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# Morse complexes and multiplicative structures 

## Hossein Abbaspour \& François Laudenbach


#### Abstract

In this article we lay out the details of Fukaya's $A_{\infty}$-structure of the Morse complexe of a manifold possibily with boundary. We show that this $A_{\infty}$-structure is homotopically independent of the made choices. We emphasize the transversality arguments that some fiber product constructions make valid.


## Contents

1. Introduction ..... 1
2. Multi-intersections towards $A_{\infty}$-structures ..... 4
3. Compactification ..... 12
4. Coherence ..... 14
5. Transition ..... 20
6. Orientations ..... 23
7. $A_{\infty}$-structure ..... 26
8. Morse concordance and homotopy of $A_{d}$-structures ..... 29
Appendix A. Basics on homotopical algebras ..... 33
References ..... 35

## 1. Introduction

We are given an $n$-dimensional compact manifold $M$ with boundary and a generic Morse function $f: M \rightarrow \mathbb{R}$, generic meaning that $f$ has no critical point on the boundary and that the restriction $f_{\partial}$ of $f$ to the boundary $\partial M$ is a Morse function. For the purpose of the present paper, it is useful to assume that $M$ is orientable.

We recall that there are two types of critical points of $f_{\partial}$, those of Neumann type and those of Dirichlet type; a critical point $x$ of $f_{\partial}$ has type Neumann (resp. Dirichlet) if $<d f(x), n(x)>$ is negative (resp. positive) where $n(x)$ is a vector in $T_{x} M$ pointing outwards. We shall denote by $c r i t_{k} f$ the set of critical points of $f$ (in the interior of $M$ ) of index $k$ and by $c r i t_{k}^{N} f_{\partial}$ (resp. $\operatorname{crit}_{k}^{D} f_{\partial}$ ) the set of critical points of $f_{\partial}$ of index $k \in\{0, \ldots, n-1\}$ which are of Neumann type (resp. Dirichlet type).

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This setting was already considered in [15] where the main idea was to introduce so-called adapted pseudo-gradients, defined as follows. Below, we use notation slightly different from the cited paper.

A vector field $X^{N}$ is said to be $N$-adapted to $f$ if the following conditions are fulfilled:

1) $X^{N} \cdot f<0$ outside critf $\cup c r i t^{N} f_{\partial}$;
2) $X^{N}$ is pointing inwards at every point of $\partial M$ except in some neighborhood of $c r i t^{N} f_{\partial}$ where $X^{N}$ is tangent to the boundary; therefore, $X^{N}$ vanishes exactly at the points of critf $\cup \operatorname{crit}^{N} f_{\partial}$;
3) near critf (resp. crit ${ }^{N} f_{\partial}$ ) the vector field $X_{N}$ (resp. $X_{N} \mid \partial M$ ) is the descending gradient of $f$ (resp. $f_{\partial}$ ) with respect to the Euclidean metric of some (unspecifized) Morse chart. ${ }^{1}$
Since the flow of $X^{N}$ is positively complete, each $x \in \operatorname{critf} \cup \operatorname{crit}^{N} f_{\partial}$ has an unstable manifold $W^{u}(x)$ whose dimension is equal to the index of $x$ and a local stable manifold $W_{l o c}^{s}(x)$. Actually, there is also a global stable manifold $W^{s}(x)$ by taking the union of the inverse images of $W_{\text {loc }}^{s}(x)$ by the positive semi-flow of $X^{N}$.

The vector field is said to be Morse-Smale when all these invariant manifolds intersect mutually transversely. Under this assumption, after choosing arbitrarily orientations of the unstable manifolds, one defines a graded complex

$$
C_{*}\left(f, X^{N}\right)=C_{*}^{N}=\left(C_{n}^{N} \xrightarrow{\partial^{N}} \cdots C_{k}^{N} \xrightarrow{\partial^{N}} \cdots C_{0}^{N}\right) .
$$

Here, $C_{k}^{N}$ is the $\mathbb{Z}$-module freely generated by $\operatorname{crit}_{k} f \cup \operatorname{crit}_{k}^{N} f_{\partial}$ and the differential $\partial^{N}$ is defined by counting with signs the connecting orbits from $x$ to $y$ when the index of $x$ equals $\operatorname{ind}(y)+1$ (notice that the local stable manifolds are co-oriented).

Similarly, a vector field $X^{D}$ is said to be $D$-adapted to $f$ when it is $N$-adapted to $-f$. Notice that $X^{D} \cdot f>0$ apart from critf $\cup c r i t^{D} f_{\partial}$. Choose such an $X^{D}$ which is Morse-Smale and choose an orientation of its unstable manifolds; they exist globally since the flow of $X^{D}$ is still positively complete. One defines a second complex

$$
C_{*}\left(f, X^{D}\right)=C_{*}^{D}=\left(C_{n}^{D} \xrightarrow{\partial^{D}} \cdots C_{k}^{D} \xrightarrow{\partial^{D}} \cdots C_{0}^{D}\right) .
$$

Here, $C_{k}^{D}$ is the $\mathbb{Z}$-module freely generated by $\operatorname{crit}_{k} f \cup \operatorname{crit}_{k-1}^{D} f_{\partial}$. Notice the shift of the grading which is justified by the equality:

$$
C_{k}^{D}(f)=C_{n-k}^{N}(-f)
$$

The differential $\partial^{D}$ is defined on a generator $x \in C_{k}^{D}$ by counting with signs the connecting orbits of $X^{D}$ from $y \in C_{k-1}^{D}$ to $x$. The main result in [15] is the following.
Theorem 1.1.

1) The homology of the complex $C_{*}\left(f, X^{N}\right)$ is isomorphic to $H_{*}(M ; \mathbb{Z})$.
2) The homology of the complex $C_{*}\left(f, X^{D}\right)$ is isomorphic to $H_{*}(M, \partial M ; \mathbb{Z})$.

The labelling, Neumann or Dirichlet, comes from similar results which have been obtained previously in Witten's theory of de Rham cohomology for manifolds with boundary (see [4, 10, 13]).

[^0]In the present article, we present an important complement to Theorem 1.1 which deals with the multiplicative structures which exist on the considered complexes. Here we follow ideas which have been developed by K. Fukaya for closed manifolds (see [8] and also [3],[23]). Indeed, in [8] Fukaya has proposed the construction of an $A_{\infty}$-category whose objects are the smooth functions on a given closed manifold $M$ and the set of the morphisms $\operatorname{Mor}(f, g)$ is $\mathbb{Z}$-module generated by the critical points of $g-f$. He describes the $A_{\infty}$-operations

$$
m_{n}: \operatorname{Mor}\left(f_{1}, f_{2}\right) \otimes \operatorname{Mor}\left(f_{1}, f_{2}\right) \cdots \otimes \operatorname{Mor}\left(f_{n-1}, f_{n}\right) \rightarrow \operatorname{Mor}\left(f_{1}, f_{n}\right)
$$

by counting points with sign (orientation) on the zero-dimensional moduli space of flow lines intersection according to the scheme provided by a generic (trivalent) rooted tree.

These operations are only partially defined, meaning that the operation $m_{n}$ is only defined for generic function $f_{i}$ 's. In particular, by taking $f_{i}=i f$, where $f \in C^{\infty}(M)$ is a generic Morse function, he suggested the existence of an $A_{\infty}$-structure on the Morse complex of $f$. Note that in this example $\operatorname{Mor}(i f,(i+1) f))$ is precisely the Morse complex of $f$. In this article, not only we give an accurate construction of the hitherto described $A_{\infty}$-structure on the Morse complex of a Morse function $f$, but also we prove that this $A_{\infty}$-structure is well-defined up to quasiisomorphism of $A_{\infty}$-algebras. It turns out that the construction of $A_{\infty}$-quasi-isomorphisms requires to extend Fukaya's $A_{\infty}$-structure to manifolds with boundary.

Theorem 1.2. Here, $M$ is supposed to be orientable and oriented. Then, each of the complexes $C_{*}^{N}$ and $C_{*}^{D}$ can be endowed with a structure of $A_{\infty}$-algebra $\mathcal{A}=\left\{m_{1}, m_{2}, \ldots\right\}$ such that $m_{1}$ is the differential of the considered complex; here $m_{d}$ denotes the d-fold product. This structure is welldefined up to "homotopy" from the data of a coherent family of Morse-Smale approximations of $X^{N}$ (resp. $X^{D}$ ).

The approximations in question will be submitted to some transversality conditions for which the possible choices are not at all unique. The coherence (Definition 4.1) will be a form of naturality of these choices with respect to a certain group of diffeomorphisms of $M$.

The basic definitions about $A_{\infty}$-structures are recalled in Appendix A. As we shall see in Section 8, the concept of homotopy of $A_{\infty}$-structures is the algebraic translation of the idea of cobordism for the geometric objects we are going to introduce further.

The main example that we have in mind is 3 -dimensional. Consider a link $L$ in the 3 -sphere $S^{3}$, equipped with the standard height function $h: S^{3} \rightarrow \mathbb{R}$. The manifold with boundary we are interested in is $M:=S^{3} \backslash U(L)$, where $U(L)$ is the interior of a small tubular neighborhood of $L$, built by means of an exponential map. In general position of $L$, the height function induces a Morse function on $L$, and hence a generic Morse function $f$ on $M$. Each maximum of $h \mid L$ gives rise to a pair of critical points of $f_{\partial}$, one of Neumann type and index 2 , and one of Dirichlet type and index 1 (hence of degree 2 in $C_{*}^{D}$ ). Each minimum of $h \mid L$ gives rise to a pair of critical points of $f_{\partial}$, one of Neumann type and index 1, and one of Dirichlet type and index 0 (hence of degree 1 in $C_{*}^{D}$ ). It is reasonable to expect that the Morse complexes of this pair $(M, f)$ informs a lot on the topology of $L$. We have not yet explored this topic systematically. As an exercise only, by using the Massey product which is derived from the third product of the $A_{\infty}$-structure on the Dirichlet type complex, one could prove à la Morse that the Borromean link is not trivial.

Sections 2 to 6 are devoted to topological preparation to multiplicative structures by means of a large use of Thom's transversality Theorem with constraints [22]. Here are some more details:
-Section 2 makes a list of transversality conditions which will be used for defining products of an $A_{\infty}$-structure. These conditions are generic.
-Section 3 deals with the compactification of the geometric objects introduced in Section 2. The simple structure of the respective compactifications guarantees the preceding transversality conditions to be open.
-Sections 5 and 4 treat refinements on transversality conditions allowing the products to satisfy the $A_{\infty}$-relations. This is the hardest part.
-Section 6 with the orientation of the moduli spaces. - In Section 7 we introduce the $A_{\infty^{-}}$ structure and prove $A_{\infty}$-relations.

- Section 8 explains why different choices in the previous constructions lead to concordant multi-intersections. That is the topological ingredient for homotopy of $A_{\infty}$-structures.

The proof of Theorem 1.2 will be achieved in Sections 7 and 8. We should say that many authors have addressed the $A_{\infty}$-structures of the Morse complex in various articles (see $[9,1,2]$ for instance). As far as we know, their construction amounts to a pre- $\infty$ category/Algebra and do not really propose a homotopy $A_{\infty}$-invariance of the construction. The other authors have treated the transversality issues using more analytical methods, such as moduli space of gradient trees, which are rather heavy when it comes to verifying the details. That is the reason why we have made efforts to provide a more topological and standard method. In particular we have tackled the difficulties relevant to the transversality issues by introducing a construction based on iterated fiber products.

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## 2. Multi-Intersections towards $A_{\infty}$-Structures

In this article, we only consider the case of the boundary relative theory dealing with the Dirichlet type critical points and adapted gradient $X^{D}$. Similar results hold true for the Neumann type complex. In contrast with [15] where critical points in critf $\cup c r i t^{D} f_{\partial}$ are only equipped with local stable manifolds, we shall here make use of global stable manifolds of critical points. Since the flow of $X^{D}$, denoted by $\bar{X}_{t}^{D}$ at time $t$, is positively complete, the following definition makes sense:

Definition 2.1. For $x \in \operatorname{critf} \cup \operatorname{crit}^{D}\left(f_{\partial}\right)$, the global stable manifold of $x$ with respect to $X^{D}$ is defined as the union

$$
W^{s}\left(x, X^{D}\right)=\bigcup_{t>0}\left(\bar{X}_{t}^{D}\right)^{-1}\left(W_{l o c}^{s}\left(x, X^{D}\right)\right)
$$

Under mild assumptions, $W^{s}\left(x, X^{D}\right)$ is a (non-proper) submanifold with boundary and its closure is a stratified set. Here is such an assumption (Morse-Model-Transversality) which is made in the rest of the paper.
(MMT) For every $x \in \operatorname{critf} \cup \operatorname{crit}^{D} f_{\partial}$ and $y \in \operatorname{crit}^{D} f_{\partial}$, the neighborhood $U_{y}$ of $y$ in $\partial M$ where $X^{D}$ is tangent to the boundary of $M$ is mapped by the flow transversely to $W_{\text {loc }}^{s}\left(x, X^{D}\right)$.

As $X^{D}$ is Morse-Smale, the transversality condition is satisfied along a small neighborhood $U$ of the local unstable manifold $W_{l o c}^{u}\left(y, X^{D}\right)$. Then, after some small perturbation of $X^{D}$ on $U_{y} \backslash U$ destroying partially the tangency of $X^{D}$ to $\partial M$, Condition (MMT) is fulfilled for the pair $(y, x)$. Thus, Condition (MMT) is generic among the $D$-adapted vector fields. The following proposition can be proved easily.

## Proposition 2.2.

1) Under Condition (MMT), the global stable manifold $W^{s}\left(x, X^{D}\right)$ is a submanifold with boundary (not closed in general).
2) If $z$ belongs to the closure of $W^{s}\left(x, X^{D}\right)$, then there exists a broken $X^{D}$-orbit from $z$ to $x$. The number of breaks defines a stratification of this closure $\operatorname{cl}\left(W^{s}\left(x, X^{D}\right)\right)$.

For the rest of this section, we consider a generic Morse function $f: M \rightarrow \mathbb{R}$ and a pseudogradient $X^{D}$ which is $D$-adapted to $f$. The transversality conditions Morse-Smale and (MMT) are assumed.

We now turn to $A_{\infty}$-structures for which we refer to B. Keller [12]. In ([8, 9]), K. Fukaya had proposed the construction of such structures on the Morse complexes of a closed manifold. We adapt his ideas to the case of $M$, a manifold with non-empty boundary. The main point is to parametrize multi-intersections by trees. First, we are going to define the trees under consideration, that we call Fukaya trees.

Definition 2.3. Let $d$ be a positive integer. A Fukaya tree of order d (or a d-tree) is a finite rooted tree with $d$ leaves which is properly $C^{1}$-embedded in the unit closed disc $\mathbb{D}$. The end points (the root and the leaves) lie on the boundary $\partial \mathbb{D}$. The rest of the tree lies in the interior of $\mathbb{D}$. By a vertex we mean an interior vertex; it is required to have a valency greater than 2. An interior edge has its two end points in int $\mathbb{D}$. Each edge is oriented from the root to the leaves.

Let $\mathcal{T}_{d}$ be the set of Fukaya $d$-trees. There is a natural $C^{1}$ topology on this set. The $\varepsilon$ neighborhood of $T_{0} \in \mathcal{T}_{d}$ is the set of Fukaya trees $T$ equipped with a simplicial map $\rho: T \rightarrow T_{0}$ such that each edge $\alpha$ of $T$ is embedded $C^{1}$-close to the embedding of $\rho(\alpha)$ (which is an edge or a vertex). This topology is a topology of infinite-dimensional manifold (Stasheff). But, up to isotopy (not ambient isotopy, due to moduli of angles), there are finitely many representatives only. From this fact it is possible to derive a stratification of $\mathcal{T}_{d}$ where the number of interior vertices is fixed on each stratum. In a generic tree all vertices have valency 3 (codimension 0 stratum). A codimension-one stratum in $\mathcal{T}_{d}$ is made of trees all of which vertices have valency 3 except one which has valency 4.

Definition 2.4. Given the pair $\left(f, X^{D}\right)$, a decoration $\mathcal{D}$ of a Fukaya tree consists of the following: with each edge e of $T$, interior or not, one associates some approximation $X_{e}$ of $X^{D}$ which will be generic among the pseudo-gradients $D$-adapted to approximations ${ }^{2}$ of $f$.

Some mutual transversality conditions will be specified in Section 5. For the time being, we just list the needed conditions. Since the successive manifolds we are going to construct have natural compactification, all required transversality conditions will be not only dense but also open.

[^1]In the construction right below, we will use the positive semi-flow $\bar{X}_{e}:[0,+\infty) \times M \rightarrow M$ of $X_{e}$. But for compactification purposes, it is more convenient to consider the graph of the semi-flow in the following sense.
Definition 2.5. The graph $G\left(\bar{X}_{e}\right)$ of the positive semi-flow $\bar{X}_{e}$ is the part of $M \times M$ made of the pairs $(x, y)$ such that $y$ belongs to the positive half-orbit of $x$, that is: there exists $t \in[0,+\infty)$ such that $y=\bar{X}_{e}(t, x)$. Since $X_{e}$ is a pseudo-gradient, this time $t$ is unique if $x$ is not a zero of $X^{e}$.

The graph contains the diagonal of $M \times M$. For a pseudo-gradient semi-flow, the graph is a non-proper $(n+1)$-dimensional submanifold, except at the points $(a, a)$ where $a$ is a zero of $X_{e}$. Its compactification will be discussed in Section 3 (in particular, Proposition 3.2).

