



**HAL**  
open science

# Infinite periodic groups of even exponents

Rémi Coulon

► **To cite this version:**

| Rémi Coulon. Infinite periodic groups of even exponents. 2018. hal-01898778v2

**HAL Id: hal-01898778**

**<https://hal.science/hal-01898778v2>**

Preprint submitted on 3 Nov 2018 (v2), last revised 12 Jan 2021 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Infinite periodic groups of even exponents

Rémi Coulon

November 3, 2018

## Abstract

We give a new proof that free Burnside groups of sufficiently large even exponents are infinite. The method can also be used to study (partially) periodic quotients of any group which admits an action on a hyperbolic space satisfying a weak form of acylindricity.

*Keywords.* Burnside groups; periodic groups, geometric group theory; small cancellation theory; hyperbolic geometry.

*MSC.* 20F50, 20F65, 20F67, 20F06

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Hyperbolic geometry</b>	<b>9</b>
2.1	General facts	9
2.2	Isometries	13
2.3	Group action	17
<b>3</b>	<b>Invariants of a group action</b>	<b>19</b>
3.1	Geometric invariants	19
3.2	Mixed invariants	22
<b>4</b>	<b>Small cancellation theory</b>	<b>27</b>
4.1	General setting	27
4.2	A few additional facts regarding the cone-off space	31
4.3	Apex stabilizer in the quotient space.	32
4.4	Lifting properties	37
4.5	The action of $\tilde{G}$ on $\tilde{X}$	43
4.6	Structure of elementary subgroups	45
4.7	Invariants of $\tilde{G}$ acting on $\tilde{X}$	56
4.7.1	Geometric invariants	56
4.7.2	Mixed invariants	58

<b>5</b>	<b>Periodic groups</b>	<b>68</b>
5.1	Induction step	68
5.2	Construction of periodic groups	72
5.3	Examples	75

# 1 Introduction

## Historical background

A group  $G$  has exponent  $n \in \mathbf{N}$  if  $g^n = 1$  for every  $g \in G$ . In 1902, Burnside asked whether every finitely generated group with finite exponent was necessarily finite [5]. Despite its simplicity, this question remained open for a long time and motivated many developments in group theory. The class of groups with exponent  $n$  forms a group variety whose free elements are the *free Burnside groups of exponent  $n$* . More concretely, the free Burnside group of rank  $r$  and exponent  $n$ , that we denote by  $\mathbf{B}_r(n)$ , admits the following presentation

$$\mathbf{B}_r(n) = \langle a_1, \dots, a_r \mid x^n, \forall x \rangle.$$

A major breakthrough in the subject was achieved by Novikov and Adian in 1968 [29]. They proved that  $\mathbf{B}_r(n)$  is infinite provided  $r \geq 2$  and  $n$  is a sufficiently large *odd* exponent. Later Ol'shanskiĭ provided an alternative proof of the same result [30]. Despite these progresses the case of *even* exponents hold up longer. It was only in the early 90's that Ivanov [24] and Lysenok [26] independently proved that free Burnside groups of sufficiently large even exponents are also infinite.

The aforementioned results rely (more or less explicitly) on a combinatorial approach of small cancellation theory (using combinatorics on words and/or van Kampen diagram). Historically, the respective works of Novikov-Adian and Ol'shanskiĭ appeared before the emergence of hyperbolicity as formalized by Gromov in [19]. However hyperbolic groups and their various generalizations provide a perfect framework offering new insights into small cancellation theory. See for instance [21, 11] for a survey on the topic. Using this geometric point of view, Delzant and Gromov revisited the Burnside problem making an *explicit* use of hyperbolic geometry [15]. Nevertheless their work only applies to odd exponents.

**Main results.** In this article we provide a new approach to the free Burnside groups with *even* exponents based on the geometrical ideas of Delzant and Gromov [15]. More precisely we prove the following statement.

**Theorem 1.1.** *Let  $r \geq 2$ . There exists a critical exponent  $N_0 \in \mathbf{N}$  such that for every integer  $n \geq N_0$ , the free Burnside group  $\mathbf{B}_r(n)$  is infinite.*

Not only is our approach substantially shorter than the one of Lysenok and Ivanov (200 and 300 pages respectively) it also gives a way to produce (partially) periodic quotients of many groups as soon as they carry a certain form

of negative curvature. Our next theorem is originally due to Ol’shanskii and Ivanov [31, 25] answering a question of Gromov [19]. Given an arbitrary group  $G$ , we write  $G^n$  for the (normal) subgroup of  $G$  generated by the  $n$ -th power of all its elements.

**Theorem 1.2.** *Let  $G$  be a non-elementary hyperbolic group. There exist  $p, N_0 \in \mathbf{N}$ , such that for every integer  $n \geq N_0$  that is a multiple of  $p$ , the quotient  $G/G^n$  is infinite. Moreover*

$$\bigcap_{n \geq 1} G^n = \{1\}.$$

More generally if  $G$  is a group acting acylindrically on a Gromov hyperbolic space  $X$  (see Section 5.2 for a precise definition) then for arbitrary large exponents  $n \in \mathbf{N}$ , we are able to produce a partially  $n$ -periodic quotient of  $G$  (Theorem 5.7), i.e. a quotient  $Q$  of  $G$  such that

- (i) every elliptic subgroup of  $G$  (for its action on  $X$ ) embeds in  $Q$ ,
- (ii) for every  $q \in Q$ , either  $q$  is the image of an elliptic element of  $G$  or  $q^n = 1$ .

In particular it applies to the mapping class group of a surface acting on the associated curve complex (Theorem 5.13) or certain amalgamated products acting on their corresponding Bass-Serre trees (Theorem 5.14). For instance if  $G = A * B$  is a free product, the corresponding quotient  $Q$  corresponds to the  $n$ -periodic product of  $A$  and  $B$ , see for instance [1]. Those statements generalize the previous work of the author [10]. Note that the same strategy could also be used to study the outer automorphism group of free Burnside groups with even exponent, extending some other work of the author [8]. Nevertheless to limit the length of the article we decided not to detail that part.

## A geometrical approach

Let us highlight a few important ideas involved in the proofs. For simplicity we restrict our attention to free Burnside groups as this case already covers all the difficulties. As shown by Ivanov and Lysenok, free Burnside groups with (sufficiently large) odd or even exponent have a considerably different algebraic structure. For instance if  $n$  is odd, every finite subgroup of  $\mathbf{B}_r(n)$  is cyclic. By contrast if  $n$  is even,  $\mathbf{B}_r(n)$  contains arbitrarily large direct products of dihedral groups. Nevertheless the global strategy to study those groups remains the same.

**A sequence of approximation groups.** Given a large exponent  $n \in \mathbf{N}$ , one produces by induction an *approximation sequence* of hyperbolic groups

$$\mathbf{F}_r = G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \cdots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \dots \quad (1)$$

whose directly limit is exactly  $\mathbf{B}_r(n)$ . At each step  $G_{k+1}$  is obtained from  $G_k$  by adding relations of the form  $h^n = 1$ , where  $h$  run over the set of all “small” loxodromic elements of  $G_k$ . The goal is to prevent this sequence to collapse to a finite group. To that end we use small cancellation theory.

**Geometric small cancellation.** Let  $S$  be a finite set and  $R$  a collection of cyclically reduced words of  $\mathbf{F}(S)$ . Assume for simplicity that  $R$  is invariant under taking cyclic permutations and inverses and write  $\ell$  for the length of its smallest element. Given  $\lambda \in (0, 1)$ , the group presentation  $\langle S | R \rangle$  satisfies the classical  $C'''(\lambda)$  small cancellation condition if every prefix  $u$  of two distinct relations in  $R$  has length at most  $|u| < \lambda\ell$ . For small values of  $\lambda$ , one understands precisely the properties of the corresponding group  $\bar{G} = \mathbf{F}(S)/\langle\langle R \rangle\rangle$ . For instance if  $\lambda \leq 1/6$ , then  $\bar{G}$  is hyperbolic. The  $C'''(\lambda)$  condition can advantageously be reformulated as follows. Let  $X$  be the Cayley graph of  $\mathbf{F}(S)$  with respect to  $S$ . The presentation  $\langle S | R \rangle$  satisfies the  $C'''(\lambda)$  condition if for every distinct  $r_1, r_2 \in R$ , the overlap between the respective axes of  $r_1$  and  $r_2$  has length less than  $\lambda\ell$ , where  $\ell$  is also the smallest translation length of an element in  $R$ .

With this idea in mind, one can extend the small cancellation theory to the context of hyperbolic groups [19, 32, 14] (or more generally of groups acting on a hyperbolic space). Let  $G$  be a group acting properly co-compactly by isometries on a hyperbolic space  $X$  and  $R$  a subset of  $G$ , which is invariant under conjugation. Roughly speaking we will say that  $R$  satisfies a small cancellation condition if given any two distinct  $r_1, r_2 \in R$ , the length  $\Delta(r_1, r_2)$  on which the respective axes of  $r_1$  and  $r_2$  fellow travel is very small compare to the translation lengths  $\|r_1\|$  and  $\|r_2\|$  (see Figure 1). In this situation the quotient  $\bar{G} = G/\langle\langle R \rangle\rangle$

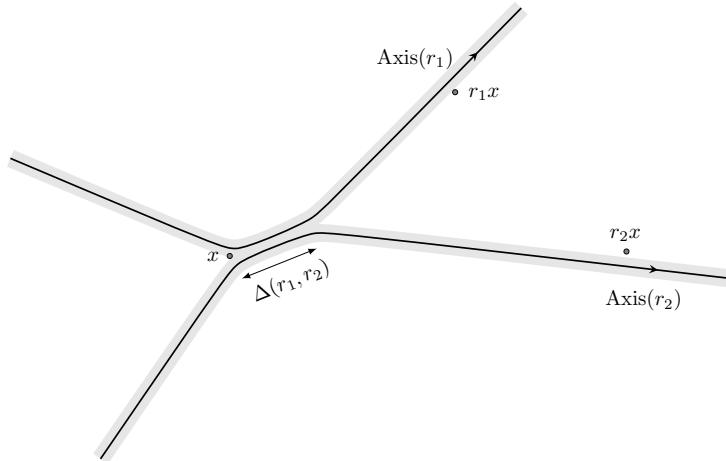


Figure 1: Overlap between two relations seen in the hyperbolic space  $X$ . The translation length  $\|r_i\|$  of  $r_i$  is roughly speaking the distance between  $x$  and  $r_ix$ .

is a non-elementary hyperbolic group. Under this hypothesis, Gromov explains in [20] how to let this group act on a hyperbolic space  $\bar{X}$  whose geometry is finer than the one of the Cayley graph of  $\bar{G}$ , see also [15, 7, 9]. Assume for instance that  $\bar{G}$  is a quotient of the form  $\bar{G} = G/\langle\langle h^n \rangle\rangle$  where  $h$  is a loxodromic element and  $n$  a (large) exponent. Then  $\bar{M} = \bar{X}/\bar{G}$  can be seen as an orbifold (whose fundamental group is  $\bar{G}$  and universal cover is  $\bar{X}$ ) and comes with an

analog of Margulis’ thin/thick decomposition for hyperbolic manifolds. The thin part corresponds to the neighborhood of a single singular point whose isotropy group is exactly the maximal finite subgroup  $\bar{F} \subset \bar{G}$  containing the image of  $h$ . The pre-image in  $\bar{X}$  of the thin part is roughly speaking the collection of all  $\bar{G}$ -translates of an  $\bar{F}$ -invariant hyperbolic disc  $\mathcal{D} \subset \bar{X}$ . Moreover there exists a natural map  $q: \bar{F} \rightarrow \mathbf{D}_n$ , where  $\mathbf{D}_n$  is a the dihedral group of order  $2n$ , such that the action of  $\bar{F}$  on  $\mathcal{D}$  is identified via  $q$  to the natural action of  $\mathbf{D}_n$  on the disc.

Adopting this point of view, we associate to each approximation group  $G_k$  in (1) a hyperbolic space  $X_k$  on which  $G_k$  acts properly co-compactly. The goal will be to prove that at each step the new relations defining  $G_{k+1}$  will satisfy a small cancellation condition in the above sense (i.e. relative to the action of  $G_k$  on the space  $X_k$ ). It has the following main advantage: almost every needed property of the relations defining  $G_k$  is captured by the hyperbolicity of  $X_k$ . Consequently when studying the quotient map  $G_k \twoheadrightarrow G_{k+1}$  one can completely forget the relations defining  $G_k$  and rely only on the geometry of  $X_k$ . This allows us to formulate – unlike in [29, 30, 26, 24] – the induction hypothesis used to build the approximation sequence (1) in a rather compact form (see Proposition 5.1).

**A Margulis’ lemma.** As mentioned above, the main challenge when building the approximation sequence (1) is to make sure that  $G_k$  is not eventually finite. This will not happen if, at each step, the relations (of the form  $h^n = 1$ ) used to define  $G_{k+1}$  from  $G_k$  satisfy a small cancellation condition. Therefore, given any two loxodromic elements  $g_1, g_2 \in G_k$  which do not generate an elementary subgroup, one needs to control *uniformly, independently of  $k$*  the ratio

$$\frac{\Delta(g_1, g_2)}{\max\{\|g_1\|, \|g_2\|\}} \quad (2)$$

where  $\Delta(g_1, g_2)$  measures the “overlap” between the respective axes of  $g_1$  and  $g_2$  in  $X_k$ . If  $X_k$  was a simply connected manifold with *pinched* negative sectional curvature, such estimate would follow from Margulis’ Lemma. However hyperbolicity only provides an upper bound for the curvature of the space. To compensate this lack, one usually uses assumptions on the action of the group (e.g. the fact that the action is proper co-compact or acylindrical). For instance, a first (naive) attempt to bound the ratio (2) could work as follows. Suppose that  $g_1, g_2 \in G_k$  are two loxodromic elements such that  $\Delta(g_1, g_2) > N \max\{\|g_1\|, \|g_2\|\}$ . It is a standard exercise of hyperbolic geometry to see that there exists a point  $x \in X_k$  such that for every  $i \in \llbracket 0, N-1 \rrbracket$ , the commutator  $[g_1, g_2^i]$  moves  $x$  by at most  $100\delta_k$  (where  $\delta_k$  is the hyperbolicity constant of  $X_k$ ), see Figure 2. In particular if  $N$  exceeds the number of elements in  $\{u \in G_k : |ux - x| \leq 100\delta_k\}$ , then  $g_1$  commutes with a power of  $g_2$ , thus  $\langle g_1, g_2 \rangle$  is non-elementary. So, roughly speaking, the ratio (2) is bounded above by the cardinality of the (almost) stabilizer of points in  $X_k$ . This strategy has a major weakness though: if  $n$  is an even exponent, the cardinality of finite

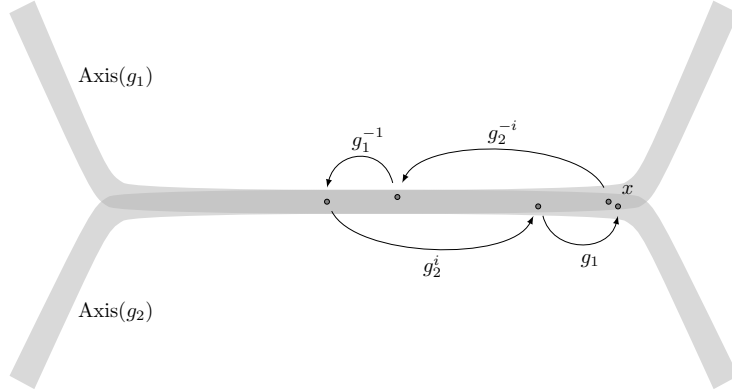


Figure 2: The point  $x$  is hardly moved by the commutator  $[g_1, g_2^i]$ .

subgroups is not uniformly bounded along the sequence  $(G_k)$ . During the process we will encounter points in  $X_k$  with arbitrary large stabilizers. Hence this method cannot be used to keep the ratio (2) uniformly bounded. Any refinement of the above argument using acylindrical actions of  $G_k$  on  $X_k$  – see for instance [13] – shall fail in the same way.

To bypass this difficulty one associates to the action of  $G_k$  on  $X_k$  several numerical invariants. Heuristically,

- (i)  $A(G_k, X_k)$  is characterized as follows: if  $S$  is any finite subset of  $G_k$  generating a non-elementary subgroup, then the set of points in  $X_k$  which are “hardly” moved by every element of  $S$  has diameter at most  $A(G_k, X_k)$  (see Definition 3.2).
- (ii)  $\nu(G_k, X_k)$  is the smallest integer  $m$  with the following property: let  $g, h \in G_k$  with  $h$  loxodromic. If  $g, hgh^{-1}, h^2gh^{-2}, \dots, h^mgh^{-m}$  generate an elementary subgroup, then so do  $g$  and  $h$  (see Definition 3.3).

If the action of  $G_k$  on  $X_k$  is acylindrical, then both  $A(G_k, X_k)$  and  $\nu(G_k, X_k)$  are finite (however the converse does not hold). The quantity  $A(G_k, X_k)$  can be thought of as a local version of the ratio (2). Indeed, its definition only involves “small” elements. Combined with the  $\nu$ -invariant one recovers the following global analogue of Margulis’ Lemma: if  $g_1, g_2 \in G_k$  do not generate an elementary subgroup, then

$$\Delta(g_1, g_2) \leq [\nu(G_k, X_k) + 2] \max \{ \|g_1\|, \|g_2\| \} + A(G_k, X_k) + 1000\delta_k,$$

see for instance [10, Proposition 3.34] or Proposition 3.4. Consequently, in order to make sure that for every  $k \in \mathbf{N}$ , the relations defining  $G_{k+1}$  from  $G_k$  satisfy a suitable small cancellation condition, it suffices to control (among others) the values of  $A(G_k, X_k)$  and  $\nu(G_k, X_k)$  all along the approximation sequence (1). This was done in [10] in the absence of even torsion.

As soon as even torsion is involved, the situation becomes much more delicate. In particular, the  $\nu$ -invariant does not behave very well when passing to a quotient (see for instance the discussion and examples at the beginning of [Section 4.7.2](#)). It results from the fact that the algebraic structure of finite subgroups of  $\mathbf{B}_r(n)$  is rather intricate, see for instance Lysenok [\[27\]](#).

**Structure of elementary subgroups.** As we mentioned earlier, if  $n$  is odd, then every maximal finite subgroup of  $\mathbf{B}_r(n)$  is isomorphic to the cyclic group  $\mathbf{Z}_n$ . Moreover finite subgroups “stabilize” along the approximation sequence [\(1\)](#). More precisely we have the following property: if  $F_0$  is a finite subgroup of some  $G_k$ , then for every  $\ell \geq k$ , every finite subgroup of  $G_\ell$  containing the image of  $F_0$  actually comes from a finite subgroup  $F$  of  $G_k$  which already contains  $F_0$ . By contrast, if  $n$  is even, finite subgroups may “grow” when taking successive quotients. Let us illustrate this fact with the following toy example.

**Example 1.3.** Assume that  $n$  is a (large) even exponent. Start with the free group  $G_0 = \mathbf{F}_2$  generated by  $a$  and  $b$  and set

$$G_1 = G_0 / \langle\langle a^n, b^n, (ba)^n \rangle\rangle.$$

In  $G_1$  the elements  $s_0 = a^{n/2}$ ,  $s_1 = b^{n/2}$  and  $s_2 = (ba)^{n/2}$  all generate a subgroup isomorphic to  $\mathbf{Z}_2$ . Consequently in the quotient

$$G_2 = G_1 / \langle\langle (s_0 s_1)^n, (s_0 s_2)^n \rangle\rangle$$

$s_0$  and  $s_1$  generates a subgroup isomorphic to the dihedral group  $\mathbf{D}_n$  of order  $2n$ . In particular,  $s_0$  commutes with the involution  $u_1 = s_0(s_0 s_1)^{n/2}$ . Similarly  $s_0$  commutes with the involution  $u_2 = s_0(s_0 s_2)^{n/2}$ . Form now the quotient

$$G_3 = G_2 / \langle\langle (u_1 u_2)^n \rangle\rangle.$$

Note that  $s_0$ ,  $u_1$  and  $u_2$  generate a subgroup isomorphic to  $\mathbf{Z}_2 \times \mathbf{D}_n$ . In this example,  $F_1 = \langle s_0 \rangle = \mathbf{Z}_2$  is a finite subgroup of  $G_1$ . Its image in  $G_2$  (respectively  $G_3$ ) embeds in  $F_2 = \langle s_0, s_1 \rangle = \mathbf{D}_n$  (respectively  $F_3 = \langle s_0, u_1, u_2 \rangle = \mathbf{Z}_2 \times \mathbf{D}_n$ ). However  $F_2$  (respectively  $F_3$ ) is not the image of a finite subgroup of  $G_1$  (respectively  $G_2$ ). We stopped our example after three steps. However one can proceed further and embeds  $F_0$  in an arbitrarily large product of the form  $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 \times \mathbf{D}_n$ .

The previous example suggests that  $G_k$  – and thus  $\mathbf{B}_r(n)$  – contains nested copies of dihedral groups. As mentioned above, one of the advantages of the space  $X_k$  (compare to the Cayley graph of  $G_k$ ) is that one “sees” some of these dihedral groups acting as the isometry group of a disc. Unfortunately this is not sufficient to capture all the properties of finite subgroups of  $G_k$ . To sort a bit this nested structure, we introduce the notion of *dihedral germ*. A dihedral germ of  $G_k$  is an elliptic subgroup  $C$  (for its action on  $X_k$ ) containing a subgroup  $C_0$  which is normalized by a loxodromic element and such that  $[C : C_0]$  is a power of 2. As suggested by the terminology, the dihedral germs are exactly the finite



subgroups of  $G_k$  that may eventually grow in  $G_{k+1}$ , i.e. be embedded in a larger finite subgroup of  $G_{k+1}$  that does not come from  $G_k$ . This typically arises as follows.

**Example 1.4.** Assume that  $A$  is a finite subgroup of  $G_k$  and  $C$  a subgroup of  $A$  of index 2 which is normalized by a loxodromic element  $h \in G_k$ . In particular,  $A$  is a dihedral germ. Suppose for simplicity that  $h^n$  is trivial in  $G_{k+1}$ . Let  $a \in A \setminus C$ . Seen in  $G_{k+1}$ , the group  $C$  has index two in both  $A = \langle C, a \rangle$  and  $B = \langle C, h^{n/2} \rangle$ . In particular  $A$  and  $B$  generate an elementary subgroup  $E$  of  $G_{k+1}$  which is (most of the time) isomorphic to  $E = A *_C B$  and such that  $E/C = \mathbf{D}_\infty$ . As an element of  $G_{k+1}$  the product  $t = ah^{n/2}$  has infinite order. However, since  $\mathbf{B}_r(n)$  is the direct limit of  $(G_k)$ , we have  $t^n = 1$  in  $G_\ell$  for some  $\ell > k + 1$ . The image in  $G_\ell$  of  $E$  is actually isomorphic to  $E/\langle t^n \rangle$  which is a finite group that strictly contains the dihedral germ  $A$ . Nevertheless there is no finite subgroup  $F$  of  $G_k$  containing  $A$  such that the canonical quotient map  $G_k \rightarrow G_\ell$  induces an isomorphism preserving  $A$  from  $F$  onto  $E/\langle t^n \rangle$ .

It turns out that the dihedral germs of  $G_k$  are exactly its 2-subgroups (when studying general periodic quotients different from free Burnside groups, those dihedral germs are slightly more complicated). A careful analysis of dihedral germs allows us to prove that every finite subgroup of  $G_k$  embeds in a direct product of the form  $\mathbf{D}_n \times \mathbf{D}_{n_2} \times \cdots \times \mathbf{D}_{n_2}$  where  $n_2$  is the largest power of 2 dividing  $n$ . In particular finite subgroups of  $G_k$  share many identities with finite dihedral groups. Those identities can be used to control a variation on the  $\nu$ -invariant which captures both geometric and algebraic features of the groups  $G_k$  (see [Definition 3.10](#)). Once we control the  $\nu$ -invariant along the sequence (1), a uniform estimate of the quantity  $A(G_k, X_k)$  follows rather easily ([Proposition 4.46](#)). Those two invariants (together with the injectivity radius of  $G_k$  acting on  $X_k$ ) provide a sufficient control to show that each group  $G_{k+1}$  is actually a small cancellation quotient of  $G_k$ . Therefore it is hyperbolic and non-elementary, which ensures that at the limit  $\mathbf{B}_r(n)$  is infinite.

**Critical exponent.** All these arguments actually only work provided  $n$  is divisible by a large power of 2, namely 128 for free Burnside groups. Nevertheless, given any two integers  $p, n \in \mathbf{N}$ , the group  $\mathbf{B}_r(pn)$  naturally maps onto  $\mathbf{B}_r(n)$ . Since Burnside groups of large odd exponents are known to be infinite, we can conclude that free Burnside groups of sufficiently large exponents are infinite. The works of Ivanov [24] and Lysenok [26] have a similar restriction. They require  $n$  to be divisible by  $2^9$  and 16 respectively. Using [27] as a “black box” our proof can be adapted for large exponents  $n$  which are only divisible by 16. However for completeness and simplicity we preferred to detail our own understanding of finite subgroups of  $\mathbf{B}_r(n)$ .

According to [Theorem 1.1](#) there exists a critical exponent  $N_0$  such that for every integer  $n \geq N_0$ , the group  $\mathbf{B}_r(n)$  is infinite. In our method,  $N_0$  is directly related to the parameters of the Small cancellation Theorem, see Equations (22)–(25). Since our approach to small cancellation is *qualitative* we do not

provide an explicit value of  $N_0$ . Nevertheless, for a general group  $G$  we stress how this critical exponent depends on the action of  $G$  on a hyperbolic space  $X$  (see [Theorem 5.4](#)). An interested reader could go through all the arguments with a *quantitative* point of view to get an estimate of  $N_0$ . However the resulting  $N_0$  would most likely be very large.

## Outline of the paper

The proof that we present here is essentially self-contained. Beside hyperbolic geometry, the arguments only rely on geometrical small cancellation theory which is now well understood. See for instance [\[11, 21\]](#) for a survey on the topic. For the benefit of the reader we did not attempt to write the shortest possible proof. In particular, we added in the course of the article numerous discussions, examples and figures to highlight the main difficulties and illustrate the important results.

In [Section 2](#) we make a short review of hyperbolic geometry. We define in [Section 3](#) all the geometric and algebraic invariants needed to control the small cancellation parameters when building the approximation sequence [\(1\)](#). In [Section 4](#) we first review the main properties of small cancellation theory. Given a group  $G$  acting on a hyperbolic space, the goal is to understand the properties of the quotient  $\bar{G}$  obtained from  $G$  by adjoining relations of the form  $h^n = 1$ , where  $n$  is a large even integer. In particular, we study the elementary subgroups of  $\bar{G}$  ([Section 4.6](#)) as well as the geometric/algebraic invariants of  $\bar{G}$  ([Section 4.7](#)). [Section 5](#) collects all the previous work. We first state and prove the induction hypothesis used to produce the approximation sequence [\(1\)](#), see [Proposition 5.1](#). Then we apply our main result ([Theorem 5.4](#)) to various examples ([Section 5.3](#)) such as free Burnside groups, periodic quotients of hyperbolic groups, etc.

**Acknowledgment.** The author is grateful to the *Centre Henri Lebesgue* ANR-11-LABX-0020-01 for creating an attractive mathematical environment. He acknowledges support from the Agence Nationale de la Recherche under Grant *Dagger* ANR-16-CE40-0006-01. He is greatly indebted to Thomas Delzant for his valuable comments and suggestions that helped improving the exposition of the article.

## 2 Hyperbolic geometry

We recall here a few basic facts about hyperbolic spaces in the sense of Gromov [\[19\]](#). A reader familiar with the subject can directly jump to [Section 3](#) where we define the important invariants associated to the action of a group on a hyperbolic space. We included precise references for the *quantitative* results. Some of them only provide a proof in the context of geodesic metric spaces. However, by relaxing if necessary the constants, which we do here, the same arguments work in the more general settings of length spaces. For the rest, we

refer the reader to Gromov’s original article [19] or the numerous literature on the subject, e.g. [6, 18].

## 2.1 General facts

**Four point inequality.** Let  $X$  be a metric length space. In this article all the paths are rectifiable and parametrized by arc length. Given two points  $x, y \in X$ , we write  $|x - y|_X$  or simply  $|x - y|$  for the distance between them. The Gromov product of three points  $x, y, z \in X$  is defined as

$$\langle x, y \rangle_z = \frac{1}{2} [|x - z| + |y - z| - |x - y|].$$

Let  $\delta \in \mathbf{R}_+^*$ . We assume that  $X$  is  $\delta$ -hyperbolic, i.e. for every  $x, y, z, t \in X$  we have

$$\langle x, y \rangle_t \geq \min \{ \langle x, z \rangle_t, \langle z, y \rangle_t \} - \delta. \quad (3)$$

or equivalently

$$|x - y| + |z - t| \leq \max \{ |x - z| + |y - t|, |x - t| + |y - z| \} + 2\delta. \quad (4)$$

In this context, the Gromov product has the following useful interpretation. For every  $x, y, z \in X$ , the quantity  $\langle y, z \rangle_x$  is roughly the distance between  $x$  and any geodesic  $[y, z]$  between  $y$  and  $z$ . More precisely, we have

$$\langle y, z \rangle_x \leq d(x, [y, z]) \leq \langle y, z \rangle_x + 4\delta,$$

see for instance [6, Chapitre 3, Lemme 2.7].

*Remark 2.1.* In this article, the metric spaces we are going to consider are not necessarily geodesic. In particular, we cannot speak of the distance between a point and a geodesic. One option to bypass this technical issue would be to replace geodesics by  $(1, \varepsilon)$ -quasi-geodesics for arbitrarily small  $\varepsilon > 0$ . However this solution is rather burdensome. Instead we prefer to work with Gromov’s products. The reader should keep in mind that an inequality of the form  $\langle y, z \rangle_x \leq d$  (respectively  $\langle y, z \rangle_x \geq d$ ) means that  $x$  is close (respectively far) from a “geodesic” between  $y$  and  $z$ .

**Boundary at infinity.** Let  $o$  be a base point in  $X$ . A sequence  $(y_n)$  of points of  $X$  *converges at infinity* if  $\langle y_n, y_m \rangle_o$  diverges to infinity as  $n$  and  $m$  tend to infinity. The set  $\mathcal{S}$  of all such sequences is endowed with an equivalence relation defined as follows: two sequences  $(y_n)$  and  $(y'_n)$  are related if

$$\lim_{n \rightarrow \infty} \langle y_n, y'_n \rangle_o = +\infty.$$

The *boundary at infinity* of  $X$ , denoted by  $\partial X$  is the quotient of  $\mathcal{S}$  by this equivalence relation. If the sequence  $(y_n)$  is an element in the class of  $\xi \in \partial X$ , we say that  $(y_n)$  *converges* to  $\xi$  and write

$$\lim_{n \rightarrow \infty} y_n = \xi.$$

The Gromov product of three points can be extended to the boundary: given  $x \in X$  and  $y, z \in X \cup \partial X$ , we define

$$\langle y, z \rangle_x = \inf \left\{ \liminf_{n \rightarrow +\infty} \langle y_n, z_n \rangle_x : \lim_{n \rightarrow \infty} y_n = y, \lim_{n \rightarrow \infty} z_n = z \right\}$$

Let  $x \in X$ . Let  $(y_n)$  and  $(z_n)$  be two sequences of points of  $X$  respectively converging to  $y$  and  $z$  in  $X \cup \partial X$ . It follows from (3) that

$$\langle y, z \rangle_x \leq \liminf_{n \rightarrow +\infty} \langle y_n, z_n \rangle_x \leq \limsup_{n \rightarrow +\infty} \langle y_n, z_n \rangle_x \leq \langle y, z \rangle_x + k\delta, \quad (5)$$

where  $k$  is the number of points in  $\{y, z\}$  which belongs to  $\partial X$ . Moreover, for every  $t \in X$ , for every  $x, y, z \in X \cup \partial X$ , the four point inequality (3) leads to

$$\langle x, z \rangle_t \geq \min \{ \langle x, y \rangle_t, \langle y, z \rangle_t \} - \delta. \quad (6)$$

The isometry group of  $X$  naturally acts on  $\partial X$  preserving Gromov's products.

