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Morphogenesis of surfaces with planar lines of curvature and application to architectural design

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Abstract

This article presents a methodology to generate surfaces with planar lines of curvature from two or three curves and tailored for architectural design. Meshing with planar quadrilateral facets and optimal offset properties for the structural layout are guaranteed. The methodology relies on the invariance of circular meshes by spherical inversion and discrete Combescure transformations, and uses parametrisation of surfaces with cyclidic patches. The shapes resulting from our methodology are called *super-canal surfaces* by the authors, as they are an extension of canal surfaces. An interesting connection to shell theory is recalled, as the shapes proposed in this paper are at equilibrium under uniform normal loading. Some applications of these shapes to architecture are shown.

Keywords: super-canal surface, fabrication-aware design, cyclidic net, architectural geometry, structural morphogenesis, façade

1. Introduction

1.1. Constructive geometry in architecture

The construction of architectural shapes is subject to technological constraints that highly impact the economy of the cladding and structure. The study that aims at expressing technological requirements as geometrical constraints is often referred to as *fabrication-aware design* in the computer science community, whereas architects or engineers speak of *shape rationalization* or *constructive geometry*. This topic, takes root in the eighteenth century and stereotomy, and the work of Gaspard Monge[1].

In glass or metal envelopes, the planarity of the panels is regarded as one of the most significant aspect in the design of technologically-feasible solutions, and motivated the creation of tailor-made morphogenesis strategies by engineering office Schlaich Bergermann und Partner [2] and

later by Gehry Technologies [3]. Triangular meshes are always covered with planar facets, but their high node valence makes the fabrication of the structure complicated [4]. They are also considered less transparent than quadrilateral layouts [3]. Developable panels are also of interest because cold-bending technologies for glass can be used at a reasonable cost, as illustrated by some projects of engineering office RFR [5, 6].

The geometry of the supporting structure is another indicator of the complexity of fabrication in free-form architecture. The most economical solution is to build with planar beams that meet exactly along axes. This topic is well-known by gridshell builders [7] and is covered from a mathematical perspective in [8], with a tool called '*mesh parallelism*'. Building a support structure with planar beams implies indeed the existence of a mesh which has all its edges parallel to the initial mesh.

These two construction constraints (planarity of panels

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and planarity of beams) can be integrated in the design of free-form architecture, either in top-down [4, 6, 9] or in bottom-up approaches [10, 11, 12, 13, 14]. The latter approaches generate design spaces where the fabrication requirements are fulfilled. They offer thus the possibility to integrate constraints of a different nature early in the design process, like structural behaviour or energy consumption. This is particularly important in the context of architectural design: fabrication is only one of the many criteria that should be rationalised or optimised in a building envelope.

1.2. Geometrically-constrained shape generation

A natural way to deal with construction constraints is to generate a design-space of shapes that satisfy the most critical fabrication-constraints. This approach, known as “*geometrically-constrained design strategy*” [15] has been used extensively in the history of architecture. Methods that guarantee planar quads include surfaces of revolution, surfaces of translation [2], scale-trans surfaces [3], moulding and Monge surfaces [16, 17, 18]. These surfaces can be generated using two curves and a rule of transformation, either translation or sweeping along Bishop’s frame. The designer controls the overall shape and its discretisation simultaneously, which makes all these shapes easily understandable and usable [19]. Accordingly, geometrically-constrained approaches using two curves like surfaces of translation are very popular in the community of structural engineers [20].

Table 1 shows the correspondance between shape generation techniques using two or three curves and their fabrication-aware counterpart. For example extrusion along a curve that yields surfaces of translation and surfaces of revolution are good examples of fabrication-aware shapes. Monge surfaces, that can be generated by sweeping a planar curve called generatrix along a rail curve, are also very interesting for architectural shape design. It can be noticed that the *sweep 2 rails* command has no

fabrication-aware equivalent.

CAD generation process	Fabrication-aware shape
Extrusion along curve	Surface of translation
Revolve	Surface of revolution
Rail Revolve	Scale-trans surface
Sweep 1 rail	Monge surface* Isoradial mesh
Sweep 2 rails	-

Table 1: Kinematic method to generate free-form surfaces and their fabrication-aware equivalent, surfaces marked with asterisks are subject to additional constraints.

The objective of this work is thus to enrich the design space accessible with geometrically-constrained design strategies by proposing new shapes constructed from two and three curves. The shapes can be generated in real-time on standard computers, which eases the exploration of this design space.

The second section of this paper discusses thus the general methodology that generates super-canal surfaces, a new family of shapes for fabrication-aware design in architecture, as well as a new algorithm for the fast computation of parallel meshes. Applying the results of [21], we also show that super-canal surfaces are remarkable with respect to shell theory: their lines of curvatures are lines of principal stress under uniform normal loading. This work thus meets fabrication with equilibrium, two major aspects of architectural design. A new method for the generation of canal surfaces from two contour curves is presented in Section 4. The fifth Section introduces some inverse problems solved in with super-canal surface. A brief discussion and conclusion sum up the contributions of the present article.

94 2. Methodology

95 2.1. Möbius geometry and cyclidic nets

96 The present methodology for shape generation relies
97 on a more general framework proposed recently for archi-
98 tectural design [22] in the following of previous work de-
99 veloped in [23, 24, 25, 26]. The main concept is to link
100 discrete objects, namely circular meshes, with a smooth
101 underlying surface. All the shapes are thus described as
102 coarse circular meshes, which support portions of Dupin’s
103 cyclides. Among remarkable features of cyclides, one may
104 mention that their lines of curvature are circles and that
105 a patch delimited by four lines of curvature on a Dupin
106 cyclide has its four vertices inscribed within a circle. The
107 formal potential of this framework is shown in [26], where
108 various fitting problems on complex shapes are solved.

109 Cyclidic nets provide thus a natural way to cover com-
110 plex shapes with circular quadrilateral meshes. Moreover,
111 as transformations mapping circular quadrilaterals to cir-
112 cular quadrilaterals also preserve cyclidic nets and the un-
113 derlying parametrisation, such transformations are of par-
114 ticular interest. Two of those will be studied in the follow-
115 ing: Möbius transformations in Section 2.3 and Combes-
116 cure transformations in Section 2.4. Starting from surfaces
117 easily described with cyclidic nets, the application of these
118 transformations creates new shapes for fabrication-aware
119 design in architecture.

120 2.2. Geometrical properties of canal surface

121 Canal surfaces are a fundamental family of surfaces in
122 the context of Möbius geometry, as this family is indeed
123 invariant by Möbius transformations. Canal surfaces are
124 defined as envelopes of spheres. They are commonly used
125 as blending surfaces, either by using contour curves [27] or
126 by joining spheres that can be manipulated by the designer
127 [28].

128 Alternatively, canal surfaces can be defined as surfaces
129 such that lines of curvatures are circles. The invariance of
130 canal surfaces under Möbius transformations is obvious,

131 because inversions preserve lines of curvatures and circles.
132 Another way to look at it is that Möbius transformations
133 preserve both spheres and angles, therefore an envelope of
134 spheres is preserved by inversions. Notice that the shape-
135 generation of canal surfaces with Dupin cyclides has been
136 studied in [29]. We use an optimisation approach intro-
137 duced in [26] to obtain shapes with smooth parametrisa-
138 tions.



Figure 1: Canal surface as an envelope of spheres whose centers are on a curve.

139 These surfaces are very easily parameterised by cyclidic
140 patches, as Dupin cyclides are particular cases of canal
141 surfaces. The C^1 continuity and generalisation to C^0 con-
142 tinuity is insured by reflections according to the method
143 described in [25].

144 2.3. Möbius transformations

145 The transformation at the core of the framework using
146 cyclidic nets is the Möbius transformation or *inversion*,
147 which is a very simple non-linear map. We recall here some
148 of its elementary properties, and introduce the notations
149 used in the following of this paper.

Möbius transformations preserve locally angles, and
are thus conformal maps. They also preserves circles.
Möbius transformations are compositions of translation,

scaling and spherical inversions. The latter transformation is defined by a center and a ratio. Consider a point \mathbf{C} , later called center of inversion, and a real number k . The inversion of center \mathbf{C} and ratio k applied to a point \mathbf{M} is a point \mathbf{M}' defined by the well-known equation:

$$\mathbf{CM}' = \frac{k}{\|\mathbf{CM}\|^2} \cdot \mathbf{CM} \quad (1)$$

In the complex plane, the inversion of ratio k with center \mathbf{C} (complex number z_C) reads as:

$$f_{k,\mathbf{C}}(z) = z_C + \frac{k}{\bar{z} - \bar{z}_C} \quad (2)$$

An elementary property of inversions is that they are involutions, which means that Möbius transformations are their own inverse transformations. This property is used in many applications shown in this paper (see Section 4).