The first projection $M \times M \rightarrow M$ induces $\sigma_{e}: G\left(\bar{X}_{e}\right) \rightarrow M$ which is called the source map. The second projection induces $\tau_{e}: G\left(\bar{X}_{e}\right) \rightarrow M$ which is called the target map. These two maps have a maximal rank, except at the points $(a, a)$ as above.

Example 2.6. Let $Q: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be the quadratic form of Morse index $k$ and rank $n$ :

$$
Q\left(x_{1}, \ldots, x_{n}\right)=-x_{1}^{2}-\ldots-x_{k}^{2}+x_{k+1}^{2}+\ldots+x_{n}^{2} .
$$

After taking local closure, the graph of the semi-flow of $\nabla Q$ looks like, for $k=1, \ldots, n$, the $\mathbb{R}$-cone over an $n$-dimensional band (that is, $\cong \mathbb{R}^{n-1} \times[0,1]$ ) bounded by two affine subspaces: one is $\left(-1, \mathbb{R}^{k-1}, 0, \ldots, 0\right) \times\left(0, \ldots, 0, \mathbb{R}^{n-k}\right) \subset \mathbb{R}^{n} \times \mathbb{R}^{n}$ and the other is the part of the diagonal over $\left\{x_{k}=-1\right\}$. For $k=0$, it is similar (change $Q$ to $-Q$ ). See Figure 1 .


Figure 1.
2.7. Multi-intersection modelled on $T$. A construction. We are given a generic Fukaya $d$-tree $T$, with a decoration $\mathcal{D}$, and $d$ entries $\left(x_{1}, \ldots, x_{d}\right)$ where each $x_{i}$ belongs to critf $\cup c r i t^{D} f_{\partial}$. The entries decorate the leaves of $T$ clockwise.

With these data we want to associate a manifold

$$
\begin{equation*}
I\left(T, \mathcal{D}, x_{1}, \ldots x_{d}\right) \subset M^{\times n(T)} \text { where } n(T)=d-1 \tag{2.1}
\end{equation*}
$$

Note that $n(T)-1=d-2$ is equal to the number of interior edges. The reason of that dimension will appear along the construction. This manifold, called the multi-intersection modelled on $T$
or the $T$-intersection of the given entries, should be considered as a generalized intersection of the stable manifolds prescribed by $\left(x_{1}, \ldots x_{d}\right)$ and the pseudo-gradients listed in $\mathcal{D}$.

The intersection process works as follows. With each edge $e$ of $T$ we are going to associate a generalized stable manifold $W^{s}(e)$ and with each interior vertex $v$ we are going to associate a multi-intersection $I(v)$ by applying the next rules inductively.

Rule 1. The edge $e_{i}$ ending at the $i$-th leaf is given the entry $x_{i}$ and is decorated by the vector field $X_{e_{i}}$. This data yields the stable manifold $W^{s}\left(x_{i}^{\prime}, X_{e_{i}}\right)$ where $x_{i}^{\prime}$ is the unique zero of $X_{e_{i}}$ close to $x_{i}$ provided by the Implicit Function Theorem. One sets

$$
W^{s}\left(e_{i}\right):=W^{s}\left(x_{i}^{\prime}, X_{e_{i}}\right)
$$

## Rule 2.

Let $v$ be a vertex which is the starting point of $e_{i}$ and $e_{i+1}$; there always exists such an $i$ except when $T$ has no vertex. It is assumed that, whatever the entries are, the intersection

$$
I(v):=W^{s}\left(e_{i}\right) \cap W^{s}\left(e_{i+1}\right)
$$

is transverse. Since there are finitely many entries, this transversality condition is easily fulfilled, at least when $x_{i} \neq x_{i+1}$. The case $x_{i}=x_{i+1}$ raises some difficulty: the decoration of $e_{i+1}$ has to differ from that of $e_{i}$ (compare Section 5).

The edge $e$ ending at $v$ is decorated by $X_{e}$. We consider the graph $G_{e}:=G\left(\bar{X}_{e}\right)$ of its positive semi-flow $\bar{X}_{e}$, in the sense of definition 2.5. Let $\tau_{e}: G_{e} \rightarrow M$ be its target map.

Rule 3. We define the generalized stable manifold $W^{s}(e)$ as the fiber product

$$
W^{s}(e):=\lim \left(G_{e} \xrightarrow{\tau_{e}} M \stackrel{j}{\longleftrightarrow} I(v)\right),
$$

where $j$ denotes the inclusion $I(v) \rightarrow M$. We have $W^{s}(e) \subset M \times M$. It is endowed with a source map which is induced by the source map $\sigma_{e}$ of $G_{e}$. Note that $\sigma_{e}$ is the restriction to $W^{s}(e)$ of the first projection $p_{1}: M \times M \rightarrow M$.

Generically on $X_{e}$, that vector field has no zero on $I(v)$. Then, since $\tau_{e}$ has maximal rank, $W^{s}(e)$ is a (non-proper) submanifold, whatever the entries are; it is said to be transversely defined.

Nevertheless, the source map $\sigma_{e}: W^{s}(e) \rightarrow M$ is not immersive in general (due to the tangencies of $X_{e}$ with $I(v)$ ). Generically on $X_{e}$, this is an immersion almost everywhere. Hence, the question is: how to make further intersections?

Let $v^{\prime}$ be the origin of the above-mentioned edge $e$. First, consider the particular case where $v^{\prime}$ is also the origin of $e_{i-1}$. It is assumed that $I(v)$ is transverse to $W^{s}\left(e_{i-1}\right)$ (a new transversality condition).

Rule 4. The multi-intersection $I\left(v^{\prime}\right)$ is defined as the fiber product

$$
I\left(v^{\prime}\right):=\lim \left(W^{s}\left(e_{i-1}\right) \stackrel{j}{\longrightarrow} M \stackrel{\sigma_{e}}{\leftarrow} W^{s}(e)\right),
$$

where $j$ stands for the inclusion $W^{s}\left(e_{i-1}\right) \rightarrow M$.


Figure 2.

It becomes natural to ask the following question: under which transversality condition this fiber product is a manifold? Observe that $I(v)$ is not moving when $X_{e}$ is perturbed; as a consequence, the requirement $W^{s}\left(e_{i-1}\right) \pitchfork I(v)$ is a necessary condition.

Assume that condition. Then, generically on $X_{e}$, the restriction $\sigma_{e} \mid W^{s}(e)$ of the source map is transverse to $W^{s}\left(e_{i-1}\right)$, whatever the entries are. Indeed, this follows from Thom's transversality theorem with constraints [22].

In that case, the fiber product is said to be transversely defined. It is a smooth submanifold $I\left(v^{\prime}\right)$ in $G_{e} \subset M \times M$ endowed with the projection $p_{v^{\prime}}: I\left(v^{\prime}\right) \rightarrow M$ induced by $\sigma_{e}$. That projection is again the restriction of the first projection $p_{1}: M \times M \rightarrow M$. This $I\left(v^{\prime}\right)$ is the desired intersection, in our particular case. We introduce some notations and then we shall ready for an inductive construction.

If $e$ is an interior edge in $T$, we denote $T(e)$ the subtree of $T$ rooted at the origin of $e$ and containing $e$. Then, the integer $n(e)$ is defined so that $n(e)-1$ is equal to the total number of interior edges of $T$ lying in $T(e)$. If $e$ is not interior and ends in a leaf, one takes $n(e)=1$.

If $v_{0}$ is an interior vertex of $T$, the integer $n\left(v_{0}\right)$ is defined so that $n\left(v_{0}\right)-1$ is equal to the total number of interior edges above $v_{0}$ with respect to the orientation of the tree from the root to the leaves. Let $e_{1}, e_{2}$ be the two edges starting from $v_{0}$ clockwise. The part of $T$ rooted at $v_{0}$ will be denoted as a bouquet $T\left(e_{1}\right) \vee T\left(e_{2}\right)$. One checks the formula

$$
\begin{equation*}
n\left(v_{0}\right)=n\left(e_{1}\right)+n\left(e_{2}\right)-1 \tag{2.2}
\end{equation*}
$$

If $e_{0}$ is the edge ending in $v_{0}$ and $e_{0}$ does not start from the root of $T$, we have

$$
\begin{equation*}
n\left(e_{0}\right)=n\left(v_{0}\right)+1 \tag{2.3}
\end{equation*}
$$

In what follows, we choose to denote the first projection of $M^{\times k} \rightarrow M$ by $p_{1}$ whatever $k$ is.
We now start with the inductive construction. Denote by $v_{1}$ and $v_{2}$ the respective end points of $e_{1}$ and $e_{2}$ (see Figure 2). The inductive assumptions are those stated from (2.4) to (2.6):
(2.4) The multi-intersection $I\left(v_{j}\right)$ is transversely defined as a submanifold of $M^{\times n\left(v_{j}\right)}$.
(2.5) The generalized stable manifold $W^{s}\left(e_{j}\right)$ is transversely defined as a submanifold of $M^{\times n\left(e_{j}\right)}$.

Each of these stable manifolds is endowed with a source map $\sigma_{j}:=\sigma_{e_{j}}$ to $M$ which is again the restriction of the first projection $p_{1}: M^{\times n\left(e_{j}\right)} \rightarrow M$.

Assume also that the two following maps are transverse, that is:

$$
\begin{equation*}
p_{1}\left|W^{s}\left(e_{1}\right) \pitchfork p_{1}\right| I\left(v_{2}\right), \tag{2.6}
\end{equation*}
$$

meaning that their product is transverse to the diagonal of $M \times M$.
Be careful that the first projections $p_{1}$ in the above formulas have not the same source in general. Under these assumptions, Thom's transversality theorem tells us that, generically on the decoration $X_{e_{2}}$, the respective source maps $\sigma_{1}$ and $\sigma_{2}$ are mutually transverse. Thus, the next rule holds:

Rule 5. The multi-intersection $I\left(v_{0}\right)$, denoted by $I\left(T\left(e_{1}\right) \vee T\left(e_{2}\right)\right)$, is transversely defined by the fiber product

$$
\begin{equation*}
I\left(v_{0}\right):=\lim \left(W^{s}\left(e_{1}\right) \xrightarrow{\sigma_{1}} M \stackrel{\sigma_{2}}{\leftarrow} W^{s}\left(e_{2}\right)\right) \tag{2.7}
\end{equation*}
$$

This is a submanifold of $M^{\times\left(n\left(e_{1}\right)+n\left(e_{2}\right)-1\right)}=M^{\times n\left(v_{0}\right)}$. It is endowed with a projection $p_{v_{0}}$ : $I\left(v_{0}\right) \rightarrow M$, which is the common value of $\sigma_{1}$ and $\sigma_{2}$ on this intersection. By convention, the base of a fiber product can always be put as the first factor of the product. Thus, this common value is the restriction of $p_{1}: M^{\times\left(n\left(e_{1}\right)+n\left(e_{2}\right)-1\right)} \rightarrow M$.

The edge $e_{0}$ ending at $v_{0}$ is decorated with a pseudo-gradient $X_{e_{0}}$. Let $G_{e_{0}}$ be the graph of its semi-flow and let $\tau_{e_{0}}: G_{e_{0}} \rightarrow M$ be the target map. Generically on $X_{e_{0}}$, its zeroes are not in the image $p_{v_{0}}\left(I\left(v_{0}\right)\right)$. Hence, $\tau_{e_{0}}$ is transverse to $p_{v_{0}} \mid I\left(v_{0}\right)$. Thus, the next rule holds:

RuLE 6. The generalized stable manifold $W^{s}\left(e_{0}\right)$ is transversely defined by

$$
\begin{equation*}
W^{s}\left(e_{0}\right):=\lim \left(G_{e_{0}} \xrightarrow{\tau_{e_{0}}} M \stackrel{p_{v_{0}}}{\leftrightarrows} I\left(v_{0}\right)\right) . \tag{2.8}
\end{equation*}
$$

This fiber product is a submanifold of $M^{\times n\left(e_{0}\right)}$. The first projection of the latter product induces the projection $\sigma_{e_{0}}: W^{s}\left(e_{0}\right) \rightarrow M$.

For allowing us to pursue the construction inductively, it is necessary to assume some transversality conditions similar to the one expressed in (2.6). This condition is, a priori, generically satisfied by approximation of all previous decorations. More details will be given later on (Proposition 4.2); actually, all generic choices may be done inductively (Section 5).

Finally, let $e_{\text {root }}=e_{\text {root }}(T)$ be the edge starting from the root of $T$ and let $v_{\text {root }}^{1}=v_{\text {root }}^{1}(T)$ be its interior vertex. Arguing inductively as above, we have a generalized intersection

$$
\begin{equation*}
I(T):=I\left(v_{\text {root }}^{1}\right) \subset M^{\times n\left(v_{\text {root }}^{1}\right)}=M^{\times n(T)} . \tag{2.9}
\end{equation*}
$$

In this notation, the decoration and entries are missing. We define the T-intersection of the entries $\left(x_{1}, \ldots, x_{d}\right)$ associated with the chosen decoration $\mathcal{D}$ by

$$
I(T)=I\left(T, \mathcal{D}, x_{1}, \ldots x_{d}\right):=I\left(v_{\text {root }}^{1}\right) \subset M^{\times n(T)}
$$

It is endowed with a projection $p_{\text {root }}: I\left(v_{\text {root }}^{1}\right) \rightarrow M$ in the same way as $I\left(v_{0}\right)$ was endowed with a projection $p_{v_{0}}$ in Rule 5 . This is again the restriction of the first projection $p_{1}: M^{\times n(T)} \rightarrow M$.

## Proposition 2.8. (Dimension formula)

$$
\begin{equation*}
\delta(T):=\operatorname{dim} I(T)=\sum_{1 \leq i \leq d}\left(\operatorname{dim} W^{s}\left(x_{i}\right)-n\right)+d-2+n . \tag{2.10}
\end{equation*}
$$

Proof. If we consider the Fukaya tree $T_{0}$ where all interior edges are collapsed, formula (2.10) where $d-2$ is erased (as there is no interior edge) reduces to the usual dimension formula for an intersection of $d$ submanifolds: it is additive up to the shift by the dimension. Each time an interior edge is created, the dimension increases by 1 since some flow is used and generates a stable manifold.

All the above-described manifolds are oriented. The orientations will be specified in Section 6. They will play an important rôle in the $A_{\infty}$-structures with integral coefficients.
2.9. Multi-intersection as a chain. In order to see the above multi-intersection $I(T)$ as a chain of degree $\delta(T)$ in the Morse complex $C_{*}^{D}(f)$, we have to define the coefficient $<I(T), x_{\text {root }}>$ for every test data $x_{\text {root }} \in \operatorname{critf} \cup \operatorname{crit}^{D} f_{\partial}$ of degree $\delta(T)$. We insist to decorate the edge $e_{\text {root }}$ with the vector field $X^{D}$ itself. Of course, this choice requires the following condition:

$$
\left\{\begin{array}{l}
\text { The projection } p_{\text {root }} \text { is transverse to the unstable manifold } W^{u}\left(x_{\text {root }}, X^{D}\right)  \tag{2.11}\\
\text { for every } x_{\text {root }} \text {. }
\end{array}\right.
$$

Here $p_{\text {root }}: I(T) \rightarrow M$ is induced by the first projection $p_{1}: M^{\times n(T)} \rightarrow M$.
Lemma 2.10. Given a Fukaya tree $T$, a generic decoration of $T$ yields a multi-intersection $I(T)$ which fulfills (2.11).

Proof. In the recursive construction of a decoration of $T$, we are allowed to add some new transversality condition at each stage. Namely, for every vertex $v$ (resp. edge $e$ ) of $T$ and every data $x_{\text {root }}$ we demand $I(v)$ (resp. $W^{s}\left(X_{e}\right)$ ) to be transversely defined and, in addition, to be transverse to $W^{u}\left(x_{\text {root }}, X^{D}\right)$. This new requirement can be easily satisfied.
2.11. T-evaluation map. By choice of the degree of the test data, the codimension of the unstable manifold $W^{u}\left(x_{\text {root }}, X^{D}\right)$ is equal to $\delta(T)$. Then, the following multi-intersection

$$
\left.<I(T), x_{\text {root }}\right)>:=\lim \left(I\left(v_{\text {root }}^{1}\right) \xrightarrow{p_{\text {root }}} M \stackrel{j}{\longleftarrow} W^{u}\left(x_{\text {root }}, X^{D}\right)\right)
$$

is 0 -dimensional, where $j$ stands for the inclusion $W^{u}\left(x_{\text {root }}, X^{D}\right) \hookrightarrow M$. As it will be explained in Section 3, it is a compact set, and hence finite, with an orientation. Here, it is important to specify the orientation convention: for every critical point $z$

$$
\begin{equation*}
\operatorname{or}\left(W^{s}(z)\right) \wedge \operatorname{or}\left(W^{u}(z)\right)=\operatorname{or}(M) \tag{2.12}
\end{equation*}
$$

Therefore, it makes sense to define $<I(T), x_{\text {root }}>$ as the algebraic counting of elements in this finite set. The map $<I(T),->$ from the right degree test data to $\mathbb{Z}$ will be called the $T$-evaluation map; it depends on the choice of the decoration of $T$.

Definition 2.12. Given a generic Fukaya tree $T$, a decoration $\mathcal{D}$ is said to be admissible if, whatever the entries and the test data are, all the multi-intersections (resp. stable manifolds) associated with vertices (resp. edges) are transversely defined as submanifolds of $M^{\times k}$ for some integer $k$, as well as the $T$-evaluation.

Admissibilty is easily seen to be a generic property. But it is useful that this property be open. This requires to compactify the various objects that we just introduced (multi-intersections and their stable manifolds). This will be done in the next section.

