.3) \quad \frac{\langle M \rangle_n}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_\theta^{(n)}} I(\theta),$$

where $I(\theta)$ is defined in (2.14),

$$(5.4) \quad \sigma_n^2 \xrightarrow[n \rightarrow \infty]{} \sigma_\infty^2 > 0,$$

$$(5.5) \quad \frac{M_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, I(\theta))$$

under $\mathbb{P}_\theta^{(n)}$.

Remark 5.1. *The process (ζ_n) obtained via (2.10) is a Markov process, but unfortunately inhomogeneous. The first step in our proofs will be to show that the first p components of (ζ_n) have the same asymptotic behavior as an autoregressive process strictly stationary and ergodic.*

Proposition 5.1. *Let $(\zeta_n^{(1)})$ be the first p components of (ζ_n) defined in (2.11), and $(\zeta_n^{(2)})$ be the last p components. Consider the process*

$$\gamma_n = A_0 \gamma_{n-1} + \ell^1 \sigma_\infty \varepsilon_n, \quad \forall n \geq 1$$

with γ_0 having the strictly stationary and ergodic distribution associated with the autoregressive relation, and $\ell^1 = (1, 0, \dots, 0)^$ a vector of length p . Then,*

$$(5.6) \quad \left\| \zeta_n^{(1)} - \gamma_n \right\| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Proof. See Section 5.1. □

Remark 5.2. Since $\rho(A_0) < 1$ the process (γ_n) admits a unique representation with the following properties : causality, stationarity and ergodicity (see [4] for more details).

This property about (γ_n) will allow us later via ergodicity arguments to obtain the a.s convergences. Since (γ_n) is a Gaussian ergodic process, $\mathbb{E}(\|\gamma_n\|^p) = K_p < \infty$ for all p .

Lemma 5.1. Consider a random vector $T_n \in \mathbb{R}^d$ such that, for all $n \geq 1$,

$$T_n \sim \mathcal{N}(0, A_n)$$

where the covariance matrix satisfies $\|A_n\| = O(n^{-\delta})$ for some $\delta > 0$. Then,

$$(5.7) \quad \|T_n\| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Proof. Following the idea of Lemma A.1 in [9], let $\varepsilon > 0$ such that

$$\begin{aligned} \mathbb{P}(\|T_n\| > \varepsilon) &= \mathbb{P}(\|T_n\|^2 > \varepsilon^2) \\ &= \mathbb{P}(\langle A_n \mu_n, \mu_n \rangle > \varepsilon^2) \leq \mathbb{P}\left(\|\mu_n\|^2 > \frac{\varepsilon^2}{\|A_n\|}\right), \end{aligned}$$

where $\|\mu_n\|^2 \sim \chi^2(d)$ which, in turn, implies

$$\mathbb{P}(\|\mu_n\|^2 > \varepsilon^2 \|A_n\|^{-1}) = \int_{\varepsilon^2 \|A_n\|^{-1}}^{\infty} c(d) x^{\frac{d}{2}-1} \exp\left(-\frac{x}{2}\right) dx \leq \int_{\varepsilon^2 \|A_n\|^{-1}}^{\infty} c(d) x^{-\beta} dx$$

for any $\beta > 0$ and n big enough, where $c(d)$ is some positive constant independent of x and n . Making use of the hypothesis on $\|A_n\|$, we get

$$\int_{\varepsilon^2 \|A_n\|^{-1}}^{\infty} c(d) x^{-\beta} dx = \frac{c(d) \varepsilon^{2(1-\beta)} \|A_n\|^{\beta-1}}{\beta-1} = O(n^{\delta(1-\beta)})$$

as soon as $\beta > 1$. Let us choose β such that $\delta(1-\beta) < -1$, i.e. $\beta > \frac{1+\delta}{\delta}$. It remains to apply Borel-Cantelli's lemma to reach the desired result. □

The following lemma will allow us to control the norm of the matrix \tilde{A}_n .

Lemma 5.2. Let \tilde{A}_n be the transition matrix as it is defined in (2.11), namely

$$\forall n \geq 1, \tilde{A}_n = \begin{pmatrix} A_0 & \beta_n A_0 \\ \beta_n Id_p & Id_p \end{pmatrix} \text{ with } \tilde{A}_0 = \begin{pmatrix} A_0 & A_0 \\ Id_p & Id_p \end{pmatrix}.$$

Then,

$$(5.8) \quad \sup_{n \in \mathbb{N}} \left\| \prod_{i=1}^n \tilde{A}_{n-i} \right\| < \infty.$$

Proof. Let $T_0 = Id_{2p}$ and, for $n \geq 1$,

$$T_n = \prod_{i=1}^n \tilde{A}_{n-i}.$$

Working block by block, it is not hard to see that

$$\begin{cases} T_{n+1}^{(11)} = A_0 T_n^{(11)} + \beta_n A_0 T_n^{(21)} \\ T_{n+1}^{(12)} = A_0 T_n^{(12)} + \beta_n A_0 T_n^{(22)} \\ T_{n+1}^{(21)} = \beta_n T_n^{(11)} + T_n^{(21)} \\ T_{n+1}^{(22)} = \beta_n T_n^{(12)} + T_n^{(21)} \end{cases}$$

from the recursive equation $T_{n+1} = \tilde{A}_n T_n$, where we use the notation

$$T_n = \begin{pmatrix} T_n^{(11)} & T_n^{(12)} \\ T_n^{(21)} & T_n^{(22)} \end{pmatrix}.$$

