Coherent confluence modulo relations and double groupoids

Benjamin Dupont, Philippe Malbos

COHERENT CONFLUENCE MODULO RELATIONS AND DOUBLE GROUPOIDS

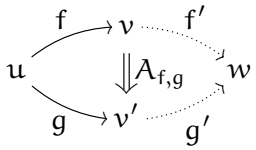
BENJAMIN DUPONT – PHILIPPE MALBOS

Abstract – A coherent presentation of an n-category is a presentation by generators, relations and relations among relations. Completions of presentations by rewriting systems give coherent presentations, whose relations among relations are generated by confluence diagrams induced by critical branchings. This paper extends this construction to presentations by polygraphs defined modulo a set of relations. Our coherence results are formulated using the structure of n-category enriched in double groupoids, whose horizontal cells represent rewriting sequences, vertical cells represent the congruence generated by relations modulo and square cells represent coherence cells induced by confluences modulo. We illustrate our method on a diagrammatic presentation of a pivotal monoidal category modulo the isotopy relations.

Keywords – Rewriting modulo, double categories, coherence of higher categories.
M.S.C. 2010 – 68Q42, 18D05.

INTRODUCTION

Coherence by confluence. Coherent presentations of a monoid extend the notion of presentations of the monoid by globular homotopy generators taking into account the relations among the relations, that is the 2-syzygies of the presentation. Explicitly, a coherent presentation is defined by a set X of generators, a set R of relations and an acyclic set Γ of 2-spheres of the free $(2,1)$-category $R^\top$ generated by the presentation (X, R), where acyclicity means that the quotient of the $(2,1)$-category $R^\top$ by the congruence generated by Γ is aspherical. This notion of coherent presentation extends to n-categories presented by $(n+1)$-polygraphs: a coherent presentation is an $(n+2, n)$-polygraph, that is an $(n+1)$-polygraph P_{n+1} extended by an acyclic cellular extension P_{n+2} of the free $(n+1, n)$-category on P_{n+1}. When the $(n+1)$-polygraph is convergent, that is confluent and terminating, it can be extended into a coherent presentation by adding generating $(n+2)$-cells defined by a family of confluence diagrams



for every critical branching (f, g) of the polygraph P_{n+1}. Coherent presentations constructed in this way generalize rewriting systems by keeping track of the cells generated by confluence diagrams. This method was initiated by Squier in [23] for monoids and generalized to n-categories in [12].

Coherence by confluence modulo. The aim of this paper is to extend these constructions to presentations of n-categories by rewriting systems defined modulo a set of fixed relations. One of the motivations is to reduce the set of critical branchings that should be considered in the analysis of coherence by confluence. For that, we introduce the notion of $(n+1)$-*polygraph modulo* as a data (R, E, S) made of an $(n+1)$-polygraph R, whose generating $(n+1)$-cells are called *primary rules*, an $(n+1)$-polygraph E such that $E_k = R_k$ for $k \leqslant n-1$ and $E_n \subseteq R_n$, whose generating $(n+1)$-cells are called *modulo rules*, and S is a cellular extension of R_n^* depending on both cellular extensions R_{n+1} and E_{n+1}. In this way, the rules are split into two parts: oriented rules in a set R_{n+1} and non-oriented equations in a set E_{n+1}. The most naive approach of rewriting modulo is to consider S to be ${}_E R_E$ consisting in rewriting on congruence classes modulo E, but this appears inefficient for analysis of confluence, see [1, Chapter 11]. Another approach of rewriting modulo has been considered by Huet in [15] where rewriting paths does involve only oriented rules and no equivalence steps, and confluence is formulated modulo equivalence. However, we will mainly use algebraic rewriting modulo by setting the inherent algebraic awioms into the modulo rules, so this notion is too restrictive for computations, see [17]. Jouannaud and Kirchner enlarged this approach in [16] by introducing rewriting modulo properties for any rewriting system between R and ${}_E R_E$. They also provided a completion procedure for confluence of the polygraph modulo ${}_E R$, which was generalized by Bachmair and Dershowitz in [2]. Several other approaches have also been developed for term rewriting systems modulo to deal with various equational theories, see [20, 25].

In Section 3, we define rewriting properties on polygraphs modulo. In particular, we recall the notion of confluence modulo as introduced by Huet, [15] and Jouannaud and Kirchner in [16]. In particular, we recall Huet's principle of double induction from [15] which we will use in many proofs, and we give a completion procedure for a polygraph modulo $(R, E, {}_E R)$.

Confluence modulo and double categories. Our coherence results modulo for an n-category are formulated using the structure of n-category enriched in double groupoids. The notion of double category was first introduced by Ehresmann in [11] as an internal category in the category of categories. Such a categorical structure gives four related categories: a vertical category, an horizontal category and categories of squares with either vertical or horizontal sources and targets, whether they compose horizontally or vertically. A square is pictured by

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ {\scriptstyle e}\downarrow & \Downarrow A & \downarrow{\scriptstyle e'} \\ u' & \xrightarrow[g]{} & v' \end{array}$$

The horizontal cells are the $(n+1)$-cells of the free $(n+1)$-category S^* generated by the $(n+1)$-polygraph S and the vertical cells are the $(n+1)$-cells of the free $(n+1,n)$-category $E^\top$ generated by the $(n+1)$-polygraph E. In this way, horizontal cells represent rewriting sequences, vertical cells represent the congruence generated by modulo relations and square cells represent *coherence cells modulo*. A *branching modulo* E of an $(n+1)$-polygraph S defined modulo an $(n+1)$-polygraph E is a triple (f,e,g), where f and g are $(n+1)$-cells of the free $(n+1)$-category S^* and e is an $(n+1)$-cell of the free $(n+1,n)$-category $E^\top$ generated by the $(n+1)$-polygraph E:

$$\begin{array}{ccc} u & \xrightarrow{f} & u' \\ {\scriptstyle e}\downarrow & & \\ v & \xrightarrow[g]{} & v' \end{array}$$

Such a branching is *confluent modulo* E if there exist $(n+1)$-cells f' and g' in S^* and an $(n+1)$-cell e' in $E^\top$ as in the following diagram:

$$\begin{array}{ccccc} u & \xrightarrow{f} & u' & \overset{f'}{\dashrightarrow} & u'' \\ {\scriptstyle e}\downarrow & & & & \vdots{\scriptstyle e'} \\ v & \xrightarrow[g]{} & v' & \underset{g'}{\dashrightarrow} & w'' \end{array}$$

Coherent confluence modulo. The notion of coherence modulo introduced in this paper is based on an adaptation of the structure of polygraph known in the globular setting, [5, 22, 24], to a cubical setting. We define a *double* $(n+2,n)$- *polygraph* as a data $P = (P^v, P^h, P^s)$ made of two $(n+1)$-polygraphs P^v and P^h with the same underlying n-polygraph, and a square extension P^s made of square of the form

$$\begin{array}{ccc} u & \xrightarrow{f} & u' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ v & \xrightarrow[g]{} & v' \end{array}$$

where f, g are $(n+1)$-cells of the free $(n+1,n)$-category $(P^v)^\top$ generated by P^v and e, e' are $(n+1)$-cells of the free $(n+1,n)$-category $(P^h)^\top$ generated by P^h. We define a *2-fold coherent presentation of an* n*-category* $\mathbf{C}$ as a double $(n+2,n)$-polygraph $P = (P^v, P^h, P^s)$ such that the coproduct of the polygraphs P^v and P^h is a presentation of the n-category $\mathbf{C}$ and that the square extension P^s is acyclic,

that is for any square S constructed on the vertical $(n+1,n)$-category $(P^v)^\top$ and the horizontal $(n+1,n)$-category $(P^h)^\top$, there exists a square $(n+2)$-cell A in the free n-category enriched in double groupoids generated by P whose boundary is S.

In Section 4, we introduce the notion of coherent confluence modulo obtained by adjunction of a square cell in each diagram of confluence modulo. Then we prove coherent confluence of an n-polygraph modulo from local coherent confluence properties. The first result of this section, Theorem 4.1.4, is a formulation of the Newman lemma for confluence modulo. Explicitly, given (R,E,S) an n-polygraph modulo such that ${}_E R_E$ is terminating, and Γ be a square extension of $(E^\top, S^*)$. If S is locally Γ-confluent modulo E then it is Γ-confluent modulo E. Then Theorem 4.2.2 gives a coherent formulation of the critical branching lemma modulo, deducing coherent local confluence from coherent confluence of some critical branchings modulo.

Coherent completion modulo. In Section 5, we show how to construct a two-fold coherent presentation of an $(n-1)$-category $\mathcal{C}$ starting with a presentation of this $(n-1)$-category by an n-polygraph modulo. Theorem 5.2.2 shows that for an n-polygraph modulo (R,E,S) and a square extension Γ on the vertical $(n+1,n)$-category $E^\top$ and horizontal $(n+1,n)$-category $S^\top$ such that E is convergent, S is Γ-confluent modulo E, $\mathrm{Irr}(E)$ is E-normalizing with respect to S, and ${}_E R_E$ is terminating, then the square extension $\Gamma \cup \mathrm{Cd}(E)$ is acyclic. From this result, when E is empty, we recover Squier's theorem for convergent n-polygraphs as given in [12, Theorem 5.2.], see also [13].

We prove that an acyclic extension of a pair $(E^\top, S^\top)$ of $(n+1,n)$-categories coming from a polygraph modulo (R,E,S) can also be obtained from an assumption of commuting normalization strategies for the polygraphs S and E. In particular, we show how to prove the property of E-normalization from an additional condition on the commutation of these normalizations strategies. The last result of this article, Theorem 5.3.3 proves that given an n-polygraph modulo (R,E,S), a square extension Γ of the pair of $(n+1,n)$-categories $(E^\top, S^\top)$, σ and ρ be normalization strategies for S and E respectively, such that E is convergent, S is Γ-confluent modulo E, ${}_E R_E$ is terminating, σ and ρ are weakly commuting, then the square extension $\Gamma \cup \mathrm{Cd}(E) \cup N(\sigma,\rho)$ is acyclic.

Finally, we illustrate in subsection 5.4 our method on diagrammatic presentation of a pivotal monoidal category modulo some isotopy relations.

Organisation of the article. In Section 1, we recall all the notions about higher-dimensional n-categories and n-polygraphs that we will need in the sequel. We refer the reader to [12] for deeper details on these notions. We recall from [11] the notions of double categories and of double groupoids. In Section 2 we define the notion of double polygraphs, giving 2-fold coherent presentations of globular n-categories. We explicit following [10] a construction of a free n-category enriched in double groupoids generated by a double n-polygraph, in which the coherence results will be formulated. In Section 3, we define the notion of polygraph modulo as presentations of categories modulo relations, and we formulate the rewriting properties of termination, branchings, confluence and local confluence for these polygraphs. Following [2] we give a completion procedure in terms of critical branchings for confluence modulo of the polygraph modulo ${}_E R$. In Section 4, we introduce the notion of coherent confluence modulo and we prove a coherent version of Newman's lemma and critical branching lemma for polygraphs modulo. Finally, in Section 5, we define the notion of coherent completion modulo, and show how to construct a 2-fold coherent presentations of n-categories presented by polygraphs modulo from a such a coherent completion. We illustrate the results of this section on a toy example of a diagrammatic rewriting system presenting a pivotal monoidal category, using rewriting modulo isotopy relations.

1. PRELIMINARIES

In this preliminary section, we give notations on higher-dimensional categories used in this paper. In particular, we recall the structure of polygraph from [5, 22, 24] and we refer the reader to [12, 14] for rewriting properties of polygraphs. We recall the notion of double categories from [11] and we refer the reader to [3, 9, 10] for deeper presentations on double categories and double groupoids.

1.1. Higher-dimensional categories and polygraphs

Throughout this paper, n denotes either a fixed natural number or ∞.

1.1.1. Higher-dimensional categories. If $\mathcal{C}$ is a (small, strict and globular) n-category, we denote by $\mathcal{C}_k$ the set of k-cells of $\mathcal{C}$. If f is a k-cell of $\mathcal{C}$, then $\partial_{-,i}(f)$ and $\partial_{+,i}(f)$ respectively denote the i-source and i-target of f, while $(k-1)$-source and $(k-1)$-target will be denoted by $\partial_-(f)$ and $\partial_+(f)$ respectively. The source and target maps satisfy the *globular relations*:

$$\partial_{\alpha,i}\partial_{\alpha,i+1} \;=\; \partial_{\alpha,i}\partial_{\beta,i+1}, \tag{1.1.2}$$

for all α, β in $\{-,+\}$. Two k-cells f and g are i-*composable* when $\partial_{+,i}(f) = \partial_{-,i}(g)$. In that case, their i-composite is denoted by $f \star_i g$, or by fg when $i = 0$. The compositions satisfy the *exchange relations*:

$$(f_1 \star_i g_1) \star_j (f_2 \star_i g_2) \;=\; (f_1 \star_j f_2) \star_i (g_1 \star_j g_2). \tag{1.1.3}$$

for all $i \neq j$ and for all cells f_1, f_2, g_1, g_2 such that both sides are defined. If f is a k-cell, we denote by 1_f its identity $(k+1)$-cell. When 1_f is composed with l-cells, we simply denote it by f for $l \geqslant k+1$.

A k-cell f of an n-category $\mathcal{C}$ is i-*invertible* when there exists a (necessarily unique) k-cell g in $\mathcal{C}$, with i-source $\partial_{+,i}(f)$ and i-target $\partial_{-,i}(f)$, called the i-*inverse of* f, that satisfies

$$f \star_i g \;=\; 1_{\partial_{-,i}(f)} \qquad \text{and} \qquad g \star_i f \;=\; 1_{\partial_{+,i}(f)}.$$

When $i = k-1$, we just say that f is *invertible* and we denote by f^- its *inverse*. As in higher-dimensional groupoids, if a k-cell f is invertible and if its i-source u and i-target v are invertible, then f is $(i-1)$-invertible, with $(i-1)$-inverse given by $v^- \star_{i-1} f^- \star_{i-1} u^-$.

For a natural number $p \leqslant n$, or for $p = n = \infty$, an (n,p)-*category* is an n-category whose k-cells are invertible for every $k > p$. When $n < \infty$, this is an n-category enriched in $(n-p)$-groupoids and, when $n = \infty$, an n-category enriched in ∞-groupoids. In particular, an (n,n)-category is an n-category, and an $(n,0)$-category is an n-*groupoid*, also called a *groupoid* for $n = 1$.

A 0-*sphere of* $\mathcal{C}$ is a pair $\gamma = (f,g)$ of 0-cells of $\mathcal{C}$ and, for $1 \leqslant k \leqslant n$, a k-*sphere of* $\mathcal{C}$ is a pair $S = (f,g)$ of k-cells of $\mathcal{C}$ such that $\partial_-(f) = \partial_-(g)$ and $\partial_+(f) = \partial_+(g)$. The k-cell f (resp. g) is called the *source* (resp. target) of S denoted by $\partial_-(S)$ (resp. $\partial_+(S)$). We will denote by $\mathrm{Sph}_k(\mathcal{C})$ the set of k-spheres of $\mathcal{C}$. If f is a k-cell of $\mathcal{C}$, for $1 \leqslant k \leqslant n$, the *boundary of* f is the $(k-1)$-sphere $(s(f), t(f))$ denoted by $\partial(f)$.

1.1.4. Cellular extensions. Suppose $n < \infty$, a *cellular extension* of an n-category $\mathcal{C}$ is a set Γ equipped with a map $\gamma : \Gamma \to \mathrm{Sph}_n(\mathcal{C})$. By considering all the formal compositions of elements of Γ, seen as $(n+1)$-cells with source and target in $\mathcal{C}$, one builds the *free* $(n+1)$*-category generated by* Γ *over* $\mathcal{C}$, denoted by $\mathcal{C}[\Gamma]$. The *size* of an $(n+1)$-cell f of $\mathcal{C}[\Gamma]$ is the number denoted by $||f||_\Gamma$, of $(n+1)$-cells of Γ it contains. We denote by $\mathcal{C}^{(1)}$ the set of n-cells in $\mathcal{C}$ of size 1. We denote by $\mathcal{C}/\Gamma$ the *quotient of* $\mathcal{C}$ *by* Γ, i.e., the n-category one gets from $\mathcal{C}$ by identification of the n-cells $\partial_-(S)$ and $\partial_+(S)$, for all n-sphere S of Γ.

If $\mathcal{C}$ is an (n,p)-category and Γ is a cellular extension of $\mathcal{C}$, then the *free* $(n+1,p)$*-category generated by* Γ *over* $\mathcal{C}$ is denoted by $\mathcal{C}(\Gamma)$ and defined as follows:

$$\mathcal{C}(\Gamma) \;=\; \mathcal{C}\left[\Gamma,\ \Gamma^-\right] / \,\mathrm{Inv}(\Gamma)$$

where Γ^- contains the same $(n+1)$-cells as Γ, with source and target reversed, and $\mathrm{Inv}(\Gamma)$ is the cellular extension of $[\Gamma,\ \Gamma^-]$ made of two $(n+2)$-cells

$$f \star_{n+1} f^- \;\to\; 1_{\partial_-(f)} \qquad \text{and} \qquad f^- \star_{n+1} f \;\to\; 1_{\partial_+(f)}$$

for every $(n+1)$-cell f in Γ.

Let $\mathcal{C}$ be an (n,p)-category, for $p < n < \infty$. A cellular extension Γ of $\mathcal{C}$ is *acyclic* if the (n,p)-category $\mathcal{C}/\Gamma$ is aspherical, i.e., such that, for every n-sphere S of $\mathcal{C}$, there exists an $(n+1)$-cell with boundary S in the $(n+1,p)$-category $\mathcal{C}(\Gamma)$.