The end of the present section is devoted to extend the description of multi-intersection to the case of non-generic Fukaya trees. First, we specifiy what is a subtree.

## Definition 2.13.

1) The ordered set of leaves in a Fukaya tree $T$ is denoted by $L(T)$. Let $T_{0}$ and $T_{1}$ be two Fukaya trees. A Fukaya embedding $j: T_{0} \rightarrow T_{1}$ is an injective, non surjective, simplicial map which sends $L\left(T_{0}\right)$ to a consecutive subset of $L\left(T_{1}\right)$ increasingly. The image $j\left(T_{0}\right)$ is called a Fukaya subtree of $T_{1}$.
2) An edge $e$ in a Fukaya tree $T$ is said to be of generation $k$ if the maximal number of edges in a monotone path of $T$ linking the origin of $e$ to a leaf is equal to $k$.
3) A vertex which is the origin of a generation-k edge and not the origin of an edge of higher generation is said to be of generation $k$.

Note that the generation- $k$ edges are ordered from the left. Indeed, there is a first leaf of $T$ which is the terminating point of a monotone path $P \subset T$ with $k$ edges. The first edge of $P$ will be the first generation- $k$ edge. Then, erase the subtree which is rooted at the origin of $P$ and contains $P$, and start again.

If $v_{0}$ is a generation- $k$ vertex in $T$, it is the origin of edges $e_{1}, \ldots, e_{m}$, where $m>1$ (at least one $e_{j}$ is of generation $k$ ). Let $v_{1}, \ldots, v_{m}$ denote the respective end points of $e_{1}, \ldots, e_{m}$ and let $T_{1}, \ldots, T_{m}$ be the subtrees rooted at $v_{0}$ containing $v_{1}, \ldots, v_{m}$ respectively. As before, the integer $n\left(v_{j}\right)$ is defined such that $n\left(v_{j}\right)-1$ is equal to the number of interior edges above $v_{j}$. The integer $n\left(e_{j}\right):=n\left(v_{j}\right)+1$ is equal to the number of interior edges in $T$ lying in $T_{j}$.
Definition 2.14. Let $f_{j}: N_{j} \rightarrow M, j \in J$, be a finite set of smooth maps from manifolds to $M$. They are said to be transverse if, for every subset $K \subset J$, the product map

$$
\prod_{j \in K} f_{j}: \prod_{j \in K} N_{j} \rightarrow M^{|K|}
$$

is transverse to the small diagonal of the target.
In that case, the fiber product $\lim _{j \in J} f_{j}$ is said to be transversely defined. This is a smooth submanifold of the product $\prod_{j \in J} N_{j}$.

Note that in the usual definition one takes $K=J$.
Let us again consider a Fukaya tree $T$, not necessarily generic, with a decoration $\mathcal{D}_{T}$, that is, the data of an approximation $X_{e}$ of $X^{D}$ for each edge $e$ in $T$. We are going to repeat the inductive definition of multi-intersection for each vertex of $T$. For that, we look at the vertex $v_{0}$ in the setting previously described.

Assume that the multi-intersections $I\left(v_{j}\right), j=1, \ldots, m$, are transversely defined and are submanifolds of $M^{\times n\left(v_{j}\right)}$. Generically on $X_{e_{j}}$, its zeroes do not lie on the image $p_{1}\left(I\left(v_{j}\right)\right)$ where
$p_{1}$ is again the first projection $p_{1}: M^{\times n\left(v_{j}\right)} \rightarrow M$. Thus, the generalized stable manifold $W^{s}\left(e_{j}, \mathcal{D}_{T}\right)$ is transversely defined as the fiber product

$$
\begin{equation*}
W^{s}\left(e_{j}, \mathcal{D}_{T}\right):=\lim \left(G_{e_{j}} \xrightarrow{\tau_{e_{j}}} M \stackrel{p_{1}}{\leftarrow} I\left(v_{j}\right)\right) \tag{2.13}
\end{equation*}
$$

Therefore, this stable manifold is a submanifold of $M^{\times n\left(e_{j}\right)}$ and the restriction of the source map $\sigma_{e_{j}}$ to this fiber product is nothing but the restriction of the first projection $p_{1}: M^{\times n\left(e_{j}\right)} \rightarrow M$.

Proposition 2.15. In the above setting, the extra following assumptions are made:

$$
\begin{equation*}
\text { The family of maps }\left(p_{1}: I\left(v_{j}\right) \rightarrow M\right)_{j \in\{1, \ldots, m\}} \text { is transverse. } \tag{2.14}
\end{equation*}
$$

Then, without changing the decoration $\mathcal{D}_{T}$ above the vertices $v_{j}$, generically on the decorating vector fields $X_{e_{j}}, j=1, \ldots, m$, the multi-intersection $I\left(v_{0}, \mathcal{D}_{T}\right)$ is transversely defined as the fiber product

$$
\begin{equation*}
I\left(v_{0}, \mathcal{D}_{T}\right):=\lim _{j}\left(W^{s}\left(e_{j}, \mathcal{D}_{T}\right) \xrightarrow{p_{1}} M\right) . \tag{2.15}
\end{equation*}
$$

Moreover, $I\left(v_{0}, \mathcal{D}_{T}\right)$ is a submanifold of $M^{\times n\left(v_{0}\right)}$, where $n\left(v_{0}\right)-1$ is equal to the number of interior edges in $T_{1} \vee \ldots \vee T_{m}$, and its projection to $M$ is the restriction of the first projection.
Proof. First, thanks to assumption (2.14), for each $j=1, \ldots, m$, generically on the gradient field $X_{e_{j}}$, the family

$$
\left(p_{1}\left|I\left(v_{1}\right), \ldots, p_{1}\right| I\left(v_{j-1}\right), p_{1} \mid W^{s}\left(e_{j}, X_{e_{j}}\right)\right)
$$

is transverse. Once, this is done, it is available to put successively $p_{1} \mid W^{s}\left(e_{j}, X_{e_{j}}\right)$ transverse to the preceding family $\left(p_{1}\left|W^{s}\left(e_{1}, X_{e_{1}}\right), \ldots, p_{1}\right| W^{s}\left(e_{j-1}, X_{e_{j-1}}\right)\right)$ by generic approximation. In all cases, Thom's transversality theorem with constraints applies. The rest of the statement is clear by counting the dimensions and applying the definition of the integers $n\left(v_{j}\right)$ and $n\left(e_{j}\right)$.

Remark 2.16. Due to the compactness which will be proved in Section 3, each genericity condition in question in the previous statement is fulfilled in an open dense subset of the space of gradient vector fields.

## 3. Compactification

In general, the stable manifolds $W^{s}\left(x_{i}, X^{D}\right)$ are not compact and the graphs $G_{e}$ are never compact. In this section, we analyse their individual compactification and how they contribute to the compactification of the multi-intersection $I(T)$. All these closures will be stratified. The codimension-one strata are of particular interest. This is explained in the subsequent propositions.

Proposition 3.1. Let $x$ be a critical point in crit $f \cup \operatorname{crit}_{k-1}^{D} f_{\partial}$ and $X^{D}$ be a Morse-Smale pseudo-gradient $D$-adapted to $f$. Then, the closure in $M$ of the stable manifold $W^{s}\left(x, X^{D}\right)$ is made of all points which are linked to $x$ by an orbit or a broken orbit of $X^{D}$. It is a stratified set whose strata of positive codimension are stable manifolds of some critical points $y \in \operatorname{crit}_{\ell} f \cup \operatorname{crit}_{\ell-1}^{D} f_{\partial}$ with $\ell<k$. The codimenion-one strata are obtained for $\ell=k-1$.

This is well known for closed manifolds [14]. The same proof gives the case of manifold with non-empty boundary. The proof of the following proposition is very similar by considering the flow in the Morse model near a critical point. In what follows, we omit to note the pseudogradient.
Proposition 3.2. Let $G \subset M \times M$ be the graph of the positive semi-flow $\bar{X}^{D}$ of the pseudogradient $X^{D}$. Then:

1) The closure $\operatorname{cl}(G)$ of $G$ in $M \times M$ is made of all pairs of points $(x, y)$ where $y$ belongs to the positive orbit of $x$ or any broken positive orbit starting from $x$.
2) This cl $(G)$ is a stratified set whose strata of positive codimension are made of pairs of points $(x, y)$ where $x$ is connected to $y$ by a broken orbit passing through a non-empty sequence of critical points in critf $\cup \operatorname{crit}^{D} f_{\partial}$.
3) The codimension-one strata are made of pairs of distinct points $(x, y)$ where $x$ belongs to the stable manifold $W^{s}(z)$ for some $z \in \operatorname{critf} \cup \operatorname{crit}^{D} f_{\partial}$ and $y$ belongs to the unstable manifold $W^{u}(z)$.

From Example 2.6 (or Proposition 6.6), we know that the strata considered in 3) above are (non-closed) boundary component of $G$. Observe that the index of $z$ has no effect on the codimension of the stratum.

Notice also that $G$ is already a manifold with boundary and corners due to the fact that $M \times M$ is such a manifold (if $\partial M \neq \emptyset$ ). Moreover, the diagonal of $M \times M$ lies in $G$ as a boundary (except over the zeroes of $X^{D}$ ). The list of strata described in the items 2) and 3) does not include them. Under the transversality assumption (MMT) introduced in Definition 2.1, the stratification of stable manifolds and graphs behave nicely with respect to $\partial M$ and all together form a stratification with conical singularities in the sense of [14].

These partial results are made more precise in the following proposition.
Proposition 3.3. Let $T$ be a Fukaya tree with d leaves and let $\mathcal{D}$ be an admissible decoration (in the sense of Definiton 2.12). Then:

1) The closure of $I_{T}\left(\mathcal{D}, x_{1}, \ldots x_{d}\right)$ in $M^{\times n(T)}$ has a natural stratification whose singularities are conical for every system of entries.
2) $A$ codimension-one stratum $H$ other than those coming from $\partial M$ or the diagonal of $M \times M$ has the following form:

There exist $j \in\{0, \ldots, d-1\}$ and a sub-tree $T_{0}$ in $T$, with $k$ leaves, a root $v_{0}$ and an initial edge $e$ ending at the vertex $v_{1}$, and there is a zero $z$ of the pseudo-gradient $X_{e} \in \mathcal{D}$ decorating e such that:

- (i) the dimension of $I_{T_{0}}\left(\mathcal{D}, x_{j+1}, \ldots, x_{j+k}\right)$ is equal to the Morse index of $z$;
- (ii) the stratum $H$ and its transverse conical structure are generated in $I(T)$ by iterated fiber products from the pair $\left(W^{s}(z) \times c\left(W^{u}(z) \underset{M}{\times} I\left(T_{0}\right)\right), W^{s}(z) \times\{z\}\right)$, where $c(-)$ stands for the cone of $(-)$.
Remark 3.4. Note that by (i) the fiber product $W^{u}(z) \times{ }_{M} I\left(T_{0}\right)$ is a finite set $S$. When taking the orientations into account, $S$ is a signed finite set and the sum of these signs is an algebraic multiplicity, say $\mu$. Therefore, $H$ can be seen with a boundary component of the closure of $I(T)$ with mutiplicity $\mu$.

Proof. We limit ourselves to the case of generic trees. In order to determine the closure of $I(T)$ one follows the recursive construction 2.7. As a result, the closure of $I(T)$ is itself the iterated fiber product of the closures of the factors. Thus, we have to check that, at each step the projection maps, restricted to strata in factors, are mutually transverse. That follows from the fact that the transversality conditions for multi-intersections are required to be fulfilled for every system of entries.

Concretely, when looking at the step mentioned on Figure 2 and assuming that the closures of $W^{s}\left(e_{1}\right), W^{s}\left(e_{2}\right)$ are stratified with conical singularities, the following is required: for each pair of strata $\left(A_{1}, A_{2}\right)$ where $A_{i}$ is a stratum of the closure of $W^{s}\left(e_{i}\right), i=1,2$, the restricted source maps $\sigma_{1} \mid A_{1}$ and $\sigma_{2} \mid A_{2}$ are transverse. By fiber product diagram chasing, we get that $B:=A_{1} \times A_{2}$ is a stratum of codimension $k_{1}+k_{2}$ in the closure of $I_{T}\left(v_{0}\right)$ where $k_{i}$ is the codimension of $A_{i}$ in $c l\left(W^{s}\left(e_{i}\right)\right)$. The singularities are conical by products and intersection with the diagonal of $M \times M$.

For looking at what happens through the stable manifold $W^{s}\left(e_{0}\right)$, we limit ourselves to the case of a stratum of codimension one neither generated by $\partial M$ nor by the diagonal $\Delta_{M}$. If $B$ from above is of codimension one in $\operatorname{cl}\left(I_{T}\left(v_{0}\right)\right)$ one can pursue the process of fiber products and $B$ generates a codimension-one stratum in $I_{T}$.

For having a new phenomenon, we need to start with $I\left(v_{0}\right)$ itself and consider, in the closure of $W^{s}\left(v_{0}\right)$, a stratum of broken orbits generated by a zero $z$ of the vector field $X_{e}$. By admissibility of the decoration, the embedding of $W^{u}(z)$ into $M$ is transverse to the projection $p_{v_{0}}: I_{T}\left(v_{0}\right) \rightarrow$ $M$. Then the part of $I_{T}\left(v_{0}\right)$ over $W^{u}(z)$ is a submanifold $S$ whose codimension is equal to the Morse index of $z$.

If the dimension condition (i) is fulfilled, $S$ is a finite set. Taking the pull-back to the graph $G_{e}$ of the semi-flow $\bar{X}_{e}$ and then the closure produces a bundle over $W^{s}(z)$ which contains $W^{s}(z) \times\{z\}$ and whose fiber is $c(S)$ (adapt [14, Lemma 4] to the fiber product setting). Therefore, $W^{s}(z) \times\{z\}$ appears as a boundary component of the closure of $W^{s}(e)$ in $M \times M$ with multiplicity and (ii) is fulfilled.

If $S$ were not zero-dimensional, the top of the above-mentioned cone would be of codimension larger than one in $c(S)$, and hence, $H$ would not be of codimension one in $\operatorname{cl}(I(T))$.

Corollary 3.5. The admissible decorations of a given Fukaya tree form a dense open set among all decorations.

Proof. Let $\mathcal{D}$ be an admissible decoration of the generic Fukaya tree $T$. At each step of the construction 2.7, one looks at the strata of highest codimension first. They are closed, hence compact. Transversality along a compact set is an open property. As the singularities of the stratification are conical, transversality along a stratum $S$ implies the following local transversality in a neighborhood $N(S)$ of $S$ in the ambiant manifold (which is some product $M^{\times k}$ ). For each stratum $S^{\prime}$ having $S$ in the closure, transversality along $S^{\prime} \cap N(S)$ holds true. But the complement in $S^{\prime}$ of an open neighborhood of the frontier $\operatorname{cl}\left(S^{\prime}\right) \backslash S^{\prime}$ is compact. We use again that transversality along a compact is an open property and the proof of the openness goes on.

## 4. Coherence

In this section, all Fukaya trees will be generic or not. The $A_{\infty}$-structure that we want to reach requires to consider all Fukaya trees and to decorate them in a coherent way. We give the precise definition right below.

Definition 4.1. 1) Two admissible decorations of $T$ are said to be isotopic if they lie in the same arcwise connected component of admissible decorations.
2) Assume that $T_{0}$ and $T_{1}$ are given admissible decorations $\mathcal{D}_{0}$ and $\mathcal{D}_{1}$ respectively. The two decorations are said to be coherent if, for any Fukaya embedding $j: T_{0} \rightarrow T_{1}$ (in the sense of Definition 2.13), the induced decoration $j^{*} \mathcal{D}_{1}$ is isotopic to $\mathcal{D}_{0}$.
3) A system of admissible decorations $\{\mathcal{D}(T)\}_{T}$ for all generic Fukaya trees is said to be coherent if for any pair $\left(T_{0}, T_{1}\right)$ of Fukaya trees the corresponding decorations $\mathcal{D}\left(T_{0}\right)$ and $\mathcal{D}\left(T_{1}\right)$ are coherent.

We consider the group $G=\operatorname{Diff} f_{0}(M)$ of smooth diffeomorphisms of $M$ isotopic to $I d_{M}$. We have chosen from the beginning a Morse-Smale vector field $X^{D}$ thanks to which some Morse complex was built which calculates the relative homology $H_{*}(M, \partial M ; \mathbb{Z})$.