**Busemann cocycles.** To every point  $\xi \in \partial X$ , we would like to associate a Busemann cocycle. However the space  $X$  is neither locally compact, nor geodesic. To that end we proceed as follows. Given any point  $\xi \in \partial X$  and a base point  $o \in X$ , we first define a map  $b: X \rightarrow \mathbf{R}$  by

$$b(x) = \langle o, \xi \rangle_x - \langle x, \xi \rangle_o$$

and then let

$$\begin{aligned} c: X \times X &\rightarrow \mathbf{R} \\ (x, y) &\rightarrow b(x) - b(y). \end{aligned}$$

The map  $c$  is obviously a cocycle, i.e.

$$c(x_1, x_3) = c(x_1, x_2) + c(x_2, x_3), \quad \forall x_1, x_2, x_3 \in X.$$

We call  $c$  a *Busemann cocycle at  $\xi$  (based at  $o$ )*. Note that  $c$  does depend on the base point  $o$ . Nevertheless for every  $x, y \in X$ ,

$$\left| c(x, y) - \left[ \langle y, \xi \rangle_x - \langle x, \xi \rangle_y \right] \right| \leq 3\delta. \quad (7)$$

In particular any two cocycles at  $\xi$  differ by at most  $6\delta$ . Moreover  $c$  is almost 1-Lipschitz, i.e.

$$|c(x, y)| \leq |x - y| + 2\delta, \quad \forall x, y \in X. \quad (8)$$

**Lemma 2.2.** *Let  $\xi \in \partial X$  and  $c$  a Busemann cocycle at  $\xi$ . For every  $x, y, y' \in X$  we have*

$$|y - y'| \leq |c(y, y')| + 2 \max \{ \langle x, \xi \rangle_y, \langle x, \xi \rangle_{y'} \} + 8\delta.$$

*Proof.* Let  $(z_n)$  be a sequence of points of  $X$  converging to  $\xi$ . It follows from the four point inequality that for every  $n \in \mathbf{N}$

$$\begin{aligned} |y - y'| &\leq ||y - z_n| - |y' - z_n|| + 2 \max \left\{ \langle x, z_n \rangle_y, \langle x, z_n \rangle_{y'} \right\} + 2\delta \\ &\leq \left| \langle y', z_n \rangle_y - \langle y, z_n \rangle_{y'} \right| + 2 \max \left\{ \langle x, z_n \rangle_y, \langle x, z_n \rangle_{y'} \right\} + 2\delta \end{aligned}$$

see for instance [9, Lemma 2.2 (ii)]. The conclusion follows by taking the limit and applying (5).  $\square$

We denote by  $\partial_h X$  the set of all Busemann cocycles obtained as above. The isometry group of  $X$  naturally acts on  $\partial_h X$ : if  $g$  is an isometry of  $X$  and  $c$  a Busemann cocycle at a point  $\xi \in \partial X$ , then the map  $gc: X \times X \rightarrow \mathbf{R}$  defined by  $(gc)(z, z') = c(g^{-1}z, g^{-1}z')$  is a Busemann cocycle at  $g\xi$ .

**Quasi-geodesics.** Let  $\kappa \in \mathbf{R}_+^*$  and  $\ell \in \mathbf{R}_+$ . A  $(\kappa, \ell)$ -quasi-isometric embedding is a map  $f: X_1 \rightarrow X_2$  between two metric spaces such that for every  $x, x' \in X_1$ ,

$$\kappa^{-1} |x - x'| - \ell \leq |f(x) - f(x')| \leq \kappa |x - x'| + \ell.$$

A  $(\kappa, \ell)$ -quasi-geodesic is a  $(\kappa, \ell)$ -quasi-isometric embedding  $\gamma: I \rightarrow X$  from an interval  $I$  of  $\mathbf{R}$  into  $X$ . Recall that all the paths we consider are rectifiable by arc length. Hence, if  $\gamma: I \rightarrow X$  is a  $(\kappa, \ell)$ -quasi-geodesic, we have the following more accurate inequalities:

$$\kappa^{-1} |s - t| - \ell \leq |\gamma(s) - \gamma(t)| \leq |s - t|, \quad \forall s, t \in I.$$

A path is an  $L$ -local  $(\kappa, \ell)$ -quasi-geodesic if its restriction to any interval of length  $L$  is a  $(\kappa, \ell)$ -quasi-geodesic. If  $\gamma: \mathbf{R}_+ \rightarrow X$  is  $(\kappa, \ell)$ -quasi-geodesic, then there exists a unique point  $\xi \in \partial X$  such that for every sequence of real numbers  $(t_n)$  diverging to infinity we have  $\lim_{n \rightarrow \infty} \gamma(t_n) = \xi$ . We view  $\xi$  as the endpoint at infinity of  $\gamma$  and write  $\xi = \gamma(\infty)$ . In this article, we mostly work with local  $(1, \ell)$ -quasi-geodesics. Therefore we use the following version of the stability of quasi-geodesics.

**Proposition 2.3** (Stability of quasi-geodesics [9, Corollaries 2.6 and 2.7]). *There exists  $L_0 \in \mathbf{R}_+$  such that for every  $\delta \in \mathbf{R}_+^*$ , for every  $\ell \in [0, 10^5 \delta]$ , the following holds. Let  $\gamma: I \rightarrow X$  be an  $L_0 \delta$ -local  $(1, \ell)$ -quasi-geodesic of a  $\delta$ -hyperbolic length space  $X$ .*

- (i) *The path  $\gamma$  is a (global)  $(2, \ell)$ -quasi-geodesic.*
- (ii) *For every  $s, t, t' \in I$ , with  $t \leq s \leq t'$ , we have  $\langle \gamma(t), \gamma(t') \rangle_{\gamma(s)} \leq \ell/2 + 5\delta$ .*
- (iii) *For every  $x \in X$ , for every  $y, y' \in X$ , lying on  $\gamma$ , we have  $d(x, \gamma) \leq \langle y, y' \rangle_x + \ell + 8\delta$ .*

*In addition the Hausdorff distance between two  $L_0 \delta$ -local  $(1, \ell)$ -quasi-geodesics with the same endpoints (eventually in  $\partial X$ ) is at most  $2\ell + 5\delta$ .*

*Remark.* Note that the parameter  $L_0$  does not depend on  $\delta$  or on the metric space  $X$ . Without loss of generality we can assume that  $L_0 \geq 1000$ , which we do here.

Although the space  $X$  is not geodesic, its boundary satisfies a visibility property: for every  $x \in X$ ,  $\xi \in \partial X$ , and  $L \geq L_0\delta$ , there exists an  $L$ -local  $(1, 11\delta)$ -quasi-geodesic  $\gamma: \mathbf{R}_+ \rightarrow X$  such that  $\gamma(0) = x$  and  $\gamma(\infty) = \xi$  [10, Lemma 2.9].

**Lemma 2.4.** *Let  $\ell \in [0, 10^5\delta]$ . Let  $\gamma: \mathbf{R}_+ \rightarrow X$  be an  $L_0\delta$ -local  $(1, \ell)$ -quasi-geodesic and  $\xi = \gamma(\infty)$  its endpoint at infinity. Let  $c$  be a Busemann cocycle at  $\xi$ . For every  $s, t \in \mathbf{R}_+$  with  $t \geq s$ , we have*

$$|c(\gamma(s), \gamma(t)) - |\gamma(s) - \gamma(t)|| \leq \ell + 15\delta.$$

*Proof.* The lemma directly follows from (7) and Proposition 2.3 (ii).  $\square$

**Quasi-convex subsets.** Let  $\alpha \in \mathbf{R}_+$ . A subset  $Y$  of  $X$  is  $\alpha$ -quasi-convex if for every  $x \in X$ , for every  $y, y' \in Y$ ,  $d(x, Y) \leq \langle y, y' \rangle_x + \alpha$ . Assume now that  $Y$  is connected by rectifiable paths. The length metric on  $Y$  induced by the restriction of  $|\cdot|_X$  to  $Y$  is denoted by  $|\cdot|_Y$ . We say that  $Y$  is *strongly quasi-convex* if it is  $2\delta$ -quasi-convex and for every  $y, y' \in Y$ ,

$$|y - y'|_X \leq |y - y'|_Y \leq |y - y'|_X + 8\delta. \quad (9)$$

A  $(1, \ell)$ -quasi-geodesic is  $(\ell + 3\delta)$ -quasi-convex [9, Proposition 2.4]. More generally, every  $L_0\delta$ -local  $(1, \ell)$ -quasi-geodesic is  $(\ell + 8\delta)$ -quasi-convex, provided  $\ell \leq 10^5\delta$ , see Proposition 2.3 (iii). If  $Y$  is an  $\alpha$ -quasi-convex subset of  $X$ , then for every  $A \geq \alpha$ , its  $A$ -neighborhood, that we denote by  $Y^{+A}$ , is  $2\delta$ -quasi-convex [9, Proposition 2.13]. Similarly, for every  $A > \alpha + 2\delta$ , the *open*  $A$ -neighborhood of  $Y$ , i.e. the set of points  $x \in X$  such that  $d(x, Y) < A$ , is strongly quasi-convex [10, Lemma 2.13].

We adopt the convention that the diameter of the empty set is zero, whereas the distance from a point to the empty set is infinite. Let  $x$  be a point of  $X$ . A point  $y \in Y$  is an  $\eta$ -projection of  $x$  on  $Y$  is  $|x - y| \leq d(x, Y) + \eta$ . A 0-projection is simply called a *projection*.

**Lemma 2.5** (Projection on a quasi-convex subset [9, Lemma 2.12]). *Let  $\alpha \in \mathbf{R}_+$  and  $Y$  an  $\alpha$ -quasi-convex subset of  $X$ . Let  $x, x' \in X$ .*

(i) *If  $p$  is an  $\eta$ -projection of  $x$  on  $Y$ , then for every  $y \in Y$ , we have  $\langle x, y \rangle_p \leq \alpha + \eta$ .*

(ii) *If  $p$  and  $p'$  are respectively  $\eta$ - and  $\eta'$ -projection of  $x$  and  $x'$  on  $Y$  then*

$$|p - p'| \leq \max\{|x - x'| - |x - p| - |x' - p'| + 2\varepsilon, \varepsilon\},$$

$$\text{where } \varepsilon = 2\alpha + \delta + \eta + \eta'$$

**Lemma 2.6** ([9, Lemma 2.13]). *Let  $Y_1, \dots, Y_m$  be a collection of subsets of  $X$  such that  $Y_j$  is  $\alpha_j$ -quasi-convex, for every  $j \in \llbracket 1, m \rrbracket$ . For all  $A \geq 0$ , we have*

$$\text{diam}(Y_1^{+A} \cap \dots \cap Y_m^{+A}) \leq \text{diam}(Y_1^{+\alpha_1+3\delta} \cap \dots \cap Y_m^{+\alpha_m+3\delta}) + 2A + 4\delta.$$

## 2.2 Isometries

An isometry  $g$  of  $X$  is either *elliptic* (its orbits are bounded) *parabolic* (its orbits admit exactly one accumulation point in  $\partial X$ ) or *loxodromic* (its orbits admit exactly two accumulation points in  $\partial X$ ). In order to measure the action of  $g$  on  $X$  we introduce the *translation length* and the *stable translation length* respectively defined by

$$\|g\|_X = \inf_{x \in X} |gx - x| \quad \text{and} \quad \|g\|_X^\infty = \lim_{n \rightarrow \infty} \frac{1}{n} |g^n x - x|.$$

If there is no ambiguity, we will omit the space  $X$  in the notation. These lengths are related by

$$\|g\|^\infty \leq \|g\| \leq \|g\|^\infty + 16\delta, \quad (10)$$

see [6, Chapitre 10, Proposition 6.4]. In addition,  $g$  is loxodromic if and only if  $\|g\|^\infty > 0$ . In such a case the accumulation points of  $g$  in  $\partial X$  are

$$g^- = \lim_{n \rightarrow \infty} g^{-n} x \quad \text{and} \quad g^+ = \lim_{n \rightarrow \infty} g^n x.$$

They are the only points of  $X \cup \partial X$ , fixed by  $g$ . We write  $\Gamma_g$  for the union of all  $L_0\delta$ -local  $(1, \delta)$ -quasi-geodesic joining  $g^-$  to  $g^+$ . The *cylinder* of  $g$ , denoted by  $Y_g$  is the *open*  $20\delta$ -neighborhood of  $\Gamma_g$ . It is a strongly quasi-convex subset of  $X$  [10, Proposition 3.13]. It can be thought of as the “smallest”  $g$ -invariant quasi-convex subset.

**Lemma 2.7.** *Let  $g$  be a loxodromic isometry of  $X$ . Let  $\gamma: \mathbf{R} \rightarrow X$  be a bi-infinite  $L_0\delta$ -local  $(1, \delta)$ -quasi-geodesic between  $g^-$  and  $g^+$ . Let  $Z$  be a non-empty  $g$ -invariant  $\alpha$ -quasi-convex subset of  $X$ . Then  $\gamma$  lies in the  $(\alpha + 8\delta)$ -neighborhood of  $Z$ . In particular the cylinder  $Y_g$  of  $g$  lies in the  $(\alpha + 28\delta)$ -neighborhood of  $Z$ .*

*Proof.* Let  $x$  be a point on  $\gamma$ . It follows from the stability of quasi-geodesics that  $\langle g^-, g^+ \rangle_x \leq 6\delta$  (Proposition 2.3). We fix a point  $z \in Z$ . Since  $Z$  is  $\alpha$ -quasi-convex, we have  $d(x, Z) \leq \langle g^{-n} z, g^n z \rangle_x + \alpha$ , for every  $n \in \mathbf{N}$ . We pass to the limit as  $n$  approaches infinity and use (5) to get

$$d(x, Z) \leq \langle g^-, g^+ \rangle_x + \alpha + 2\delta \leq \alpha + 8\delta. \quad \square$$

The next lemma is a variation on the quasi-convexity of the distance function in a  $\delta$ -hyperbolic space.

**Lemma 2.8** ([9, Lemma 2.26]). *Let  $x, x'$  and  $y$  be three points of  $X$ . If  $g$  is an isometry of  $X$ , then  $|gy - y| \leq \max\{|gx - x|, |gx' - x'|\} + 2\langle x, x' \rangle_y + 6\delta$ .*

Given a set  $S$  of isometries and  $d \in \mathbf{R}_+$ , we denote by  $\text{Mov}(S, d)$  the set of points which are moved by  $S$  by a distance less than  $d$ , i.e.

$$\text{Mov}(S, d) = \{x \in X : \forall g \in S, |gx - x| < d\}. \quad (11)$$

If the set  $S = \{g\}$  is reduced to a single isometry, we simply write  $\text{Mov}(g, d)$  for  $\text{Mov}(S, d)$ . The *characteristic subset* of  $g$  (or *axis* of  $g$ ) is the set

$$A_g = \text{Mov}(g, \|g\| + 8\delta) = \{x \in X : |gx - x| < \|g\| + 8\delta\}. \quad (12)$$

Note that this definition does not require  $g$  to be a loxodromic element.

**Lemma 2.9.** *Let  $S$  be a set of isometries and  $d > 7\delta$ . If  $\text{Mov}(S, d)$  is non-empty, then  $\text{Mov}(S, d)$  is  $10\delta$ -quasi-convex. Moreover it satisfies the following properties.*

(i) *For every  $x \in X \setminus \text{Mov}(S, d)$ , we have*

$$\sup_{g \in S} |gx - x| \geq 2d(x, \text{Mov}(S, d)) + d - 14\delta.$$

(ii) *Let  $x \in X$  and  $A \in \mathbf{R}_+$ . If  $|gx - x| \leq d + 2A$  for every  $g \in S$ , then  $x$  is  $(A + 7\delta)$ -close to  $\text{Mov}(S, d)$ .*

*Remark.* The converse of (ii) is obvious. Indeed the open  $A$ -neighborhood of  $\text{Mov}(S, d)$  is contained in  $\text{Mov}(S, d + 2A)$ , for every  $A \in \mathbf{R}_+$ . Although the objects are defined in a slightly different way, the proof works verbatim as in [9, Proposition 2.28]. Nevertheless for completeness we reproduce it here.

*Proof.* We first prove Point (i) when  $S$  is reduced to a single element, say  $g$ , in which case  $\text{Mov}(S, d)$  is  $\langle g \rangle$ -invariant. Let  $x \in X \setminus \text{Mov}(g, d)$ . Let  $\eta > 0$  and  $y$  be an  $\eta$ -projection of  $x$  on  $\text{Mov}(g, d)$ . Since  $x$  does not belong to  $\text{Mov}(g, d)$ , one observes that  $|gy - y| \geq d - 2\eta$ . We now fix  $\varepsilon \in (0, \eta)$  such that  $|gy - y| + \varepsilon < d$  and choose a  $(1, \varepsilon)$ -quasi-geodesic  $\gamma: I \rightarrow X$  joining  $y$  to  $gy$ , so that  $\gamma$  is entirely contained in  $\text{Mov}(g, d)$ . In particular  $y$  and  $gy$  are respective  $\eta$ -projections of  $x$  and  $gx$  on  $\gamma$ , which is  $(\varepsilon + 3\delta)$ -quasi-convex. Consequently Lemma 2.5 yields

$$d - 2\eta \leq |gy - y| \leq \max\{|gx - x| - 2|x - y| + 8\eta + 14\delta, 4\eta + 7\delta\}. \quad (13)$$

Recall that  $d > 7\delta$ . Taking  $\eta > 0$  arbitrarily small leads to

$$|gx - x| \geq 2d(x, \text{Mov}(g, d)) + d - 14\delta. \quad (14)$$

We now prove Point (i) for a general set  $S$ . Let  $x \in X \setminus \text{Mov}(S, d)$ . Note that  $\text{Mov}(S, d)$  is exactly the intersection of all  $\text{Mov}(g, d)$  where  $g$  runs over  $S$ . Hence there exists  $g \in S$  such that  $x$  does not belong to  $\text{Mov}(g, d)$ . Applying (14) we get

$$|gx - x| \geq 2d(x, \text{Mov}(g, d)) + d - 14\delta \geq 2d(x, \text{Mov}(S, d)) + d - 14\delta.$$

and Point (i) follows. Point (ii) is a direct consequence of Point (i). We are left to prove that  $\text{Mov}(S, d)$  is quasi-convex. Let  $y$  and  $y'$  be two points of  $\text{Mov}(S, d)$ . Let  $x$  be a point of  $X$ . Lemma 2.8 yields for every  $g \in S$ ,

$$|gx - x| \leq \max\{|gy - y|, |gy' - y'|\} + 2\langle y, y' \rangle_x + 6\delta < d + 2\langle y, y' \rangle_x + 6\delta.$$

It follows then from Point (ii) that  $d(x, \text{Mov}(S, d)) \leq \langle y, y' \rangle_x + 10\delta$ .  $\square$

The next statement explicits the relation between the cylinder  $Y_g$  of a loxodromic isometry  $g$  and its axis  $A_g$ .



**Lemma 2.10.** *Let  $g$  be a loxodromic isometry of  $X$ . Let  $\gamma: \mathbf{R} \rightarrow X$  be a bi-infinite  $L_0\delta$ -local  $(1, \delta)$ -quasi-geodesic joining  $g^-$  to  $g^+$ .*

- (i) *The path  $\gamma$  is contained in the  $18\delta$ -neighborhood of the axis  $A_g$ . In particular,  $|\gamma y - y| \leq \|g\| + 84\delta$ , for every  $y \in Y_g$ .*
- (ii) *Conversely, if  $\|g\| > L_0\delta$ , then  $A_g$  lies in the  $10\delta$ -neighborhood of  $\gamma$ . In particular  $A_g$  is contained in  $Y_g$ .*

*Proof.* The first part of the statement is a direct application of [Lemma 2.7](#). The second half is a consequence of the stability of quasi-geodesics, see [[9](#), Lemma 2.33].  $\square$

**Lemma 2.11.** *Let  $\xi \in \partial X$  and  $g$  be an isometry of  $X$  fixing  $\xi$ . There exists  $\varepsilon \in \{\pm 1\}$ , such that for every  $x \in X$ , we have  $|c(gx, x) + \varepsilon\|g\|^\infty| \leq 6\delta$ .*

*Proof.* Let  $x \in X$ . Let  $n \in \mathbf{N}$ . Observe that

$$c(g^n x, x) = \sum_{k=0}^{n-1} c(g^{k+1}x, g^k x) = \sum_{k=0}^{n-1} g^{-k} c(gx, x).$$

Since  $g$  fixes  $\xi$ , for every  $k \in \mathbf{N}$ , the map  $g^{-k}c$  is a Busemann cocycle at  $\xi$ , and therefore differs from  $c$  by at most  $6\delta$ . Thus  $|c(g^n x, x) - nc(gx, x)| \leq 6n\delta$ . As Busemann cocycles are almost 1-Lipschitz, we get

$$|c(gx, x)| \leq \frac{1}{n} |c(g^n x, x)| + 6\delta \leq \frac{1}{n} |g^n x - x| + \left(6 + \frac{2}{n}\right) \delta.$$

Taking the limit yields  $|c(gx, x)| \leq \|g\|^\infty + 6\delta$ . In particular, the result holds if  $g$  is either elliptic or parabolic.

Assume now that  $g$  is loxodromic. There exists  $\varepsilon \in \{\pm 1\}$  such that  $\xi$  is the attractive point of  $g^\varepsilon$ . Moreover,  $\|g^n\| > L_0\delta$ , for every sufficiently large  $n \in \mathbf{N}$ . We fix such an exponent  $n$  and write  $h = g^{\varepsilon n}$ . Let  $y \in X$  be a point such that  $|hy - y| \leq \|h\| + \delta$ . We choose a  $(1, \delta)$ -quasi-geodesic  $\gamma: [0, T] \rightarrow X$  joining  $y$  to  $hy$  and extend  $\gamma$  to a bi-infinite path  $\gamma: \mathbf{R} \rightarrow X$  as follows: for every  $t \in [0, T)$ , for every  $n \in \mathbf{Z}$ , we let  $\gamma(nT + t) = h\gamma(t)$ . It follows from our choice of  $y$  that  $\gamma$  is an  $L_0\delta$ -local  $(1, 2\delta)$ -quasi-isometry from  $h^-$  to  $h^+ = \xi$ . Applying [Lemma 2.4](#), we get

$$|c(y, hy) - |hy - y|| \leq 17\delta. \tag{15}$$

Observe that  $h^{-1}c$  and  $c$  are two cocycles at  $\xi$  (since  $h$  fixes  $\xi$ ), hence they differ by at most  $6\delta$ . The cocycle property yields  $|c(g^n x, x) + \varepsilon c(y, hy)| \leq 12\delta$ . Thus (15) becomes  $|c(g^n x, x) + \varepsilon|g^n y - y|| \leq 29\delta$ . Recall that  $c(g^n x, x)$  and  $nc(gx, x)$  differs by at most  $6n\delta$ . Hence

$$\left|c(gx, x) + \frac{\varepsilon}{n} |g^n y - y|\right| \leq \frac{1}{n} |c(g^n x, x) + \varepsilon|g^n y - y|| + 6\delta \leq \left(6 + \frac{29}{n}\right) \delta.$$

The result follows by taking the limit as  $n$  approaches infinity.  $\square$

**Lemma 2.12.** *Let  $\xi \in \partial X$  and  $g$  be an isometry of  $X$  fixing  $\xi$ . Let  $L \geq \max\{L_0\delta, \|g\|^\infty\}$  and  $\ell \in [0, 10^5\delta]$ . Let  $\gamma: \mathbf{R}_+ \rightarrow X$  be an  $L$ -local  $(1, \ell)$ -quasi-geodesic ray starting at  $x = \gamma(0)$  and ending at  $\gamma(\infty) = \xi$ . There exists  $\varepsilon \in \{\pm 1\}$  such that for every  $t \geq 2|gx - x| + 4\ell + 16\delta$  we have*

$$|\gamma(t + \varepsilon \|g\|^\infty) - g\gamma(t)| \leq 5\ell + 59\delta.$$

*Proof.* Let  $p = g\gamma(t_0)$  be a projection of  $x$  on  $g\gamma$ , so that  $\langle x, gx \rangle_p \leq \ell + 8\delta$  and  $\langle x, \xi \rangle_p \leq \ell + 8\delta$  (Lemma 2.5). Since  $\gamma$  is also a global  $(2, \ell)$ -quasi-geodesic, we get

$$|gx - x| \geq |gx - p| - \langle x, gx \rangle_p \geq |g\gamma(0) - g\gamma(t_0)| - \langle x, gx \rangle_p \geq \frac{1}{2}t_0 - 2\ell - 8\delta.$$

Hence  $t_0 \leq 2|gx - x| + 4\ell + 16\delta$ . Let  $c$  be a Busemann cocycle at  $\xi$ . According to Lemma 2.11 there exists  $\varepsilon \in \{\pm 1\}$  such that for every  $x \in X$ , we have  $|c(gx, x) + \varepsilon\|g\|^\infty| \leq 6\delta$ . Let  $t \geq 2|gx - x| + 4\ell + 16\delta$ , so that  $t \in [t_0, \infty)$ . For simplicity we write  $y_1 = \gamma(t)$  and  $y_2 = \gamma(t + \varepsilon\|g\|^\infty)$ . Since  $\gamma$  is an  $L$ -local  $(1, \ell)$ -quasi-geodesic, we have  $c(y_1, y_2)$  differs from  $\varepsilon\|g\|^\infty$  by at most  $2\ell + 15\delta$  (Lemma 2.4). Hence

$$|c(gy_1, y_2)| \leq |c(gy_1, y_1) + c(y_1, y_2)| \leq 2\ell + 21\delta.$$

Recall that  $t \geq t_0$ , hence  $\langle p, \xi \rangle_{gy_1} \leq \ell/2 + 5\delta$ . The triangle inequality combined with (5) yields

$$\langle x, \xi \rangle_{gy_1} \leq \langle x, \xi \rangle_p + \langle p, \xi \rangle_{gy_1} + 2\delta \leq 3\ell/2 + 15\delta.$$

Applying Lemma 2.2 we get

$$|gy_1 - y_2| \leq |c(gy_1, y_2)| + 2 \max\{\langle x, \xi \rangle_{gy_1}, \langle x, \xi \rangle_{y_2}\} + 8\delta \leq 5\ell + 59\delta. \quad \square$$

The same arguments can be used to prove the following lemma.

**Lemma 2.13.** *Let  $g$  be a loxodromic isometry. Let  $L \geq \max\{L_0\delta, \|g\|^\infty\}$  and  $\ell \in [0, 10^5\delta]$ . Let  $\gamma: \mathbf{R}_+ \rightarrow X$  be an  $L$ -local  $(1, \ell)$ -quasi-geodesic from  $g^-$  to  $g^+$ . Then for every  $t \in \mathbf{R}$ , for every  $n \in \mathbf{Z}$ , we have*

$$|\gamma(t + n\|g\|^\infty) - g^n\gamma(t)| \leq 5\ell + 59\delta.$$

## 2.3 Group action

**Classification of actions.** Let  $G$  be a group acting by isometries on  $X$ . Its *limit set*  $\Lambda(G)$  is the set of accumulation points in  $\partial X$  of some (hence any) orbit of  $G$ . The action of  $G$  on  $X$  is *elliptic* (respectively *parabolic*, *loxodromic*, *non-elementary*) if  $\Lambda(G)$  is empty (respectively contains exactly 1 point, exactly 2 points, at least 3 points). If there is no ambiguity regarding the action, we simply say that  $G$  is *elliptic* (respectively *parabolic*, *loxodromic*, *non-elementary*).

**Elliptic action.** Even though  $X$  is not necessarily locally compact, a group  $G$  is elliptic if and only if its orbits are bounded [10, Proposition 3.5]. Elliptic groups actually have very small orbits.

**Lemma 2.14** ([9, Corollary 2.38]). *Let  $G$  be an elliptic group of isometries of  $X$ . If  $Y$  is an  $G$ -invariant  $\alpha$ -quasi-convex subset of  $X$ , then  $\text{Mov}(G, 11\delta)$  intersects the  $\alpha$ -neighborhood of  $Y$ . In particular,  $\text{Mov}(G, 11\delta)$  is not empty.*

**Loxodromic action.** Let  $G$  be a loxodromic group. In particular it contains a loxodromic isometry, say  $g$  [10, Proposition 3.6]. Note that  $g^-$  and  $g^+$  are the two points of  $\Lambda(G)$ . Moreover every element of  $G$  preserves  $\{g^-, g^+\}$ . We denote by  $G^+$  the subgroup of  $G$  fixing pointwise  $\{g^-, g^+\}$ . It has index at most 2 in  $G$ . If  $G = G^+$  we say that  $G$  preserves the orientation. The cylinder  $Y$  of  $G$  is the cylinder of some (hence any) loxodromic element contained in  $G$ . By construction  $\partial Y = \{g^-, g^+\} = \Lambda(G)$ .

**Lemma 2.15.** *Let  $G$  be a loxodromic group of isometries of  $X$  and  $Y$  its cylinder. Let  $E$  be an elliptic normal subgroup of  $G$ . Then  $Y$  is contained in  $\text{Mov}(E, 88\delta)$ .*

*Proof.* Since  $E$  is a normal subgroup,  $\text{Mov}(E, 11\delta)$  is a  $G$ -invariant  $10\delta$ -quasi-convex subset (Lemma 2.9). It follows from Lemma 2.7 that  $Y$  is contained in the  $38\delta$ -neighborhood of  $\text{Mov}(E, 11\delta)$ , hence in  $\text{Mov}(E, 88\delta)$ .  $\square$

**Lemma 2.16.** *Let  $G$  be a loxodromic group of isometries of  $X$  and  $Y$  its cylinder. Let  $E$  be an elliptic group preserving  $\partial Y$ . If  $E$  contains an element permuting the two points of  $\partial Y$ , then  $Y \cap \text{Mov}(E, 11\delta)$  is non-empty and its diameter is at most  $141\delta$ .*

*Proof.* Let  $\eta$  and  $\xi$  be the two points of  $\Lambda(G)$ . By assumption  $E$  contains an element  $u$  that permutes  $\eta$  and  $\xi$ . Since the cylinder  $Y$  is still  $u$ -invariant, it contains a point in  $\text{Mov}(u, 11\delta)$  (Lemma 2.14). Let  $c$  be a Busemann cocycle at  $\xi$ . Note that  $u^{-1}c$  is a Busemann cocycle at  $\eta$ . Recall that  $Y$  is contained in the  $27\delta$ -neighborhood of any  $L_0\delta$ -local  $(1, \delta)$ -quasi-geodesics from  $\eta$  to  $\xi$ . Combining Lemma 2.4 with the fact that Busemann cocycles are almost 1-Lipschitz, we get

$$2|x - y| \leq |c(x, y) - u^{-1}c(x, y)| + 256\delta, \quad \forall x, y \in Y.$$

It follows from the cocycle property that  $c(x, y) - u^{-1}c(x, y) = c(x, ux) - c(y, uy)$ . Since  $c$  is almost 1-Lipschitz, the previous inequality becomes

$$2|x - y| \leq |c(x, ux)| + |c(y, uy)| + 256\delta \leq |x - ux| + |y - uy| + 260\delta.$$

This inequalities hold for every  $x, y \in Y$ . Hence the diameter of  $Y \cap \text{Mov}(u, 11\delta)$  is at most  $141\delta$ .  $\square$

**Non elementary action.** The next lemma is an improved version of the classical ping-pong argument. It provides a simple criterion to ensure that a group is non-elementary.