1.1.5. Polygraphs. Recall that an n*-polygraph* is a data $P = (P_0, P_1, \dots, P_n)$ made of a set P_0 and, for every $0 \leqslant k < n$, a cellular extension P_{k+1} of the free k-category

$$P_k^* \;:=\; P_0[P_1]\dots[P_k].$$

For $0 \leqslant k \leqslant n-1$, we will denote by $P_{\leqslant k}$ the underlying k-polygraph $(P_0, P_1, \dots, P_k)$. We will denote by P^* (resp. $P^\top$) the free n-category (resp. $(n, n-1)$-category) generated by an n-polygraph P. We will denote by $\overline{P}$ the $(n-1)$*-category presented by the polygraph* P, that is $\overline{P} := P_{n-1}^*/P_n$.

Given two n-polygraph P and Q, a *morphism* of n-polygraphs from P to Q is a pair (ξ_{n-1}, f_n) where ξ_{n-1} is a morphism of $(n-1)$-polygraphs from P_{n-1} to Q_{n-1}, and where f_n is a map from P_n to Q_n such that the following diagrams commute:

$$\begin{array}{ccc} P_{n-1}^* & \xleftarrow{s_{n-1}^P} & P_n \\ {\scriptstyle F_{n-1}(\xi_{n-1})}\downarrow & & \downarrow{\scriptstyle f_n} \\ Q_{n-1}^* & \xleftarrow[s_{n-1}^Q]{} & Q_n \end{array} \qquad \begin{array}{ccc} P_{n-1}^* & \xleftarrow{t_{n-1}^P} & P_n \\ {\scriptstyle F_{n-1}(\xi_{n-1})}\downarrow & & \downarrow{\scriptstyle f_n} \\ Q_{n-1}^* & \xleftarrow[t_{n-1}^Q]{} & Q_n \end{array}$$

Equivalently, it is a sequence of maps $(f_k : P_k \to Q_k)_k$ indexed by integer $0 \leqslant k \leqslant n-1$ such that the conditions

$$f_k s_k^P = s_k^Q f_{k+1} \quad \text{and} \quad f_k t_k^P = t_k^Q f_{k+1}$$

holds for all $0 \leqslant k \leqslant n-1$.

For $p \leqslant n$, an (n,p)-*polygraph* is a data P made of an n-polygraph $(P_0, \dots, P_n)$, and for every $p \leqslant k < n$, a cellular extension P_{k+1} of the free (k,p)-category

$$P_k^\top := P_p^*(P_{n+1}) \cdots (P_k).$$

Note that an (n,n)-polygraph is an n-polygraph.

Given an n-polygraph P, recall from [12, Proposition 2.1.5] that every n-cell f in P^* with size $k \geqslant 1$ has a decomposition

$$f = C_1[\gamma_1] \star_{n-1} \cdots \star_{n-1} C_k[\gamma_k].$$

where $\gamma_1, \dots, \gamma_k$ are n-cells in P and $C_1, \dots, C_k$ are whiskers of P^*.

1.1.6. Contexts in n-categories. A *context* of an n-category $\mathcal{C}$ is a pair (S, C) made of an $(n-1)$-sphere S of $\mathcal{C}$ and an n-cell C in $\mathcal{C}[S]$ such that $||C||_S = 1$. We often denote simply by C, such a context. Recall from [12, Proposition 2.1.3] that every context of $\mathcal{C}$ has a decomposition

$$f_n \star_{n-1} (f_{n-1} \star_{n-2} \cdots (f_1 \star_0 S \star_0 g_1) \cdots \star_{n-2} g_{n-1}) \star_{n-1} g_n,$$

where S is an $(n-1)$-sphere and, for every k in $\{1, \dots, n\}$, f_k and g_k are n-cells of $\mathcal{C}$. Moreover, one can choose these cells so that f_k and g_k are (the identities of) k-cells. A *whisker of* $\mathcal{C}$ is a context with a decomposition

$$f_{n-1} \star_{n-2} \cdots (f_1 \star_0 S \star_0 g_1) \cdots \star_{n-2} g_{n-1}$$

such that, for every k in $\{1, \dots, n-1\}$, f_k and g_k are k-cells.

1.2. Double groupoids

In this subsection, we recall the notion of double category introduced in [11]. It can be defined as an internal category in the category Cat of all (small) categories and functors. Recall that given $\mathcal{V}$ be a category with finite limits, an *internal category* $\mathbf{C}$ in $\mathcal{V}$ is a data $(\mathbf{C}_1, \mathbf{C}_0, \partial_-^{\mathbf{C}}, \partial_+^{\mathbf{C}}, \circ_{\mathbf{C}}, i_{\mathbf{C}})$, where

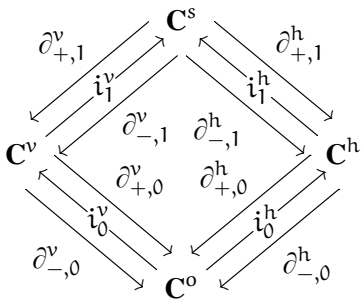
$$\partial_-^{\mathbf{C}}, \partial_+^{\mathbf{C}} : \mathbf{C}_1 \longrightarrow \mathbf{C}_0, \qquad i_{\mathbf{C}} : \mathbf{C}_0 \longrightarrow \mathbf{C}_1, \qquad \circ_{\mathbf{C}} : \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \longrightarrow \mathbf{C}_1$$

are morphisms of $\mathcal{V}$ satisfying the usual axioms of a category, and where $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$ denotes the pullback in $\mathcal{V}$ over morphisms $\partial_-^{\mathbf{C}}$ and $\partial_+^{\mathbf{C}}$. An internal functor from $\mathbf{C}$ to $\mathbf{D}$ is a pair of morphisms $\mathbf{C}_1 \to \mathbf{D}_1$ and $\mathbf{C}_0 \to \mathbf{D}_0$ in $\mathcal{V}$ commuting in the obvious way. We denote by $\mathsf{Cat}(\mathcal{V})$ the category of internal categories in $\mathcal{V}$ and their functors.

1.2.1. Double categories and double groupoids. The category of *double categories* is defined as the category $\mathsf{Cat}(\mathsf{Cat})$, and the category of *double groupoids* is defined as the category $\mathsf{Cat}(\mathsf{Grpd})$ of internal categories in the category Grpd of groupoids and their functors. A double groupoid is thus a double category whose the four related categories are groupoids and the source, target, composition and identity maps are morphisms of groupoids. Explicitly, a double category is an internal category $(\mathbf{C}_1, \mathbf{C}_0, \partial_-^{\mathbf{C}}, \partial_+^{\mathbf{C}}, \circ_{\mathbf{C}}, i_{\mathbf{C}})$ in Cat, that gives four related categories:

$$\mathbf{C}^{sv} := (\mathbf{C}^s, \mathbf{C}^v, \partial_{-,1}^v, \partial_{+,1}^v, \diamond^v, i_1^v), \qquad \mathbf{C}^{sh} := (\mathbf{C}^s, \mathbf{C}^h, \partial_{-,1}^h, \partial_{+,1}^h, \diamond^h, i_1^h),$$
$$\mathbf{C}^{vo} := (\mathbf{C}^v, \mathbf{C}^o, \partial_{-,0}^v, \partial_{+,0}^v, \circ^v, i_0^v), \qquad \mathbf{C}^{ho} := (\mathbf{C}^h, \mathbf{C}^o, \partial_{-,0}^h, \partial_{+,0}^h, \circ^h, i_0^h),$$

where $\mathbf{C}^{sh}$ is the category $\mathbf{C}_1$ and $\mathbf{C}^{vo}$ is the category $\mathbf{C}_0$. The sources, target and identity maps pictured in the following diagram



satisfy the following relations:

i) $\partial^h_{\alpha,0}\partial^h_{\beta,1} = \partial^v_{\beta,0}\partial^v_{\alpha,1}$, for all α, β in $\{-,+\}$,

ii) $\partial^\mu_{\alpha,1} i^\eta_1 = i^\mu_0 \partial^\eta_{\alpha,0}$, for all α in $\{-,+\}$ and μ, η in $\{v, h\}$,

iii) $i^v_1 i^v_0 = i^h_1 i^h_0$,

iv) $\partial^\mu_{\alpha,1}(A \diamond^\mu B) = \partial^\mu_{\alpha,1}(A) \circ^\mu \partial^\mu_{\alpha,1}(B)$, for all $\alpha \in \{-,+\}$, $\mu \in \{v, h\}$ and any squares A, B such that both sides are defined,

v) *middle four interchange law* :

$$(A \diamond^v A') \diamond^h (B \diamond^h B') = (A \diamond^h B) \diamond^v (A' \diamond^h B'), \tag{1.2.2}$$

for any cells A, A', B, B' in $\mathbf{C}^s$ such that both sides are defined.

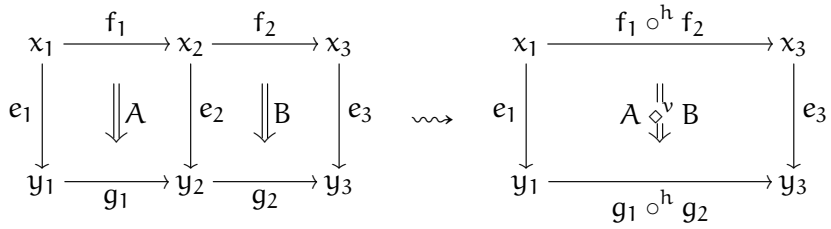
Elements of $\mathbf{C}^o$ are called *point cells*, the elements of $\mathbf{C}^h$ and $\mathbf{C}^v$ are respectively called *horizontal cells* and *vertical cells* and pictured by

$$x_1 \xrightarrow{f} x_2 \qquad \begin{array}{c} x_1 \\ e \big\downarrow \\ x_2 \end{array}$$

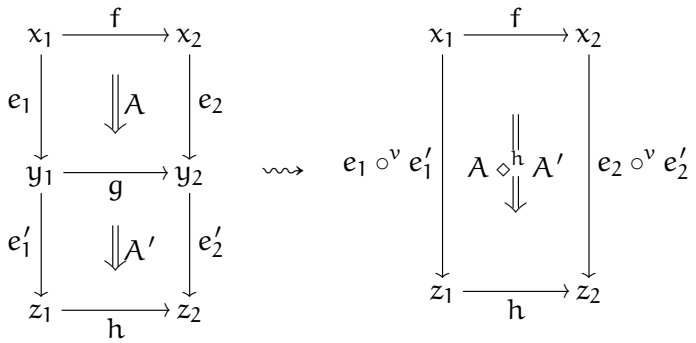
Following relations **i)**, the elements of $\mathbf{C}^s$ are called *square cells* and can be pictured by squares as follows:

$$\begin{array}{ccc} & \partial^h_{-,1}(A) & \\ \partial^v_{-,1}(A) \big\downarrow & \Downarrow A & \big\downarrow \partial^v_{+,1}(A) \\ & \partial^h_{+,1}(A) & \end{array} \qquad \begin{array}{ccc} x_1 & \xrightarrow{f} & x_2 \\ i^v_0(x_1) \big\downarrow & \Downarrow i^h_1(f) & \big\downarrow i^v_0(x_2) \\ x_1 & \xrightarrow[f]{} & x_2 \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{i^h_0(x)} & x \\ e \big\downarrow & \Downarrow i^v_1(e) & \big\downarrow e \\ y & \xrightarrow[i^h_0(y)]{} & y \end{array}$$

The compositions $\diamond^v$ (resp. $\diamond^h$) are called respectively *vertical* and *horizontal compositions*, and can be pictured as follows

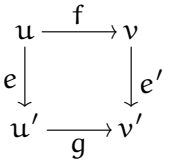


for all x_i, y_i in $\mathbf{C}^o$, f_i, g_i in $\mathbf{C}^h$, e_i in $\mathbf{C}^v$ and A, B in $\mathbf{C}^s$,



for all x_i, y_i, z_i in $\mathbf{C}^o$, f, g, h in $\mathbf{C}^h$, e_i, e'_i in $\mathbf{C}^v$ and A, A' in $\mathbf{C}^s$.

1.2.3. Squares. A *square* of a double category $\mathbf{C}$ is a quadruple (f, g, e, e') such that f, g are horizontal cells and e, e' are vertical cells that compose as follows:



The *boundary* of a square cell A in $\mathbf{C}$ is the square $(\partial_{-,h}(A), \partial_{+,h}(A), \partial_{-,v}(A), \partial_{+,v}(A))$, denoted by $\partial(A)$. We will denote by $\mathrm{Sqr}(\mathbf{C})$ the set of square cells of $\mathbf{C}$.

1.2.4. n-categories enriched in double categories. The coherence results for rewriting systems modulo presented in this paper are formulated using the notion of n-categories enriched in double groupoids. Let us expand this notion for $n > 0$. Consider the category $\mathsf{Cat}(\mathsf{Grpd})$ equipped with the cartesian product defined by

$$\mathbf{C} \times \mathbf{D} = (\mathbf{C}_1 \times \mathbf{D}_1, \mathbf{C}_0 \times \mathbf{D}_0, s_{\mathbf{C}} \times t_{\mathbf{C}}, c_{\mathbf{C}} \times c_{\mathbf{D}}, i_{\mathbf{C}} \times i_{\mathbf{D}}),$$

for any double groupoids $\mathbf{C}$ and $\mathbf{D}$. The terminal double groupoid $\mathbf{T}$ has only one point cell, denoted by $\bullet$, and identities $i^v_0(\bullet)$, $i^h_0(\bullet)$, $i^v_1 i^h_0(\bullet) = i^h_1 i^v_0(\bullet)$.

An n-category enriched in double groupoids is an n-category $\mathcal{C}$ such that for any x, y in $\mathcal{C}_{n-1}$ the homset $\mathcal{C}_n(x,y)$ has a double groupoid structure, whose point cells are the n-cells in $\mathcal{C}_n(x,y)$. We will denote by $\mathcal{C}^v_{n+1}$ (resp. $\mathcal{C}^h_{n+1}$, $\mathcal{C}^s_{n+2}$) the sets $\mathcal{C}_n(x,y)^v$ (resp. $\mathcal{C}_n(x,y)^h$, $\mathcal{C}_n(x,y)^s$) for all x, y in $\mathcal{C}_{n-1}$. An $(n+2)$-cell A in $\mathcal{C}^s_{n+2}$ can be represented by the following diagrams:

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ {\scriptstyle e}\downarrow & \Downarrow A & \downarrow{\scriptstyle e'} \\ u' & \xrightarrow[g]{} & v' \end{array}$$

with $u, u', v, v' \in \mathcal{C}_n$, $f, g \in \mathcal{C}^h_{n+1}$ and $e, e' \in \mathcal{C}^v_{n+1}$. The compositions and identities are induced by the functors of double categories

$$\star^{x,y,z}_{n-1} : \mathcal{C}_n(x,y) \times \mathcal{C}_n(y,z) \to \mathcal{C}_n(x,z), \qquad 1_x : \mathbf{T} \to \mathcal{C}_n(x,x),$$

for all $(n-1)$-cells x, y, z. The $(n-1)$-composite of an $(n+2)$-cell A in $\mathcal{C}_n(x,y)$ with an $(n+2)$-cell B in $\mathcal{C}_n(y,z)$ such that

$$\begin{array}{ccc} u_1 & \xrightarrow{f_1} & v_1 \\ {\scriptstyle e_1}\downarrow & \Downarrow A & \downarrow{\scriptstyle e'_1} \\ u'_1 & \xrightarrow[g_1]{} & v'_1 \end{array} \qquad \begin{array}{ccc} u_2 & \xrightarrow{f_2} & v_2 \\ {\scriptstyle e_2}\downarrow & \Downarrow B & \downarrow{\scriptstyle e'_2} \\ u'_2 & \xrightarrow[g_2]{} & v'_2 \end{array}$$

is defined by composition along $\star_{n-1}$ and denoted by:

$$\begin{array}{ccc} u_1 \star_{n-1} u_2 & \xrightarrow{f_1 \star_{n-1} f_2} & v_1 \star_{n-1} v_2 \\ {\scriptstyle e_1 \star_{n-1} e_2}\downarrow & \Downarrow A \star_{n-1} B & \downarrow{\scriptstyle e'_1 \star_{n-1} e'_2} \\ u'_1 \star_{n-1} u'_2 & \xrightarrow[g_1 \star_{n-1} g_2]{} & v'_1 \star_{n-1} v'_2 \end{array}$$

By functoriality, the $(n-1)$-composition satisfies the following exchange relations:

$$(A \diamond^\mu A') \star_{n-1} (B \diamond^\mu B') \;=\; (A \star_{n-1} B) \diamond^\mu (A' \star_{n-1} B'), \tag{1.2.5}$$

$$(A \diamond^\mu A') \star_{n-1} (B \diamond^\eta B') \;=\; ((A \star_{n-1} B) \diamond^\mu (A' \star_{n-1} B)) \diamond^\eta ((A \star_{n-1} B') \diamond^\mu (A' \star_{n-1} B')). \tag{1.2.6}$$

Using middle four interchange law (1.2.2), the identity (1.2.6) is equivalent to the following identity

$$(A \diamond^\mu A') \star_{n-1} (B \diamond^\eta B') \;=\; ((A \star_{n-1} B) \diamond^\eta (A \star_{n-1} B')) \diamond^\mu ((A' \star_{n-1} B) \diamond^\eta (A' \star_{n-1} B'))$$

for all $\mu \neq \eta$ in $\{v, h\}$ and any $(n+2)$-cells A, A', B, B' such that both sides are defined.