Notation 4.2. Denote by $\Sigma$ the union of unstable manifolds $W^{u}\left(x, X^{D}\right)$ of positive codimension, $x \in \operatorname{crit} f \cup \operatorname{crit}^{D} f_{\partial}$. And similarly, $\Sigma^{*}$ denote the union of stable manifolds $W^{s}\left(x, X^{D}\right)$ of positive codimension. Both $\Sigma$ and $\Sigma^{*}$ are stratified submanifolds of $M$ with conical singularities. Let $G_{\Sigma}$ denote the connected component of $I d$ in the subgroup of $G$ which consists of diffeomorphisms preserving the stratified set $\Sigma$. Recall that $\mathbb{T}_{1}$ (resp. $\mathbb{T}_{2}$ ) denotes the unique tree with one leaf (resp. two leaves).
Proposition 4.3. Let $d$ be a fixed positive integer. Then, for every $g$ in some open and dense subset of $G_{\Sigma}$, the sequence $\mathcal{S}=\left(\mathcal{D}_{1}^{k}\right)_{k=1}^{d}$ of decorations $\mathbb{T}_{1}$ defined by

$$
\begin{equation*}
\mathcal{D}_{1}^{1}=X^{D}, \mathcal{D}_{1}^{2}=g_{*} X^{D}, \ldots, \mathcal{D}_{1}^{j}=g_{*}^{\circ(j-1)} X^{D}, \ldots, \mathcal{D}_{1}^{d}=g_{*}^{\circ(d-1)} X^{D} \tag{4.1}
\end{equation*}
$$

has the following properties:

1) The family $\left\{W^{s}\left(\mathcal{D}_{1}^{1}\right), \ldots, W^{s}\left(\mathcal{D}_{1}^{d}\right)\right\}$ is transverse whatever the entries are. Moreover, this family is transverse to $\Sigma$, that is, to every unstable manifold contained in $\Sigma$.
2) For every integer $j, 1 \leq j<d$, the decoration of $\mathbb{T}_{2}$ with $\mathcal{D}^{j}\left(\mathbb{T}_{2}\right):=\left(\mathcal{D}_{1}^{j}, \mathcal{D}_{1}^{j+1}, g_{*}^{\circ(j-1)} X^{D}\right)$, for the left branch, the right branch and the trunk respectively, is admissible. The decoration $\mathcal{D}^{j}\left(\mathbb{T}_{2}\right)$ is isotopic to $\mathcal{D}^{1}\left(\mathbb{T}_{2}\right)$ in the space of admissible decorations of $\mathbb{T}_{2}$. In particular, the family $\left\{\mathcal{D}^{j}\left(\mathbb{T}_{2}\right)\right\}_{j \leq d-1}$ is coherent.

First, we need some lemma involving a more simple setup.
Lemma 4.4. Let $N^{\prime}$ (possibly empty) and $N$ be two transverse compact submanifolds of $M$ of positive codimension. Let $G_{N^{\prime}}$ denote the subgroup of $G$ made of diffeomorphisms leaving $N^{\prime}$ invariant. Then, the following property is generic for $g \in G_{N^{\prime}}$ :
$\hat{a} \breve{A} \check{T}$ for every positive integer $d$, the family of embeddings $\left\{g^{\circ j}: N \rightarrow M\right\}_{j=0}^{d-1}$ is transverse (in the sense of Definition 2.14);
$\hat{a} \breve{A} \check{T}$ this family is transverse to $N^{\prime}$.
Moreover, for a given d, the corresponding property is open and dense.

Proof. We recall the following classical fact wich is a part of the so-called Kupka-Smale Theorem (see J. Palis \& W. de Melo's book [18, Chap. 3]): For a generic $g \in G_{N^{\prime}}$, all periodic points of $g$ whose periods are less than $d$ are non-degenerate. In particular, they are isolated. Therefore, generically, they do not lie in $N$.

First, we assume $N^{\prime}$ is empty; in that case $G_{N^{\prime}}=G$. As usual for proving a transversality theorem with constraints, it is sufficient to prove that the statement holds when replacing $g$ with a smooth finite dimensional family in $G$ passing though $g$. Indeed, Sard's theorem says that, if the statement holds for a family, it holds for almost every element in that family. We do it when $d=3$; the general case is similar with more complicated notations. We have to prove that generically the following intersections are transverse:
(i) The triple intersection $\Theta_{g}:=N \cap g(N) \cap g^{\circ 2}(N)$ is transverse;
(ii) The intersections $N \cap g(N)$ and $N \cap g^{\circ 2}(N)$ are transverse.

We limit ourselves to prove the first item. The desired transversality condition is the following: The map

$$
\begin{array}{ccc}
\tilde{g}: N \times N \times N & \longrightarrow & M \times M \times M \\
(x, y, z) & \mapsto & \left(x, g(y), g^{\circ 2}(z)\right) \tag{4.2}
\end{array}
$$

is transverse to the so-called small diagonal $\delta \subset M \times M \times M$. This property is open by compactness of $N$. We have only to prove that it is satisfied on a dense set of $G$. We start with an element $g \in G$ having its periodic points apart from $N$. We search for a family $\Gamma \subset G$ passing through $g$ and having the triple transversality condition (4.2). We first need to find a set of parameters.

If $x_{0} \in \Theta_{g}$, there exists $\left(y_{0}, z_{0}\right) \in N \times N$ such that $x_{0}=g\left(y_{0}\right)$ and $x_{0}=g^{\circ 2}\left(z_{0}\right)$. By assumption on $g$, these three points are mutually distinct. Then, there is a finite coverings of $\Theta_{g}$ by Euclidean closed balls $\left\{B_{j}\right\}_{j=1}^{q}$ with the following property: denote by $B_{j}^{\prime}$ and $B_{j}^{\prime \prime}$ the respective preimages of $B_{j}$ by $g$ and $g^{\circ 2}$; if $B_{j}$ is small enough, $B_{j}, B_{j}^{\prime}$ and $B_{j}^{\prime \prime}$ are mutually disjoint. Each of these balls and their preimages are thought of as contained in an open chart. Our space of parameters will be $\left(\mathbb{R}^{n}, 0\right)^{q} \times\left(\mathbb{R}^{n}, 0\right)^{q}$, where $n=\operatorname{dim} M$ and $\left(\mathbb{R}^{n}, 0\right)$ stands for a small neighborhood of the origin in $\mathbb{R}^{n}$.

For $s_{j}$ a small vector in the coordinates around $B_{j}$, the deformation $g_{s_{j}}$ of $g$ is defined by $g_{s_{j}}(y)=g(y)+s_{j}$ when $y \in B_{j}^{\prime}$ and $g_{s_{j}}(y)=g(y)$ when $y$ lies outside some small neighborhood of $B_{j}^{\prime}$ given in advance.

Similarily, for $t_{j}$ a small vector in the coordinates around $B_{j}^{\prime}$, the deformation $g_{t_{j}}$ is defined by $g_{t_{j}}(z)=g(z)+t_{j}$ when $z \in B_{j}^{\prime \prime}$ and $g_{t_{j}}(z)=g(z)$ when $z$ lies outside some small given neighborhood of $B_{j}^{\prime \prime}$. Notice that, for $z \in B_{j}^{\prime \prime}$,

$$
\left(g_{t_{j}}\right)^{\circ 2}(z)=g\left(g(z)+t_{j}\right) .
$$

If $s=\left(s_{1}, \ldots, s_{q}\right)$ and $t=\left(t_{1}, \ldots, t_{q}\right)$ are two $q$-tuples of small vectors in $\mathbb{R}^{n}$, the announced family $\Gamma=g_{s, t}$ is any family in $G$ which coincides with $g_{s_{j}}$ (resp. $g_{t_{j}}$ ) on the $s_{j}$-axis (resp. on the $t_{j}$-axis). As a map, $\Gamma$ reads as the following:

$$
\begin{array}{rlc}
\Gamma: N \times N \times N \times\left(\mathbb{R}^{n}, 0\right)^{q} \times\left(\mathbb{R}^{n}, 0\right)^{q} & \longrightarrow & M \times M \times M \\
(x, y, z, s, t) & \longmapsto\left(x, g_{s, t}(y),\left(g_{s, t}\right)^{\circ 2}(z)\right) \tag{4.3}
\end{array}
$$

We are going to show that this map is transverse to the small diagonal $\delta$. More precisely, we are going to check that, for every triple $(x, y, z)$ such that $x=g(y)=g^{\circ 2}(z)$, the restricted map

$$
(s, t) \in\left(\mathbb{R}^{n}, 0\right)^{q} \times\left(\mathbb{R}^{n}, 0\right)^{q} \longmapsto\left(g_{s, t}(y),\left(g_{s, t}\right)^{\circ 2}(z)\right)
$$

is a submersion valued in $M \times M$. The triple $(x, y, z)$ belongs to some triple of balls $\left(B_{j}, B_{j}^{\prime}, B_{j}^{\prime \prime}\right)$. Thus, we restrict to $s=s_{j}$ and $t=t_{j}$, the other translation coordinates being equal to 0 . The jacobian matrix $J$ at $(x, y, z, 0,0)$ reads:

$$
J=\left(\begin{array}{cc}
I d_{T_{x} M} & 0 \\
0 & d g(g(z))
\end{array}\right)
$$

where $d g(g(z)): T_{y} M \rightarrow T_{x} M$ is the tangent map to $g$ at the point $y=g(z)$. Finally, $J$ is an epimorphism. That finishes the proof that $\Gamma$ is transverse to the small diagonal in the case when $N^{\prime}$ is empty; if $N^{\prime} \cap N=\emptyset$, the proof is the same. We are left with the case $N^{\prime} \pitchfork N \neq \emptyset$.

Let $K$ denote the transverse intersection $N^{\prime} \cap N$. We first have to put the triple intersection $K \cap g(K) \cap g^{02}(K)$ in transverse position in $N^{\prime}$. This can be done according to the preceding proof by replacing $(N, M)$ with $\left(K, N^{\prime}\right)$. Just observe that the restriction $G_{N^{\prime}} \rightarrow \operatorname{Diff} f_{0}\left(N^{\prime}\right)$ is a fibration. Automatically, the transversality we get along $N^{\prime}$ extends to transversality in some neighborhood of $N^{\prime}$.

Now, we are reduced to put in transverse position the triple intersections which do not approach $N^{\prime}$. Notice that, when $x=g(y)=g^{\circ 2}(z) \notin N^{\prime}$, then $y$ and $z$ are not in $N^{\prime}$ since $g$ preserves $N^{\prime}$. For such triple points, the preceding proof works as if $N^{\prime}$ were empty. This finishes the proof of Lemma 4.4.

We notice that the statement of Lemma 4.4 extends to the case where $N$ and $N^{\prime}$ are submanifolds with conical singularities. We briefly indicate how the proof has to be changed. First, we consider the case when $N^{\prime}=\emptyset$. The domain of the map $\Gamma$ in formula (4.3) is a stratified set. Thus we have to check that when $A$ and $B$ are two submanifolds with conical singularities of $M$, then $A \times B$ is a submanifold with conical singularities in $M \times M$. Namely, if some stratum $S \subset A$ (resp. $S^{\prime} \subset B$ ) has a $\operatorname{link}^{3} L$ (resp. $L^{\prime}$ ), the link of the product $S \times S^{\prime}$ is the join $L * L^{\prime}$. Now, if the domain of $\Gamma$ is stratified with conical singularities, if $\sigma$ is one of the strata and if $\Gamma \mid \sigma$ is transverse to the small diagonal of the target, then the same holds near $\sigma$ for each stratum $\tau$ having $\sigma$ in its closure. Therefore, we are allowed to argue on each stratum of the domain of $\Gamma$ inductively on the increasing dimension.

In the case where $N \pitchfork N^{\prime} \neq \emptyset$, we look at each smooth stratum $\sigma$ of $N^{\prime}$. Considering $K:=N \pitchfork \sigma$ as a submanifold with conical singularities in $\sigma$, we are able to realize the required transversality if it is already realized along every stratum in the frontier $\bar{\sigma} \backslash \sigma$. Again, arguing inductively, we are done with the stratified extension of Lemma 4.4.

## Proof of Proposition 4.3.

1) The Morse-Smale property of the vector field $X^{D}$ implies that $\Sigma^{*}$ and $\Sigma$ are two transverse submanifolds with conical singularities. Then, we are allowed to apply Lemma 4.4 in its generalized form by taking $N=\Sigma^{*}$ and $N^{\prime}=\Sigma$. It tells us that for a generic $g \in G_{\Sigma}$ and for
${ }^{3}$ For $x \in S$, the intersection of $A$ with a neighborhood of $x$ in $M$ reads $S \times c L$ where $L$ is a submanifold with conical singularities in the $r$-sphere, $r=\operatorname{dim} M-\operatorname{dim} S-1$, and where $c L$ stands for the cone on $L$ in the $(r+1)$-ball.
any entries, the intersection $W^{s}\left(g_{*}^{\circ(j-1)} X^{D}\right) \cap W^{s}\left(g_{*}^{\circ j} X^{D}\right)=: W^{s}\left(\mathcal{D}_{1}^{j}\right) \cap W^{s}\left(\mathcal{D}_{1}^{j+1}\right)$ is transverse. Moreover, by 2) of Lemma 4.4, this intersection is transverse to $\Sigma$. This proves the first item.
2) Let us prove first the admissibility of $\mathcal{D}^{1}\left(\mathbb{T}^{2}\right)$. Set $I_{1}^{1}:=W^{s}\left(\mathcal{D}_{1}^{1}\right) \pitchfork W^{s}\left(\mathcal{D}_{1}^{2}\right)$. Consider the intersection $W^{u}\left(X^{D}\right) \pitchfork I_{1}^{1}$. Its transversality follows from the first item. Thus, the inverse image $p_{2}^{-1}\left(I_{1}^{1}\right)$ by the target map $p_{2}$ is transverse to $p_{1}^{-1}\left(W^{u}\left(X^{D}\right)\right)$ in $M \times M$ at any point of the diagonal.

The intersection $G\left(\bar{X}^{D}\right) \cap p_{2}^{-1}\left(I_{1}^{1}\right)$ is transverse since there are no zeroes of $X^{D}$ in $I_{1}^{1}$ (the stable manifold associated with the trunk is transversely defined). We have to check that $p_{1}^{-1}\left(W^{u}\left(X^{D}\right)\right)$ intersects transversely $G\left(\bar{X}^{D}\right) \cap p_{2}^{-1}\left(I_{1}^{1}\right)$. A point $(x, y)$ in that intersection satifies: $x \in W^{u}\left(X^{D}\right), y \in I_{1}^{1}$ and $y=\bar{X}_{t}^{D}(x)$ for some $t \geq 0$. As $W^{u}\left(X^{D}\right)$ is positively invariant by the flow, we have $y \in W^{u}\left(X^{D}\right)$, that is, $y \in W^{u}\left(X^{D}\right) \cap I_{1}^{1}$.

The lifted flow $\bar{X}^{D} \times I d_{M}$ to $M \times M$ preserves the three submanifolds the intersection of which we are looking at. As said above, their mutual transversality holds in $(y, y)$, and hence it holds in $(x, y)$. This is the desired admissibility of $\mathcal{D}^{1}\left(\mathbb{T}^{2}\right)$. The admissibility of $\mathcal{D}^{j}\left(\mathbb{T}^{2}\right)$ will follow from the isotopy argument which is given right below.

For the second part of 2 ), observe that $g^{\circ(j-1)}$ is a common factor in the three decorations which appear in $\mathcal{D}^{j}\left(\mathbb{T}^{2}\right)$. Recall $g$ is isotopic to $I d_{M}$ by an isotopy $\left(g_{t}\right)_{t \in[0,1]}$ which keeps $\Sigma$ invariant. Decompose $g^{\circ j}=g^{\circ(j-1)} \circ g$, apply the isotopy to the left factor, namely $g_{t}^{\circ(j-1)}$, and keep the right factor $g$ unchanged. Along the isotopy, all the required transversality conditions are preserved since the triple of decorations are moved by an ambient isotopy. Therefore, this is an isotopy of admissible decorations, which is the statement of the second item.

We are going to generalize Proposition 4.3 to any forest in the following sense.

Definition 4.5. 1) The height of a tree is the generation of its root (see Definition 2.13). In this subsection, the trees are drawn in the upper half-plane with roots in $\mathbb{R} \times\{0\}$. By definition, the root of a tree $T$ belongs to one edge only; this edge is called the trunk of $T$ (denoted by $e_{\text {root }}(T)$ in Subsection (2.7). The other edges of $T$ are its branches.
2) $A$ forest of height $h$ is a finite union of pairwise disjoint trees of height $\leq h$ with at least one tree of height $h$.

Any height- $h$ forest $F$ yields a height- $(h-1)$ forest $c(F)$ just by erasing a small neighborhood of all generation- $h$ trunks and descending the newly created roots to $\mathbb{R} \times\{0\}$. This process can be iterated and yields successively $c^{2}(F), c^{3}(F)$ and so forth.
4.6. Labeling of the edges and $g$-standard decoration. Denote by $\mathcal{F}(d)$ the set of forests with $d$ leaves at most. For $F \in \mathcal{F}(d)$, the $j$-th leaf of $F$ determines a unique maximal monotone path $P^{j}(F)$ in $F$ ending to that leaf. If $e$ is an edge of $F$, it will be labeled $e_{h}^{j}(F)$ (or $e_{h}^{j}$ when no possible confusion) if $h$ is the generation of $e$ and $j$ labels the leftmost path $P^{j}(F)$ containing $e$ (see Figure 3). Observe (e.g. on this example) that for a given $h>1$, any $j \in[1, d]$ cannot appear as label of an edge of $F$.


Figure 3. A height-3 forest with 6 leaves
Recall the group $G_{\Sigma}$ of diffeomorphisms of $M$ which are isotopic to $I d_{M}$ by an isotopy keeping $\Sigma$ invariant. For $g \in G_{\Sigma}$, the $g$-standard decoration of $F$ consists of decorating $e_{h}^{j}(F)$ with the vector field $g_{*}^{\circ(j-1)} X^{D}$. The multi-intersection associated with the vertex (of generation $h-1$ ) which lies at the end of $e_{h}^{j}(F)$ will be denoted by $I_{h}^{j}(F)$.

Remark 4.7. If $T$ is a subtree of $F$, there are two natural decorations on $T$ : the first one is $\mathcal{D}^{g}(T)$, the own $g$-standard decoration of $T$; the second one is $\mathcal{D}^{g}(F)_{\mid T}$ which is induced on $T$ from the $g$-standard decoration of $F$. The question of coherence amounts to compare these two decorations on $T$. In this aim, what follows is helpful: if $e_{1}^{j}(F)$ is the leftmost edge of generation 1 sitting in $T$, then $g_{*}^{\circ(j-1)}$ is a common factor to every edge decoration in $\mathcal{D}^{g}(F)_{\mid T}$.