Since $\rho(A_0) < 1$, we know that there exists a matrix norm $\|\cdot\|_* = \sup(|\cdot u|_*; u \in \mathbb{C}^p, |u|_* = 1)$ satisfying $\|A_0\|_* < 1$ (see Proposition 2.3.15 of [11]). It follows that

$$\begin{cases} \|T_{n+1}^{(11)}\|_* \leq a_0 \|T_n^{(11)}\|_* + |\beta_n| a_0 \|T_n^{(21)}\|_* \\ \|T_{n+1}^{(21)}\|_* \leq |\beta_n| \|T_n^{(11)}\|_* + \|T_n^{(21)}\|_* \end{cases}$$

where, for a better readability, we set $a_0 = \|A_0\|_*$. From the previous relations, there is some constant $C > 0$ such that

$$(5.9) \quad \|T_{n+1}^{(11)}\|_* + \|T_n^{(21)}\|_* \leq C \prod_{i=1}^n \|H_i\|,$$

where

$$\forall i \geq 1, H_i = \begin{pmatrix} a_0 & |\beta_i| a_0 \\ |\beta_i| & 1 \end{pmatrix}.$$

Now, to evaluate $\|H_i\|$, one is going to study the spectrum of $H_i^* H_i$. Let

$$P_i(\lambda) = \lambda^2 - \lambda(1 + a_0^2)(1 + \beta_i^2) + a_0^2(1 - \beta_i^2)^2$$

be the characteristic polynomial of $H_i^* H_i$ defined for all $\lambda \in \mathbb{C}$. Then, a straightforward calculation gives

$$(5.10) \quad \Delta_i = [(1 - a_0)^2 + \beta_i^2(1 + a_0)^2][(1 + a_0)^2 + \beta_i(1 - a_0)^2] > 0$$

as discriminant of the equation $P_i(\lambda) = 0$, thus leading to real eigenvalues

$$\lambda_{k,i} = \frac{(1 - a_0^2)(1 + \beta_i^2) \pm \sqrt{\Delta_i}}{2}, \quad k = 1, 2.$$

Since we recall that $\beta_n \rightarrow 0$, a Taylor expansion of $\sqrt{\Delta_i}$ enables to write

$$\lambda_{k,i} = \frac{(1 + a_0^2)(1 + \beta_i^2)}{2} \pm \frac{1 - a_0^2}{2} [1 + K_i \beta_i^2 + o(\beta_i^2)]$$

for some easily identifiable $|K_i| = O(1)$, as $i \rightarrow \infty$. Consequently,

$$\lambda_{1,i} = a_0^2 + O(\beta_i^2) \quad \text{and} \quad \lambda_{2,i} = 1 + O(\beta_i^2)$$

which clearly gives $\|H_i\|^2 = 1 + O(\beta_i^2)$. This rate together with (5.9) and (5.2) are sufficient to get $\|T_{n+1}^{(11)}\|_* + \|T_{n+1}^{(21)}\|_* < \infty$. Since the same reasoning holds for $\|T_{n+1}^{(12)}\|_* + \|T_{n+1}^{(22)}\|_*$, the desired result is proved through the definition of T_n . \square

The next lemma is interested in the a.s. convergence of $(\beta_n \zeta_n)$.

Lemma 5.3. *Consider the process (ζ_n) defined in (2.11). Then,*

$$(5.11) \quad \|\beta_n \zeta_n\| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Proof. The autoregressive relation leads to

$$\zeta_n = \sum_{k=1}^{n-1} \left(\prod_{i=1}^k \tilde{A}_{n-i} \right) \ell \sigma_{n-k} \varepsilon_{n-k} + \ell \sigma_n \varepsilon_n.$$

Thus,

$$\begin{aligned} \mathbb{E}(\beta_n^2 \zeta_n \zeta_n^*) &= \beta_n^2 \sum_{k=1}^n P_{k,n} \ell \sigma_{n-k}^2 \mathbb{E}(\varepsilon_{n-k}^2) \ell^* P_{k,n}^* + \beta_n^2 \ell \ell^* \sigma_n^2 \mathbb{E}(\varepsilon_n^2) \\ &= \beta_n^2 \left(\sum_{k=1}^n P_{k,n} \ell \sigma_{n-k}^2 \ell^* P_{k,n}^* + \ell \sigma_n^2 \ell^* \right) \end{aligned}$$

where, for $n-1 \geq k \geq 1$, $P_{k,n} = \tilde{A}_{n-1} \dots \tilde{A}_{n-k}$. From Lemma 5.2,

$$\|\mathbb{E}(\beta_n^2 \zeta_n \zeta_n^*)\| \leq \beta_n^2 \left(\sum_{k=1}^n \|P_{k,n}\|^2 \sigma_{n-k}^2 + \sigma_n^2 \right) \leq K n \beta_n^2$$

for some $K > 0$ and a sufficiently large n . Thus, from hypothesis (H_3) ,

$$\|\mathbb{E}(\beta_n^2 \zeta_n \zeta_n^*)\| \leq \frac{K}{n^\delta}$$

for some $\delta > 0$, and Lemma 5.1 gives the result. \square

We can now use the previous Lemmas to prove the Proposition 5.1.