It can finally be noticed that the ratio k is nothing more than a scaling factor. The position of the point \mathbf{C} is the parameter that has a true impact on the shape deformation.

2.4. Combescure transformations

It has just been seen that Möbius transformations allow to modify the overall appearance of circular meshes by preserving the circumcircles of all quads. Another transformation that has the same property is the mesh parallelism transformation. Two meshes are said *parallels* if they have the same connectivity and if all their edges are parallel. The transformation mapping one mesh to the other is called a *Combescure transformation* [8]. By definition, Combescure transformations preserve discrete angles. Therefore they map circular meshes to circular meshes. Combined with Möbius transformations, they offer a range of possibilities to deform circular meshes.

Two meshes related by a Combescure transformation, with respective edges (\mathbf{e}_i) and (\mathbf{e}'_i), have to satisfy a linear equation:

$$\forall i, \mathbf{e}_i \wedge \mathbf{e}'_i = \mathbf{0} \quad (3)$$

Solutions for this equation are usually found using Singular Value Decomposition (SVD) [8]. We introduce here a different original approach, restricted to quadrilateral meshes, but that offers a better performance than SVD. This technique takes inspiration from the one employed in [14], which is applied to the form-finding of planar quadrilaterals meshes.

2.5. Efficient computation of Combescure transformations

Let us consider two parallel quadrilaterals, like the ones shown in Figure 2. Up to a translation, prescribing the lengths of two sides l_0 and l_3 (thick lines on the figure) is sufficient to determine a unique quadrilateral with internal angles $\alpha, \beta, \gamma, \delta$. The last point C (white dot on the figure) is found by intersecting two lines (dashed lines on the figure). For the sake of simplicity, we consider planar quadrilaterals in the reference plane (ABD): the equations are written in a frame centred in A and represented by the blue arrows in the figure. The intersection is found by solving the following equation:

$$\begin{pmatrix} l_0 + l_3 \cos \alpha \\ l_3 \sin \alpha \end{pmatrix} = \begin{pmatrix} \cos \beta & \cos(\alpha - \delta) \\ -\sin \beta & \sin(\alpha - \delta) \end{pmatrix} \cdot \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \quad (4)$$

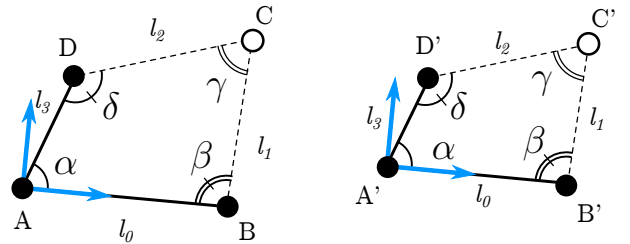


Figure 2: Two quads related by a Combescure transformation.

In the same way, prescribing the lengths of all edges on two intersecting lines, as shown in Figure 3 is sufficient to determine the entire parallel mesh. In this image, the thick lines correspond to edges which have prescribed lengths. Starting from a quadrilateral with two prescribed lengths, it is possible to apply equation (4) and find the last point of the quadrilateral (white dot). It is then possible to apply

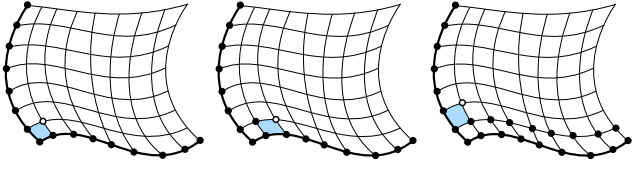


Figure 3: Propagation method for the computation of a Combescure transformation with quadrangles.

this procedure to the next quadrangle in the same row, and so forth, up to completion of each strip.

This iterative procedure is computationally efficient. The number of operations and the use of memory is proportional to the number of faces in the mesh, as the solution of the propagation requires NM applications of equation (4) for a mesh of N times M facets. The computation time also varies linearly with the number of faces, as discussed in [14]. This technique is thus more efficient than SVD, which requires assembling of matrices. The computational gain is especially important for large meshes and makes the method proposed in this paper suited for real-time applications.

2.6. Super-canal surface

We call *super-canal surfaces* the surfaces that are images of canal surfaces by arbitrary compositions of Combescure transformations and Möbius transformations. This name recalls the term *supercyclide* to name projective transforms of Dupin cyclides by Pratt [30]. We choose to use the same prefix even if the transformations at stake in this paper are different from the ones studied by Pratt.

The image of a circular quad-mesh by Combescure transformations and inversions remains a circular quad-mesh, but both transformations affect differently the overall shape, creating interesting formal possibilities. The two operations do not commute, so specifying the order of application of Combescure and Möbius transformation has an influence on the properties of the final shape.

Hence, the methodology proposed in the following is to reconstruct a super-canal surface from two curves assuming a composition of applications of inversions or Combescure

transforms. Rather than playing with canal surfaces and transformations, the principle of the method relies on a reverse approach which aims at finding an initially unknown canal surface that would satisfy two prescribed boundaries (see Section 5.1).

3. Super-canal surfaces

3.1. A general framework for shape generation

The method exposed above translates into a simple framework that requires two perpendicular curves as input. Indeed, canal surfaces do not have umbilical points (except poles), and consequently, their lines of curvature are necessary perpendicular. The designer thus chose a rule of construction for the surface, i.e. a specific combination of Combescure and Möbius transformations. The concatenation of transformations provides more design freedom to the end-user than the utilization of one specific transformation: this is discussed in the next sections. The identified families are proposed in Figure 4. The nomenclature for the different surfaces follows:

- the letter C denotes that the initial shape was subjected to a Combescure transformation;
- the letter M denotes that the initial shape was subjected to a Möbius transformation;
- the order of the letters gives the order of composition of the transformations: CM means that the initial shape was subject to a Möbius transformation, then a Combescure transformation;
- the name of the initial shape subject to the transformations stands at the end: for example a M -revolution surface is an inversion of a surface of revolution.

Many surfaces well-identified in the literature can be generated with this method as illustrated in Figure 4. All the common surfaces used for geometrically-constrained

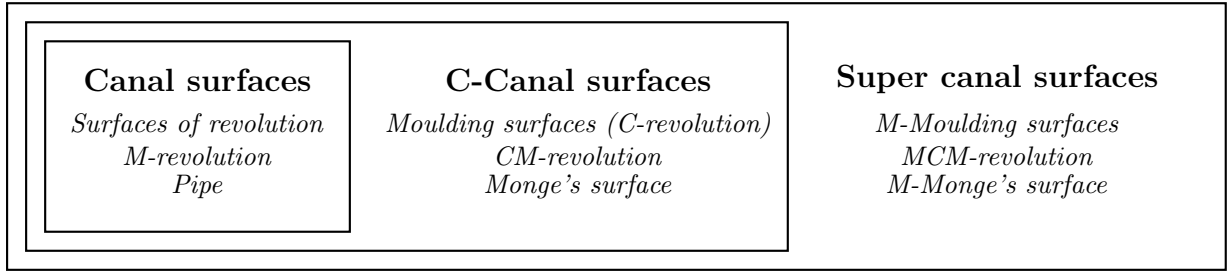


Figure 4: Super canal surfaces

254 methods mentioned in Section 1.2 fall into the category 281
255 of super-canal surfaces, with the exception of scale-trans
256 surfaces. The curves used in surfaces of translation and
257 scale-trans surfaces do not correspond in general to lines of
258 curvatures and cannot be approached by circular meshes.
259 Therefore, they do not have any specific offset properties.

260 It appears that moulding surfaces and Monge sur-
261 faces discussed in [17] are a subset of the shapes gener-
262 ated by Combescure transformations of canal surfaces.
263 From a practical point of view, shapes with a family of
264 planar curves are of great interest in construction. For
265 that reason, we restrict the examples of application to
266 *CM*-surfaces, where the families of circles are trans-
267 formed into planar curves.