**Lemma 2.17** ([10, Lemma 3.24]). *Let  $A \geq 0$ . Let  $x \in X$ . Let  $G$  be a group of isometries of  $X$  generated by two elements  $u$  and  $v$  such that*

$$(i) \ 2\langle u^{\pm 1}x, v^{\pm 1}x \rangle_x < \min\{|ux - x|, |vx - x|\} - A - 8\delta,$$

$$(ii) \ 2\langle ux, u^{-1}x \rangle_x < |ux - x| + A,$$

$$(iii) \ 2\langle vx, v^{-1}x \rangle_x < |vx - x| + A.$$

*Then  $G$  is non-elementary.*

**Gentle action.** In order to complete the description of loxodromic groups started above, we introduce a (harmless) additional assumption.

**Definition 2.18.** The action of  $G$  on  $X$  is *gentle* if every loxodromic subgroup  $H$  preserving the orientation splits as a semi-direct product  $H = F \rtimes \mathbf{Z}$  where  $F$  consists exactly of *all* elliptic elements of  $H$ .

If every loxodromic subgroup is virtually cyclic, then the action of  $G$  is automatically gentle. From now on we assume that the action of  $G$  on  $X$  is gentle. Let  $H$  be a loxodromic subgroup of  $G$  and  $H^+$  the subgroup of  $H$  fixing pointwise  $\Lambda(H)$ . Let  $F$  be the set of all elliptic elements of  $H^+$ . It follows from our assumption that  $F$  is an elliptic normal subgroup of  $H$  and is maximal for these properties. The quotient  $H/F$  is either isomorphic to  $\mathbf{Z}$  if  $H$  preserves the orientation (i.e.  $H = H^+$ ) or the infinite dihedral group  $\mathbf{D}_\infty$  otherwise. Observe that if a loxodromic subgroup  $H \subset G$  is generated by two elliptic subgroups, then  $H$  cannot preserve the orientation.

**Lemma 2.19.** *Let  $H$  be a loxodromic subgroup of  $G$ . If  $p: H \rightarrow \mathbf{D}_\infty$  is a morphism whose kernel is elliptic, then this kernel is exactly the maximal normal elliptic subgroup  $F$  of  $H$ .*

*Proof.* By assumption the kernel of  $p$  is an elliptic normal subgroup of  $H$ , hence it is contained in  $F$ . Let us prove the other inclusion. We fix a loxodromic element  $h \in H$ . According to our assumption, the pre-image under  $p$  of any finite subgroup of  $\mathbf{D}_\infty$  is elliptic (as a finite extension of an elliptic subgroup). Hence  $p(h)$  belongs to  $\mathbf{Z}^* \subset \mathbf{D}_\infty$ . Note also that  $p(F)$  does not contain any element of  $\mathbf{Z} \setminus \{0\}$ . Indeed otherwise there would exist  $m \in \mathbf{Z} \setminus \{0\}$  and  $u \in F$  such that  $p(h^m) = p(u)$ . Since the kernel of  $p$  is contained in  $F$ , the element  $h^m$  should belong to  $F$  which contradicts the fact that  $h$  is loxodromic. Hence  $p(F)$  is a finite subgroup of  $\mathbf{D}_\infty$ . As  $h$  normalizes  $F$ , its image  $p(h)$  normalizes  $p(F)$  which forces  $p(F)$  to be trivial.  $\square$

Given a loxodromic element  $g \in G$ , we write  $E(g)$  for the subgroup of  $G$  preserving  $\{g^-, g^+\}$ . It is the maximal elementary subgroup of  $G$  containing  $g$  [10, Lemma 3.28]. The group  $E^+(g)$  stands for the maximal subgroup of  $E(g)$  fixing pointwise  $\{g^-, g^+\}$ .

**Definition 2.20.** (Primitive element) Let  $g \in G$  be a loxodromic element. Let  $F$  be the maximal normal elliptic subgroup of  $E(g)$ . We say that  $g$  is *primitive* if its image in  $E^+(g)/F \cong \mathbf{Z}$  generates the group.

### 3 Invariants of a group action

Let  $X$  be a  $\delta$ -hyperbolic length space and  $G$  a group acting gently by isometries on  $X$ . Note that for the moment, we have not made any serious assumption on the group  $G$  or the space  $X$ . In order to study the action of  $G$  on  $X$  we define several numerical invariants. Those quantities will be useful later to estimate the small cancellation parameters needed to run the induction leading to Burnside groups. We define two types of invariants. The first kind, namely the *injectivity radius*  $\text{inj}(G, X)$ , the *acylindricity constant*  $A(G, X)$  as well as the  $\nu$ -invariant  $\nu(G, X)$  are purely *geometric*. Those invariants already appeared in [15, 9, 10]. Unfortunately they are not sharp enough to handle even torsion. More precisely the  $\nu$ -invariant, does not behave well when passing to quotient. Therefore we also define (among others) a strong variation  $\nu_{\text{stg}}(G, X)$  of the  $\nu$ -invariant, which has a mixed nature: it reflects both the *geometric* and *algebraic* features of  $G$ .

#### 3.1 Geometric invariants

**Definition 3.1** (Injectivity radius). The *injectivity radius* of  $G$  on  $X$  is the quantity

$$\text{inj}(G, X) = \inf \{ \|g\|_X^\infty : g \in G \text{ loxodromic} \}$$

**Definition 3.2** (Acylindricity). The *acylindricity parameter*  $A(G, X)$  is defined as

$$A(G, X) = \sup_{S \subset G} \text{diam}(\text{Mov}(S, 2L_0\delta)),$$

where  $S$  runs over all subsets of  $G$  generating a *non-elementary* subgroup.

We adopt the following terminology borrowed from Lysenok [26]. A *chain of length  $m$*  is a tuple  $\mathcal{C} = (g_0, \dots, g_m)$  of elements of  $G$  for which there exists  $h \in G$  such that for every  $k \in \llbracket 0, m-1 \rrbracket$ , we have  $g_{k+1} = hg_k h^{-1}$ . The element  $h$  is called a *conjugating element* of  $\mathcal{C}$ . Note that such an element is not necessarily unique.

**Definition 3.3** ( $\nu$ -invariant). The quantity  $\nu(G, X)$  is the smallest integer  $\nu$  with the following property: if  $\mathcal{C} = (g_0, \dots, g_\nu)$  is a chain of length  $\nu$  generating an elementary subgroup and  $h$  a *loxodromic* conjugating element of  $\mathcal{C}$ , then  $\langle g_0, h \rangle$  is elementary.

**Proposition 3.4.** *Let  $g$  and  $h$  be two elements of  $G$  which generate a non-elementary subgroup.*

(i) *If  $\|g\| \leq L_0\delta$ , then*

$$\text{diam}(\text{Mov}(g, L_0\delta)^{+13\delta} \cap A_h^{+13\delta}) \leq \nu(G, X) \|h\| + A(G, X) + 76\delta.$$

(ii) Without any assumption on  $g$  we have

$$\text{diam} (A_g^{+13\delta} \cap A_h^{+13\delta}) \leq \|g\| + \|h\| + \nu(G, X) \max\{\|g\|, \|h\|\} + A(G, X) + 191\delta.$$

*Remark.* This statement is proved in [10] closely following the ideas of Delzant and Gromov [15]. For completeness, we reproduce it here.

*Proof.* For simplicity we let  $\nu = \nu(G, X)$ . Assume first that  $\|g\| \leq L_0\delta$ . If  $\|h\| \leq L_0\delta$ , then the intersection of the  $13\delta$ -neighborhood of  $\text{Mov}(g, L_0\delta)$  and  $A_h$  respectively is contained in  $\text{Mov}(\{g, h\}, 2L_0\delta)$ , and thus its diameter is at most  $A(G, X)$ . Consequently, we can assume that  $\|h\| > L_0\delta$ . In particular,  $h$  is loxodromic. Assume that contrary to our claim

$$\text{diam} (\text{Mov}(g, L_0\delta)^{+13\delta} \cap A_h^{+13\delta}) > \nu \|h\| + A(G, X) + 76\delta.$$

Fix  $L > \max\{L_0\delta, \|h\|^\infty, A(G, X) + 2\delta\}$  and choose an  $L$ -local  $(1, \delta)$ -quasi-geodesic  $\gamma: \mathbf{R} \rightarrow X$  from  $h^-$  to  $h^+$ . Since  $\|h\| > L_0\delta$ , the axis  $A_h$  is contained in the  $10\delta$ -neighborhood of  $\gamma$  (Lemma 2.10). It follows from Lemma 2.6 that

$$\text{diam} (\text{Mov}(g, L_0\delta)^{+13\delta} \cap \gamma^{+12\delta}) > \nu \|h\| + A(G, X) + 26\delta.$$

In particular there exists  $x = \gamma(s)$  and  $x' = \gamma(s')$  lying in the  $25\delta$ -neighborhood of  $\text{Mov}(g, L_0\delta)$  such that  $|x - x'| > \nu \|h\| + A(G, X) + 2\delta$ . Without loss of generality we can assume that  $s < s'$ , so that

$$s' - s \geq |x - x'| > \nu \|h\| + A(G, X) + 2\delta.$$

Since  $\text{Mov}(g, L_0\delta)$  is  $10\delta$ -quasi-convex (Lemma 2.9), its  $25\delta$ -neighborhood is  $2\delta$ -quasi-convex. Consequently  $\gamma$  restricted to  $[s, s']$  lies entirely in the  $33\delta$ -neighborhood of  $\text{Mov}(g, L_0\delta)$ . We now fix  $t = s + A(G, X) + 2\delta$ . Let  $r \in [s, t]$  and  $k \in \llbracket 0, \nu \rrbracket$ . Note that  $r_k = r + k\|h\|^\infty$  belongs to  $[s, s']$ . Thus  $|g\gamma(r_k) - \gamma(r_k)| \leq (L_0 + 66)\delta$ . Applying Lemma 2.13 we obtain

$$|h^{-k}gh^k\gamma(r) - \gamma(r)| \leq |gh^k\gamma(r) - h^k\gamma(r)| \leq |g\gamma(r_k) - \gamma(r_k)| + 128\delta < 2L_0\delta.$$

In other words the restriction of  $\gamma$  to  $[s, t]$  is contained in  $\text{Mov}(S, 2L_0\delta)$ , where  $S$  is the set  $S = \{g, h^{-1}gh, \dots, h^{-\nu}gh^\nu\}$ . Consequently the diameter of  $\text{Mov}(S, 2L_0\delta)$  is larger than  $A(G, X)$ , and thus  $S$  generate an elementary subgroup. Recall that  $h$  is loxodromic. It follows from the definition of  $\nu$  that  $g$  and  $h$  generate an elementary subgroup which contradicts our assumption.

We now focus on the general case. According to the previous discussion we can assume that  $\|h\| \geq \|g\| > L_0\delta$ . Suppose again that contrary to our claim that

$$\text{diam} (A_g^{+13\delta} \cap A_h^{+13\delta}) > \|g\| + (\nu + 1) \|h\| + A(G, X) + 191\delta.$$

For simplicity we let

$$\ell = \nu \|h\| + A(G, X) + 117\delta \quad \text{and fix } L > \ell + \|g\| + \|h\|.$$

Let  $\gamma_g: \mathbf{R} \rightarrow X$  (respectively  $\gamma_h: \mathbf{R} \rightarrow X$ ) be an  $L$ -local  $(1, \delta)$ -quasi-geodesic from  $g^-$  to  $g^+$  (respectively  $h^-$  to  $h^+$ ). Since  $\|g\| > L_0\delta$ , the set  $A_g$  lies in the  $10\delta$ -neighborhood of  $\gamma_g$  (Lemma 2.10). The same holds for  $h$ . It follows from Lemma 2.6 that

$$\text{diam}(\gamma_g^{+12\delta} \cap \gamma_h^{+12\delta}) > \|g\| + (\nu + 1)\|h\| + A(G, X) + 141\delta.$$

We fix two points  $x, y \in X$  lying in the  $12\delta$ -neighborhood of both  $\gamma_g$  and  $\gamma_h$  such that

$$|x - y| > \|g\| + (\nu + 1)\|h\| + A(G, X) + 141\delta.$$

Up to changing the origin of  $\gamma_g$  and  $\gamma_h$  we can assume that  $\gamma_g(0)$  and  $\gamma_h(0)$  are projections of  $x$  on  $\gamma_g$  and  $\gamma_h$  respectively. We write  $\gamma_g(s)$  and  $\gamma_h(s')$  for projections of  $y$  on  $\gamma_g$  and  $\gamma_h$  respectively. Note that  $s, s' \in (L, \infty)$ . We now claim that  $|\gamma_g(r) - \gamma_h(r)| \leq 87\delta$ , for every  $r \in [0, L]$ . Since  $\gamma_g$  is an  $L$ -local  $(1, \delta)$ -quasi-geodesic, we have  $\langle \gamma_g(0), \gamma_g(s) \rangle_{\gamma_g(r)} \leq 6\delta$  (Proposition 2.3), thus  $\langle x, y \rangle_{\gamma_g(r)} \leq 30\delta$ . Similarly  $\langle x, y \rangle_{\gamma_h(r)} \leq 30\delta$ . On the other hand, the quantities  $|\gamma_g(r) - x|$  and  $|\gamma_h(r) - x|$  differ by at most  $25\delta$ . It follows from the four point inequality – see for instance [9, Lemma 2.2 (2)] – that  $|\gamma_g(r) - \gamma_h(r)| \leq 87\delta$ , which completes the proof of our claim.

According to Lemma 2.13,  $g$  (respectively  $h$ ) acts on  $\gamma_g$  (respectively  $\gamma_h$ ) almost like a translation of length  $\|g\|^\infty$  (respectively  $\|h\|^\infty$ ). Hence for every  $r \in [0, \ell]$ ,

$$|gh\gamma_h(r) - hg\gamma_h(r)| \leq 603\delta < L_0\delta,$$

compare with Figure 2. Consequently the path  $\gamma_h$  restricted to  $[0, \ell]$  is not only contained in the  $18\delta$ -neighborhood of  $A_h$  (Lemma 2.10) but also in  $\text{Mov}(u, L_0\delta)$  where  $u = g^{-1}h^{-1}gh$ . Thus, applying Lemma 2.6 we get

$$\text{diam}(\text{Mov}(u, L_0\delta)^{+13\delta} \cap A_h^{+13\delta}) > \ell - 41\delta \geq \nu\|h\| + A(G, X) + 76\delta.$$

It follows from the previous discussion that  $u$  and  $h$  generates an elementary subgroup. Hence  $g^{-1}hg$  and  $h$  generate an elementary subgroup. Since  $h$  is loxodromic,  $g$  fixes  $h^-$  and  $h^+$ , therefore  $g$  and  $h$  generate an elementary subgroup, which contradicts our assumption.  $\square$

The next statement should be seen as an analogue of the Margulis lemma for manifolds with pinched negative curvature. The invariant  $\nu(G, X)$  is used here to compensate the fact that the curvature of  $X$  is not necessarily bounded from below.

**Corollary 3.5.** *Let  $S$  be a subset of  $G$ . If  $S$  does not generate an elementary subgroup, then for every  $d \in \mathbf{R}_+$ , we have*

$$\text{diam}(\text{Mov}(S, d)) \leq A(G, X) + [\nu(G, X) + 3]d + 209\delta.$$

*Proof.* For simplicity we write  $\nu$  for  $\nu(G, X)$ . Without loss of generality we can assume that  $S$  is finite and  $\text{Mov}(S, d)$  is non-empty. Recall that for every  $g \in S$

the set  $\text{Mov}(g, d)$  is contained in the  $(d/2+7\delta)$ -neighborhood of  $A_g$  (Lemma 2.9). It follows from Lemma 2.6 that

$$\text{diam}(\text{Mov}(S, d)) \leq \text{diam}\left(\bigcap_{g \in S} A_g^{+(d/2+7\delta)}\right) \leq \text{diam}\left(\bigcap_{g \in S} A_g^{+13\delta}\right) + d + 18\delta. \quad (16)$$

Assume first that no element of  $S$  is loxodromic. In particular  $\|g\| \leq 16\delta$ , for every  $g \in S$ , hence the  $13\delta$ -neighborhood of  $A_g$  is contained in  $\text{Mov}(g, 2L_0\delta)$ . Consequently (16) yields

$$\text{diam}(\text{Mov}(S, d)) \leq A(G, X) + d + 18\delta.$$

Assume now that  $S$  contains a loxodromic element, say  $h$ . It follows from our assumption that there exists  $g \in S$  such that  $g$  and  $h$  do not generate an elementary subgroup. Hence applying Proposition 3.4, Inequality (16) becomes

$$\text{diam}(\text{Mov}(S, d)) \leq \|g\| + \|h\| + \nu \max\{\|g\|, \|h\|\} + A(G, X) + d + 209\delta.$$

Nevertheless, as  $\text{Mov}(S, d)$  is non-empty,  $d \geq \max\{\|g\|, \|h\|\}$ . It follows that

$$\text{diam}(\text{Mov}(S, d)) \leq A(G, X) + (\nu + 3)d + 209\delta. \quad \square$$

**Lemma 3.6.** *Let  $P$  be a parabolic subgroup of  $G$ . Let  $\xi \in \partial X$  be the unique accumulation point of  $P$ . If  $A(G, X)$  and  $\nu(G, X)$  are finite, then  $\text{Stab}(\xi)$  is parabolic as well.*

*Proof.* For simplicity we let  $E = \text{Stab}(\xi)$ . Assume that contrary to our claim that  $E$  is not parabolic. In particular  $E$  contains a loxodromic element  $h$ . Let  $g \in P$ . We write  $g_k = h^k g h^{-k}$ , for every  $k \in \mathbf{N}$  and let  $S = \{g_0, \dots, g_\nu\}$ , where  $\nu = \nu(G, X)$ . Note that  $S$  fixes  $\xi$ . Since  $S$  is finite  $\text{Mov}(S, 2L_0\delta)$  has infinite diameter (Lemma 2.12). It follows from the definition of  $A(G, X)$  that the subgroup of  $G$  generated by  $S$  is elementary. However  $h$  is loxodromic. By the very definition of  $\nu(G, X)$ , the elements  $g$  and  $h$  generate an elementary subgroup. Consequently  $P$  is contained in the maximal elementary subgroup  $E(h)$  of  $G$  containing  $h$ . Nevertheless the subgroups of  $E(h)$  are either elliptic or loxodromic, which contradicts our assumption.  $\square$

### 3.2 Mixed invariants

As explained in the previous proposition, the combination of the acylindricity parameter and the  $\nu$ -invariant provides a useful substitute to the Margulis lemma. Nevertheless this invariant does not behave well when passing to quotient (see Section 4.7). To bypass this difficulty we consider a stronger version of the  $\nu$ -invariant whose mixed nature combines both geometric and algebraic features of  $G$ . More precisely the algebraic part captures the properties of a special class of elementary subgroups that we define now.



**Dihedral germs and dihedral pairs.** Recall that  $\mathbf{D}_\infty$  stands for the infinite dihedral group. Given  $m \in \mathbf{N}$ , we denote by

$$\mathbf{D}_m = \langle \mathbf{s}, \mathbf{r} \mid \mathbf{s}^2, (\mathbf{sr})^2, \mathbf{r}^m \rangle$$

the dihedral group of order  $2m$  and by  $\mathbf{Z}_m$  the cyclic group of order  $m$ . Note that  $\mathbf{D}_1 = \mathbf{Z}_2$ . By convention  $\mathbf{D}_0$  is the trivial group. We think of  $\mathbf{D}_m$  as the isometry group of the plane preserving a regular  $m$ -gon. This motivates the following terminology. The subgroup  $\langle \mathbf{r} \rangle$  is a normal subgroup called the *rotation subgroup*. Its elements are also called *orientation preserving*. The *signature* is the morphism  $\varepsilon: \mathbf{D}_m \rightarrow \mathbf{Z}_2$ , where  $\mathbf{Z}_2$  is the quotient of  $\mathbf{D}_m$  by the rotation subgroup. An element of  $\mathbf{D}_m$  that does not preserve the orientation is called a *reflection*.

We adopt a similar terminology for  $\mathbf{D}_\infty$ . In particular its *rotation subgroup* (or *translation subgroup*) is the maximal subgroup isomorphic to  $\mathbf{Z}$ , which we also denote sometimes  $\mathbf{Z}_\infty$  to be coherent with the notations of finite groups.

If  $m \neq 2$ , the rotation subgroup of  $\mathbf{D}_m$  is algebraically completely determined: it is the unique cyclic subgroup of order  $m$ . Otherwise it should be thought an implicit piece of information attached to  $\mathbf{D}_2$ .

**Definition 3.7.** A subgroup  $C$  of  $G$  is called a *dihedral germ* if it contains an elliptic subgroup  $C_0$  which is normalized by a loxodromic element and such that  $[C : C_0]$  is a power of 2.

Note that dihedral germs are elliptic. Being a dihedral germ is invariant under conjugation. Without any further assumption on the structure of loxodromic subgroups, it is not true in general that being a dihedral germ is invariant by taking subgroup.

**Definition 3.8.** A *dihedral pair* is a pair  $(E, C)$  such that  $C$  is a dihedral germ which is also normal in  $E$  and  $E/C$  embeds in a dihedral group (finite or infinite). A subgroup  $E$  of  $G$  has *dihedral shape* if there exists a subgroup  $C$  such that  $(E, C)$  is a dihedral pair.

Every subgroup with dihedral shape is elementary. Indeed such a group is virtually the extension of an elliptic subgroup by a cyclic group. Note that the morphism from  $E/C$  to a dihedral group is in general not unique.

**Lemma 3.9.** *Let  $E$  be a loxodromic subgroup of  $G$  and  $C$  a subgroup of  $E$ . Then  $(E, C)$  is a dihedral pair if and only if  $C$  is the maximal elliptic normal subgroup of  $E$ .*

*Proof.* Assume that  $(E, C)$  is a dihedral pair. In particular  $E/C$  embeds in a dihedral group. Note that this dihedral group cannot be finite. Indeed otherwise  $E$  would be a finite extension of the elliptic subgroup  $C$ , hence an elliptic subgroup as well. Consequently  $E/C$  embeds in  $\mathbf{D}_\infty$ . It follows from [Lemma 2.19](#) that  $C = F$ . The converse statement is obvious.  $\square$

**Strong  $\nu$ -invariant.**

**Definition 3.10** (strong  $\nu$ -invariant). The quantity  $\nu_{\text{stg}}(G, X)$  is the smallest integer  $\nu$  with the following property: if  $\mathcal{C} = (g_0, \dots, g_\nu)$  is a chain generating an elementary subgroup and  $h$  a conjugating element of  $\mathcal{C}$  such that

- either  $h$  is loxodromic,
- or  $\langle g_0, \dots, g_{\nu-1} \rangle$  is contained in a dihedral germ,

then  $\langle g_0, h \rangle$  is elementary with dihedral shape.

One observes easily that  $\nu(G, X) \leq \nu_{\text{stg}}(G, X)$ . Let us mention an example where these two invariants are not equal.

**Example 3.11.** Observe first that if  $G = G_1 * G_2$  is a free product acting on its Bass-Serre tree  $T$ , then  $\nu(G, T) \leq 2$ . Consider indeed  $g, h \in G$  with  $h$  loxodromic such that the subgroup  $E = \langle g, hgh^{-1}, h^2gh^{-2} \rangle$  is elementary. Without loss of generality we can assume that  $g$  is non trivial. We first claim that the subgroup  $E_0 = \langle g, hgh^{-1} \rangle$  cannot be elliptic. Assume on the contrary that  $E_0$  fixes a point say  $x \in T$ . As  $G$  is a free product,  $x$  is the *unique* fixed point of  $g$ . Nevertheless  $hgh^{-1}$  also fixes  $x$  (as it belongs to  $E_0$ ), hence  $g$  fixes  $h^{-1}x$ . This forces  $h$  to fix  $x$  which contradicts the fact that  $h$  is loxodromic and completes the proof of the claim. Since  $T$  is a tree it does not admit any finitely generated parabolic subgroup, hence  $E_0$  is loxodromic. Observe now that the elementary subgroup  $E$  is generated by  $E_0$  and  $hE_0h^{-1}$ . Consequently  $h$  necessarily belongs to the maximal elementary loxodromic subgroup containing  $E_0$ . Therefore  $g$  and  $h$  generates an elementary subgroup. This proves that  $\nu(G, T) \leq 2$  as announced.

In this setting, every elliptic subgroup which is normalized by a loxodromic element is trivial. Hence a subgroup of  $G$  is a dihedral germ if and only if it is a finite 2-group. Let us now consider a more precise example. We fix  $m \in \mathbf{N}$  and let  $A = \mathbf{Z}_2^{m+1}$ . For every  $i \in \llbracket 0, m \rrbracket$ , we write  $g_i$  for a generator of the  $(i+1)$ -th factor  $\mathbf{Z}_2$  in  $A$ . We denote by  $G_1$  the following HNN extension of  $A$

$$G_1 = \langle A, h \mid hg_ih^{-1} = g_{i+1}, \forall i \in \llbracket 0, m-1 \rrbracket \rangle.$$

Let  $G = G_1 * \mathbf{Z}$ . It follows from the construction that

$$\langle g_0, hg_0h^{-1}, \dots, h^m g_0 h^{-m} \rangle = \langle g_0, \dots, g_m \rangle = A$$

is a dihedral germ. On the other hand the subgroup  $\langle g_0, h \rangle$  corresponds to  $G_1$  which is not virtually cyclic, thus it cannot have dihedral shape. This shows that  $\nu_{\text{stg}}(G, T) \geq m$ . In particular if  $m > 2$ , then  $\nu(G, T) < \nu_{\text{stg}}(G, T)$ .

Note that in this example, the difference between  $\nu(G, X)$  and  $\nu_{\text{stg}}(G, X)$  comes from the algebraic structure of elliptic subgroups. It emphasizes the fact that  $\nu_{\text{stg}}(G, X)$  is not a purely geometric invariant.

**Model collections.** As we will see later, controlling the strong  $\nu$ -invariant is a key ingredient to handle even torsion and which was not needed to study free Burnside groups with odd exponents. It requires a fine understanding of the structure of dihedral pairs. We complete this section by a last notion designed to describe those subgroups.

A *model collection* is a family  $\mathcal{E}$  of (abstract) torsion groups. Its *exponent*  $\mu = \mu(\mathcal{E})$  is the smallest positive integer such that  $\mathbf{g}^\mu = 1$ , for every  $\mathbf{E} \in \mathcal{E}$ , for every  $\mathbf{g} \in \mathbf{E}$ .

**Definition 3.12.** Let  $p \in \mathbf{N}$  and  $\mathcal{E}$  be a model collection. We say that a dihedral pair  $(E, C)$  has *type*  $(\mathcal{E}, p)$  if there exist  $k \in \mathbf{N}$  and a morphism  $\varphi: E \rightarrow \mathbf{E}$ , where  $\mathbf{E} \in \mathcal{E}$  such that the map and  $\varphi$  extends to an embedding from  $E$  into  $E/C \times \mathbf{D}_p^k \times \mathbf{E}$ .

*Remark.* A reader only interested in *free* Burnside groups can read the entire article by taking for  $\mathcal{E}$  the collection that consists only of the trivial group. The exponent of this trivial model collection is 1.

We now fix an integer  $p \in \mathbf{N}$  and a model collection  $\mathcal{E}$  and write  $\mu = \mu(\mathcal{E})$  for its exponent. Saying that  $(E, C)$  has type  $(\mathcal{E}, p)$  means that, up to a residual factor  $\mathbf{E} \in \mathcal{E}$ , the group  $E$  essentially embeds into a direct product of dihedral groups. In particular we can exploit the *algebraic identities* of dihedral groups to recover information about  $E$ . The next two statements give simple but essential examples of this idea. Other applications will arise later in the article.

**Proposition 3.13.** Let  $(E, C)$  be a dihedral pair with type  $(\mathcal{E}, p)$ . Let  $\mathcal{C} = (g_0, g_1, \dots, g_{\mu+2})$  be a chain of  $G$  and  $h$  a conjugating element of  $\mathcal{C}$ . If  $g_0$  and  $h$  belong to a subgroup  $E$ , then  $g_{\mu+2}$  belongs to  $\langle g_0, g_1, \dots, g_{\mu+1} \rangle$ .

*Proof.* By assumption, there exist  $k \in \mathbf{N}$ , a group  $\mathbf{E} \in \mathcal{E}$ , and a morphism  $\varphi: E \rightarrow \mathbf{E}$ , where  $\mathbf{E} \in \mathcal{E}$  such that  $\varphi$  extends to an embedding  $E \hookrightarrow E/C \times \mathbf{D}_p^k \times \mathbf{E}$ . For every  $i \in \llbracket 0, 3 \rrbracket$  we let

$$u_i = g_i g_{i+1} \cdots g_{i+\mu-1}.$$

Note that it suffices to prove that  $u_3 u_2^{-1} u_0 u_1^{-1} = 1$ . To that end, we have to check that this identity holds in every factor of  $E/C \times \mathbf{D}_p^k \times \mathbf{E}$ . It was observed by Lysenok [26, Proposition 15.10] that if  $\mathbf{x}$  and  $\mathbf{y}$  are two elements of  $\mathbf{D}_\infty$  then

$$(\mathbf{y}^3 \mathbf{x} \mathbf{y}^{-3}) (\mathbf{y}^2 \mathbf{x}^{-1} \mathbf{y}^{-2}) \mathbf{x} (\mathbf{y} \mathbf{x}^{-1} \mathbf{y}^{-1}) = [\mathbf{y}^2, [\mathbf{y}, \mathbf{x}]] = 1.$$

Hence the identity  $u_3 u_2^{-1} u_0 u_1^{-1} = 1$  holds in  $E/C$  as well as in any factor  $\mathbf{D}_p$ . On the other hand we observe that  $u_0 = (g_0 h)^\mu h^{-\mu}$ . By the very definition of the exponent  $\mu$ , the element  $\varphi(u_0) \in \mathbf{E}$  is trivial and thus so are its conjugates  $\varphi(u_1)$ ,  $\varphi(u_2)$  and  $\varphi(u_3)$ . Hence the identity  $u_3 u_2^{-1} u_0 u_1^{-1} = 1$  also holds in  $\mathbf{E}$ , and the proof is complete.  $\square$

Let  $n \in \mathbf{N}$ . Let  $\mathbf{E} \in \mathcal{E}$  and  $\mathbf{\Pi} = \mathbf{D}_{p_1} \times \cdots \times \mathbf{D}_{p_k} \times \mathbf{E}$  be a direct product of dihedral groups with  $\mathbf{E}$  where  $p_i$  divides  $n$  for every  $i \in \llbracket 1, k \rrbracket$ . The

signature  $\varepsilon_i: \mathbf{D}_{p_i} \rightarrow \mathbf{Z}_2$  induces a morphism  $\mathbf{\Pi} \rightarrow (\mathbf{Z}_2)^k$ , whose kernel is  $\mathbf{\Pi}_+ = \mathbf{Z}_{p_1} \times \cdots \times \mathbf{Z}_{p_k} \times \mathbf{E}$  is the *pure rotation subgroup*. Given a subgroup  $\mathbf{A}$  of  $\mathbf{\Pi}$ , its *reflection rank* is the dimension of the image of  $\mathbf{A}$  in  $\mathbf{\Pi}/\mathbf{\Pi}_+$  (seen as a  $\mathbf{Z}_2$ -vector space). The next lemma is a variation on Ivanov [24, Lemma 16.2].