The next lemma is the technical part for proving that, generically, the $g$-standard decoration is admissible and coherent.

Lemma 4.8. Given the number $d$ of leaves, for every generation $h, 1 \leq h \leq d$, there exists an open dense subset $\mathcal{G}_{h}(d) \subset G_{\Sigma}$ such that the following two properties hold for every height-h forest $F \in \mathcal{F}(d)$ and for every $g \in \mathcal{G}_{h}(d)$ :

$$
\begin{align*}
& \left\{\begin{array}{l}
\text { For every possible } j \text {, the map } p_{1} \mid W^{s}\left(e_{h}^{j}(F), g_{*}^{\circ(j-1)} X^{D}\right) \text { is transverse to } \Sigma \text { and to } \\
\text { the family }\left\{p_{1} \mid W^{s}\left(e_{k}^{\ell}(F), g_{*}^{\circ(\ell-1)} X^{D}\right)\right\}_{(k, \ell)} \text { where } k<h \text { and }(k, \ell) \text { runs among } \\
\text { all labels of branches of } F \text {, except those in the tree whose trunk is } e_{h}^{j}(F) .
\end{array}\right.  \tag{4.4}\\
& \left\{\begin{array}{l}
\text { The family of maps }\left\{p_{1} \mid W^{s}\left(e_{h}^{j}(F), g_{*}^{\circ(j-1)} X^{D}\right)\right\}_{j}, j \text { running among } \\
\text { all possible labels, is a transverse family and, in addition, transverse to } \Sigma .
\end{array}\right.
\end{align*}
$$

In particular, if $g \in \mathcal{G}_{h}(d)$ the $g$-standard decoration makes all multi-intersections of $F$ transversely defined; that is, this decoration is admissible.

Proof. We prove this statement inductively on $h$. For $h=1$, this is the first item of Proposition 4.3; let $\mathcal{G}_{1}(d)$ denote the open dense set in question in that proposition. As a finite intersection of open dense sets is still so, we may fix $h$ and $F \in \mathcal{F}(d)$ in what follows.

Let us prove the statement for $h=2$. About genericity of the condition (4.4), we limit ourselves to prove the genericity of transversality of $p_{1} \mid W^{s}\left(e_{2}^{j}(F), g_{*}^{\circ(j-1)} X^{D}\right)$ to one $W^{s}\left(\mathcal{D}_{1}^{k}\right)$ only, $k \neq j, j+1$. As openness of the condition is clear, we focuse on denseness. Then, we start with $g_{1} \in \mathcal{G}_{1}(d)$ and we try to approximate it by $g \in G_{\Sigma}$ fulfilling the required transversality.

Denote by $\Phi_{t}^{j}$ the positive semi-flow of $g_{*}^{\circ(j-1)} X^{D}$ and take the $g$-standard decoration of every considered edge. In that setting, (4.4) amounts to say that the equation whose unknowns are $(x, t) \in W^{s}\left(\mathcal{D}_{1}^{k}\right) \times \mathbb{R}_{+}:$

$$
\begin{equation*}
\Phi_{t}^{j}(x) \in I_{2}^{j}(F) \tag{4.6}
\end{equation*}
$$

is regular. This equation depends on $g \in \mathcal{G}_{\Sigma}$ which plays the role of a parameter; it is hidden in the domain of $x$ and the intersection $I_{2}^{j}(F)$. By Proposition 4.3, for $g$ close to $g_{1}$ the submanifolds $W^{s}\left(\mathcal{D}_{1}^{k}\right)$ and $I_{2}^{1}(F)$ are mutually transverse and their intersection is transverse to $\Sigma$. That allows us to search maximal rank for the equation (4.6) by slightly deforming $g$ in $G_{\Sigma}$. Note: firstly, $W^{s}\left(\mathcal{D}_{1}^{k}\right)$ can be thought of as being compact since all the entries of degree less $\operatorname{dim} M$ are considered; and secondly, generically on $g$ there are no zeroes of $g_{*}^{\circ(j-1)} X^{D}$ lying in $I_{2}^{j}(F)$. As a consequence, there are no solutions of (4.6) when $t$ is large enough; (by reasoning on each stratum of $\Sigma^{*}$ âĂŤ see Notation 4.2 âĂ̄̌ by decreasing codimension yields a uniform bound for a possible $t$ when $g$ varies near $g_{1}$ ).

When the parameter $g$ is incorporated to the unknowns, the equation (4.6) extended to all $g$ in a neighborhood of $g_{1}$ is of maximal rank (already with respect to $g$ ) for ( $x, t$ ) close to some $\left(x_{0}, t_{0}\right)$. By the above-noted compactness, we have only finitely many $\left(x_{0}, t_{0}\right)$ to consider. Therefore, we have a finite dimensional family $G_{0} \subset G_{\Sigma}$ passing through $g_{1}$ such that the equation (4.6) extended to $G_{0}$ is of maximal rank. Then, by Sard's Theorem, $g_{1}$ can be approximate by some $g \in G_{0}$ where (4.6) has a maximal rank. Moreover, the transversality to $\Sigma$ of the considered family (with the $g$-standard decoration) is for free as the approximation of $g_{1}$ is made inside $G_{\Sigma}$.

Regarding the condition (4.5), we limit ourselves to transversality of stable manifolds associated with two trunks of generation 2 only, say $e_{2}^{j}(F)$ and $e_{2}^{k}(F), k>j$. For any $i$, let $\Phi_{t}^{i}$ denote the flow of $g_{*}^{\circ(i-1)} X^{D}$. For a given $g \in G_{\Sigma}$, (4.5) reads:

$$
\begin{equation*}
\Phi_{t_{1}}^{j}(x) \in I_{2}^{j}(F), \Phi_{t_{2}}^{k}(x) \in I_{2}^{k}(F), \tag{4.7}
\end{equation*}
$$

the unknows being $\left(x, t_{1}, t_{2}\right) \in M \times\left(\mathbb{R}_{+}\right)^{\times 2}$. The regularity of (4.7) could be discussed as we did for (4.6). In particular, it is important to check there are no solution when one $t_{i}, i \in\{1,2\}$ approaches $+\infty$. It is not needed to say more about that condition.

About the induction from generation $h-1$ to generation $h$, it is not different from the passage from $h=1$ to $h=2$, except that having more complicated notation. More details seem not to be needed.

Proposition 4.9. Every forest $F \in \mathcal{F}(d)$ decorated with a $g$-standard decoration is admissible and coherent if $g$ belongs to $\mathcal{G}_{d}(d)$ from Lemma 4.8.

Proof. The admissibility of such a decoration holds by Lemma 4.8. Concerning the question of coherence, suppose we are given $F \in \mathcal{F}(d)$ and a height- $h$ sub-tree $T \subset F$. Let $d^{\prime}<d$ the number of leaves of $T$. Take any $g \in \mathcal{G}_{d}(d)$. This determines the $g$-standard decoration $\mathcal{D}^{g}(F)$ of $F$, together with the induced decoration $\left.\mathcal{D}^{g}(F)\right|_{T}$ on $T$, and also a decoration $\mathcal{D}^{g}(T)$ of $T$ as $\mathcal{G}_{d}(d) \subset \mathcal{G}_{h}(d)$ for every $h \leq d$. The question is to check that these two decorations of $T$ are isotopic in the space of admissible decorations (see Definition 4.1).

Let $e_{1}^{j}(F)$ be the leftmost generation-1edge of $T \subset F$. As noted in Remark 4.7, $g_{*}^{\circ(j-1)}$ is a common factor in the $g$-standard decoration of all edges of $T$ : if $e_{q}^{\ell}(F)$ is an edge of $T$, then the decoration of $e_{q}^{\ell}(F)$ reads $g_{*}^{\circ(\ell-1)} X^{D}=g_{*}^{\circ(j-1)}\left(g_{*}^{\circ(\ell-j)} X^{D}\right)$.

Remember that $g$ is isotopic to $I d$ among the diffeomorphisms which preserve $\Sigma$. Let $\left(g_{t}\right)_{t \in[0,1]}$ such an isotopy, with $g_{0}=g$ and $g_{1}=I d_{M}$. Apply $\left(g_{t}\right)_{t}$ to the factor $g^{\circ(j-1)}$ while keeping the other factors unchanged. This ambient isotopy of $M$ carries decorations of all edges $T$ accordingly. Every multi-intersection remains transversely defined and transverse to $\Sigma$. Thus, admissibility is kept throughout the isotopy.

Call $\mathcal{D}_{1}(T)$ the decoration of $T$ at time $t=1$ of the isotopy $\left(g_{t}^{\circ(j-1)}\right)_{t}$. By construction, the generation-1 edges of $T$ are decorated by $X^{D}, g_{*} X^{D}, \ldots, g_{*}^{\circ\left(d^{\prime}-1\right)} X^{D}$ consecutively. Since the monotone paths descending from the leaves to the root of $T$ determined the decoration of $T$ at time $t=0$, this remains true at time $t=1$ of the isotopy. As a consequence, $\mathcal{D}_{1}(T)$ is the $g$-standard decoration of $T$. The coherence property is proved.

## 5. Transition

Proving $A_{\infty}$-relations in Section 7 requires analysing the transition phenomenon from $I\left(T^{\prime}\right)$ to $I\left(T^{\prime \prime}\right)$, where $T^{\prime}$ and $T^{\prime \prime}$ are two generic Fukaya trees with $d$ leaves on each side of a codimensionone stratum in the space $\mathcal{T}_{d}$ of Fukaya trees with $d$ leaves (see Definition 2.3).
5.1. Setting of the transition. We consider two Fukaya trees $T^{\prime}$ and $T^{\prime \prime}$ which differ only in the star of $v$ (see Figure 4). The intermediate Fukaya tree $T$ has codimension one since $v$ has valency 4. Up to isotopy, $T^{\prime}$ and $T^{\prime \prime}$ are the only two possible deformations from $T$ to a generic Fukaya tree. The edge $e^{\prime}$ (resp. $e^{\prime \prime}$ ) is collapsed in $T^{\prime} \rightarrow T$ (resp. $T^{\prime \prime} \rightarrow T$ ). The counting of interior edges gives $n\left(T^{\prime}\right)=n\left(T^{\prime \prime}\right)=n(T)+1$.

The collapse $e^{\prime} \searrow v$ carries any admissible decoration $\mathcal{D}_{T^{\prime}}$ to a unique admissible decoration of $T$, just by forgetting the decoration of $e^{\prime}$. It will be denoted by $\mathcal{D}_{T^{\prime} \rightarrow T}$. One will say that $\mathcal{D}_{T^{\prime} \rightarrow T}$ is induced by $\mathcal{D}_{T^{\prime}}$ through the collapse of $e^{\prime}$. Note that the diagram

$$
\begin{equation*}
W^{s}\left(e^{\prime}\right) \rightarrow M \leftarrow W^{s}\left(e_{3}\right) \tag{5.1}
\end{equation*}
$$

contains the diagram

$$
\begin{equation*}
I\left(v^{\prime}\right) \rightarrow M \leftarrow W^{s}\left(e_{3}\right) \tag{5.2}
\end{equation*}
$$

as a codimension-one subdiagram by intersecting $W^{s}\left(e^{\prime}\right)$ with the diagonal of $M \times M$. Then, the diagram (5.2) defines the generalized triple intersection $I(v)$ transversely.

Lemma 5.2. Let $e^{\prime}$ be an interior edge of a Fukaya tree $T^{\prime}$ and let $T:=T^{\prime} / e^{\prime}$ be the tree resulting of the collapse of $e^{\prime}$. Then, the induction map is a continuous. In particular, path connected components of admissible decorations are sent to admissible path components.

Proof. This statement follows from the fact that all transversality conditions are open since they are to be satisfied along compact sets which are the stratified compactifications of the


Figure 4.
involved multi-intersections. If $T^{\prime}$ reduces to $T$ by collapsing an edge, the transversality conditions for admissibility in $T^{\prime}$ are contained in those for admissiblity in $T$.

Definition 5.3. Let $T^{\prime}$ and $T^{\prime \prime}$ be two trees reducing by collapse onto $T$. The two admissible decorations $\mathcal{D}_{T^{\prime}}$ and $\mathcal{D}_{T^{\prime \prime}}$ are said to be transition compatible if they induce two isotopic admissible decorations on the codimension-one tree $T$.

Proposition 5.4. Let $T^{\prime} \rightarrow T \leftarrow T^{\prime \prime}$ be a transition of Fukaya trees in $\mathcal{F}(d)$. Let $g$ be an element of $\mathcal{G}_{d}(d)$ from Lemma 4.8. Then the $g$-standard decorations $\mathcal{D}_{g}\left(T^{\prime}\right)$ and $\mathcal{D}_{g}\left(T^{\prime \prime}\right)$ of $T^{\prime}$ and $T^{\prime \prime}$ respectively induce the same decorations on $T$, namely the $g$-standard decoration of $T$.

Proof. According to the definition of the $g$-standard decoration, the collapse of an edge only changes the generation level of some edges, but not their upper indices. Therefore, the $g$ standard decoration of any edge existing in $T$ is the same as it was before the collapse.

The following proposition will be useful in Section 6.
Proposition 5.5. In this setting, the multi-intersection $I(T)$ has a natural smooth embedding $j^{\prime}: I(T) \rightarrow I\left(T^{\prime}\right)$ (resp. $j^{\prime \prime}: I(T) \rightarrow I\left(T^{\prime \prime}\right)$ ) as a boundary. These embeddings extend to the closure cl $(I(T))$ in a way compatible with the stratifications. Then, $I\left(T^{\prime}\right) \underset{I(T)}{\cup} I\left(T^{\prime \prime}\right)$ is a (piecewise smooth) manifold which is equipped with a natural stratified compactification.

Notice that this amalgamation is not contained in $M^{\times(n(T)+1)}$. It only can be immersed thereto, with a fold along $I(T)$.

Proof. It is sufficient to focus on the subtrees $T\left(v_{0}\right), T^{\prime}\left(v_{0}\right)$ and $T^{\prime \prime}\left(v_{0}\right)$ rooted at $v_{0}$ (Figure 4). Since $T^{\prime}$ and $T^{\prime \prime}$ play the same rôle with respect to $T$, we look only at $T\left(v_{0}\right)$ and $T^{\prime}\left(v_{0}\right)$. For short, $v_{0}$ will named $v$.

On the one hand, the multi-intersection $I_{T}(v)$ is contained in $M^{\times n(v, T)}$, where $n(v, T)-1$ is equal to the number of interior edges of $T$ lying above $v$. We have

$$
\begin{equation*}
I_{T}(v)=\left(W^{s}\left(e_{1}\right) \times W^{s}\left(e_{2}\right)\right) \underset{M}{\times} W^{s}\left(e_{3}\right), \tag{5.3}
\end{equation*}
$$

where the fiber product is associative. On the other hand, $I_{T^{\prime}}(v)$ is contained in $M^{\times n\left(v, T^{\prime}\right)}$ and the graph $G_{e^{\prime}}$ of the semi-flow associated with the decoration of $e^{\prime}$ is contained in the product of the first two factors. Thus, there is a diagonal map

$$
\left\{\begin{array}{ccc}
J: M^{\times n(v, T)} & \rightarrow & M^{\times n\left(v, T^{\prime}\right)}  \tag{5.4}\\
(x, y, z, \ldots) & \mapsto & (x, x, y, z, \ldots)
\end{array}\right.
$$

Observe that $I_{T^{\prime}}\left(v^{\prime}\right)$ is canonically isomorphic to $W^{s}\left(e_{1}\right) \underset{M}{\times} W^{s}\left(e_{2}\right)$. Therefore, we have:

$$
\begin{equation*}
I_{T}(v)=J^{*}\left(\left(G_{e^{\prime}} \times I_{T^{\prime}}\left(v^{\prime}\right)\right) \underset{M}{\times} W^{s}\left(e_{3}\right)\right) . \tag{5.5}
\end{equation*}
$$

As a consequence, $J$ induces the desired embedding $j^{\prime}$, which is a boundary because the diagonal is a boundary of $G_{e^{\prime}}$.

In the preceding formulas we neglected to mention the admissible decorations which is always a $g$-standard decoration for a generic $g$ in $G_{\Sigma}$. But, by Proposition 5.4 we know that each edge has the same decoration when it contributes to the multi-intersection (5.3) or to the multiintersection (5.5).

### 5.6. From $d$ leaves to $d+1$ leaves.

So far, in Sections 4 and 5 we have worked with trees or forests having a bounded number of leaves. We now explain how to decorate them without having any bound on the number of leaves.

Proposition 4.9 and Proposition 5.4 state the existence of admissible decorations of Fukaya trees with $d$ leaves having the two properties: coherence and transition compatibility. Recall the group $G_{\Sigma}$ defined in Notation 4.2. And recall from Lemma 4.8 the open dense subset $\mathcal{G}_{h}(d)$ of $G_{\Sigma}$. Since increasing the number of leaves increases the number of forests, and for each of them the number of transversality requirements, the set $\mathcal{G}_{h}(d+1)$ is open and dense in $\mathcal{G}_{h}(d)$. Therefore, $\mathcal{G}_{d+1}(d+1)$ is open dense in $\mathcal{G}_{d}(d)$. As $G_{\Sigma}$ endowed with the $C^{\infty}$ topology is a Baire space, the intersection $\mathcal{G}_{\infty}:=\bigcap_{d}^{\infty} \mathcal{G}_{d}(d)$ is non-empty, and even dense. One can choose to decorate every Fukaya tree with its $g$-standard decoration where $g$ belongs to this intersection. It satisfies the conclusion of the two above-mentioned propositions for every $d$. Therefore, the following proposition holds true.

