THE KELLER-SEGEL SYSTEM ON THE 2D-HYPERBOLIC SPACE

Patrick Maheux, Vittoria Pierfelice

THE KELLER-SEGEL SYSTEM ON THE 2D-HYPERBOLIC SPACE

PATRICK MAHEUX & VITTORIA PIERFELICE

ABSTRACT. In this paper, we shall study the parabolic-elliptic Keller-Segel system on the Poincaré disk model of the 2D-hyperbolic space. We shall investigate how the negative curvature of this Riemannian manifold influences the solutions of this system. As in the 2D-Euclidean case, under the sub-critical condition $\chi M < 8\pi$, we shall prove global well-posedness results with any initial L^1-data. More precisely, by using dispersive and smoothing estimates we shall prove Fujita-Kato type theorems for local well-posedness. We shall then use the logarithmic Hardy-Littlewood-Sobolev estimates on the hyperbolic space to prove that the solution cannot blow-up in finite time. For larger mass $\chi M > 8\pi$, we shall obtain a blow-up result under an additional condition with respect to the flat case, probably due to the spectral gap of the Laplace-Beltrami operator. According to the exponential growth of the hyperbolic space, we find a suitable weighted moment of exponential type on the initial data for blow-up.

1. INTRODUCTION

In the last forty years, various models of the Keller-Segel system (also called Patlak-Keller-Segel) for chemotaxis have been widely studied due to important applications in biology. Most of the results of these analytical investigations focus on the fact that the global existence or the blow-up of the solutions of these problems is a space dependent phenomenon. Historically, the key papers for this family of models are the original contribution [12] E. F. Keller and L. A. Segel, and a work by C. S. Patlack [16]. The optimal results in the Euclidean space $\mathbb{R}^2$ are obtained by A. Blanchet, J. Dolbeault and B. Perthame (see [6],[3]). Moreover, a large series of results, mostly in the bounded domain case has been obtained by T. Nagai, T. Senba and T. Suzuki (see [15],[18],[19],[22],[10]). The literature on this subject is huge and we shall not attempt to give a complete bibliography.

In this paper, we study the (parabolic-elliptic)-Keller-Segel system (1.1) on the classical model of Riemannian manifold of constant negative curvature -1, namely the 2D-hyperbolic space. We present our study in the Poincaré disk model $\mathbb{B}^2$. Let

$$\begin{cases} n : [0,+\infty) \times \mathbb{B}^2 \to \mathbb{R}^+ \\ \qquad (t,x) \to n_t(x) = n(t,x) \end{cases}$$

be a non-negative function satisfying the following Keller-Segel system of equations

$$\begin{cases} \frac{\partial}{\partial t} n_t(x) = \Delta_{\mathbb{H}} n_t(x) - \chi div_{\mathbb{H}} \left(n_t(x) \nabla_{\mathbb{H}} c_t(x) \right), \quad x \in \mathbb{B}^2, \quad t > 0, \\ -\Delta_{\mathbb{H}} c_t(x) = n_t(x), \quad x \in \mathbb{B}^2, \quad t > 0, \\ n(t=0,x) = n_0(x) \geq 0, \quad x \in \mathbb{B}^2. \end{cases} \tag{1.1}$$

Date: October 19, 2018.

2010 *Mathematics Subject Classification.* 35R01, 47J35, 58J35, 43A85, 35J08, 35B45, 35D35, 35B44.

Key words and phrases. Keller-Segel equations, non compact Riemannian manifolds, negative curvature, hyperbolic space, Green function, logarithmic Hardy-Littlewood-Sobolev inequality, entropy method, dispersive estimates, smoothing estimates, global well-posedness.

We shall denote n_t or $n(t)$ indifferently. The subscript $\mathbb{H}$ on differential operators refers to the operators associated with the Riemannian metric of the Poincaré disk. More details will be given below in Section 2.
We shall understand the second condition in (1.1) on the function c_t as

$$c_t(x) = (-\Delta_{\mathbb{H}})^{-1} n_t(x) = \int_{\mathbb{B}^2} G_{\mathbb{H}}(x,y)\, n_t(y)\, dV_y, \quad x \in \mathbb{B}^2,$$

where $G_{\mathbb{H}}$ is the (Dirichlet)-Green function of $-\Delta_{\mathbb{H}}$ given by

$$G_{\mathbb{H}}(x,y) = -\frac{1}{2\pi} \log(\tanh \rho(x,y)/2), \quad x, y \in \mathbb{B}^2.$$

We denote by $\rho(x,y)$ the hyperbolic distance between x and y in $\mathbb{B}^2$ (see Section 2).

The Cauchy problem for the analogous Keller-Segel system (1.1) in $\mathbb{R}^2$ is now very well-understood (see [6],[3],[2]). The natural framework in dimension two is to work in L^1 which is the Lebesgue space invariant by the scaling of the equation and with non-negative solutions. The mass $M = \int_{\mathbb{R}^2} n_t dx$ is then a preserved quantity. Nevertheless because of the scaling critical aspect of L^1, the conservation of the mass is not enough to ensure global well-posedness. A simple "virial" type argument that we shall recall in Section 4 allows to prove that the solution blows up if the initial mass is such that $\chi M > 8\pi$. When $\chi M < 8\pi$, global existence results were proven in [3] by using the gradient flow structure of the equation. More precisely, the Keller-Segel system can be seen as the gradient flow of the free energy

$$F[n] = \int n \log n - \frac{\chi}{2} \int n\, c$$

and thus, if the initial data has finite entropy then we get a control of the free energy for all times. The use of the sharp logarithmic Hardy-Littlewood-Sobolev inequality in $\mathbb{R}^2$ then allows to get an a priori estimate in $LlogL$ of the solution of (1.1) which is enough to propagate any higher L^p regularity. In the case $\chi M = 8\pi$, it was latter shown in [2] that concentration occurs in infinite time.

As we see in this brief reminder, the results in $\mathbb{R}^2$ are sharp but use very deeply the structure of the system and the dilation structure of the Euclidean space. If the system is perturbed a little bit, for example by replacing the Poisson equation by the equation $-\Delta c + \alpha c = n$ with $\alpha > 0$, or by replacing the Poisson equation by the parabolic equation $\partial_t c - \Delta c = n$, then the results are much less complete, see for instance [4]. In the same way, we can expect that any change in the geometry will also change some results. This type of problems was already studied in bounded domains of $\mathbb{R}^2$ with various boundary conditions. The aim of our work is to investigate the influence of the geometry and in particular the curvature on these results. Note that, for larger mass, an additional condition for blow-up appears with respect to the Euclidean case. The main blow-up result of our Theorem 4.1 will be the following.

Theorem 1.1. *There exists a weight $p : \mathbb{B}^2 \mapsto [0, +\infty[$ such that, if the following two conditions:*

$$\chi M > 8\pi,$$

with $M := \int_{\mathbb{B}^2} n_0\, dV$, and

$$\int_{\mathbb{B}^2} p\, n_0\, dV < M \left(\sqrt{\frac{\chi M}{8\pi}} - 1 \right)$$

are satisfied, then a smooth solution $n : [0, T^*) \times \mathbb{B}^2 \to \mathbb{R}^+$ *of the Keller-Segel system* (1.1) *with initial condition* n_0 *can exist only on a finite interval* $[0, T^*)$.

Note that in $\mathbb{R}^2$, the sufficient criterion for blow-up is obtained by studying the 2-moment

$$I(t) = \int_{\mathbb{R}^2} |x|^2 n(t,x)\, dx.$$

The first difficulty we had to face was to find the appropriate substitute for the weight $|x|^2$ on the hyperbolic space. According to the exponential growth of the hyperbolic space, here the good quantity to use is the following weight of exponential type

$$p(\rho) := p(x) := \frac{2|x|^2}{1-|x|^2} = 2\sinh^2(\rho/2) = \cosh\rho - 1 \geq 0, \quad x \in \mathbb{B}^2,$$

where $\rho := \rho(x, 0)$ is the distance from $x \in \mathbb{B}^2$ to 0. Because of the hyperbolic geometry, as we shall see in Section 4, the proof is more involved than in the Euclidean case. Our second main result is the following global well-posedness in the case $\chi M < 8\pi$.

Theorem 1.2. *For every* $n_0 \in L^1_+(\mathbb{B}^2)$, *with* $I_0 = \int_{\mathbb{B}^2} p\, n_0\, dV < \infty$ *and* $\chi M < 8\pi$, *we have global well-posedness on* $X_{T,q} \cap \mathcal{C}(\mathbb{R}^+, L^1_+(\mathbb{B}^2, (1+p)dV))$, *for every* $T > 0$ *of the Keller-Segel system* (1.1), *where*

$$X_{T,q} = \{n : [0,+\infty) \times \mathbb{B}^2 \mapsto \mathbb{R} \mid \sup_{[0,T]} t^{(1-\frac{1}{q})} \|n_t\|_{L^q(\mathbb{B}^2)} < +\infty\},$$

with $\frac{4}{3} < q < 2$.

A crucial ingredient in our proof is a logarithmic Hardy-Littlewood-Sobolev type inequality on the hyperbolic space, that we deduce from a Hardy-Littlewood-Sobolev type inequality on $\mathbb{B}^2$, see [13]. To build the solution, by using dispersive and smoothing estimates, we shall propose a different approach than the one used in [3] which is based on the construction of weak solutions by compactness arguments. More precisely, we shall use the fixed point method popularized by Kato for parabolic equations (including the Navier-Stokes system) to prove local well-posedness in space $X_{T,q} \cap \mathcal{C}_T L^1$, see (3.3). We shall then use a priori estimates which can be deduced from the free energy dissipation to prove that the solution cannot blow-up in finite time.

Note that in the hyperbolic space, the case

$$\chi M > 8\pi \quad \text{and} \quad \int_{\mathbb{B}^2} p\, n_0\, dV \geq M\left(\sqrt{\frac{\chi M}{8\pi}} - 1\right)$$

is not covered by Theorem 1.1 and Theorem 1.2 above. It will be interesting to analyze this case further. A similar situation occurs in $\mathbb{R}^2$ in the case studied by V. Calvez and L. Corrias in [4], where the Poisson equation is replaced by $-\Delta c + \alpha c = n$ with $\alpha > 0$. Note that this case shares some similarities with our case since the Laplacian has a spectral gap on the hyperbolic space, $\sigma(\Delta_{\mathbb{H}}) = [1/4, +\infty)$.

2. The hyperbolic space and some useful formulas

In this section, we shall recall the main geometric and analytic objects on the Poincaré disk and some useful formulae that we need. It is well-known that the Poincaré disk is one of the models of the hyperbolic space, which is a non-compact Riemannian manifold with constant negative curvature **-1**. Of course, our results can be translated in the other models as far as they are isometric to $\mathbb{B}^2$. For more details, we refer to Analysis and

Riemannian geometry textbooks [20],[7],[11] for example.

Let $\mathbb{B}^2 = \{x \in \mathbb{R}^2, |x| < 1\}$ the 2-dimensional hyperbolic disk endowed with its Riemannian metric

$$ds^2 = \frac{4(dx_1^2 + dx_2^2)}{(1-|x|^2)^2}.$$

The hyperbolic distance between $x \in \mathbb{B}^2$ and $0 = (0,0) \in \mathbb{B}^2$ is given by

$$\rho := \rho(x,0) = \log\left(\frac{1+|x|}{1-|x|}\right)$$

(equivalent to $|x| = \tanh(\rho/2)$). More generally, the hyperbolic distance between x and y in the disk is given by

$$\rho(x,y) = \rho(T_x(y),0) = \rho(T_y(x),0) = \log\left(\frac{1+|T_x(y)|}{1-|T_x(y)|}\right),$$

where $T_x(y)$ is the Möebius transformation

$$T_x(y) = \frac{|y-x|^2 x - (1-|x|^2)(y-x)}{1-2x\cdot y + |x|^2|y|^2},$$

with $x \cdot y = x_1 y_1 + x_2 y_2$ denoting the scalar product on $\mathbb{R}^2$ (see [20]). We have several useful relations

$$|T_x(y)|^2 = \frac{|x-y|^2}{1-2x\cdot y + |x|^2|y|^2},$$

$$T_y(T_y(x)) = x,$$

$$\sinh\frac{\rho(T_x(y))}{2} = \frac{|T_x(y)|}{\sqrt{1-|T_x(y)|^2}} = \frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}},$$

$$\cosh\frac{\rho(T_x(y))}{2} = \frac{1}{\sqrt{1-|T_x(y)|^2}} = \frac{\sqrt{1-2x\cdot y+|x|^2|y|^2}}{\sqrt{(1-|x|^2)(1-|y|^2)}}.$$

We can consider three different systems of coordinates: $x = (x_1, x_2)$ (cartesian coordinates), (r,θ) with $r = |x|$ (polar coordinates) and (ρ,θ) (spherical coordinates) with $|x| = \tanh(\rho/2)$ where $|x|$ is the Euclidean norm. So, we can write

$$x = |x|(\cos\theta, \sin\theta) = |x|e^{i\theta} = \tanh(\rho/2)e^{i\theta} = \tanh(\rho/2)(\cos\theta,\sin\theta).$$

We shall denote indifferently $f(x) = f(r,\theta) = f(re^{i\theta}) = f(\rho,\theta)$ for simplicity. We shall also use indifferently in the same equation both variables x and ρ.