**Lemma 3.14.** *Let  $\mathbf{A}$  be a subgroup of  $\mathbf{\Pi}$  and  $r$  its reflection rank. We assume that  $2^{r+3}\mu$  divides  $n$ . For every  $\mathbf{h} \in \mathbf{\Pi}$ , normalizing  $\mathbf{A}$ , there exists  $\mathbf{a} \in \mathbf{A}$  with the following properties.*

- (i)  $\mathbf{h}^{n/4}\mathbf{a}^{-1}$  centralizes  $\mathbf{A}$
- (ii) for every  $\mathbf{b} \in \mathbf{\Pi}$ ,  $[\mathbf{h}^{n/4}\mathbf{a}^{-1}, \mathbf{b}]$  centralizes  $\mathbf{\Pi}$ .

*Proof.* By assumption, there exist  $\mathbf{s}_1, \dots, \mathbf{s}_r \in \mathbf{A}$  such that  $\mathbf{A}$  is generated by  $\mathbf{s}_1, \dots, \mathbf{s}_r$  and  $\mathbf{A} \cap \mathbf{\Pi}_+$ . We let

$$\mathbf{a} = \prod_{(\varepsilon_1, \dots, \varepsilon_r) \in \{0,1\}^r} [\mathbf{r}, \mathbf{s}_1^{\varepsilon_1} \dots \mathbf{s}_r^{\varepsilon_r}], \quad \text{where } \mathbf{r} = \mathbf{h}^{2^{-(r+2)}n}.$$

Since  $\mathbf{h}$  normalizes  $\mathbf{A}$ , the element  $\mathbf{a}$  belongs to  $\mathbf{A}$ . It is sufficient to check that in each factor of  $\mathbf{\Pi}$  the image of  $\mathbf{h}^{n/4}\mathbf{a}^{-1}$  satisfies the announced properties. Since the exponent of  $\mathbf{E}$  divides  $2^{-(r+2)}n$ , the elements  $\mathbf{h}^{n/4}$  and  $\mathbf{a}$  are trivial in  $\mathbf{E}$ . Thus so is  $\mathbf{h}^{n/4}\mathbf{a}^{-1}$ . Hence (i) and (ii) hold in  $\mathbf{E}$ .

We now focus on the dihedral factors. Let  $i \in \llbracket 1, k \rrbracket$ . Assume first that the image of  $\mathbf{A}$  in  $\mathbf{D}_{p_i}$  is contained in the rotation group  $\mathbf{Z}_{p_i}$ . Note that 2 divides  $2^{-(r+2)}n$ . Hence (the image of)  $\mathbf{r}$  lies in the rotation group of  $\mathbf{D}_{p_i}$ . Thus  $\mathbf{a}$  is trivial in  $\mathbf{D}_{p_i}$ , while  $\mathbf{h}^{n/4}$  belongs to  $\mathbf{Z}_{p_i}$ . Consequently (the image of)  $\mathbf{h}^{n/4}\mathbf{a}^{-1}$  centralizes  $\mathbf{Z}_{p_i}$  and thus (the image of)  $\mathbf{A}$ . Moreover for every  $\mathbf{b} \in \mathbf{\Pi}$ , (the image of)  $[\mathbf{h}^{n/4}\mathbf{a}^{-1}, \mathbf{b}]$  which coincide with (the image of)  $[\mathbf{h}^{n/4}, \mathbf{b}]$ , centralizes  $\mathbf{D}_{p_i}$ . Assume now that the image of  $\mathbf{A}$  in  $\mathbf{D}_{p_i}$  contains a reflection. By construction every element of  $\mathbf{A} \cap \mathbf{\Pi}_+$  is mapped to the rotation subgroup  $\mathbf{Z}_{p_i}$ . Without loss of generality we can assume that  $\mathbf{s}_1$  is mapped to a reflection of  $\mathbf{D}_{p_i}$ . Let  $\Omega$  be the subset of all tuples  $(\varepsilon_1, \dots, \varepsilon_r) \in \{0,1\}^r$  such that  $\mathbf{s}_1^{\varepsilon_1} \dots \mathbf{s}_r^{\varepsilon_r}$  is mapped to a reflection in  $\mathbf{D}_{p_i}$ . As previously the image of  $\mathbf{r}$  in  $\mathbf{D}_{p_i}$  is a rotation. Hence, seen in  $\mathbf{D}_{p_i}$ , we have

$$[\mathbf{r}, \mathbf{s}_1^{\varepsilon_1} \dots \mathbf{s}_r^{\varepsilon_r}] = \begin{cases} \mathbf{r}^2 & \text{if } (\varepsilon_1, \dots, \varepsilon_r) \in \Omega \\ 1 & \text{otherwise} \end{cases}$$

Consequently we get in  $\mathbf{D}_{p_i}$

$$\mathbf{a} = \mathbf{r}^{2|\Omega|} = \mathbf{h}^{2^{-(r+1)}|\Omega|n}.$$

Observe that the cardinality of  $\Omega$  is  $2^{r-1}$ . Indeed the map sending  $(\varepsilon_1, \dots, \varepsilon_r)$  to  $(\varepsilon_1 + 1, \dots, \varepsilon_r)$  induces a bijection from  $\Omega$  onto  $\{0,1\}^r \setminus \Omega$ . It follows that  $\mathbf{h}^{n/4}$  and  $\mathbf{a}$  coincide in  $\mathbf{D}_{p_i}$ . Thus (i) and (ii) hold in  $\mathbf{D}_{p_i}$ .  $\square$

## 4 Small cancellation theory

Let us recall the main strategy to study free Burnside group  $\mathbf{B}_r(n)$ . Starting from the free group  $\mathbf{F}_r$ , we are going to build a sequence of non-elementary hyperbolic groups

$$\mathbf{F}_r = G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \cdots$$

whose directly limit is exactly  $\mathbf{B}_r(n)$ . The group  $G_{k+1}$  is obtained from  $G_k$  by adjoining relations of the form  $h^n = 1$  where  $h$  runs over a subset of “small” loxodromic elements of  $G_k$ . The main difficulty is to make sure that  $G_{k+1}$  remains a non-elementary hyperbolic group although the exponent  $n$  has been fixed in advance. We achieve this by using a *geometric* approach of small cancellation theory.

In this section we focus on a *single* step  $G_k \twoheadrightarrow G_{k+1}$ . We present first an overview small cancellation theory and develop later the required additional material. For proving infiniteness of Burnside groups we only need to consider relations of the form  $h^n = 1$ . Nevertheless we will work in a slightly more general setting as the intermediate results can be of independent interest.

### 4.1 General setting

Let  $X$  be a  $\delta$ -hyperbolic length space and  $G$  a group acting gently by isometries on  $X$ . Let  $\mathcal{Q}$  be a collection of pairs  $(H, Y)$  where  $H$  is a loxodromic subgroup of  $G$  and  $Y$  its cylinder (see [Section 2.3](#) for the definitions). We assume that  $\mathcal{Q}$  is invariant under the action of  $G$  defined by  $g \cdot (H, Y) = (gHg^{-1}, gY)$ , for every  $(H, Y) \in \mathcal{Q}$  and every  $g \in G$ . We denote by  $K$  the (normal) subgroup of  $G$  generated by all  $H$  where  $(H, Y)$  runs over  $\mathcal{Q}$ . The goal is to study the quotient  $\tilde{G} = G/K$ . To that end, we defined two parameters  $\Delta(\mathcal{Q}, X)$  and  $\text{inj}(\mathcal{Q}, X)$  which play the role of the lengths of the largest piece and the smallest relation respectively.

$$\begin{aligned} \Delta(\mathcal{Q}, X) &= \sup \{ \text{diam}(Y_1^{+5\delta} \cap Y_2^{+5\delta}) : (H_1, Y_1) \neq (H_2, Y_2) \in \mathcal{Q} \}, \\ \text{inj}(\mathcal{Q}, X) &= \inf \{ \|h\| : h \in H, (H, Y) \in \mathcal{Q} \}. \end{aligned}$$

*Remark.* As explained above, we will later focus on a particular set of relations. More precisely the collection  $\mathcal{Q}$  will be of the form

$$\mathcal{Q} = \{(Y_h, \langle h^n \rangle) : h \in S\}$$

where  $n$  is a large integer and  $S$  a subset of “small” loxodromic elements of  $G$ , which is invariant under conjugation. Assuming that  $\Delta(\mathcal{Q}, X)$  is finite will automatically imply that  $\langle h^n \rangle$  is normal in  $\text{Stab}(Y_h)$ , for every  $h \in S$ .

We now fix once for all a number  $\rho \in \mathbf{R}_+^*$ . Its value will be made precise later (see [Theorem 4.7](#)). It should be thought of as a very large length.

**Cones.** Let  $(H, Y) \in \mathcal{Q}$ . The *cone of radius  $\rho$  over  $Y$* , denoted by  $Z_\rho(Y)$  or simply  $Z(Y)$ , is the quotient of  $Y \times [0, \rho]$  by the equivalence relation that identifies all the points of the form  $(y, 0)$ . The equivalence class of  $(y, 0)$ , denoted by  $v$ , is called the *apex* or *cone point* of  $Z(Y)$ . By abuse of notation, we still write  $(y, r)$  for the equivalence class of  $(y, r)$ . The map  $\iota: Y \rightarrow Z(Y)$  that sends  $y$  to  $(y, \rho)$  provides a natural embedding from  $Y$  to  $Z(Y)$ . The *radial projection*  $p: Z(Y) \setminus \{v\} \rightarrow Y$  is the map sending  $(y, r)$  to  $y$ . We denote by  $|\cdot|_Y$  the length metric on  $Y$  induced by the restriction of  $|\cdot|$  to  $Y$ . This cone  $Z(Y)$  can be endowed with a metric as described below.

**Proposition 4.1.** [4, Chapter I.5, Proposition 5.9] *The cone  $Z(Y)$  is endowed with a metric characterized in the following way. Let  $x = (y, r)$  and  $x' = (y', r')$  be two points of  $Z(Y)$  then*

$$\operatorname{ch} |x - x'|_{Z(Y)} = \operatorname{ch} r \operatorname{ch} r' - \operatorname{sh} r \operatorname{sh} r' \cos \theta(y, y'), \quad (17)$$

where  $\theta(y, y')$  is the angle at the apex defined by

$$\theta(y, y') = \min \left\{ \pi, \frac{|y - y'|_Y}{\operatorname{sh} \rho} \right\}.$$

The distance between two points  $x = (y, r)$  and  $x' = (y', r')$  of  $Z(Y)$  has the following geometric interpretation. Consider a geodesic triangle in the hyperbolic plane  $\mathbf{H}_2$  such that the lengths of two sides are respectively  $r$  and  $r'$  and the angle between them is  $\theta(y, y')$ . According to the law of cosines,  $|x - x'|$  is exactly the length of the third side of the triangle (see Figure 3).

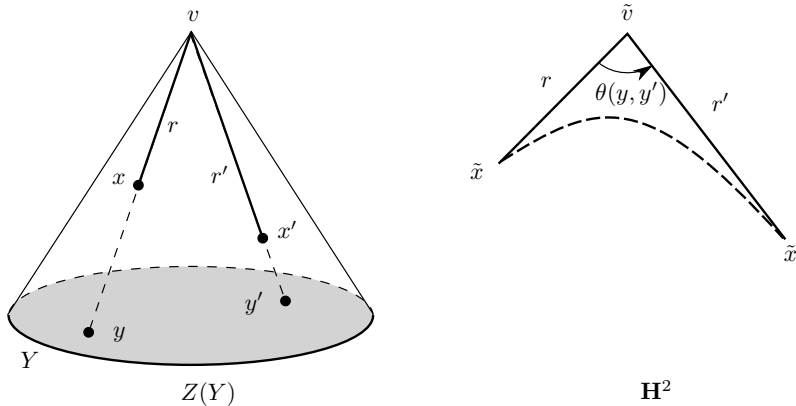


Figure 3: Geometric interpretation of the distance in the cone.

**Example 4.2.** If  $Y$  is a circle whose perimeter is  $2\pi \operatorname{sh} \rho$  and endowed with the length metric, then  $Z(Y)$  is the closed hyperbolic disc of radius  $\rho$ . If  $Y$  is the real line, then  $Z(Y) \setminus \{v\}$  is the universal cover of the punctured hyperbolic disc of radius  $\rho$ .

In order to compare the metric of  $Y$  and  $Z(Y)$ , we use the map  $\mu: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  characterized as follows

$$\operatorname{ch} \mu(t) = \operatorname{ch}^2 \rho - \operatorname{sh}^2 \rho \cos \left( \min \left\{ \pi, \frac{t}{\operatorname{sh} \rho} \right\} \right), \quad \forall t \in \mathbf{R}_+,$$

so that for every  $y, y' \in Y$  we have

$$|\iota(y) - \iota(y')|_{Z(Y)} = \mu(|y - y'|_Y).$$

The next proposition summarizes the properties of  $\mu$ .

**Proposition 4.3.** *The map  $\mu$  is continuous, concave, and non-decreasing. In addition, for all  $t \in [0, \pi \operatorname{sh} \rho]$ , we have  $t \leq \pi \operatorname{sh}(\mu(t)/2)$ .*

We complete this part with a useful tool to compare two cones.

**Lemma 4.4.** *Let  $f: Y_1 \rightarrow Y_2$  a  $(1, \ell)$  quasi-isometric embedding between two metric spaces. The map  $Z(Y_1) \rightarrow Z(Y_2)$  sending  $(y, r)$  to  $(f(y), r)$  is again a  $(1, \ell)$ -quasi-isometric embedding.*

*Proof.* The result is a direct consequence of the geometric interpretation of the metric on the cones.  $\square$

**The cone-off space  $\dot{X}$ .** The cone-off of radius  $\rho$  over  $X$  relative to  $\mathcal{Q}$  denoted by  $\dot{X}_\rho(\mathcal{Q})$  (or simply  $\dot{X}$ ) is obtained by attaching for every  $(H, Y) \in \mathcal{Q}$ , the cone  $Z(Y)$  on  $X$  along  $Y$  according to  $\iota$ . The subset of  $X$  consisting of all apices of the cones is denoted by  $\mathcal{V}$ . We endow  $\dot{X}$  with the largest pseudo-metric  $|\cdot|_{\dot{X}}$  for which all the maps  $X \rightarrow \dot{X}$  and  $Z(Y) \rightarrow \dot{X}$  – where  $(H, Y)$  runs over  $\mathcal{Q}$  – are 1-Lipschitz. It turns out that this pseudo-distance is a length metric on  $\dot{X}$  [9, Proposition 5.10]. The next lemmas detail the relationship between the metrics of  $X$  and  $\dot{X}$ .

**Lemma 4.5** ([9, Lemma 5.8]). *For every  $x, x' \in X$ , we have*

$$\mu(|x - x'|_X) \leq |x - x'|_{\dot{X}} \leq |x - x'|_X.$$

**Lemma 4.6** ([9, Lemma 5.7]). *Let  $(H, Y) \in \mathcal{Q}$ . Let  $x \in Z(Y)$ . Let  $d(x, Y)$  be the distance between  $x$  and  $\iota(Y)$  computed with  $|\cdot|_{Z(Y)}$ . For all  $x' \in \dot{X}$ , if  $|x - x'|_{\dot{X}} < d(x, Y)$  then  $x'$  belongs to  $Z(Y)$ . Moreover  $|x - x'|_{\dot{X}} = |x - x'|_{Z(Y)}$ .*

Let  $v$  be the apex of  $Z(Y)$ . It follows from the lemma that, as a set, the ball  $B(v, \rho)$  (for the metric of  $\dot{X}$ ) is nothing but  $Z(Y) \setminus \iota(Y)$ . Moreover the metrics  $|\cdot|_{\dot{X}}$  and  $|\cdot|_{Z(Y)}$  coincide on  $B(v, \rho/3)$ .

**The quotient space  $\bar{X}$ .** The action of  $G$  on  $X$  naturally extends to an action on  $\dot{X}$  as follows. Let  $(H, Y) \in \mathcal{Q}$ . For every  $g \in G$ , for every  $x = (y, r)$  in  $Z(Y)$ , we define  $gx$  to be the point of  $Z(gY)$  given by  $gx = (gy, r)$ . The space  $\bar{X}$  is the quotient  $\bar{X} = \dot{X}/K$ . The metric on  $\dot{X}$  induces a pseudo-metric on  $\bar{X}$ . We write  $\zeta: \dot{X} \rightarrow \bar{X}$  for the canonical projection from  $\dot{X}$  to  $\bar{X}$ . The quotient  $\bar{G}$  naturally acts by isometries on  $\bar{X}$ . We denote by  $\bar{\mathcal{V}}$  the image of  $\mathcal{V}$  in  $\bar{X}$ . For every  $x \in \dot{X}$ , we usually write  $\bar{x}$  for its image in  $\bar{X}$ .

**Small cancellation theorem.** The next statement is a combination of Proposition 6.4, Proposition 6.7, Corollary 3.12 and Proposition 3.15 in [9]

**Theorem 4.7.** *There exists  $\delta_0, \delta_1, \Delta_0, \rho_0 \in \mathbf{R}_+^*$ , which do not depend on  $X, G$  or  $\mathcal{Q}$ , with the following property. Assume that  $\rho \geq \rho_0$ . If  $\delta \leq \delta_0$ ,  $\Delta(\mathcal{Q}, X) \leq \Delta_0$  and  $\text{inj}(\mathcal{Q}, X) \geq 10\pi \text{sh } \rho$ , then the following holds*

- (i) *The cone-off space  $\dot{X}$  is  $\dot{\delta}$ -hyperbolic with  $\dot{\delta} \leq \delta_1$ .*
- (ii) *The quotient space  $\bar{X}$  is  $\bar{\delta}$ -hyperbolic with  $\bar{\delta} \leq \delta_1$ .*
- (iii) *Let  $(H, Y) \in \mathcal{Q}$ . Let  $\bar{v}$  be the image in  $\bar{X}$  of the apex  $v$  of  $Z(Y)$ . The subgroup  $\text{Stab}(\bar{v}) \subset \bar{G}$  is isomorphic to the quotient  $\text{Stab}(Y)/H$ . Moreover the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $B(v, \rho/2)/H$  onto  $B(\bar{v}, \rho/2)$ .*
- (iv) *For every  $r \in (0, \rho/20]$ , for every  $x \in \dot{X}$ , if  $d(x, \mathcal{V}) \geq 2r$ , then the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $B(x, r)$  onto  $B(\bar{x}, r)$ .*
- (v) *For every  $x \in \dot{X}$  for every  $g \in K \setminus \{1\}$ , we have  $|gx - x|_{\dot{X}} \geq \min\{2r, \rho/5\}$ , where  $r = d(v, \mathcal{V})$ . In particular,  $K$  acts freely on  $\dot{X} \setminus \mathcal{V}$ . Moreover, the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  induces a covering map  $\dot{X} \setminus \mathcal{V} \rightarrow \bar{X} \setminus \bar{\mathcal{V}}$ .*

*Remark.* Note that the constants  $\delta_0$  and  $\Delta_0$  (respectively  $\rho_0$ ) can be chosen arbitrarily small (respectively large). From now on, we will always assume that  $\rho_0 > 10^{20} L_0 \delta_1$  whereas  $\delta_0, \Delta_0 < 10^{-10} \delta_1$ . These estimates are absolutely not optimal. We chose them very generously to ensure that all the inequalities which we might need later will be satisfied. What really matters is their orders of magnitude recalled below.

$$\max\{\delta_0, \Delta_0\} \ll \delta_1 \ll \rho \ll \pi \text{sh } \rho.$$

Another important point to remember is the following. The constants  $\delta_0, \Delta_0$  and  $\pi \text{sh } \rho$  are used to describe the geometry of  $X$  whereas  $\delta_1$  and  $\rho$  refers to the one of  $\dot{X}$  or  $\bar{X}$ . From now on and until the end of Section 4 we assume that  $X, G$  and  $\mathcal{Q}$  are as in Theorem 4.7. In particular,  $\dot{X}$  and  $\bar{X}$  are respectively  $\dot{\delta}$ - and  $\bar{\delta}$ -hyperbolic. Up to increasing one constant or the other, we can actually assume that  $\dot{\delta} = \bar{\delta}$ . Nevertheless we still keep two distinct notations, to remember which space we are working in.

*Notations.* In this section we work with three metric spaces namely  $X$ , its cone-off  $\dot{X}$  and the quotient  $\bar{X}$ . Since the map  $X \hookrightarrow \dot{X}$  is an embedding we use the same letter  $x$  to designate a point of  $X$  and its image in  $\dot{X}$ . We write  $\bar{x}$  for its image in  $\bar{X}$ . Unless stated otherwise, we keep the notation  $|\cdot|$  (without mentioning the space) for the distances in  $X$  or  $\bar{X}$ . The metric on  $\dot{X}$  will be denoted by  $|\cdot|_{\dot{X}}$ .



## 4.2 A few additional facts regarding the cone-off space

**Radial projection.** The *radial projection*  $p: \dot{X} \setminus \mathcal{V} \rightarrow X$  is defined as follows. Its restriction to  $X$  is the identity. Given any  $(H, Y) \in \mathcal{Q}$ , the restriction of  $p$  to  $Z(Y) \setminus \{v\}$ , where  $v$  stands for the apex of  $Z(Y)$ , coincides with the radial projection defined in the previous paragraph. This map is  $G$ -equivariant. Observe that  $|x - p(x)|_{\dot{X}} \leq \rho$ , for every  $x \in \dot{X} \setminus \mathcal{V}$ .

*Remark.* Recall that neither  $X$ ,  $\dot{X}$  or  $\bar{X}$  are supposed to be geodesic. As explained in [Remark 2.1](#), the Gromov product  $\langle x, y \rangle_z$  roughly represents the distance between  $z$  and any “geodesic” joining  $x$  and  $y$  though. In particular, in the next statements, assumptions of the form  $\langle z, z' \rangle_v > d$  mean that the “geodesic” in  $\dot{X}$  between  $z$  and  $z'$  stay sufficiently far away from the apex  $v$ .

**Proposition 4.8.** *Let  $x, x' \in X$  such that  $\langle x, x' \rangle_v > 0$ , for every  $v \in \mathcal{V}$  (here the Gromov product is computed in  $\dot{X}$ ). Then*

$$|x - x'|_{\dot{X}} \leq |x - x'|_X \leq \frac{\pi \operatorname{sh} \rho}{2\rho} |x - x'|_{\dot{X}}.$$

*Proof.* In this proof all the Gromov products are computed in  $\dot{X}$ . The first inequality directly follows from the fact that the embedding  $X \rightarrow \dot{X}$  is 1-Lipschitz. Let us focus on the second inequality. Let  $\eta > 0$  and  $\gamma: [a, b] \rightarrow \dot{X}$  be a  $(1, \eta)$ -quasi-geodesic from  $x$  to  $x'$ . According to our assumption, up to decreasing  $\eta$  we can assume that for every  $(H, Y) \in \mathcal{Q}$ , the diameter of  $\gamma \cap Z(Y)$  is less than  $2\rho$ . Consequently there exists a partition  $t_0 = a \leq t_1 \leq \dots \leq t_m = b$  of  $[a, b]$  such that

- (i)  $\gamma(t_i)$  belongs to  $X$  for every  $i \in \llbracket 0, m \rrbracket$ ;
- (ii)  $|\gamma(t_{i+1}) - \gamma(t_i)|_{\dot{X}} < 2\rho$ , for every  $i \in \llbracket 0, m-1 \rrbracket$ .

[Lemma 4.5](#) combined with the concavity of the map  $\mu$  tells us that

$$\begin{aligned} \frac{2\rho}{\pi \operatorname{sh} \rho} |x - x'| &\leq \frac{2\rho}{\pi \operatorname{sh} \rho} \sum_{i=0}^{m-1} |\gamma(t_{i+1}) - \gamma(t_i)| \leq \sum_{i=0}^{m-1} \mu \left( |\gamma(t_{i+1}) - \gamma(t_i)| \right) \\ &\leq \sum_{i=0}^{m-1} |\gamma(t_{i+1}) - \gamma(t_i)|_{\dot{X}} \\ &\leq |x - x'|_{\dot{X}} + \eta. \end{aligned}$$

This inequality holds for every sufficiently small  $\eta > 0$ , hence the result.  $\square$

**Corollary 4.9.** *Let  $Z$  be a subset of  $\dot{X}$  such that  $\langle z, z' \rangle_v > 2\delta$  for every  $z, z' \in Z$  and  $v \in \mathcal{V}$  (here the Gromov product is computed in  $\dot{X}$ ). Then the radial projection  $p: \dot{X} \setminus \mathcal{V} \rightarrow X$  when restricted to  $Z$  is a quasi-isometric embedding.*

*Proof.* In this proof all the Gromov products are computed in  $\dot{X}$ . Let  $z, z' \in Z$ . Let  $y, y' \in X$  be the radial projections of  $z$  and  $z'$  respectively. It follows from

the triangle inequality that  $|z - z'|_{\dot{X}}$  and  $|y - y'|_{\dot{X}}$  differs by at most  $2\rho$ . In view of [Proposition 4.8](#) it is sufficient to prove that  $\langle y, y' \rangle_v > 0$  for every  $v \in \mathcal{V}$ . The four point inequality [\(3\)](#) applied in  $\dot{X}$  gives

$$\langle y, y' \rangle_v \geq \min \left\{ \langle y, z \rangle_v, \langle z, z' \rangle_v, \langle z', y' \rangle_v \right\} - 2\dot{\delta}.$$

Assume that  $\langle y, z \rangle_v \leq 2\dot{\delta}$ . Then  $z$  necessarily belongs to the cone  $Z(Y)$  for some  $(H, Y) \in \mathcal{Q}$ . Indeed otherwise  $z = y$  is a point of  $X$ , and thus  $\langle y, z \rangle_v \geq 2\rho$ . It follows from the definition of the radial projection and [Lemma 4.6](#) that  $z$  lies on a geodesic between  $y$  and the apex of  $Z(Y)$ . As the distance between two apices is at least  $2\rho$ , the point  $v$  is necessarily the apex of  $Z(Y)$ . Hence

$$\langle z, z \rangle_v = |v - z|_{\dot{X}} = \langle y, z \rangle_v$$

is bounded above by  $2\dot{\delta}$ , which contradicts our assumption. We prove in the same way that  $\langle y', z' \rangle_v > 2\dot{\delta}$ . On the other hand, according to our assumption we have  $\langle z, z' \rangle_v > 2\dot{\delta}$ . Thus  $\langle y, y' \rangle_v > 0$ .  $\square$

### Parabolic subgroups.

**Lemma 4.10.** *Let  $P$  be a subgroup of  $G$ . If  $P$  is parabolic for its action on  $\dot{X}$ , then so is its action on  $X$ .*

*Proof.* Since the embedding  $X \rightarrow \dot{X}$  is 1-Lipschitz,  $P$  cannot be elliptic for its action on  $X$ . Hence it suffices to prove that  $P$  does not contain any loxodromic element (for its action on  $X$ ). We denote by  $\xi$  the unique point of  $\Lambda(P) \subset \partial\dot{X}$ . Let  $\gamma : \mathbf{R}_+ \rightarrow \dot{X}$  be an  $L_0\dot{\delta}$ -local  $(1, 11\dot{\delta})$ -quasi-geodesic ray whose endpoint at infinity is  $\xi$ . Let  $g \in P$ . By [Lemma 2.12](#), there is  $t_0 \in \mathbf{R}_+$  such that for every  $t \geq t_0$ ,  $|g\gamma(t) - \gamma(t)|_{\dot{X}} \leq 114\dot{\delta}$ . Since  $\gamma$  is infinite, there exists  $t \geq t_0$  such that  $\gamma(t)$  belongs to  $X$ . It follows then from [Lemma 4.5](#) that

$$\mu(|g\gamma(t) - \gamma(t)|) \leq |g\gamma(t) - \gamma(t)|_{\dot{X}} \leq 114\dot{\delta} < 2\rho.$$

Thus  $|g\gamma(t) - \gamma(t)| \leq \pi \operatorname{sh}(57\dot{\delta})$  ([Proposition 4.3](#)). Consequently  $\|g\| \leq \pi \operatorname{sh}(57\dot{\delta})$ , for every  $g \in P$ . In particular,  $P$  does not contain any loxodromic element for its action on  $X$ .  $\square$

### 4.3 Apex stabilizer in the quotient space.

As we mentioned in the introduction the quotient space  $\bar{M} = \bar{X}/\bar{G}$  can be seen as an orbifold, and  $\bar{G}$  its fundamental group [[15](#)]. Although this is not the point of view we adopted here, it is a great source of inspiration. According to [Theorem 4.7 \(iii\)](#), for every  $(H, Y) \in \mathcal{Q}$ , the quotient  $\operatorname{Stab}(Y)/H$  embeds in  $\bar{G}$ , which basically means that  $\bar{M}$  is developable, so that its universal cover is  $\bar{X}$ . This orbifold  $\bar{M}$  also comes with an analog of Margulis' thin/thick decomposition for hyperbolic manifolds. The thin part corresponds to the neighborhood of the cone points (or more precisely their images in  $\bar{M}$ ). In this section we study the structure of  $\bar{X}$  around the apices. In particular we prove that the isotropy group of such a point locally acts as a dihedral group on a hyperbolic disc.

**Local classification of isometries.** Let  $(H, Y) \in \mathcal{Q}$ . Recall that  $\text{Stab}(Y)$  is the maximal loxodromic subgroup containing  $H$ . We write  $\text{Stab}^+(Y)$  for the subgroup of  $\text{Stab}(Y)$  fixing pointwise  $\partial Y$ . Its index in  $\text{Stab}(Y)$  is at most 2. Since the action of  $G$  on  $X$  is gentle, the set  $F$  of all elliptic elements of  $\text{Stab}^+(Y)$  is a normal subgroup of  $\text{Stab}(Y)$ . Moreover  $\text{Stab}^+(Y)/F$  is isomorphic to  $\mathbf{Z}$  while  $\text{Stab}(Y)/F$  embeds in  $\mathbf{D}_\infty$ . In other words we have a short exact sequence

$$1 \rightarrow F \rightarrow \text{Stab}(Y) \xrightarrow{q} \mathbf{L} \rightarrow 1,$$

where  $\mathbf{L}$  is either  $\mathbf{Z}$  or  $\mathbf{D}_\infty$ .