282 3.2. Input for design with super-canal surfaces 283

269 In the following of [3], we propose to design super-canal 284
270 surfaces from two curves. The simplest way to parame- 285
271 terise a canal surface is to take a strip of circles as input 286
272 parameters, as pictured in Figure 5. A two parameters 287
273 family of cyclidic nets can be supported on the circular 288
274 mesh: the choice of those parameters can be done to fulfill 289
275 some design requirements, like the shape smoothness, eval- 290
276 uated with conformal Willmore energy [31, 22]. In the ex- 291
277 ample of Figure 5, eight circles in the same plane are used 292
278 to generate a canal surface. Only the portion of the canal 293
279 surface above the construction plane is shown. Note also 294
280 that the resulting surface, made of cyclidic patches, is a 295

C^1 surface with curvature discontinuities between patches.

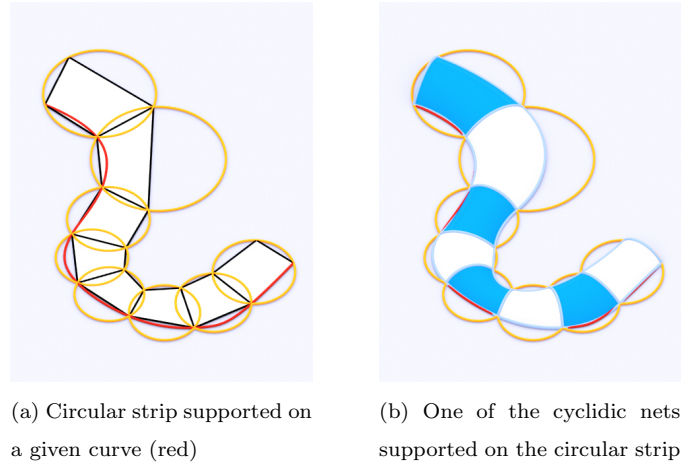


Figure 5: A canal surface created from a coarse circular strip

282 To define the strip of circles, the user can draw manu-
283 ally a collection of circles, or entirely parametrised it by a
boundary curve and the radii of circles or a target length
for each border. The latter parametrisation is depicted in
Figure 6, whose input data follows:

1. a list of points on a curve in space;
2. one point \mathbf{P} in space defining the first circle;
3. a function describing the lengths of each edge cross-
ing the strip (thick orange lines on Figure 6).

It is then possible to construct one unique circular strip
passing through the input points by propagation, in the
manner of [32]. The construction of a circular strip re-
stricts the two boundaries to be lines of curvature of the
resulting surface. Section 4.1 will show how this condition

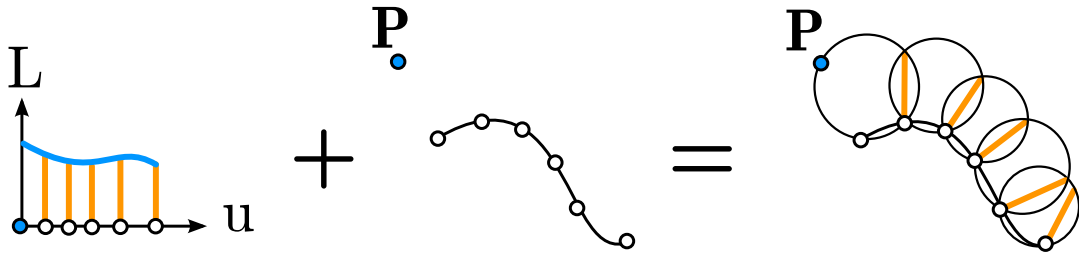


Figure 6: Parameters creating a circular strip that can support a canal surface

296 can be relaxed, while keeping the parametrisation of the 309
 297 shapes by cyclidic nets and circular strips.

298 3.3. Shape smoothing

Some input data might lead to visually unpleasent results, therefore we use the strategy proposed in [26] and take the position of the vertices and the orientation of the normal vector to the cyclidic net at one node as variables for smoothness optimisation. The objective is to fit exactly one input curve. To this end, the points on this curves are parametrised by the vector \mathbf{u} . The other parameters governing the shape of the canal surface are the lengths of the edges crossing the circular strip \mathbf{L} (see Figure 6). The cyclidic net is then generated by the choice of an orthog-
 312 onal frame, parametrised by two angles λ and θ , which
 313 are angles defining a spherical coordinate system. The
 314 smoothness functional \mathcal{F} is finally defined as a quadratic
 315 function of the radii of the edges of the cyclidic net R_{edge} :
 316

$$\mathcal{F}(\mathbf{u}, \mathbf{L}, \lambda, \theta) = \sum_{\text{edges}} \frac{1}{R_{edge}^2(\mathbf{u}, \mathbf{L}, \lambda, \theta)} \quad (5)_{317}$$

299 The computation of the function is not hard, and its min-
 300 imisation gives satisfying results and is done in real-time.
 301 The user can specify additional constraints, like the angle
 302 made by the normal and a reference plane. In the latter³²¹
 303 case, the degrees of freedom λ and θ become coupled, and³²²
 304 the normal rotates along a cone.³²³

305 Practically, the minimisation is here done by the means³²⁴
 306 of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algo-³²⁵
 307 rithm. Figure 7 shows the smoothing of a canal surface³²⁶
 308 based on the proposed energy. The parameters are the

circle radii. Note that only local changes are introduced
 310 after optimisation, and that the areas where the facets
 311 were degenerated have disappeared.

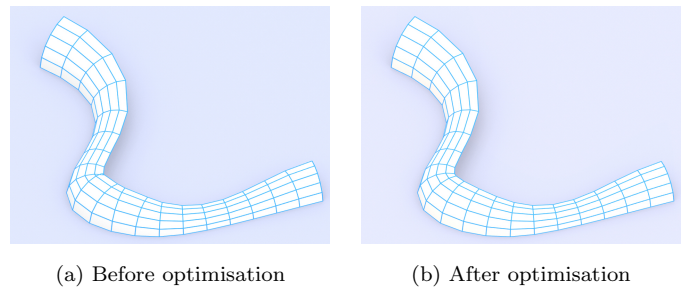


Figure 7: Optimisation of the smoothness of a canal surface.

3.4. Mechanical properties of super-canal surfaces

This Section discusses briefly the mechanical behaviour of super-canal surfaces. C-canal surfaces play indeed a particular role in shell theory, as Rogers and Schief proved that their lines of curvatures are also lines of principal stresses under a uniform external load [21]. This result was also proven for canal surfaces before in [33].

This induces two remarkable features for the behaviour of the shapes previously presented:

- principal stresses lines following principal curvature lines, the natural mesh of C-canal surfaces is an optimal mechanical layout for a grid structure;
- all closed shapes generated by this method are in equilibrium under uniform pressure and therefore suited for pneumatic structures.

Before showing the shape generation framework, we should make a comment on potential applications for shallow roof structures. A normal pressure load is surely very close to a uniform distributed load for surfaces with moderate curvature. It can be concluded that shallow canal surfaces are close to funicular shapes under uniformly distributed load. This kind of consideration has been documented for shallow arches: shallow circular arcs, parabolas or catenary have similar geometry and mechanical behaviour, especially buckling capacity. For more comments on this topic, the reader can refer to [34].

Furthermore, it should be recalled that in practice temporary actions are not negligible compared to the self-weight of a structure, and gridshells or thin shells are often designed with respect to non-symmetrical loads, for which lines of curvature are not principal stress lines. The finding of a structural optimum for different load case combinations is far from obvious, but its computation is not necessarily a practical design objective: just like fabrication, structural performance is not the only criterion taken into account by the architects and engineers. The integration of principal stress under self-weight can be seen as a simple way to generate a good, but not necessarily optimal structural pattern, while creating a rich design narrative referring to pioneering works of structural artists. Lines of principal stress have been used by Pier Luigi Nervi for the design of concrete ribbed slabs. Nervi did not solve an optimisation problem, but used a simple guiding principle for his design, which resulted indeed in highly efficient structures. The meshing of super-canal surfaces by their lines of curvatures combines thus constructability with structural efficiency.

4. Application to shape modelling

4.1. Generation of canal surfaces

The previous section discussed how canal surfaces can be parametrised with circular strips supporting cyclidic

nets. This generation method leads however to a strong formal restriction, as it forces the two boundaries of the strip to be lines of curvature of the resulting canal surface. The practical consequence is that the second curve is restricted to be on a developable surface passing through the first curve, whereas the designer would prefer to define it independently. This section introduces thus an original algorithm for the shape generation of canal surfaces from two curves where only one of the two curves is a line of curvature of the canal surface. The problem is illustrated in Figure 8 and it will be shown that it admits a one parameter family of solutions.