Let us denote by $g_x(X,Y)$ the metric tensor on two vector fields

$$X(x) = X_1(x)\frac{\partial}{\partial x_1} + X_2(x)\frac{\partial}{\partial x_2}, \quad Y(x) = Y_1(x)\frac{\partial}{\partial x_1} + Y_2(x)\frac{\partial}{\partial x_2},$$

evaluated at $x \in \mathbb{B}^2$, which is given by

$$g_x(X,Y) = \left(\frac{2}{1-|x|^2}\right)^2 \sum_{i=1}^{2} X_i Y_i.$$

The Riemannian element of volume (measure) of the exponential growth is given by

$$dV(x) = \left(\frac{2}{1-|x|^2}\right)^2 dx = \sinh\rho\, d\rho d\theta,$$

where $dx = dx_1 dx_2$ is the Lebesgue measure on $\mathbb{B}^2$. In cartesian coordinates, we now define classical differential operators. First of all, the gradient $\nabla_{\mathbb{H}}$ with respect to the Riemannian structure is defined by

$$\nabla_{\mathbb{H}} f(x) := \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e f(x), \quad x \in \mathbb{B}^2,$$

where $\nabla_e f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})$ is the Euclidean gradient. For a radial function f, we note $f(x) = f(\rho(x))$ and have

$$\nabla_{\mathbb{H}} \left(f(\rho(x))\right) = \frac{f'(\rho)}{2\cosh^2(\rho/2)} \frac{x}{|x|} = \frac{f'(\rho)}{\sinh(\rho)} x,$$

where $f'(\rho)$ is the derivative of f with respect to ρ. In particular

$$g_x(\nabla_{\mathbb{H}} f, \nabla_{\mathbb{H}} f) = \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e f \cdot \nabla_e f,$$

where $\nabla_e f \cdot \nabla_e g = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2}$. For a radial function f, we have

$$|\nabla_{\mathbb{H}} f\left(\rho(x)\right)|_g^2 = g_x(\nabla_{\mathbb{H}} f, \nabla_{\mathbb{H}} f) = |f'(\rho)|^2.$$

The divergence of the vector field $Z(x) = Z_1(x)\frac{\partial}{\partial x_1} + Z_2(x)\frac{\partial}{\partial x_2}$ is defined by

$$div_{\mathbb{H}} Z(x) = \nabla_{\mathbb{H}} \cdot Z(x) := \frac{4}{(1-|x|^2)} \sum_{i=1}^{2} x_i Z_i(x) + \sum_{i=1}^{2} \partial_i Z_i(x), \quad x \in \mathbb{B}^2.$$

We also define the Laplace-Beltrami operator on $\mathbb{B}^2$

$$\Delta_{\mathbb{H}} f(x) := \left(\frac{1-|x|^2}{2}\right)^2 \Delta_e f(x), \quad x \in \mathbb{B}^2,$$

where Δ_e is the Euclidean Laplacian

$$\Delta_e = \frac{\partial^2}{\partial^2 x_1} + \frac{\partial^2}{\partial^2 x_2},$$

with $x = (x_1, x_2) \in \mathbb{B}^2$. For radial function $f(\rho)$, the Laplace-Beltrami operator takes the form

$$\Delta_{\mathbb{H}} f(\rho) = f''(\rho) + \coth(\rho) f'(\rho) = \frac{1}{\sinh\rho} \frac{\partial}{\partial\rho} \left(\sinh\rho \frac{\partial}{\partial\rho} f\right).$$

The (Dirichlet)-Green function of $-\Delta_{\mathbb{H}}$ is given by

$$G_{\mathbb{H}}(x,y) = -k_1 \log\tanh(\rho(x,y)/2) = -\frac{k_1}{2}\log|T_x(y)|^2, \tag{2.2}$$

with $k_1 = \frac{1}{2\pi}$ (see [20]).

For any function u and vector field Y defined on $\mathbb{B}^2$, we define in the usual way

$$\|u\|_{L^q} = \left(\int_{\mathbb{B}^2} |u(x)|^q dV_x\right)^{\frac{1}{q}}, \quad \|Y\|_{L^q} = \left(\int_{\mathbb{B}^2} |g_x(Y,Y)|^{\frac{q}{2}} dV_x\right)^{\frac{1}{q}}.$$

3. Local well-posedness

3.1. **Auxiliary estimates.** Let us first state the dispersive and smoothing estimates that we shall use for the heat equation associated with the Laplace-Beltrami operator. The proof of these estimates can be found for example by V. Pierfelice in [17].

Lemma 3.1. *For every $1 \leq p \leq q$, we have the estimate*

$$\|e^{t\Delta_{\mathbb{H}}}u_0\|_{L^q(\mathbb{B}^2)} \leq c_1(t)^{\frac{1}{p}-\frac{1}{q}}e^{-t\gamma_{p,q}}\|u_0\|_{L^p(\mathbb{B}^2)}, \quad t>0,$$

where $\gamma_{p,q} = \delta(\frac{1}{p}-\frac{1}{q}+\frac{8}{q}(1-\frac{1}{p})) \geq 0$ and $c_1(t) = CMax\,(1,t^{-1})$ for some δ, $C>0$.

We shall also use the next lemma.

Lemma 3.2. *For every $1 \leq p \leq q$, and every vector field Y on $\mathbb{B}^2$, we have the estimate*

$$\|e^{t\Delta_{\mathbb{H}}}div_{\mathbb{H}}Y\|_{L^q(\mathbb{B}^2)} \leq c_1(t)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}e^{-t\frac{\gamma_{p,q}+\gamma_{q,q}}{2}}\|Y\|_{L^p(\mathbb{B}^2)}, \quad t>0,$$

where $\gamma_{p,q} = \delta(\frac{1}{p}-\frac{1}{q}+\frac{8}{q}(1-\frac{1}{p})) \geq 0$ and $c_1(t) = CMax\,(1,t^{-1})$ for some δ, $C>0$.

We shall also need to estimate $c := -\Delta_{\mathbb{H}}^{-1}n$. To do so, in the next Lemma, we use the boundedness of the Riesz transorm, Sobolev embedding and Poincaré inequality on the hyperbolic space.

Lemma 3.3. *For every $s>1$, $1<q<2$ such that $\frac{1}{s} = \frac{1}{q}-\frac{1}{2}$, we have the estimate*

$$\|\nabla_{\mathbb{H}}c\|_{L^s(\mathbb{B}^2)} \lesssim \|n\|_{L^q(\mathbb{B}^2)}.$$

Proof. We want to estimate $\nabla_{\mathbb{H}}c = -\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1}n$. Note that we can write

$$\nabla_{\mathbb{H}}c = -(\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-\frac{1}{2}})\Delta_{\mathbb{H}}^{-\frac{1}{2}}n$$

therefore, by using the continuity of the Riesz transform on L^s (see [21] for example), we obtain

$$\|\nabla_{\mathbb{H}}c\|_{L^s} \lesssim \|\Delta_{\mathbb{H}}^{-\frac{1}{2}}n\|_{L^s}.$$

Next, by using the Sobolev embedding $W^{1,q} \subset L^s$ and the Poincaré inequality on the hyperbolic space, we obtain

$$\|\nabla_{\mathbb{H}}c\|_{L^s} \lesssim \|\Delta_{\mathbb{H}}^{-\frac{1}{2}}n\|_{W^{1,q}} \lesssim \|\nabla_H\Delta_{\mathbb{H}}^{-\frac{1}{2}}n\|_{L^q}.$$

By using again the continuity of the Riesz transform on L^q, we finally obtain that

$$\|\nabla_{\mathbb{H}}c\|_{L^s} \lesssim \|n\|_{L^q},$$

which is the desired estimate.

□

3.2. **Local well-posedness results.** Let $1 \leq q \leq +\infty$ and $T>0$ be fixed. We recall the following Banach space

$$X_{T,q} = \{n : [0,+\infty)\times\mathbb{B}^2 \mapsto \mathbb{R} \mid \sup_{[0,T]} t^{(1-\frac{1}{q})}\|n_t\|_{L^q(\mathbb{B}^2)} < +\infty\}, \tag{3.3}$$

with norm

$$\|n\|_{X_{T,q}} = \sup_{[0,T]} t^{(1-\frac{1}{q})}\|n_t\|_{L^q(\mathbb{B}^2)}.$$

Theorem 3.4. *For every $n_0 \in L^1$ and $\frac{4}{3} < q < 2$, there exists $T>0$ such that there is a unique solution n of the Keller-Segel system (1.1) in $X_{T,q} \cap \mathcal{C}([0,T], L^1(\mathbb{B}^2))$.*

Remark 3.5. *Note that as usual, the arguments in the proof can also be used to get global well-posedness for sufficiently small data in L^1.*

Proof. Without lost of generality, we can assume that $\chi = 1$ in (1.1). By using Duhamel formula, solving the system (1.1) is equivalent to look for

$$n = e^{t\Delta_{\mathbb{H}}} n_0 + B(n, n), \tag{3.4}$$

with

$$B(n,n) = -\int_0^t e^{(t-s)\Delta_{\mathbb{H}}} \mathrm{div}_{\mathbb{H}}(n(s)\nabla_{\mathbb{H}} c(s))\, ds, \qquad c = -\Delta_{\mathbb{H}}^{-1} n.$$

We shall use the following classical variant of the Banach fixed point Theorem:

Lemma 3.6. *Consider X a Banach space and B a bilinear operator $X \times X \longrightarrow X$ such that*

$$\forall u,\, v \in X, \quad \|B(u,v)\|_X \leq \gamma \|u\|_X \, \|v\|_X,$$

then, for every $u_1 \in X$, such that $4\gamma\|u_1\|_X < 1$, the sequence defined by

$$u_{n+1} = u_1 + B(u_n, u_n), \quad u_0 = 0$$

converges to the unique solution of

$$u = u_1 + B(u,u)$$

such that $2\gamma\|u\|_X < 1$.

We can always assume that $T \leq 1$ such that, in the following, we will use Lemma 3.1 and Lemma 3.2 with $c(t) = C/t$.

Step 1. Existence of solutions in $X_{T,q}$.
1.1. Smallness of $u_1 = e^{t\Delta_{\mathbb{H}}} n_0$ in $X_{T,q}$ for some $T > 0$. On one hand, by applying Lemma 3.1, we have

$$N_T(n_0) := \|e^{t\Delta_{\mathbb{H}}} n_0\|_{X_{T,q}} \leq c\|n_0\|_{L^1(\mathbb{B}^2)}, \tag{3.5}$$

for any $n_0 \in L^1(\mathbb{B}^2)$ and $q \geq 1$. On the other hand, we have

$$\lim_{t\to 0} t^{(1-\frac{1}{q})} \|e^{t\Delta_{\mathbb{H}}} n_0\|_{L^q(\mathbb{B}^2)} = 0,$$

for any $n_0 \in L^q \cap L^1$ and $q > 1$. Hence $\lim_{T\to 0} N_T(n_0) = 0$. Then, by density of $L^q \cap L^1$ in L^1, we get

$$\lim_{T\to 0} \|e^{t\Delta_{\mathbb{H}}} n_0\|_{X_{T,q}} = 0, \tag{3.6}$$

for any $n_0 \in L^1(\mathbb{B}^2)$. So, for all $\gamma > 0$ and all $n_0 \in L^1(\mathbb{B}^2)$, there exists $T > 0$ such that $4\gamma\|e^{t\Delta_{\mathbb{H}}} n_0\|_{X_{T,q}} < 1$.