As  $\text{inj}(\mathcal{Q}, X) \geq 10\pi \text{sh } \rho$ , any non-trivial element of  $H$  is loxodromic. Thus its image in  $\mathbf{L}$  is  $n\mathbf{Z}$  for some  $n \in \mathbf{N} \setminus \{0\}$ . We write  $\mathbf{L}_n$  for  $\mathbf{L}/n\mathbf{Z}$ , i.e.  $\mathbf{L}_n = \mathbf{D}_n$  if  $\mathbf{L} = \mathbf{D}_\infty$  and  $\mathbf{L}_n = \mathbf{Z}_n$ , if  $\mathbf{L} = \mathbf{Z}$ . Let  $v$  be the apex of  $Z(Y)$  and  $\bar{v}$  its image in  $\bar{X}$ . Recall that, according to the small cancellation theorem ([Theorem 4.7](#)) the subgroup  $\text{Stab}(\bar{v})$  is isomorphic to  $\text{Stab}(Y)/H$ . After taking the quotient by  $H$  we get the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & F & \longrightarrow & \text{Stab}(Y) & \longrightarrow & \mathbf{L} \longrightarrow 1 \\ & & \downarrow \wr & & \downarrow \pi & & \downarrow \\ 1 & \longrightarrow & \bar{F} & \longrightarrow & \text{Stab}(\bar{v}) & \longrightarrow & \mathbf{L}_n \longrightarrow 1 \end{array}$$

where the horizontal lines are short exact sequences. Note that  $\text{Stab}(\bar{v}) \rightarrow \mathbf{L}_n$  is a well-defined map (up to post-composing by an inner automorphism of  $\mathbf{L}_n$  and/or  $\pm \text{Id}$ ). Indeed if  $(H', Y')$  is another pair of  $\mathcal{Q}$  such that  $\bar{v}$  is this image of the apex  $v'$  of  $Z(Y')$ , then there exists an element  $u \in K$  such that  $(H', Y') = (uHu^{-1}, uY)$ . Thus the maps  $q: \text{Stab}(Y) \rightarrow \mathbf{L}$  and  $q': \text{Stab}(Y') \rightarrow \mathbf{L}$  differ at the source by a conjugation and at the target by an automorphism of  $\mathbf{L}$ .

By analogy with singularities the integer  $n$  is called the *order* of the cone point  $\bar{v}$ . As we explained before  $\mathbf{L}_n$  can be either  $\mathbf{Z}_n$  or  $\mathbf{D}_n$ . In any case it embeds in  $\mathbf{D}_n$ . We call the map  $q_{\bar{v}} \rightarrow \mathbf{D}_n$  obtained in this way the *geometric realization of  $\text{Stab}(\bar{v})$* . Although it is not made explicit in the notation, we allow for the moment the order to be different from one apex to the other. Recall that the elements of  $\mathbf{D}_n$  are called *rotations* or *reflections* according to their action on the regular  $n$ -gon (see [Section 3.2](#)). This allows us to define similar notion for the elements of  $\text{Stab}(\bar{v})$ . More precisely, we say that an element  $\bar{g} \in \text{Stab}(\bar{v})$  is a *rotation* (respectively a *reflection*, *almost trivial*) at  $\bar{v}$  if its image under  $q_{\bar{v}}$  is a rotation (respectively a reflection, trivial). A rotation at  $\bar{v}$  is *strict* if it does not belong to  $\bar{F}$ . A *central half-turn at  $\bar{v}$*  is a strict rotation at  $\bar{v}$  which is an involution and centralizes  $\text{Stab}(\bar{v})$  (note that the existence of such a half-turn forces  $n$  to be even). Given a reflection  $\mathbf{x} \in \mathbf{L}_n$ , the pre-image under  $q_{\bar{v}}$  of  $\langle \mathbf{x} \rangle$  is called a *reflection group at  $\bar{v}$* .

*Remark.* Being a reflection at  $\bar{v}$  is a *local property*. Given two distinct apices  $\bar{v}, \bar{v}' \in \bar{\mathcal{V}}$ , an element  $\bar{g} \in \bar{G}$  can be simultaneously a reflection at  $\bar{v}$  and almost trivial at  $\bar{v}'$ . For instance, consider the hyperbolic group

$$G = \langle a, b, c \mid a^2, b^2, [b, c] \rangle = \mathbf{Z}_2 * (\mathbf{Z}_2 \times \mathbf{Z})$$

where the left factor  $\mathbf{Z}_2$  is generated by  $a$ , whereas the right factor  $\mathbf{Z}_2 \times \mathbf{Z}$  is generated by  $b$  and  $c$ . We consider the action of  $G$  on its Bass-Serre tree and blow up every vertex associated to (a conjugate of)  $\mathbf{Z}_2 \times \mathbf{Z}$  to a line (on which  $\mathbf{Z}_2$  acts trivially). The resulting space  $X$  is a tree on which  $G$  acts properly co-compactly by isometries. Fix now a large integer  $n$  and define

$$\bar{G} = G / \langle\langle (ab)^n, c^n \rangle\rangle$$

One checks easily that  $\bar{G}$  is a small cancellation quotient of  $G$ . Let  $\bar{v}, \bar{v}' \in \bar{X}$  be the apices of the cones attached to the relations  $(ab)^n$  and  $c^n$  respectively. One observes that the image  $\bar{b}$  of  $b$  in  $\bar{G}$  is a reflection at  $\bar{v}$  but almost trivial at  $\bar{v}'$ . This subtlety is a source of difficulty when studying the strong  $\nu$ -invariant  $\nu_{\text{stg}}(\bar{G}, \bar{X})$ .

From now on, we make the following assumption.

**Assumption 4.11** (Central half-turn). *For every apex  $\bar{v} \in \bar{\mathcal{V}}$ , if the image of the geometric realization map  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  has even torsion, then  $\text{Stab}(\bar{v})$  contains central half-turn at  $\bar{v}$ .*

*Remark.* Let us explain quickly how such an assumption can be satisfied. Later, when building the approximation sequence of  $\mathbf{B}_r(n)$ , we will see that every loxodromic subgroup of  $G$  can be assumed to embed in a product of the form  $\mathbf{D}_\infty \times \mathbf{D}_n \times \cdots \times \mathbf{D}_n$ . In particular if  $g \in \text{Stab}(Y)$  is a primitive element of  $E$ , then  $g^{n/2}$  is almost central: it commutes with every element in  $E^+$  and anti-commutes with the ones of  $E \setminus E^+$  (i.e.  $ug^{n/2}u^{-1} = g^{-n/2}$ , for every  $u \in E \setminus E^+$ ). Consequently, if  $H$  is the subgroup generated by  $h = g^n$ , then the image of  $g^{n/2}$  in  $\text{Stab}(Y)/H$  is a central half-turn.

**Geometric realization.** As suggested by the above terminology, the projection  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  captures how  $\text{Stab}(\bar{v})$  acts geometrically on the ball  $B(\bar{v}, \rho)$ . To make this idea more precise, we are going to build a quasi-isometry between  $B(\bar{v}, \rho)$  and a comparison hyperbolic cone  $\mathcal{D}$  (endowed with the obvious action of  $\mathbf{D}_n$ ) which is almost  $q_{\bar{v}}$ -equivariant.

We first define a morphism  $\mathbf{L} \rightarrow \text{Isom}(\mathbf{R})$ . Let  $\xi$  be one of the endpoints at infinity of  $Y$ . Let  $h_0$  be a primitive element of  $\text{Stab}(Y)$  whose attractive point is  $\xi$ .

- If  $\mathbf{L}$  is cyclic, then we map the positive generator  $\mathbf{t}$  of  $\mathbf{L}$  to the translation by  $\|h_0\|^\infty$ .
- If  $\mathbf{L}$  is the dihedral group  $\mathbf{D}_\infty = \langle \mathbf{x}, \mathbf{y} \mid \mathbf{x}^2, \mathbf{y}^2 \rangle$ , then we map  $\mathbf{x}$  to the symmetry at 0 and  $\mathbf{y}$  to the symmetry at  $\|h_0\|^\infty/2$ . In particular  $\mathbf{t} = \mathbf{xy}$  is mapped to the translation by  $\|h_0\|^\infty$ .

Note that the resulting morphism  $\mathbf{L} \rightarrow \text{Isom}(\mathbf{R})$  does not depend on the choice of  $h_0$ . The quotient  $\mathbf{R}/n\mathbf{Z}$  is a circle whose perimeter is  $\ell = n\|h_0\|^\infty$ . We denote by  $\mathcal{D}$  the cone of radius  $\rho$  over  $\mathbf{R}/n\mathbf{Z}$  and write  $o$  of its apex. The action of  $\mathbf{L}$  on  $\mathbf{R}$  induces an action by isometries of  $\mathbf{L}_n$  on  $\mathcal{D}$  which fixes  $o$ . Observe that

the space  $\mathcal{D}$  is a hyperbolic cone (i.e. with constant sectional curvature equal to  $-1$  everywhere except maybe at the apex) whose total angle at the apex  $o$  is

$$\Omega = \frac{n\|h_0\|^\infty}{\text{sh } \rho}.$$

It follows from the small cancellation assumption that  $\Omega > 10\pi$ . Said differently  $\mathcal{D}$  can be decomposed into  $n$  copies of a sector of the hyperbolic disc of radius  $\rho$  whose angle is  $\|h_0\|^\infty/\text{sh } \rho$ , so that  $\mathbf{D}_n$  is the group of isometries of  $\mathcal{D}$  preserving this decomposition (see Figure 4).

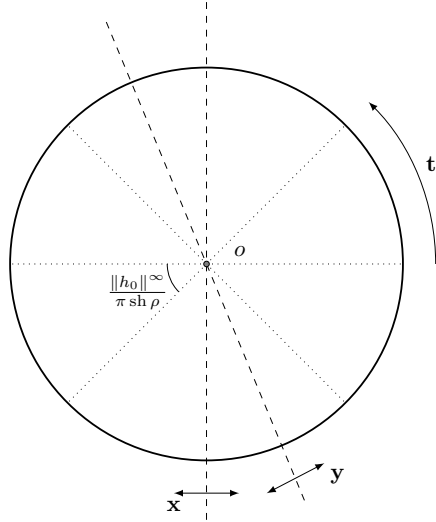


Figure 4: The comparison  $\mathcal{D}$  cone for  $n = 8$ .

Let us now compare the hyperbolic cone  $\mathcal{D}$  to the ball  $B(\bar{v}, \rho)$ . Let  $c_\xi$  be a Busemann cocycle at  $\xi$ . Recall that  $H \cap F$  is trivial, hence  $H$  is cyclic. This allows us to build an  $H$ -invariant cocycle  $c: X \times X \rightarrow \mathbf{R}$  which is at bounded distance from  $c_\xi$ . Indeed as  $H$  is amenable there exists an  $H$ -invariant mean  $M: \ell^\infty(H) \rightarrow \mathbf{R}$ . For every  $x, y \in X$ , we write  $f_{x,y}: H \rightarrow \mathbf{R}$  for the map sending  $h$  to  $hc_\xi(x, y)$  and define  $c(x, y)$  as the mean of  $f_{x,y}$ . One checks easily that  $c$  is an  $H$ -invariant cocycle. Recall that  $H$  fixes  $\xi$ , hence  $hc_\xi$  and  $c_\xi$  differ by at most  $6\delta$ , for every  $h \in H$ . Consequently  $c$  and  $c_\xi$  differs by at most  $6\delta$  as well. In particular, Lemma 2.11 yields  $|c(hx, x)| = \|h\|^\infty$ , for every  $x \in X$  and  $h \in H$ .

We now fix an arbitrary base point  $y_0 \in Y$ . If  $\mathbf{L}$  is the infinite dihedral group, we choose  $y_0$  in  $\text{Mov}(A, 11\delta)$  where  $A \subset \text{Stab}(Y)$  is the pre-image of  $\langle \mathbf{x} \rangle$  by  $q$ . Recall that  $Y$  is contained in the  $27\delta$ -neighborhood of any  $L_0\delta$ -local  $(1, \delta)$ -quasi-geodesic joining the endpoints of  $Y$ . It follows from Lemma 2.4 that the map  $\varphi: Y \rightarrow \mathbf{R}$  sending  $y$  to  $c(y_0, y)$  is an  $H$ -equivariant  $(1, 150\delta)$ -quasi-

isometric embedding. Moreover, this application is almost  $q$ -equivariant, in the sense that for every  $y \in Y$ , for every  $g \in \text{Stab}(Y)$ , we have

$$|\varphi(gy) - q(g)\varphi(y)| \leq 200\delta.$$

Consequently  $\varphi$  induces a map  $\bar{\varphi}: Y/H \rightarrow \mathbf{R}/n\mathbf{Z}$ , such that for every  $\bar{g} \in \text{Stab}(Y)/H$ , for every  $\bar{y} \in Y/H$ ,

$$|\bar{\varphi}(\bar{g}\bar{y}) - q_{\bar{v}}(\bar{g})\bar{\varphi}(\bar{y})| \leq 200\delta.$$

By [Lemma 4.4](#),  $\bar{\varphi}$  induces a  $(1, 150\delta)$ -quasi-isometric embedding  $Z(Y/H) \rightarrow \mathcal{D}$ , that we again still denote  $\bar{\varphi}$ , so that for every  $\bar{g} \in \text{Stab}(Y)/H$ , for every  $\bar{x} \in Z(Y/H)$ , we have

$$|\bar{\varphi}(\bar{g}\bar{x}) - q_{\bar{v}}(\bar{g})\bar{\varphi}(\bar{x})|_{\mathcal{D}} \leq 200\delta. \quad (18)$$

Roughly speaking, this means that  $\text{Stab}(Y)/H$  acts on  $Z(Y/H)$  as  $\mathbf{L}_n$  does on  $\mathcal{D}$ .

Note that  $Z(Y/H)$  – which is actually isometric to  $Z(Y)/H$  – is endowed here with the metric defined by (17). Although, as a set of points,  $Z(Y)/H$  can be identified with the closed ball of  $\bar{X}$  of radius  $\rho$  centered at  $\bar{v}$  ([Theorem 4.7](#)), the distance we considered so far is not the exactly the one coming from  $\bar{X}$ . Nevertheless the embedding  $Z(Y) \rightarrow \dot{X}$  is 1-Lipschitz. It follows that the map  $\bar{\varphi}: B(\bar{v}, \rho) \rightarrow \mathcal{D}$  induced by  $\bar{\varphi}: Z(Y/H) \rightarrow \mathcal{D}$  is such that for every  $\bar{x}, \bar{x}' \in B(\bar{v}, \rho)$  we have

$$|\bar{x} - \bar{x}'|_{\bar{X}} \leq |\bar{\varphi}(\bar{x}) - \bar{\varphi}(\bar{x}')|_{\mathcal{D}} + 150\delta \quad (19)$$

As we observed previously, the metrics of  $Z(Y)$  and  $\dot{X}$  coincide on  $B(v, \rho/3)$ . It follows that the metric on  $Z(Y/H)$  and  $\bar{X}$  coincide on  $B(\bar{v}, \rho/3)$ . Hence the map  $\bar{\varphi}: B(\bar{v}, \rho) \rightarrow \mathcal{D}$  is a  $(1, 150\delta)$ -quasi-isometric embedding when restricted to  $B(\bar{v}, \rho/3)$ .

**Proposition 4.12.** *Let  $\bar{v} \in \bar{\mathcal{V}}$ .*

- (i) *If  $\bar{g} \in \text{Stab}(\bar{v})$  is almost trivial at  $\bar{v}$ , then  $B(\bar{v}, \rho)$  is contained in  $\text{Mov}(\bar{g}, \bar{\delta})$ .*
- (ii) *If  $\bar{A}$  is a reflection group at  $\bar{v}$ , then there exists a point  $\bar{x} \in \text{Mov}(\bar{A}, \bar{\delta})$  with  $|\bar{v} - \bar{x}| = \rho$  such that for every  $\bar{z} \in \text{Mov}(\bar{A}, \bar{\delta}) \cap B(\bar{v}, \rho/3)$  we have*

$$\min \{ \langle \bar{x}, \bar{v} \rangle_{\bar{z}}, \langle \bar{g}\bar{x}, \bar{v} \rangle_{\bar{z}} \} \leq \bar{\delta},$$

*where  $\bar{g}$  is a central half-turn at  $\bar{v}$ .*

- (iii) *If  $\bar{g} \in \text{Stab}(\bar{v})$  is a strict rotation at  $\bar{v}$ , then there exists  $k \in \mathbf{Z}$ , such that  $\text{Mov}(\bar{g}^k, \bar{\delta})$  is contained in the  $\bar{\delta}$ -neighborhood of  $\bar{v}$ . In particular  $\bar{v}$  is the unique vertex fixed by  $\bar{g}$ .*

*Remark 4.13.* Roughly speaking Point (ii) is saying that any point of  $B(\bar{v}, \rho/3)$  that is fixed by  $A$  lies on the geodesic  $[\bar{x}, \bar{g}\bar{x}]$  – which goes through  $\bar{v}$  by Point (iii). Nevertheless, in our setting,  $\bar{X}$  does not need to be geodesic. Thus a rigorous statement is the one formulated above.

*Remark 4.14.* It follows from Point (iii) that if  $\bar{F}$  is an elliptic subgroup of  $\bar{G}$  containing a strict rotation at  $\bar{v}$ , then  $\text{Mov}(\bar{F}, 11\bar{\delta})$  is contained in  $B(\bar{v}, \bar{\delta})$ . In particular  $\bar{F}$  is a subgroup of  $\text{Stab}(\bar{v})$ .

*Proof.* We use the comparison map  $\bar{\varphi}: B(\bar{v}, \rho) \rightarrow \mathcal{D}$  defined during the previous discussion. Assume first that  $\bar{g}$  is almost trivial at  $\bar{v}$ , i.e.  $q_{\bar{v}}(\bar{g}) = 1$ . In other words  $q_{\bar{v}}(\bar{g})$  acts trivially on  $\mathcal{D}$ . Combining (18) and (19) we get  $|\bar{g}\bar{x} - \bar{x}|_{\bar{X}} \leq 350\bar{\delta}$ , for every  $\bar{x} \in B(\bar{v}, \rho)$ . Hence  $B(\bar{v}, \rho)$  is contained in  $\text{Mov}(\bar{g}, \bar{\delta})$ , which completes the proof of (i).

Assume now that  $\bar{g}$  is a strict rotation at  $\bar{v}$ . For simplicity, we let  $\mathbf{r} = q_{\bar{v}}(\bar{g})$ . Since  $\mathbf{r}$  is a non trivial rotation, one checks easily that there exists  $k \in \mathbf{Z}$  such that  $\mathbf{r}^k$  acts on  $\mathcal{D}$  as a rotation centered at  $o$  whose angle belongs to  $[\Omega/4, 3\Omega/4]$ . We noticed before that thanks to the small cancellation assumption  $\Omega > 10\pi$ . In particular for every  $\bar{x} \in \mathcal{D}$ , the angle at  $o$  between  $\bar{x}$  and  $\mathbf{r}^k\bar{x}$  is larger than  $\pi$ . Consequently

$$|\mathbf{r}^k\bar{x} - \bar{x}|_{\mathcal{D}} = 2|\bar{x} - o|_{\mathcal{D}}.$$

Recall that  $\varphi$  induces an almost  $q_{\bar{v}}$ -equivariant  $(1, 150\bar{\delta})$ -quasi-isometric embedding from  $B(\bar{v}, \rho/3)$  into  $\mathcal{D}$ . Hence for every  $\bar{x} \in B(\bar{v}, \rho/3)$ ,

$$|\bar{g}^k\bar{x} - \bar{x}|_{\bar{X}} \geq 2|\bar{x} - \bar{v}|_{\bar{X}} - 750\bar{\delta}.$$

In particular  $\text{Mov}(\bar{g}^k, \bar{\delta}) \cap B(\bar{v}, \rho/3)$  is contained in  $B(\bar{v}, \bar{\delta})$ . Since  $\text{Mov}(\bar{g}^k, \bar{\delta})$  is  $10\bar{\delta}$ -quasi-convex (Lemma 2.9) the set  $\text{Mov}(\bar{g}^k, \bar{\delta})$  is entirely contained in  $B(\bar{v}, \bar{\delta})$ , which completes the proof of (iii).

We are left to prove Point (ii). Let  $\bar{A}$  be a reflection group at  $\bar{v}$ . Without loss of generality we can assume that  $q_{\bar{v}}(A) = \langle \mathbf{x} \rangle$ . It follows from Assumption 4.11 that  $n$  is even and  $\text{Stab}(\bar{v})$  contains a central-half turn  $\bar{g}$ . We write  $\mathbf{r}$  for its image in  $\mathbf{L}_n$ . Recall that  $y_0$  is a base point in  $Y \cap \text{Mov}(A, 11\bar{\delta})$  chosen to define the map  $\bar{\varphi}$ . Let  $\bar{y}_0$  its image in  $\bar{X}$ . It follows from the construction that the set of fixed point of  $\mathbf{x}$  is exactly the geodesic of  $\mathcal{D}$  between  $\bar{\varphi}(\bar{y}_0)$  and  $\mathbf{r}\bar{\varphi}(\bar{y}_0)$ . Note that this geodesic passes through  $o$  as the angle  $\Omega$  at the apex of  $\mathcal{D}$  is larger than  $2\pi$ . Consequently for every  $d \geq 0$ , for every  $\bar{x} \in B(\bar{v}, \rho/3)$ , such that  $|\mathbf{x}\bar{\varphi}(\bar{x}) - \bar{\varphi}(\bar{x})| \leq d$ , we have either  $\langle o, \bar{\varphi}(\bar{y}_0) \rangle_{\bar{\varphi}(\bar{x})} \leq d/2$  or  $\langle o, \mathbf{r}\bar{\varphi}(\bar{y}_0) \rangle_{\bar{\varphi}(\bar{x})} \leq d/2$ . We carry again this observation in  $\bar{X}$  using the map  $\bar{\varphi}: B(\bar{v}, \rho) \rightarrow \mathcal{D}$  to get the conclusion of (ii).  $\square$

*Vocabulary.* In view of the previous statement, we can say that an element  $\bar{g} \in \bar{G}$  is a *strict rotation* if there is an apex  $\bar{v}$  such that  $\bar{g}$  is a strict rotation at  $\bar{v}$ . Indeed in such a case,  $\bar{g}$  cannot be almost trivial or a reflection at any other vertex. Note that being a strict rotation is invariant under conjugation.

## 4.4 Lifting properties

In Theorem 4.7 (iv) we mention a very important fact: small cancellation does not affect the small scale geometry of the space. More precisely the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  is an isometry when restricted on small ball lying sufficiently far

away from apices. This is a key ingredient to lift several figures from  $\bar{X}$  to  $\dot{X}$ . We complete this picture with other properties of the map  $\zeta: \dot{X} \rightarrow \bar{X}$ . Exceptionally, in this section all the distances are measured either in  $\dot{X}$  or  $\bar{X}$ .

The first step is to explain how one can lift isometrically in  $\dot{X}$  a quasi-convex subset  $\bar{Z} \subset \bar{X}$  as well as its (partial) stabilizer, provided it stays far away from the apex set  $\bar{\mathcal{V}}$ .

**Lemma 4.15.** *Let  $x, y \in \dot{X}$  such that  $\langle x, y \rangle_v > 3L_0\dot{\delta}$ , for every  $v \in \mathcal{V}$ . Then  $|x - y|_{\dot{X}} = |\bar{x} - \bar{y}|$ .*

*Remark.* Recall that the space  $\dot{X}$  is not necessarily geodesic. Nevertheless, our assumption means that any ‘‘geodesic’’ from  $x$  to  $y$  stays sufficiently far away from the apex  $v$ .

*Proof.* Since the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  is 1-Lipschitz,  $|\bar{x} - \bar{y}| \leq |x - y|$ . Let us prove the converse inequality. We first claim that  $\langle \bar{x}, \bar{y} \rangle_{\bar{v}} \geq (3L_0 - 9)\bar{\delta}$ , for every  $\bar{v} \in \bar{\mathcal{V}}$ . To that end we fix a  $(1, \dot{\delta})$ -quasi-geodesic  $\gamma_1: [a_1, b_1] \rightarrow \dot{X}$  joining  $x$  to  $y$ . According to our assumption, for every  $v \in \mathcal{V}$ ,  $d(v, \gamma_1) \geq \langle x, y \rangle_v > 3L_0\dot{\delta}$ . It follows from [Theorem 4.7 \(iv\)](#) that the image  $\bar{\gamma}_1: [a_1, b_1] \rightarrow \bar{X}$  of  $\gamma_1$  in  $\bar{X}$  is an  $L_0\bar{\delta}$ -local  $(1, \bar{\delta})$ -quasi-geodesic joining  $\bar{x}$  to  $\bar{y}$ . Let  $v \in \mathcal{V}$ . Applying the stability of quasi-geodesics to  $\bar{\gamma}_1$  ([Proposition 2.3](#)) we get

$$3L_0\bar{\delta} \leq \inf_{g \in K} d(gv, \gamma_1) \leq d(\bar{v}, \bar{\gamma}_1) \leq \langle \bar{x}, \bar{y} \rangle_{\bar{v}} + 9\bar{\delta},$$

which completes the proof of our first claim.

Let  $\eta \in (0, \dot{\delta})$ . There exists a pre-image  $y' \in \dot{X}$  of  $\bar{y}$  such that  $|x - y'| \leq |\bar{x} - \bar{y}| + 2\eta$ . In particular  $\langle x, y' \rangle_v \geq \langle \bar{x}, \bar{y} \rangle_{\bar{v}} - \eta$ , for every  $v \in \mathcal{V}$ . We claim that  $y$  and  $y'$  are very close. Let  $\gamma_2: [a_2, b_2] \rightarrow \dot{X}$  be a  $(1, \eta)$ -quasi-geodesic joining  $y$  to  $y'$ . Let  $v \in \mathcal{V}$ . Applying the four point inequality ([3](#)) in  $\dot{X}$ , we observe that

$$\langle y, y' \rangle_v \geq \min \{ \langle x, y \rangle_v, \langle x, y' \rangle_v \} - \dot{\delta} \geq \min \{ \langle x, y \rangle_v, \langle \bar{x}, \bar{y} \rangle_{\bar{v}} - \eta \} - \dot{\delta} \geq (3L_0 - 11)\dot{\delta}.$$

Reasoning as previously we see that the image  $\bar{\gamma}_2: [a_2, b_2] \rightarrow \bar{X}$  of  $\gamma_2$  in  $\bar{X}$  is an  $L_0\bar{\delta}$ -local  $(1, \eta)$ -quasi-geodesic from  $\bar{y}$  to  $\bar{y}'$ . Thus it is also a (global)  $(2, \eta)$ -quasi-geodesic ([Proposition 2.3](#)) joining  $\bar{y}$  to itself. Therefore  $|y - y'| \leq |b_2 - a_2| \leq 2\eta$ . It follows for the definition of  $y'$  that  $|x - y| \leq |\bar{x} - \bar{y}| + 4\eta$ . This holds for every sufficiently small  $\eta \in \mathbf{R}_+^*$ , hence  $|x - y| \leq |\bar{x} - \bar{y}|$ .  $\square$

**Lemma 4.16.** *Let  $Z$  be a subset of  $\dot{X}$  such that  $\langle z, z' \rangle_v > 4L_0\dot{\delta}$ , for every  $z, z' \in Z$  and every  $v \in \mathcal{V}$ . The map  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto its image  $\bar{Z}$ . In addition, the following holds.*

- (i) *Let  $\bar{g} \in G$  and  $z_1, z_2 \in Z$  such that  $\bar{g}\bar{z}_1 = \bar{z}_2$ . Then there exists unique pre-image  $g \in G$  of  $\bar{g}$  such that  $gz_1 = z_2$ . Moreover for every  $z, z' \in Z$ , if  $\bar{g}\bar{z} = \bar{z}'$ , then  $gz = z'$ .*
- (ii) *The projection  $\pi: G \rightarrow \bar{G}$  induces an isomorphism from  $\text{Stab}(Z)$  onto  $\text{Stab}(\bar{Z})$ .*



*Remark.* The statement apply in particular if  $Z$  is  $\alpha$ -quasi-convex and satisfies  $d(v, Z) > \alpha + 4L_0\delta$ , for every  $v \in \mathcal{V}$ . This slightly weaker version will be more flexible for later use, though.

*Proof.* By [Lemma 4.15](#), the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$ . Let  $\bar{g} \in G$  and  $z_1, z_2 \in Z$  such that  $\bar{g}z_1 = \bar{z}_2$ . By the very definition of  $\bar{X}$ , there exists a pre-image  $g \in G$  of  $\bar{g}$ , such that  $gz_1 = z_2$ . Uniqueness follows from the fact that  $K$  acts freely on  $\dot{X} \setminus \mathcal{V}$  – see [Theorem 4.7 \(v\)](#). We now prove that  $g$  satisfies the announced property. Let  $z, z' \in Z$  such that  $\bar{g}z = \bar{z}'$ . Let  $v \in \mathcal{V}$ . Applying the four point inequality [\(3\)](#) in  $\dot{X}$  we have

$$\langle gz, z' \rangle_v \geq \min \{ \langle gz, gz_1 \rangle_v, \langle z_2, z' \rangle_v \} - \delta \geq \min \{ \langle z, z_1 \rangle_{g^{-1}v}, \langle z_2, z' \rangle_v \} - \delta$$

Note that  $z_1, z_2, z$  and  $z'$  belongs to  $Z$ . Hence

$$\langle z, z_1 \rangle_{g^{-1}v} > 4L_0\delta \quad \text{and} \quad \langle z_2, z' \rangle_v > 4L_0\delta.$$

Consequently  $\langle gz, z' \rangle_v > 3L_0\delta$ , for every  $v \in \mathcal{V}$ . It follows from [Lemma 4.15](#) that  $|gz - z'| = |\bar{g}z - \bar{z}'| = 0$ . This completes the proof of [\(i\)](#). Point [\(ii\)](#) follows directly from [\(i\)](#).  $\square$

**Lemma 4.17.** *Let  $\bar{Z}$  be a subset of  $\bar{X}$  such that  $\langle \bar{z}, \bar{z}' \rangle_{\bar{v}} > 4L_0\bar{\delta}$ , for every  $\bar{z}, \bar{z}' \in \bar{Z}$  and every  $\bar{v} \in \bar{\mathcal{V}}$ . Let  $\bar{z}_0$  be a point of  $\bar{Z}$  and  $z_0 \in \dot{X}$  a pre-image of  $\bar{z}_0$ . Then there exists a unique subset  $Z$  of  $\dot{X}$  containing  $z_0$  such that the projection  $\zeta: \dot{X} \rightarrow \bar{X}$ , induces an isometry from  $Z$  onto  $\bar{Z}$ . Moreover,  $\langle z, z' \rangle_v > 4L_0\delta$  for every  $z, z' \in Z$  and every  $v \in \mathcal{V}$ .*

*Remark.* Note that [Lemma 4.16](#) applies to the lifted set  $Z$ . Hence, we can lift any isometry  $\bar{g} \in \bar{G}$  which (partially) preserves  $\bar{Z}$  to an isometry  $g \in G$  with the same properties. [Lemma 4.17](#) holds in particular if  $\bar{Z}$  is  $\alpha$ -quasi-convex and satisfies  $d(\bar{v}, \bar{Z}) > \alpha + 4L_0\bar{\delta}$ , for every  $\bar{v} \in \bar{\mathcal{V}}$ , in which case one can prove that  $Z$  is quasi-convex as well. Nevertheless we will not use this fact here.