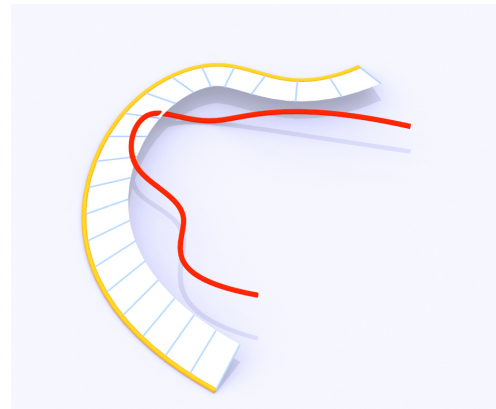


Figure 8: Input data for the curve-fitting problem. Line of curvature (orange), line to fit (red), and surface (white) containing the centers of the spheres.

Preliminary considerations

The relevant definition of canal surfaces in this case is to consider them as the envelope of a family of spheres. Remarkable properties of canal surfaces, and of lines of curvature in general can be mentioned:

1. Canal surfaces are envelopes of spheres, and as such, the spheres generating the surface meet tangentially with any curve of a canal surface.
2. The envelope of the lines directed by the normal of the surface along a line of curvature is a developable surface.

From remark 1, we get that the locus of the centers of the spheres generating the canal surface is on the surface

388 generated by the normals of the surface. From remark
 389 2, we get immediately that this is a developable surface.
 390 Actually, it is a specific case of Monge surface [17]: once
 391 one normal has been chosen, the other normals are de-
 392 termined uniquely so that the envelope is indeed a devel-
 393 opable surface. The locus of the centres of the spheres is
 394 therefore controlled by one orientation parameter. This
 395 is illustrated in Figure 9: choosing the orientation of the
 396 normal is equivalent to choosing a surface tangent to the
 resulting canal surface.

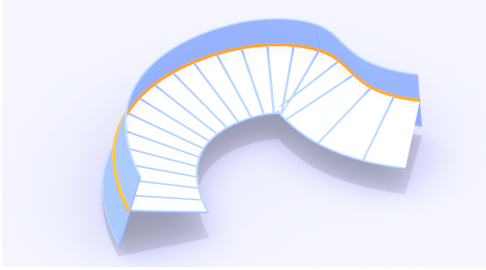


Figure 9: Line of curvature: one developable surface containing the centres of the spheres (white). The developable surface perpendicular to it (blue) is tangent to the resulting canal surface (not shown in the Figure).

397

398 Computation of the locus of centres

399 Consider now that a normal vector and a line of curva-
 400 ture have been specified for the canal surface. The locus
 401 of centres is on a developable surface. So far we did not
 402 use any property of the second curve. We notice however
 403 that the centres of the spheres are on the *bisector surface*
 404 *of the two curves*. Such surface is defined as the envelope⁴¹⁹
 405 of the points which are equidistant to both curves. They⁴²⁰
 406 have been studied in [36] for example. ⁴²¹

407 Therefore, the centres of the spheres can be found by⁴²²
 408 intersecting the bisector surface of the two curves and the⁴²³
 409 developable surface constructed from the normals. Both⁴²⁴
 410 surfaces are not bounded, and it seems intuitive that they⁴²⁵
 411 will have an intersection in non-degenerate cases. The⁴²⁶
 412 construction of the whole bisector surface is however not⁴²⁷
 413 necessary, as it is meaningful to consider a finite collec-⁴²⁸
 414 tion of spheres that will construct the cyclidic net that⁴²⁹

parametrise the canal surface.

Consider hence the first curve discretised with n sub-
 divisions, as depicted in Figure 10. The centres of the
 spheres belong to n lines on the developable surface. Let
 \mathbf{P}_k be the k^{th} point on the first curve, \mathbf{C}_k the centre of
 the bi-tangent sphere on the corresponding line and \mathbf{C}'_k
 the closest point to \mathbf{C}_k on the second curve. By default,
 \mathbf{C}_k is not on the bisector surface. Therefore, the following
 functional is introduced and minimised.

$$F = \sum_{k=0}^n (\|\mathbf{C}_k \mathbf{P}_k\| - \|\mathbf{C}_k \mathbf{C}'_k\|)^2 \quad (6)$$

416 The positions of the \mathbf{C}_k are encoded with independent
 417 unique parameters. Each term of the sum can thus be
 418 minimised individually by the means of Newton's method.

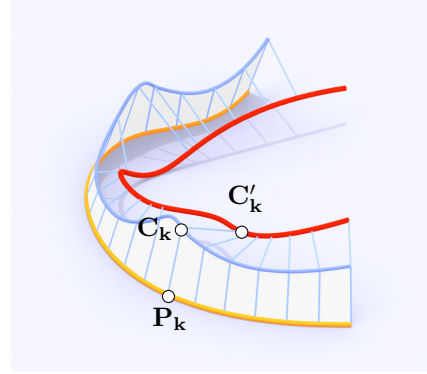


Figure 10: The curve fitting problem: the locus of the sphere centres (in dark blue) belongs to the developable surface chosen by the user, and each \mathbf{C}_k belongs to a straight line of this surface. The locus of centres is equidistant to both input curves.

Algorithm for spheres generation

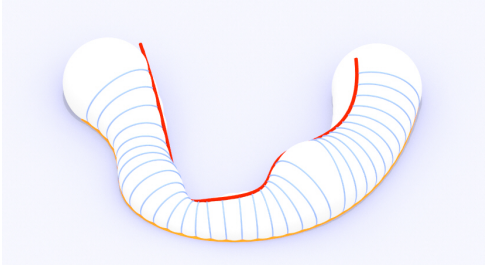
The algorithm for the generation of a canal surface from two curves follows:

1. Select two curves, one of them being a line of curvature on the final surface.
2. Choose an orientation of the canal surface: specifying one orientation restricts the locus of centres to be in a uniquely defined developable surface.
3. Discretise the line of curvature with points \mathbf{P}_k , and generate the lines containing the centres of the spheres on the developable surface.

430 4. Initialise the \mathbf{C}_k with $\mathbf{C}_k = \mathbf{P}_k$.
 431 5. Minimise Equation(6) with Newton's method.
 432 The result is a collection of points corresponding to sphere
 433 centres. The radius R_k of each sphere is given by the
 434 distance $\|\mathbf{C}_k \mathbf{P}_k\|$.

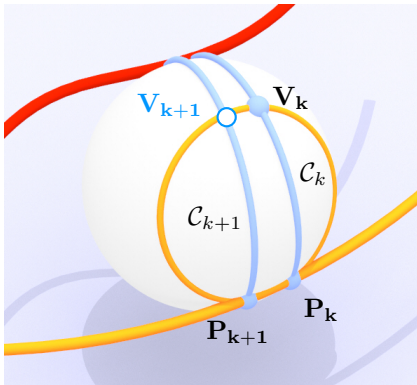
435 *Generation of a supporting cyclidic net*

436 We have seen that given two curves and a supplement-
 437 ary condition, it is possible to define one unique family
 438 of spheres that optimally fits the two curves. Consider
 439 now the circles \mathcal{C}_k defined as the intersection of successive
 440 spheres $\mathcal{S}_k, \mathcal{S}_{k+1}$, like shown in Figure 11. \mathbf{P}_k is the point
 441 of \mathcal{C}_k on the input curve.



465 Figure 11: A family of spheres (white) fitting two curves (red and
 466 orange), and their successive intersection (blue).

442 It is clear that for any k , \mathcal{C}_k and \mathcal{C}_{k+1} both belong
 443 to the sphere \mathcal{S}_{k+1} . Consider Figure 12: choosing one
 444 point \mathbf{V}_k on \mathcal{C}_k there is exactly one point \mathbf{V}_{k+1} on \mathcal{C}_{k+1}
 445 so that $\mathbf{P}_k \mathbf{V}_k \mathbf{V}_{k+1} \mathbf{P}_{k+1}$ is inscribed within a circle. The



478 Figure 12: Two circles \mathcal{C}_k and \mathcal{C}_{k+1} : by choosing one point \mathbf{V}_k on \mathcal{C}_k ,
 479 one defines a circle and a one parameter family of cyclidic patches.

445
 446 process can be applied iteratively to generate a circular

447 strip, supporting a cyclidic net. The circles \mathcal{C}_k can be
 448 edges of the resulting cyclidic strip because they belong to
 449 the same sphere [25].