1.2.7. 2-categories as double categories. From a 2-category $\mathcal{C}$, one can construct two canonical double categories, by defining the vertical or horizontal cells to be only identities in $\mathcal{C}$. In this way, 2-categories can be considered as special cases of double categories. The *quintet construction* gives an other way to associate a double category, called the *double category of quintets in* $\mathcal{C}$ and denoted by $\mathbf{Q}(\mathcal{C})$ to a 2-category $\mathcal{C}$. The vertical and horizontal categories of $\mathbf{Q}(\mathcal{C})$ are both equal to $\mathcal{C}$, and there is a square cell

$$\begin{array}{ccc} u & \xrightarrow{f} & u' \\ {\scriptstyle g}\downarrow & \Downarrow A & \downarrow{\scriptstyle k} \\ v & \xrightarrow[h]{} & v' \end{array}$$

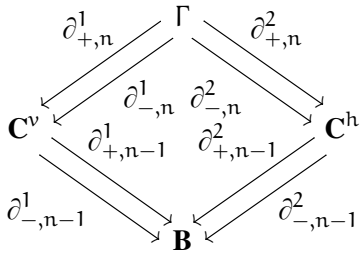
in $\mathbf{Q}(\mathcal{C})$ whenever there is a 2-cell $A : f \star_1 k \Rightarrow g \star_1 h$ in $\mathcal{C}$. This defines a functor $\mathbf{Q} : \mathsf{Cat}_2 \to \mathsf{DbCat}$. Similarly, for $n \geqslant 2$ one can associate to an n-category an $(n-2)$-category enriched in double categories by a quintet construction.

2. Two-fold coherent presentations

In this section, we introduce the notion of two fold coherent presentations for n-categories. This notion is constructed using the structure of double polygraph introduced in Section 2.1 and that extends to double categories the known structure of polygraph on strict globular categories from [5, 22, 24]. We give a construction of the free n-category enriched in double categories generated by a double polygraph.

2.1. Double polygraphs

2.1.1. Square extensions. Let $(\mathbf{C}^v, \mathbf{C}^h)$ be a pair of n-categories with the same underlying $(n-1)$-category $\mathbf{B}$. A *square extension* of the pair $(\mathbf{C}^v, \mathbf{C}^h)$ is a set Γ equipped with four maps $\partial^{\mu}_{\alpha,n}$, with $\alpha \in \{-,+\}$, $\mu \in \{1,2\}$, as depicted by the following diagram:



and satisfying the following relations:

$$\partial^v_{\alpha,n-1}\partial^v_{\beta,n} = \partial^h_{\beta,n-1}\partial^h_{\alpha,n},$$

for all α, β in $\{-,+\}$. The point cells of a square A in Γ are the $(n-1)$-cells of $\mathbf{B}$ of the form

$$\partial^{\mu}_{\alpha,n-1}\partial^{\eta}_{\beta,n}(A)$$

with α, β in $\{-,+\}$, and η, μ in $\{h, v\}$. Note that by construction these four $(n-1)$-cells have the same $(n-2)$-source and $(n-2)$-target in $\mathbf{B}$ respectively denoted by $\partial_{-,n-2}(A)$ and $\partial_{+,n-2}(A)$.

A pair of categories $(\mathbf{C}^v, \mathbf{C}^h)$ has two canonical square extensions, the empty one, and the full one denoted by $\mathrm{Sqr}(\mathbf{C}^v, \mathbf{C}^h)$ that contains all squares on $(\mathbf{C}^v, \mathbf{C}^h)$. We will denote by $\mathrm{Sph}(\mathbf{C}^v, 1)$ (resp. $\mathrm{Sph}(1, \mathbf{C}^h)$) the 2-square extension of $(\mathbf{C}^v, \mathbf{C}^h)$ made of all squares of the form

$$\begin{array}{ccc} u & \xrightarrow{=} & u \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ v & \xrightarrow[=]{} & v \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} u & \xrightarrow{f} & u' \\ \| & & \| \\ \downarrow & & \downarrow \\ u & \xrightarrow[g]{} & u' \end{array}\right)$$

for all n-cells e, e' in $\mathbf{C}^v$ (resp. n-cells in f, g in $\mathbf{C}^h$).

The *Peiffer square extension* of the pair $(\mathbf{C}^v, \mathbf{C}^h)$ is the square extension of $(\mathbf{C}^v, \mathbf{C}^h)$, denoted by $\mathrm{Peiff}(\mathbf{C}^v, \mathbf{C}^h)$, containing the squares of the form

$$\begin{array}{ccc} u \star_i v & \xrightarrow{f \star_i v} & u' \star_i v \\ {\scriptstyle u \star_i e}\downarrow & & \downarrow{\scriptstyle u' \star_i e} \\ u \star_i v' & \xrightarrow[f \star_i v']{} & u' \star_i v' \end{array} \qquad \begin{array}{ccc} w \star_i u & \xrightarrow{w \star_i f} & w \star_i u' \\ {\scriptstyle e' \star_i u}\downarrow & & \downarrow{\scriptstyle e' \star_i u'} \\ w' \star_i u & \xrightarrow[w' \star_i f]{} & w' \star_i u' \end{array}$$

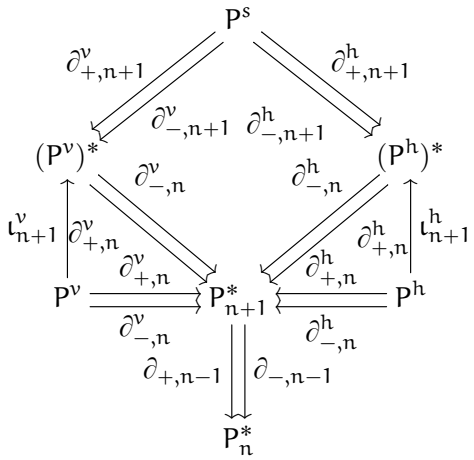
for all n-cells e, e' in $\mathbf{C}^v$ and n-cell f in $\mathbf{C}^h$.

2.1.2. Double polygraphs. For $n \geqslant 0$, we define a *double* n- *polygraph* as a data $P = (P^v, P^h, P^s)$ made of

i) two $(n+1)$-polygraphs P^v and P^h such that $P^v_{\leqslant n} = P^h_{\leqslant n}$,

ii) a square extension P^s of the pair of free $(n+1)$-categories $((P^v)^*, (P^h)^*)$.

Such a data can be pictured by the following diagram



For $0 \leqslant k \leqslant n+1$, the k-cells of the $(n+1)$-polygraphs P^v and P^h are called *generating* k*-cells of* P. The elements of P^s are called *generating square* $(n+2)$*-cells* of P.

2.1.3. The category of double n-polygraphs. Given two double n-polygraphs $P = (P^v, P^h, P^s)$ and $Q = (Q^v, Q^h, Q^s)$, a *morphism of double n-polygraphs* from P to Q is a triple (f^v, f^h, f^s) made of two morphisms of $(n+1)$-polygraphs

$$f^v : P^v \to Q^v, \qquad f^h : P^h \to Q^h,$$

and a map $f^s : P^s \to Q^s$ such that the following diagrams commute:

$$\begin{array}{ccc} P^\mu_{n+1} & \xleftarrow{\partial^{\mu,P}_{-,n-1}} & P^s \\ {\scriptstyle f^\mu_{n+1}}\downarrow & & \downarrow{\scriptstyle f^s} \\ Q^\mu_{n+1} & \xleftarrow[\partial^{\mu,Q}_{-,n-1}]{} & Q^s \end{array} \qquad \begin{array}{ccc} P^\mu_{n+1} & \xleftarrow{\partial^{\mu,P}_{+,n-1}} & P^s \\ {\scriptstyle f^\mu_{n+1}}\downarrow & & \downarrow{\scriptstyle f^s} \\ Q^\mu_{n+1} & \xleftarrow[\partial^{\mu,Q}_{+,n-1}]{} & Q^s \end{array}$$

for μ in $\{v, h\}$. We will denote by DbPol_n the category of double n-polygraphs and their morphisms.

Let us explicit two full subcategories of DbPol_n used in the sequel to formulate coherence and confluence results for polygraphs modulo. We define a *double* $(n+2, n)$*-polygraph* as a double n-polygraph whose square extension P^s is defined on the pair of $(n+1, n)$-categories $((P^v)^\top, (P^h)^\top)$. We denote by $\mathsf{DbPol}_{(n+2,n)}$ the category of double $(n+2, n)$-polygraphs. In some situations, we will also consider double n-polygraphs whose square extension is defined on the pair of $(n+1)$-categories $((P^v)^\top, (P^h)^*)$ (resp. $((P^v)^*, (P^h)^\top)$). We will respectively denote by DbPol^v_n (resp. DbPol^h_n) the full subcategories of DbPol_n they form.

2.2. Two-fold coherent presentations

Recall that a *coherent presentation* of an n-category $\mathcal{C}$ is an $(n+2, n)$-polygraph P such that the n-category $\mathcal{C}$ is isomorphic to the quotient n-category P^*_n/P_{n+1} and P_{n+2} is an acyclic extension of the $(n+1, n)$-category $P^\top_{n+1}$. In this subsection, we define a similar notion for n-categories enriched in double categories.

Let us first explicit the construction of a free n-category enriched in double categories generated by a double n-polygraph. The question of the construction of free double categories was considered in several works, [7–10]. In particular, Dawson and Pare give in [10] constructions of free double categories generated by double graphs and double reflexive graphs. Such free double categories always exists, and they show how to describe their cells explicitly in geometric terms. However, they show that free double categories generated by double graphs can not describe many of the possible compositions in free double categories. They fixed this problem by considering double reflexive graphs as generators. In this work, we consider this approach but using double n-polygraphs as generating data.

2.2.1. Free n-category enriched in double categories. Consider a double n-polygraph $P = (P^v, P^h, P^s)$. We construct the *free n-category enriched in double categories on* P, denoted by $P^\square$, as follows:

i) the underlying n-category of $P^\square$ is the free n-category P^*_n,

ii) for all $(n-1)$-cells x and y of P^*_n, the hom-double category $P^\square(x, y)$ is constructed as follows

a) the point cells of $\mathrm{P}^{\square}(x,y)$ are the n-cells in $\mathrm{P}^{*}_{n}(x,y)$,

b) the vertical cells of $\mathrm{P}^{\square}(x,y)$ are the $(n+1)$-cells of the free $(n+1)$-category $(\mathrm{P}^{v})^{*}$,

c) the horizontal cells of $\mathrm{P}^{\square}(x,y)$ are the $(n+1)$-cells of the free $(n+1)$-category $(\mathrm{P}^{h})^{*}$,

d) the set of square cells of $\mathrm{P}^{\square}(x,y)$ is defined recursively and contains

- the square cells A of P^{s} such that $\partial_{-,n-1}(A) = x$ and $\partial_{+,n-1}(A) = y$,
- the square cells $C[A]$ for any context C of the n-category P^{*}_{n} and A in P^{s}, such that $\partial_{-,n-1}(C[A]) = x$ and $\partial_{+,n-1}(C[A]) = y$,
- identities square cells $i^{h}_{1}(f)$ and $i^{v}_{1}(e)$, for any $(n+1)$-cells f in $(\mathrm{P}^{h})^{*}$ and $(n+1)$-cell e in $(\mathrm{P}^{v})^{*}$ whose $(n-1)$-source (resp. $(n-1)$-target) in P^{*}_{n} is x (resp. y),
- all formal pastings of these elements with respect to $\diamond^{h}$-composition and $\diamond^{v}$-composition.

e) two square cells constructed as such formal pastings are identified by the associativity, and identity axioms of compositions $\diamond^{v}$ and $\diamond^{h}$ and middle four interchange law given in (1.2.2),

iii) for all $(n-1)$-cells x, y, z of P^{*}_{n}, the composition functor

$$\star_{n-1} : \mathrm{P}^{\square}(x,y) \times \mathrm{P}^{\square}(y,z) \longrightarrow \mathrm{P}^{\square}(x,z)$$

is defined for any

$$\begin{array}{ccc} u_1 & \xrightarrow{f_1} & v_1 \\ {\scriptstyle e_1}\downarrow & \Downarrow A_1 & \downarrow{\scriptstyle e'_1} \\ u'_1 & \xrightarrow[g_1]{} & v'_1 \end{array} \quad \text{in } \mathrm{P}^{\square}(x,y) \text{ and } \quad \begin{array}{ccc} u_2 & \xrightarrow{f_2} & v_2 \\ {\scriptstyle e_2}\downarrow & \Downarrow A_2 & \downarrow{\scriptstyle e'_2} \\ u'_2 & \xrightarrow[g_2]{} & v'_2 \end{array} \quad \text{in } \mathrm{P}^{\square}(y,z),$$

by

$$\begin{array}{ccc} u_1 \star_{n-1} u_2 & \xrightarrow{f_1 \star_{n-1} f_2} & v_1 \star_{n-1} v_2 \\ {\scriptstyle e_1 \star_{n-1} e_2}\downarrow & \Downarrow A_1 \star_{n-1} A_2 & \downarrow{\scriptstyle e'_1 \star_{n-1} e'_2} \\ u'_1 \star_{n-1} u'_2 & \xrightarrow[g_1 \star_{n-1} g_2]{} & v'_1 \star_{n-1} v'_2 \end{array}$$

where the square cell $A_1 \star_{n-1} A_2$ is defined recursively using exchanges relations (1.2.5-1.2.6) from functoriality of the composition $\star_{n-1}$, and the middle four identities (1.2.2),

iv) for all $(n-1)$-cell x of P^{*}_{n}, the identity map $\mathbf{T} \longrightarrow \mathrm{P}^{\square}(x,x)$, where $\mathbf{T}$ is the terminal double groupoid mapping on $i^{v}_{1}i^{h}_{1}(x) = i^{h}_{1}i^{v}_{1}(x)$:

$$\begin{array}{ccc} x & \xrightarrow{i^{h}_{0}(x)} & x \\ {\scriptstyle i^{v}_{1}(x)}\downarrow & \Downarrow & \downarrow{\scriptstyle i^{v}_{1}(x)} \\ x & \xrightarrow[i^{h}_{0}(x)]{} & x \end{array}$$

2.2.2. By similar constructions of the free n-category enriched in double categories on a double n-polygraph $P = (P^v, P^h, P^s)$ given in 2.2.1, we can construct the following categories:

i) the free n-category enriched in double groupoids generated by a double $(n+2, n)$-polygraph $P = (P^v, P^h, P^s)$, that we denote by $P^{\top\top}$;

ii) the free n-category enriched in double category, whose vertical category is groupoid, generated by a n-polygraph (P^v, P^h, P^s), whose square extension is defined on the pair of $(n+1)$-categories $((P^v)^\top, (P^h)^*)$.

2.2.3. Acyclicity. Let $P = (P^v, P^h, P^s)$ be a double $(n+2, n)$-polygraph. The square extension P^s of the pair of $(n+1, n)$-categories $((P^v)^\top, (P^h)^\top)$ is *acyclic* if for any square S over $((P^v)^\top, (P^h)^\top)$ there exists a square $(n+2)$-cell A in the free n-category enriched in double groupoids $P^{\top\top}$ such that $\partial(A) = S$. For example, the set of squares over $((P^v)^\top, (P^h)^\top)$ forms an acyclic extension.

2.2.4. Two-fold coherent presentations of n-categories. Recall that a *presentation of an n-category* $\mathbf{C}$ is an $(n+1)$-polygraph P whose presented category $\overline{P}$ is isomorphic to $\mathbf{C}$. We define a *2-fold coherent presentation of* $\mathbf{C}$ as a double $(n+2, n)$-polygraph (P^v, P^h, P^s) satisfying the two following conditions:

i) the $(n+1)$-polygraph $(Q, P^v_{n+1} \coprod P^h_{n+1})$ is a presentation of $\mathbf{C}$, where Q is the underlying n-polygraph of P^v and P^h,

ii) the square extension P^s is acyclic.

3. Polygraphs modulo

In this section, we introduce the notion of polygraph modulo and we formulate the rewriting properties of termination, confluence and local confluence for these polygraphs.

3.1. Polygraphs modulo

3.1.1. Cellular extensions modulo. Consider two n-polygraphs E and R such that $E_{\leqslant n-2} = R_{\leqslant n-2}$ and $E_{n-1} \subseteq R_{n-1}$. One defines the cellular extension

$$\gamma^{E,R} : {}_E R \to \mathrm{Sph}_{n-1}(R^*_{n-1})$$

where ${}_E R$ is the set defined by the following pullback in Set:

$$\begin{array}{ccc} E^\top_n \times_{R^*_{n-1}} R^{*(1)}_n & \xrightarrow{\pi_2} & R^{*(1)}_n \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \partial_{-,n-1}} \\ E^\top_n & \xrightarrow[\partial_{+,n-1}]{} & R^*_{n-1} \end{array}$$

and the map $\gamma^{{}_{\mathsf{E}}\mathsf{R}}$ defined by $\gamma^{{}_{\mathsf{E}}\mathsf{R}}(e,f) = (\partial_{-,n-1}(e), \partial_{+,n-1}(f))$ for all e in $\mathsf{E}^{\top}$ and f in $\mathsf{R}^{*(1)}_{n}$. Similarly, one defines the cellular extension

$$\gamma^{\mathsf{R}_{\mathsf{E}}} : \mathsf{R}_{\mathsf{E}} \to \mathrm{Sph}_{n-1}(\mathsf{R}^{*}_{n-1})$$

where R_{E} is the set defined by the following pullback in Set:

$$\begin{array}{ccc}
\mathsf{R}^{*(1)}_{n} \times_{\mathsf{R}^{*}_{n-1}} \mathsf{E}^{\top}_{n} & \xrightarrow{\pi_2} & \mathsf{E}^{\top}_{n} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \partial_{-,n-1}} \\
\mathsf{R}^{*(1)}_{n} & \xrightarrow[\partial_{+,n-1}]{} & \mathsf{R}^{*}_{n-1}
\end{array}$$

and the map $\gamma^{\mathsf{R}_{\mathsf{E}}}$ defined by $\gamma^{\mathsf{R}_{\mathsf{E}}}(f,e) = (\partial_{-,n-1}(f), \partial_{+,n-1}(e))$ for all e in $\mathsf{E}^{\top}$ and f in $\mathsf{R}^{*(1)}_{n}$. Finally, one defines the cellular extension

$$\gamma^{{}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}}} : {}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}} \to \mathrm{Sph}_{n-1}(\mathsf{R}^{*}_{n-1}),$$

where ${}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}}$ is the set defined by the following composition of pullbacks in Set:

$$\begin{array}{ccccc}
\mathsf{E}^{\top}_{n} \times_{\mathsf{R}^{*}_{n-1}} \mathsf{R}^{*(1)}_{n} \times_{\mathsf{R}^{*}_{n-1}} \mathsf{E}^{\top}_{n} & \xrightarrow{(\pi_2,\pi_3)} & \mathsf{R}^{*(1)}_{n} \times_{\mathsf{R}^{*}_{n-1}} \mathsf{E}^{\top}_{n} & \xrightarrow{\pi_2} & \mathsf{E}^{\top}_{n} \\
\downarrow{\scriptstyle (\pi_1,\pi_2)} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \partial_{-,n-1}} \\
\mathsf{E}^{\top}_{n} \times_{\mathsf{R}^{*}_{n-1}} \mathsf{R}^{*(1)}_{n} & \xrightarrow{\pi_2} & \mathsf{R}^{*(1)}_{n} & \xrightarrow[\partial_{+,n-1}]{} & \mathsf{R}^{*}_{n-1} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \partial_{-,n-1}} & & \\
\mathsf{E}^{\top}_{n} & \xrightarrow[\partial_{+,n-1}]{} & \mathsf{R}^{*}_{n-1} & &
\end{array}$$

and the map $\gamma^{{}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}}}$ is defined by $\gamma^{{}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}}}(e,f,e') = (\partial_{-,n-1}(e), \partial_{+,n-1}(e'))$.