*Proof.* We define  $Z$  as the set of points  $z \in \dot{X}$  being the pre-image of a point  $\bar{z} \in \bar{Z}$  and such that  $|z - z_0| = |\bar{z} - \bar{z}_0|$ . Let  $\bar{z} \in \bar{Z}$ . We claim that  $\bar{z} \in \bar{Z}$  has a pre-image in  $Z$ . There exists a pre-image  $z \in \dot{X}$  of  $\bar{z}$  such that  $|z - z_0| \leq |\bar{z} - \bar{z}_0| + \bar{\delta}$ . In particular  $\langle z, z_0 \rangle_v \geq \langle \bar{z}, \bar{z}_0 \rangle_{\bar{v}} - \bar{\delta} > 3L_0\delta$ , for every  $v \in \mathcal{V}$ . It follows from [Lemma 4.15](#) that  $|z - z_0| = |\bar{z} - \bar{z}_0|$ . Hence  $z$  belongs to  $Z$ , which completes the proof of our claim. In other words the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  maps  $Z$  onto  $\bar{Z}$ .

We now prove that  $\langle z, z' \rangle_v > 3L_0\delta$ , for every  $z, z' \in Z$  and every  $v \in \mathcal{V}$ . It follows from the very definition of  $Z$  that  $\langle z, z_0 \rangle_v \geq \langle \bar{z}, \bar{z}_0 \rangle_{\bar{v}}$  and  $\langle z', z_0 \rangle_v \geq \langle \bar{z}', \bar{z}_0 \rangle_{\bar{v}}$ . Combining the four point inequality [\(3\)](#) with our assumption on  $\bar{Z}$ , we get

$$\langle z, z' \rangle_v \geq \min \{ \langle z, z_0 \rangle_v, \langle z', z_0 \rangle_v \} - \delta \geq \min \{ \langle \bar{z}, \bar{z}_0 \rangle_{\bar{v}}, \langle \bar{z}', \bar{z}_0 \rangle_{\bar{v}} \} - \bar{\delta} > 3L_0\delta.$$

It now follows from [Lemma 4.15](#) that the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto its image, i.e.  $\bar{Z}$ . This proves the existence of the set  $Z$ .

The uniqueness directly follows from the definition of  $Z$ . Since  $Z \rightarrow \bar{Z}$  is an isometry,  $\langle z, z' \rangle_v \geq \langle \bar{z}, \bar{z}' \rangle_{\bar{v}} > 4L_0\bar{\delta}$ , for every  $z, z' \in Z$ , for every  $v \in \mathcal{V}$ .  $\square$

The previous statements explain how to lift in  $\dot{X}$  a quasi-convex subset  $\bar{Z} \subset \bar{X}$ , as well as its (partial) stabilizer, as soon as it stays away from the apex set  $\bar{\mathcal{V}}$ . In particular it applies to any  $(1, \eta)$ -quasi-geodesic path of  $\bar{X}$  that avoid the cone points. We focus now on a more delicate operation which consists in lifting paths of  $\bar{X}$  (and their almost stabilizers) going through one or several apices. The next statement follows [10, Proposition 5.13].

**Proposition 4.18.** *Let  $x$  and  $y$  be two points of  $X$ . Let  $\gamma: [a, b] \rightarrow \dot{X}$  be a path from  $x$  to  $y$  whose image  $\bar{\gamma}: [a, b] \rightarrow \bar{X}$  is a  $(1, \bar{\delta})$ -quasi-geodesic. Let  $S$  be a subset of  $G$  and  $\bar{S}$  its image in  $\bar{G}$ . We assume that  $|gx - x|_{\dot{X}} \leq \rho/100$  and  $|\bar{g}\bar{y} - \bar{y}| \leq \rho/100$ , for every  $g \in S$ . In addition we suppose that for every apex  $\bar{v} \in \bar{\mathcal{V}}$  satisfying  $\langle \bar{x}, \bar{y} \rangle_{\bar{v}} \leq \rho/4$ , the set  $\bar{S}$  lies in the local kernel at  $\bar{v}$ . Then  $|gy - y|_{\dot{X}} = |\bar{g}\bar{y} - \bar{y}|$  for every  $g \in S$ .*

*Proof.* Since  $\bar{\gamma}$  is a  $(1, \bar{\delta})$ -quasi-geodesic and the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  is 1-Lipschitz, the path  $\dot{\gamma}$  is a  $(1, \bar{\delta})$ -quasi-geodesic. Let  $v_1, \dots, v_m$  be the apices of  $\mathcal{V}$  which are  $\rho/5$ -close to  $\gamma$ . For every  $j \in \llbracket 1, m \rrbracket$ , we denote by  $\gamma(c_j)$  a projection of  $v_j$  on  $\gamma$ . By reordering the apices we can always assume that  $c_1 \leq c_2 \leq \dots \leq c_m$ . For simplicity of notation we put  $c_0 = a$  and  $c_{m+1} = b$ . Let  $j \in \llbracket 1, m \rrbracket$ . Since  $\gamma$  is a  $(1, \bar{\delta})$ -quasi-geodesic, we can find  $b_{j-1} \in (c_{j-1}, c_j]$  and  $a_j \in [c_j, c_{j+1})$  with the following properties.

- (i)  $|v_j - \gamma(b_{j-1})|_{\dot{X}} = 9\rho/10$  and  $|v_j - \gamma(a_j)|_{\dot{X}} = 9\rho/10$ ,
- (ii)  $\gamma \cap B(v_j, 4\rho/5)$  is contained in  $\gamma((b_{j-1}, a_j))$

In addition, we let  $a_0 = a$ ,  $b_m = a_{m+1} = b$  (see Figure 5). We claim that for

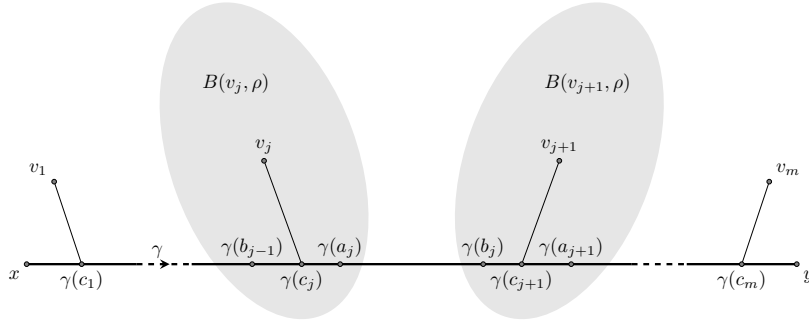


Figure 5: The cones intersecting  $\gamma$ .

every  $j \in \llbracket 0, m+1 \rrbracket$ , for every  $g \in S$ , we have

$$|\bar{g}\bar{\gamma}(a_j) - \bar{\gamma}(a_j)| = |g\gamma(a_j) - \gamma(a_j)|.$$

The proof is by induction on  $j$ . If  $j = 0$  then  $\gamma(a_j) = x$ . The claim follows from the fact that the map  $\zeta : \dot{X} \rightarrow \bar{X}$  induces an isometry from  $B(x, \rho/20)$  onto  $B(\bar{x}, \rho/20)$  – see [Theorem 4.7 \(iv\)](#). Assume now that our claim is true for some  $j \in \llbracket 0, m \rrbracket$ . Since  $\gamma$  is a local quasi-geodesic,  $a_j \leq b_j$ . We denote by  $\gamma_j$  the restriction of  $\gamma$  to  $[a_j, b_j]$  and by  $\bar{\gamma}_j$  its image in  $\bar{X}$ . In addition we write  $Z_j$  (respectively  $\bar{Z}_j$ ) for the  $\rho/50$ -neighborhood of  $\gamma_j$  (respectively  $\bar{\gamma}_j$ ). It follows from the construction that  $\gamma_j$  and thus  $\bar{\gamma}_j$  stay away from any cone point. Consequently  $\zeta : \dot{X} \rightarrow \bar{X}$  is an isometry when restricted to the  $Z_j$  ([Lemma 4.16](#)). Moreover, it induces an isometry from  $Z_j$  onto  $\bar{Z}_j$  ([Lemma 4.17](#)).

Let  $g \in S$ . By assumption its image  $\bar{g}$  in  $\bar{G}$  moves  $\bar{x}$  and  $\bar{y}$  by at most  $\rho/100$ . Recall that we required  $\bar{\gamma}$  to be  $(1, \bar{\delta})$ -quasi-geodesic. It follows from [Lemma 2.8](#) that  $\bar{g}$  moves  $\bar{\gamma}(a_j)$  and  $\bar{\gamma}(b_j)$  by at most  $\rho/50$ . In particular  $\bar{g}\bar{\gamma}(a_j)$  and  $\bar{g}\bar{\gamma}(b_j)$  belongs to  $\bar{Z}_j$ . Nevertheless, according to our induction assumption,  $g\gamma(a_j)$  is the (unique) lift of  $\bar{g}\bar{\gamma}(a_j)$  that belongs to  $Z_j$ . Applying [Lemma 4.16 \(i\)](#) we observe that  $g\gamma(b_j)$  is the unique pre-image in  $Z_j$  of  $\bar{g}\bar{\gamma}(b_j)$ . In particular

$$|g\gamma(b_j) - \gamma(b_j)|_{\dot{X}} = |\bar{g}\bar{\gamma}(b_j) - \bar{\gamma}(b_j)| \leq \rho/50.$$

If  $j = m$ , then  $a_{m+1} = b_m$ , thus the claim holds for  $j + 1$ . Otherwise,  $|v_{j+1} - \gamma(b_j)| = 9\rho/10$ , thus  $g$  necessarily belongs to  $\text{Stab}(v_{j+1})$ . Moreover

$$\langle \bar{x}, \bar{y} \rangle_{\bar{v}_{j+1}} \leq d(\bar{v}_{j+1}, \bar{\gamma}) + 4\bar{\delta} \leq d(v_{j+1}, \gamma) + 4\delta \leq \rho/4,$$

see for instance by [\[9, Lemma 2.4\]](#). Hence by assumption,  $\bar{g}$  is trivial at  $\bar{v}_{j+1}$ . Therefore it moves all the points of  $B(v_{j+1}, \rho)$  by a distance at most  $\bar{\delta}$  ([Proposition 4.12](#)). In particular,  $|g\gamma(a_{j+1}) - \gamma(a_{j+1})|_{\dot{X}} \leq \bar{\delta}$ . However, the map  $\zeta : \dot{X} \rightarrow \bar{X}$  induces an isometry from the ball  $B(\gamma(a_{j+1}), \rho/20)$  onto its image, hence  $|g\gamma(a_{j+1}) - \gamma(a_{j+1})|_{\dot{X}} = |\bar{g}\bar{\gamma}(a_{j+1}) - \bar{\gamma}(a_{j+1})|$ . This proves our claim for  $j + 1$ . The statement of the lemma follows from our claim for  $j = m + 1$ .  $\square$

In the previous statement, we assumed that any isometry  $\bar{g} \in \bar{S}$  which hardly move the endpoints of  $\bar{\gamma}$ , is actually in the local kernel of every vertex  $\bar{v}$  lying close to  $\bar{\gamma}$ . We now explore the situation where some element  $\bar{g}$  might be a reflection at  $\bar{v}$ .

**Proposition 4.19.** *Let  $x$  and  $y$  be two points of  $X$ . Let  $S$  be a subset of  $G$  and  $\bar{S}$  its image in  $\bar{G}$ . We assume that  $|gx - x| \leq \rho/100$  and  $|\bar{g}\bar{y} - \bar{y}| \leq \rho/100$ , for every  $g \in S$ . In addition we suppose that for every apex  $\bar{v} \in \bar{\mathcal{V}}$ , the set  $\bar{S} \cap \text{Stab}(\bar{v})$  is contained in a reflection group at  $\bar{v}$ . Then there exists an element  $u \in G$  with the following properties.*

- (i)  $\bar{u}$  commutes with every element in  $\bar{S}$ ;
- (ii)  $|guy - uy|_{\dot{X}} = |\bar{g}\bar{y} - \bar{y}|$  for every  $g \in S$ ;
- (iii) either  $\bar{u}$  is trivial, or  $\bar{S}$  lies in a reflection group at some apex  $\bar{v} \in \bar{\mathcal{V}}$ .

*Remark 4.20.* Note that if  $\bar{S} \cap \text{Stab}(\bar{v})$  is not contained in a reflection group at  $\bar{v}$ , then there is a strict rotation at  $\bar{v}$  which is the product of at most two elements of  $\bar{S}$ . Indeed given any reflection  $\mathbf{x}$  of  $\mathbf{D}_n$ , the geometric realization  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  cannot map  $\bar{S} \cap \text{Stab}(\bar{v})$  into  $\langle \mathbf{x} \rangle$ . Consequently the image of  $\bar{S} \cap \text{Stab}(\bar{v})$  either contains a non-trivial rotation or two distinct reflections, whence the claim. This observation will be useful later to check that the assumptions of the proposition are fulfilled.

*Proof.* Assume first that for every apex  $\bar{v} \in \bar{\mathcal{V}}$  satisfying  $\langle \bar{x}, \bar{y} \rangle_{\bar{v}} \leq \rho/4$ , the set  $\bar{S}$  lies in the local kernel at  $\bar{v}$ . We fix  $\varepsilon > 0$  and  $u \in K$  such that  $|x - uy|_{\dot{X}} \leq |\bar{x} - \bar{y}| + \varepsilon$ . In addition we take a  $(1, \varepsilon)$ -quasi-geodesic  $\gamma: [a, b] \rightarrow \dot{X}$  from  $x$  to  $uy$ . It follows from our choice of  $u$  that the image  $\bar{\gamma}: [a, b] \rightarrow \dot{X}$  of  $\gamma$  is a  $(1, 2\varepsilon)$ -quasi-geodesic from  $\bar{x}$  to  $\bar{y}$ . Hence if  $\varepsilon$  is sufficiently small, [Proposition 4.18](#) applies, which completes the proof.

Assume now that there exists a vertex  $\bar{v} \in \bar{\mathcal{V}}$  satisfying  $\langle \bar{x}, \bar{y} \rangle_{\bar{v}} \leq \rho/4$  such that the set  $\bar{S}$  does not lie in the local kernel at  $\bar{v}$ . Any element  $\bar{g} \in \bar{S}$  moves  $\bar{x}$  and  $\bar{y}$  by at most  $\rho/100$ . Hence  $\bar{g}$  moves  $\bar{v}$  by at most  $\rho$  ([Lemma 2.8](#)). It follows that  $\bar{S}$  is contained in  $\text{Stab}(\bar{v})$ . According to our assumption  $\bar{S}$  is contained in a reflection group at  $\bar{v}$ . We now denote by  $\mathcal{U}$  the set of all elements  $u \in G$  whose image  $\bar{u}$  in  $\bar{G}$  commutes with  $\bar{S}$ . This set is non-empty as it contains the identity. We chose  $u_0 \in \mathcal{U}$  such that  $|\bar{u}_0 \bar{y} - \bar{x}| \leq |\bar{u} \bar{y} - \bar{x}| + \bar{\delta}$ , for every  $u \in \mathcal{U}$ .

We are going to prove that for every  $\bar{v} \in \bar{\mathcal{V}}$ , if  $\langle \bar{x}, \bar{u}_0 \bar{y} \rangle_{\bar{v}} \leq \rho/4$ , then  $\bar{S}$  lies in the local kernel at  $\bar{v}$ . Consider indeed an apex  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\langle \bar{x}, \bar{u}_0 \bar{y} \rangle_{\bar{v}} \leq \rho/4$ . As  $\bar{u}_0$  commutes with  $\bar{S}$ , the distance  $|\bar{g} \bar{u}_0 \bar{y} - \bar{u}_0 \bar{y}| = |\bar{g} \bar{y} - \bar{y}|$  is bounded above by  $\rho/100$  for every  $\bar{g} \in \bar{S}$ . We prove as above that  $\bar{S}$  is contained in a reflection group at  $\bar{v}$ . Suppose now that contrary to our claim  $\bar{S}$  is not in the local kernel at  $\bar{v}$ . We fix  $\bar{s} \in \bar{S}$  a reflection at  $\bar{v}$ . According to [Proposition 4.12 \(ii\)](#), there exists a point  $\bar{z}_0 \in \bar{X}$  with  $|\bar{v} - \bar{z}_0| = \rho$  such that for every  $\bar{z} \in \text{Mov}(\bar{S}, \bar{\delta}) \cap B(\bar{v}, \rho/3)$  we have

$$\min \{ \langle \bar{z}_0, \bar{v} \rangle_{\bar{z}}, \langle \bar{h} \bar{z}_0, \bar{v} \rangle_{\bar{z}} \} \leq \bar{\delta},$$

where  $\bar{h}$  is a central half-turn at  $\bar{v}$ . Observe that  $\bar{u}_1 = \bar{h} \bar{u}_0$  belongs to  $\mathcal{U}$ . Indeed, as  $\bar{h}$  centralizes  $\text{Stab}(\bar{v})$ , it commutes with  $\bar{S}$ . We now claim that  $|\bar{u}_1 \bar{y} - \bar{x}| < |\bar{u}_0 \bar{y} - \bar{x}| - 2\rho/15$ . Let  $\bar{\gamma}: [a, b] \rightarrow \bar{X}$  be a  $(1, \bar{\delta})$ -quasi-geodesic from  $\bar{x}$  to  $\bar{u}_0 \bar{y}$ . Let  $\bar{\gamma}(t)$  be the projection of  $\bar{v}$  onto  $\bar{\gamma}$ . By assumption  $\langle \bar{x}, \bar{u}_0 \bar{y} \rangle_{\bar{v}} \leq \rho/4$ , hence  $|\bar{\gamma}(t) - \bar{v}| \leq \rho/4 + 4\bar{\delta}$ , see for instance [\[9, Lemma 2.4\]](#). Let

$$\begin{aligned} s_- &= \sup \{ s \in [a, t] : |\bar{\gamma}(s) - \bar{v}| \geq \rho/3 \}, \\ s_+ &= \inf \{ s \in [t, b] : |\bar{\gamma}(s) - \bar{v}| \geq \rho/3 \}, \end{aligned}$$

so that  $\bar{y}_- = \bar{\gamma}(s_-)$  and  $\bar{y}_+ = \bar{\gamma}(s_+)$  are at distance exactly  $\rho/3$  from  $\bar{v}$  and

$$|\bar{y}_- - \bar{y}_+| \geq \rho/6 - 8\bar{\delta}.$$

Since any element of  $\bar{S}$  moves the endpoint of  $\bar{\gamma}$  by at most  $\rho/100$ , the path  $\bar{\gamma}$  restricted to  $[s_-, s_+]$  is contained in  $\text{Mov}(\bar{S}, \rho/50)$  ([Lemma 2.8](#)) hence in

the  $(\rho/100 + 7\bar{\delta})$ -neighborhood of  $\text{Mov}(\bar{S}, \bar{\delta})$  (Lemma 2.9). Hence for every  $s \in (s_-, s_+)$  we have

$$\min \left\{ \langle \bar{z}_0, \bar{v} \rangle_{\bar{\gamma}(s)}, \langle \bar{h}\bar{z}_0, \bar{v} \rangle_{\bar{\gamma}(s)} \right\} \leq \rho/100 + 8\bar{\delta}.$$

See Figure 6. By continuity it also applies to  $s_-$  and  $s_+$ . Recall that  $|\bar{y}_- - \bar{y}_+| \geq$

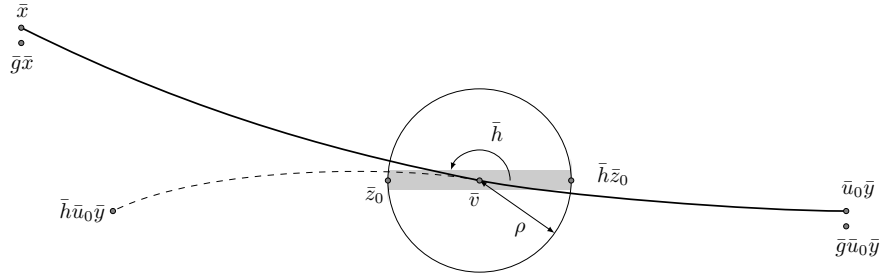


Figure 6: The path  $\bar{\gamma}$  going through the ball  $B(\bar{v}, \rho)$ . The grey area corresponds to  $\text{Mov}(\bar{S}, \bar{\delta}) \cap B(\bar{v}, \rho)$ .

$\rho/10$ . Up to permuting  $\bar{z}_0$  and  $\bar{h}\bar{z}_0$ , it forces

$$\langle \bar{z}_0, \bar{v} \rangle_{\bar{y}_-} \leq \rho/100 + 8\bar{\delta}, \quad \text{and} \quad \langle \bar{z}_0, \bar{v} \rangle_{\bar{h}\bar{y}_+} = \langle \bar{h}\bar{z}_0, \bar{v} \rangle_{\bar{y}_+} \leq \rho/100 + 8\bar{\delta}.$$

Hence  $|\bar{y}_- - \bar{h}\bar{y}_+| \leq \rho/50 + 18\bar{\delta}$  [9, Lemma 2.2 (ii)]. On the other hand, since  $\bar{y}_-$  and  $\bar{y}_+$  lie on a  $(1, \bar{\delta})$ -quasi-geodesic between  $\bar{x}$  and  $\bar{u}_0\bar{y}$  we have

$$\begin{aligned} |\bar{x} - \bar{u}_0\bar{y}| &\geq |\bar{x} - \bar{y}_-| + |\bar{y}_- - \bar{y}_+| + |\bar{y}_+ - \bar{u}_0\bar{y}| - \bar{\delta} \\ &\geq |\bar{x} - \bar{y}_-| + |\bar{y}_+ - \bar{u}_0\bar{y}| + \rho/6 - 7\bar{\delta} \end{aligned}$$

Combined with the triangle inequality, it yields

$$|\bar{x} - \bar{u}_1\bar{y}| \leq |\bar{x} - \bar{y}_-| + |\bar{y}_- - \bar{h}\bar{y}_+| + |\bar{y}_+ - \bar{u}_0\bar{y}| \leq |\bar{x} - \bar{u}_0\bar{y}| - 2\rho/15.$$

which completes the proof our claim and contradicts the minimality of  $u_0$ . Consequently for every  $\bar{v} \in \bar{\mathcal{V}}$ , if  $\langle \bar{x}, \bar{u}_0\bar{y} \rangle_{\bar{v}} \leq \rho/4$ , then  $\bar{S}$  lies the local kernel at  $\bar{v}$ . The conclusion now follows from Proposition 4.18.  $\square$

## 4.5 The action of $\bar{G}$ on $\bar{X}$

We now study the general properties of the action of  $\bar{G}$  on  $\bar{X}$ .

**Proposition 4.21.** *The action of  $\bar{G}$  on  $\bar{X}$  is gentle.*

*Remark.* Recall that the action of  $\bar{G}$  on  $\bar{X}$  is gentle if every loxodromic subgroup  $\bar{E}$  preserving the orientation splits as  $\bar{E} = \bar{F} \rtimes \mathbf{Z}$ , where  $\bar{F}$  is the set of all elliptic elements of  $\bar{E}$ .

*Proof.* Let  $\bar{E}$  be a loxodromic subgroup of  $\bar{G}$ , preserving the orientation. We write  $\bar{Z}$  for the cylinder of  $\bar{E}$  (see Section 2.3 for the definition). Assume first that there exists an apex  $\bar{v} \in \bar{\mathcal{V}}$  such that  $d(\bar{v}, \bar{Z}) \leq 5L_0\bar{\delta}$ . Let  $\bar{F}$  be the set of all elliptic elements of  $\bar{E}$ . Since  $\bar{E}$  preserves the orientation,  $\bar{Z}$  is contained in  $\text{Mov}(\bar{F}, 88\bar{\delta})$  (Lemma 2.15). It follows that  $|\bar{u}\bar{v} - \bar{v}| < 2\rho$ , for every  $\bar{u} \in \bar{F}$ . Consequently  $\bar{F}$  is coincide with  $\bar{E} \cap \text{Stab}(\bar{v})$  and is therefore a (normal) subgroup of  $\bar{E}$ . We now prove that  $\bar{E}/\bar{F}$  is cyclic. To that end, we fix a loxodromic element  $\bar{g}_0 \in \bar{E}$  such that  $\|\bar{g}_0\| \leq \|\bar{g}\| + \bar{\delta}$ , for every loxodromic element  $\bar{g} \in \bar{E}$ . Since  $\bar{g}_0$  is a loxodromic element of  $\bar{E}$ , it sends  $\bar{v}$  to a distinct apex. It follows that  $\|\bar{g}_0\| \geq \rho$ . Let  $\bar{g} \in \bar{E}$ . The element  $\bar{g}_0$  acts on  $\bar{Z}$  by translation of length approximately  $\|\bar{g}_0\|$  (Lemma 2.10). Hence there exists  $k \in \mathbf{Z}$  such that  $\|\bar{g}_0^k \bar{g}\| \leq \|\bar{g}_0\|/2 + \rho/10 < \|\bar{g}_0\| - \rho/10$ . It follows from our choice of  $\bar{g}_0$ , that  $\bar{g}_0^k \bar{g}$  is elliptic and thus belongs to  $\bar{F}$ . Hence  $\bar{E}/\bar{F}$  is a cyclic group generated by the image of  $\bar{g}_0$ .

Assume now that  $d(\bar{v}, \bar{Z}) > 5L_0\bar{\delta}$  for every apex  $\bar{v} \in \bar{\mathcal{V}}$ . Since  $\bar{Z}$  is  $2\bar{\delta}$ -quasi-convex, there exists a subset  $Z$  of  $\dot{X}$  such that the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$  (Lemma 4.17). It follows then from Lemma 4.16 that there exists a subgroup  $E$  of  $G$  such that  $\pi: G \rightarrow \bar{G}$  induces an isomorphism from  $E$  onto  $\bar{E}$ . By construction  $\langle z, z' \rangle_v > 4L_0\bar{\delta}$ , for every  $z, z' \in Z$ , for every apex  $v \in \mathcal{V}$  (the Gromov product is computed in  $\dot{X}$  here). Consequently the radial projection  $p: \dot{X} \setminus \mathcal{V} \rightarrow X$  induces a  $E$ -equivariant quasi-isometry from  $Z$  onto  $p(Z)$  (Corollary 4.9). This produces a  $\pi_E$ -equivariant quasi-isometry from  $p(Z)$  to  $\bar{Z}$  where  $\pi_E$  stands for the map  $\pi$  restricted to  $E$ . Since  $\bar{E}$  is loxodromic and preserves the orientation, the same holds for  $E$ . Moreover if  $F$  (respectively  $\bar{F}$ ) stands for the set of elliptic elements of  $E$  (respectively  $\bar{E}$ ), then  $\pi$  sends  $F$  onto  $\bar{F}$ . As the action of  $G$  is gentle,  $E$  splits as  $E = F \rtimes \mathbf{Z}$ . Hence  $\bar{E}$  splits as well as  $E = \bar{F} \rtimes \mathbf{Z}$ .  $\square$

**Lemma 4.22.** *The set  $\bar{\mathcal{V}}$  contains at least two apices.*

*Proof.* In this proof all the distances are measured in  $\dot{X}$  or  $\bar{X}$ . Let  $v_1$  be an apex of  $\mathcal{V}$  and  $\bar{v}_1$  its image in  $\bar{X}$ . We denote by  $v_2$  another apex such that  $|v_1 - v_2| \leq |v_1 - v| + \bar{\delta}$ , for every  $v \in \mathcal{V} \setminus \{v_1\}$ . We are going to prove that the image  $\bar{v}_2$  of  $v_2$  in  $\bar{X}$  is distinct from  $\bar{v}_1$ . Let  $\gamma: [a_1, a_2] \rightarrow \dot{X}$  be a  $(1, \bar{\delta})$ -quasi-geodesic from  $v_1$  to  $v_2$ . Let  $b_1 = a_1 + \rho/4$  and  $b_2 = a_2 - \rho/4$ . For simplicity we write  $x_1 = \gamma(b_1)$  and  $x_2 = \gamma(b_2)$ . Note that  $|x_1 - x_2| \geq 3\rho/2$ . Moreover,  $\langle x_1, x_2 \rangle_v > 3L_0\bar{\delta}$ , for every  $v \in \mathcal{V}$ . Indeed otherwise  $v$  would be a cone distinct from  $v_1$  but much closer to  $v_1$  than  $v_2$ . According to Lemma 4.15, we have  $|\bar{x}_1 - \bar{x}_2| = |x_1 - x_2|$ , hence  $|\bar{x}_1 - \bar{x}_2| \geq 3\rho/2$ . Combined with the triangle inequality we obtain  $|\bar{v}_1 - \bar{v}_2| \geq \rho$ .  $\square$

**Proposition 4.23.** *Assume that for every  $\bar{v} \in \bar{\mathcal{V}}$ , the order of  $\bar{v}$  is at least 3. Then the action of  $\bar{G}$  on  $\bar{X}$  is non-elementary.*

*Proof.* According to Lemma 4.22 that  $\bar{\mathcal{V}}$  contains two distinct apices say  $\bar{v}_1$  and  $\bar{v}_2$ . We fix a point  $\bar{x} \in \bar{X}$  such that  $\langle \bar{v}_1, \bar{v}_2 \rangle_{\bar{x}} \leq \bar{\delta}$  whereas  $|\bar{x} - \bar{v}_1| \geq \rho$  and  $|\bar{x} - \bar{v}_2| \geq \rho$ . Reasoning as in Proposition 4.12 (iii) we see that  $\text{Stab}(\bar{v}_i)$

contains a rotation  $\bar{g}_i$  at  $\bar{v}_i$  such that  $\text{Mov}(\bar{S}_i, \bar{\delta})$  is contained in  $B(\bar{v}_i, \bar{\delta})$ , where  $\bar{S}_i = \{\bar{g}_i, \bar{g}_i^2\}$ , see Figure 7. Applying Lemma 2.9, we get

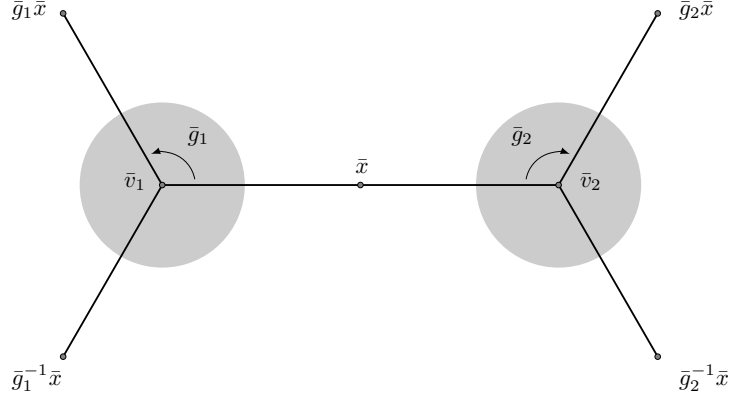


Figure 7: The element  $\bar{g}_1$  and  $\bar{g}_2$  acting on  $\bar{x}$ . The shaded discs represent  $B(\bar{v}_1, \rho)$  and  $B(\bar{v}_2, \rho)$  respectively.

$$2 \langle \bar{g}_i \bar{x}, \bar{g}_i^{-1} \bar{x} \rangle_{\bar{x}} \leq |\bar{g}_i \bar{x} - \bar{x}| + 30\bar{\delta} \quad \text{while} \quad |\bar{g}_i \bar{x} - \bar{x}| \geq 2|\bar{x} - \bar{v}_i| - 15\bar{\delta} \geq 2\rho - 15\bar{\delta}$$

Combined with the four point inequality (3) it yields  $\langle \bar{g}_1^{\pm 1} \bar{x}, \bar{g}_2^{\pm 1} \bar{x} \rangle_{\bar{x}} \leq 3\bar{\delta}$ . It follows from Lemma 2.17 that  $\bar{g}_1$  and  $\bar{g}_2$  generate a non-elementary subgroup. Hence  $\bar{G}$  is non-elementary.  $\square$

## 4.6 Structure of elementary subgroups

An important step to study further the action of  $\bar{G}$  on  $\bar{X}$  (and its invariants) is to understand the algebraic structure of its elementary subgroups. As the map  $X \rightarrow \bar{X}$  is 1-Lipschitz, the projection  $\pi: G \rightarrow \bar{G}$  maps every elementary subgroup of  $G$  to an elementary subgroup of  $\bar{G}$ . More precisely it sends every elliptic (respectively parabolic, loxodromic) to an elliptic (respectively elliptic or parabolic, elementary) subgroup of  $\bar{G}$ . However the nature of these subgroups (i.e. whether they are elliptic, parabolic, or loxodromic) may change. Indeed given any element  $g \in G$ , there always exists  $h \in K$  such that  $gh$  is a loxodromic element of  $G$ . Thus  $\langle gh \rangle$  is loxodromic whereas its image  $\langle \bar{g} \rangle$  in  $\bar{G}$  could be anything. New elementary subgroups may also appear in  $\bar{G}$ . This motivates the following definition.