Proposition 5.7. Let $\mathcal{T}$ be the collection of Fukaya trees. Every tree $T \in \mathcal{T}$ can be given an admissible decoration $\mathcal{D}_{T}$ such that the family $\mathcal{D}:=\left\{\mathcal{D}_{T}\right\}_{T \in \mathcal{T}}$ is coherent and transition compatible.

## 6. Orientations

The matter of orientation is a question of Linear Algebra. Some conventions have to be chosen.

### 6.1. Orientation, co-orientation and boundary.

1) Let $E$ be a vector subspace of an oriented vector space $V$. Let $\nu(E, V)$ be a complement to $E$ in $V$. Then, the orientation and the co-oriention of $E$ will be related as follows:

$$
\begin{equation*}
\operatorname{or}(\nu(E, V)) \wedge \operatorname{or}(E)=\operatorname{or}(V) \tag{6.1}
\end{equation*}
$$

2) Let $E$ be a half-space with boundary $B$. Let $\varepsilon$ be a vector in $\nu(B, E)$ pointing outwards, where $\nu(B, E)$ is a complement to $B$ in span $(E)$. Then, the orientations of $B$ and $E$ will be related as follows:

$$
\begin{equation*}
\varepsilon \wedge \operatorname{or}(B)=\operatorname{or}(E) \tag{6.2}
\end{equation*}
$$

When $E$ is oriented, this orientation of $B$ is called the boundary orientation; it is denoted by $\operatorname{or}_{\partial}(B, E)$; one also says that $B$ is the oriented boundary of $E$. Notice that, when $E \subset V$, the choices 1) and 2) are compatible if we choose $\nu(B, V)=\nu(E, V) \oplus \varepsilon \mathbb{R}$.
6.2. Orientation and fiber product. Let $E_{1}, E_{2}, V$ be three oriented vector spaces and, for $i=1,2$, let $f_{i}: E_{i} \rightarrow V$ be a linear map. Assume that $f_{1} \times f_{2}: E_{1} \times E_{2} \rightarrow V \times V$ is transverse to the diagonal $\Delta$. Then the fiber product $E_{12}:=E_{1} \times E_{2}$ is well-defined as the preimage of $\Delta$ by $f_{1} \times f_{2}$.

The first factor of $V \times V$ is seen as a complement to $\Delta$ in $V \times V$. So, the orientation of $V$ defines a co-orientation of the diagonal. The transversality yields a canonical isomorphism $\nu\left(E_{12}, E_{1} \times E_{2}\right) \cong \nu(\Delta, V \times V)$. Thus, $E_{12}$ is co-oriented in $E_{1} \times E_{2}$. Eventually, it is oriented according to (6.1).

Proposition 6.3. In the case when a fiber product with three factors is defined, the orientation is associative, that is: $\left(E_{1} \underset{V}{\times} E_{2}\right) \underset{V}{\times} E_{3}$ and $E_{1} \underset{V}{\times}\left(E_{2} \underset{V}{\times} E_{3}\right)$ have the same orientation.
Proof. It is sufficient to look at the small diagonal $\delta_{3}$ in $V \times V \times V$. In the first case it is seen as the diagonal of $\Delta \times V$ and in the second case it is seen as the diagonal $V \times \Delta$. In both cases, its co-orientation is induced by the orientation of the first $V \times V$.

In the setting of 6.2, we have the following formulas.
Proposition 6.4. 1) Let $E_{1}$ be an oriented linear half-space with oriented boundary $B_{1}$ and let $E_{2}$ be an oriented vector space. Assume that the restriction $f_{1} \times f_{2} \mid\left(B_{1} \times E_{2}\right)$ is transverse to $\Delta$. Then, the fiber product $B_{12}:=B_{1} \times E_{2}$ is the boundary of $E_{12}$ and its orientation coincides with the boundary orientation, that is:

$$
\begin{equation*}
\operatorname{or}_{\partial}\left(B_{12}, E_{12}\right)=\operatorname{or}\left(B_{1} \underset{V}{\times} E_{2}\right) . \tag{6.3}
\end{equation*}
$$

2) Let $E_{2}$ be now an oriented linear half-space with an oriented boundary $B_{2}$ and let $E_{1}$ be an oriented vector space. Assume that the restriction $f_{1} \times f_{2} \mid\left(E_{1} \times B_{2}\right)$ is transverse to $\Delta$. Then, the fiber product $B_{12}:=E_{1} \underset{V}{\times} B_{2}$ is the boundary of $E_{12}$. The orientations are related as follows:

$$
\begin{equation*}
\operatorname{or}_{\partial}\left(B_{12}, E_{12}\right)=(-1)^{\operatorname{dim} E_{1}} \operatorname{or}\left(E_{1} \underset{V}{\times} B_{2}\right) . \tag{6.4}
\end{equation*}
$$

Proof. In both cases the co-orientation of $B_{12}$ in the boundary $\partial\left(E_{1} \times E_{2}\right)$ is induced by the co-orientation of the diagonal $\Delta$. So, the only difference depends on the boundary orientation of $\partial\left(E_{1} \times E_{2}\right)$. In the first case, the boundary orientation is the product orientation or ${ }_{\partial}\left(B_{1}, E_{1}\right) \wedge$ or $\left(E_{2}\right)$. In the second case, we have:

$$
\operatorname{or}_{\partial}\left(\partial\left(E_{1} \times E_{2}\right), E_{1} \times E_{2}\right)=(-1)^{\operatorname{dim} E_{1}} \operatorname{or}\left(E_{1}\right) \wedge o r_{\partial}\left(B_{2}, E_{2}\right) .
$$

Remark 6.5. Of course, all of that was previously said in the linear case applies word to word in the non-linear case to fiber products of manifolds with boundary when they are defined, that is, under some transversality assumptions. The intersection of two transverse submanifolds is a particular case of the previous discussion.

Orientation and graph of semi-flow. Let $e$ be an interior edge in a decorated Fukaya tree $(T, \mathcal{D})$ and let $X_{e}$ be the pseudo-gradient associated with $e$. Let $G_{e}$ be the graph of its positive semi-flow $\bar{X}_{e}$. The source map $\sigma_{e}$ makes $G_{e}$ a $[0,+\infty)$-bundle over $M$. By convention, $G_{e}$ will be oriented like $\operatorname{or}([0, \infty)) \wedge \operatorname{or}(M)$. Recall also the target map $\tau_{e}: G_{e} \rightarrow M,(t, x) \mapsto \bar{X}_{e}^{t}(x)$.

Proposition 6.6. Let $z \in \operatorname{critf} \cup \operatorname{crit}^{D} f_{\partial}$ and let $H$ be the codimension-one stratum that $z$ generates in the closure of $G_{e}$ in $M^{\times n(e)}$. Denote by $\dot{W}^{s}(z)$ the stable manifold punctured at $z$; and similarly for $\dot{W}^{u}(z)$. We have

$$
\begin{equation*}
H \cong \dot{W}^{s}(z) \times \dot{W}^{u}(z) \tag{6.5}
\end{equation*}
$$

as oriented manifolds if $H$ is oriented as a boundary component of $G_{e}$. Moreover, the right handside of (6.6) is a (punctured) sub-product of $M \times M$.

Proof. First, recall that $W^{u}(z)$ is oriented arbitrarily; it is also co-oriented so that $\operatorname{co-or}\left(W^{u}(z)\right) \wedge \operatorname{or}\left(W^{u}(z)\right)=\operatorname{or}(M)$. By convention (2.12), the stable manifold is oriented by the co-orientation of the unstable manifold. Thus, the right hand side of (6.5) has the orientation of $M$.

Now, take a pair $(x, y) \in \dot{W}^{s}(z) \times \dot{W}^{u}(z)$ and a small $\varepsilon>0$. Set $a=x+\varepsilon z \vec{y}$ in the affine structure of the Morse model around $z$. The orbit of $a$ intersects the affine line $y+\mathbb{R} \overrightarrow{z x}$ in exactly one point $a^{\prime}$ at some time $t^{\prime}$; we have $\bar{X}_{e}\left(t^{\prime}, a\right)=a^{\prime}$. So, for some small enough $\delta$ and $0<\varepsilon<\delta$, we have a collar map

$$
\begin{array}{rllc}
C:(0, \delta) \times \dot{W}^{s}(z) \times \dot{W}^{u}(z) & \longrightarrow & G_{e} \\
(\varepsilon, x, y) & \mapsto & \left(a, a^{\prime}\right)
\end{array}
$$

which extends to a diffeomorphism $\{0\} \times \dot{W}^{s}(z) \times \dot{W}^{u}(z) \cong H$. By a computation in the Morse model, it is seen that, fixing $\varepsilon$, the map $\tau_{e} \circ C_{\varepsilon}:(x, y) \mapsto a^{\prime}$ is direct. Moreover, making $\varepsilon$ decrease (which is the outgoing direction along the boundary) makes $t^{\prime}$ increase. Altogether, we have the desired isomorphism of orientations.

## Orientation and multi-intersection.

Let $T$ be a generic tree with $d$ leaves, an admissible decoration $\mathcal{D}$ and entries $x_{1}, \ldots, x_{d}$. Let $e$ be an interior edge of $T$ and let $z$ be a zero of the pseudo-gradient $X_{e} \in \mathcal{D}$. Assume the index of $z$ is convenient so that $z$ generates a codimension-one stratum $H$ in closure of the generalized stable manifold $W^{s}(e)$ and think of $H$ as a boundary component with algebraic multiplicity $\mu$ (see Remark 3.4). Denote by $j$ the rank of the leaf which lies just to the left of the set of leaves above $e$.

Proposition 6.7. Consider the above data and orient $H$ as a boundary component of $W^{s}(e)$ with multiplicity $\mu$. Then $H$ contributes to a boundary component of the multi-intersection $I(T)$ with multipliciy $\mu$ and the sign $(-1)^{\varepsilon_{j}}$ given by

$$
\begin{equation*}
\varepsilon_{j}=n+j-1+\sum_{i=1}^{j}\left(\operatorname{dim} W^{s}\left(x_{i}\right)-n\right) . \tag{6.6}
\end{equation*}
$$

Proof. It consists of a generalization to fiber product of the sign given in the case of a product by Proposition 6.4. Note that the sign we are interested in is invariant by sliding the edges (compare the transition move on Figure 4) as long as they remain to the left of $e$.

After a well-chosen sequence of transitions, $T$ has the following form: $T\left(v_{1}\right)=T_{\alpha} \vee T_{\beta}$. Here, $v_{1}$ is the first interior vertex above the root of $T$ and $T\left(v_{1}\right)$ stands for the union of edges above $v_{1}$; the $\vee$ means the bouquet; $T_{\alpha}$ is a tree with $j$ leaves and the edge root of $T_{\beta}$ is $e$. In that case, the multi-intersection becomes a basic fiber product and its left factor has a dimension equal to $\operatorname{dim} I_{T_{\alpha}}+1$. By the dimension formula (2.10), this is the desired $\varepsilon$.

Orientation and gluing. We are in the setting of 5.1. We will prove the following statement:
Proposition 6.8. The two multi-intersections $I\left(T^{\prime}\right)$ and $I\left(T^{\prime \prime}\right)$, equipped with their natural orientations, give $I(T)$ two opposite boundary orientations. In other words, the amalgamation $I\left(T^{\prime}\right) \underset{I(T)}{\cup} I\left(T^{\prime \prime}\right)$ is made in the category of oriented manifolds.
Proof. First we observe that the decoration has no effect on orientation matter. So, without loss of generality, we may assume $X_{e^{\prime}}=X_{e^{\prime \prime}}$ (notation of Figure 4). We are now going to use formulas (5.3) and (5.4) from the proof of Proposition 5.5. Each factors in the iterated fiber product diagram (5.3) is contained in some $M^{\times q}$ and the maps in the diagram of fiber product are induced by the first coordinate in each factor. In coordinates, a point $a \in I(v, T)$ reads:

$$
\begin{gather*}
a=\left\{\left(x_{1}, \ldots, x_{k}\right)\left(y_{1}, \ldots, y_{\ell}\right)\left(z_{1}, \ldots, z_{m}\right)\right\}  \tag{6.7}\\
x_{1}=y_{1}=z_{1}
\end{gather*}
$$

where each coordinate $x_{i}, y_{i}$ or $z_{i}$ denotes a point in $M$. Any point $a^{\prime}$ in $I\left(v, T^{\prime}\right)$ reads

$$
\begin{gather*}
a^{\prime}=\left\{\left(s^{\prime}, t^{\prime}\right)\left(x_{1}, \ldots, x_{k}\right)\left(y_{1}, \ldots, y_{\ell}\right)\left(z_{1}, \ldots, z_{m}\right)\right\}  \tag{6.8}\\
t^{\prime}=x_{1}=y_{1}, s^{\prime}=z_{1}
\end{gather*}
$$

where the new coordinates $\left(s^{\prime}, t^{\prime}\right)$ are those of $G_{e^{\prime}}$, source and target. Similarly, any point $a^{\prime \prime}$ in $I\left(v, T^{\prime \prime}\right)$ reads:

$$
\begin{gather*}
a^{\prime \prime}=\left\{\left(x_{1}, \ldots, x_{k}\right)\left(s^{\prime \prime}, t^{\prime \prime}\right)\left(y_{1}, \ldots, y_{\ell}\right)\left(z_{1}, \ldots, z_{m}\right)\right\}  \tag{6.9}\\
s^{\prime \prime}=x_{1}, t^{\prime \prime}=y_{1}=z_{1}
\end{gather*}
$$

where the coordinates $\left(s^{\prime \prime}, t^{\prime \prime}\right)$ are those of $G_{e^{\prime}}$, source and target. When comparing formulas (6.8) and (6.9) along $I(v, T)$, we have $s^{\prime \prime}=t^{\prime}, t^{\prime \prime}=s^{\prime}$. This corresponds to reversing the time of the flow of $X_{e^{\prime}}=X_{e^{\prime \prime}}$. Then, the time is the only variable whose orientation is changed. This is the reason for the change of orientation of $I(T)$ as boundary of $I\left(T^{\prime}\right)$ or $I\left(T^{\prime \prime}\right)$. The change of the place of the couple (source, target) has no effect on the orientation since it is an equidimensional couple.

## 7. $A_{\infty}$-STRUCTURE

In this section, we exhibit how one can construct an $A_{\infty}$-structure on the Morse complex $A=C_{*}\left(f, X^{D}\right)$ whose first operation $m_{1}$ coincide with the differential $\partial^{D}$. The grading is now defined by setting $|x|:=n-\operatorname{dim} W^{s}(x)$ for every critical point $x$. Note that this grading is cohomological, that is, the degree of the differential $m_{1}=\partial^{D}$ is +1 .

By Proposition 5.7, one can decorate all the Fukaya trees in a coherent and transitioncompatible manner. We fix one such a family of admissible decorations $\mathcal{D}=\left\{D_{T}\right\}_{T \in \mathcal{T}}$. So, for any tree $T$ with $d \geq 2$ leaves and any sequence of $(d+1)$ critical points $x_{1}, \ldots x_{d}, x_{d+1}$ (with possible repetition) we have the multi-intersection $I_{T}\left(x_{1}, \ldots x_{d} ; x_{d+1}\right)$ defined by the following fiber product (compare 2.11):

$$
\begin{equation*}
I_{T}\left(x_{1}, \ldots, x_{d} ; x_{d+1}\right):=\lim \left(I\left(v_{\text {root }}^{1}\right) \xrightarrow{p_{\text {root }}} M \stackrel{j}{\longleftarrow} W^{u}\left(x_{d+1}, X^{D}\right)\right) \tag{7.1}
\end{equation*}
$$

We recall that $v_{\text {root }}^{1}$ is the terminal vertex of the edge originating from the root and the associated generalized intersection is defined inductively in Section 2. Since $\mathcal{D}_{T}$ is admissible, this set is a manifold. Using the dimension formula, Proposition 2.8, we conclude that its dimension is

$$
\begin{equation*}
\operatorname{dim} I_{T}\left(x_{1}, \ldots, x_{d} ; x_{d+1}\right)=d-2-\sum_{i=1}^{k}\left|x_{i}\right|+\left|x_{d+1}\right| . \tag{7.2}
\end{equation*}
$$

Therefore the dimension of $I_{T}\left(x_{1}, \cdots x_{d} ; x_{d+1}\right)$ is zero if and only if

$$
\left|x_{d+1}\right|=\sum_{i=1}^{k}\left|x_{i}\right|+2-d
$$

In what follows, we denote by $\mathcal{T}^{0}$ (resp. $\mathcal{T}_{d}^{0}$ ) the set of generic Fukaya trees (resp. with $d$ leaves). For $d \geq 2$, we define the linear maps $m_{d}: A^{\otimes d} \rightarrow A$ by

$$
\left\{\begin{array}{l}
m_{d}\left(x_{1}, \ldots, x_{d}\right):=  \tag{7.3}\\
(-1)^{(d-1)\left|x_{d}\right|+(d-2)\left|x_{d-2}\right| \cdots+\left|x_{d-1}\right|} \sum_{T \in \mathcal{T}_{d}^{0}}\left[\sum_{|y|=2-d+\sum\left|x_{i}\right|} \# I_{T}\left(x_{1}, \ldots, x_{d} ; y\right) y\right] .
\end{array}\right.
$$

One should think of $I_{T}\left(x_{1}, \ldots, x_{d} ; y\right)$ as an oriented zero-dimensional manifold and $\# I_{T}\left(x_{1}, \ldots, x_{d} ; y\right)$ is the algebraic number of signed points in this manifold. It is clear from the definition that the degree of $m_{d}$ is $2-d$.