1.2. Boundedness of $B(n,n)$. Next we shall study the continuity of $B(n,n)$ on $X_{T,q}$. By using successively Lemma 3.2 and the Hölder inequality, we obtain

$$\begin{aligned}
\|B(n,n)\|_{X_{T,q}} &\lesssim \sup_{[0,T]} t^{(1-\frac{1}{q})} \int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}} \|n(s)\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1} n(s)\|_{L^p(\mathbb{B}^2)}\, ds \\
&\lesssim \sup_{[0,T]} t^{(1-\frac{1}{q})} \int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}} \|n(s)\|_{L^q(\mathbb{B}^2)} \|\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1} n(s)\|_{L^r(\mathbb{B}^2)}\, ds,
\end{aligned}$$

with $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$ and $p\le q$. By using Lemma 3.3 with $c=-\Delta_{\mathbb{H}}^{-1}n$, we have

$$\lesssim \sup_{[0,T]} t^{(1-\frac{1}{q})}\int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\|n(s)\|^2_{L^q(\mathbb{B}^2)}\,ds,$$

with $\frac{1}{r}=\frac{1}{q}-\frac{1}{2}$. Thus for $1\le p\le q$ such that $\frac{1}{p}=\frac{2}{q}-\frac{1}{2}$ and hence $q\ge\frac{4}{3}$, we have

$$\|B(n,n)\|_{X_T}\lesssim I_1\|n\|^2_{X_T}, \tag{3.7}$$

where

$$I_1=\sup_{[0,T]} t^{(1-\frac{1}{q})}\int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\frac{1}{s^{2(1-\frac{1}{q})}}\,ds. \tag{3.8}$$

By using the change of variables $s=tw$, we find

$$I_1=\sup_{[0,T]} t^{(1-\frac{1}{q})}\int_0^1 \frac{1}{t^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}(1-w)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\frac{t\,dw}{t^{2(1-\frac{1}{q})}w^{2(1-\frac{1}{q})}}=\int_0^1\frac{1}{(1-w)^{\frac{1}{q}}}\frac{1}{w^{2(1-\frac{1}{q})}}\,dw$$

assuming $\frac{1}{q}<1$ i.e. $q>1$ and $2(1-\frac{1}{q})<1$ i.e. $q<2$, we ensure $I_1<\infty$ and hence

$$\|B(n,n)\|_{X_{T,q}}\lesssim I_1\|n\|^2_{X_{T,q}},\quad \text{for } \frac{4}{3}\le q<2. \tag{3.9}$$

1.3. Conclusion. By using Lemma 3.6 with $X=X_{T,q}$ with $T>0$ given by Step 1.1, (3.6) and (3.9), we find a solution of (1.1) in $X_{T,q}$ i.e for Keller-Segel problem with initial data n_0 in L^1 for T sufficiently small.

Step 2. Proof of $n\in L^\infty([0,T],L^1(\mathbb{B}^2))$ and Conclusion.
Since by Lemma 3.1 we have

$$\|e^{t\Delta_{\mathbb{H}}}n_0\|_{L^\infty([0,T],L^1(\mathbb{B}^2))}\le\|n_0\|_{L^1(\mathbb{B}^2)}, \tag{3.10}$$

it remains to estimate $\|B(n,n)\|_{L^\infty([0,T],L^1(\mathbb{B}^2)}$. By using successively Lemma 3.2 and the Hölder inequality, we obtain

$$\begin{aligned}\|B(n,n)\|_{L^\infty([0,T],L^1(\mathbb{B}^2))}&\lesssim \sup_{[0,T]}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}}\|n(s)\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1}n(s)\|_{L^1(\mathbb{B}^2)}\,ds\\&\lesssim \sup_{[0,T]}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}}\|n(s)\|_{L^q(\mathbb{B}^2)}\|\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1}n(s)\|_{L^{q'}(\mathbb{B}^2)}\,ds.\end{aligned}$$

Thus we have

$$\|B(n,n)\|_{L^\infty([0,T],L^1(\mathbb{B}^2))}\lesssim \sup_{[0,T]}\left(\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}s^{1-\frac{1}{q}}}\|\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1}n(s)\|_{L^{q'}(\mathbb{B}^2)}\,ds\right)\|n\|_{X_{T,q}}. \tag{3.11}$$

By Lemma 3.3 we have

$$\|\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1}n(s)\|_{L^{q'}(\mathbb{B}^2)}\lesssim\|n(s)\|_{L^\eta(\mathbb{B}^2)},\quad \text{with } \frac{1}{\eta}=\frac{3}{2}-\frac{1}{q}.$$

Since $\frac{4}{3}<q<2$, we obtain that $1<\eta<q$ and hence we can use the following interpolation inequality

$$\|n(s)\|_{L^\eta(\mathbb{B}^2)}\lesssim\|n(s)\|^\theta_{L^1(\mathbb{B}^2)}\|n(s)\|^{1-\theta}_{L^q(\mathbb{B}^2)},\quad \text{with } \theta=\frac{(\frac{3}{2}-\frac{2}{q})}{(1-\frac{1}{q})}$$

to obtain

(3.12)
$$\|B(n,n)\|_{L^\infty([0,T],L^1(\mathbb{B}^2))} \lesssim \left(\sup_{[0,T]}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}s^{1-\frac{1}{q}}}\|n(s)\|^\theta_{L^1(\mathbb{B}^2)}\|n(s)\|^{1-\theta}_{L^q(\mathbb{B}^2)}\,ds\right)\|n\|_{X_{T,q}}.$$

Thus we have

$$\|B(n,n)\|_{L^\infty([0,T],L^1(\mathbb{B}^2))} \lesssim I_2\|n\|^\theta_{L^\infty([0,T],L^1(\mathbb{B}^2))}\|n\|^{2-\theta}_{X_{T,q}}, \tag{3.13}$$

where

$$I_2 = \sup_{[0,T]}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\frac{1}{s^{(2-\theta)(1-\frac{1}{q})}}\,ds = \sup_{[0,T]}\int_0^t \frac{ds}{(t-s)^{\frac{1}{2}}s^{\frac{1}{2}}}. \tag{3.14}$$

As before, by using the change of variables $s = tw$, we find

$$I_2 = \sup_{[0,T]}\int_0^1 \frac{tdw}{t\,(1-w)^{\frac{1}{2}}w^{\frac{1}{2}}}$$

which is finite. This yields

$$\|n\|_{L^\infty([0,T],L^1(\mathbb{B}^2))} \lesssim \|n_0\|_{L^1(\mathbb{B}^2)} + \|n\|^\theta_{L^\infty([0,T],L^1(\mathbb{B}^2))}\|n\|^{2-\theta}_{X_{T,q}},$$

with $0<\theta<1$. By using the Young inequality we have

$$\|n\|_{L^\infty([0,T],L^1(\mathbb{B}^2))} \lesssim \|n_0\|_{L^1(\mathbb{B}^2)} + \|n\|^{\frac{2-\theta}{1-\theta}}_{X_{T,q}},$$

which proves that $n\in L^\infty([0,T],L^1(\mathbb{B}^2))\cap X_{T,q}$. By a classical argument we deduce that $n\in\mathcal{C}([0,T],L^1(\mathbb{B}^2))\cap X_{T,q}$.

We can also deduce the uniqueness of $n\in\mathcal{C}([0,T],L^1(\mathbb{B}^2))\cap X_{T,q}$ from similar arguments. This produces automatically local well-posedeness result.

□

Similar to Theorem 3.4, we shall obtain the next result of existence and uniqueness of the solution of the Keller-Segel system (1.1) with initial data n_0 in $L^q(\mathbb{B}^2)$ for all q such that $\frac{4}{3}\le q<2$ and a uniform positive lower bound on the existence time T independent of the initial data n_0 in any fixed ball of L^q.

Theorem 3.7. *For every $n_0\in L^q(\mathbb{B}^2)$ and $\frac{4}{3}\le q<2$, there exists $T>0$ such that there exists a unique solution $n\in\mathcal{C}([0,T],L^q(\mathbb{B}^2))$ of the Keller-Segel system (1.1). Moreover for every $R>0$ there exists $T(R)>0$ such that $T\ge T(R)$ for every $\|n_0\|_{L^q(\mathbb{B}^2)}\le R$.*

Proof. We shall again use Lemma 3.6 to solve (1.1). We can always assume that $T\le 1$ such that in the following we will use Lemma 3.1 and Lemma 3.2 with $c(t)=C/t$. We have

$$\|e^{t\Delta_{\mathbb{H}}}n_0\|_{\mathcal{C}([0,T],L^q(\mathbb{B}^2))} \le \|n_0\|_{L^q(\mathbb{B}^2)}. \tag{3.15}$$

To study the continuity of $B(n_t,n_t)$ on $\mathcal{C}([0,T],L^q(\mathbb{B}^2))$ we use successively Lemma 3.2 and the Hölder inequality

$$\begin{aligned}\|B(n,n)\|_{\mathcal{C}([0,T],L^q(\mathbb{B}^2))} &\lesssim \sup_{[0,T]}\int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\|n(s)\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1}n(s)\|_{L^p(\mathbb{B}^2)}\,ds\\ &\lesssim \sup_{[0,T]}\int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\|n(s)\|_{L^q(\mathbb{B}^2)}\|\nabla_{\mathbb{H}}\Delta_{\mathbb{H}}^{-1}n(s)\|_{L^r(\mathbb{B}^2)}\,ds,\end{aligned}$$

with $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$, $p\le q$. By using Lemma 3.3 with $c=-\Delta_{\mathbb{H}}^{-1}n$, we have

$$\lesssim \sup_{[0,T]}\int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\|n(s)\|^2_{L^q(\mathbb{B}^2)}\,ds,$$

with $\frac{1}{r}=\frac{1}{q}-\frac{1}{2}$, so $\frac{4}{3}\le q<2$. Thus for $1\le p\le q$ such that $\frac{1}{p}=\frac{2}{q}-\frac{1}{2}$, we have

$$\|B(n,n)\|_{\mathcal{C}([0,T],L^q(\mathbb{B}^2))}\lesssim I_3\|n\|^2_{\mathcal{C}([0,T],L^q(\mathbb{B}^2))}, \tag{3.16}$$

where

$$I_3=\sup_{[0,T]}\int_0^t \frac{1}{(t-s)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\,ds. \tag{3.17}$$

As before, by using the change of variables $s=tw$, we find

$$I_3=\sup_{[0,T]}\int_0^1 \frac{tdw}{t^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}(1-w)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}=\sup_{[0,T]}t^{(1-\frac{1}{q})}\int_0^1\frac{1}{(1-w)^{\frac{1}{q}}}\,dw$$

which is finite for $q>1$. Hence there exists a constant $C_1>0$ independent of T and n such that

$$\|B(n,n)\|_{\mathcal{C}([0,T],L^q(\mathbb{B}^2))}\le C_1T^{(1-\frac{1}{q})}\|n\|^2_{\mathcal{C}([0,T],L^q(\mathbb{B}^2))} \tag{3.18}$$

for any q such that $\frac{4}{3}\le q<2$.

Let $X=\mathcal{C}([0,T],L^q(\mathbb{B}^2))$ and $\gamma:=C_1T^{(1-\frac{1}{q})}$. Imposing the conditions $0<T\le 1$ and $4C_1T^{(1-\frac{1}{q})}\|n_0\|_{L^q(\mathbb{B}^2)}<1$, this implies $4\gamma\|e^{t\Delta_{\mathbb{H}}}n_0\|_{\mathcal{C}([0,T],L^q(\mathbb{B}^2))}<1$ by (3.15). By using Lemma 3.6 with $u_1:=e^{t\Delta_{\mathbb{H}}}n_0$ and Duhamel formula (3.4), we obtain a unique solution $n\in\mathcal{C}([0,T],L^q(\mathbb{B}^2))$ of the Keller-Segel system (1.1). This produces automatically local well-posedeness result.

We now prove the uniformity result. Choose $T(R):=(8C_1R)^{\frac{-q}{q-1}}$ such that $T(R)\le 1$ for R positive and large enough. The same arguments as above prove the existence of a unique solution n defined on the fixed interval $[0,T(R)]$ for all initial conditions $n_0\in L^q(\mathbb{B}^2)$ such that $\|n_0\|_{L^q(\mathbb{B}^2)}\le R$. This finishes the proof of our theorem. □

4. Blow-up

In the case of $\mathbb{R}^2$, the blow-up for the Keller-Segel system is quite easy to prove. In fact, under the assumption $\int_{\mathbb{R}^2}(1+|x|^2)n_0(x)\,dx<\infty$, we have the following "virial" type identity

$$\int_{\mathbb{R}^2}|x|^2n(t,x)\,dx=\int_{\mathbb{R}^2}|x|^2n_0(x)\,dx+\frac{4M}{8\pi}(8\pi-\chi M)t,\quad \forall t>0, \tag{4.19}$$

where $M=\int_{\mathbb{R}^2}n_0(x)\,dx$. If $\chi M>8\pi$ and t large enough, the right-hand side of (4.19) is negative, contradicting the non-negative left-hand side of the equation. Thus the solution cannot exist for $t>T^*$ for some finite T^*.

In this section, our goal is to study the blow-up phenomenon for the solution of (1.1) on $\mathbb{B}^2$. Because of the geometry of the hyperbolic space, our main difficulty here is to find an appropriate weight to obtain a "virial" type argument for blow-up. Thanks to our choice of a weight of exponential type, we are able to replace the identity (4.19) by the inequality (4.22) below. This inequality will allow to prove blow up for $M=\int_{\mathbb{B}^2}n_0(x)\,dV>8\pi/\chi$ under an additional condition on the moment, with a suitable weight p of exponential type, $\int_{\mathbb{B}^2}p\,n_0dV$. As noted before, an additional condition for blow-up on the 2-moment

$\int_{\mathbb{R}^2} |x|^2 n_0\, dx$ was also needed in [4][Thm.1.2 eq.(1.2)] where in particular the Keller-Segel system on $\mathbb{R}^2$ with the Laplacian replaced by the operator $-\Delta + \alpha$, $\alpha > 0$ was studied.