**Definition 4.24.** Let  $\bar{E}$  be an elliptic (respectively parabolic, loxodromic) subgroup of  $\bar{G}$ . We say that  $\bar{E}$  can be lifted if there exists an elliptic (respectively parabolic, loxodromic) subgroup  $E$  of  $G$  (for its action on  $X$ ) such that the quotient map  $\pi: G \rightarrow \bar{G}$  induces an isomorphism from  $E$  onto  $\bar{E}$ . In this situation  $E$  is a lift of  $\bar{E}$ .

Note that in this definition we ask  $E$  and  $\bar{E}$  to have the same nature. The idea is that the subgroups of  $\bar{G}$  that can be lifted are as “easy” as the elementary subgroups of  $G$ . Complicated algebraic structures necessarily come for the “new” elementary subgroups. In the next paragraphs we discuss whether an elementary subgroup of  $\bar{G}$  can be lifted. If not we use the geometry of small cancellation theory to describe their properties.

**Comparing lifts.** We start by proving that the lift of an elliptic subgroup of  $\bar{G}$  is essentially unique. More precisely, if  $F_1$  and  $F_2$  are two lifts of the same elliptic subgroup  $\bar{F} \subset \bar{G}$ , then  $F_1$  and  $F_2$  are conjugated. This result is a particular case of a more general statement (see [Corollary 4.27](#)) that allows to consider both elliptic and parabolic subgroups.

**Lemma 4.25.** *Let  $S$  be a subset of  $G$  such that  $\text{Mov}(S, \rho/10)$  is non-empty. Then the projection  $\pi: G \rightarrow \bar{G}$  is one-to-one when restricted to  $S$ . In particular if  $E$  is an elliptic or a parabolic subgroup, the projection  $\pi$  induces an isomorphism from  $E$  onto its image.*

*Proof.* The first part of the statement is a consequence of [Theorem 4.7 \(v\)](#). Since  $E$  is elliptic or parabolic,  $\text{Mov}(\{1, g\}, 17\delta)$  is non-empty for every  $g \in E$  – see for instance [\(10\)](#). Hence the result.  $\square$

**Proposition 4.26.** *Let  $E$  be an elliptic or a parabolic subgroup of  $G$  (for its action on  $X$ ). Let  $S_1$  be a subset of  $E$  such that  $\text{Mov}(S_1, \rho/100)$  is non-empty and  $\bar{S}_1$  its image in  $\bar{G}$ . Let  $\bar{h} \in \bar{G}$ . Let  $S_2$  be a pre-image in  $G$  of  $\bar{h}\bar{S}_1\bar{h}^{-1}$  such that  $\text{Mov}(S_2, \rho/100)$  is non-empty. Then there exists  $h_0 \in G$  with the following properties*

- (i) *For every  $g \in S_1$ , the element  $h_0gh_0^{-1}$  is the (unique) pre-image of  $\bar{h}\bar{g}\bar{h}^{-1}$  in  $S_2$ .*
- (ii) *If  $\bar{h}$  is loxodromic, then either  $h_0$  is loxodromic, or  $\bar{S}_1$  is contained in a reflection group.*

*Remark.* Observe that  $h_0$  is not necessarily the pre-image of  $\bar{h}$ .

*Proof.* Let  $h \in G$  be an arbitrary pre-image of  $\bar{h}$ . We fix two points  $x_1, x_2 \in X$  lying respectively in  $\text{Mov}(S_1, \rho/100)$  and  $\text{Mov}(S_2, \rho/100)$ . Note that both  $\bar{x}_1$  and  $\bar{h}^{-1}\bar{x}_2$  belongs to  $\text{Mov}(\bar{S}_1, \rho/100)$ . We claim that  $\bar{S}_1 \cap \text{Stab}(\bar{v})$  is contained in a reflection group at  $\bar{v}$ , for every  $\bar{v} \in \bar{\mathcal{V}}$ . Assume on the contrary that it is not the case. There exists  $g \in E$  whose image  $\bar{g}$  is a strict rotation ([Remark 4.20](#)). According to [Proposition 4.12 \(iii\)](#) there exists  $k \in \mathbf{N}$  such that  $\text{Mov}(\bar{g}^k, \bar{\delta})$  is contained in  $B(\bar{v}, \bar{\delta})$ . On the other hand, since  $g$  belongs to  $E$ , the element  $g^k$  is elliptic or parabolic (as an isometry of  $X$ ). Hence there exists  $x \in X$  such that  $|g^kx - x| \leq 17\delta$  – see for instance [\(10\)](#) – and thus  $|\bar{g}^k\bar{x} - \bar{x}| \leq \bar{\delta}$ . This contradicts the previous point and complete the proof of our claim. It follows



from [Proposition 4.19](#) applied with  $x = x_1$  and  $y = h^{-1}x_2$  that there exists  $u \in G$ , such that  $\bar{u}$  centralizes  $\bar{S}_1$  and

$$|guh^{-1}x_2 - uh^{-1}x_2|_{\bar{X}} = |\bar{g}\bar{h}^{-1}\bar{x}_2 - \bar{h}^{-1}\bar{x}_2|, \quad \forall g \in S_1. \quad (20)$$

Moreover either  $\bar{u}$  is trivial or  $\bar{S}_1$  lies in a reflection group. We let  $h_0 = hu^{-1}$ . Let  $g \in S_1$  and  $g'$  the (unique) pre-image of  $\bar{h}\bar{g}\bar{h}^{-1}$  in  $S_2$ . It follows from (20) that  $h_0gh_0^{-1}$  and  $g'$  are two pre-images of  $\bar{h}\bar{g}\bar{h}^{-1}$  that move  $x_2$  by at most  $\rho/100$ . Thus  $h_0gh_0^{-1} = g'$ , which proves (i). Assume now that  $\bar{h}$  is loxodromic. If  $\bar{u}$  is trivial, then  $h_0$  is a pre-image of  $\bar{h}$ , hence a loxodromic element (recall that  $\zeta: X \rightarrow \bar{X}$  is 1-Lipschitz). On the contrary if  $\bar{u}$  is not trivial, then  $\bar{S}_1$  is contained in a reflection group.  $\square$

**Corollary 4.27.** *Let  $F_1$  and  $F_2$  be two subgroups of  $G$ . We assume that  $F_1$  is elliptic and  $F_2$  generated by a set  $S_2$  such that  $\text{Mov}(S_2, \rho/100)$  is non-empty. Let  $\bar{F}_1$  and  $\bar{F}_2$  be their respective images in  $\bar{G}$ . If  $\bar{F}_1 = \bar{F}_2$ , then there exists  $u \in G$  whose image in  $\bar{G}$  centralizes  $\bar{F}_1$  and such that  $F_2 = uF_1u^{-1}$ .*

*Remark.* Note that the assumption on  $F_2$  is automatically satisfied if  $F_2$  is elliptic. In particular if  $F_1$  and  $F_2$  are two elliptic subgroups of  $G$  whose images in  $\bar{G}$  coincide, then they are conjugate.

*Proof.* Let  $\bar{S}$  be the image of  $S_2$  in  $\bar{F}_1 = \bar{F}_2$  and  $S_1$  the pre-image of  $\bar{S}$  in  $F_1$ . Note that  $\text{Mov}(S_1, \rho/100)$  is non-empty ([Lemma 2.14](#)). According to [Proposition 4.26](#) applied with  $S_1$  and  $S_2$ , there exists  $u \in G$  such that for every  $s \in S_1$ , the element  $usu^{-1}$  is the pre-image of  $\bar{s}$  in  $S_2$ . In particular  $\bar{u}$  commutes with  $\bar{S}$ , hence  $\bar{F}_1$ . Moreover since  $S_2$  generates  $F_2$ , the group  $u^{-1}F_2u$  is contained in  $F_1$ . Nevertheless the projection  $\pi: G \rightarrow \bar{G}$  is one-to-one when restricted to  $F_1$  ([Lemma 4.25](#)). Thus  $F_2 = uF_1u^{-1}$ .  $\square$

**Corollary 4.28.** *Let  $F_1$  and  $F_2$  be two subgroups of  $G$ . We assume that  $F_1$  is elliptic and  $F_2$  generated by a set  $S_2$  such that  $\text{Mov}(S_2, \rho/100)$  is non-empty. Let  $\bar{F}_1$  and  $\bar{F}_2$  be their respective images in  $\bar{G}$ . If  $\bar{F}_1$  and  $\bar{F}_2$  are conjugated in  $\bar{G}$ , then so are  $F_1$  and  $F_2$  in  $G$ .*

**Lifting elliptic subgroups.** We now characterize the elliptic subgroups of  $\bar{G}$  that can be lifted and explore the structure of the one that cannot be lifted.

**Proposition 4.29** (Lifting elliptic subgroups). *An elliptic subgroup  $\bar{F}$  of  $\bar{G}$  cannot be lifted if and only if it contains a strict rotation. In this case,  $\bar{F}$  fixes an apex  $\bar{v} \in \bar{\mathcal{V}}$ . Moreover,  $\text{Mov}(\bar{F}, \bar{\delta})$  is contained in  $B(\bar{v}, \bar{\delta})$ . In particular,  $\bar{v}$  is the only apex fixed by  $\bar{F}$ .*

*Proof.* Recall that  $\text{Mov}(\bar{F}, 11\bar{\delta})$  is a non-empty  $10\bar{\delta}$ -quasi-convex subset of  $\bar{X}$  ([Lemmas 2.14](#) and [2.9](#)). Assume first that there exists a point  $\bar{x} \in \text{Mov}(\bar{F}, 11\bar{\delta})$  such that  $d(\bar{x}, \bar{\mathcal{V}}) \geq \rho/3$ . By [Proposition 4.12 \(iii\)](#),  $\bar{F}$  does not contain a strict rotation. We are going to prove that  $\bar{F}$  can be lifted. We write  $\bar{Z}$  for the  $\bar{F}$ -orbit of  $\bar{x}$ . It is  $\bar{F}$ -invariant and its diameter is at most  $11\bar{\delta}$ . It follows that

$\langle \bar{z}, \bar{z}' \rangle_{\bar{v}} \geq \rho/4$ , for every  $\bar{z}, \bar{z}' \in \bar{Z}$ , for every  $\bar{v} \in \bar{\mathcal{V}}$ . According to Lemmas 4.17 and 4.16, there exists a subgroup  $F$  of  $G$  and an  $F$ -invariant subset  $Z$  of  $\bar{X}$  with the following properties: the map  $\zeta: \bar{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$ ; the projection  $\pi: G \rightarrow \bar{G}$  induces an isomorphism from  $F$  onto  $\bar{F}$ ; moreover  $\langle z, z' \rangle_v > 4L_0\delta$  for every  $z, z' \in Z$ , for every  $v \in \mathcal{V}$  (the Gromov product in computed in  $\bar{X}$  here). In particular  $Z$  is bounded. It follows from Corollary 4.9 that the radial projection  $p(Z)$  is a bounded  $F$ -invariant subset of  $X$ . Hence  $F$  is an elliptic subgroup of  $G$  (for its action on  $X$ ), lifting  $\bar{F}$ .

Assume now that  $d(\bar{x}, \bar{\mathcal{V}}) < \rho/3$  for every  $\bar{x} \in \text{Mov}(\bar{F}, 11\bar{\delta})$ . Since the set  $\text{Mov}(\bar{F}, 11\bar{\delta})$  is  $10\bar{\delta}$ -quasi-convex, there exists  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\text{Mov}(\bar{F}, 11\bar{\delta})$  is contained in  $B(\bar{v}, \rho/3)$ . In particular,  $\bar{F}$  fixes  $\bar{v}$ . In addition,  $\bar{F}$  cannot be lifted. Indeed if  $F$  was a lift of  $\bar{F}$ , then the image in  $\bar{X}$  of  $\text{Mov}(F, 11\delta) \subset X$  would be contained in  $\text{Mov}(\bar{F}, 11\bar{\delta}) \setminus B(\bar{v}, \rho/3)$ . We now claim that  $\bar{F}$  contains a strict rotation  $\bar{g}$  at  $\bar{v}$ . If it was not the case, then  $\bar{F}$  would be contained in a reflection group at  $\bar{v}$ . Thus, there would exist a point  $\bar{x}' \in \text{Mov}(\bar{F}, 11\bar{\delta})$  such that  $|\bar{x}' - \bar{v}| > \rho/2$ , see Proposition 4.12 (ii), which contradicts the previous observation. It follows then from Proposition 4.12 (iii) that  $\text{Mov}(\bar{F}, \bar{\delta})$  is contained in  $B(\bar{v}, \bar{\delta})$ .  $\square$

**Corollary 4.30.** *Let  $(H, Y) \in \mathcal{Q}$ . Let  $\bar{v}$  be the image in  $\bar{X}$  of the apex  $v$  of  $Z(Y)$ . Let  $\bar{C}$  be a subgroup of  $\text{Stab}(\bar{v})$ . If  $\bar{C}$  can be lifted, then it admits a lift which is contained in  $\text{Stab}(Y)$ .*

*Proof.* Let  $F$  be the maximal elliptic normal subgroup of  $\text{Stab}(Y)$  and  $\bar{F}$  its image in  $\bar{G}$ . Recall that we have the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & F & \longrightarrow & \text{Stab}(Y) & \longrightarrow & \mathbf{L} \longrightarrow 1 \\ & & \downarrow \zeta & & \downarrow \pi & & \downarrow \\ 1 & \longrightarrow & \bar{F} & \longrightarrow & \text{Stab}(\bar{v}) & \xrightarrow{q_{\bar{v}}} & \mathbf{L}_n \longrightarrow 1 \end{array}$$

where  $(\mathbf{L}, \mathbf{L}_n)$  is either  $(\mathbf{Z}, \mathbf{Z}_n)$  or  $(\mathbf{D}_\infty, \mathbf{D}_n)$ . Since  $\bar{C}$  can be lifted, its image under  $q_{\bar{v}}$  does not contain a non-trivial rotation (Proposition 4.29). The result follows from diagram chasing.  $\square$

We continue with a study of dihedral germs. Let  $F$  be an elliptic subgroup of  $G$  and  $\bar{F}$  its image in  $\bar{G}$ . Observe that if  $F$  is a dihedral germ, then the same does not necessarily hold for  $\bar{F}$ . Indeed it may happen that the only loxodromic elements that are normalizing (a finite index subgroup of)  $F$  became elliptic in  $\bar{G}$ . Nevertheless the converse statement holds. This is the aim of the next lemmas.

**Lemma 4.31.** *Let  $(H, Y) \in \mathcal{Q}$ . Every elliptic subgroup of  $\text{Stab}(Y)$  is a dihedral germ.*

*Proof.* For simplicity we let  $E = \text{Stab}(Y)$ . Let  $E^+$  be the maximal subgroup of  $E$  preserving the orientation and  $F$  the maximal elliptic subgroup of  $E^+$ . Let

$C$  be an elliptic subgroup of  $E$ . The intersection  $C_0 = C \cap E^+$  is a subgroup of  $C$  with index at most 2. Let  $h$  be a (loxodromic) element in  $H$ . Let  $c \in C_0$ . It follows from the small cancellation assumption that  $H$  is a normal subgroup of  $E$ . In particular  $chc^{-1} = h^k$  for some  $k \in \mathbf{Z}$ . Recall that  $E^+/F$  is isomorphic to  $\mathbf{Z}$ . Pushing the previous identity in  $\mathbf{Z}$ , we get that  $k = 1$ . In other words  $h$  commutes with  $C_0$ . Hence  $C$  is a dihedral germ.  $\square$

**Lemma 4.32.** *Let  $C$  be an elliptic subgroup of  $G$  (for its action on  $X$ ). Let  $\bar{v} \in \bar{\mathcal{V}}$ . If the image of  $C$  in  $\bar{G}$  is contained in a reflection group at  $\bar{v}$ , then  $C$  is a dihedral germ.*

*Proof.* Let  $\bar{C}$  be the image of  $C$  in  $\bar{G}$ . Let  $(H, Y) \in \mathcal{Q}$  such that the apex  $v$  of the cone  $Z(Y)$  is a pre-image of  $\bar{v}$ . There exists an elliptic subgroup  $C'$  of  $\text{Stab}(Y)$  such that the projection  $\pi: G \rightarrow \bar{G}$  maps  $C'$  onto  $\bar{C}$  (Corollary 4.30). In other words  $C$  and  $C'$  are two lifts of  $\bar{C}$ , hence they are conjugated (Corollary 4.27). Being a dihedral germ is invariant under conjugacy. The conclusion follows from Lemma 4.31.  $\square$

**Lemma 4.33** (Lifting dihedral germs). *Let  $C$  be an elliptic subgroup of  $G$  and  $\bar{C}$  (for its action on  $X$ ) its image in  $\bar{G}$ . If  $\bar{C}$  is a dihedral germ, then so is  $C$ .*

*Proof.* By assumption there exists a subgroup  $\bar{C}_0$  of  $\bar{C}$  which is normalized by a loxodromic element, say  $\bar{h}$ , and such that  $[\bar{C} : \bar{C}_0] = 2^k$  for some  $k \in \mathbf{N}$ . We write  $C_0$  for the pre-image of  $\bar{C}_0$  in  $C$ . Note that  $[C : C_0] = 2^k$ . It follows from Proposition 4.26 applied with  $S_1 = S_2 = C_0$  that there exists  $h_0 \in G$  normalizing  $C_0$ . Moreover either  $h_0$  is loxodromic or  $\bar{C}_0$  is contained in a reflection group at some apex  $\bar{v} \in \bar{\mathcal{V}}$ . If  $h_0$  is loxodromic, then  $C$  is automatically a dihedral germ. Assume now that  $\bar{C}_0$  is contained in a reflection group at  $\bar{v}$ . It follows from Lemma 4.32 that  $C_0$  is a dihedral germ. Hence it contains a subgroup  $C_1$  which is normalized by a loxodromic element of  $G$  and such that  $[C_0 : C_1] = 2^m$  for some  $m \in \mathbf{N}$ . Thus  $[C : C_1] = 2^{k+m}$  and  $C$  is a dihedral germ.  $\square$

The next lemma is formally not needed. However it illustrates the role played by dihedral germs. As we observed earlier, every elliptic subgroup  $F$  of  $G$  yields an elliptic subgroup  $\bar{F}$  of  $\bar{G}$ . However it could happen that  $\bar{F}$  is strictly contained in an elliptic subgroup, which does not already come from a subgroup of  $G$  containing  $F$ . In this case  $F$  is necessarily a dihedral germ. As suggested by the name, dihedral germs are exactly the elliptic subgroups of  $G$  which can eventually “grow” when passing to the quotient  $\bar{G}$ .

**Lemma 4.34.** *Let  $C$  be an elliptic subgroup of  $G$  (for its action on  $X$ ) and  $\bar{C}$  its image in  $\bar{G}$ . Assume that there exists an elliptic subgroup  $\bar{A}$  containing  $\bar{C}$  which cannot be lifted. Then  $C$  is a dihedral germ.*

*Proof.* This is just a reformulation of Lemma 4.32. Indeed according to Proposition 4.29, there exists  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\bar{A}$  is contained in  $\text{Stab}(\bar{v})$ . Since  $\bar{C}$  can be lifted, it does not contain a strict rotation at  $\bar{v}$  (Proposition 4.29) and thus lies in a reflection group at  $\bar{v}$ .  $\square$

We complete our discussion on elliptic subgroups with some preparatory work for the study of loxodromic. If such group  $\bar{E}$  does not preserve the orientation, it can be decomposed as  $\bar{E} = \bar{A} *_{\bar{C}} \bar{B}$ , where  $\bar{C}$  has index 2 in both  $\bar{A}$  and  $\bar{B}$ . As the cylinder of  $\bar{E}$  is contained in  $\text{Mov}(\bar{C}, 88\bar{\delta})$  (Lemma 2.15),  $\bar{C}$  can also be lifted. We describe in this context what is the structure of  $\bar{A}$  or  $\bar{B}$ .

**Lemma 4.35.** *Let  $\bar{A}$  be an elliptic subgroup of  $\bar{G}$ . Assume that  $\bar{A}$  contains a subgroup  $\bar{C}$  of index 2 that can be lifted. Let  $\bar{a} \in \bar{A} \setminus \bar{C}$ . Then there exists  $\bar{u} \in \bar{G}$  such that*

- (i)  $\langle \bar{C}, \bar{u} \rangle$  is an elliptic subgroup that can be lifted;
- (ii)  $\bar{a}\bar{u}^{-1}$  centralizes  $\bar{C}$ ; and
- (iii)  $\bar{a}^2 = \bar{u}^2$ .

*Remark.* Observe that if  $\bar{u}$  is trivial, then  $\bar{A}$  is isomorphic to  $\bar{C} \times \langle \bar{a} \rangle = \bar{C} \times \mathbf{Z}_2$ . In general the map  $\bar{a} \mapsto \bar{u}$  extends to an (abstract) isomorphism from  $\bar{A}$  onto  $\langle \bar{C}, \bar{u} \rangle$ .

*Proof.* If  $\bar{A}$  can be lifted, then the statement obviously holds. Assume now that  $\bar{A}$  cannot be lifted. There exists  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\bar{A}$  is contained in  $\text{Stab}(\bar{v})$  (Proposition 4.29). Let  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  be corresponding the geometric realization. Since  $\bar{C}$  can be lifted,  $q_{\bar{v}}(\bar{C})$  is either trivial or equal to  $\langle \mathbf{x} \rangle$  where  $\mathbf{x}$  is a reflection of  $\mathbf{D}_n$ . Let  $\mathbf{t}$  be a generator of the rotation group  $\mathbf{Z}_n \subset \mathbf{D}_n$ . Recall that  $\bar{C}$  has index 2 in  $\bar{A}$  and  $\bar{A}$  cannot be lifted in  $G$ . It follows that  $n$  is even and

$$q_{\bar{v}}(\bar{A}) = \langle q_{\bar{v}}(\bar{C}), \mathbf{t}^{n/2} \rangle.$$

In particular,  $\mathbf{t}^{n/2}q_{\bar{v}}(\bar{a})$  belongs to  $q_{\bar{v}}(\bar{C})$ . According to Assumption 4.11,  $\text{Stab}(\bar{v})$  contains a central half-turn at  $\bar{v}$  that we denote by  $\bar{g}$ . Note that  $q_{\bar{v}}$  maps  $\bar{g}$  to  $\mathbf{t}^{n/2}$ . We let  $\bar{u} = \bar{g}\bar{a}$ . We observe that  $\bar{a}\bar{u}^{-1} = \bar{g}$  centralizes  $\bar{C}$  and  $\bar{a}^2 = \bar{u}^2$ . By construction  $\langle \bar{C}, \bar{u} \rangle$  is contained in a reflection group at  $\bar{v}$ , hence can be lifted, which completes the proof.  $\square$

Another crucial ingredient to describe loxodromic subgroups of  $\bar{G}$ , is to understand the normalizer of elliptic subgroups that can be lifted, see for instance Proposition 4.41. This is the purpose of the next proposition.

**Proposition 4.36** (Lifting normalizer). *Let  $F$  be an elliptic subgroup of  $G$  and  $\bar{F}$  its image in  $\bar{G}$ . For every  $\bar{h} \in \text{Norm}(\bar{F})$  there exists  $h_0 \in \text{Norm}(F)$  such that  $\bar{h}^{-1}\bar{h}_0$  centralizes  $\bar{F}$ . If in addition  $\bar{h}^2$  belongs to  $\bar{F}$ , then one can choose  $h_0$  such that  $h_0^2 \in F$  and  $\bar{h}^2 = \bar{h}_0^2$ .*

*Remark 4.37.* The result can be reformulated in the following way. Let  $N(F)$  and  $C(F)$  the the respective normalizer and centralizer of  $F$  in  $G$ . We define  $N(\bar{F})$  and  $C(\bar{F})$  in the same way. The projection  $\pi: G \rightarrow \bar{G}$  does not necessarily map  $N(F)$  onto  $N(\bar{F})$ . Nevertheless it induces an epimorphism from  $N(F)/C(F)$  onto  $N(\bar{F})/C(\bar{F})$ . Actually Lemma 4.25 implies that this map is an isomorphism, but we will not need this fact here.

*Proof.* Applying [Proposition 4.26](#) with  $S_1 = S_2 = F$ , we see that there exists  $h_0 \in G$  such that for every  $g \in G$ , the element  $h_0gh_0^{-1}$  is the pre-image of  $\bar{h}\bar{g}\bar{h}^{-1}$  in  $F$ . In particular  $h_0$  normalizes  $F$  and  $\bar{h}^{-1}h_0$  centralizes  $\bar{F}$ . Let us now focus on the second part of the statement. We assume that  $\bar{h} \in \bar{G}$  normalizes  $\bar{F}$  and  $\bar{h}^2 \in \bar{F}$ . In particular  $\bar{F}' = \langle \bar{F}, \bar{h} \rangle$  is an elliptic subgroup of  $\bar{G}$ . Assume first that  $\bar{F}'$  can be lifted in  $G$ . Let  $F'$  be a lift of  $\bar{F}'$  and  $h$  the pre-image of  $\bar{h}$  in  $F'$ . There exists  $u \in G$  such that  $\bar{u}$  centralizes  $\bar{F}$  and  $hFh^{-1} = uFu^{-1}$  ([Corollary 4.27](#)). We choose  $h_0 = u^{-1}hu$ . Since  $h^2$  belongs to  $F$ , the element  $h_0^2$  is a pre-image in  $F'$  of  $\bar{h}^2$ , hence  $h_0^2 = h^2$ . If  $\bar{F}'$  cannot be lifted, then the lemma is a consequence of [Lemma 4.35](#).  $\square$

**Lifting parabolic subgroups.** The first result is not needed for the rest of the study. However, we believe that it may help the reader by clarifying the structure of parabolic subgroups in  $G$ . As we mentioned earlier, since  $\zeta: X \rightarrow \bar{X}$  is 1-Lipschitz, the image by the projection  $\pi: G \rightarrow \bar{G}$  of a parabolic subgroup of  $G$  is either elliptic or parabolic. The next statement tells us that the former case does not happen.

**Lemma 4.38.** *Let  $E$  be an elementary subgroup of  $G$  and  $\bar{E}$  its image in  $\bar{G}$ . Assume that there exists  $d \in \mathbf{R}_+$  and a subset  $S$  generating  $E$  such that  $\text{Mov}(S, d)$  is non-empty. If  $E$  is parabolic (for its action on  $X$ ) then  $\bar{E}$  is parabolic (for its action on  $\bar{X}$ ).*

*Remark.* Note that the assumption automatically holds if  $E$  is finitely generated.

*Proof.* Since  $E$  is parabolic, it has a unique fixed point in  $\partial X$ . Thus according to [Lemma 2.12](#) we can assume that  $\text{Mov}(S, \rho/100)$  is non-empty. We just need to prove that  $\bar{E}$  cannot be elliptic. Assume on the contrary that it is. We distinguish two cases. Suppose first that  $\bar{E}$  can be lifted and let  $E'$  be a lift of  $\bar{E}$  (recall that by definition  $E'$  is elliptic). Applying [Corollary 4.27](#) with  $S_2 = S$ , we get that  $E$  and  $E'$  are conjugate, which contradicts the fact that  $E$  and  $E'$  have different nature. Suppose now that  $\bar{E}$  cannot be lifted. In particular, there exists  $g \in E$  whose image  $\bar{g}$  in  $\bar{E}$  is a strict rotation ([Proposition 4.29](#)). By [Proposition 4.12 \(iii\)](#), there exists  $k \in \mathbf{Z}$  such that  $\text{Mov}(\bar{g}^k, \bar{\delta})$  is contained in  $B(\bar{v}, \bar{\delta})$  for some apex  $\bar{v} \in \bar{\mathcal{V}}$ . On the other hand  $g^k$  cannot be loxodromic, hence  $\|g^k\|_X \leq 16\delta$ . Thus there exists  $x \in X$  such that  $|g^kx - x| \leq 17\delta$ , thus  $\bar{x}$  belongs to  $\text{Mov}(\bar{g}^k, \bar{\delta}) \setminus B(\bar{v}, \bar{\delta})$ , which yields another contradiction.  $\square$

**Proposition 4.39** (Lifting parabolic subgroups). *Let  $\bar{P}$  be parabolic subgroup of  $\bar{G}$ . Assume that there exist  $d \in \mathbf{R}_+$  and a subset  $\bar{S}$  generating  $\bar{P}$  such that  $\text{Mov}(\bar{S}, d)$  is non-empty. Then  $\bar{P}$  can be lifted.*

*Proof.* Let  $\bar{\xi}$  be the unique point of  $\partial \bar{X}$  fixed by  $\bar{P}$  and  $\bar{x}$  be a point in the set  $\text{Mov}(\bar{S}, d)$ . Let  $\bar{\gamma}: \mathbf{R}_+ \rightarrow \bar{X}$  be an  $L_0\bar{\delta}$ -local  $(1, 11\bar{\delta})$ -quasi-geodesic ray starting at  $\bar{x}$  whose endpoint at infinity is  $\bar{\xi}$ . According to [Lemma 2.12](#), there exists  $t_0$  such that for every  $t \geq t_0$ , for every  $g \in S$ , we have  $|\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| \leq 114\bar{\delta}$ . It follows that  $d(\bar{\gamma}(t), \bar{\mathcal{V}}) \geq \rho/2$ , for every  $t \geq t_0$ . Indeed otherwise, there would

exit  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\bar{S}$ , and thus  $\bar{P}$ , is contained in  $\text{Stab}(\bar{v})$ , which contradicts our assumption.