3.1.2. Polygraphs modulo. A n-*polygraph modulo* is a data $(\mathsf{R},\mathsf{E},\mathsf{S})$ made of

i) an n-polygraph R, whose generating n-cells are called *primary rules*,

ii) an n-polygraph E such that $\mathsf{E}_k = \mathsf{R}_k$ for $k \leqslant n-2$ and $\mathsf{E}_{n-1} \subseteq \mathsf{R}_{n-1}$, whose generating n-cells are called *modulo rules*,

iii) S is a cellular extension of R^{*}_{n-1} such that the inclusions of cellular extensions

$$\mathsf{R} \subseteq \mathsf{S} \subseteq {}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}}$$

holds.

By abuse of notation, the n-polygraphs modulo $(\mathsf{R},\mathsf{E},{}_{\mathsf{E}}\mathsf{R})$, $(\mathsf{R},\mathsf{E},\mathsf{R}_{\mathsf{E}})$ and $(\mathsf{R},\mathsf{E},{}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}})$ will be denoted by ${}_{\mathsf{E}}\mathsf{R}$, R_{E} and ${}_{\mathsf{E}}\mathsf{R}_{\mathsf{E}}$ respectively. Given an n-polygraph modulo $(\mathsf{R},\mathsf{E},\mathsf{S})$, we will consider in the sequel the following categories:

i) the free n-category $R^*_{n-1}[R_n, E_n \coprod E_n^{-1}]/\mathrm{Inv}(E_n, E_n^{-1})$, denoted by $R^*(E)$.

ii) the free n-category generated by S, denoted by S^*,

iii) the free $(n, n-1)$-category generated by S, denoted by $S^\top$.

For any n-cell f in S^* (resp. $S^\top$), the *size* of f is defined as the positive integer $||f||_{R_n}$ corresponding to the number of n-cells of R_n contained in A, and denoted by $\ell(f)$.

3.2. Termination and normal forms

In this subsection, we introduce the property of termination and the notion of normal form for polygraphs modulo. We explain how to prove termination of polygraphs modulo using a termination order compatible modulo rules. Finally, we recall the double induction principle introduced by Huet in [15] that we will use in many proofs in the sequel.

3.2.1. Termination. Recall that an n-polygraph *terminates* if it has no infinite rewriting sequence. The n-polygraph modulo (R, E, S) is *terminating* when S is terminating. Note that, when $S \neq R$, the following conditions hold

i) ${}_E R$ is terminating,

ii) R_E is terminating,

iii) ${}_E R_E$ is terminating,

iv) S is terminating.

An order relation $\prec$ on R^*_{n-1} is *compatible with* S *modulo* E if it satisfy the following conditions:

i) $v \prec u$, for any $(n-1)$-cells u, v in R^*_{n-1} such that there exists an n-cell $u \to v$ in S,

ii) if $v \prec u$ for $(n-1)$-cells u, v in R^*_{n-1}, then $v' \prec u'$ holds for any $(n-1)$-cells u', v' in R^*_{n-1} such that $v' \approx v$ and $u' \approx u$.

A *termination order for* S *modulo* E is a well-founded order relation compatible with S modulo E.

In this work, many constructions will be based on the termination of the n-rsm R_E, which can be proved by constructing a termination order for one of the n-rsm ${}_E R$, R_E and ${}_E R_E$. It can be also proved by constructing a termination order for R compatible with E.

3.2.2. Normal forms. A $(n-1)$-cell u in R^*_{n-1} is S-*reduced* if it cannot be reduced by n-cells of S. A S-*normal form with respect to* S for a $(n-1)$-cell u in R^*_{n-1} is a S-reduced $(n-1)$-cell v such that u can be reduced to v with respect to S. We will denote by $\mathrm{Irr}(S)$ the set of irreducible $(n-1)$-cells with respect to S. We denote by $\mathrm{NF}(S, u)$ the set of normal forms of a $(n-1)$-cell u with respect to S.

If S is terminating, every $(n-1)$-cell has at least one normal form with respect to S. Moreover, *Noetherian induction* allows definitions and proofs of properties of $(n-1)$-cells by induction on the maximum size of the n-cells leading to normal forms.

3.2.3. Noetherian induction. If S is terminating at f, then it is normalizing at f, that is, every n-cell has at least one normal form. In that case, one can prove properties using *Noetherian induction*. For that, one proves the property on normal forms; then one fixes an n-cell f, one assumes that the result holds for every g such that f reduces into g and one proves that, under those hypotheses, the n-cell f satisfies the property.

Let us recall the double Noetherian induction principle introduced by Huet in [15] to prove the equivalence between confluence modulo and local confluence modulo under a termination hypothesis. We construct an auxiliary n-polygraph S^{II} as follows. One defines

$$S^{II}_k = S_k \times S_k \text{ for } 0 \leqslant k \leqslant n-1,$$

and S^{II}_n contains an n-cell $(u,v) \to (u',v')$, for all $(n-1)$-cells u, u', v, v' in any of the following situation:

i) there exists an n-cell $u \to u'$ in S^* and $v = v'$;

ii) there exists an n-cell $v \to v'$ in S^* and $u = u'$;

iii) there exist n-cells $u \to u'$ and $u \to v'$ in S^*;

iv) there exist n-cells $v \to u'$ and $v \to v'$ in S^*;

v) there exist n-cells e_1, e_2 and e_3 in $E^\top$ as in the following diagram

$$u \xrightarrow{e_1} v \xrightarrow{e_2} u' \xrightarrow{e_3} v'$$

such that $\ell(e_1) > \ell(e_3)$.

Note that this definition implies that, if there exists n-cells $u \to u'$ and $v \to v'$ in S^*, then there is an n-cell $(u,v) \to (u',v')$ in S^{II} given by the following composition:

$$(u,v) \to (u',v) \to (u',v')$$

Following [15, Proposition 2.2], if S_E is terminating, then so is S^{II}.

In the sequel, we will apply this Noetherian induction on S^{II} with the following property:

> *for any n-cells $f : u \to u'$, $g : v \to v'$ in S^* and $e : u \to v$ in $E^\top$, there exists n-cells $f' : u' \to u''$, $g' : v' \to w''$ in S^* and $e' : u'' \to w''$ in $E^\top$, and a square $(n+1)$-cell A in a given $(n-1)$-category enriched in groupoids, as depicted in the following diagram:*

$$\begin{array}{ccccc} u & \xrightarrow{f} & u' & \overset{f'}{\cdots\!\to} & u'' \\ {\scriptstyle e}\downarrow & & \Downarrow A & & \downarrow{\scriptstyle e'} \\ v & \xrightarrow[g]{} & v' & \underset{g'}{\cdots\!\to} & w'' \end{array}$$

In the language of Section 4, this means that any branching (f, e, g) of source (x, y) of S modulo E is coherently confluent modulo E.

3.3. Confluence modulo

In this subsection, we define properties of confluence and local confluence modulo for an n-polygraph modulo (R, E, S), and we explicit a classification of branchings of S modulo E.

3.3.1. Branchings. A *branching* of the n-polygraph S is a pair (f, g), where f and g are n-cells of $S^*$and such that $\partial^h_{-,(n-1)}(f) = \partial^h_{-,(n-1)}(g)$. Such a branching is depicted by

$$\begin{array}{ccc} u & \xrightarrow{f} & u' \\ \| & & \downarrow \\ u & \xrightarrow[g]{} & v' \end{array} \tag{3.3.2}$$

and will be denoted by $(f, g) : u \Rightarrow (u', v')$. The $(n-1)$-cell u is called the *source* of this branching. We do not distinguish the branchings (f, g) and (g, f).

A *branching modulo* E of the n-polygraph modulo S is a triple (f, e, g) where f and g are n-cells of S^* with f non trivial and e is an n-cell of $E^\top$. Such a branching is depicted by

$$\begin{array}{ccc} u & \xrightarrow{f} & u' \\ e\downarrow & & \\ v & \xrightarrow[g]{} & v' \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} u & \xrightarrow{f} & u' \\ e\downarrow & & \\ v & & \end{array} \right) \tag{3.3.3}$$

when g is non trivial (resp. trivial) and denoted by

$$(f, e, g) : (u, v) \Rightarrow (u', v') \qquad (\text{resp. } (f, e) : u \Rightarrow (u', v)\ ).$$

The pair of $(n-1)$-cells (u, v) (resp. (u, u)) is called the *source* of this branching modulo E. Note that any branching (f, g) is a branching modulo E (f, e, g) where $e = i^v_1(\partial^h_{-,(n-1)}(f)) = i^v_1(\partial^h_{-,(n-1)}(g))$.

3.3.4. Confluence and confluence modulo. A *confluence* of the n-polygraph S is a pair (f', g') of n-cells of S^* such that $\partial^h_{+,(n-1)}(f') = \partial^h_{+,(n-1)}(g')$. Such a confluence is depicted by

$$\begin{array}{ccc} u' & \xrightarrow{f'} & w \\ & & \downarrow \| \\ v' & \xrightarrow[g']{} & w \end{array}$$

and denoted by $(f', g') : (u', v') \Rightarrow w$. A *confluence modulo* E of the n-polygraph S is a triple (f', e', g'), where f', g' are n-cells of S^* and e' is an n-cell of $E^\top$ such that $\partial^h_{+,(n-1)}(f') = \partial^v_{-,(n-1)}(e')$ and $\partial^h_{+,(n-1)}(g') = \partial^v_{+,(n-1)}(e')$. Such a confluence is depicted by

$$\begin{array}{ccc} u' & \xrightarrow{f'} & w \\ & & \downarrow e' \\ v' & \xrightarrow[g']{} & w' \end{array}$$

and denoted by $(f', e', g') : (u', v') \Rightarrow (w, w')$.

A branching as in (3.3.2) is *confluent* (resp. *confluent modulo* E) if there exists n-cells f', g' in S^* and e' in $E^\top$ as in the following diagrams:

$$\begin{array}{ccccc} u & \xrightarrow{f} & u' & \overset{f'}{\dashrightarrow} & w \\ \| & & & & \downarrow \| \\ u & \xrightarrow[g]{} & v' & \underset{g'}{\dashrightarrow} & w' \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccccc} u & \xrightarrow{f} & u' & \overset{f'}{\dashrightarrow} & w \\ \| & & & & \vdots e' \\ u & \xrightarrow[g]{} & v' & \underset{g'}{\dashrightarrow} & w' \end{array} \quad \right).$$

A branching modulo E as in (3.3.3) is *confluent modulo* E if there exists n-cells f', g' in S^* and e' in $E^\top$ as in the following diagram:

$$\begin{array}{ccccc} u & \xrightarrow{f} & u' & \overset{f'}{\dashrightarrow} & w \\ e \downarrow & & & & \vdots e' \\ v & \xrightarrow[g]{} & v' & \underset{g'}{\dashrightarrow} & w' \end{array}.$$

We say that the n-polygraph modulo S is *confluent* (resp. *confluent modulo* E) if all of its branchings are confluent (resp. confluent modulo E). Note that when S is confluent, every $(n-1)$-cell of S^* has at most one normal form with respect to S. Under the confluence modulo hypothesis, a $(n-1$−-cell may admit several S-normal forms, which are all equivalent modulo E.

Finally, let us recall the notion of confluence modulo introduced by Jouannaud and Kirchner in [16]. We say that the n-polygraph modulo S is

i) *JK confluent modulo* E, if any branching is confluent modulo E,

ii) *JK coherent modulo* E, if for any branching modulo E $(f, e) : u \Rightarrow (u', v)$ is confluent modulo E:

$$\begin{array}{ccccc} u & \xrightarrow{f} & v & \overset{f'}{\dashrightarrow} & v' \\ e \downarrow & & & & \vdots e' \\ u' & & \underset{g'}{\dashrightarrow} & & w \end{array}$$

in such a way that g' is a non-trivial n-cell in S^*.

3.3.5. Local branchings. A branching (f, g) of the n-polygraph S is *local* if f, g are n-cells of $S^{*(1)}$. A branching (f, e, g) modulo E is *local* if f is an n-cell of $S^{*(1)}$, g is an n-cell of S^* and e an n-cell of $E^\top$ such that $\ell(g) + \ell(e) = 1$. Local branchings belong to one of the following families:

i) *local aspherical* branchings of the form:

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ \| \downarrow & & \downarrow \| \\ u & \xrightarrow[f]{} & v \end{array}$$

where f is an n-cell of $S^{*(1)}$;

ii) *local Peiffer* branchings of the form:

$$\begin{array}{ccc} u \star_i v & \xrightarrow{f \star_i v} & u' \star_i v \\ {\scriptstyle \|}\downarrow & & \\ u \star_i v & \xrightarrow[u \star_i g]{} & u \star_i v' \end{array}$$

where $0 \leqslant i \leqslant n-2$, f and g are n-cells of $S^{*(1)}$,

iii) *local Peiffer modulo* of the forms:

$$\begin{array}{ccc} u \star_i v & \xrightarrow{f \star_i v} & u' \star_i v \\ {\scriptstyle u \star_i e}\downarrow & & \\ u \star_i v' & & \end{array} \qquad \begin{array}{ccc} w \star_i u & \xrightarrow{w \star_i f} & w \star_i u' \\ {\scriptstyle e' \star_i u}\downarrow & & \\ w' \star_i u & & \end{array} \tag{3.3.6}$$

where $0 \leqslant i \leqslant n-2$, where f is an n-cell of $S^{*(1)}$ and e, e' are n-cell of $E^{\top(1)}$;

iv) *overlapping branchings* are the remaining local branchings:

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ {\scriptstyle \|}\downarrow & & \\ u & \xrightarrow[g]{} & v' \end{array}$$

where f and g are n-cells of $S^{*(1)}$,

v) *overlapping branchings modulo* are the remaining local branchings modulo:

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ {\scriptstyle e}\downarrow & & \\ v' & & \end{array} \tag{3.3.7}$$

where f is an n-cell of $S^{*(1)}$ and e is an n-cell of $E^{\top(1)}$.

Let (f, g) (resp. (f, e, g)) be a branching (resp. branching modulo E) of the n-polygraph S with source u (resp. (u, v)) and a whisker $C[\partial u]$ of $R^*_{n_1}$ composable with u and v, the pair $(C[f], C[g])$ (resp. triple $(C[f], C[e], C[g])$) is a branching (resp. branching modulo E) of the n-polygraph S. If the branching (f, e, g) is local, then the branching $(C[f], C[e], C[g])$ is local. We denote by $\sqsubseteq$ the order relation on branchings modulo E of S defined by $(f, e, g) \sqsubseteq (f', e', g')$ when there exists a whisker C such that $(C[f], C[e], C[g]) = (f', e', g')$ hold. A branching (resp. branching modulo E) is *minimal* if it is minimal for the order relation $\sqsubseteq$. A branching (resp. branching modulo E) is *critical* if it is an overlapping branching that is minimal for the relation $\sqsubseteq$.

3.3.8. Local confluence modulo. The n-polygraph modulo S is *locally confluent modulo* E if any of its local branchings modulo E is confluent modulo E. Note that following [16], there exists a local version of JK-confluence modulo E and JK coherence modulo E, given by properties **a)** and **b)** of Proposition 4.2.1, and we will prove in the next section that all these notions are equivalent.

3.4. Completion procedure for ${}_{\mathsf{E}}\mathsf{R}$

In this subsection, we give a completion procedure for an n-polygraph modulo of the form $(\mathsf{R}, \mathsf{E}, {}_{\mathsf{E}}\mathsf{R})$, when ${}_{\mathsf{E}}\mathsf{R}$ is not confluent modulo E as n-polygraph modulo E. The procedure computes an n-polygraph $\check{\mathsf{R}}$ such that ${}_{\mathsf{E}}\check{\mathsf{R}}$ is confluent modulo E as an n-polygraph modulo E.

3.4.1. Completion of ${}_{\mathsf{E}}\mathsf{R}$ modulo E. Note that the property of JK coherence is trivially satisfied for ${}_{\mathsf{E}}\mathsf{R}$. Indeed, any branching (f, e) of ${}_{\mathsf{E}}\mathsf{R}$ modulo E is trivially confluent modulo E as follows: we split the set of critical branchings of ${}_{\mathsf{E}}\mathsf{R}$ modulo E into three sets as follows:

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle =} \\ v' & \xrightarrow[e^- \cdot f]{} & v \end{array} \tag{3.4.2}$$

where $e^- \cdot f$ is a rewriting step of ${}_{\mathsf{E}}\mathsf{R}$.