5.1. Proof of Proposition 5.1.

Proof. The direct calculation shows that

$$\zeta_n^{(1)} - \gamma_n = A_0(\zeta_{n-1}^{(1)} - \gamma_{n-1}) + \beta_{n-1} A_0 \zeta_{n-1}^{(2)} + \ell^1(\sigma_n - \sigma_\infty) \varepsilon_n.$$

From Lemma 5.3, $\|\beta_{n-1} A_0 \zeta_{n-1}^{(2)}\| \xrightarrow[n \rightarrow \infty]{a.s.} 0$ and, from (5.4) and the normality of (ε_n) ,

$\|\ell^1(\sigma_n - \sigma_\infty) \varepsilon_n\| \xrightarrow[n \rightarrow \infty]{a.s.} 0$. Hence, using the same norm as the proof of Lemma 5.2, for all $\eta > 0$, there exists a random n_0 such that, for all $n \geq n_0$,

$$\|\zeta_n^{(1)} - \gamma_n\|_* \leq \|A_0\|_* \|\zeta_{n-1}^{(1)} - \gamma_{n-1}\|_* + \eta \quad a.s.$$

Since $\|A_0\|_* < 1$, we conclude that $\|\zeta_n^{(1)} - \gamma_n\| \xrightarrow[n \rightarrow \infty]{a.s.} 0$. \square

Lemma 5.4. *Under $\mathbb{P}_{\theta_0}^{(n)}$ we have,*

$$(5.12) \quad 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_n, X^{(n)})}{\mathcal{L}(\theta_0, X^{(n)})} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \chi_p^2.$$

Proof. By using the decomposition in the proof of Theorem 3.4, we have,

$$\begin{aligned} 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_n, X^{(n)})}{\mathcal{L}(\theta_0, X^{(n)})} \right) &= 2 \langle \sqrt{n}(\hat{\theta}_n - \theta_0), \frac{M_n}{\sqrt{n}} \rangle - \langle \sqrt{n}(\hat{\theta}_n - \theta_0), I(\theta_0) \sqrt{n}(\hat{\theta}_n - \theta_0) \rangle \\ &\quad - \langle \sqrt{n}(\hat{\theta}_n - \theta_0), (\frac{\langle M \rangle_n}{n} - I(\theta_0)) \sqrt{n}(\hat{\theta}_n - \theta_0) \rangle \\ &= \langle \langle M \rangle_n^{-\frac{1}{2}} M_n, \langle M \rangle_n^{-\frac{1}{2}} M_n \rangle. \end{aligned}$$

Thus (5.12) follows immediately from (5.3) and (5.5). \square

The following lemma can be seen as a matrix Toeplitz lemma, in some sense (see Theorem 1.1 in [15]).

Lemma 5.5. *Let $(B_{k,n})$ and (A_n) be two sequences of square matrices such that*

- $\|A_n - A\| \xrightarrow{n \rightarrow \infty} 0$,
- $\|\sum_{k=1}^n B_{k,n} - B\| \xrightarrow{n \rightarrow \infty} 0$,
- $\sum_{k=1}^n \|B_{k,n}\|$ is bounded with respect to n ,
- for all $n_0 > 0$, $\sum_{k=1}^{n_0} \|B_{k,n}\| \xrightarrow{n \rightarrow \infty} 0$.

Then,

$$(5.13) \quad \left\| \sum_{k=1}^n A_k B_{k,n} - AB \right\| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We have,

$$\sum_{k=1}^n A_k B_{k,n} - AB = \sum_{k=1}^n A_k B_{k,n} - \sum_{k=1}^n AB_{k,n} + \sum_{k=1}^n AB_{k,n} - AB.$$

For all $\varepsilon > 0$, one can find $n_0 > 0$ such that, for all $n \geq n_0$, $\|A_n - A\| < \varepsilon$. Thus,

$$\begin{aligned} \left\| \sum_{k=1}^n A_k B_{k,n} - AB \right\| &\leq \left\| \sum_{k=1}^n (A_k - A) B_{k,n} \right\| + \left\| \sum_{k=1}^n A (B_{k,n} - B) \right\| \\ &\leq \sum_{k=1}^{n_0-1} \|A_k - A\| \|B_{k,n}\| + \varepsilon \sum_{k=n_0}^n \|B_{k,n}\| + \|A\| \left\| \sum_{k=1}^n B_{k,n} - B \right\|. \end{aligned}$$