450 *Comment*

451 The proposed method allows for the construction of
 452 a canal surface that fits optimally two input curves. The
 453 surface can be parametrised instantly with cyclidic patches
 454 and covered with a circular mesh as detailed in [25]. Con-
 455 sider indeed the collection of circles (\mathcal{C}_n), and two consecu-
 456 tive circles \mathcal{C}_n and \mathcal{C}_{n+1} . By construction, these circles are
 457 co-spherical, and thus, any planar quad with vertices on
 458 \mathcal{C}_n and \mathcal{C}_{n+1} is also a circular quad. This simplifies greatly
 459 the meshing process. The meshing algorithm is illustrated
 460 in Figure 13.

- 461 • Choose a discretisation of the first circle \mathcal{C}_1 , the k^{th}
 462 point of the n^{th} circle is noted $\mathbf{P}_{n,k}$
- 463 • The $\mathbf{P}_{n,1}$ are chosen so that they all belong to the
 464 fitted curve which is a line of curvature of the result-
 ing surface;
- 465 • Starting from $k = 1$ and $n = 1$, generate the plane
 466 \mathcal{P} going through $\mathbf{P}_{n,k}, \mathbf{P}_{n,k+1}$ and $\mathbf{P}_{n+1,k}$ (step 1
 in Figure 13);
- 467 • The point $\mathbf{P}_{n+1,k}$ is the intersection between \mathcal{P} and
 468 \mathcal{C}_{n+1} (step 2 in Figure 13);
- 469 • Iterate over k (step 3 in Figure 13);
- 470 • Iterate on \mathcal{C}_{n+1} and \mathcal{C}_{n+2} (step 4 in Figure 13);

471
 472
 473 The tool recalls the two-rails sweep commonly used in
 474 CAD software. One curve is a line of curvature of the re-
 475 sulting shape. It provides proper alignment of the mesh
 476 with the borders, which often dictate the mechanical be-
 477 haviour of the structure.

478 *4.2. Generation of closed canal surfaces*

479 The proposed construction can be extended to closed
 480 strips with several limitations. The first one has been dis-
 481 cussed in [22]: a closed cyclidic net gives a smooth closed

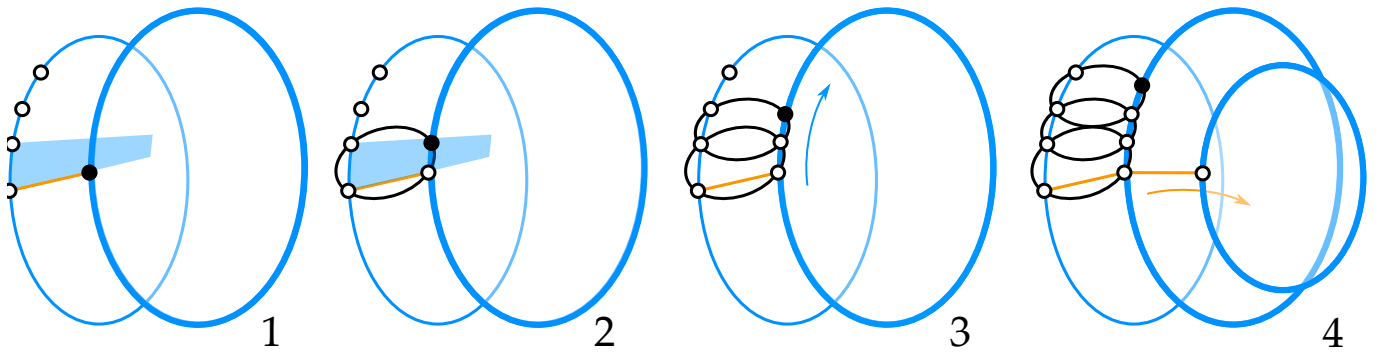


Figure 13: Meshing between two co-spherical circles (axonometry).

482 surface if and only if the discrete guide curve is a pseudo-
 483 spherical curve (it is parallel to a curve which has all its
 484 vertices inscribed within a sphere). The second condition
 485 corresponds to the possibility of drawing the last circle of
 486 the strip. Consider Figure 14: the first circle of the strip
 487 is written C_0 , the penultimate circle C_f , the initial point
 488 \mathbf{P} and the first and last point of the curve \mathbf{P}_0 and \mathbf{P}_f
 489 respectively. There are two cases:

- C_0 and C_f belong to the same sphere, then the circle
 going through \mathbf{P} , \mathbf{P}_0 and \mathbf{P}_f intersects the circle
 C_f in two points. This circle is the solution we are
 looking for and is represented with dashed lines on
 Figure 14.
- In the other cases, the spheres (C_0, \mathbf{P}_f) and (C_f, \mathbf{P}_0)
 are distinct. Their intersection is a circle intersecting
 the circle C_0 and C_f in two different points. This
 circle is the only solution that allows the closing of
 the circular strip, and it does not intersect C_0 in \mathbf{P} .

In the first case, only the intersection of the last circle
 and C_f is unknown. In the second case, the position of \mathbf{P}
 cannot be specified arbitrarily (as in section 2. for open
 strips). Compared to open strips, there is therefore a loss
 of at most two degrees of freedom for the control of the
 shape.

Figure 15 shows a rendering of a facade covered with
 a canal surface (the structural system supporting the can-
 tilvering facade is not shown). Being able to model closed

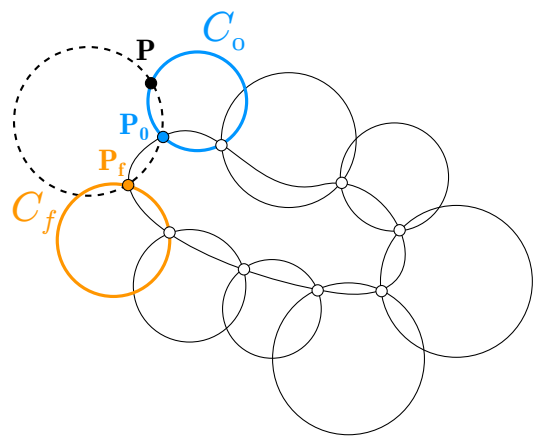


Figure 14: Problem of a closing strip

surfaces is crucial for architectural shapes, as façades are
 usually closed.

4.3. Practical applications

The method presented in this paper has been used dur-
 ing a one week workshop in 2015. Architecture and engi-
 neering students had to design and build a 30 m² free-form
 pavilion, the only material available was polystyrene in
 flat rectangular sheets. The shape is a super-canal surface
 meshed with circular quadrilaterals. The pavilion, shown
 in Figure 16 is a grid structure with a torsion-free beam
 layout. The offset was computed with a reflection rule sim-
 ilar to the one generating cyclidic nets. An optimisation
 was performed in order to minimise the height gap at the
 nodes between beams of constant height. The fast compu-
 tation of the space of solutions was key to the success of

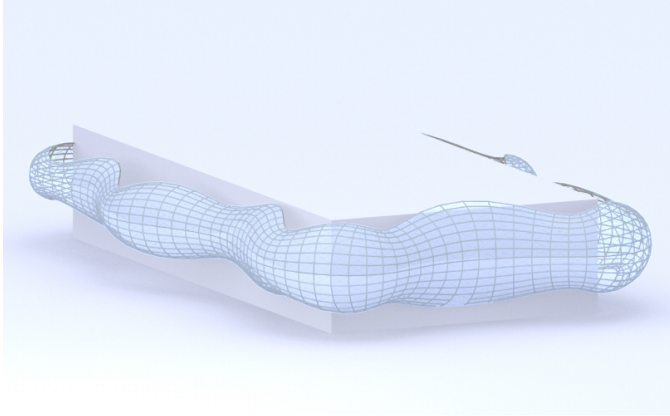


Figure 15: A visualisation of a façade as a canal surface, covered with a circular mesh.



Figure 17: A model of a canal surface with planar quadrangles used for bracing.

this operation within a limited time frame (5 days).

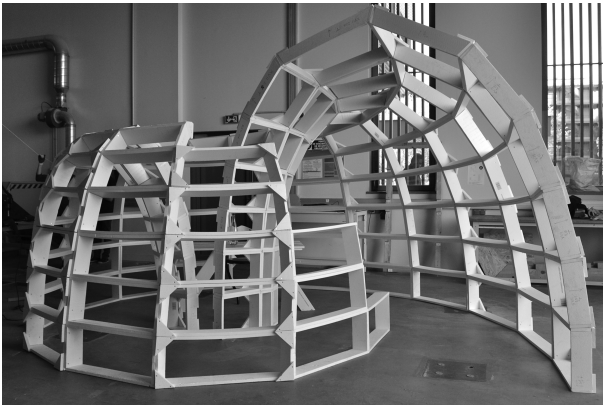


Figure 16: A prototype built with torsion free-nodes on a super-canal surface.