Following the critical branching lemma for rewriting modulo, enounced in a coherent version and given in the next section in Theorem 4.2.2, we describe a completion procedure for confluence of ${}_{\mathsf{E}}\mathsf{R}$ modulo E in terms of critical branchings, similar to the Knuth-Bendix completion in a non modulo setting. From (3.4.2) and Theorem 4.2.2, when ${}_{\mathsf{E}}\mathsf{R}$ is terminating, ${}_{\mathsf{E}}\mathsf{R}$ is confluent modulo E if and only if all critical branchings (f, g) of ${}_{\mathsf{E}}\mathsf{R}$ modulo E with f in ${}_{\mathsf{E}}\mathsf{R}^{*(1)}$ and g in $\mathsf{R}^{*(1)}$ are confluent modulo E, as depicted by:

$$\begin{array}{ccccc} u & \xrightarrow[f \in {}_{\mathsf{E}}\mathsf{R}]{} & v & \overset{f' \in ({}_{\mathsf{E}}\mathsf{R})^*}{\cdots\cdots\cdots\cdots\cdots} & v' \\ {\scriptstyle =}\downarrow & & & & \vdots{\scriptstyle e'} \\ u & \xrightarrow[g \in \mathsf{R}]{} & w & \underset{g' \in ({}_{\mathsf{E}}\mathsf{R})^*}{\cdots\cdots\cdots\cdots\cdots} & w' \end{array}$$

We denote by $\mathrm{CP}({}_{\mathsf{E}}\mathsf{R}, \mathsf{R})$ the set of such critical branchings.

3.4.3. Completion procedure. Let us consider R and E two n-polygraphs such that $\mathsf{E}_{\leqslant n-2} = \mathsf{R}_{\leqslant n-2}$ and $\mathsf{E}_{n-1} \subseteq \mathsf{R}_{n-1}$, and $\prec$ a termination order compatible with R modulo E. In this paragraph, we describe a procedure to compute a completion $\check{\mathsf{R}}$ of the n-polygraph R such that ${}_{\mathsf{E}}\check{\mathsf{R}}$ is confluent modulo E. We denote by $\hat{u}^{{}_{\mathsf{E}}\mathsf{R}}$ a normal form of an element u in R^*_{n-1} wrt ${}_{\mathsf{E}}\mathsf{R}$.

Input:
- R and E 2-polygraphs over a 1-polygraph X.
- $\prec$ a termination order for R compatible with E,
  which is total on the set of ${}_{E}R$-irreducible elements.

begin
  $\mathcal{A} \leftarrow \varnothing$
  $\mathcal{C} \leftarrow \mathrm{CP}({}_{E}R, R)$
  while $\mathcal{C} \neq \varnothing$ **do**
    Pick any branching $c = (f : u \Rightarrow v, g : u \Rightarrow w)$ in $\mathcal{C}$, with f in ${}_{E}R^*$ and g in R^*;
    Reduce v to $\hat{v}^{E,R}$ a ${}_{E}R$-normal form;
    Reduce w to $\hat{w}^{E,R}$ a ${}_{E}R$-normal form;
    $\mathcal{C} \leftarrow \mathcal{C} \backslash \{c\}$
    if $\hat{v}^{E,R} \not\approx_E \hat{w}^{E,R}$ **then**
      if $\hat{w}^{E,R} \prec \hat{v}^{E,R}$ **then**
        $R \leftarrow R \cup \{\hat{v}^{E,R} \overset{\alpha}{\Rightarrow} \hat{w}^{E,R}\}$
      end
      if $\hat{v}^{E,R} \prec \hat{w}^{E,R}$ **then**
        $R \leftarrow R \cup \{\hat{w}^{E,R} \overset{\alpha}{\Rightarrow} \hat{v}^{E,R}\}$
      end
    end
    $\mathcal{C} \leftarrow \mathcal{C} \cup \{({}_{E}R, R)\text{-critical branchings created by } \alpha\}$
  end
end

This procedure may not be terminating. However, it does not fail because of the hypothesis that $\prec$ is total on the set of ${}_{E}R$-irreducible elements.

3.4.4. Proposition. *When it terminates, this completion procedure returns a rewriting system* $\check{R}$ *such that* ${}_{E}\check{R}$ *is confluent modulo* E.

Proof. The proof of correctness of this procedure is a consequence of the inference system given by Bachmair and Dershowitz in [2] in order to get a set of rules $\check{R}$ such that ${}_{E}\check{R}$ is confluent modulo E. Given two rewriting systems R and E and a termination order $>$ compatible with R modulo E, their inference system is given by the following six elementary rules:

1) Orienting an equation:

$$(A \cup \{s = t\}, R) \qquad \rightsquigarrow \qquad (A, R \cup \{s \to t\}) \text{ if } s > t.$$

2) Adding an equational consequence:

$$(A, R) \qquad \rightsquigarrow \qquad (A \cup \{s = t\}, R) \text{ if } s \longleftarrow^*_{R \cup E} u \longrightarrow^*_{R \cup E} t.$$

3) Simplifying an equation:

$$(A \cup \{s = t\}, R) \qquad \rightsquigarrow \qquad (A \cup \{u = t\}, R) \text{ if } s \overset{E,R}{\to} u.$$

4) Deleting an equation:

$$(A \cup \{s = t\}, R) \qquad \rightsquigarrow \qquad (A, R) \text{ if } s \approx_E t.$$

5) Simplifying the right-hand side of a rule:

$$(A, R \cup \{s \to t\}) \qquad \rightsquigarrow \qquad (A, R \cup \{s \to u\}) \text{ if } t \xrightarrow{E R} u.$$

6) Simplifying the left-hand side of a rule:

$$(A, R \cup \{s \to t\}) \qquad \rightsquigarrow \qquad (A \cup \{u = t\}, R) \text{ if } s \xrightarrow{E R} u.$$

The correctness of Procedure 3.4.3 is a consequence of the following arguments:

i) For any critical branching $(f : u \to v, g : u \to w)$ in $\mathrm{CP}({}_E R, R)$, we can add an equation $v = w$ using the rule *Adding an equational consequence*, and simplify it to $\hat{v}^{E R} = \hat{w}^{E R}$ using the rule *Symplifying and equation*.

ii) If $\hat{v}^{E R} \approx_E \hat{w}^{E R}$, we can delete the equation using the rule *Deleting an equation*.

iii) Otherwise, we can always orient it using the rule *Orienting an equation*.

Thus, each step of this completion procedure comes from one of the inference rules given by Bachmair and Dershowitz. Following [2], it returns a set R so that ${}_E R$ is confluent modulo E. □

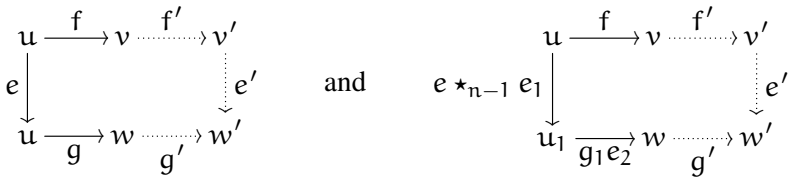
4. Coherent confluence modulo

In this section, we introduce the property of coherent confluence modulo defined by the adjunction of a square cell for each confluence diagram modulo. Under a termination hypothesis, Theorem 4.1.4 shows how to deduce coherent confluence modulo for a polygraph modulo from coherent local confluence modulo. This result is a coherent version of Newman's lemma that states the equivalence between local confluence and confluence under a termination hypothesis, [21]. Theorem 4.2.2 formulates a coherent version of the critical branching lemma, it shows how to deduce local coherent confluence modulo from the coherent confluence modulo of some critical branchings.

4.1. Coherent Newman's lemma modulo

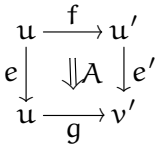
4.1.1. Biaction of $E^\top$ on $\mathrm{Sqr}(E^\top, S^*)$. Let (R, E, S) be an n-polygraph modulo. Let Γ be a square extension of the pair of n-categories $(E^\top, S^*)$. As the inclusion $R \subseteq S \subseteq {}_E R_E$ of cellular extensions holds, any n-cell f in S^* can be decomposed in $f = e_1 \star_{n-1} f_1 \star_{n-1} e_2 \star_{n-1} f_2$ with f_1 in $R^{*(1)}$, f_2 in S^* such that $\ell(f_2) = \ell(f) - 1$, e_1 and e_2 in $E^\top$ possibly identities, and $\star_{n-1}$ denoting for the composition along $(n-1)$-cells in the free n-category generated by $R \cup E$.

Thus, a branching (f, e, g) of S modulo E with a choice of a generating confluence (f', e', g') may correspond to different squares in $\mathrm{Sqr}(E^\top, S^*)$. For instance, if g can be decomposed $g = e_1 \star_{n-1} g_1 \star_{n-1} e_2$, the following squares in $\mathrm{Sqr}(E^\top, S^*)$ correspond to the same branching of S modulo E:



When computing a coherent presentation of S modulo E, one does not want to consider two different elements in a coherent completion of S modulo E, as defined in 5.1, to tile these squares which are not equal in the free n-category enriched in double category generated by the double $(n-1)$-polygraph $(E, S, \Gamma \cup \mathrm{Peiff}(E^\top, S^*))$.

In order to avoid these redundant squares, we define a *biaction* of $E^\top$ on $\mathrm{Sqr}(E^\top, S^*)$. For any n-cells e_1 and e_2 in $E^\top$ and any $(n+1)$-cell



in $\mathrm{Sqr}(E^\top, S^*)$ satisfying the following conditions

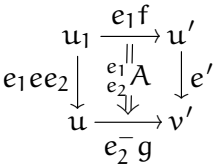
i) $\partial_{+,n-1}(e_1) = \partial^h_{-,n-1}\partial^-, n^v(A)$,

ii) $\partial_{-,n-1}(e_2) = \partial^h_{+,n-1}\partial^-, n^v(A)$,

iii) $e_1 \partial^h_{-,n}(A) \in S$,

iv) $e_2^- \partial^h_{+,n}(A) \in S$,

we define the square $(n+1)$-cell ${}^{e_1}_{e_2}A$ as follows:



where $u_1 = \partial_{-,n-1}(e_1)$. For a square extension Γ of $(E^\top, S^*)$, we denote by $E \rtimes \Gamma$ the set containing all elements ${}^{e_1}_{e_2}A$ with A in Γ and e_1, e_2 in $E^\top$, whenever it is well defined. For any e_1,e_2 in $E^\top$ and A,A' in Γ, the following equalities hold whenever both sides are defined:

i) ${}^{e_1'}_{e_2'}({}^{e_1}_{e_2}A) = {}^{e_1'e_1}_{e_2e_2'}A$;

ii) ${}^{e_1}_{e_2}(A \diamond^v A') = ({}^{e_1}_{e_2}A) \diamond^v A'$;

iii) ${}^{e_1}_{e_2}(A \diamond^h A') = ({}^{e_1}_{1}A) \diamond^h ({}^{1}_{e_2}A')$.

4.1.2. Coherent confluence modulo. Let (R, E, S) be an n-polygraph modulo. Let Γ be a square extension of the pair of n-categories $(E^\top, S^*)$. We will consider

i) $(E, S, \Gamma \cup \mathrm{Peiff}(E^\top, S^*))^{\mathbb{T}}$ the free $(n-1)$-category enriched in double groupoids generated by the double $(n+1, n-1)$-polygraph (E, S, Γ)

ii) $(E, S, \Gamma \cup \mathrm{Peiff}(E^\top, S^*))^{\mathbb{T},\nu}$ the free $(n-1)$-category enriched in double categories whose vertical n-cells are invertible generated by the double $(n-1)$-polygraph $(E, S, \Gamma \cup \mathrm{Peiff}(E^\top, S^*))$ in DbPol^{ν}_{n}.

A branching modulo E as in (3.3.3) is Γ-*confluent modulo* E if there exists n-cells f', g' in S^*, e' in $E^\top$ and a $(n+1)$-cell A in $(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*))^{\mathbb{T},\nu}$ as in the following diagram:

$$\begin{array}{ccccc} u & \xrightarrow{f} & u' & \overset{f'}{\dashrightarrow} & w \\ {\scriptstyle e}\downarrow & & \Downarrow A & & \downarrow{\scriptstyle e'} \\ v & \xrightarrow[g]{} & v' & \underset{g'}{\dashrightarrow} & w' \end{array} .$$

We say that S is Γ-*confluent* (resp. *locally* Γ-confluent, resp. *critically* Γ-*confluent*) *modulo* E if every branching (resp. local branching, resp. critical branching) modulo E is Γ-confluent modulo E. Note that when $\Gamma = \mathrm{Sqr}(E^\top, S^*)$ (resp. $\Gamma = \mathrm{Sph}(S^*)$), the property of Γ-confluence modulo E corresponds to the property of confluence modulo E (resp. confluence) given in 3.3.

In the sequel, proofs of confluence modulo results will be based on the Huet double Noetherian induction principle on the n-polygraph $S^{\amalg}$ defined in 3.2.3 and the property $\mathcal{P}$ on $R^*_{n-1} \times R^*_{n-1}$ defined, for any u, v in R^*_{n-1}, by

$$\mathcal{P}(u, v) : \text{any branching } (f, e, g) \text{ of } S \text{ modulo } E \text{ of source } (u, v) \text{ is } \Gamma\text{-confluent modulo } E.$$

4.1.3. Proposition (Coherent half Newman's modulo lemma). *Let* (R, E, S) *be an* n*-polygraph modulo such that* ${}_E R_E$ *is terminating, and* Γ *be a square extension of* $(E^\top, S^*)$*. If* S *is locally* Γ*-confluent modulo* E *then the two following conditions hold*

i) *any branching* (f, e) *of* S *modulo* E *with* f *in* $S^{*(1)}$ *and* e *in* $E^\top$ *is* Γ*-confluent modulo* E*,*

ii) *any branching* (f, e) *of* S *modulo* E *with* f *in* S^* *and* e *in* $E^{\top(1)}$ *is* Γ*-confluent modulo* E*,*

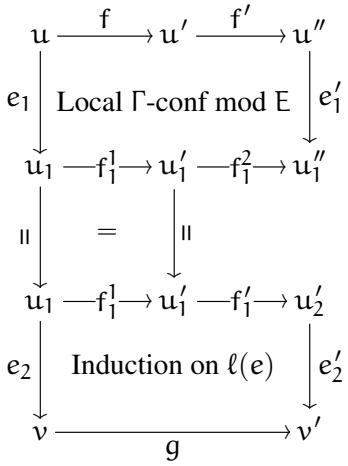
Proof. We prove condition **i)**, the proof of condition **ii)** is similar. Let us assume that S is locally Γ-confluent modulo E, we proceed by double induction.

We denote by u the source of the branching (f, e). If u is irreducible with respect to S, then f is an identity n-cell, and the branching is trivially Γ-confluent.

Suppose that f is not an identity and assume that for any pair (u', v') of $(n-1)$-cells in R^*_{n-1} such that there is an n-cell $(u, u) \to (u', v')$ in $S^{\amalg}$, any branching (f', e', g') of source (u', v') is Γ-confluent modulo E. Prove that the branching (f, e) is Γ-confluent modulo E.

We proceed by induction on $\ell(e) \geqslant 1$. If $\ell(e) = 1$, (f, e) is a local branching of S modulo E and it is Γ-confluent modulo E by local Γ-confluence of S modulo E. Now, let us assume that for $k \geqslant 1$, any branching (f'', e'') of S modulo E such that $\ell(e'') = k$ is Γ-confluent modulo E, and let us consider a branching (f, e) of S modulo E such that $\ell(e) = k + 1$, with source u. We choose a decomposition

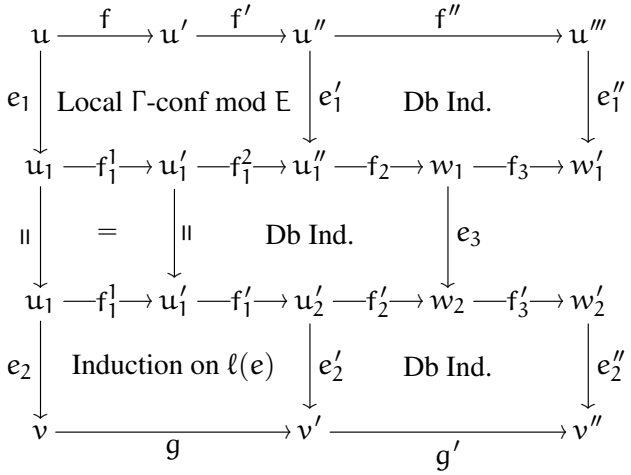
$e = e_1 \star_{n-1} e_2$ with e_1 in $E^{\top(1)}$ and e_2 in $E^{\top}$. Using local Γ-confluence on the branching (f, e_1) of source u, there exist n-cells f' and f_1 in S^*, an n-cell $e'_1 : t_{n-1}(f') \to t_{n-1}(f_1)$ in $E^{\top}$ and a $(n+1)$-cell A in $(E, S, \Gamma)^{\mathbb{T},\nu}$ such that $\partial^h_{-,n}(A) = f \star_{n-1} f'$ and $\partial^h_{+,n}(A) = f_1$. Then, we choose a decomposition $f_1 = f^1_1 \star_{n-1} f^2_1$ with f^1_1 in $S^{*(1)}$ and f^2_1 in S^*. Using the induction hypothesis on the branching (f^1_1, e_2) of S modulo E of source $u_1 := t_{n-1}(e_1) = s_{n-1}(e_2)$, there exist n-cells f'_1 and g in S^*, an n-cell $e_2 : t_{n-1}(f'_1) \to t_{n-1}(g)$ in $E^{\top}$ and a $(n+1)$-cell B in $(E, S, \Gamma)^{\mathbb{T},\nu}$ such that $\partial^h_{-,n}(B) = f^1_1 \star_{n-1} f'_1$ and $\partial^h_{+,n}(B) = g$. This can be represented by the following diagram:



Now, there is an n-cell $(u, u) \to (u'_1, u'_1)$ in $S^{\mathbb{I}}$ given by the composition

$$(u, u) \to (u_1, u_1) \to (u_1, u'_1) \to (u'_1, u'_1)$$

where the first step exists because $\ell(e_1) > 0$ and the remaining composition is as in 3.2.3. Then, we apply double induction on the branching (f^2_1, f'_1) of S modulo E of source (u'_1, u'_1): there exist n-cells f_2 and f'_2 in S^* and an n-cell $e_3 : t_{n-1}(f_2) \to t_{n-1}(f'_2)$ in $E^{\top}$. By a similar argument, we can apply double induction on the branchings $(f_2, (e'_1)^-)$ and (f'_2, e'_2) of S modulo E, so that there exist n-cells f'',f_3, f'_3 and g' in S^* and n-cells $e''_1 : t_{n-1}(f'') \to t_{n-1}(f_3)$ and $e''_2 : t_{n-1}(f'_3) \to t_{n-1}(g')$ as in the following diagram:



We can then repeat the same process using double induction on the branching (f_3, e_3, f_3') of S modulo E of source (w_1, w_2) and so on, and this process terminates in finitely many steps, otherwise it leads to an infinite rewriting sequence wrt S starting from u_1, which is not possible since ${}_E R_E$, and thus S, is terminating. This yields the Γ-confluence of the branching (f, e). □

4.1.4. Theorem (Coherent Newman's lemma modulo). *Let* (R, E, S) *be an* n*-polygraph modulo such that* ${}_E R_E$ *is terminating, and* Γ *be a square extension of* $(E^\top, S^*)$*. If* S *is locally* Γ*-confluent modulo* E *then it is* Γ*-confluent modulo* E*.*

Proof. Prove that any branching (f, e, g) of S modulo E is Γ-confluent modulo E. Let us choose such a branching and denote by (u, v) its source. We assume that any branching (f', e', g') of S modulo E of source (u', v') such that there is an n-cell $(u, v) \to (u', v')$ in $S^{\amalg}$ is Γ-confluent modulo E. We follow the proof scheme used by Huet in [15, Lemma 2.7]. Let us denote by $n := \ell(f)$ and $m := \ell(g)$. We assume without loss of generality that $n > 0$ and we fix a decomposition $f = f_1 \star_{n-1} f_2$ with f_1 in $S^{*(1)}$ and f_2 in S^*.