The combination of the whole hypotheses enables to show that the right-hand side of latter expression can be made arbitrarily small, as n tends to infinity. \square

Remark 5.3. *The last result can be easily extended when hypotheses holds a.s.*

Remark 5.4. *Given two sequences of vectors (u_n) and $(v_n) \in \mathbb{R}^d$. We have $u_n v_n^* = \bar{u}_n \bar{v}_n^*$, where*

$$\bar{u}_n = \begin{pmatrix} u_{11} & 0 & \dots & \dots & 0 \\ u_{21} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ u_{(d-1)1} & 0 & \dots & \dots & 0 \\ u_{d1} & 0 & \dots & \dots & 0 \end{pmatrix}.$$

More precisely, \bar{u}_n is $d \times d$ matrix with u_n for first column and 0 elsewhere. Then $\|u_n v_n^*\| \leq \|\bar{u}_n\| \|\bar{v}_n^*\|$, and, since we work in finite dimension, $\|\bar{u}_n - \bar{u}\| \xrightarrow{n \rightarrow \infty} 0$ if and only if each component of $\bar{u}_n - \bar{u}$ converges to 0.

Lemma 5.6. Θ is an open subset of \mathbb{R}^p .

Proof. Let

$$P(z) = 1 - \theta_1 z - \dots - \theta_p z^p,$$

a polynomial function defined on \mathbb{C} . (H_1) is equivalent at the following condition (see [4]) :

$$(5.14) \quad \text{if } |z| \leq 1 \text{ then } |P(z)| > 0.$$

Let

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \mathbb{R}^p \text{ and } Q_\alpha(z) = 1 - (\theta_1 + \alpha_1)z - \dots - (\theta_p + \alpha_p)z^p,$$

then, for all $|z| \leq 1$,

$$\begin{aligned} |Q_\alpha(z)| &= |P(z) - \alpha_1 z - \dots - \alpha_p z^p| \\ &\geq |P(z)| - |\alpha_1 z + \dots + \alpha_p z^p| \\ &\geq |P(z)| - |\alpha_1| - \dots - |\alpha_p|. \end{aligned}$$

Condition (5.14) ensures that

$$\inf_{z \in D(0;1)} |P(z)| = \delta_{min} > 0.$$

It remains to choose $|\alpha_1| + \dots + |\alpha_p| < \delta_{min}$ in order to reach the desired result. \square

Remark 5.5. The last Lemma ensures that if $\theta \in \Theta$ then for all $u \in \mathbb{R}^p$ and for n big enough, $\theta + \frac{u}{\sqrt{n}} \in \Theta$. Take the notation of the last proof and choose $\alpha \in \mathbb{R}^p$ such that $\|\alpha\| < \delta_{min}$, then, for any rotation R from \mathbb{R}^p to \mathbb{R}^p and n big enough, $\theta + \frac{Ru}{\sqrt{n}} \in \Theta$.

6. PROOFS OF THE MAINS RESULTS.

6.1. Proof of Theorem 3.1.

Proof. We have,

$$\begin{aligned} \frac{\langle M \rangle_n}{n} &= \frac{1}{n} \sum_{i=1}^n \frac{(\zeta_{i-1}^{(1)} + \beta_{i-1} \zeta_{i-1}^{(2)} - \gamma_{i-1} + \gamma_{i-1})(\zeta_{i-1}^{(1)} + \beta_{i-1} \zeta_{i-1}^{(2)} - \gamma_{i-1} + \gamma_{i-1})^*}{\sigma_i^2} \\ &= S_n + r_n \end{aligned}$$

where

$$S_n = \frac{1}{n} \sum_{k=1}^n \frac{\gamma_{i-1} \gamma_{i-1}^*}{\sigma_i^2}.$$

The remainder term r_n is shown to be negligible via Cesàro's theorem as well as the ergodicity of (γ_n) , Lemma 5.3, Proposition 5.1, Lemma 5.5 and Remark 5.4. A direct application of the ergodic theorem together with Lemma 5.5 leads to

$$S_n \xrightarrow[n \rightarrow \infty]{a.s.} \frac{1}{\sigma_\infty^2} \mathbb{E}(\gamma_0 \gamma_0^*),$$

and finally thanks to (5.3),

$$S_n \xrightarrow[n \rightarrow \infty]{a.s.} I(\theta).$$

Thus,

$$(6.1) \quad \frac{\langle M \rangle_n}{n} \xrightarrow[n \rightarrow \infty]{a.s.} I(\theta).$$

By using the fact that ε_n is independent of γ_{n-1} and similar arguments as in this proof, we have,

$$(6.2) \quad \frac{M_n}{n} \xrightarrow[n \rightarrow \infty]{a.s.} 0,$$

leading to the strong consistency

$$\widehat{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta.$$

□

From now on, let $\bar{\mathcal{F}}_n$ be the σ -algebra $\bar{\mathcal{F}}_n = \bar{\mathcal{F}}_n(X_0, \dots, X_n, \gamma_0, \dots, \gamma_n)$ where, (γ_n) is the process defined in Proposition 5.1.