Let us recall the expression of the weight p that we shall use in our blow-up argument

$$p(\rho) := p(x) := \frac{2|x|^2}{1-|x|^2} = 2\sinh^2(\rho/2) = \cosh\rho - 1 \geq 0, \quad x \in \mathbb{B}^2. \tag{4.20}$$

Note that the expression of the weight $p = 2\sinh^2(\rho/2) = \cosh\rho - 1$ is the same in any isometric representation of the Poincaré disk (for instance in the Poincaré upper-half space model). Next, we shall need the following relations $\frac{p}{2} + 1 = \frac{1}{1-|x|^2}$ and

$$\Delta_{\mathbb{H}}\, p = 2p + 2. \tag{4.21}$$

Our blow-up result is the following statement.

Theorem 4.1. *Let $n : [0, T^*) \times \mathbb{B}^2 \to \mathbb{R}^+$ be a solution of the Keller-Segel system (1.1) with $T^* \leq +\infty$ such that $n \in \mathcal{C}([0, T^*), L^1_+(\mathbb{B}^2, (1+p)dV)$. Then we have*

(1) *For all $t \in [0, T^*)$,*

$$\left(\int_{\mathbb{B}^2} p\, n_t\, dV + M\right)^2 \leq \left(\left[\int_{\mathbb{B}^2} p\, n_0\, dV + M\right]^2 - \frac{\chi}{8\pi} M^3\right) e^{4t} + \frac{\chi}{8\pi} M^3, \tag{4.22}$$

with $M = \int_{\mathbb{B}^2} n_0(x)\, dV_x$.

(2) *If the two conditions $\chi M > 8\pi$ and*

$$\int_{\mathbb{B}^2} p\, n_0\, dV < \lambda^*(M), \tag{4.23}$$

where

$$\lambda^*(M) = M\left(\sqrt{\frac{\chi M}{8\pi}} - 1\right), \tag{4.24}$$

are satisfied, then the solution n_t can exist only on a finite interval $[0, T^*)$ with

$$T^* \leq T_{bl} := \frac{1}{4}\log\left[\frac{M^2}{8\pi}(\chi M - 8\pi)\left[\frac{\chi M^3}{8\pi} - \left(M + \int_{\mathbb{B}^2} p\, n_0\, dV\right)^2\right]^{-1}\right]. \tag{4.25}$$

In particular, a smooth solution n_t does not exist for $t > T_{bl}$.

Proof. 1. Let $\mathcal{I}(t) = \int_{\mathbb{B}^2} p\, n_t\, dV$. Formally, the derivative of $\mathcal{I}(t)$ is given by

$$\mathcal{I}'(t) = \int_{\mathbb{B}^2} p\frac{\partial}{\partial t} n_t(x)\, dV = \int_{\mathbb{B}^2} p\, \Delta_{\mathbb{H}}\, n_t(x)\, dV_x - \chi \int_{\mathbb{B}^2} p \nabla_{\mathbb{H}} \cdot (n_t(x) \nabla_{\mathbb{H}} c_t(x))\, dV_x$$

$$= \int_{\mathbb{B}^2} \Delta_{\mathbb{H}}\, p\, n_t(x)\, dV_x + \chi \int_{\mathbb{B}^2} g_x(\nabla_{\mathbb{H}}\, p(x), n_t(x)\nabla_{\mathbb{H}} c_t(x))\, dV_x.$$

The second integral has been integrated by parts. Thus we obtain

$$\mathcal{I}'(t) = \int_{\mathbb{B}^2} \Delta_{\mathbb{H}}\, p\, n_t(x)\, dV_x + \chi \int_{\mathbb{B}^2} \left(\frac{2}{1-|x|^2}\right)^2 \nabla_{\mathbb{H}} p(x) \cdot \nabla_{\mathbb{H}} c_t(x) n_t(x)\, dV_x.$$

Here again $X \cdot Y$ denotes the Euclidean scalar product. We express the hyperbolic gradient with the Euclidean gradient and get

$$\mathcal{I}'(t) = \int_{\mathbb{B}^2} \Delta_{\mathbb{H}}\, p\, n_t(x)\, dV_x + \chi \int_{\mathbb{B}^2} \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e p(x) \cdot \nabla_e c_t(x) n_t(x)\, dV_x.$$

Using the Green kernel $G_{\mathbb{H}}$ of $-\Delta_{\mathbb{H}}$ to express c_t, we obtain for the second integral

$$\int_{\mathbb{B}^2} \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e p(x) \cdot \nabla_e c_t(x) n_t(x)\, dV_x =$$

$$\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e p(x) \cdot \nabla_e^x G_{\mathbb{H}}(x,y) n_t(x) n_t(y)\, dV_x dV_y,$$

where we recall that

$$G_{\mathbb{H}}(x,y) = -k_1 \log\left[\tanh(\rho(x,y)/2)\right] = -\frac{k_1}{2}\log|T_x(y)|^2, \quad k_1 = \frac{1}{2\pi}.$$

To simplify our notation, we set $H(x,y) = \log|T_x(y)|^2$, where $T_x(y)$ is the Möebius transformation (see Section 2). This leads to the following

$$\mathcal{I}'(t) = \int_{\mathbb{B}^2} \Delta_{\mathbb{H}}\, p\, n_t(x)\, dV_x - \frac{\chi k_1}{2}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e p(x) \cdot \nabla_e^x H(x,y)\, d\mu(x,y),$$

where $d\mu(x,y)$ is the symmetric measure given by $d\mu(x,y) := n_t(x)n_t(y)\, dV_x dV_y$. Note that $\mu = \mu_t$ depends on t. For the second integral, we set

$$J := \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e p(x) \cdot \nabla_e^x H(x,y)\, d\mu(x,y).$$

By changing the role of x and y in the integral (using the symmetry of the measure and the fact that $|T_x(y)| = |T_y(x)|$), we also have

$$J = \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \left(\frac{1-|y|^2}{2}\right)^2 \nabla_e p(y) \cdot \nabla_e^y H(x,y)\, d\mu(x,y).$$

Hence, by symmetrization, the derivative of $\mathcal{I}(t)$ takes the next form

$$\mathcal{I}'(t) = \int_{\mathbb{B}^2} \Delta_{\mathbb{H}}\, p\, n_t(x)\, dV_x - \frac{\chi_0}{2}(2J),$$

with $\chi_0 := \frac{\chi k_1}{2}$ and $2J = \int\int L(x,y)\, d\mu(x,y)$, where

$$L(x,y) := \left(\frac{1-|x|^2}{2}\right)^2 \nabla_e p(x) \cdot \nabla_e^x H(x,y) + \left(\frac{1-|y|^2}{2}\right)^2 \nabla_e p(y) \cdot \nabla_e^y H(x,y).$$

2. *Computation of* $\nabla_e^x H(x,y)$. We write the square of the Euclidean norm of the Möebius transformation as

$$|T_x(y)|^2 = \frac{|x-y|^2}{V},$$

with $V = V(x,y) := 1 - 2x\cdot y + |x|^2|y|^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$.
By a straightforward computation, we have

$$\nabla_e^x |T_x(y)|^2 = \frac{2}{V^2}\left[(1-|x|^2)(1-|y|^2)(x-y) + |x-y|^2(1-|y|^2)x\right].$$

Hence, we obtain

$$\nabla_e^x H(x,y)|^2 = \nabla_e^x |T_x(y)|^2/|T_x(y)|^2 = \frac{2(1-|x|^2)(1-|y|^2)}{|x-y|^2 V}(x-y) + \frac{2(1-|y|^2)x}{V}.$$

We deduce that L has the following simple form with the choice of p given by (4.20)

$$L(x,y) = \frac{2}{V}\left[1 - |x|^2|y|^2\right].$$

Note that, in the Euclidean case $\mathbb{R}^2$, the function L is a positive constante (and $\Delta_e|x|^2$ also) and leads immediately to the blow-up equation (4.19). On the Poincaré disk, besides

the fact that L is not a constant, we have to deal with a new difficulty i.e. $\inf L = 0$. Finally, we have

$$\mathcal{I}'(t) = \int_{\mathbb{B}^2} \Delta_{\mathbb{H}}\, p\, n_t(x)\, dV_x - \chi_0 \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{2\,(1-|x|^2|y|^2)}{V}\, d\mu(x,y).$$

By the relation $\Delta_{\mathbb{H}}\, p = 2p+2$, the function $\mathcal{I}(t)$ satisfies the following differential equation

$$\mathcal{I}'(t) = 2\,\mathcal{I}(t) + 2M - \chi_0 \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{2\,(1-|x|^2|y|^2)}{V}\, d\mu(x,y). \tag{4.26}$$

3. *Dealing with* K. Let $K := \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{(1-|x|^2|y|^2)}{V}\, d\mu(x,y)$.
In order to obtain an upper bound on $\mathcal{I}'(t)$ in (4.26) i.e. a lower bound on K, we use the fact that

$$V = V(x,y) := 1 - 2\,x\cdot y + |x|^2|y|^2 \le (1+|x||y|)^2\,.$$

Hence, we have

$$K \ge \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{(1-|x|^2|y|^2)}{(1+|x||y|)^2}\, d\mu(x,y) = \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{1-|x||y|}{1+|x||y|}\, d\mu(x,y).$$

Then, by Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} M^2 &= \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} n_t(x) n_t(y)\, dV_x dV_y = \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} d\mu(x,y) \\ &= \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \left(\frac{1-|x||y|}{1+|x||y|}\right)^{1/2} \left(\frac{1+|x||y|}{1-|x||y|}\right)^{1/2} d\mu(x,y) \\ &\le \left(\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{1-|x||y|}{1+|x||y|}\, d\mu(x,y)\right)^{1/2} \left(\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{1+|x||y|}{1-|x||y|}\, d\mu(x,y)\right)^{1/2}. \end{aligned}$$

Finally, we have

$$M^2 \le \sqrt{K} \left(\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{1+|x||y|}{1-|x||y|}\, d\mu(x,y)\right)^{1/2}. \tag{4.27}$$

To obtain an upper bound on $\mathcal{I}'(t)$ in (4.26) i.e. a lower bound on K (since $-\chi_0 < 0$), it remains to prove an upper bound on

$$Q := \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{1+|x||y|}{1-|x||y|}\, d\mu(x,y)$$

as follows. Using the fact that for any $x, y \in \mathbb{B}^2$

$$1 - |x||y| \ge \sqrt{1-|x|^2}\sqrt{1-|y|^2},$$

we deduce that

$$Q \le \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{1}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}\, d\mu + \int_{\mathbb{B}^2}\int_{\mathbb{B}^2} \frac{|x||y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}\, d\mu.$$

Now, we can introduce again the weight $p(x) = \frac{2|x|^2}{1-|x|^2}$ using the two next formulae

$$\sqrt{p/2} = \frac{|x|}{\sqrt{1-|x|^2}}, \quad \sqrt{p/2+1} = \frac{1}{\sqrt{1-|x|^2}}.$$

This implies for Q

$$Q \le \left(\int_{\mathbb{B}^2} \left[\left(\frac{p}{2}+1\right) n_t\right]^{\frac{1}{2}} \sqrt{n_t}\, dV_x\right)^2 + \left(\int_{\mathbb{B}^2} \sqrt{\frac{p\, n_t}{2}}\sqrt{n_t}\, dV_x\right)^2.$$

Again by Cauchy-Schwarz inequality applied on both integrals, we have

$$Q \leq M\left(\int_{\mathbb{B}^2}\left(\frac{p}{2}+1\right)n_t\,dV_x+\int_{\mathbb{B}^2}\frac{p}{2}n_t\,dV_x\right)=M\left(\int_{\mathbb{B}^2}pn_t\,dV_x+M\right)=M\left(\mathcal{I}(t)+M\right).$$

From (4.27), we finally get

$$M^4 \leq KQ \leq KM\left(\mathcal{I}(t)+M\right),$$

or equivalently a lower bound on K

$$K \geq \frac{M^3}{\mathcal{I}(t)+M}. \tag{4.28}$$

4. *A differential inequality* for $\mathcal{I}(t)$.
From (4.26) and (4.28), we deduce a differential inequality for $\mathcal{I}(t)=\int_{\mathbb{B}^2}p\,n_t\,dV$,

$$\mathcal{I}'(t) \leq 2(\mathcal{I}(t)+M)-\chi_0 K \leq 2(\mathcal{I}(t)+M)-\chi_0\frac{M^3}{\mathcal{I}(t)+M}.$$

We set $\varphi(t)=\mathcal{I}(t)+M$, the preceding inequality reads as

$$\varphi'(t) \leq 2\varphi(t)-\chi_0\frac{M^3}{\varphi(t)}.$$

Thus, we have

$$\frac{1}{2}\left(\varphi^2\right)'(t) \leq 2\varphi^2(t)-\chi_0 M^3.$$

Now, by setting $\psi(t)=\varphi^2(t)$, we deduce the following simple differential inequality

$$\psi'(t)-4\psi(t) \leq -2\chi_0 M^3.$$

By multiplying the last inequality by e^{-4t}, we obtain

$$\left[e^{-4t}\psi(t)\right]' \leq \frac{1}{2}\chi_0 M^3\left[e^{-4t}\right]'.$$

We integrate on $(0,t)$ and we get for any $t\in(0,T^*)$

$$e^{-4t}\psi(t)-\psi(0) \leq \frac{1}{2}\chi_0 M^3\left[e^{-4t}-1\right].$$

This leads to the following

$$\psi(t) \leq \left[\psi(0)-\frac{1}{2}\chi_0 M^3\right]e^{4t}+\frac{1}{2}\chi_0 M^3. \tag{4.29}$$

Since $\psi(t)=\left(\mathcal{I}(t)+M\right)^2$, where $\mathcal{I}(t)=\int_{\mathbb{B}^2}p\,n_t\,dV$ and $k_1=(2\pi)^{-1}$ ($\chi_0=\chi(4\pi)^{-1}$), we get (4.22), i.e.

$$\left(\int_{\mathbb{B}^2}pn_t\,dV+M\right)^2 \leq \left(\left[\int_{\mathbb{B}^2}pn_0\,dV+M\right]^2-\frac{\chi}{8\pi}M^3\right)e^{4t}+\frac{\chi}{8\pi}M^3.$$

5. *Blow-up conditions.* Now assume that the solution n_t exists for any $t>0$ (i.e. $T^*=+\infty$) and assume also that

$$\left[\int_{\mathbb{B}^2}pn_0\,dV+M\right]^2-\frac{\chi}{8\pi}M^3<0 \tag{4.30}$$

holds true. We obtain a contradiction since the right-hand side term of (4.22) tends to $-\infty$ as $t\to+\infty$ and $\left(\int_{\mathbb{B}^2}pn_t\,dV+M\right)^2$ remains non-negative for any $t>0$.