If $m = 0$, by Proposition 4.1.3 on the branching (f_1, e) of S modulo E, there exist n-cells f_1' and g' in S^*, an n-cell $e' : t_{n-1}(f_1') \to t_{n-1}(g')$ and a $(n+1)$-cell A in $(E, S, \Gamma)^{\top, v}$ such that $\partial^h_{-,n}(A) = f_1 \star_{n-1} f_1'$ and $\partial^h_{+,n}(A) = g'$. Then, since there is an n-cell $(u, u) \to (u_1, u_1)$ in $S^{\amalg}$ with $u_1 := t_{n-1}(f_1)$, we can apply double induction on the branching (f_2, f_1') of S modulo E as in the following diagram:

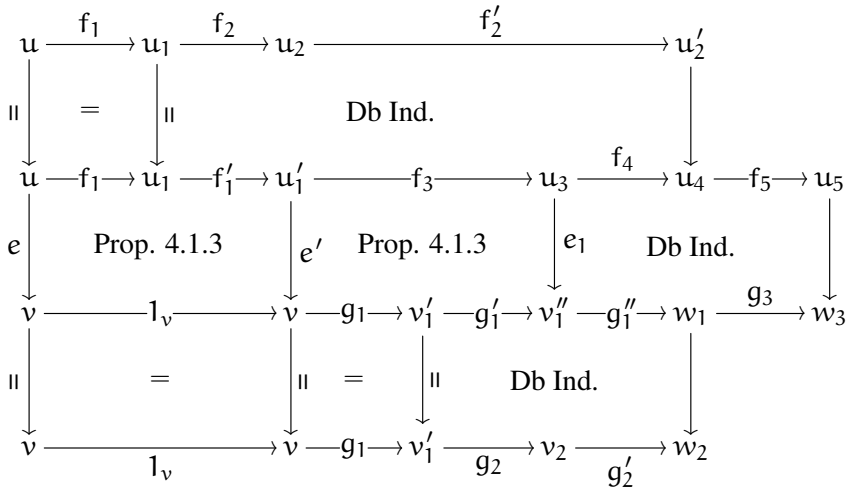
$$\begin{array}{ccccc}
u & \xrightarrow{f_1} & u_1 & \xrightarrow{f_2} & u_2 \xrightarrow{f_2'} u_2' \\
\| \downarrow & = & \downarrow \| & \text{Db Ind.} & \downarrow \\
u & \xrightarrow{f_1} & u_1 & \xrightarrow{f_1'} & u_2 \xrightarrow{f_1''} u_2' \\
e \downarrow & & \text{Prop. 4.1.3} & & \downarrow e' \\
v & & \xrightarrow[g']{} & & v'
\end{array}$$

We finish the proof of this case with a similar argument than in 4.1.3, using repeated double inductions that can not occur infinitely many times since S is terminating.

Now, assume that $m > 0$ and fix a decomposition $g = g_1 \star_{n-1} g_2$ of g with g_1 in $S^{*(1)}$ and g_2 in S^*. By Step 1 on the branching (f_1, e) of S modulo E, there exist n-cells f_1' and h_1 in S^*, an n-cell $e_1 : t_{n-1}(f_1') \to t_{n-1}(h_1)$ in $E^\top$ and a $(n+1)$-cell A in $(E, S, \Gamma)^{\top, v}$ such that $\partial^h_{-,n}(A) = f_1 \star_{n-1} f_1'$ and $\partial^h_{+,n}(A) = h_1$. We distinguish two cases whether h_1 is trivial or not.

If h_1 is trivial, the Γ-confluence of the branching (f, e, g) of S modulo E is given by the following

diagram

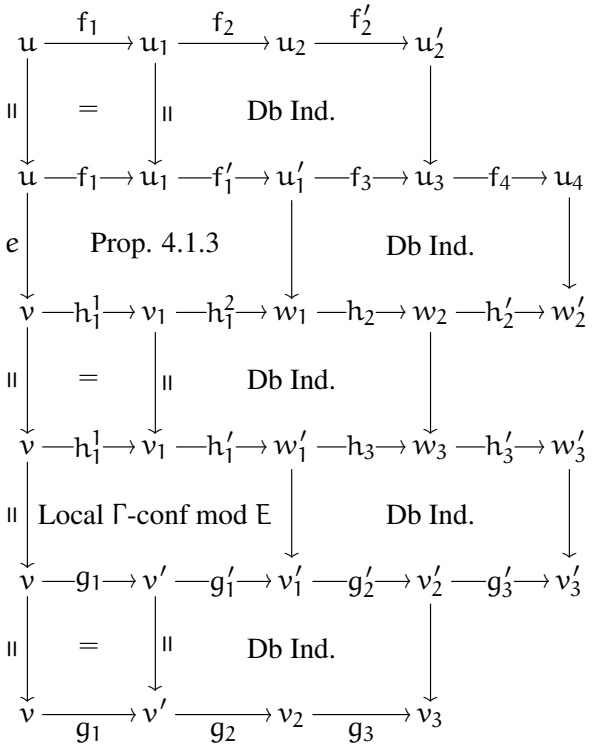


where the branchings (f_1, e) and (g_1, e') of S modulo E are Γ-confluent by Proposition 4.1.3, double induction applies on the branchings $(f_2, f_1' \star_{n-1} f_3)$, (g_1', g_2) and (f_4, e_1, g_1'') since there are n-cells

$$(u, v) \to (u, u) \to (u_1, u_1)\ ,\ (u, v) \to (v, v) \to (v, v_1') \to (v_1', v_1') \text{ and } (u, v) \to (u_3, v) \to (u_3, v_1'')$$

in $S^{\amalg}$ and one can check that this process of double induction can be repeated, terminating in a finite number of steps since S is terminating and yields a Γ-confluence of the branching (f, e, g) modulo E.

If h_1 is not trivial, let us fix a decomposition $h_1 = h_1^1 \star_{n-1} h_1^2$ with h_1^1 in $S^{*(1)}$ and h_1^2 in S^*. The Γ-confluence of the branching (f, e, g) of S modulo E is given by the following diagram:



where the branching (f_1, e) modulo E is Γ-confluent by Proposition 4.1.3, the branching (h_1^1, g_1) is Γ-confluent by assumption of local Γ-confluence of S, and one can check that double induction applies on the branchings (f_2, f_1'), (h_1^2, h_1'), (g_1', g_2), (f_3, h_2) and (h_3, g_2'). This process of double induction can be repeated, terminating in a finite number of steps since S is terminating and yields a Γ-confluence of the branching (f, e, g) modulo E.

□

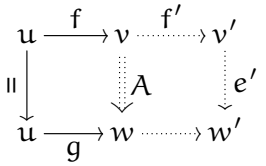
4.2. Coherent critical branching lemma modulo

In this subsection, we show that to prove coherent local confluence of an n-polygraph modulo, we do not need to consider all the local branchings. We show how to prove coherent local confluence of an n-polygraph modulo from coherent confluence of critical branchings.

4.2.1. Proposition. *Let* (R, E, S) *be an* n*-polygraph modulo such that* ${}_E R_E$ *is terminating. Then* S *is* Γ*-locally confluent modulo* E*, if and only if the two following conditions hold:*

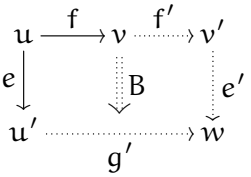
a) *any local branching* $(f, g) : u \Rightarrow (v, w)$ *with* f *in* $S^{*(1)}$ *and* g *in* $R^{*(1)}$ *is* Γ*-confluent modulo* E*:*

a):



b) *any local branching* $(f, e) : u \Rightarrow (v, u')$ *modulo* E *with* f *in* $S^{*(1)}$ *and* e *in* $E^{\top(1)}$ *is* Γ*-confluent modulo* E*:*

b)*:*

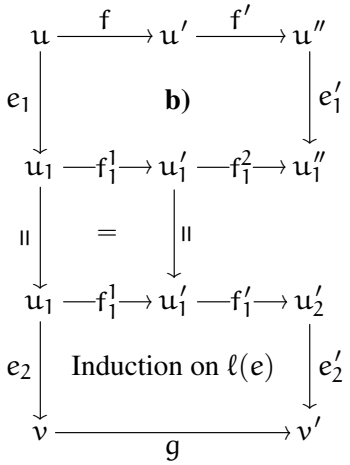


Proof. We prove this result using Huet double Noetherian induction principle on the n-polygraph $S^{\amalg}$ defined in 3.2.3 and the property $\mathcal{P}$ on $R^*_{n-1} \times R^*_{n-1}$ defined by: for any u, v in R^*_{n-1},

$$\mathcal{P}(u, v) : \text{any branching } (f, e, g) \text{ of } S \text{ modulo } E \text{ of source } (u, v) \text{ is } \Gamma\text{-confluent modulo } E.$$

The only part is trivial because properties **a)** and **b)** correspond to Γ-confluence of some local branchings of S modulo E. Conversely, assume that S satisfy properties **a)** and **b)** and let us at first prove that any branching (f, e) with f in $S^{*(1)}$ and e in $E^{\top}$ is Γ-confluent. We consider such a branching (f, e) of source u and assume that any branching (f', e', g') of S modulo E of source (u', v') such that there exist an n-cell $(u, u) \to (u', v')$ in $S^{\amalg}$ is Γ-confluent modulo E. We proceed by induction on $\ell(e)$. If $\ell(e) = 1$, the Γ-confluence of the branching (f, e) is given by property **b)**. Now, suppose this result proved for any branching (f'', e'') with f'' in $S^{*(1)}$ and e'' in $E^{\top}$ such that $\ell(e'') = k \geqslant 1$ and assume that $\ell(e) = k + 1$. Let us choose a decomposition $e = e_1 \star_{n-1} e_2$ with e_1 in $E^{\top(1)}$ and e_2 in $E^{\top}$ such that $\ell(e_2) = k$. By property **b)**, the local branching (f, e_1) is Γ-confluent modulo E: there exist n-cells f' and f_1 in S^*, an n-cell $e' : t_{n-1}(f') \to t_{n-1}(f_1)$ and a $(n+1)$-cell A in $(E, S, \Gamma)^{\overline{\amalg},\nu}$ such that $\partial^h_{-,n}(A) = f \star_{n-1} f'$ and $\partial^h_{+,n}(A) = f_1$.

We choose a decomposition $f_1 = f_1^1 \star_{n-1} f_1^2$ with f_1^1 in $S^{*(1)}$ and f_1^2 in S^*, and by induction hypothesis, the branching (f_1^1, e_2) is Γ-confluent modulo E: there exist n-cells f_1' and g in S^*, an n-cell $e'' : t_{n-1}(f_1') \to t_{n-1}(g)$ and a $(n+1)$-cell B in $(E, S, \Gamma)^{\overline{\amalg},\nu}$ such that $\partial^h_{-,n}(B) = f_1^1 \star_{n-1} f_1'$ and $\partial^h_{+,n}(B) = g$:



We then finish this part of the proof with a similar argument than in the proof of Proposition 4.1.3.

Now, let us prove that any local branching is Γ-confluent modulo E. We consider a local branching (f, e, g) of S modulo E, and assume without loss of generality that f is a non-trivial n-cell in $S^{*(1)}$. There

are two cases: either g is trivial, and the local branching (f, e) of S modulo E is Γ-confluent by **b)**, or e is trivial. In that case, if g is in $R^{*(1)}$, then Γ-confluence of the branching (f, g) is given by **a)**. Otherwise, let us choose a decomposition $g = e_1 \star_{n-1} g' \star_{n-1} e_2$ with e_1,e_2 in $E^\top$ and g' in $R^{*(1)}$. Now, let us prove the confluence of the branching

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ {\scriptstyle e_1}\downarrow & & \\ u' & \xrightarrow[g'e_2]{} & v' \end{array}$$

of S modulo E, where $g'e_2$ is an n-cell in $S^{*(1)}$. We will then prove the Γ-confluence of the branching (f, g) using the biaction of $E^\top$ on $\mathrm{Sqr}(E^\top, S^*)$. Using first part of the proof on the branching (f, e_1) of S modulo E, there exist n-cells f' and f_1 in S^*, an n-cell $e' : t_{n-1}(f') \to t_{n-1}(f_1)$ and a $(n+1)$-cell A in $(E, S, \Gamma)^{\top,\nu}$ such that $\partial^h_{-,n}(A) = f \star_{n-1} f'$ and $\partial^h_{+,n}(A) = f_1$. Using property **a)** on the local branching $(g', g'e_2) \in R^{*(1)} \times S^{*(1)}$ and the trivial confluence given by the right vertical cell e_2, there exist a $(n+1)$-cell B in $(E, S, \Gamma)^{\top,\nu}$ such that $\partial^h_{-,n}(B) = g'$ and $\partial^h_{+,n}(B) = g'e_2$. Let us choose a decomposition $f_1 = f_1^1 \star_{n-1} f_1^2$ with f_1^1 in $S^{*(1)}$ and f_1^2. By property **a)** on the local branching (f_1^1, g'), there exist n-cells f_1' and g_1' in S^*, an n-cell $e'' : t_{n-1}(f_1') \to t_{n-1}(g_1')$ and a $(n+1)$-cell C in $(E, S, \Gamma)^{\top,\nu}$ such that $\partial^h_{-,n}(C) = f_1^1 \star_{n-1} f_1'$ and $\partial^h_{+,n}(C) = g' \star_{n-1} g_1'$ as depicted on the following diagram:

$$\begin{array}{ccccc}
u & \xrightarrow{f} & u' & \xrightarrow{f'} & u'' \\
{\scriptstyle e_1}\downarrow & & \Downarrow A & & \downarrow{\scriptstyle e_1'} \\
u_1 & \xrightarrow{f_1^1} & u_1' & \xrightarrow{f_1^2} & u_1'' \\
\| & = & \downarrow \| & & \\
u_1 & \xrightarrow{f_1^1} & u_1' & \xrightarrow{f_1'} & u_2' \\
\| & & \Downarrow C & & \downarrow{\scriptstyle e_2'} \\
v & \xrightarrow{g'} & v_1 & \xrightarrow[g_2']{} & v_2 \\
\| & & \Downarrow B & \downarrow{\scriptstyle e_2} & \\
v & \xrightarrow[g'e_2]{} & v' & &
\end{array}$$

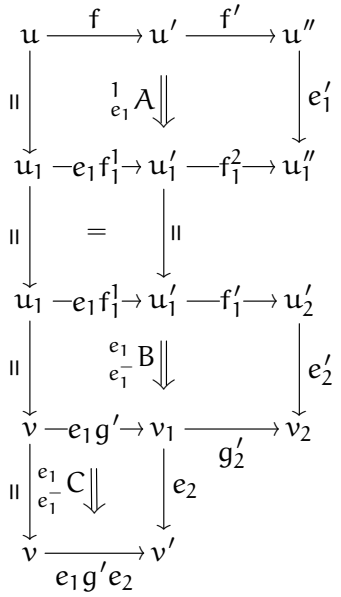
There are n-cells $(u, u) \to (u_1', u_1')$ and $(u, u) \to (v_1, v_1)$ in $S^{\square}$ given by the following compositions

$$(u, u) \to (u_1, u_1) \to (u_1, u_1') \to (u_1', u_1')$$

$$(u, u) \to (u_1, u_1) \to (u_1, v) \to (v, v) \to (v, v_1) \to (v_1, v_1)$$

so that we can apply double induction on the branchings (f_1^2, f_1') and (g_2', e_2) of S modulo E, and we finish the proof of Γ-confluence of the branching $(f, e_1, g'e_2)$ using repeated double inductions, terminating in a finite number of steps since S is terminating.