Remark 6.1. (M_n) is a $\bar{\mathcal{F}}_n$ -martingale and the introduction of $\bar{\mathcal{F}}_n$ is necessary in the following proof.

6.2. Proof of Theorem 3.2.

Proof. In order to prove Theorem 3.2, we will use the quadratic strong law for martingales (see Theorem 2.1 in [7]). Take $V_n = \sqrt{n}Id_p$, a sequence of regular matrices in the sense of Chaabane and Maouia. Now, we study the asymptotic behavior of

$$(6.3) \quad \frac{[M]_n}{n} = \frac{1}{n} \sum_{i=1}^n \left(\frac{(\zeta_{i-1}^{(1)} + \beta_{i-1} \zeta_{i-1}^{(2)} - \gamma_{i-1} + \gamma_{i-1})(\zeta_{i-1}^{(1)} + \beta_{i-1} \zeta_{i-1}^{(2)} - \gamma_{i-1} + \gamma_{i-1})^*}{\sigma_i^2} \right) \varepsilon_i^2.$$

By using similar arguments as in the previous proof we have,

$$(6.4) \quad \frac{[M]_n}{n} \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E} \left(\frac{\gamma_1 \gamma_1^*}{\sigma_\infty^2} \right).$$

Let us now look at the Lindeberg's condition, we have to show that, for all $\varepsilon > 0$,

$$(6.5) \quad L_n = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left(\|\Delta M_i\|^2 \mathbb{1}_{\{\|\Delta M_i\| \geq \varepsilon \sqrt{n}\}} \middle| \bar{\mathcal{F}}_{i-1} \right) \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Let $M > 0$ and

$$L_{n,M} = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left(\|\Delta M_i\|^2 \mathbb{1}_{\{\|\Delta M_i\| \geq M\}} \middle| \bar{\mathcal{F}}_{i-1} \right).$$

From (5.1) and the definition of a_n , we have $\Delta M_1 = M_1$ and, for $n \geq 2$,

$$\Delta M_n = \frac{(\zeta_{n-1}^{(1)} + \beta_{n-1} \zeta_{n-1}^{(2)}) \varepsilon_n}{\sigma_n}.$$

It follows that

$$\begin{aligned} \|\Delta M_n\|^2 &= \left\| \frac{(\zeta_{n-1}^{(1)} - \gamma_{n-1} + \beta_{n-1} \zeta_{n-1}^{(2)} + \gamma_{n-1}) \varepsilon_n}{\sigma_n} \right\|^2 \\ &\leq \frac{2 \left\| \zeta_{n-1}^{(1)} - \gamma_{n-1} + \beta_{n-1} \zeta_{n-1}^{(2)} \right\|^2 \varepsilon_n^2}{\sigma_n^2} + \frac{2 \|\gamma_{n-1}\|^2 \varepsilon_n^2}{\sigma_n^2}. \end{aligned}$$

Then,

$$L_{n,M} \leq 2R_{1,n} + 2R_{2,n} + 2R_{3,n},$$

where,

$$R_{1,n} = \frac{1}{n} \sum_{i=1}^n \frac{\left\| \zeta_{i-1}^{(1)} - \gamma_{i-1} + \beta_{i-1} \zeta_{i-1}^{(2)} \right\|^2}{\sigma_i^2},$$

$$R_{2,n} = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left(\frac{\|\gamma_{i-1}\|^2}{\sigma_i^2} \varepsilon_i^2 \mathbb{1}_{\left\{ \frac{\|\gamma_{i-1}\| \varepsilon_i}{\sigma_i} \geq \frac{M}{2} \right\}} \middle| \bar{\mathcal{F}}_{i-1} \right),$$

and,

$$R_{3,n} = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left(\frac{\|\gamma_{i-1}\|^2}{\sigma_i^2} \varepsilon_i^2 \mathbb{1}_{\left\{ \frac{\|\zeta_{i-1}^{(1)} - \gamma_{i-1} + \beta_{i-1} \zeta_{i-1}^{(2)}\| \varepsilon_i}{\sigma_i} \geq \frac{M}{2} \right\}} \middle| \bar{\mathcal{F}}_{i-1} \right).$$

The same reasoning as above shows that $R_{1,n}$ tend to 0, *a.s.* Let us focus on the more intricate terms $R_{2,n}$ and $R_{3,n}$. First, we know from (5.4) that, for some $0 < \tilde{m} < \sigma_\infty$, there exists n_0 such that, for $n \geq n_0$, $|\sigma_n| \geq \tilde{m}$. Hence,

$$R_{2,n} \leq R_{2,n}^{(0)} + \frac{1}{n} \sum_{i=n_0}^n \mathbb{E} \left(\frac{\|\gamma_{i-1}\|^2}{\tilde{m}^2} \varepsilon_i^2 \mathbb{1}_{\left\{ \frac{\|\gamma_{i-1}\| \varepsilon_i}{\tilde{m}} \geq \frac{M}{2} \right\}} \middle| \bar{\mathcal{F}}_{i-1} \right),$$