Remark 7.1. The coherence of the system of decorations we have used to define the $m_{k}$ 's implies the following: for every tree $T^{\prime} \in \mathcal{T}_{k}^{0}$ embedded in $T$ and every critical point $y$ of
degree $2-k+\sum_{1}^{k}\left|x_{i}\right|$, the value $\# I_{T^{\prime}}\left(x_{1}, \ldots, x_{k} ; y\right)$ is the same when it is calculated with the decoration $\mathcal{D}_{T^{\prime}}$ or with the decoration induced by $\mathcal{D}_{T}$ on $T^{\prime}$ (Compare Definition 4.1).
7.2. Geometric definition of the first operation. So far, we do not have considered trees with only leaf. Nevertheless, for $x \in \operatorname{critf} \cup c r i t^{D} f_{\partial}$, one can define $m_{1}(x)$ geometrically in the following way. From the compactification of $W^{s}\left(x, X^{D}\right)$ one extracts the frontier $F^{s}(x)$ which is the complement of $W^{s}\left(x, X^{D}\right)$ in its closure ${ }^{4}$. As $X^{D}$ is Morse-Smale, $F^{s}(x)$ is transverse to $W^{u}\left(y, X^{D}\right)$. When $|y|=|x|+1$, one defines the 0-dimensional intersection manifold $I(x ; y):=$ $F^{s}(x) \cap W^{u}\left(y, X^{D}\right)$. As it is oriented, it is made of a finite set of points with signs. Then we define

$$
\begin{equation*}
m_{1}(x)=\sum_{|y|=|x|+1} \# I(x ; y) \tag{7.4}
\end{equation*}
$$

Theorem 7.3. $\left(A, m_{1}, m_{2}, \ldots\right)$ is an $A_{\infty}$-algebra.
Proof. The $A_{\infty}$-relations read for every $d>0$ :

$$
\begin{equation*}
\sum_{j, k, l}(-1)^{j+k \ell} m_{j+1+\ell}\left(1^{\otimes j} \otimes m_{k} \otimes 1^{\otimes l}\right)=0 \tag{7.5}
\end{equation*}
$$

where the sum is taken over all non-negative integers $j, k, l$ such that $j+k+l=d$.
When putting entries $\left(x_{1}, \ldots, x_{d}\right)$, new signs appear according to Koszul's rule:

$$
\begin{align*}
& \left(1^{\otimes j} \otimes m_{k}\right)\left(x_{1}, \ldots, x_{j}, x_{j+1}, \ldots, x_{j+k}\right)= \\
& (-1)^{\left(\left|x_{1}\right|+\cdots+\left|x_{j}\right|\right)\left|m_{k}\right|}\left(x_{1}, \ldots, x_{j}, m_{k}\left(x_{j+1}, \ldots, x_{j+k}\right)\right) \tag{7.6}
\end{align*}
$$

and Identity (7.5) becomes:

$$
\begin{equation*}
\sum_{j+k+l=n}(-1)^{\varepsilon} m_{j+1+l}\left(x_{1}, \ldots, x_{j}, m_{k}\left(x_{j+1}, \ldots, x_{j+k}\right), x_{j+k+1}, \ldots, x_{d}\right)=0 \tag{7.7}
\end{equation*}
$$

where $\varepsilon=j+k l+(2-k)\left(\sum_{i=1}^{j}\left|x_{i}\right|\right)$.
By the very definition of the $m_{k}$ 's and Remark 7.1, the above $A_{\infty}$-relations are equivalent to the following identities

$$
\begin{equation*}
\sum_{j, T_{\alpha}, T_{\beta}} \sum_{y}(-1)^{\varepsilon^{\prime}} \# I_{T_{\alpha}}\left(x_{j+1}, \ldots, x_{j+k} ; y\right) \# I_{T_{\beta}}\left(x_{1}, \ldots, x_{j}, y, x_{j+k+1}, \ldots x_{d} ; z\right)=0 \tag{7.8}
\end{equation*}
$$

for all $d \geq 1$ and all sequence $\left(x_{1}, \ldots, x_{d}, y, z\right)$ of critical points with $|y|=2-k+\sum_{i=j+1}^{j+k}\left|x_{i}\right|$, $|z|=3-d+\sum_{i=1}^{d}\left|x_{i}\right|$ and $\varepsilon^{\prime}=\left|x_{1}\right|+\cdots\left|x_{j}\right|-j$. In this sum, $T_{\alpha}$ is a generic tree with $k$ leaves and $T_{\beta}$ is a generic tree with $d-k+1$ leaves. By (7.2), the manifold $I_{T_{\beta}}\left(x_{1}, \ldots, x_{j}, y, x_{j+k+1}, \ldots x_{d} ; z\right)$ is 0 -dimensional.

Note that, from $|z|=3-d+\sum_{i=1}^{d}\left|x_{i}\right|$, it follows that for all generic tree $T$, the multiintersection $I_{T}\left(x_{1}, \ldots, x_{j+k+l} ; z\right)$ is one-dimensional. The proof of (7.10) will follow from the analysis of frontier of this oriented manifold in its compactification.

We fix a generic tree $T$ with $d$ leaves and consider the compact 1-dimensional submanifold with conical singularities $\operatorname{cl}\left(I_{T}\left(x_{1}, \ldots, x_{d} ; z\right)\right) \subset M^{\times d}$. By blowing up the singular points, such a manifold can be thought of as a manifold with boundary where some boundary points are

[^2]identified. Such a point $P$ is equipped with a sign which is the sum of the boundary-orientation signs of the preimages of $P$ in the above blowing up, which is oriented; the signs are calculated at the blown-up level. Therefore, we have:
\[

$$
\begin{equation*}
\sum_{P \in \partial\left(c l\left(I_{T}\left(x_{1}, \ldots, x_{d} ; z\right)\right)\right)} \operatorname{sign}(P)=0 \tag{7.9}
\end{equation*}
$$

\]

By Proposition 3.3, the boundary components of the closure $\operatorname{cl}\left(I_{T}\left(x_{1}, \ldots, x_{d} ; z\right)\right)$ are divided in two types:

Type A: The boundary components coming from the broken orbits in the compactification of the generalized stable manifold $W^{s}\left(e, \mathcal{D}_{T}\right)$. Here, $e$ is an interior edge in the tree $T$ (see (2.8)). Such a codimension-one stratum involves some critical point $y$ and its stable/unstable invariant manifolds with respect to the pseudo-gradient given by the decoration of $e$. Therefore, it is of the form $I_{T_{\alpha}}\left(x_{j+1}, \ldots, x_{j+k} ; y\right) \times I_{T_{\beta}}\left(x_{1}, \ldots, x_{j}, y, x_{j+k+1}, \ldots x_{d} ; z\right), 0 \leq k \leq d$. The first (resp. second) factor in this product comes from the unstable (resp. stable) manifold of $y$. The tree $T$ is equal to the connected sum

$$
T=T_{\alpha} \#_{j+1} T_{\beta}
$$

where the root of $T_{\alpha}$ is glued to the $(j+1)$-th leaf of $T_{\beta}$.
Type B: The boundary components of the form $I_{T / e}\left(x_{1}, \ldots, x_{d} ; z\right)$ where $e$ is an interior edge of $T$. They are induced by the diagonal of $M \times M$ except over the zeroes of the vector field $\mathcal{D}_{T}(e)$ which decorates $e$.

Type C: The boundary components which are induced by the boundary of $M$. In general, a stable manifolds has orbits coming from $\partial M$. Actually, the type-C components are empty in the considered multi-intersection $I_{T}\left(x_{1}, \ldots, x_{d} ; z\right)$. Indeed, by construction, the unstable manifold $W^{u}\left(z, X^{D}\right)$ lies in the interior of $M$ except very near $z$. Thus, the multi-intersection $I_{T}\left(x_{1}, \ldots, x_{d} ; z\right)$, that is, the evaluation $<I_{T}\left(x_{1}, \ldots, x_{d}\right), z>$ (compare Subsection 2.11), has no type-C boundary components.

Therefore the identity (7.9) splits into two sums

$$
\begin{equation*}
S_{A}+S_{B}=0 \tag{7.10}
\end{equation*}
$$

where $S_{A}$ (resp. $S_{B}$ ) is the contribution of the type-A (resp. type-B) components. Note that $S_{A}$ is exactly the left handside of Equation (7.10) since $T, j$ and $k$ determine $T_{\alpha}$ and $T_{\beta}$. Therefore, we are reduced to prove the nullity of $S_{B}$.

By Proposition 5.5, a type $B$ boundary component $I_{T / e}\left(x_{1}, \cdots, x_{d} ; z\right)$ appears as a boundary component of exactly one another one-dimensional intersection submanifold $I_{T^{\prime}}\left(x_{1}, \cdots, x_{d} ; z\right)$ where $T^{\prime}$ is the unique generic tree, distinct from $T$, obtained from $T / e$ by an expansion at its unique degree- 4 vertex (See Figure 4). Moreover, by Proposition 6.8, the induced orientations are opposite. Therefore, in the sum $S_{B}$ these two terms cancel each other out.

Checking of the signs. We apply Proposition 6.7 which gives us the following sum of chains of geometric nature (without evaluating):

$$
\begin{equation*}
\partial I_{T}\left(x_{1}, \ldots, x_{d}\right)=\sum_{j, k, y}(-1)^{\varepsilon_{j}} \# I_{T_{\alpha}}\left(x_{j+1}, \ldots, x_{j+k} ; y\right) \partial I_{T_{\alpha}}\left(x_{1}, \ldots, y, x_{j+k+1}, \ldots, x_{d}\right) . \tag{7.11}
\end{equation*}
$$

Here, $j$ varies from 1 to $d-1 ; k$ from 1 to $d-j ; y$ is a critical point such that $|y|=2-k+\sum\left|x_{i}\right|$ and the geometric sign is the one given by formula 6.6 , that is,

$$
\varepsilon_{j}=n+j-1+\sum_{i=1}^{j}\left(\operatorname{dim} W^{s}\left(x_{i}\right)-n\right)=n+j-1-\sum_{i=1}^{j}\left|x_{j}\right| .
$$

## 8. Morse concordance and homotopy of $A_{d}$-Structures

We have seen that the operations $m_{1}, m_{2}, \ldots$ which define an $A_{\infty}$-structure on the complex $A:=C_{*}\left(f, X^{D}\right)$ are determined by the choice of a family of coherent decorations for every Fukaya tree $T$. Recall that a decoration of an edge $e$ is a vector field $X_{e}$ approximating $X^{D}$. In particular, it lies in the same connected component of Morse-Smale vector fields.

Assume we have two coherent and transition-compatible families $\left\{\mathcal{D}_{T}\right\}_{T}$ and $\left\{\mathcal{D}_{T}^{\prime}\right\}_{T}$ decorating all generic Fukaya trees. In general, these two families give rise to two distinct $A_{\infty}$-structures $\left(m_{1}, m_{2}, \ldots\right)$ and $\left(m_{1}^{\prime}, m_{2}^{\prime}, \ldots\right)$. We are going to show that these two structures can be linked by a homotopy thanks to multi-intersections over the product manifold $\hat{M}:=M \times[0,1]$ which is a manifold with boundary and corners. Note that the complex $C_{*}^{D}\left(f, X^{D}\right)$ is kept unchanged; in particular, $m_{1}=m_{1}^{\prime}$. A multi-intersection $\hat{I}_{T}$ over $\hat{M}$ associated with a decoration $\hat{\mathcal{D}}_{T}$ will be thought of as a cobordism from its trace over $M \times\{0\}$ to its trace over $M \times\{1\}$. Such a family of cobordisms will be called a geometric homotopy. The expression Morse concordance emphasizes the fact that the underlying manifold is a product $M \times[0,1]$ equipped with a function without critical points in its interior. Here, we are inspired by Conley's continuation map [5] that we have extended to $A_{\infty}$-case. The case of the Morse complex is discussed in [21] and [11] as a prelude to the (infinite dimensional) case of Floer homology. In fact, Andreas Floer [7] had first evoked the idea for the infinite dimensional Morse Theory. The invariance of the Morse homology was proved earlier using other methods.

### 8.1. Construction of a Morse concordance.

For simplicity, we first restrict to the case where $\partial M$ is empty. Then, $\hat{M}:=M \times[0,1]$ is a manifold with boundary. The general case will be sketched in Remark 8.4. When $\partial M=\emptyset$, we consider a Morse-Smale (positive) pseudo-gradient $X$ adapted to the Morse function $f$.

We first build a Morse function $\hat{f}$ on $\hat{M}$ with no critical points in the interior of $\hat{M}$ whose restriction to $M_{i}:=M \times\{i\}, i=0,1$, reads $\left.\hat{f}\right|_{M_{i}}=f+c_{i}$ where $c_{i}$ is a constant. More precisely, one requires the critical points of $\left.\hat{f}\right|_{M_{0}}$ to be of Dirichlet type and those of $\left.\hat{f}\right|_{M_{1}}$ to be of Neumann type. The pseudo-gradient vector fields adapted to $\hat{f}$ are required to be tangent to the boundary. This needs a slight modification with respect to the Morse theory we have considered so far. This modification is also emphasized when speaking of Morse concordance; that does not exactly deal with a Morse theory on a manifold with boundary.

Let $h: \mathbb{R} \rightarrow \mathbb{R}$ be the Morse function defined by $h(t)=(2 t-1)^{3}-3(2 t-1)$; its critical points are $t=0,1$. For $(x, t) \in \hat{M}$, set $\hat{f}(x, t)=f(x)+h(t)$. If $a$ is a critical point in $M_{0}$ (resp. in $M_{1}$ ), we have:

$$
\begin{equation*}
\operatorname{Ind}(\hat{f}, a)=\operatorname{Ind}(f, a)+1 \quad(\text { resp. }=\operatorname{Ind}(f, a)) \tag{8.1}
\end{equation*}
$$

If $\hat{X}$ is a pseudo-gradient on $\hat{M}$ adapted to $\hat{f}$ and tangent to $M_{0} \cup M_{1}$, when $a$ lies in $M_{0}$ (resp. in $M_{1}$ ), the stable manifold $W^{s}(a, \hat{X})$ (resp. the unstable manifold $W^{u}(a, \hat{X})$ ) meets the interior of $\hat{M}$; on the contrary, the unstable (resp. stable) manifold lies entirely in $M_{0}$ (resp. in $M_{1}$ ).

The critical points of $\hat{f}$ in $M_{0}$ will serve as entries; those lying in $M_{1}$ will be used as test data. Consider now an edge $e \subset T$ and its two decorations $X_{e} \in \mathcal{D}_{T}$ and $X_{e}^{\prime} \in \mathcal{D}_{T}^{\prime}$. It is possible to join them by a path $\left(X_{e}^{t}\right)_{t \in[0,1]}$ of Morse-Smale vector fields and form the vector field $\hat{X}_{e}$ on $\hat{M}$ defined by

$$
\begin{equation*}
\hat{X}_{e}(x, t):=X_{e}^{t}(x)+\nabla h(t) . \tag{8.2}
\end{equation*}
$$

This is a baby case of a method initiated by A. Floer ${ }^{5}$. Assume moreover that the path $\left(X_{e}^{t}\right)_{t}$ is stationary for $t$ close to 0 and 1 in order that $X_{e}$ is adapted to $\hat{f}$ near each critical point. Generically on the collection of paths $\left(X_{e}^{t}\right)_{t \in[0,1]}, e \subset T$, some transversality conditions are fulfilled which allow us to construct recursively:

- generalized stable manifolds $\hat{W}^{s}(e)$ associated with the edges $e$ of $T$,
- multi-intersections $\hat{I}(v)$ of stable manifolds associated with the vertices $v$ of $T$,
both of them being transversely defined. In other words, the decoration $\hat{\mathcal{D}}_{T}$ made of the collection $\left\{\hat{X}_{e}\right\}_{e \subset T}$ is admissible (cf. Definition 2.12).

Then, for every vertex $v$ in the considered Fukaya tree, the manifolds $\hat{I}\left(v, \hat{\mathcal{D}}_{T}\right)$ are transverse to $p_{1}^{-1}\left(M_{i}\right), i=0,1$, where $p_{1}$ denotes the first projection $\hat{M}^{\times n(v)} \rightarrow \hat{M}$. Thus, we have proved the following:
Proposition 8.2. For every vertex of $T$, the multi-intersections $\hat{I}\left(v, \hat{\mathcal{D}}_{T}\right)$ is a cobordism from $I\left(v, \mathcal{D}_{T}\right)$ to $I\left(v, \mathcal{D}_{T}^{\prime}\right)$. This cobordism extends to a stratified cobordism between their respective compactifications.

For the definition of the $A_{\infty}$-operations, it was crucial that the family of chosen decorations have two properties: coherence and transition compatibility.

Proposition 8.3. The set of decorations $\left\{\hat{\mathcal{D}}_{T}\right\}_{T}$, where $T$ runs among the Fukaya trees can be chosen in order to be:

1) coherent.
2) transition compatible;

Proof. 1) The problem of coherence can be solved by using the same method as in Section 4 and performing it "over $M \times[0,1]$ ", that is, replacing the group $G$ of diffeomorphisms of $M$, isotopic to $I d_{M}$, by the groupoid of smooth paths in $G$ (viewed as isotopies of $M$ ). The key Lemma 4.4 of that section extends to this new setting.

[^3]2) The group theoretical approach of coherence has made transition compatibility "over $M$ " automatically fulfilled (Proposition 5.4). The same is true over $M \times[0,1]$ when coherence is solved by a groupoidic approach, as it can be done.