Then we say that a blow-up appears i.e. the solution does not exist globally in time (i.e. $T^* < +\infty$).

Remark 4.2. *The inequality* (4.30) *is equivalent to*

$$0 \leq \int_{\mathbb{B}^2} p(x) n_0(x)\, dV_x < M\left(\sqrt{\frac{\chi M}{8\pi}} - 1\right) := \lambda^*(M). \tag{4.31}$$

In fact, the inequality (4.30) *implies two conditions. First, the right-hand side of the equation above should be positive i.e.* $\chi M > 8\pi$ *and, secondly, the p-moment at the initial time* $t = 0$ *should satisfy* (4.31)*. Note that* $M > 8\pi/\chi$ *is the same condition for blow-up than the Euclidean case* $\mathbb{R}^2$*. But here, as pointed out before, it appears the additional condition* (4.31) *for blow-up on our weighted moment.*

6. *Estimate of the blow-up time* T_{bl}. In case of blow-up i.e. under the conditions $M > 8\pi/\chi$ and (4.31), the existence time T^* of any solution (n_t) of the Keller-Segel system is bounded by the next bound T_{bl} obtained as follows. From (4.29) and

$$M^2 \leq \psi(t) = (\mathcal{I}(t) + M)^2,$$

we have

$$M^2 \leq -\left[\frac{1}{2}\chi_0 M^3 - \left(M + \int_{\mathbb{B}^2} p\, n_0\, dV\right)^2\right] e^{4t} + \frac{1}{2}\chi_0 M^3,$$

or, equivalently for any $t < T^*$ with $\frac{1}{2}\chi_0 = \frac{\chi}{8\pi}$,

$$t \leq T_{bl} := \frac{1}{4}\log\left[\frac{M^2}{8\pi}(\chi M - 8\pi)\left[\frac{\chi M^3}{8\pi} - \left(M + \int_{\mathbb{B}^2} p\, n_0\, dV\right)^2\right]^{-1}\right]$$

which leads to (4.25). This completes the proof of Theorem 4.1. □

5. A priori control of the Entropy of a positive solution

In this section, we prove lower and upper bounds of the entropy of the solution of the Keller-Segel problem (1.1) on the hyperbolic space. In particular, we shall prove that $n_t \log n_t \in L^1$ is locally uniformly bounded in time under the condition $\chi M < 8\pi$. The main ingredients in our proof are the use of the upper bound of the p-moment (4.22) and the logarithmic Hardy-Littlewood-Sobolev inequality on $\mathbb{B}^2$ (5.39).

5.1. Lower bound on the Entropy of the solution of Keller-Segel system.

We prove a lower bound on the entropy of the solution of the Keller-Segel problem (1.1) on the hyperbolic space by using the tool of relative entropy as in [3].

Lemma 5.1. *Assume that a function* $f : \mathbb{B}^2 \to [0, +\infty)$ *is such that* $M := \int_{\mathbb{B}^2} f\, dV$ *and* $\int_{\mathbb{B}^2} pf\, dV$ *are finite. Then, for all* $s > 0$*, we have*

$$\int_{\mathbb{B}^2} f \log f\, dV \geq -\frac{1}{s}\int_{\mathbb{B}^2} pf\, dV + M \log\left(\frac{M}{2\pi s}\right). \tag{5.32}$$

Here $p = p(\rho) = 2\sinh^2(\rho/2)$ *denotes the weight defined in* (4.20)*.*

Proof. Let $s > 0$ and define

$$q(x) := q_s(x) = \frac{1}{2\pi s}\exp\left(-\frac{p(x)}{s}\right) = \frac{1}{2\pi s}\exp\left(\frac{-2\sinh^2(\rho/2)}{s}\right).$$

By using the volume element dV in spherical coordinates, we easily see that q_s is a density of a probability measure on $\mathbb{B}^2$ with respect to the mesure dV. A lower bound on the relative entropy of f with respect to q is given by

$$\int_{\mathbb{B}^2} f \log f \, dV - \int_{\mathbb{B}^2} f \log q \, dV = \int_{\mathbb{B}^2} \frac{f}{q} \log\left(\frac{f}{q}\right) q \, dV \geq \Psi\left(\int_{\mathbb{B}^2} \frac{f}{q} q \, dV\right) = M \log M.$$

The last inequality is obtained by Jensen's inequality applied to the convex function $\Psi(u) = u \log u$ and the probability measure qdV. We deduce that

$$\int_{\mathbb{B}^2} f \log f \, dV \geq \int_{\mathbb{B}^2} \left(-\frac{p}{s} - \log(2\pi s)\right) f \, dV + M \log M,$$

which is nothing else than (5.32). □

By using this general Lemma 5.1 and the estimates (4.22) obtained in Section 4, we obtain the following lower bound on the entropy of the positive solution (n_t) of Keller-Segel system (1.1).

Lemma 5.2. *Let $n : [0,T) \times \mathbb{B}^2 \to \mathbb{R}^+$ be a solution of the Keller-Segel system (1.1) with $T \leq +\infty$ such that $n \in \mathcal{C}([0,T), L^1_+(\mathbb{B}^2, (1+p)dV)$. Assume that $M := \int_{\mathbb{B}^2} n_t \, dV = \int_{\mathbb{B}^2} n_0 \, dV$ and $\int_{\mathbb{B}^2} p \, n_0 \, dV$ are finite. Then, for all $0 < t < T$ and $s > 0$ we have the following lower bound on the entropy of the positive solution (n_t) of Keller-Segel system (1.1)*

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV \geq -\frac{1}{s}\left[\left(\int_{\mathbb{B}^2} p \, n_0 \, dV)\right) e^{2t} + M(e^{2t} - 1)\right] + M \log\left(\frac{M}{2\pi s}\right). \tag{5.33}$$

With $s = e^{2t}$, we deduce for all $t > 0$,

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV \geq -\int_{\mathbb{B}^2} p \, n_0 \, dV - M + Me^{-2t} + M \log\left(\frac{M}{2\pi}\right) - 2Mt. \tag{5.34}$$

Proof. Set $\mathcal{I}(t) := \int_{\mathbb{B}^2} p n_t \, dV$, by using the estimates (4.22) we have

$$(\mathcal{I}(t) + M)^2 \leq (I(0) + M)^2 e^{4t} + \frac{1}{2}\chi_0 M^3 (1 - e^{4t}) \leq (I(0) + M)^2 e^{4t}.$$

Thus, we deduce the next upper bound on the p-moment $\mathcal{I}(t)$ for any $t > 0$,

$$\mathcal{I}(t) = \int_{\mathbb{B}^2} p n_t \, dV \leq \left(\int_{\mathbb{B}^2} p n_0 \, dV\right) e^{2t} + M(e^{2t} - 1).$$

By Lemma 5.1 with $f = n_t$, we get

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV \geq -\frac{1}{s}\int_{\mathbb{B}^2} p n_t \, dV + M \log\left(\frac{M}{2\pi s}\right),$$

since $M := \int_{\mathbb{B}^2} n_t \, dV = \int_{\mathbb{B}^2} n_0 \, dV$ for all $t > 0$. From the upper bound on $\mathcal{I}(t)$ just above, we immediately deduce (5.33) and its consequence (5.34) when choosing $s = e^{2t}$. □

5.2. **Upper bound on the Entropy of the solution of Keller-Segel system.** The aim of this section is to obtain upper bound estimates for the entropy of the positive solution (n_t) of Keller-Segel system (1.1). To do so, we prove a suitable logarithmic Hardy-Littlewood-Sobolev inequality on $\mathbb{B}^2$, involving the entropy of the function and the Green kernel on $\mathbb{B}^2$.

5.2.1. *Logarithmic Hardy-Littlewood-Sobolev type inequality on* $\mathbb{B}^2$. We start by recalling a Hardy-Littlewood-Sobolev type inequality proved recently on the hyperbolic space $\mathbb{B}^n$ by G. Lu and Q. Yang, see [13].

Theorem 5.3. [13]. *Let* $n \in \mathbb{N}^*$, $0 < \lambda < n$ *and* $p = p(\lambda) = \frac{2n}{2n-\lambda}$. *For all* $f, g \in L^p(\mathbb{B}^n)$, *we have*

$$\int_{\mathbb{B}^n}\int_{\mathbb{B}^n} \frac{|f(x)g(y)|}{[2\sinh(\rho(x,y)/2)]^\lambda}\, dV_x dV_y \leq C_{n,\lambda}||f||_{L^p(\mathbb{B}^n)}||g||_{L^p(\mathbb{B}^n)}, \tag{5.35}$$

where $C_{n,\lambda}$ *is given by*

$$C_{n,\lambda} = \pi^{\lambda/2}\frac{\Gamma(n/2-\lambda/2)}{\Gamma(n-\lambda/2)}\left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{-1+\lambda/n} \tag{5.36}$$

and $\rho(x,y)$ *is the hyperbolic metric.*

By the same derivation argument at $\lambda = 0$ used by Carlen and Loss [5], we deduce the following version of logarithmic Hardy-Littlewood-Sobolev inequality on $\mathbb{B}^n$.

Theorem 5.4. *Let* $n \in \mathbb{N}^*$. *For all* $f \in L^1(\mathbb{B}^n)$ *with* $f \geq 0$, *we have*

$$\int_{\mathbb{B}^n} f(x)\log f(x)\, dV_x + \frac{n}{M}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n} f(x)f(y)\log\left[2\sinh(\rho(x,y)/2)\right]\, dV_x dV_y \geq -C_n(M), \tag{5.37}$$

with $M := \int_{\mathbb{B}^n} f(x)\, dV_x$ *and*

$$C_n(M) = -M\log M + nM\frac{d}{d\lambda}C_{n,\lambda}|_{\lambda=0}$$
$$= -M\log M + nM\left[\frac{1}{2}\log\pi + \frac{1}{n}\log\frac{\Gamma(n/2)}{\Gamma(n)} + \frac{1}{2}(\Psi(n) - \Psi(n/2))\right],$$

where Ψ *is the logarithmic derivative of the Gamma function. In particular,* $C_2(M) := M\log(e\pi M)$.

The constants $C_{n,\lambda}$ and $C_n(M)$ are sharp and exactly the same as in the Euclidean space $\mathbb{R}^n$, see [13] and [5].

Now recall that, for non-negative functions on $\mathbb{R}^2$, the classical logarithmic Hardy-Littlewood-Sobolev inequality can be expressed as

$$\int_{\mathbb{R}^2} f(x)\log f(x)\, dx - \frac{4\pi}{M}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)f(y)G_e(x,y)\, dxdy \geq -M\log(e\pi M), \tag{5.38}$$

with $M = \int_{\mathbb{R}^2} f(x)\, dx$ and the Euclidean Green kernel given by

$$G_e(x,y) = -\frac{1}{2\pi}\log|x-y|.$$

In particular, note that the classical inequality (5.38) involves naturally the Green kernel, which is the important link with the Keller-Segel system on $\mathbb{R}^2$. Here, because the expression of the Green kernel $G_{\mathbb{H}}$ on $\mathbb{B}^2$ is given by

$$G_{\mathbb{H}}(x,y) = -\frac{1}{2\pi}\log\left[\tanh(\rho(x,y)/2)\right],$$

the inequality (5.37) is not enough to be connected with the Keller-Segel system (1.1). So, in the following Theorem, we state a suitable logarithmic Hardy-Littlewood-Sobolev inequality on $\mathbb{B}^2$, which allow us to obtain upper bound estimates for the entropy of the solution of the Keller-Segel system (1.1).