where obviously, $R_{2,n}^{(0)} \xrightarrow[n \rightarrow \infty]{a.s.} 0$. The process $(\gamma_{n-1} \varepsilon_n)$ is ergodic since (ε_n) is *i.i.d.* and since there exist ϕ independent of n such that $\gamma_{n-1} = \phi(\varepsilon_{n-1}, \varepsilon_{n-2}, \dots)$ (See theorem 5.3.8 in [18]). Thus,

$$\lim_{n \rightarrow \infty} R_{2,n} \leq \mathbb{E} \left(\frac{\|\gamma_0\|^2}{\tilde{m}^2} \varepsilon_1^2 \mathbb{1}_{\left\{ \frac{\|\gamma_0\|}{\tilde{m}} |\varepsilon_1| \geq \frac{M}{2} \right\}} \middle| \bar{\mathcal{F}}_{i-1} \right),$$

and letting $M \rightarrow \infty$ leads to $R_{2,n} \xrightarrow[n, M \rightarrow \infty]{a.s.} 0$. Then, by Cauchy-Schwarz's inequality,

$$\begin{aligned} R_{3,n}^2 &\leq \frac{1}{n^2} \sum_{i=1}^n \mathbb{E} \left(\frac{\|\gamma_{i-1}\|^4}{\sigma_i^4} \varepsilon_i^4 \middle| \bar{\mathcal{F}}_{i-1} \right) \sum_{i=1}^n \mathbb{E} \left(\mathbb{1}_{\left\{ \frac{\|\zeta_{i-1}^{(1)} - \gamma_{i-1} + \beta_{i-1} \zeta_{i-1}^{(2)}\| |\varepsilon_i|}{|\sigma_i|} \geq \frac{M}{2} \right\}} \middle| \bar{\mathcal{F}}_{i-1} \right) \\ &\leq \frac{3}{n^2} \sum_{i=1}^n \frac{\|\gamma_{i-1}\|^4}{\sigma_i^4} \sum_{i=1}^n \left[\mathbb{1}_{\left\{ \frac{\|\zeta_{i-1}^{(1)} - \gamma_{i-1} + \beta_{i-1} \zeta_{i-1}^{(2)}\|}{|\sigma_i|} \geq \frac{\sqrt{M}}{\sqrt{2}} \right\}} + \mathbb{P} \left(|\varepsilon_1| \geq \frac{\sqrt{M}}{\sqrt{2}} \right) \right]. \end{aligned}$$

Since $\|\zeta_{i-1}^{(1)} - \gamma_{i-1} + \beta_{i-1} \zeta_{i-1}^{(2)}\| \xrightarrow[n \rightarrow \infty]{a.s.} 0$ by Proposition 5.1 and Lemma 5.3, and since (ε_n) is Gaussian, another application of the ergodic theorem is sufficient to ensure that $R_{3,n} \xrightarrow[n \rightarrow \infty]{a.s.} 0$, letting again $M \rightarrow \infty$. Thus,

$$L_{n,M} \xrightarrow[n, M \rightarrow \infty]{a.s.} 0 \text{ and so } L_n \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

By the quadratic strong law for martingales,

$$(6.6) \quad \frac{1}{p \log n} \sum_{k=1}^n \left(1 - \frac{k^p}{(k+1)^p} \right) \frac{M_k M_k^*}{k} \xrightarrow[n \rightarrow \infty]{a.s.} I(\theta).$$

Let $\lambda^{(k)} = \rho(M_k M_k^*) = \|M_k M_k^*\| \leq \text{Tr}(M_k M_k^*)$ because the matrices $M_k M_k^*$ are positives. Put $v_i = (0, \dots, 0, 1, 0, \dots, 0)^*$ a vector of length p with 1 at the i -th coordinate and 0 elsewhere. Then,

$$(6.7) \quad \frac{1}{p \log n} \sum_{k=1}^n \left(1 - \frac{k^p}{(k+1)^p} \right) \frac{\lambda^{(k)}}{k} \leq \frac{1}{p \log n} \sum_{i=1}^p \sum_{k=1}^n \left(1 - \frac{k^p}{(k+1)^p} \right) \frac{\langle v_i, M_k M_k^* v_i \rangle}{k} \text{ a.s.}$$

Since the right member of (6.7) is bounded, the conditions of Lemma 5.5 are satisfied, and we have,

$$(6.8) \quad \frac{1}{\log n} \sum_{k=1}^n (\hat{\theta}_k - \theta)(\hat{\theta}_k - \theta)^* = \frac{1}{\log n} \sum_{k=1}^n \langle M \rangle_k^{-1} M_k M_k^* \langle M \rangle_k^{-1}.$$

Since $\left(1 - \frac{k^p}{(k+1)^p} \right) \sim \frac{p}{k}$ by using (6.6), (6.8), (6.1) together with Lemma 5.5, we obtain,

$$(6.9) \quad \frac{1}{\log n} \sum_{k=1}^n (\hat{\theta}_k - \theta)(\hat{\theta}_k - \theta)^* \xrightarrow[n \rightarrow \infty]{a.s.} I(\theta)^{-1}.$$