Now, we get the Γ-confluence of the branching (f, g) of S by the following diagram:

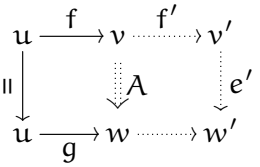


since the top rectangle is by definition tiled by the $(n+1)$-cell $\overset{1}{}_{e_1}A$, the bottom rectangle is tiled by the $(n+1)$-cell $\overset{e_1}{}_{e_1^-}B$ and the remaining rectangle is tiled by the $(n+1)$-cell $\overset{e_1}{}_{e_1^-}C$. The rest of the diagram is tiled in the same way than above. □

4.2.2. Theorem (Coherent critical branching lemma modulo). *Let (R, E, S) be an n-polygraph modulo such that ${}_E R_E$ is terminating. Then S is Γ-locally confluent modulo E, if and only if the two following conditions hold*

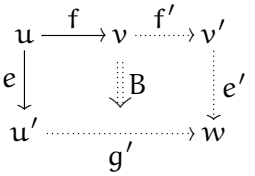
a_0) *any critical branching $(f, g) : u \Rightarrow (v, w)$ with f in $S^{*(1)}$ and g in $R^{*(1)}$ is Γ-confluent modulo E:*

a_0):



b_0) *any critical branching $(f, e) : u \Rightarrow (v, u')$ modulo E with f in $S^{*(1)}$ and e in $E^{\top(1)}$ is Γ-confluent modulo E:*

b_0):



Proof. By Proposition 4.2.1, the local Γ-confluence is equivalent to both conditions **a)** and **b)**. Let us prove that the condition **a)** (resp. **b)**) holds if and only if the condition **a_0)** (resp. **b_0)**) holds. One

implication is trivial. Suppose that conditions $\mathbf{b_0}$) holds and prove condition **b)**. The proof of the other implication is similar.

We examine all the possible forms of local branchings modulo given in 3.3.5. Local Peiffer branching modulo of the forms (3.3.6) are trivially confluent modulo:

$$\begin{array}{ccc} u \star_i v & \xrightarrow{f \star_i v} & u' \star_i v \\ {\scriptstyle u \star_i e}\downarrow & & \downarrow{\scriptstyle u' \star_i e} \\ u \star_i v' & \xrightarrow[f \star_i v']{} & u' \star_i v' \end{array} \qquad \begin{array}{ccc} w \star_i u & \xrightarrow{w \star_i f} & w \star_i u' \\ {\scriptstyle e' \star_i u}\downarrow & & \downarrow{\scriptstyle e' \star_i u'} \\ w' \star_i u & \xrightarrow[w' \star_i f]{} & w' \star_i u' \end{array}$$

and Γ-confluent modulo by definition of Γ-confluence. The other local branchings modulo are overlapping branchings modulo $(f, e) : u \Rightarrow (u', v)$ of the form (3.3.7), where f is an n-cell of $S^{*(1)}$ and e is an n-cell of $E^{\top(1)}$. There exists a whisker C on R^*_{n-1} and a critical branching $(f', e') : u_0 \Rightarrow (u'_0, v_0)$ such that $f = C[f']$ and $e = C[e']$. Following condition $\mathbf{b_0}$) the branching (f', e') is Γ-confluent, that is there exists a Γ-confluence modulo E:

$$\begin{array}{ccccc} u & \xrightarrow{f'} & v & \overset{f''}{\cdots\!\!>} & v' \\ {\scriptstyle e'}\downarrow & & \Downarrow{\scriptstyle A} & & \downarrow{\scriptstyle e''} \\ u' & & \underset{g'}{\cdots\!\!>} & & w \end{array}$$

inducing a Γ-confluence for (f, e):

$$\begin{array}{ccccc} C[u] & \xrightarrow{C[f']} & C[v] & \overset{C[f'']}{\cdots\!\!>} & v' \\ {\scriptstyle C[e']}\downarrow & & \Downarrow{\scriptstyle C[A]} & & \downarrow{\scriptstyle C[e'']} \\ C[u'] & & \underset{C[g']}{\cdots\!\!>} & & w \end{array}$$

This proves the condition **b)**. □

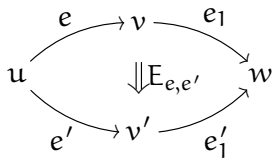
5. Coherent completion modulo

In this section, we show how to construct a two-fold coherent presentation of an $(n-1)$-category $\mathcal{C}$ starting with a presentation of this $(n-1)$-category by an n-polygraph modulo. We explain how the results presented in this section generalize to n-polygraph modulo the coherence results from n-polygraphs as given in [12, 13].

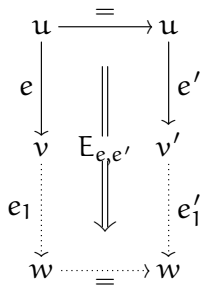
5.1. Coherent completion modulo

In this subsection, we recall the notion of coherent completion of a convergent n-polygraph and introduce the notion of coherent completion modulo for polygraphs modulo, given by adjunction of a square cell for any confluence diagram of critical branching modulo.

5.1.1. Coherent completion. Recall from [12] that a convergent n-polygraph can be extended into a coherent presentation of the category it presents. Explicitly, given a convergent n-polygraph E, we consider a family of generating confluences of E as a cellular extension of the free $(n+1,n)$-category $E^\top$ that contains exactly one $(n+1)$-cell

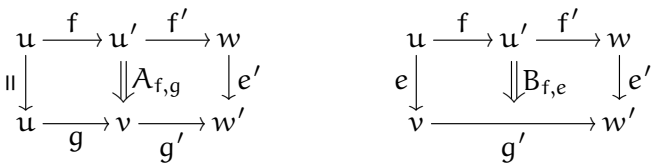


for every critical branching (f,g) of E. Any $(n+1,n)$-polygraph obtained from E by adjunction of a chosen family of generating confluences of E is a coherent presentation of the $(n-1)$-category $\overline{E}$, [12]. Note that this result was originally proved by Squier in the case $n=2$ in [23]. From such a $(n+1,n)$-polygraph we can define a double $(n+2,n)$-polygraph $(E,\varnothing,\mathrm{Cd}(E))$, where $\mathrm{Cd}(E)$ is the square extension of the $(n+1,n)$-categories $(E^\top,1)$ seen as an n-category enriched in double groupoids that contains exactly one square $(n+1)$-cell



for every critical branching (e,e') of E.

5.1.2. Coherent completion modulo. Let (R,E,S) be an n-polygraph modulo. A *coherent completion modulo* E of S is a square extension of the pair of $(n+1,n)$-categories $(E^\top,S^\top)$ whose elements are the square cells $A_{f,g}$ and $B_{f,e}$ of the following form:



(5.1.3)

for any critical branchings (f,g) and (f,e) of S modulo E, where f,g and e are n-cells of $S^{*(1)}$, $R^{*(1)}$ and $E^{\top(1)}$ respectively. Note that such completion is not unique in general and depends on the n-cells f',g',e' chosen to obtain the confluence of the critical branchings.

5.2. Coherence by E-normalization

In this subsection, we show how to obtain an acyclic square extension a the pair of categories $(E^\top,S^\top)$ coming from a polygraph modulo (R,E,S), under an assumption of normalization of S with respect to$E$.

5.2.1. Normalisation in polygraphs modulo. Let us recall the notion of normalization strategy in an n-polygraph P. Denote by $\mathcal{C}$ the $(n-1)$-category presented by P. Consider a section $s : \mathcal{C} \to P_n^*$ of the canonical projection $\pi : P_n^* \to \mathcal{C}$, that sends any $(n-1)$-cell u in $\mathcal{C}$ on a $(n-1)$-cell in P_{n-1}^* denoted by $\hat{u}$ such that $\pi(\hat{u}) = u$. A *normalization strategy for* P *with respect to* s is a map

$$\sigma : P_{n-1}^* \to P_n^*$$

that sends every $(n-1)$-cell u of P_{n-1}^* to a $(n+1)$-cell

$$\sigma_u : u \to \hat{u}.$$

Let (R, E, S) be an n-polygraph modulo. The n-polygraph modulo S is *normalizing* if any $(n-1)$-cell u admits at least one normal with respect to S, that is $\mathrm{NF}(S, u)$ is not empty.

A set X of $(n-1)$-cells in R_{n-1}^* is E-*normalizing with respect to* S if for any u in X, the set $\mathrm{NF}(S, u) \cap \mathrm{Irr}(E)$ is not empty. The n-polygraph modulo S is E-*normalizing* if it normalizing and R_{n-1}^* is E-normalizing. When S is E-normalizing, a E-*normalization strategy* σ for S, associates to each $(n-1)$-cell u in R_{n-1}^* an n-cell $\sigma_u : u \to \hat{u}$ in S^*, where $\hat{u}$ belongs to $\mathrm{NF}(S, u) \cap \mathrm{Irr}(E)$.

5.2.2. Theorem. *Let* (R, E, S) *be* n*-polygraph modulo, and* Γ *be a square extension of the pair of* $(n+1, n)$*-categories* $(E^\top, S^\top)$*. If the following conditions hold*

i) E *is convergent,*

ii) S *is* Γ*-confluent modulo* E*,*

iii) $\mathrm{Irr}(E)$ *is* E*-normalizing with respect to* S*,*

iv) ${}_E R_E$ *is terminating,*

then the square extension $\Gamma \cup \mathrm{Cd}(E)$ *is acyclic.*

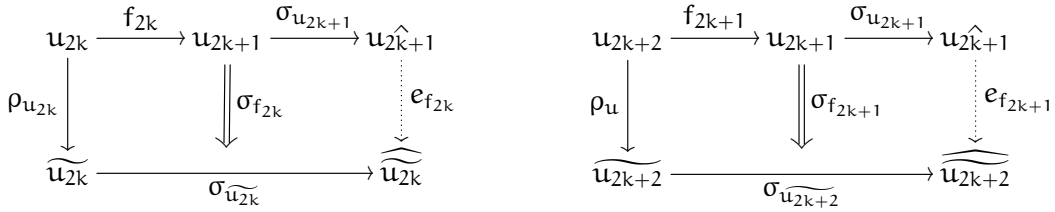
Proof. Let Γ be a square extension of $(E^\top, S^\top)$. We will denote by $\mathbf{C}$ the free n-category enriched in double groupoid $(E, S, \Gamma \cup \mathrm{Cd}(E))^{\top\top}$ generated by the double $(n+2, n)$-polygraph $(E, S, \Gamma \cup \mathrm{Cd}(E))$. We will denote by $\tilde{u}$ the unique normal form of a $(n-1)$-cell u in R_{n-1}^* with respect to E and we fix a normalization strategy $\rho_u : u \to \tilde{u}$ for E.

By termination of ${}_E R_E$, the n-polygraph modulo S is normalizing. Let us fix a E-normalization strategy $\sigma_u : u \to \hat{u}$ for S. Let us consider a square

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ u' & \xrightarrow[g]{} & v' \end{array} \tag{5.2.3}$$

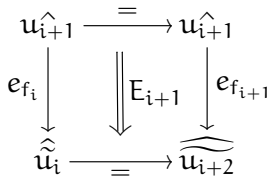
in $\mathbf{C}$. By definition the n-cell f in $S^\top$ can be decomposed (in general in a non unique way) into a zigzag sequence $f_0 \star_{n-1} f_1^- \star_{n-1} \cdots \star_{n-1} f_{2n} \star_{n-1} f_{2n+1}^-$ with source u and target v where the $f_{2k} : u_{2k} \to u_{2k+1}$ and $f_{2k+1} : u_{2k+2} \to u_{2k+1}$, for all $0 \leqslant k \leqslant n$ are n-cell of S^*, with $u_0 = u$ and $u_{2n+2} = v$.

By Γ-confluence modulo E there exist an n-cells e_{f_i} in $E^\top$ and $(n+1)$-cell σ_{f_i} in $\mathbf{C}$ as in the following diagrams:



for all $0 \leqslant k \leqslant n$. By definition of the normalization strategy σ, for any $0 \leqslant i \leqslant 2n+1$, the $(n-1)$-cell $\widehat{\widetilde{u}}$ is a normal form with respect to E, and by convergence of the n-polygraph E we have the equality $\widehat{\widetilde{u}}_i = \widetilde{\widehat{u}_{i+1}}$.

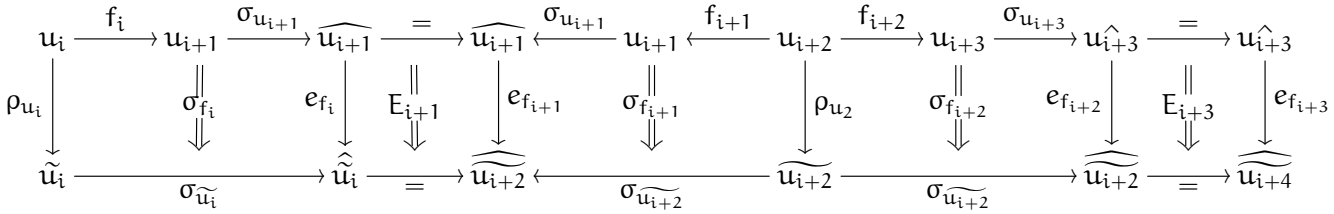
Moreover, for any $1 \leqslant i \leqslant 2n+1$, there exists a square $(n+1)$-cell in $\mathbf{C}$ as in the following diagram:



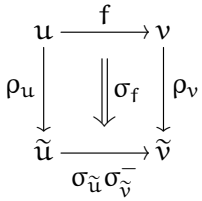
We define a square $(n+1)$-cell σ_f in $\mathbf{C}$ as the following $\diamond^v$-composition:

$$\sigma_{f_0} \diamond^v E_1 \diamond^v \sigma_{f_1} \diamond^v \sigma_{f_2} \diamond^v \dots \diamond^v \sigma_{f_{2n}} \diamond^v E_{2n+1} \diamond^v \sigma_{f_{2n+1}}$$

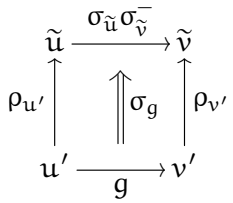
For an even integer $i \geqslant 0$



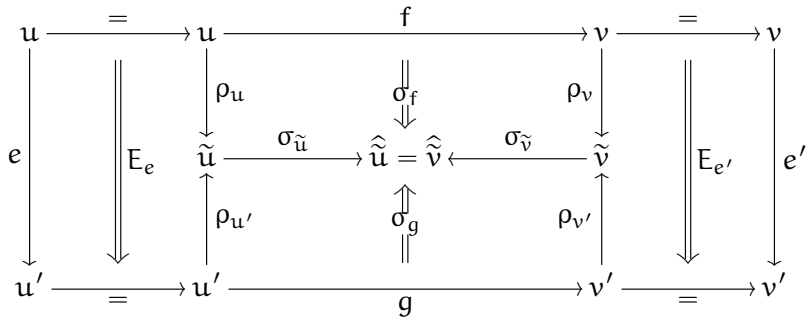
In this way, we have constructed a square $(n+1)$-cell



Similarly, we construct a square $(n+1)$-cell σ_g as follows:



using that $\widetilde{u} = \widetilde{u'}$ and $\widetilde{v} = \widetilde{v'}$ by convergence of E. We obtain a square $(n+1)$-cell $E_e \diamond^v (\sigma_f \diamond^h \sigma_g^-) \diamond^v E_{e'}$ filling the square (5.2.3), as in the following diagram:



□

5.2.4. Corollary. *Let* (R, E, S) *be an* n*-polygraph modulo such that*

i) E *is convergent,*

ii) S *is confluent modulo* E*,*

iii) ${}_E R_E$ *is terminating,*

iv) $\mathrm{Irr}(E)$ *is* E*-normalizing with respect to* S*.*

For any coherent completion Γ *of* S *modulo* E *and any coherent completion* $\mathrm{Cd}(E)$ *of* E*, the square extension* $\Gamma \cup \mathrm{Cd}(E)$ *is acyclic.*

Note that, when E is empty in Corollary 5.2.4, we recover Squier's theorem [23, Theorem 5.2] for convergent n-polygraphs, [12, Proposition 4.3.4].

5.2.5. Decreasing orders for E-normalization. Let (R, E, S) be an n-polygraph modulo. We describe a way to prove that the set $\mathrm{Irr}(E)$ is E-normalizing, laying on the definition of a termination order for R.

Given an n-polygraph P, one defines a *decreasing order operator* for P as a family of functions

$$\Phi_{p,q} : P^*_{n-1}(p, q) \to \mathbb{N}^{m(p,q)}$$

indexed by pairs of $(n-2)$-cells p and q in P^*_{n-2} satisfying the following conditions:

i) For any $(n-1)$-cells u and v in $P^*_{n-1}(p,q)$ such that there exists a n-cell $f : u \to v$ in P^*, the function $\Phi_{p,q}$ satisfy $\Phi_{p,q}(u) > \Phi_{p,q}(v)$, where $>$ is the lexicographic order on $\mathbb{N}^{m(p,q)}$. We denote by $>_{\text{lex}}$ the partial order on P^*_{n-1} defined by $u >_{\text{lex}} v$ if and only if u and v have same source p and target q and $\Phi_{p,q}(u) > \Phi_{p,q}(v)$.

ii) For any u and v in P^*_{n-1} and any whisker C on P^*_{n-1}, $u >_{\text{lex}} v$ implies that $C[u] >_{\text{lex}} C[v]$.

iii) The normal forms in $P^*_{n-1}(p,q)$ with respect to P are sent to the tuple $(0,\dots,0)$ in $\mathbb{N}^{m(p,q)}$.

Note that if an n-polygraph P admits a decreasing order operator, it is terminating. Actually, such a decreasing order is a terminating order for P which is similar to a monomial order, but that we do not require to be total.

5.2.6. Proving coherence modulo using a decreasing order. Consider an n-polygraph modulo (R,E,S) such that E is terminating. A decreasing order operator Φ for E is *compatible with* R if for any n-cell $f : u \to v$ in R^*, then $\Phi_{p,q}(u) \geqslant \Phi_{p,q}(v)$.