Theorem 5.5. *For all $f \in L^1(\mathbb{B}^2)$ with $f \geq 0$, we have*

$$\int_{\mathbb{B}^2} f(x)\log f(x)\,dV_x - \frac{4\pi}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} f(x)f(y)G_{\mathbb{H}}(x,y)\,dV_xdV_y \tag{5.39}$$

$$\geq -K_2(M) - 2\int_{\mathbb{B}^2}\rho(x)f(x)\,dV_x,$$

with $M := \int_{\mathbb{B}^2} f(x)\,dV_x$ and $K_2(M) := M\log(4e\pi M)$.

Proof. Let $d = \rho(x,y)$ to avoid confusion with $\rho(x) := \rho(x,0)$. From Theorem 5.4 with $n = 2$, we have

$$\int_{\mathbb{B}^2} f(x)\log f(x)\,dV_x + 2M\log 2 + \frac{2}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} f(x)f(y)\log\left[\sinh(d/2)\right]\,dV_xdV_y \geq -C_2(M)$$

because

$$\frac{2}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} f(x)f(y)\log 2\,dV_xdV_y = 2M\log 2.$$

This can be rewritten as

$$\int_{\mathbb{B}^2} f(x)\log f(x)\,dV_x + \frac{2}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} f(x)f(y)\log\left[\tanh(d/2)\right]\,dV_xdV_y \tag{5.40}$$

$$+\frac{2}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} f(x)f(y)\log\left[\cosh(d/2)\right]\,dV_xdV_y \geq -C_n(M) - 2M\log 2.$$

Now by using the triangle inequality for the metric ρ, we have the following

$$\log\left[\cosh(\rho(x,y)/2\right] \leq \rho(x,y)/2 \leq \rho(x,0)/2 + \rho(y,0)/2 = \frac{1}{2}\left[\rho(x)+\rho(y)\right],$$

by multiplying the last inequality by $\frac{2}{M}f(x)f(y) \geq 0$, integrating over $\mathbb{B}^2\times\mathbb{B}^2$ and using Fubini's Theorem, we obtain

$$\frac{2}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} f(x)f(y)\log\left[\cosh(\rho(x,y)/2\right]\,dV_xdV_y \tag{5.41}$$

$$\leq \frac{1}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} f(x)f(y)\left[\rho(x)+\rho(y)\right]\,dV_xdV_y$$

$$= \frac{2}{M}\left(\int_{\mathbb{B}^2} f(y)\,dV_y\right)\left(\int_{\mathbb{B}^2}\rho(x)f(x)\,dV_x\right) = 2\int_{\mathbb{B}^2}\rho(x)f(x)\,dV_x.$$

Then the inequality (5.39) follows from inequalities (5.40) and (5.41) above. This completes the proof. □

5.2.2. *Entropy upper bound.* The next results are used for the study of the entropy upper bound of the solution of the Keller-Segel system (1.1). Here, we consider the Lyapunov functional defined by

$$F[n_t] = \int_{\mathbb{B}^2} n_t\,\log n_t\,dV - \frac{\chi}{2}\int_{\mathbb{B}^2} n_t\,c\,dV$$

similar to the Euclidean one given in [3].

From now we shall perform a priori estimates in the setting of smooth enough solutions, the one constructed in Theorem 3.7. In particular, for such a solution we can define T^* as the maximal existence time which is characterized by if $T^* < +\infty$, then $\sup_{t\in[0,T^*[}\|n(t)\|_{L^q(\mathbb{B}^2)} = +\infty$.

Proposition 5.6. *Let n_t be a solution of the Keller-Segel system (1.1) as above. Assume that $F[n_0]$ is finite, then we have*

$$\frac{\partial}{\partial t}F[n_t] = -\int_{\mathbb{B}^2} n_t|\nabla_{\mathbb{H}}\log n_t - \chi\nabla_{\mathbb{H}}c|^2\, dV \leq 0,$$

for all $t \in (0, T^*)$. In particular, $F[n_t] \leq F[n_0]$ for all $0 < t < T^*$.

Proof. The proof is similar to the Euclidean one and relies on the existence of an explicit expression of the Green kernel on $\mathbb{B}^2$. □

Next, by using the inequality (4.22) we obtain an upper bound on the p-moment of the solution of the Keller-Segel system (1.1).

Proposition 5.7. *Let n_t be a solution of the Keller-Segel system (1.1) as above. Then, for all $0 < t < T^*$, we have the following upper bound of the p-moment*

$$\int_{\mathbb{B}^2} p\, n_t\, dV \leq C_+(p, n_0)e^{2t} + \lambda^*(M), \tag{5.42}$$

with

$$C_+(p, n_0) = \left(\int_{\mathbb{B}^2} p\, n_0\, dV - \lambda^*(M)\right)_+^{1/2} \left(\int_{\mathbb{B}^2} p\, n_0\, dV + M + M\sqrt{\frac{\chi M}{8\pi}}\right)^{1/2}, \tag{5.43}$$

where $\lambda^*(M)$ is given by (4.24) and $M = \int_{\mathbb{B}^2} n_0\, dV$.

Proof. We recall here (4.22)

$$\left(\int_{\mathbb{B}^2} p\, n_t\, dV + M\right)^2 \leq \left(\left[\int_{\mathbb{B}^2} p\, n_0\, dV + M\right]^2 - \frac{\chi}{8\pi}M^3\right)e^{4t} + \frac{\chi}{8\pi}M^3,$$

valid for all $t \in (0, T^*)$ with $T^* \leq +\infty$. By taking the square root on both sides and using the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, a, b \geq 0$, we obtain

$$\int_{\mathbb{B}^2} p\, n_t\, dV \leq \left(\left[\int_{\mathbb{B}^2} p\, n_0\, dV + M\right]^2 - \left(M\sqrt{\frac{\chi M}{8\pi}}\right)^2\right)_+^{1/2} e^{2t} + \sqrt{\frac{\chi}{8\pi}M^3} - M.$$

Hence, we have

$$\int_{\mathbb{B}^2} p\, n_t\, dV \leq \left(\int_{\mathbb{B}^2} p\, n_0\, dV - \lambda^*(M)\right)_+^{1/2} \left(\int_{\mathbb{B}^2} p\, n_0\, dV + M + M\sqrt{\frac{\chi M}{8\pi}}\right)^{1/2} e^{2t} + \lambda^*(M).$$

So, we obtain

$$\int_{\mathbb{B}^2} p\, n_t\, dV \leq C_+(p, n_0)e^{2t} + \lambda^*(M),$$

where $C_+(p, n_0)$ and $\lambda^*(M)$ are given respectively by (5.43) and (4.24). This concludes the proof of the proposition. □

Remark 5.8. *Note that if $\int_{\mathbb{B}^2} pn_0\, dV - \lambda^*(M) \leq 0$, it implies that $\lambda^*(M) \geq 0$, hence $\chi M \geq 8\pi$. Under the condition of blow-up or critical case we get $C_+(p, n_0) = 0$ and so, we have the following uniform bounds in time of the p-moment*

$$\int_{\mathbb{B}^2} p\, n_t\, dV \leq \lambda^*(M),$$

for any $t \in (0, T^*)$.

Now, we shall control the ρ-moment of the solution n_t of the Keller-Segel system (1.1) $\int_{\mathbb{B}^2} \rho\, n_t\ dV$, which appears in the lower bound of our logarithmic Hardy-Littlewood-Sobolev inequality on $\mathbb{B}^2$ (5.39) with $f = n_t$.

Proposition 5.9. *For any solution n_t of the Keller-Segel system (1.1) with $T^* \leq +\infty$, we have*

$$0 \leq \int_{\mathbb{B}^2} \rho\, n_t\ dV \leq K_+ + 2Mt, \quad 0 < t < T^*,$$

with $M = \int_{\mathbb{B}^2} n_0\ dV$, $\rho = \rho(x, 0)$ is the distance from $x \in \mathbb{B}^2$ to the center 0 of $\mathbb{B}^2$ and

$$K_+ = 2M \log\left(\sup\left(2\frac{C_+}{2M}, 2\sqrt{\frac{\lambda_+^*(M)}{2M}} + 1\right)\right),$$

with C_+ given by (5.43) of Proposition 5.7 .

Proof. We apply Jensen's inequality with $\Phi(u) = \sinh(u)$ as convex function and $d\mu = \frac{n_t}{M}\ dV$ as probability measure. For any $0 < t < T^* \leq +\infty$, we get

$$\int_{\mathbb{B}^2} \rho\, n_t\ dV \leq 2M \sinh^{-1}\left(\frac{1}{\sqrt{2M}}\left(\int_{\mathbb{B}^2} p\, n_t\ dV\right)^{1/2}\right).$$

By using the inequality $\sinh^{-1}(u) \leq \log(2u + 1), u \geq 0$ and Proposition 5.7, we obtain

$$\int_{\mathbb{B}^2} \rho\, n_t\ dV \leq 2M \sinh^{-1}\left(\sqrt{\frac{C_+}{2M}}e^t + \sqrt{\frac{\lambda_+^*(M)}{2M}}\right)$$

$$\leq 2M \log\left(2\sqrt{\frac{C_+}{2M}}e^t + 2\sqrt{\frac{\lambda_+^*(M)}{2M}} + 1\right) \leq K_+ + 2Mt,$$

with $K_+ = 2M \log\left(\sup(2\sqrt{\frac{C_+}{2M}}, 2\sqrt{\frac{\lambda_+^*(M)}{2M}} + 1)\right)$. The proof is completed. □

The following Lemma is nothing else than the Lemma 8 of [3] written in a general measure theory context.

Lemma 5.10. (1) *Let (E, ν) be a measure space. Assume that $u, q : E \to [0, +\infty[$ satisfy the following conditions $M := \int u\ d\nu$, $\int u \log u\ d\nu <$ and $\int(-\log q)u\ d\nu < \infty$ are finite. The measure qdV is a density of probability i.e. $\int q\ d\nu = 1$. Then for $v = u1_{0\leq u\leq 1}$, we have*

$$\int u|\log u|\ d\nu \leq \int u \log u\ d\nu + \frac{2}{e} + 2\int(-\log q)v\ d\nu.$$

(2) *Let $E = \mathbb{B}^2$ and $\nu = dV$ the hyperbolic measure. Let p the weight given by (4.20). Then, we have for any $s > 0$*

$$\int_{\mathbb{B}^2} u|\log u|\ d\nu \leq \int_{\mathbb{B}^2} u \log u\ d\nu + \frac{2}{e} + 2M \log(2\pi s) + \frac{1}{s}\int_{\mathbb{B}^2} pu\ d\nu. \tag{5.44}$$

Finally, we prove the upper bound of the entropy of a positive solution. Note that, we are able to prove that $n_t \log n_t \in L^1$ is locally uniformly bounded in time under the condition $\chi M < 8\pi$.

Proposition 5.11. *Let n_t be a positive solution of Keller-Segel system (1.1) and T^* the maximal existence time of the solution. Assume that $\int_{\mathbb{B}^2} p\, n_0\, dV$ and $F[n_0]$ are finite.*

(1) *For any* $0 < t \le T < T^*$ *and* $\chi M < 8\pi$, *we have the following estimate*

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV \le \left(1 - \frac{\chi M}{8\pi}\right)^{-1} C(n_0, T), \tag{5.45}$$

with

$$C(n_0, T) := F[n_0] + \frac{\chi M}{8\pi}\left(K_2(M) + 2K_+ + 4MT\right),$$

where $K_2(M)$ *and* K_+ *are defined respectively in Proposition 5.9 and Theorem 5.5.*

(2) *Moreover, for all* $0 < t \le T < T^*$, $s > 0$ *and* $\chi M < 8\pi$, *we have*

$$\int_{\mathbb{B}^2} n_t |\log n_t| \, dV \le \left(1 - \frac{\chi M}{8\pi}\right)^{-1} C(n_0, T) + \frac{2}{e} + 2M \log(2\pi s) + \frac{1}{s}\int_{\mathbb{B}^2} p\, n_t \, dV \tag{5.46}$$

and its consequence

$$\int_{\mathbb{B}^2} n_t |\log n_t| \, dV \le \left(1 - \frac{\chi M}{8\pi}\right)^{-1} C(n_0, T) + \frac{2}{e} + 2M \log(2\pi s) + s^{-1} C_+ e^{2T} + s^{-1}\lambda^*(M), \tag{5.47}$$

with $C_+ = C_+(p, n_0)$ *given by* (5.43).

Proof. 1) The proof follows the same lines as in Lemma 7 (upper bound) of [3]. We provide the details.