□

6.3. Proof of Proposition 3.1.

Proof. To prove the law of iterated logarithm, one is going to apply Lemma C.2 of [1]. We have already established (6.1), so it remains to show that

$$\sum_{n=1}^{\infty} \left(\frac{\|a_n^* \zeta_n\|}{|\sigma_{n+1}| \sqrt{n}} \right)^{\beta} < \infty \text{ a.s.}$$

for some $\beta > 2$. From Proposition 5.1 and Lemma 5.3, there exists a sequence (τ_n) such that

$$\|a_n^* \zeta_n\| = \|\gamma_n + \tau_n\| \text{ and } \tau_n \rightarrow 0 \text{ a.s.}$$

Consequently, for a sufficiently large n ,

$$\left(\frac{\|a_n^* \zeta_n\|}{|\sigma_{n+1}| \sqrt{n}} \right)^{\beta} \leq \frac{\|\tau_n + \gamma_n\|^{\beta}}{\tilde{m}^{\frac{\beta}{2}} n^{\frac{\beta}{2}}} \leq \frac{(1 + \|\gamma_n\|)^{\beta}}{\tilde{m}^{\frac{\beta}{2}} n^{\frac{\beta}{2}}} \text{ a.s.}$$

where \tilde{m} is chosen as in the preceding proof. Since (γ_n) is a Gaussian ergodic process, $\|\gamma_n\| = O(n^{\alpha})$ a.s for all $0 < \alpha < \frac{1}{2}$. The last inequality leads for n sufficiently large to

$$\left(\frac{\|a_n^* \zeta_n\|}{|\sigma_{n+1}| \sqrt{n}} \right)^{\beta} \leq \frac{C}{\tilde{m}^{\frac{\beta}{2}} n^{\beta \frac{1-2\alpha}{2}}} \text{ a.s.}$$

for a constant $C > 0$. The desired result follows when $\beta > \frac{2}{1-2\alpha} > 2$. Hence,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(\frac{n}{2 \log \log n} \right)^{\frac{1}{2}} v^* \langle M \rangle_n^{-1} M_n &= - \liminf_{n \rightarrow \infty} \left(\frac{n}{2 \log \log n} \right)^{\frac{1}{2}} v^* \langle M \rangle_n^{-1} M_n \\ &= (v^* I(\theta)^{-1} v)^{\frac{1}{2}} \text{ a.s.} \end{aligned}$$

Let $\hat{\theta}_n^i$ be the i -th component of $\hat{\theta}_n$ and $v_i = (0, \dots, 0, 1, 0, \dots, 0)^*$ a vector of length p with 1 at the i -th coordinate and 0 elsewhere. The inequality above implies immediatly (3.3). \square

6.4. Proof of Theorem 3.4.

Proof. Let

$$\phi_n(\theta_0) = \frac{Id_p}{\sqrt{n}}, u = (u_1, u_2, \dots, u_p)^*,$$

$$U_n = \begin{pmatrix} u_1 & u_2 & \dots & u_p & \beta_n u_1 & \beta_n u_2 & \dots & \beta_n u_p \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \text{ and } \ell^* U_n = u^* a_n^*.$$

Then,

$$\log \left(\frac{\mathcal{L}(\theta_0 + \phi_n(\theta_0)u, X^{(n)})}{\mathcal{L}(\theta_0, X^{(n)})} \right) = - \frac{1}{2} \sum_{k=1}^n \frac{\ell^*(2\zeta_k - 2\tilde{A}_{k-1}\zeta_{k-1} - \frac{U_{k-1}}{\sqrt{n}}\zeta_{k-1}) \ell^*(-\frac{U_{k-1}}{\sqrt{n}}\zeta_{k-1})}{\sigma_k^2}$$

$$\begin{aligned}
&= -\frac{1}{2} \sum_{k=1}^n \frac{\ell^*(2\ell\sigma_k\varepsilon_k - \frac{U_{k-1}}{\sqrt{n}}\zeta_{k-1})\ell^*(-\frac{U_{k-1}}{\sqrt{n}}\zeta_{k-1})}{\sigma_k^2} \\
&= \sum_{k=1}^n \left(\frac{u^* a_{k-1}^* \zeta_{k-1} \varepsilon_k}{\sqrt{n}\sigma_k} - \frac{1}{2} \frac{u^* a_{k-1}^* \zeta_{k-1} \zeta_{k-1}^* a_{k-1} u}{n\sigma_k^2} \right) \\
&= \langle u, \frac{M_n}{\sqrt{n}} \rangle - \frac{1}{2} \langle u, \frac{\langle M \rangle_n}{n} u \rangle \\
&= \langle u, \frac{M_n}{\sqrt{n}} \rangle - \frac{1}{2} \langle u, I(\theta_0)u \rangle - \frac{1}{2} \langle u, (\frac{\langle M \rangle_n}{n} - I(\theta_0)), u \rangle.
\end{aligned}$$