The tools presented in this paper were used for shape generation as well as fabrication. Hundreds of polystyrene elements were cut according to the 3D model and assembled. The planarity of the panels was considered for use as bracing elements and was validated on a 5m^2 model, shown in Figure 17. Flat panels used as bracing elements improve the overall stability and stiffness ¹.

Another exploration was performed with a timber structure, shown in Figure 18. The structure is a plated shell structures: the facets are connected along their edges without additional stiffeners. The small-scale pavilion il-

¹More details and pictures can be found on <http://www.thinkshell.fr/building-freeform-2015/>.

illustrates thus the potential offered by planar panels rather than offset properties of circular meshes, although the discrete normals of circular meshes have been used to generate planar cuts between the plates.



Figure 18: A timber plated shell structure covered with circular quadrilaterals generated with the method proposed in this paper.

The construction of those prototypes validates the use of the numerical tools presented in this paper. The user feedback allowed us to identify the most relevant way to model super-canal surfaces. In particular, the students found important to control at least one boundary curve. This explains why the method of generation of canal surfaces presented in this work focuses on the prescription of a boundary curve, and not on the curve supporting the centers of the sphere for example.

5. Application to inverse problems

5.1. Generation of *M*-revolution surfaces

The most well-known canal surfaces are surfaces of revolution. They indeed correspond to the case of a straight

553 generatrix. Surfaces of revolutions have many interest-
 554 ing properties for applications in architecture. They are
 555 isothermic surfaces, which means that they can be discre-
 556 tised as Edge-Offset Meshes. Yet, isothermic surfaces are
 557 preserved by Combescure and Möbius transformations and
 558 they thus inherit this property.

559 In particular, we discuss here of a particular subset of
 560 'super-surfaces of revolutions', where the center of inver-
 561 sion and the axis of revolution are in a horizontal plane,
 562 as shown in Figure 19. It is clear that the parallels of the
 563 surface of revolution are vertical in this case. Since inver-
 564 sions preserve circles and angles, we can deduce that this
 565 family of curvature lines remain vertical after inversion.
 566 Combescure transformations preserve planarity: applying
 567 another Combescure transformation yields a surface with⁵⁹¹
 568 planar arches. This additional property is particularly in-⁵⁹²
 569 teresting for applications to structural system with contin-⁵⁹³
 570 uous arches and secondary structure. A specific method⁵⁹⁴
 571 has therefore been developed to generate these surfaces.
 572 It consists of solving the inverse problem detailed in the
 573 following.

574 The input data for the problem are displayed in Fig-
 575 ure 19. The user prescribes one planar curve, one circle
 576 in the same plane comprising the ends \mathbf{P}_1 and \mathbf{P}_4 of the
 577 curve, and two points \mathbf{P}_2 and \mathbf{P}_3 on this circle. The objec-
 578 tive is here to reconstruct the initial surface of revolution,
 579 therefore the problem is to find a center of inversion \mathbf{C} so
 580 that the image of the quadrangle $P_1P_2P_3P_4$ is an isosceles
 581 trapezoid.

582 Isosceles trapezoids are the only cyclic quadrilaterals
 583 that have parallel opposite edges. Notice that the prob-
 584 lem is planar and can thus be formulated with complex
 585 numbers. The parallelism corresponds to the fact the di-⁵⁹⁵
 586 rection vectors are co-linear (identical up to a scaling by
 587 a real number t). Assigning the complex numbers $z_1, z_2,$ ⁵⁹⁶
 588 z_3 and z_4 to the points $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ and \mathbf{P}_4 , and writing⁵⁹⁷
 589 $z_{j,C}$ the complex number associated to the image of z_j by
 590 an inversion of center \mathbf{C} , we obtain equation (7):

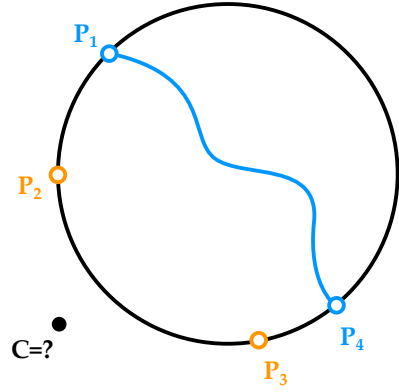


Figure 19: Problem for the practical design with inversion of surfaces of revolutions.

$$\frac{z_{2,C} - z_{1,C}}{z_{3,C} - z_{4,C}} = t \in \mathbb{R} \quad (7)$$

We can use the equation (2) to express equation (7) with respect to the z_j and obtain equation (8). It is independent of the ratio of inversion k : the position of the center of inversion is the only value of interest in this problem.

$$\frac{\left(\frac{z_2 - z_1}{(z_1 - z_C)(z_2 - z_C)} \right)}{\left(\frac{z_3 - z_4}{(z_3 - z_C)(z_4 - z_C)} \right)} = t \in \mathbb{R} \quad (8)$$

After simplifications, this equation leads to a second order equation in z_C . The general form of (8) can be written as:

$$A_t z_C^2 + B_t z_C + D_t = 0 \quad (9)$$

with

$$\begin{cases} A_t = z_2 - z_1 + t \cdot (z_4 - z_3) \\ B_t = -(1+t)z_1z_3 + (t-1)z_1z_4 \\ \quad + (1+t)z_2z_4 + (1-t)z_2z_3 \\ D_t = z_3z_4(z_2 - z_1) + tz_1z_2(z_4 - z_3) \end{cases}$$

The case of $A_t = 0$ can occur only when the quad $P_1P_2P_3P_4$ is already an isosceles trapezoid. In the other cases, for each value of t , there are two complex solutions giving two positions for the center of inversion in the complex plane. It is thus possible to solve this inverse problem with a straight-forward solution based on complex analysis.

602 An illustration of this problem is shown on Figure 20.⁶²⁸
 603 On this image, all the facets are inscribed within circles.⁶²⁹
 604 The free-form shape is thus covered with planar facets and
 605 torsion-free nodes. Since the circle shown in Figure 20 is
 606 in the horizontal plane, it is noticed that one family of
 607 lines of curvature consists of planar vertical arches. The
 608 solution proposed here can easily be extended to the case
 609 of a spherical guide curve with two successive inversions.
 610 Likewise, it is possible to apply this method to moulding
 611 or Monge surfaces.

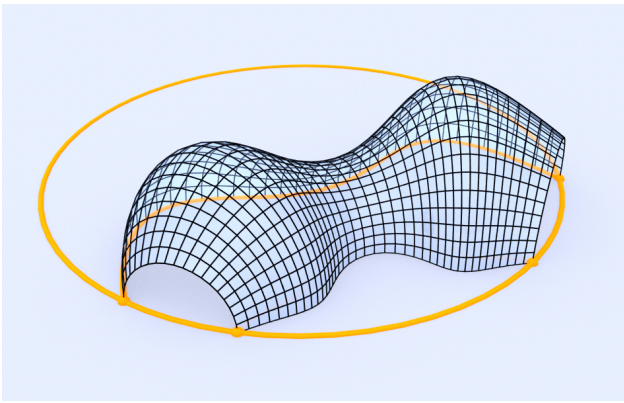


Figure 20: Surface generated by inversion of a surface of revolution,⁶³³
 constructed from one curve and two points on a circle.

612 5.2. C-canal surfaces

613 We proposed an extension of the generation method
 614 proposed in section 4.1 by adding a Combescure trans-⁶³⁷
 615 form so that the can be any planar curve, and that the⁶³⁸
 616 final surface is a C-canal surface. Figure 21a shows the⁶³⁹
 617 three input data for the generation of a C-canal surface,⁶⁴⁰
 618 while Figures 21b and 21c show two possible outputs. Like⁶⁴¹
 619 canal surfaces, the user can specify one curve, a collection⁶⁴²
 620 of lengths defining indirectly a second curve, and a planar⁶⁴³
 621 cross-section that is obtained by Combescure transforma-⁶⁴⁴
 622 tion of a circle. The inputs controlled by the designer are⁶⁴⁵
 623 thus the same as the ones described in Figure 6, with the⁶⁴⁶
 624 control of one curve in addition.⁶⁴⁷

625 The lengths of the edges is specified for the C-canal⁶⁴⁸
 626 surface, but at the beginning, only the canal surface can⁶⁴⁹
 627 be computed. An optimisation procedure is thus required⁶⁵⁰

to find the canal surface that will fit the input data after
 Combescure transformation.