Step 1: Use of the decay of the functional $F[n_t]$.
From Proposition 5.6, we have $F[n_t] \le F[n_0]$ for all $0 < t \le T < T^*$ i.e.

$$F[n_t] = \int_{\mathbb{B}^2} n_t \log n_t \, dV - \frac{\chi}{2}\int_{\mathbb{B}^2} n_t c_t \, dV \le F[n_0]$$

which can be written as

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV - \frac{\chi}{2} < n_t, (-\Delta_{\mathbb{H}})^{-1} n_t > \le F[n_0],$$

with $< f, g >= \int_{\mathbb{B}^2} f g \, dV$. Then we deduce the following inequality

$$\left(1 - \frac{\chi M}{8\pi}\right)\int_{\mathbb{B}^2} n_t \log n_t \, dV + \frac{\chi M}{8\pi}\left(\int_{\mathbb{B}^2} n_t \log n_t \, dV - \frac{4\pi}{M} < n_t, (-\Delta_{\mathbb{H}})^{-1} n_t >\right) \le F[n_0]. \tag{5.48}$$

Step 2: Use Logarithmic of Hardy-Littlewood-Sobolev inequality on $\mathbb{B}^2$. From logarithmic Hardy-Littlewood-Sobolev inequality on the hyperbolic space proved in Theorem 5.5 with $f = n_t$, we have

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV - \frac{4\pi}{M}\int_{\mathbb{B}^2}\int_{\mathbb{B}^2} n_t(x) n_t(y) G_{\mathbb{H}}(x, y) \, dV_x dV_y \ge -K_2(M) - 2\int_{\mathbb{B}^2} \rho\, n_t \, dV.$$

This can be written as

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV - \frac{4\pi}{M} < n_t, (-\Delta_{\mathbb{H}})^{-1} n_t > \ge -K_2(M) - 2\int_{\mathbb{B}^2} \rho\, n_t \, dV, \tag{5.49}$$

Combining (5.48) and (5.49), we obtain

$$\left(1 - \frac{\chi M}{8\pi}\right)\int_{\mathbb{B}^2} n_t \log n_t \, dV \le F[n_0] + \frac{\chi M}{8\pi}\left(K_2(M) + 2\int_{\mathbb{B}^2} \rho\, n_t \, dV\right).$$

Step 3. Upper bound on the ρ-moment. By Proposition 5.9, we have

$$0 \le \int_{\mathbb{B}^2} \rho\, n_t \, dV \le K_+ + 2Mt, \quad 0 < t \le T < T^*.$$

This implies that

$$\left(1 - \frac{\chi M}{8\pi}\right) \int_{\mathbb{B}^2} n_t \log n_t \, dV \le F[n_0] + \frac{\chi M}{8\pi} \left(K_2(M) + 2K_+ + 4Mt\right) \tag{5.50}$$

from which we conclude that

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV \le \left(1 - \frac{\chi M}{8\pi}\right)^{-1} \left[F[n_0] + \frac{\chi M}{8\pi} \left(K_2(M) + 2K_+ + 4MT\right)\right]$$

for $0 < t \le T < T^*$ and $\chi M < 8\pi$.

2) To prove (5.46), we apply 2) of Lemma 5.10 with $u = n_t$ and the first part of this proposition. To deduce (5.47), we use the upper bound of p-moment (5.42) of Proposition 5.7. Now, the proof is completed. □

6. L^q-BOUNDS ON SOLUTIONS

The main argument of the proof of the L^q-bounds of the solution (n_t) under the conditions $n_0 \in L^1_+$ and $n_0 \log n_0 \in L^1$ is the control of $M_t(K) := \int_{\mathbb{B}^2} (n_t - K)_+ \, dV$ for K large enough. More precisely, we have the next result.

Proposition 6.1. *Under the assumptions of Proposition 5.11, for any positive solution n_t of Keller-Segel system (1.1), we have for all $T < T^*$ the following*

$$\sup_{0 \le t \le T} M_t(K) \le \frac{1}{\log K} C(T) \tag{6.51}$$

for a positive continuous non decreasing function $C(T)$ on $[0, +\infty)$

Proof. Let $T < T^*$. We have, for any $K > 1$ and $0 \le t \le T$, the following

$$M_t(K) = \int_{\mathbb{B}^2} (n_t - K)_+ \, dV \le \int_{n_t \ge K} n_t \, dV \le \frac{1}{\log K} \int_{n_t \ge K} n_t \log n_t \, dV.$$

So, by applying (5.47) of Proposition 5.11, we have

$$M_t(K) \le \frac{1}{\log K} \int_{\mathbb{B}^2} n_t |\log n_t| \, dV \le \frac{1}{\log K} C(T).$$

The function $C(T)$ is the bound of the inequality (5.47) (for any fixed $s > 0$). □

Now, we adapt Proposition 3.3 of [3] to control the L^q-bound of the solution (n_t).

Theorem 6.2. *Assume that $n_0 \in L^1_+(\mathbb{B}^2, (1+p)dV)$ and $n_0 \log n_0 \in L^1(\mathbb{B}^2, dV)$ with $\chi M < 8\pi$ and the weight p given by (4.20). Assume in addition that $n_0 \in L^q(\mathbb{B}^2)$ for some $1 < q \le 2$. Then the solution n_t of the Keller-Segel system (1.1) satisfies the following estimate*

$$\sup_{0 \le t \le T} ||n_t||_{L^q(\mathbb{B}^2)} \le N_q(T) < +\infty,$$

where $T < T^*$ and $N_q(T)$ is a continuous function on $[0, +\infty)$.

Proof. We mention only the main steps. The first idea is to prove a differential inequality of the form

$$\phi'(t) \leq c_1\phi(t) + c_2 \tag{6.52}$$

for $\phi(t) := ||(n_t - K)_+||^q_{L^q(\mathbb{B}^2)}$ for K large enough ($c_1 > 0$) and apply Gronwall lemma from which we deduce that

$$\phi(t) \leq \left(\phi(0) + \frac{c_2}{c_1}\right) e^{c_1 t} - \frac{c_2}{c_1} \leq \left(\phi(0) + \frac{c_2}{c_1}\right) e^{c_1 T} - \frac{c_2}{c_1}, \quad 0 \leq t \leq T < T^*.$$

Step 1. Reduction to $\phi(t)$ estimate. From theoretical measure theory considerations, we easily obtain

$$||n_t||^q_{L^q(\mathbb{B}^2)} = \int_{n_t<\lambda K} n_t^q + \int_{n_t\geq\lambda K} n_t^q \leq (\lambda K)^{q-1} M + \int_{n_t\geq\lambda K} n_t^q$$

for all $K, \lambda > 1$ and $0 \leq t \leq T < T^*$. So, by using $n_t^q \leq \left(\frac{\lambda}{\lambda-1}\right)^{q-1} (n_t - K)^q$ when $n_t \geq \lambda K$, we get

$$||n_t||^q_{L^q(\mathbb{B}^2)} \leq (\lambda K)^{q-1} M + \left(\frac{\lambda}{\lambda-1}\right)^{q-1} \int_{\mathbb{B}^2} (n_t - K)^q_+ \, dV.$$

Step 2. To bound $||n_t||^q_{L^q(\mathbb{B}^2)}$, we only have to deal with the second term by proving (6.52). We set $\phi(t) := ||(n_t - K)_+||^q_{L^q(\mathbb{B}^2)}$. We use multipliers method to estimate $\phi'(t)$. After integrations by parts, we get

$$\begin{aligned}\phi'(t) = \frac{-4(q-1)}{q} \int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} \left((n_t - K)_+^{q/2}\right)|^2 \, dV + (2q-1)\chi K \int_{\mathbb{B}^2} (n_t - K)^q_+ \, dV \\ + q\chi K^2 \int_{\mathbb{B}^2} (n_t - K)^{q-1}_+ \, dV + (q-1)\chi \int_{\mathbb{B}^2} (n_t - K)^{q+1}_+ \, dV. \end{aligned} \tag{6.53}$$

As shown in [3], we have

$$\int_{\mathbb{B}^2} (n_t - K)^{q-1}_+ \, dV \leq \frac{M}{K} + \int_{\mathbb{B}^2} (n_t - K)^q_+ \, dV.$$

Thus, we get

$$\begin{aligned}\phi'(t) \leq \left[(2q-1)\chi K + q\chi K^2\right] \phi(t) + q\chi K M \\ + \left[\frac{-4(q-1)}{q} \int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} \left((n_t - K)_+^{q/2}\right)|^2 \, dV + (q-1)\chi \int_{\mathbb{B}^2} (n_t - K)^{q+1}_+ \, dV\right].\end{aligned}$$

Step 3. Use of Gagliardo-Nirenberg inequality. In order to show (6.52), it remains to show that

$$R(n_t) := \frac{-4(q-1)}{q} \int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} \left((n_t - K)_+^{q/2}\right)|^2 \, dV + (q-1)\chi \int_{\mathbb{B}^2} (n_t - K)^{q+1}_+ \, dV \leq 0$$

for K large enough and uniformly for any $0 \leq t \leq T < T^*$. For that purpose, we use Gagliardo-Nirenberg inequality valid on the hyperbolic space (see [14], [23] , [9]), for every $1 < q \leq 2$ we have

$$\int_{\mathbb{B}^2} |v|^{2(1+1/q)} \, dV \leq \kappa_p \left(\int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} v|^2 \, dV\right) \left(\int_{\mathbb{B}^2} |v|^{2/q} \, dV\right).$$

This inequality applied to $v = (n_t - K)_+^{q/2}$ leads to

$$\int_{\mathbb{B}^2} (n_t - K)^{q+1}_+ \, dV \leq \kappa_q \left(\int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} \left((n_t - K)_+^{q/2}\right)|^2 \, dV\right) M_t(K),$$

with $M_t(K) = \int_{\mathbb{B}^2} (n_t - K)_+ \, dV$. Hence,

$$R(n_t) \leq \left[\frac{-4(q-1)}{q} + (q-1)\chi\kappa_q M_t(K) \right] \left(\int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} \left((n_t - K)_+^{q/2} \right)|^2 \, dV \right),$$

and by using the inequality (6.51) of Proposition 6.1, we have

$$R(n_t) \leq \left[\frac{-4(q-1)}{q} + \frac{1}{\log K}(q-1)\chi\kappa_q C(T) \right] \left(\int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} \left((n_t - K)_+^{q/2} \right)|^2 \, dV \right),$$

for all $0 \leq t \leq T < T^*$. Thus there exists $K = K(T) > 1$ such that

$$\frac{-4(q-1)}{q} + \frac{1}{\log K}(q-1)\chi\kappa_q C(T) \leq 0.$$

This implies the inequality (6.52) with

$$c_1 = c_1(T) = (2q-1)\chi K + q\chi K^2, \qquad c_2 = c_2(T) = q\chi K M.$$

The proof is now completed. □

Remark. Note that that the explicit value of the constant $\kappa_q > 0$ in Gagliardo-Nirenberg inequality has no importance in our estimates as in [3].

7. Global well-posedness

Next, we prove global existence in time of the solution under the subcritical condition $\chi M < 8\pi$. Let be p the weight defined as in (4.20).

Theorem 7.1. *For every $n_0 \in L^1_+(\mathbb{B}^2)$, with $I_0 = \int_{\mathbb{B}^2} p\, n_0 \, dV < \infty$ and $\chi M < 8\pi$ we have global well-posedness on $X_{T,q} \cap \mathcal{C}(\mathbb{R}^+, L^1_+(\mathbb{B}^2, (1+p)dV)$ for every $T > 0$ of the Keller-Segel system (1.1).*

Proof. By Theorem 3.4 if $n_0 \in L^1(\mathbb{B}^2)$ there exists $T > 0$ such that there is a unique solution of the Keller-Segel system (1.1) $n_t \in X_{T,q} \cap \mathcal{C}([0,T], L^1(\mathbb{B}^2))$, where $X_{T,q} = \{n_t \,|\, \sup_{t\in[0,T]} t^{(1-\frac{1}{q})} \|n(t)\|_{L^q(\mathbb{B}^2)} < +\infty\}$, with $\frac{4}{3} < q < 2$. Then there exists $T_1 \in]0,T]$ such that $n(T_1) \in L^q(\mathbb{B}^2) \cap L^1_+(\mathbb{B}^2)$ for $\frac{4}{3} < q < 2$. We can take $n(T_1)$ as initial data and we use Theorem 3.7 to continue the solution. From Theorem 3.7, we obtain a maximal solution on $[T_1, T^*[$ such that $n_t \in \mathcal{C}([T_1, T^*[, L^q(\mathbb{B}^2))$. Moreover if $T^* < +\infty$, we must have $\sup_{t\in[T_1,T^*[} \|n(t)\|_{L^q(\mathbb{B}^2)} = +\infty$. Let us now assume that $T^* < +\infty$, it suffices to prove that $\sup_{t\in[T_1,T^*[} \|n(t)\|_{L^q(\mathbb{B}^2)} < +\infty$ to get a contradiction and thus obtain a global solution. Since $n(T_1) \in L^q(\mathbb{B}^2) \cap L^1_+(\mathbb{B}^2, (1+p)dV)$ for $\frac{4}{3} < q < 2$, $\int_{\mathbb{B}^2} n(T_1) \log n(T_1) dV$ is well-defined. Consequently, we can use Theorem 6.2 to obtain that

$$\sup_{0\leq t<T^*} ||n_t||_{L^q(\mathbb{B}^2)} \leq N_q(T^*) < +\infty.$$

This ends the proof. □

8. A priori estimates of the entropy and L^q-norms of solutions

In this section, we shall derive some interesting a priori estimates on the entropy and L^q-norms of the solutions of the Keller-Segel system (1.1). To do so, we use the classical multiplier's method and some functional inequalities, as for example the Poincaré-Sobolev inequality by Mugelli-Talenti [14] to have (8.55) and also Log-Sobolev inequality by Beckner [1] to prove (8.56). Let $I(n) = \int_{\mathbb{B}^2} \frac{|\nabla_{\mathbb{H}} n|^2}{n} \, dV$ be the Fisher information and the mass

$M = \int_{\mathbb{B}^2} n_t \, dV = \int_{\mathbb{B}^2} n_0 \, dV$, where $(n_t)_{0<t<T^*}$ is the non-negative solution of the Keller-Segel system (1.1).