Let $R_n(\theta_0, n) = \frac{1}{2} \langle u, (I(\theta_n) - \frac{\langle M \rangle_n}{n})u \rangle$, then $|R_n(\theta_0, n)| \leq \frac{1}{2} \|I(\theta_0) - \frac{\langle M \rangle_n}{n}\| R^2$ and the LAN property in θ_0 follows immediatly from (5.5) and (5.3). \square

Remark 6.2. In [13], the LAN property is defined for statistical experiments admitting decomposition (3.5) with $J(\theta_0) = Id_d$. Is not a real restriction since the decomposition (3.5) can be reformulated as the Definition 2.1, chapter 2 in [13] by letting $v = J(\theta_0)^{-\frac{1}{2}}u$ or equivalently considering a new sequence of matrix rate given by $\tilde{\phi}_n(\theta_0) = \phi_n(\theta_0)J(\theta_0)^{-\frac{1}{2}}$.

6.5. Proof of Proposition 3.2.

Proof. The result follows from (2.16). More precisely the following condition to obtained the equality in Lemma 13.1, chapter 2 of [13]

$$(6.10) \quad I(\theta_0)^{\frac{1}{2}} \sqrt{n}(\hat{\theta}_n - \theta_0) - I(\theta_0)^{-\frac{1}{2}} \frac{M_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}^{(n)}} 0$$

is satisfied since the MLE is asymptotically efficient in Fisher's sense. \square

Definition 6.1. A test ϕ_n^1 is asymptotically equivalent to $\tilde{\phi}_n^1$ if

$$(6.11) \quad \phi_n^1 - \tilde{\phi}_n^1 \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\theta_0}^{(n)}} 0.$$

We can now state the theorem characterizing the tests AUMPI(α) established in Theorem 3 of [8].

Theorem 6.1. Take notation of definition 3.1 and denote by C_α the α -quantile of χ_d^2 . Then, the test

$$(6.12) \quad \phi_n = \mathbb{1}_{\{(J(\theta_0)^{-1}Z_n(\theta_0), Z_n(\theta_0)) \geq C_\alpha\}}$$

is AUMPI(α) to test $\theta = \theta_0$ against $\theta \neq \theta_0$ and any other asymptotically equivalent test is AUMPI(α).

6.6. Proof of Theorem 3.5.

Proof. We have just to show that the test $\tilde{\phi}_n$ is asymptotically equivalent to the test given in Theorem 6.1 under $\mathbb{P}_{\theta_0}^{(n)}$ (the null hypothesis). We have,

$$(6.13) \quad |\tilde{\phi}_n - \phi_n| = \tilde{\phi}_n + \phi_n - 2\tilde{\phi}_n\phi_n.$$

Denoting $\delta_n = \langle I(\theta_0)^{-1} \frac{M_n}{\sqrt{n}}, \frac{M_n}{\sqrt{n}} \rangle - 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_n, X^{(n)})}{\mathcal{L}(\theta_0, X^{(n)})} \right)$, we have for all $\varepsilon > 0$,

$$(6.14) \quad |\tilde{\phi}_n - \phi_n| \leq \tilde{\phi}_n + \phi_n - 2 \mathbb{1} \left\{ 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_n, X^{(n)})}{\mathcal{L}(\theta_0, X^{(n)})} \right) \geq C_\alpha + \varepsilon \right\} \mathbb{1}_{\{\delta_n \geq -\varepsilon\}}.$$

It's easy to see that $\delta_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} 0$ under $\mathbb{P}_{\theta_0}^{(n)}$, then Slutsky's Theorem together with Lemma 5.4 and the continuous mapping Theorem's (see Theorem 2.1 in [3]) gives,

$$(6.15) \quad 2 \mathbb{1} \left\{ 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_n, X^{(n)})}{\mathcal{L}(\theta_0, X^{(n)})} \right) \geq C_\alpha + \varepsilon \right\} \mathbb{1}_{\{\delta_n \geq -\varepsilon\}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} 2 \mathbb{1}_{\{Z \geq C_\alpha + \varepsilon\}}$$

under $\mathbb{P}_{\theta_0}^{(n)}$ where $Z \sim \chi_p^2$. Since the second member of (6.14) is uniformly bounded with respect to n by 1, we have,

$$(6.16) \quad \limsup_{n \rightarrow \infty} \mathbb{E}_{\theta_0}^{(n)} (|\tilde{\phi}_n - \phi_n|) \leq 2\mathbb{P}(Z \geq C_\alpha) - 2\mathbb{P}(Z \geq C_\alpha + \varepsilon),$$

and the previous inequality leads to

$$(6.17) \quad \mathbb{E}_{\theta_0}^{(n)} (|\tilde{\phi}_n - \phi_n|) \xrightarrow[n \rightarrow \infty]{} 0.$$

The last condition ensures that the tests $\tilde{\phi}_n$ and ϕ_n are asymptotically equivalent. \square

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