Writing \mathbf{L} the target lengths for the curves crossing the
 C-canal surface (see Figure 6), we generate first the canal
 surface $\mathbf{F}(\mathbf{u}, \mathbf{L})$. There is one Combescure transformation
 f that maps the first circle of the canal surface to the
 transverse input curves chosen while preserving one input
 curve. After the Combescure transformation, the resulting
 lengths \mathbf{L}' on the C-canal surface differ from \mathbf{L} . However,
 Figure 21 shows that a canal surface and a C-canal sur-
 face related by a Combescure transformation have similar
 boundaries, even if they do not perfectly coincide. There-
 fore, a local optimization algorithm (in our case BFGS)
 can be used to minimise the error:

$$E(L_k) = \sum_k (L'_k - L_k)^2 \quad (10)$$

630 The optimisation is done for each L_k successively. This
 631 prevents from computing the whole Combescure transfor-
 632 mation at each iteration, but only the strip where the er-
 633 ror is evaluated. With this precaution, the computation
 634 remains lightweight and stable. This optimisation proce-
 635 dure can be extended to the fitting of two curves, like done
 636 in 4.1.

5.3. Meshing of super-canal surfaces

A key feature of the proposed method is that it oper-
 ates fundamentally on smooth surfaces. It is therefore in-
 dependent from the mesh density. Notice for example that
 the solution of equation (9) does not require any knowl-
 edge on the discretisation of the curves, but only the four
 prescribed points. Therefore, re-meshing of super-canal
 surfaces is extremely simple and detailed below.

It has already been pointed out that inversions are in-
 volutions. Combescure transformations are linear maps
 and can easily be inverted with the algorithm proposed in
 Section 2.4. The computation of inverse transformation
 is thus extremely light. These properties are used exten-
 sively to remesh super-canal surface and is illustrated in

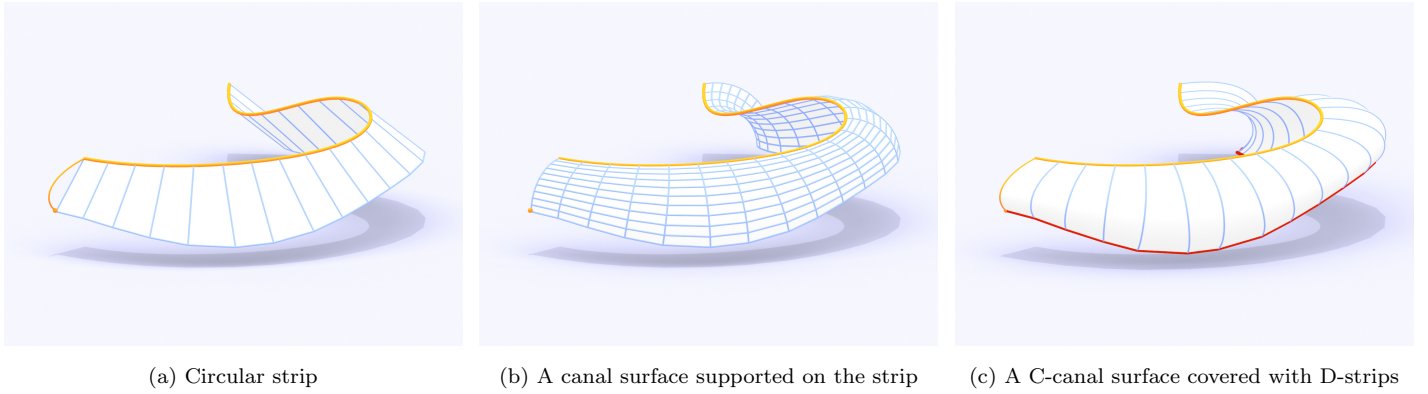


Figure 21: Generation of a C-canal surface

651 Figure 22. Given a discretisation on the guide curves, it is possible to find their image by a composition of Combescure and Möbius transformations f so that they fit with the boundaries of a canal surface. The meshing on the canal surface is done using cyclidic patches, like explained in Section 4.1. The inverse transformation f^{-1} is then computed and maps the mesh so that it fits the reference curves.

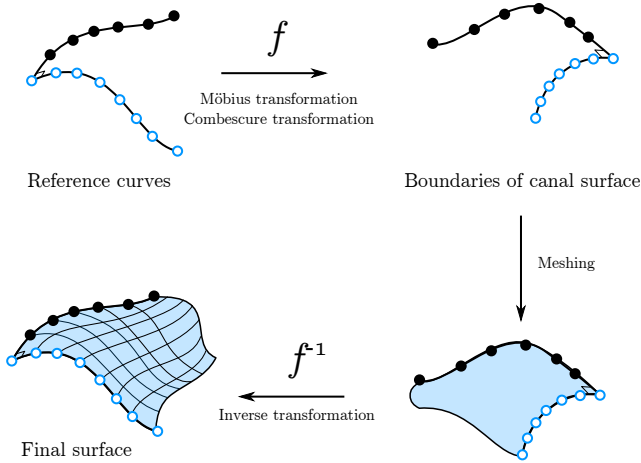


Figure 22: Remeshing procedure for a super-canal surface.

6. Discussion

6.1. Algorithmic performance

The algorithms of shape generation have been implemented in GrasshopperTM, an environment of visual

programming compatible with the modelling software RhinocerosTM. In this section, we discuss the performance of the three operations used in our method:

- the computation of circular strips and the meshing of discrete canal surfaces;
- the computation of Möbius transformations and the solution of the inverse problems;
- the computation of Combescure transformations.

Circular strips are defined using a propagation algorithm. The problem solved at each step is the intersection of a sphere and a circle. The resulting computation time varies linearly with the number of subdivisions of the guide curve.

The computation of Möbius transformations is straight forward, as equation (1) is applied to each point of the mesh. Likewise the solution of equation (9) is obvious and requires no special numerical treatment. For that reason, Möbius transformations of meshes are as fast as the computation of simple affine transformations, like scaling or translation.

A case-study for the computation of Combescure transformations was performed. Like discussed in Section 2.4, the computation time varies linearly with the number of panels in the structure. The computation time is inferior to 20 milliseconds for a mesh with 10,000 faces, a high

		Isothermic	Planar curves	Stress lines	Circular mesh
Canal surface	revolution	Yes	Yes	Yes	Yes
	M-revolution				
	General case	No			
C-canal surface	C-revolution (moulding)	Yes			
	CM-revolution				
	General case	No			
MC-canal surface	M-moulding	Yes	No	No	
	MCM-revolution				
	General case	No			

Table 2: Properties of super-canal surfaces.

688 number of faces for applications in architecture. Notice⁷¹³
689 that no pre-factorisation is required for the computation⁷¹⁴
690 of the Combescure transformation: the computation time⁷¹⁵
691 are the one experienced by the user. This is in accordance⁷¹⁶
692 with the will to offer a maximal flexibility for the design.⁷¹⁷
693 Applying successive Combescure transformations is there⁷¹⁸
694 fore possible in real-time applications with our algorithm.⁷¹⁹
695 Finally, we notice that Combescure transformations re⁷²⁰
696 quire more time than the other operations and generally⁷²¹
697 governs the overall performance of the method. Notice also⁷²²
698 that Combescure transformations and Möbius transforma⁷²³
699 tions have to be applied twice in the reverse engineering
700 methods presented in the previous section.

701 6.2. Properties of the structural layout

702 Table 2 sums up the different properties of the sur-
703 faces created with our framework. As one applies Möbius
704 and Combescure transformations and extends the formal
705 freedom, some properties are lost a priori. All the shapes
706 can nevertheless be parametrised as circular meshes. As
707 an example, it can be noticed that the lines of curvature
708 are not necessarily lines of principal stresses under uniform
709 pressure for MC-canal surfaces. In the most general case,⁷²⁴
710 there is also no guaranty that there is a family of planar
711 curves.

712 Among other remarkable properties, it may be noticed

that the images of surfaces of revolution are isothermic surfaces, so that it is possible to parametrise them with conformal squares. Optimisation of the parametrisation of isothermic surfaces towards visually pleasant meshes could thus be done in the manner of [37].