Remark 8.4. When $M$ has a non-empty boundary and we look (for instance) at the Dirichlet critical points, $\hat{M}$ has corners modelled on $\mathbb{R}^{n-1} \times Q$ where $Q$ is a quadrant in the plane and there are critical points of $\hat{f}$ lying in the corners. The only matter is to define what an adapted pseudo-gradient is in order that the stable manifolds are well defined. One solution consits of demanding the pseudo-gradient to be pointing inwards along $\partial M \times[0,1]$, except near the critical points in the corners where it is tangent to $\partial M \times[0,1]$. The rest of the previous discussion is similar.

We are going to see that the above geometric cobordisms lead to a quasi-morphism of the $A_{\infty}$-structure defined thanks to the set of decorations $\left\{D_{T}\right\}_{T}$ to the one defined by $\left\{D_{T}^{\prime}\right\}_{T}$. The required uniqueness up to homotopy will follow.

### 8.5. Construction of $A_{\infty}$-quasi-isomorphism.

We now construct a quasi-isomorphism $\left\{f_{d}\right\}_{d \geq 1}$ between the $A_{\infty}$-structures $\left(m_{i}\right)_{d \geq 1}$ and $\left(m_{d}^{\prime}\right)_{d \geq 1}$ on $A=C_{*}\left(f, X^{D}\right)$ corresponding to the decorations $\mathcal{D}_{T}$ and $\mathcal{D}_{T}^{\prime}$ as they were introduced in Section 7. In fact the construction of $f_{d}: A^{\otimes d} \rightarrow A$ is very similar to that of the $m_{i}$ 's.

For $d+1$ critical points $x_{1}, \ldots x_{d}, y$ of $f$, we define the multi-intersection submanifold of $\hat{M}^{d}$

$$
\begin{equation*}
\left.\hat{I}_{T}\left(x_{1}, \ldots, x_{d} ; y\right):=\lim \left(I\left(v_{\text {root }}^{1}\right) \xrightarrow{p_{\text {root }}} \hat{M} \stackrel{j}{\longleftarrow} W^{u}\left((y, 1), \hat{X}^{D}\right)\right)\right) \tag{8.3}
\end{equation*}
$$

which is defined using the decoration $\left\{\hat{\mathcal{D}}_{T}\right\}_{T \in \mathcal{T}_{0}}$. Here, the inputs of $\hat{I}_{T}$ are the $\left(x_{i}, 0\right)$ 's and the output is $(y, 1)$.

For $d \geq 1$, we define $f_{d}: A^{\otimes} \rightarrow A$ by

$$
\left\{\begin{array}{l}
f_{d}\left(x_{1}, \ldots, x_{d}\right):=  \tag{8.4}\\
(-1)^{(d-1)\left|x_{d}\right|+(d-2)\left|x_{d-2}\right| \cdots+\left|x_{d-1}\right|} \sum_{T \in \mathcal{T}_{d}^{0}}\left[\sum_{\|(y, 1)\|=2-d+\sum\left\|\left(x_{i}, 0\right)\right\|} \# \hat{I}_{T}\left(x_{1}, \ldots, x_{d} ; y\right) y\right] .
\end{array}\right.
$$

where degree $\|$.$\| is defined with respect to \hat{f}$ as a Morse fonction on $\hat{M}$.
Note that condition $\|(y, 1)\|=2-d+\sum\left\|\left(x_{i}, 0\right)\right\|$ is the necessary and sufficient condition for zero dimensionality of $\# \hat{I}_{T}\left(x_{1}, \ldots, x_{d} ; y\right)$. Moreover, by observing that

$$
\begin{equation*}
\|(y, 1)\|=n+1-\operatorname{Ind}(\hat{f},(y, 1))=n+1-\operatorname{Ind}(f, y)=|y|+1 \tag{8.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(x_{i}, 0\right)\right\|=n+1-\operatorname{Ind}\left(\hat{f},\left(x_{i}, 0\right)\right)=n+1-\left(\operatorname{Ind}\left(f, x_{i}\right)+1\right)=\left|x_{i}\right| \tag{8.6}
\end{equation*}
$$

we conclude that the degree of $f_{d}$ is $1-d$, (i.e. one lower than $m_{d}$ and $m_{d}^{\prime}$ ). Let us also recall that when $\partial M \neq \emptyset$ and $x \in \operatorname{crit}^{D}\left(f_{\partial}\right)$, we have $\operatorname{dim} W^{s}\left(x, X^{D}\right)=\operatorname{Ind}\left(f_{\partial}, x\right)+1$.
Proposition 8.6. The collection $\left(f_{1}, \ldots, f_{d}, \ldots\right)$ defines a quasi-isomorphism of $A_{\infty}$-structures.

Proof. It is easily checked that $f_{1}: A \rightarrow A$ is the identity. Then, as soon as the morphism relations are fulfilled, we get a quasi-isomorphism. Let us recall these relations from the appendix:

$$
\begin{equation*}
\sum_{j+k+l=d}(-1)^{l+j k} f_{j+l+1}\left(1^{\otimes j} \otimes m_{k} \otimes 1^{\otimes l}\right)=\sum_{k=1}^{d} \sum_{i_{1}+\cdots+i_{k}=d}(-1)^{\epsilon_{i_{1}}, \cdots i_{k}} m_{k}^{\prime}\left(f_{i_{1}} \otimes \cdots \otimes f_{i_{k}}\right) \tag{8.7}
\end{equation*}
$$

where $\epsilon_{i_{1}, \cdots i_{k}}=\sum_{j}\left(\left(1-i_{j}\right) \sum_{1 \leq k \leq j} i_{k}\right)$. These relations are implied by geometric information given by the decoration family $\left\{\hat{\mathcal{D}}_{T}\right\}_{T}$, namely, for every $d>0$,

$$
\left\{\begin{array}{r}
\sum_{j, T_{\alpha}, T_{\beta}} \sum_{(y, 0)}(-1)^{j-\sum_{i=1}^{j}\left|x_{j}\right|} \# I_{T_{\alpha}}^{0}\left(x_{j+1}, \ldots, x_{j+k} ; y\right) \# \hat{I}_{T_{\beta}}\left(x_{1}, \ldots, x_{j}, y, x_{j+k+1}, \ldots, x_{d} ; z\right)=  \tag{8.8}\\
\sum_{k=1}^{d} \sum_{\substack{T_{\gamma} \in \mathcal{T}_{k}^{0} \\
i_{1}+\cdots+i_{k}=d}}\left(\# I_{T_{\gamma}}^{1}\left(y_{1}, \ldots, y_{k} ; z\right) \prod_{j=1}^{k} \# \hat{I}\left(x_{i_{j-1}+1}, \ldots, x_{i_{j}} ; y_{j}\right)\right),
\end{array}\right.
$$

Here, $I_{T}^{0}(-)$ (resp. $\left.I_{T}^{1}(-)\right)$ stands for the multi-intersection calculated with the family $\left\{\mathcal{D}_{T}\right\}$ on $M_{0}$ (resp. $\left\{\mathcal{D}_{T}^{\prime}\right\}_{T}$ on $M_{1}$ ); the connected-sum tree $T=T_{\alpha} \#_{j+1} T_{\beta}$ is a generic tree with $d$ leaves. The proof of (8.8) is similar to the proof of (7.10) with some new phenomena.

By degree arguments, one knows that the multi-intersection $\hat{I}_{T}\left(x_{1}, \ldots, x_{d} ; z\right)$ is one-dimensional. So, we have to analyze its compactification. We already know that the collapse of an edge of $T$ contributes to zero because such a boundary component appears twice in the considered sum with opposite orientations. The boundary component $\partial M \times[0,1]$ contributes also to zero as the vector field $\hat{X}$ points inwards except in a very small neighbourhood of crit ${ }^{D} f \times\{0,1\}$.

The first new phenomenon is the following. The breaking of an orbit of $\hat{X}_{e}$ involves in the same time the boundary of $\hat{M}$ : if it breaks in $y \in M_{0}$, the unstable manifold $W^{u}\left(y, \hat{X}_{e}\right)$ coincide with $W^{u}\left(y, X_{e}\right)$. This explains the factor $\# I_{T_{\alpha}}^{0}\left(x_{j+1}, \ldots, x_{j+k} ; y\right)$ in the left handside of (8.8). Moreover, a double breaking involving two critical points one of both lying in $M_{0}$ is not generic in the frontier of $\hat{I}_{T}\left(x_{1}, \ldots, x_{d}\right)$ because $W^{s}\left(y, \hat{X}_{e}\right) \times W^{u}\left(y, X_{e}\right)$ generates a smooth boundary component of $\hat{I}_{T}\left(x_{1}, \ldots, x_{d}\right)$.

The second new phenomenon is that, if the breaking happens at $y \in M_{1}$ and $d>1$, then the breaking cannot happen alone. Indeed, $W^{s}\left(y, \hat{X}_{e}\right)$ is contained in $M_{1}$; therefore, it has an empty intersection with any other stable manifold (or generalized stable manifold) which, by construction, lies in $\operatorname{int}(\hat{M}) \cup M_{0}$. Assume the root of $e$ is not the root of $T$ and let $e^{\prime}$ be the other edge of $T$ having the same root as $e$. Then, we have proved that the generalized stable manifold $W^{s}\left(e^{\prime}, \hat{X}_{e^{\prime}}\right)$ must also be contained $M_{1}$ or (over $M_{1}$ through $p_{1}$ in the fiber product construction). By iterating this argument, one proves the following claim.

CLAIM. If $d>1$, any non-empty connected component $C$ of the frontier ${ }^{6}$ of $\hat{I}_{T}\left(x_{1}, \ldots, x_{d}\right)$ which involves the breaking of an orbit at a zero in $M_{1}$ and no breaking in $M_{0}$ gives rise to the following decomposition of $T$ : there exist $k>0$, some edges $e_{1}, \ldots, e_{k}$ in $T$ separating the root of $T$ from all leaves and points $y_{1}, \ldots, y_{k}$ in $M_{1}$ which are respectively zeroes of $X_{e_{j}}^{\prime}, j=1, \ldots, k$,

[^4]such that $C$ is contained in the multi-intersection $I_{T}^{1}\left(y_{1}, \ldots, y_{k}\right)$.
In particular, except when some $y_{j}$ is of maximal Morse index (which has a neutral effect), $C$ is of codimension $k$ in the compactification of $\hat{I}_{T}\left(x_{1}, \ldots, x_{d}\right)$. If $k>1$, such a $C$ does not adhere to any smooth boundary component. This phenomenon is compatible with the fact that the singularities of the compactification are conical. This claim gives the geometric signification of the right handside of (8.8) and finishes the proof up to sign. It explains that a one-dimensional intersection of $\hat{I}_{T}\left(x_{1}, \ldots, x_{d}\right)$ with $W^{u}\left(z, X^{D}\right)$ cannot generically avoid to have singular points in such a stratum. All other configurations of orbit breaking are generically avoidable, and hence, do not appear in the counting of (8.8).

## Appendix A. Basics on homotopical algebras

In this appendix we review the basic terminology and result of the theory of $A_{\infty}$-algebras. We refer the reader to Kenji Lefèvre-Hasegawa's thesis [24] for a comprehensive treatment. However, here we use the sign convention introduced [12].

Here, k is a unitary ring.
Definition A.1. An $A_{p}$-algebra is a k -module equipped with a collection of k -module maps $m_{i}: A^{\otimes i} \rightarrow A, 1 \leq i \leq p$, of degree $2-i$ satisfying the identities

$$
\begin{equation*}
\sum_{j+k+l=i}(-1)^{j+k l} m_{j+l+1}\left(1^{\otimes j} \otimes m_{k} \otimes 1^{\otimes l}\right)=0 \tag{A.1}
\end{equation*}
$$

for all $p \geq i \geq 1$.
Similarly an $A_{\infty}$-algebra is a graded k -module $A$ together with a collection of k -module maps $m_{i}: A^{\otimes i} \rightarrow A, i \geq 1$, of degree $2-i$ such for all $p,\left(A,\left\{m_{i}\right\}_{1 \leq i \leq p}\right)$ is an $A_{p}$-algebra.
Remark A.2. According to the sign convention in [24] one should put $(-1)^{j k+l}$ instead of $(-1)^{j+k l}$. It turns out that these two definitions are equivalent. Indeed if $\left(m_{1}, m_{2}, \cdots\right)$ is an $A_{\infty}$-structure according to the sign convention of [24], then $\left(m_{1},(-1)^{\binom{2}{2}} m_{2}, \cdots(-1)^{\binom{i}{2}} m_{i}, \cdots\right)$ is an $A_{\infty}$-structure by our sign convention. The sign conventions in [24] is justified by the cobar construction. The signs in [12] correspond to that of the opposite algebra in [24].

Let $\left(A,\left\{m_{i}\right\}_{1 \leq i \leq p}\right)$ and $\left(A^{\prime},\left\{m_{i}^{\prime}\right\}_{1 \leq i \leq p}\right)$ be two $A_{p}$-algebras. An $A_{p}$-morphism from $\left(A,\left\{m_{i}\right\}_{1 \leq i \leq p}\right)$ to $\left(A^{\prime},\left\{m_{i}^{\prime}\right\}_{1 \leq i \leq p}\right)$ consists of a collection of maps $f_{i}: A^{\otimes i} \rightarrow A^{\prime}, 1 \leq i \leq p$, with the $\left|f_{i}\right|=1-i$ satisfying the conditions

$$
\begin{equation*}
\sum_{j+k+l=i}(-1)^{l+j k} f_{j+l+1}\left(1^{\otimes j} \otimes m_{k} \otimes 1^{\otimes l}\right)=\sum_{k=1}^{i} \sum_{i_{1}+\cdots+i_{k}=i}(-1)^{\epsilon_{i_{1}, \cdots i_{k}}} m_{k}^{\prime}\left(f_{i_{1}} \otimes \cdots \otimes f_{i_{k}}\right) \tag{A.2}
\end{equation*}
$$

where $\epsilon_{i_{1}, \cdots i_{k}}=\sum_{j=1}^{r}(r-j)\left(i_{j}-1\right)$.
Remark A.3. If we follow the sign convention of [24], then equation of A. 2 transforms into

$$
\begin{equation*}
\sum_{j+k+l=i}(-1)^{l+j k} f_{j+l+1}\left(1^{\otimes j} \otimes m_{k} \otimes 1^{\otimes l}\right)=\sum_{k=1}^{i} \sum_{i_{1}+\cdots+i_{k}=i}(-1)^{\epsilon_{i_{1}, \cdots i_{k}}} m_{k}^{\prime}\left(f_{i_{1}} \otimes \cdots \otimes f_{i_{k}}\right) \tag{A.3}
\end{equation*}
$$

where $\epsilon_{i_{1}, \cdots i_{k}}=\sum_{j}\left(\left(1-i_{j}\right) \sum_{1 \leq k \leq j} i_{k}\right)$.
If $\left(m_{i}\right)$ and $\left(f_{i}\right)$ satisfy the equation (A.2), then $\left(m_{1},(-1)^{\binom{2}{2}} m_{2}, \cdots(-1)^{\binom{i}{2}} m_{i}, \cdots\right)$ and

$$
\left(f_{1},(-1)^{\binom{2}{2}} f_{2}, \cdots(-1)^{\binom{i}{2}} f_{i}, \cdots\right)
$$

satisfies (A.3).
A collection of k-module maps $f=\left\{f_{i}\right\}_{i \geq 1}: A^{\otimes i} \rightarrow A^{\prime}$ is said to be a morphism of $A_{\infty^{-}}$ algebras if for all $p,\left\{f_{i}\right\}_{1 \leq i \leq p}$ is a morphism of $A_{p}$-algebras.

An $A_{\infty}$-morphism $f=\left\{f_{i}\right\}_{i \geq 1}$ is said to be a quasi-isomorphism if the cochain complex map $f_{1}$ is a quasi-isomorphism.

Definition A.4. Let $A$ and $A^{\prime}$ be two $A_{\infty}$-algebras with the corresponding differentials $D$ and $D^{\prime}$ on the bar constructions $B A$ and $B A^{\prime}$. Suppose that $f=\left\{f_{i}\right\}, g=\left\{g_{i}\right\}: A \rightarrow A^{\prime}$ are two $A_{\infty}$-morphisms and $F$ and $G$ are the coalgebra morphisms corresponding to $f$ and $g$. Then $a$ homotopy between $f$ and $g$ is a $(F, G)$-coderivation $H: B A \rightarrow B A^{\prime}$ such that

$$
\begin{equation*}
F-G=D^{\prime} H-H D . \tag{A.4}
\end{equation*}
$$

Theorem A.5. (Prouté [19], see also [24]) We suppose that k is a field.
(1) For connected $A_{\infty}$-algebras, homotopy is an equivalence relation (Theorem 4.27).
(2) A quasi-isomorphism of $A_{\infty}$-algebras is a homotopy equivalence (Theorem 4.24).

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[^0]:    ${ }^{1}$ In order to control the compactification of forthcoming moduli spaces, it is easier to reinforce the nondegeneracy condition of the zeroes of $X^{N}\left(\right.$ resp. $\left.X^{D}\right)$ that was considered in [15].

[^1]:    ${ }^{2}$ By abuse, sometimes one speaks of generic pseudo-gradients adapted to $f$ though the zeroes have moved.

[^2]:    ${ }^{4}$ Here, $X^{D}$ could be replaced with any $C^{1}$-approximation.

[^3]:    ${ }^{5}$ Floer [7] has introduced this method for finding the so-called continuation morphism which connects two (Floer) complexes built from different data.

[^4]:    ${ }^{6}$ The frontier of a multi-intersection consists of its compactification with the multi-intersection in question removed.