In that case, the set $\mathrm{Irr}(E)$ is E-normalizing with respect to R, since if u in R^*_{n-1} is a normal form with respect to E, $\Phi_{p,q}(u) = (0,\dots,0)$ in $\mathbb{N}^{m(p,q)}$ and by compatibility with R, for any n-cell $f : u \to v$ in R^*, we get $\Phi_{p,q}(v) = (0,\dots,0)$ so v is still a normal form with respect to E. We can also prove that $\mathrm{Irr}(E)$ is E-normalizing with respect to ${}_E R$ using this method, provided for any $(n-1)$-cell u in $\mathrm{Irr}(E)$ irreducible by R, any $(n-1)$-cell u' such that there is an n-cell $u \to u'$ in $E^\top$ is also irreducible by R. This is for instance the case if R is *left-disjoint from* E, that is for any $(n-1)$-cell u in $s(R)$, we have $G_R(u) \cap E_n = \varnothing$ where:

- $s(R)$ is the set of $(n-1)$-sources in R^*_{n-1} of generating n-cells in R_n,
- for any u in R^*_{n-1}, $G_R(u)$ is the set of generating n-cells in R_n contained in u.

With these conditions, we can apply Theorem 5.2.2 to obtain acyclic extensions of R or ${}_E R$.

5.3. Coherence by commutation

In this subsection, we prove that an acyclic extension of a pair $(E^\top, S^\top)$ coming from a polygraph modulo (R,E,S) can be obtained from an assumption of commuting normalization strategies for the polygraphs S and E. In particular, with further assumptions about this commutation we show to prove E-normalization.

5.3.1. Commuting normalization strategies. Let (R,E,S) be an n-polygraph modulo. Let σ (resp. ρ) a normalization strategy with respect to S (resp. with respect to E). The normalization strategies σ and ρ are *weakly commuting* if for any u in R^*_{n-1}, there exists an n-cell η_u in S^* as in the following diagram:

$$\begin{array}{ccc} u & \xrightarrow{\sigma_u} & \hat{u} \\ {\scriptstyle \rho_u}\downarrow & & \downarrow{\scriptstyle \rho_{\hat{u}}} \\ \tilde{u} & \xrightarrow[\eta_u]{} & \tilde{\hat{u}} \end{array} \tag{5.3.2}$$

Given weakly commuting normalization strategies σ and ρ, we denote by $N(\sigma,\rho)$ the square extension of the pair $(E^\top, S^\top)$ made of square of the form (5.3.2), for every $(n-1)$-cell u in R^*_{n-1}.

We say that the normalization strategies σ and ρ are *commuting* if $\eta_u = \sigma_{\widetilde{u}}$ holds, for all $(n-1)$-cell u in R^*_{n-1}. Note that, σ and ρ commutes if and only if the equality $\widehat{\widetilde{u}} = \widetilde{\widehat{u}}$ hold for all $(n-1)$-cells of R^*_{n-1}.

5.3.3. Theorem. *Let* (R, E, S) *be an* n*-polygraph modulo,* Γ *be a square extension of the pair of* $(n+1, n)$*-categories* $(E^\top, S^\top)$*,* σ *and* ρ *be normalization strategies for* S *and* E *respectively. If the following four conditions hold*

i) E *is convergent,*

ii) S *is* Γ*-confluent modulo* E*,*

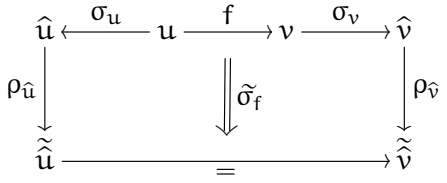
iii) ${}_E R_E$ *is terminating,*

iv) σ *and* ρ *are weakly commuting,*

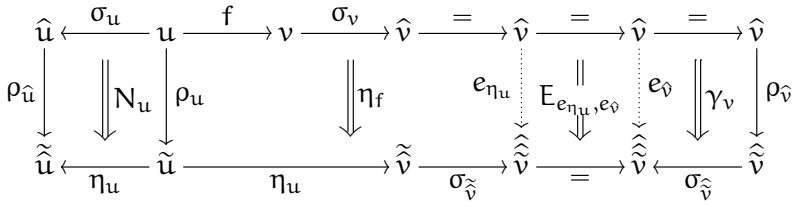
then the square extension $\Gamma \cup \mathrm{Cd}(E) \cup N(\sigma, \rho)$ *is acyclic.*

Proof. Denote by $\mathbf{C}$ the free n-category enriched in double groupoids $(E, S, \Gamma \cup \mathrm{Cd}(E) \cup N(\sigma, \rho))^{\mathbb{T}}$. For u in R^*_{n-1}, we denote by N_u the square $(n+1)$-cell in $\mathbf{C}$ corresponding to the square (5.3.2).

We prove that for any n-cell $f : u \to v$ in S^*, there exists a square $(n+1)$-cell $\widetilde{\sigma_f}$ of the following form

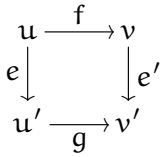


in $\mathbf{C}$. The square $(n+1)$-cell $\widetilde{\sigma_f}$ is obtained as the following composition:



where the n-cell e_{η_u} and the square $(n+1)$-cell η_f (resp. the n-cell $e_{\hat{v}}$ and the square $(n+1)$-cell γ_v) exist in $\mathbf{C}$ by Γ-confluence modulo E of S, and the square $(n+1)$-cell lives in $\mathrm{Cd}(E)^{\mathbb{T}}$.

Now, let consider a square



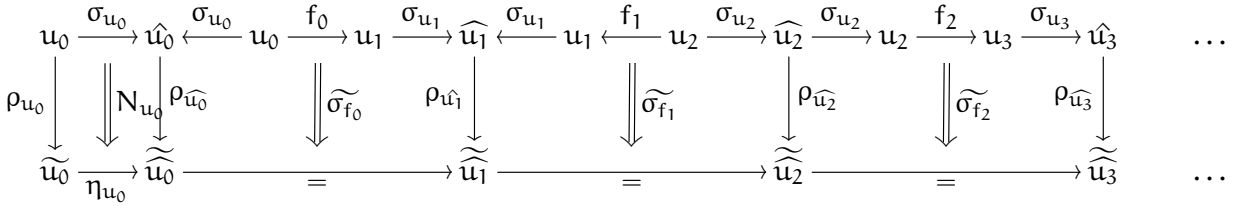
(5.3.4)

in $\mathbf{C}$. By definition the n-cell f in $S^\top$ can be decomposed (in general in a non unique way) into a zigzag sequence $f_0 \star_{n-1} f_1^- \star_{n-1} \cdots \star_{n-1} f_{2n} \star_{n-1} f_{2n+1}^-$ with source u and target v where the $f_{2k} : u_{2k} \to u_{2k+1}$

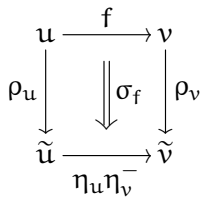
and $f_{2k+1} : u_{2k+2} \to u_{2k+1}$, for all $0 \leqslant k \leqslant n$ are n-cell of S^*, with $u_0 = u$ and $u_{2n+2} = v$. We define a square $(n+1)$-cell σ_f as the following vertical composition:

$$N_u \diamond^v \widetilde{\sigma_{f_0}} \diamond^v \widetilde{\sigma_{f_1}} \diamond^v \dots \diamond^v \widetilde{\sigma_{f_{2n+1}}} \diamond^v N_v$$

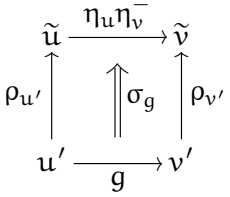
as depicted on the following diagram



In this way, we have constructed a square $(n+1)$-cell



Similarly, we construct a square $(n+1)$-cell σ_g as follows:

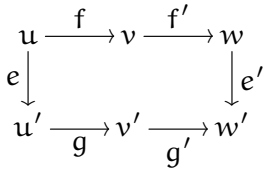


using that $\widetilde{u} = \widetilde{u'}$ and $\widetilde{v} = \widetilde{v'}$ by convergence of E. We obtain a square $(n+1)$-cell filling the square (5.3.4), as in the proof of Theorem 5.2.2. □

Moreover, when σ and ρ are commuting, $\mathrm{Irr}(E)$ is E-normalizing with respect to S since $\widehat{\widetilde{u}} = \widetilde{\widehat{u}}$ implies that the normal form $\widehat{\widetilde{u}}$ with respect to S also is a normal form with respect to E. Then Theorem 5.2.2 applies, to prove that $\Gamma \cup \mathrm{Sph}(E^\top, 1)$ is acyclic.

5.3.5. Remark. One can recover the fact that with the hypothesis of Theorem 5.3.3 and the assumption that the equality $\eta_u = \sigma_{\widetilde{u}}$ holds for any u in R^*_{p-1}, we do not need the square $(n+1)$-cells N_u in the coherent extension, using the following lemma on the square (5.3.2).

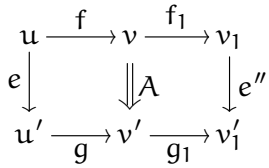
5.3.6. Lemma. *Let S be an n-polygraph modulo such that ${}_E R_E$ is terminating, and Γ be a square extension of the pair of $(n+1,n)$-categories $(E^\top, S^\top)$ such that S is Γ-confluent modulo E. Then any square in $(E, S, \Gamma)^{\top\!\!\top, v}$ of the form*



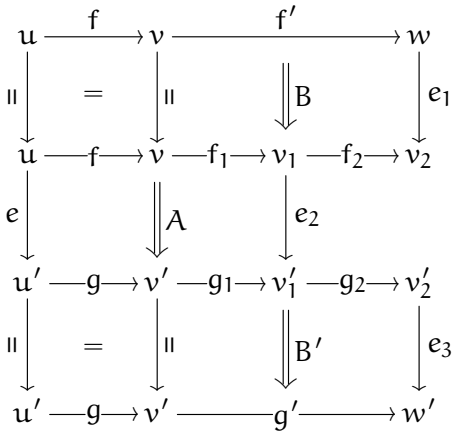
(5.3.7)

such that w and w' are normal forms with respect to S is the boundary of a square $(n+1)$-cell in $(E,S,\Gamma)^{\top\!\!\top,\nu}$.

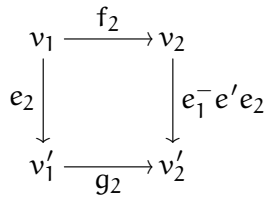
Proof. Let us consider a square as in (5.3.7). By Γ-confluence of S modulo E on the branching (f,e,g), there exists a Γ-confluence as in the following diagram:



By Γ-confluence on the branchings (f',f_1) and (g_1,g') of S, there exist square $(n+1)$-cells B and B' as follows:



Then, we use Huet's double induction as in Section 4 to prove that the square



is the boundary of a square $(n+1)$-cell in $(E,S,\Gamma)^{\top\!\!\top,\nu}$. □

5.4. A diagrammatic rewriting toy example

In this subsection, we present an application of Theorem 5.2.2 on a toy example in the context of diagrammatic rewrting. We consider a presentation of a pivotal monoidal category, seen as a pivotal 2-category with only one 0-cell presented by a 3-polygraph. The pivotal structure can be interpreted

by isotopies relations on the 2-cells of the 2-category. Such relations produce many critical branching with primary rules of the presentation. In this example, by using the structure of polygraph modulo, we show how manage such isotopy rules with respect the primary rules in the computation of a coherent presentation of the given monoidal category. In particular, we illustrate the method with a kind of relation arising in many presentation of monoidal categories, relation (5.4.7), see for instance Khovanov-Lauda's 2-category [18] which categorifies quantum groups associated with symmetrizable Kac-Moody algebras, or in the definition of Heisenberg category as given in [4, 19].

5.4.1. A toy example. We consider the 3-polygraph P defined by the following data:

i) only one 0-cell,

ii) two 1-cells $\wedge$ and $\vee$,

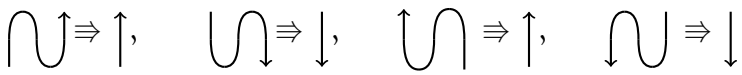
iii) eight 2-cells pictured by

(5.4.2)

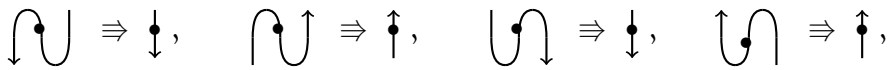
(5.4.3)

iv) the 3-cells of P are given by:

a) the three family of 3-cells of the 3-polygraph of pearls from [12]:



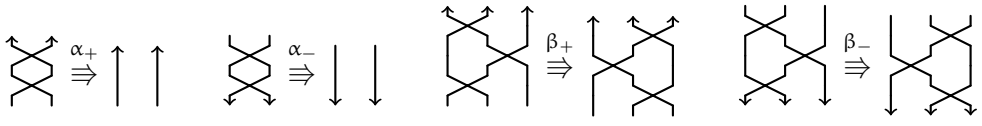
(5.4.4)



(5.4.5)

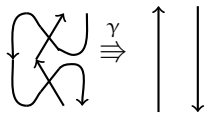
(5.4.6)

b) the 3-cells of the 3-polygraph of permutations for both upward and downward orientations of strands:



(5.4.7)

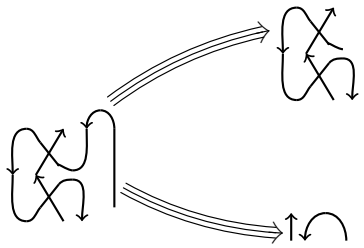
c) a 3-cell



(5.4.8)

Note that the relations (5.4.4 – 5.4.6) are isotopy relations, corresponding to the fact that the generating 1-cells $\vee$ and $\wedge$ are biadjoints in the 2-category $\overline{\mathsf{P}}$ presented by P, and cups and caps 2-cells are units and counits for these adjunctions. Relations implying dots also ensure that the dot 2-cell is a cyclic 2-morphism in the sense of [6] for the biadjunction $\vee \vdash \wedge \vdash \vee$ making $\overline{\mathsf{P}}$ a *pivotal 2-category*.

5.4.9. Confluence of polygraph P**.** The 3-polygraph P is not confluent since the branching



(5.4.10)

is not confluent. Moreover, solving this obstruction to confluence by using a Knuth-Bendix completion may create a great number of relations, making analysis of confluence from critical branchings inefficient. To tackle this issue, we use rewriting modulo isotopy.

5.4.11. Confluence modulo isotopy. We consider the 3-polygraph E defined by the following data

i) $\mathsf{E}_{\leqslant 1} = \mathsf{P}_{\leqslant 1}$,

ii) it has six 2-cells given in (5.4.3) and the dot 2-cells in 5.4.2,

iii) the isotopy 3-cells (5.4.4 – 5.4.6) of the 3-polygraph of pearls.

Let R be a 3-polygraph such that $\mathsf{R}_{\leqslant 2} = \mathsf{P}_{\leqslant 2}$ where P is the 3-polygraph of 5.4.1, and whose 3-cells are given by $(\alpha_\pm, \beta_\pm, \gamma)$ of (5.4.7 – 5.4.8), and let us consider the 3-polygraph modulo ${}_{\mathsf{E}}\mathsf{R}$. Following 3.4.1, the only critical branchings we have to consider are those of the form (f, g) with f in ${}_{\mathsf{E}}\mathsf{R}^{*(1)}$ and g in $\mathsf{R}^{*(1)}$. The branching (5.4.10) is not such a branching because the top 3-cell belongs to $\mathsf{E}^\top$, and the top-right 2-cell is not reducible by R. Moreover, one can check that the only critical branchings we have to consider are given by pairs (f, g) of 3-cells both in $\mathsf{R}^{*(1)}$. The 3-cell γ in $\mathsf{R}^{*(1)}$ does not overlap with $\alpha_\pm$ or $\beta_\pm$, so the only critical branchings we have to consider are those of the 3-polygraph of permutations described in [12, 5.4.4], with either upward or downward orientated strands.

5.4.12. Decreasing order operator for E**-normalization.** The 3-polygraph R is left-disjoint from E, since no caps and cups 2-cells appear in the sources of the generating 3-cells of R. Following 5.2.6, we prove that $\mathrm{Irr}(\mathsf{E})$ is E-normalizing with respect to ${}_{\mathsf{E}}\mathsf{R}$ using a decreasing order operator Φ for E compatible with R.

5.4.13. Lemma. *Let* E *and* R *be the* 3*-polygraphs defined above. There exists a decreasing operator order* Φ *for* E *compatible with* R*.*

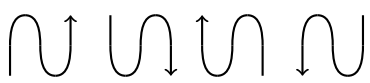
Proof. For any 1-cells p and q in R_1^*, we set $m(p, q) = 2$ and for any 2-cell u of source p and target q in R_2^*, $\Phi_{p,q}(u) = (\mathrm{ldot}(u), \mathrm{I}(u))$ where:

i) $\mathrm{ldot}(u)$ counts the number of left-dotted caps and cups, adding for such cap and cup the number of dots on it. In particular, for any n in $\mathbb{N}^*$, we have

$$\mathrm{ldot}\left(\begin{array}{c} n \\ \cap \end{array} \right) = \mathrm{ldot}\left(\begin{array}{c} n \\ \cup \end{array} \right) := n + 1$$

for both orientations of strands.

ii) $I(u)$ counts the number of instances of one of the following 2-cells of R_2^* in u:



For any 3-cell $u \Rrightarrow v$ in E, we have $\Phi(u) > \Phi(v)$ and that $\Phi(u,u) = (0,0)$ for any u in $\mathrm{Irr}(E)$. Moreover, Φ is compatible with R because rewritings with respect to R do not make the dot 2-cell move around a cup or a cap, or create sources of isotopies. □

As a consequence of Theorem 5.2.2, we deduce an acyclic square extension of the pair of $(3,2)$-categories $(E^\top, {}_E R^\top)$. This square extension is made of the ten elements given by the diagrams of the homotopy basis for the 3-polygraph of permutations from [12, Section 5.4.4] for both upward and downward orientations of strands and the 16 elements given by the diagrams of the homotopy basis or the 3-polygraph of pearls in [12, Section 5.5.3] for both orientations of strands form.

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BENJAMIN DUPONT
bdupont@math.univ-lyon1.fr
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex, France

PHILIPPE MALBOS
malbos@math.univ-lyon1.fr
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex, France