Figure 23 shows a M-moulding surface. It was generated to fit two input curves, in the manner of M-revolution surfaces. The shape is visually not different from C-canal surfaces, but the curves are not planar, which increases the complexity. The analysis of Table 2 shows that C-canal surfaces are probably the best trade-off between design freedom and properties of the structural layout.

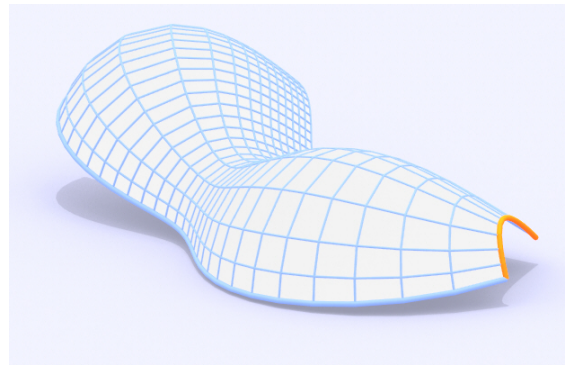


Figure 23: A M-moulding surface passing through two prescribed curves (thick lines). Only the orange curve is rigorously planar.

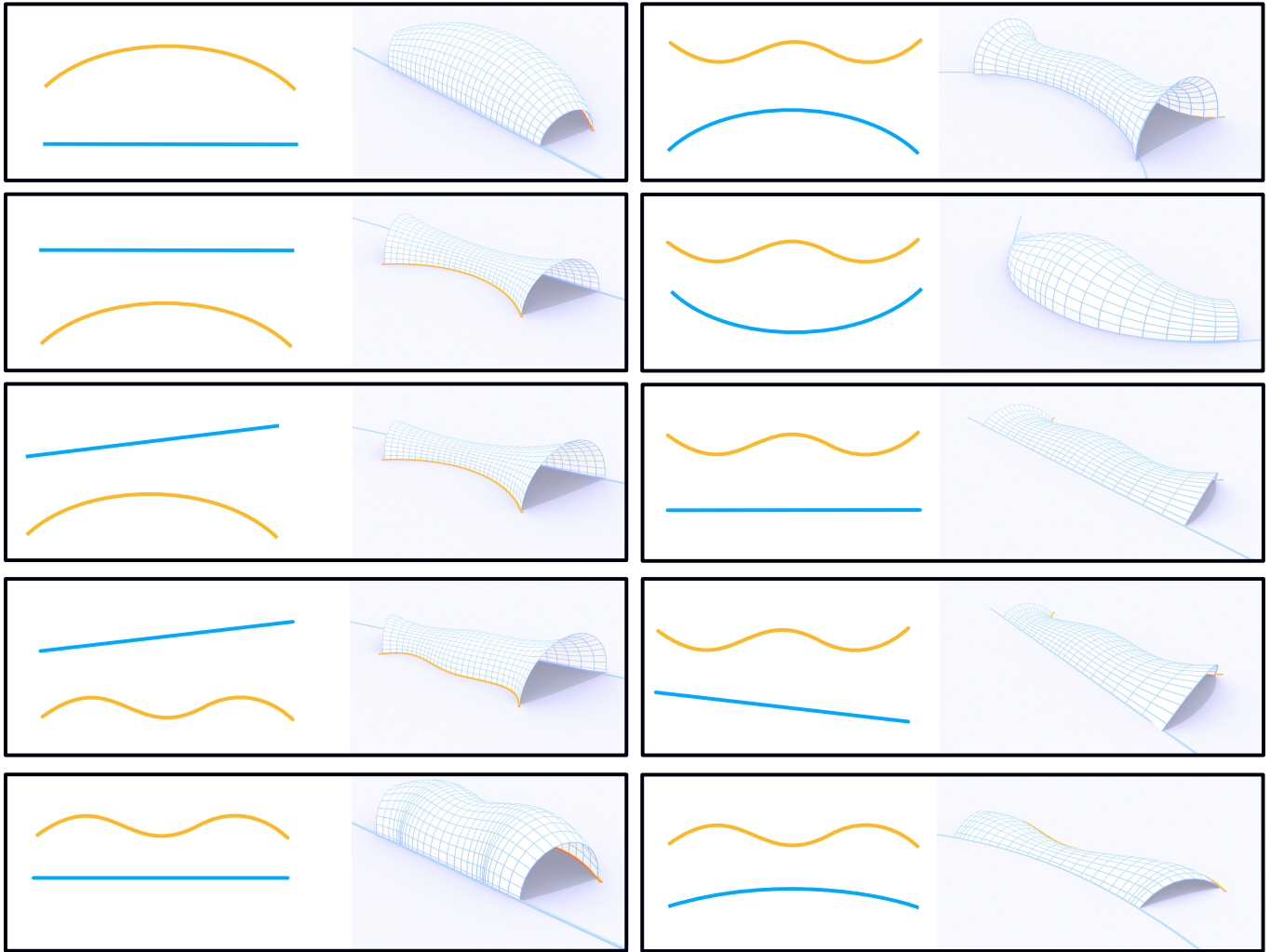


Figure 24: Canal surfaces obtained by fitting two planar curves, the orange curve is a line of curvature of the surface.

725 *6.3. Shape explorations: potential and limitations*

726 The generation super-canal surfaces is subject to mod-
 727 elling limitations discussed in this section. Figure 24 dis-
 728 plays some canal surfaces generated by solving the two-
 729 curves fitting problem. The orange curve is a line of cur-
 730 vature of the resulting surface. The examples show that
 731 it is possible to properly fit a line with a doubly-curved
 732 shape, which should be of interest for practical applica-
 733 tions.

734 These examples also highlight some limitations of
 735 super-canal surfaces. Consider for example the shape in
 736 Figure 25, where there is a noticeable shrinkage of panels.
 737 This concentration of lines of curvature is linked to the

properties of the *evolute* of the guiding curve.

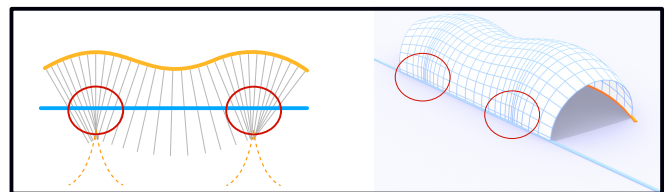


Figure 25: Concentration of lines of curvatures (circled in red) and plane view

739 The evolute of a curve is the locus of the center of its
 740 osculating circles. The lines obtained by sweeping along a
 741 curve intersect along the evolute, as shown in Figure 25. In
 742 Figure 25, it appears that the second curve used as an in-
 743 put for the fitting problem (in blue) is close to the evolute

of the first curve (in orange). As a consequence, the planes containing the circles used for the shape generation (the grey lines in the top view) converge towards the evolute and result in panel shrinkage. This practical limitation also exists for more general super-canal surfaces. The designer must limit the curvature of the control curves in order to avoid self-intersections of the surface he or she is generating.

7. Conclusion

This paper presented a new family of shapes for the rationalization of free-form structures and envelopes. It enriches the formal vocabulary of geometrically-constrained design approaches, and has many relevant applications, from pneumatic structures to gelzed gridshells. A connection to shell theory was recalled and showed that the surfaces created with this method are at equilibrium under normal uniform load. Moreover, the lines of curvatures that are used for discretisation of super-canal surfaces correspond to lines of principal stresses, making the meshes efficient for both fabrication and structural performance. The tools developed for the shape generation were used in the practical context of a workshop for architecture and engineering students.

The methodology for shape generation relies heavily on Möbius geometry, the geometry of circles in space. It studies the transformations of shapes by Combescure and Möbius transformations, and in that sense, it is a generalisation of Möbius geometry, which is only interested in the latter one. Like many other geometrically-constrained shapes, super-canal surfaces are generated from three curves. The underlying construction rule is more sophisticated than simple affine transformation, like translation or scaling. It also gives more degrees of freedom than scale-trans surfaces. Lack of design tools for designers and architects for complex structures has been shown by William Baker in a plenary talk at the Symposium of the IASS in 2015, which is a real prejudice to

the construction industry. Super-canal surfaces transcribe complex geometrical notions into tools easily usable as a design tool by architects and engineers as they provide insight on the buildability, the mechanical behavior under normal load of free-form structures. This illustrates the interest of using Möbius geometry for geometrical modelling in architecture. Other families of shapes could arise from this framework.

The shapes proposed in this paper could be combined with more general modelling techniques, for example see [38], who studies deformations of circular meshes by combination of compatible Möbius transformations.

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