Proposition 8.1. *Let $(n_t)_{0<t<T^*}$ be a non-negative solution of the Keller-Segel system (1.1), where $T^* \leq +\infty$ is the maximal existence time of the solution. Then*

(1) *For all $t \in (0, T^*)$, we have*

$$\frac{\partial}{\partial t}\mathrm{Ent}(n_t) \leq \left[-1 + \frac{\chi M}{4\pi}\right] I(n_t) - \frac{\chi M^2}{4\pi}. \tag{8.54}$$

(2) *If $\chi M \leq 4\pi$ then $t \to \mathrm{Ent}(n_t)$ is non-increasing on $(0, T^*)$ and we have*

$$\mathrm{Ent}(n_t) \leq -\frac{\chi M^2}{4\pi} t + \mathrm{Ent}(n_0). \tag{8.55}$$

for all $t \in (0, T^*)$. In particular if $T^* = +\infty$, $\lim_{t\to+\infty} \mathrm{Ent}(n_t) = -\infty$.

(3) *Under the condition $\chi M < 4\pi$, we have*

$$\mathrm{Ent}(n_t) \leq -\frac{\chi M^2}{4\pi} t \tag{8.56}$$
$$-M \log\left[\exp\left(-\frac{1}{M}\mathrm{Ent}(n_0)\right) + \frac{4\pi e}{M}\left(\frac{4\pi}{\chi M} - 1\right)\left(1 - e^{-\frac{\chi M t}{4\pi}}\right)\right].$$

for all $t \in (0, T^*)$.

Remark. It can be shown that the inequality (8.56) is stronger than (8.55). But the asymptotic behavior of the upper bound is the same i.e. $-\frac{\chi M^2}{4\pi} t$ as $t \to +\infty$ (if $T^* = +\infty$).

Proof. 1). By multiplying the first equation in the Keller-Segel system (1.1) by $\log n_t$ and integrating by parts, we get

$$\frac{\partial}{\partial t}\mathrm{Ent}(n_t) = -I(n_t) + \chi \int_{\mathbb{B}^2} n_t^2 \, dV.$$

By using the following Poincaré-Sobolev inequality of Mugelli-Talenti, with $u = n_t$,

$$\left(\int_{\mathbb{B}^2} |u| \, dV\right)^2 + 4\pi \int_{\mathbb{B}^2} |u|^2 \, dV \leq \left(\int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} u| \, dV\right)^2, \tag{8.57}$$

and by the Cauchy-Schwarz inequality, we deduce that

$$M^2 + 4\pi \int_{\mathbb{B}^2} n_t^2 \, dV \leq \left(\int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} n_t| \, dV\right)^2 \leq \left(\int_{\mathbb{B}^2} n_t \, dV\right) I(n_t) = M I(n_t).$$

Then, we have

$$\frac{\partial}{\partial t}\mathrm{Ent}(n_t) \leq -I(n_t) + \frac{\chi}{4\pi}\left[4\pi \int_{\mathbb{B}^2} n_t^2 \, dV\right] \leq -I(n_t) + \frac{\chi}{4\pi}\left(M I(n_t) - M^2\right).$$

We conclude that (8.54) holds true.

2) Since $I(n_t) \geq 0$ and $\chi M \leq 4\pi$, from (8.54) we obtain $\frac{\partial}{\partial t}\mathrm{Ent}(n_t) \leq -\frac{\chi M^2}{4\pi}$ Then the conclusion is obtained by integration.

3) W. Beckner proved in [1] that the following Log-Sobolev inequality holds on the hyperbolic space with $||u||_{L^2(\mathbb{B}^2)} = 1$,

$$\int_{\mathbb{B}^2} |u|^2 \log |u| \, dV \leq \frac{1}{2} \log\left(\frac{1}{\pi e} \int_{\mathbb{B}^2} |\nabla_{\mathbb{H}} u|^2 \, dV\right).$$

We set $u = \sqrt{n_t/M}$ and get

$$\int_{\mathbb{B}^2} n_t \log n_t \, dV \leq M \log\left(\frac{1}{4\pi e} I(n_t)\right),$$

for all $t \in (0, T^*)$. This implies that

$$-I(n_t) \leq -4e\pi \exp\left(\frac{1}{M}\mathrm{Ent}(n_t)\right).$$

By using 1), we deduce that

$$\frac{\partial}{\partial t}\left(\mathrm{Ent}(n_t) + \frac{\chi M^2}{4\pi} t\right) \leq \left[-1 + \frac{\chi M}{4\pi}\right] I(n_t) \leq -4e\pi \left[1 - \frac{\chi M}{4\pi}\right] \exp\left(\frac{1}{M}\mathrm{Ent}(n_t)\right).$$

We set $\Psi(t) = \mathrm{Ent}(n_t) + \frac{\chi M^2}{4\pi} t$. Then Ψ satisfies

$$\Psi'(t) \exp\left(-\frac{1}{M}\Psi(t)\right) \leq -4e\pi \left[1 - \frac{\chi M}{4\pi}\right] \exp\left(-\frac{\chi M}{4\pi} t\right).$$

This differential inequality is easily integrated and leads to (8.56). This completes the proof of the proposition. □

Then, in the following proposition, we deduce some L^q-estimates of the solutions of the Keller-Segel system (1.1).

Proposition 8.2. *Let $(n_t)_{0<t<T^*}$ be a non-negative solution of the Keller-Segel system (1.1) where $T^* \leq +\infty$ is the maximal existence time of the solution. Then*

(1) *For all $t \in (0, T^*)$ and all $1 < q < \infty$, we have*

$$\frac{1}{q(q-1)} \frac{\partial}{\partial t} ||n_t||^q_{L^q(\mathbb{B}^2)} \leq \left[-1 + \frac{\chi M}{8\pi} \frac{(q+1)^2}{2q}\right] \int n_t^{q-2} |\nabla_{\mathbb{H}} n_t|^2 \, dV. \tag{8.58}$$

(2) *If $\chi M \leq 4\pi h(q)$ with $h(q) = \frac{4q}{(q+1)^2}$, then*

$$\frac{\partial}{\partial t} ||n_t||^q_{L^q(\mathbb{B}^2)} \leq 0.$$

So, the map $t \to ||n_t||^q_q$ is non-increasing on $(0, T^*)$ and we have

$$||n_t||_{L^q(\mathbb{B}^2)} \leq ||n_0||_{L^q(\mathbb{B}^2)} \tag{8.59}$$

for all $t \in (0, T^*)$.

Proof. 1) By the classical multiplier method, we have that

$$\frac{1}{q(q-1)} \frac{\partial}{\partial t} ||n_t||^q_{L^q(\mathbb{B}^2)} = -\int_{\mathbb{B}^2} n_t^{q-2} |\nabla_{\mathbb{H}} n_t|^2 \, dV + \frac{\chi}{q} \int_{\mathbb{B}^2} n_t^{q+1} \, dV.$$

By applying the inequality (8.57) to $u = n_t^{(q+1)/2}$, we obtain that

$$4\pi \left(\int_{\mathbb{B}^2} n_t^{q+1} \, dV\right) \leq \frac{(q+1)^2}{4} M \left(\int_{\mathbb{B}^2} n_t^{q-2} |\nabla_{\mathbb{H}} n_t|^2 \, dV\right),$$

which implies (8.58).
2) This is deduced from (8.58) of 1). The proof is completed. □

Remark 8.3. *By using the second statement above, if $\chi M \leq 4\pi h(q) < 4\pi$ then in this case we have a L^q-norm control of the solution of the Keller-Segel system (1.1), which allow us to obtain a global well-posedness on $X_{T,q} \cap \mathcal{C}(\mathbb{R}^+, L^1_+(\mathbb{B}^2))$ without weighted additional assumption on initial data in $L^1(\mathbb{B}^2)$.*

References

[1] Beckner, William. *Geometric asymptotics and the logarithmic Sobolev inequality.* Forum Math. 11 (1999) no. 1, 105-137.

[2] Blanchet, Adrien; Carrillo, José Antonio; Masmoudi, Nader. *Infinite time aggregation for the critical Patlak-Keller-Segel model in* $\mathbb{R}^2$. Communications on Pure and Applied Mathematics, Volume 61, Issue 10.

[3] Blanchet, Adrien; Dolbeault, Jean; Perthame, Benoît. *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solution.* Electron. J. Differential Equations (2006) no. 44.

[4] Calvez, Vincent; Corrias, Lucilla. *The parabolic-parabolic Keller-Segel model in* $\mathbb{R}^2$. Commun. Math. Sci. 6 (2008) no. 2, 417-447.

[5] Carlen, Eric Anders; Loss, Michael. *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on* S^n. Geom. Funct. Anal. 2 (1992) no. 1, 90-104.

[6] Dolbeault, Jean; Perthame, Benoît. *Optimal critical mass in the two dimensional Keller-Segel model in* $\mathbb{R}^2$. C.R.Acad.Sci. (2004).

[7] Gallot, Sylvestre; Hulin, Dominique; Lafontaine, Jacques. *Riemannian geometry*, third edition. Universitext. Springer-Verlag, Berlin (2004).

[8] Grigor'yan, Alexander. *Heat kernel and analysis on manifolds.* AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA (2009).

[9] Hebey, Emmanuel. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lectures in Mathematics, New York (2000) AMS.

[10] Jäger, Willi; Luckhaus, Stephan. *On explosions of solutions to a system of partial differential equations modelling chemotaxis* , Trans. Amer. Math. Soc., 329 (1992) 819-824.

[11] Jost, Jürgen. *Riemannian geometry and geometric analysis*, Universitext. Springer-Verlag, Berlin, fifth edition (2008).

[12] Keller, Evelyn Fox; Segel, Lee Aaron. *Initiation of slide mold aggregation viewed as an instability*, J. Theor. Biol., 26 (1970) 399-415.

[13] Lu, Guozhen; Yang, Qiaohua. *Hardy-Sobolev-Maz'ya inequalities for higher order derivatives on half-spaces.* 32pp. ArXiv:1703.08171.

[14] Mugelli, Francesco; Talenti, Giorgio. *Sobolev inequalities in 2-dimensional hyperbolic space.* General inequalities, 7 (Oberwolfach, 1995), 201-216, Internat. Ser. Numer. Math., 123, Birkhuser, Basel, 1997.

[15] Nagai, Toshitaka. *Global existence and blow-up of solutions to a chemotaxis system* Nonlinear Analysis 47 (2001) 777-787.

[16] Patlak, Clifford S. *Random walk with persistence and external bias*, Bull. Math. Biophys., 15 (1953) 311-338.

[17] Pierfelice, Vittoria. *The incompressible Navier-Stokes equations on non-compact manifolds.* The Journal of Geometric Analysis 27 (2017) no. 1, 577-617.

[18] Senba, Takasi; Suzuki, Takashi. *Weak solutions to a parabolic-elliptic system of chemotaxis*, J. Funct. Anal., 191 (2002) 17-51.

[19] Senba, Takasi; Suzuki, Takashi. *Applied analysis*, Imperial College Press, London 2004. Mathematical methods in natural science.

[20] Stoll, Manfred. *Harmonic and subharmonic function theory on the hyperbolic ball.* London Mathematical Society Lecture Note Series, 431. Cambridge University Press, Cambridge (2016).

[21] Strichartz, Robert. *Analysis of the Laplacian on the Complete Riemannian manifolds.* Journal of Functional Analysis 52 (1983) 48-79.

[22] Suzuki, Takashi. *Free energy and self-interacting particles*, Progress in Nonlinear Differential Equations and their Applications, 62, Birkhäuser Boston Inc., Boston, MA, 2005.

[23] Varopoulos, Nicholas Theodor. *The heat kernel on Lie groups,* Rev. Mat. Iberoamericana 12, (1996) no. 1, 147-186.

Institut Denis Poisson (IDP), Université d'Orléans, Université de Tours, CNRS UMR 7013, Bâtiment de Mathématiques, B.P. 6759, 45067 Orléans Cedex 2, France

E-mail address: Patrick.Maheux@univ-orleans.fr & Vittoria.Pierfelice@univ-orleans.fr