Let  $\bar{Z}$  be a the  $\rho/10$ -neighborhood of  $\bar{\gamma}$  restricted to  $[t_0, \infty)$ . It is a  $2\bar{\delta}$ -quasi-convex subset. According to our claim that  $\langle \bar{z}, \bar{z}' \rangle_{\bar{v}} > 4L_0\bar{\delta}$  for every  $\bar{z}, \bar{z}' \in \bar{Z}$  and  $\bar{v} \in \bar{\mathcal{V}}$ . It follows from [Lemma 4.17](#) that there exist a subset  $Z$  of  $\dot{X}$  such that the map  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$ . Moreover  $\langle z, s' \rangle_v > 4L_0\delta$  for every  $z, z' \in Z$  and  $v \in \mathcal{V}$  (the Gromov product are computed in  $\dot{X}$  here). In particular, there exists an  $L_0\delta$ -local  $(1, 11\delta)$ -quasi-geodesic ray  $\gamma: [t_0, \infty) \rightarrow \dot{X}$  contained in  $Z$  such that  $\pi \circ \gamma = \bar{\gamma}$  and  $Z$  is its  $\rho/10$  neighborhood. We write  $\xi \in \partial\dot{X}$  for the endpoint at infinity of  $\gamma$ .

We now claim that  $\bar{P}$  is contained in the image of  $\text{Stab}(\xi)$  by the projection  $\pi: G \rightarrow \bar{G}$ . Let  $\bar{g} \in \bar{P}$ . As we observed,  $|\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| \leq 114\bar{\delta}$ , for every  $t \geq t_0$ . It follows from [Lemma 4.16](#) applied to  $Z$  that there exists a pre-image  $g \in G$  of  $\bar{g}$  such that for every  $t \geq t_0$ , we have  $|g\gamma(t) - \gamma(t)|_{\dot{X}} \leq 114\delta$ . In particular,  $g$  fixes  $\xi$ , which completes the proof our first claim.

We denote by  $P$  the pre-image of  $\bar{P}$  in  $\text{Stab}(\xi)$ . We now claim that  $P$  is parabolic for its action on  $X$ . To that end, it suffices to show that  $P$  is parabolic for its action on  $\dot{X}$  ([Lemma 4.10](#)). We are going to prove that  $P$  is not elliptic and does not contain a loxodromic element. As  $\bar{P}$  is parabolic, it has unbounded orbits. The map  $\zeta: \dot{X} \rightarrow \bar{X}$  being 1-Lipschitz, the group  $P$  has unbounded orbits, and thus cannot be elliptic (for its action on  $\dot{X}$ ). Let  $g \in P$  and  $\bar{g}$  its image in  $\bar{P}$ . According to [Lemma 2.12](#) that there exists  $t_1 \geq t_0$  and  $\varepsilon \in \{\pm 1\}$  such that for every  $t \geq t_1$  we have  $|g\gamma(t) - \gamma(t + \varepsilon\|g\|_{\dot{X}}^\infty)|_{\dot{X}} \leq 114\delta$ . In particular if  $t$  is sufficiently large both  $\gamma(t)$  and  $g\gamma(t)$  belong to  $Z$ . Since  $\zeta: \dot{X} \rightarrow \bar{X}$  is an isometry when restricted to  $Z$ , we get

$$\|g\|_{\dot{X}} \leq |g\gamma(t) - \gamma(t)|_{\dot{X}} \leq |\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| \leq 114\bar{\delta}.$$

This inequality holds for every  $g \in P$ . Therefore  $P$  cannot contain a loxodromic element for its action on  $\dot{X}$ , which completes the proof of our second claim. As  $P$  is a parabolic subgroup, the projection  $\pi: G \rightarrow \bar{G}$  is one-to-one when restricted to  $P$  ([Lemma 4.25](#)). Hence  $P$  is a lift of  $\bar{P}$ .  $\square$

**Lifting loxodromic subgroups.** It  $\bar{G}$  does not contain any element of order 2, then one can prove that all its loxodromic subgroups can be lifted. This is typically what is happening when studying Burnside groups of odd exponent. As shown by the next example this is unfortunately no more the case in the presence of even torsion.

**Example 4.40.** Assume for instance that  $\bar{G}$  is the group defined by

$$\bar{G} = \mathbf{Z}_n * \mathbf{Z}_n = \langle a, b \mid a^n = b^n = 1 \rangle.$$

If  $n$  is a sufficiently large even integer, it is a small cancellation quotient of the free group  $\mathbf{F}_2$  generated by  $a$  and  $b$ . The subgroup  $\langle a^{n/2}, b^{n/2} \rangle \cong \mathbf{D}_\infty$  is loxodromic but cannot be lifted.

In general we show that every loxodromic subgroup of  $\bar{G}$  is an (abstract) subdirect product of an elementary subgroup of  $G$  and either  $\mathbf{Z}$  or  $\mathbf{D}_\infty$ .

**Proposition 4.41** (Lifting loxodromic subgroups). *Let  $\bar{E}$  be a loxodromic subgroup of  $\bar{G}$  and  $\bar{F}$  the maximal normal elliptic subgroup of  $\bar{E}$ . There exist a lift  $F$  of  $\bar{F}$ , an elementary subgroup  $E'$  of  $G$  containing  $F$ , and an epimorphism  $\theta: \bar{E} \twoheadrightarrow E'$  with the following properties.*

- (i)  $(E', F)$  is a dihedral pair.
- (ii) The morphism  $\pi \circ \theta$  is the identity when restricted to  $\bar{F}$ .
- (iii) The map  $\theta$  induces an embedding from  $\bar{E}$  into  $\bar{E}/\bar{F} \times E'$ .

*Proof.* Since the action of  $\bar{G}$  on  $\bar{X}$  is gentle (Proposition 4.21), the group  $\bar{E}$  fits in a short exact sequence

$$1 \rightarrow \bar{F} \rightarrow \bar{E} \xrightarrow{q} \mathbf{L} \rightarrow 1,$$

where  $\mathbf{L}$  is either  $\mathbf{Z}$  or  $\mathbf{D}_\infty$ . The cylinder  $\bar{Z}$  of  $\bar{E}$  is contained in  $\text{Mov}(\bar{F}, 88\bar{\delta})$  (Lemma 2.15). In particular  $\text{Mov}(\bar{F}, 88\bar{\delta}) \cap \zeta(X)$  is non-empty. It follows from Proposition 4.29 that  $\bar{F}$  admits a lift in  $G$  that we denote by  $F$ . The subgroup  $\bar{F}$  is a dihedral germ, hence so is  $F$  (Lemma 4.33). We now claim that there exists an elementary subgroup  $E'$  of  $G$  containing  $F$  as a normal subgroup such that the canonical section  $\bar{F} \rightarrow F$  extends to an epimorphism  $\theta: \bar{E} \twoheadrightarrow E'$ . To that end we distinguish two cases.

**Case 1.** Assume that  $\bar{E}$  preserves the orientation, i.e.  $\mathbf{L} = \mathbf{Z}$ . Then  $\bar{E}$  splits as a semi-direct product  $\bar{E} = \bar{F} \rtimes \mathbf{L}$ . Let  $\bar{h}$  be a primitive element of  $\bar{E}$  (i.e. an element whose image under  $q$  generates  $\mathbf{L}$ ). According to Proposition 4.36 there exists  $h_0 \in \text{Norm}(F)$  such that  $\bar{h}^{-1}\bar{h}_0$  centralizes  $\bar{F}$ . Let  $E'$  be the subgroup of  $G$  generated by  $F$  and  $h_0$ . As an extension of an elliptic subgroup by a cyclic group,  $E'$  is elementary. Moreover the canonical section  $\bar{F} \rightarrow F$  extends to an epimorphism  $\theta: \bar{E} \twoheadrightarrow E'$  sending  $\bar{h}$  to  $h_0$ .

**Case 2.** Assume that  $\bar{E}$  does not preserve the orientation, so that  $\mathbf{L} = \mathbf{D}_\infty$ . Let  $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}_\infty$  be two reflections generating  $\mathbf{L}$ . Let  $\bar{a}_1, \bar{a}_2 \in \bar{E}$  be pre-images of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  respectively. By construction  $\bar{a}_i$  normalizes  $\bar{F}$  and  $\bar{a}_i^2$  belongs to  $\bar{F}$ . According to Proposition 4.36 there exists  $b_i \in \text{Norm}(F)$  with the following properties:  $\bar{a}_i^{-1}\bar{b}_i$  centralizes  $\bar{F}$ ;  $b_i^2$  belongs to  $F$ ; and  $\bar{a}_i^2 = \bar{b}_i^2$ . Let  $E'$  be the subgroup of  $G$  generated by  $F$ ,  $a_1$ , and  $a_2$ . As an extension of an elliptic subgroup by a cyclic group,  $\langle b_1 b_2, F \rangle$  is elementary. Nevertheless its index in  $E'$  is at most 2, thus  $E'$  is elementary. Moreover the canonical isomorphism  $\bar{F} \rightarrow F$  extends to an epimorphism  $\theta: \bar{E} \twoheadrightarrow E'$  sending  $\bar{a}_i$  to  $b_i$ .

In both cases we have build the map announced in the claim. As  $\theta$  extends the canonical section  $\bar{F} \rightarrow F$ , the composition  $\pi \circ \theta$  is the identity when restricted to  $\bar{F}$ . Moreover  $\theta$  induces an epimorphism  $\bar{E}/\bar{F} \twoheadrightarrow E'/F$ . Any quotient of a dihedral group is still a dihedral group. Hence  $(E', F)$  is a dihedral pair. One checks easily that the map  $\bar{E} \rightarrow \bar{E}/\bar{F} \times E'$  induced by  $\theta$  is an embedding.  $\square$

In the remainder of this section we revisit the previous statement and explore further the structure certain loxodromic subgroups that cannot be lifted. As suggested by [Example 4.40](#) such a group  $\bar{E}$  often does not preserve the orientation. In particular it splits as  $\bar{E} = \bar{A} *_{\bar{C}} \bar{B}$  where  $\bar{C}$  is the maximal elliptic normal subgroup of  $\bar{E}$  and has index 2 in both  $\bar{A}$  and  $\bar{B}$ . It  $\bar{E}$  could be lifted, then obviously so would  $\bar{A}$  and  $\bar{B}$ . Nevertheless the converse is false. This is the purpose of the first proposition. The second one discusses the case where  $\bar{A}$  or  $\bar{B}$  cannot be lifted.

**Proposition 4.42.** *Let  $A$  and  $B$  be two elliptic subgroups of  $G$ . Denote by  $\bar{A}$  and  $\bar{B}$  their respective images in  $\bar{G}$ . Assume that their intersection  $\bar{C} = \bar{A} \cap \bar{B}$  has index 2 in both  $\bar{A}$  and  $\bar{B}$  so that  $\bar{E} = \bar{A} *_{\bar{C}} \bar{B}$  is elementary. There exists  $u \in G$  with the following properties.*

- (i) *The image  $\bar{u}$  of  $u$  in  $\bar{G}$  centralizes  $\bar{C}$ .*
- (ii) *The subgroup  $A \cap uBu^{-1}$  contains the pre-image of  $\bar{C}$  in  $A$ , and  $\langle A, uBu^{-1} \rangle$  is elementary.*

*Proof.* Recall that the quotient map  $\pi: G \rightarrow \bar{G}$  induces an isomorphism from  $A$  and  $B$  onto  $\bar{A}$  and  $\bar{B}$  respectively ([Lemma 4.25](#)). Let  $C_A$  (respectively  $C_B$ ) be the pre-image of  $\bar{C}$  in  $A$  (respectively  $B$ ). According to [Corollary 4.27](#), there exists  $u \in G$  whose image  $\bar{u}$  in  $\bar{G}$  centralizes  $\bar{C}$  such that  $C_A = uC_Bu^{-1}$ . It follows that  $C_A \subset A \cap uBu^{-1}$ . Moreover this elliptic subgroup has index 2 in both  $A$  and  $uBu^{-1}$ , hence  $E_u$  is elementary.  $\square$

Before moving to the case where  $\bar{A}$  or  $\bar{B}$  cannot be lifted, let us illustrate the previous statement with an example.

**Example 4.43.** Let  $G$  be the group defined by

$$G = \langle a_1, a_2, b, c \mid a_1^2, a_2^2, b^2, c^2, [a_1, c], [a_2, c] \rangle = (\mathbf{D}_\infty \times \mathbf{Z}_2) *_{\mathbf{Z}_2} \mathbf{D}_\infty.$$

acting on its Cayley graph  $X$ . In this description the elements  $a_1$ ,  $a_2$  and  $c$  (respectively  $b$  and  $c$ ) generate the factor  $\mathbf{D}_\infty \times \mathbf{Z}_2$  (respectively  $\mathbf{D}_\infty$ ). In particular the amalgamated subgroup is  $\mathbf{Z}_2 = \langle c \rangle$ . Set  $s = a_1a_2$  and  $r = bc$ . If  $n$  is a sufficiently large integer the group  $\bar{G} = G / \langle\langle s^n, r^n \rangle\rangle$  is a small cancellation quotient of  $G$ .

Assume in addition that  $n$  is even. It follows that  $\bar{u} = \bar{r}^{n/2}$  commutes with  $\bar{c}$ . Consequently the subgroup  $\bar{E}$  generated by  $\bar{a}_1$ ,  $\bar{u}^{-1}\bar{a}_1\bar{u}$  and  $\bar{c}$  is loxodromic and isomorphic to  $\mathbf{D}_\infty \times \mathbf{Z}_2$ . Note that  $\bar{E}$  cannot be lifted in  $G$ . Indeed since  $\mathbf{Z}_2 = \langle c \rangle$  is malnormal in  $\mathbf{D}_\infty = \langle b, c \rangle$  every loxodromic subgroup of  $G$  is isomorphic to either  $\mathbf{Z}$  or  $\mathbf{D}_\infty$ . Observe that  $\bar{E}$  also splits as  $\bar{E} = \bar{A} *_{\bar{C}} \bar{B}$  where  $\bar{A} = \langle \bar{a}_1, \bar{c} \rangle$ ,  $\bar{B} = \langle \bar{u}^{-1}\bar{a}_1\bar{u}, \bar{c} \rangle$ , and  $\bar{C} = \langle \bar{c} \rangle$ . As  $\bar{u}$  commutes with  $\bar{b}$ , the group  $\bar{B}$  is actually  $\bar{B} = \bar{u}^{-1}\bar{A}\bar{u}$ . We are in a configuration where both  $\bar{A}$  and  $\bar{B}$  are elliptic subgroups which can be lifted.

As described in [Proposition 4.42](#), a partial conjugation by  $\bar{u}$  maps  $\bar{E}$  to a new elementary subgroup  $\bar{E}_u = \bar{A}$  which is not necessarily loxodromic. In this



precise example, it turns out that  $\bar{E}_u$  can be lifted in  $G$ . This is not always the case though. Indeed we can run the same construction with  $\bar{E}' = \langle \bar{a}_1, \bar{u}^{-1}\bar{a}_2\bar{u}, \bar{c} \rangle$ . It is a loxodromic subgroup of  $\bar{G}$  isomorphic to  $\mathbf{D}_\infty \times \mathbf{Z}_2$ . On the other hands it splits as  $\bar{E}' = \bar{A}_{\bar{C}}\bar{B}'$  where  $\bar{B}' = \langle \bar{u}^{-1}\bar{a}_2\bar{u}, \bar{c} \rangle$ . In this case  $\bar{E}'_u = \langle \bar{a}_1, \bar{a}_2, \bar{c} \rangle$  is an elliptic subgroup of  $\bar{G}$ , which is isomorphic to  $\mathbf{D}_n \times \mathbf{Z}_2$ , and thus *cannot* be lifted. Nevertheless, in both cases,  $\bar{E}_u$  or  $\bar{E}'_u$  is the image (by the natural quotient map) of an elementary subgroup of  $G$ .

**Proposition 4.44.** *Let  $\bar{E}$  be a loxodromic subgroup of  $\bar{G}$  that splits as  $\bar{E} = \bar{A} *_{\bar{C}} \bar{B}$ , where  $\bar{C}$  is the maximal elliptic normal subgroup of  $\bar{E}$  and has index 2 in both  $\bar{A}$  and  $\bar{B}$ . Assume that there exists  $\bar{v} \in \bar{V}$  such that  $\bar{A}$  contains a strict rotation at  $\bar{v}$ . Let  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  be the associated geometric realization map and  $\mathbf{r} \in \mathbf{D}_n$  a generator of the rotation group. If  $n$  is divisible by 4, then one of the following holds*

(i)  $q_{\bar{v}}(\bar{C})$  is trivial and

$$q_{\bar{v}}(\bar{A}) = \langle \mathbf{r}^{n/2} \rangle$$

(ii)  $q_{\bar{v}}(\bar{C})$  is a reflection group generated by say  $\mathbf{x} \in \mathbf{D}_n$  and

$$q_{\bar{v}}(\bar{A}) = \langle \mathbf{x}, \mathbf{r}^{n/4}\mathbf{x}\mathbf{r}^{-n/4} \rangle$$

Suppose now that there exists a subgroup  $\bar{E}_0 \subset \bar{A}$  and an element  $\bar{h} \in \bar{G}$  such that  $\bar{h}\bar{E}_0\bar{h}^{-1}$  is contained in  $\bar{A}$  and  $\bar{A} = \langle \bar{E}_0, \bar{C} \rangle$ . Then either  $\bar{E}_0$  contains a strict rotation, in which case  $\bar{h}$  fixes  $\bar{v}$  or the first case above fails and  $q_{\bar{v}}$  maps  $\bar{E}_0$  onto  $\langle \mathbf{r}^{n/4}\mathbf{x}\mathbf{r}^{-n/4} \rangle$

*Remark.* Our assumption on  $\bar{A}$  exactly means that  $\bar{A}$  cannot be lifted (Proposition 4.29). It follows from Remark 4.14 that  $\bar{A}$  is contained in  $\text{Stab}(\bar{v})$ , hence the image under  $q_{\bar{v}}$  of  $\bar{C}$  or  $\bar{A}$  is well defined. It is important to note that in the second part of the statement  $\bar{h}$  is not necessarily an element of  $\text{Stab}(\bar{v})$ . In particular if  $\bar{h}$  does not fix  $\bar{v}$ , the last conclusion tells us that  $\bar{C}$  and  $\bar{E}_0$  are two reflection group at  $\bar{v}$ ; geometrically we can think that one is the conjugate of the other by a quarter-turn.

*Proof.* The first part of the proof is essentially a variation on Lemma 4.35. Since the cylinder of  $\bar{E}$  is contained in  $\text{Mov}(\bar{C}, 88\delta)$  (Lemma 2.15) the subgroup  $\bar{C}$  can be lifted and thus does not contain a strict rotation (Proposition 4.29). In particular,  $\bar{C}$  is either almost trivial at  $\bar{v}$  or a reflection group at  $\bar{v}$ . Assume first that  $q_{\bar{v}}$  maps  $\bar{C}$  to the trivial group. By assumption  $\bar{A}$  contains a strict rotation at  $\bar{v}$  whereas  $[\bar{A} : \bar{C}] = 2$ , which forces

$$q_{\bar{v}}(\bar{A}) = \langle \mathbf{r}^{n/2} \rangle.$$

Assume now that  $q_{\bar{v}}$  maps  $\bar{C}$  to a reflection group generated by say  $\mathbf{x} \in \mathbf{D}_n$ . Reasoning as above we observe that

$$q_{\bar{v}}(\bar{A}) = \langle \mathbf{x}, \mathbf{r}^{n/2} \rangle = \langle \mathbf{x}, \mathbf{r}^{n/4}\mathbf{x}\mathbf{r}^{-n/4} \rangle$$

This completes the first part of the statement.

Let us focus now on the second half of the proposition. Suppose first that  $\bar{E}_0$  contains a strict rotation (which is necessarily at  $\bar{v}$ ). Then  $\bar{h}\bar{E}_0\bar{h}^{-1}$  contains a strict rotation at  $\bar{h}\bar{v}$ . However as a subgroup of  $\bar{A}$ , it fixes  $\bar{v}$ . Strict rotations having a single fixed vertex (Proposition 4.12) it yields  $\bar{h}\bar{v} = \bar{v}$ . Suppose now that  $\bar{E}_0$  does not contain a strict rotation. In particular  $q_{\bar{v}}$  maps both  $\bar{E}_0$  and  $\bar{C}$  to subgroups of  $\mathbf{D}_n$  which are trivial or reflection groups. Since  $\bar{E}_0$  and  $\bar{C}$  generates  $\bar{A}$ , the only possible configuration for  $q_{\bar{v}}(\bar{A})$  to contain a non-trivial rotation is the one where  $q_{\bar{v}}(\bar{E}_0)$  and  $q_{\bar{v}}(\bar{C})$  are two *distinct* reflection groups. Consequently the first case above fails, and  $q_{\bar{v}}(\bar{E}_0) = \langle \mathbf{r}^{n/4}\mathbf{xr}^{-n/4} \rangle$ , see Figure 8  $\square$

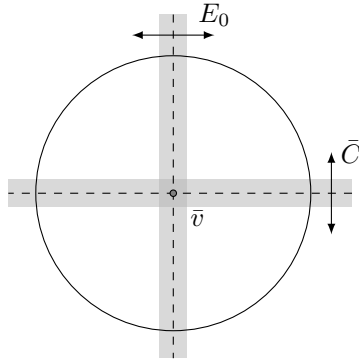


Figure 8: The action of  $\bar{E}$  on  $B(\bar{v}, \rho)$ . One assumes here that  $\bar{E}_0$  does not contain a strict rotation. The shaded areas represent  $\text{Mov}(\bar{C}, \bar{\delta})$  and  $\text{Mov}(\bar{E}_0, \bar{\delta})$  respectively.

## 4.7 Invariants of $\bar{G}$ acting on $\bar{X}$

This section is devoted to the study of the numerical invariants associated to the action of  $\bar{G}$  on  $\bar{X}$ , namely  $\text{inj}(\bar{G}, \bar{X})$ ,  $\Delta(\bar{G}, \bar{X})$  and  $\nu_{\text{stg}}(\bar{G}, \bar{X})$  (see Section 3 for the definitions). As we explained earlier, the first two are purely geometric, whereas the last one has a mixed nature and captures both geometric and algebraic features of  $\bar{G}$ .

### 4.7.1 Geometric invariants

**The injectivity radius.**

**Proposition 4.45.** *Let  $N$  be a subgroup of  $G$  containing  $K$  and  $\bar{N}$  its image in  $\bar{G}$ . We denote by  $\ell$  the infimum over the stable translation length (in  $X$ ) of loxodromic elements of  $N$  which do not belong to  $\text{Stab}(Y)$  for some  $(H, Y) \in \mathcal{Q}$ . Then*

$$\text{inj}(\bar{N}, \bar{X}) \geq \min\{\kappa\ell, \bar{\delta}\},$$

where  $\kappa = \bar{\delta}/\pi \operatorname{sh}(25\bar{\delta})$ .

*Remark.* By convention, if  $\bar{N}$  does not contain any loxodromic element then  $\operatorname{inj}(\bar{N}, \bar{X})$  is infinite, in which case the statement is void.

*Proof.* Let  $\bar{g}$  be a loxodromic element of  $\bar{N}$ . We need to show that  $\|\bar{g}\|^\infty \geq \min\{\kappa\bar{\ell}, \bar{\delta}\}$ . By (10) we have  $m\|\bar{g}\|^\infty \geq \|\bar{g}^m\| - 16\bar{\delta}$ , for every  $m \in \mathbf{N}$ . Therefore it suffices to find an integer  $m$  such that

$$\|\bar{g}^m\| \geq m \min\{\kappa\bar{\ell}, \bar{\delta}\} + 16\bar{\delta}.$$

We denote by  $m$  the largest integer satisfying  $m \min\{\kappa\bar{\ell}, \bar{\delta}\} \leq 2\bar{\delta}$ . Assume that  $\|\bar{g}^m\|$  is smaller than  $m \min\{\kappa\bar{\ell}, \bar{\delta}\} + 16\bar{\delta}$ . In particular,  $\|\bar{g}^m\| \leq 18\bar{\delta}$ . Thus  $\operatorname{Mov}(\bar{g}^m, 20\bar{\delta})$  is non empty. Moreover  $d(\bar{x}, \bar{\mathcal{V}}) \geq \rho - 10\bar{\delta}$ , for every  $\bar{x} \in \operatorname{Mov}(\bar{g}^m, 20\bar{\delta})$ . Indeed if it was not the case,  $\bar{g}^m$  would fix an apex  $\bar{v} \in \bar{\mathcal{V}}$  which contradicts the fact that  $\bar{g}$  is loxodromic. Hence  $\bar{Z} = \operatorname{Mov}(\bar{g}^m, 50\bar{\delta})$  contains a point in  $\zeta(X)$ . Note also that  $\bar{Z}$  is a  $10\bar{\delta}$ -quasi-convex subset lying in the  $22\bar{\delta}$ -neighborhood of  $\operatorname{Mov}(\bar{g}^m, 20\bar{\delta})$  (Lemma 2.9). Thus  $\langle \bar{z}, \bar{z}' \rangle_{\bar{v}} > \rho/2$ , for every  $\bar{z}, \bar{z}' \in \bar{Z}$  and  $\bar{v} \in \bar{\mathcal{V}}$ . By Lemma 4.17, there exists a subset  $Z$  of  $\bar{X}$  such that the map  $\zeta : \bar{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$  and the projection  $\pi : G \rightarrow \bar{G}$  induces an isomorphism from  $\operatorname{Stab}(Z)$  onto  $\operatorname{Stab}(\bar{Z})$ . Observe that  $\bar{g}$  preserves  $\bar{Z}$ . We denote by  $g$  the preimage of  $\bar{g}$  in  $\operatorname{Stab}(Z)$ . Since the kernel  $K$  is contained in  $N$ , the element  $g$  belongs to  $N$ . By construction  $g$  is a loxodromic element which does not belong to any  $\operatorname{Stab}(Y)$  where  $(H, Y) \in \mathcal{Q}$ . Hence  $\|g\|^\infty \geq \ell$ . As we noticed before  $\bar{Z}$  contains a point in  $\bar{z} \in \zeta(X)$ . Let  $z \in X$  be its pre-image in  $Z$ . It follows from Lemma 4.5, that

$$\mu(|g^m z - z|_X) \leq |g^m z - z|_{\bar{X}} \leq |\bar{g}^m \bar{z} - \bar{z}|_{\bar{X}} \leq 50\bar{\delta} < 2\rho.$$

By Proposition 4.3,

$$m\ell \leq m\|g\|^\infty \leq |g^m z - z|_X \leq \pi \operatorname{sh}(25\bar{\delta}) = \kappa^{-1}\bar{\delta},$$

which contradicts the maximality of  $m$ . □

### Acylicity.

**Proposition 4.46.** *The acylicity parameter for the action of  $\bar{G}$  on  $\bar{X}$  satisfies*

$$A(\bar{G}, \bar{X}) \leq A(G, X) + [\nu(G, X) + 4] \pi \operatorname{sh}(4L_0\bar{\delta}).$$

*Proof.* Let  $\bar{S}$  be a subset of  $\bar{G}$  generating a non-elementary subgroup. The goal is to bound from above the diameter of  $\bar{Z}_0 = \operatorname{Mov}(\bar{S}, 2L_0\bar{\delta})$ . Without loss of generality we can assume that this set is non-empty. Observe that  $d(\bar{x}, \bar{\mathcal{V}}) \geq \rho - L_0\bar{\delta}$  for every  $\bar{x} \in \bar{Z}_0$ . Indeed if it was not the case, every element of  $\bar{S}$  would fix a common apex  $\bar{v} \in \bar{\mathcal{V}}$ , contradicting the fact that  $\bar{S}$  generate a non-elementary subgroup. We denote by  $\bar{Z}$  the  $2L_0\bar{\delta}$ -neighborhood of  $\bar{Z}_0$ . It is a  $2\bar{\delta}$ -quasi-convex. Hence  $\langle \bar{z}, \bar{z}' \rangle_{\bar{v}} > \rho/2$ , for every  $\bar{z}, \bar{z}' \in \bar{Z}$  and  $\bar{v} \in \bar{\mathcal{V}}$ .

According to [Lemma 4.17](#) there exists a subset  $Z$  of  $\dot{X}$  such that the projection  $\zeta: \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$ . Moreover  $\langle z, z' \rangle_v > 4L_0\bar{\delta}$ , for every  $z, z' \in Z$  and  $v \in \mathcal{V}$  (the Gromov product is computed here in  $\dot{X}$ ). We write  $Z_0$  for the pre-image of  $\bar{Z}_0$  in  $Z$ . Let  $\bar{g} \in \bar{S}$ . By construction  $\bar{g}\bar{z}$  belongs to  $\bar{Z}$  for every  $\bar{z} \in \bar{Z}_0$ . Consequently there exists a (unique)  $g \in G$  such that for every  $z \in Z_0$  we have  $|gz - z|_{\dot{X}} = |\bar{g}\bar{z} - \bar{z}|$  ([Lemma 4.16](#)). We denote by  $S$  the set of all  $g \in G$  obtained in this way. Note that  $S$  does not generate an elementary subgroup, otherwise so would  $\bar{S}$ . Let  $\bar{z} \in \bar{Z}_0$  and  $z \in \dot{X}$  its pre-image in  $Z_0$ . Let  $y = p(z)$  be the radial projection of  $z$ . Since  $\bar{Z}$  lies in the  $3L_0\bar{\delta}$ -neighborhood of  $\zeta(\dot{X})$  we have  $|z - y|_{\dot{X}} \leq 3L_0\bar{\delta}$ . Combining the triangle inequality with [Lemma 4.5](#), we get for every  $g \in S$ ,

$$\mu(|gy - y|) \leq |gy - y|_{\dot{X}} \leq |gz - z|_{\dot{X}} + 6L_0\bar{\delta} \leq |\bar{g}\bar{z} - \bar{z}| + 6L_0\bar{\delta} \leq 8L_0\bar{\delta} < 2\rho.$$

By [Proposition 4.3](#), we get that  $y$  lies in  $\text{Mov}(S, d)$  where  $d = \pi \text{sh}(4L_0\bar{\delta})$ . Assume that  $\bar{z}'$  is another point in  $\bar{Z}_0$ . As previously we denote by  $z' \in \dot{X}$  its pre-image in  $Z_0$  and by  $y' = p(z')$  the radial projection of  $z'$ . In particular,  $y'$  also belongs to  $\text{Mov}(S, d)$  and  $|z' - y'|_{\dot{X}} \leq 3L_0\bar{\delta}$ . It follows from the triangle inequality that

$$|\bar{z} - \bar{z}'| \leq |z - z'|_{\dot{X}} \leq |y - y'| + 6L_0\bar{\delta} \leq \text{diam}(\text{Mov}(S, d)) + 6L_0\bar{\delta}.$$

This inequality holds for every  $\bar{z}, \bar{z}' \in \text{Mov}(\bar{S}, 2L_0\bar{\delta})$ . Hence

$$\text{diam}(\text{Mov}(\bar{S}, 2L_0\bar{\delta})) \leq \text{diam}(\text{Mov}(S, d)) + 6L_0\bar{\delta}.$$

The conclusion now follows from [Corollary 3.5](#).  $\square$

#### 4.7.2 Mixed invariants

In view of [Proposition 4.46](#) the  $\nu$ -invariant of  $G$  can be used to control the acylindricity invariant  $A(\bar{G}, \bar{X})$  of the quotient  $\bar{G}$ . If we want to iterate the procedure we need to control as well the  $\nu$ -invariant of  $\bar{G}$ . Let us start with an informal discussion to emphasize the difficulties that may arise along the way. For simplicity let us assume that  $G$  (hence  $\bar{G}$ ) does not contain any parabolic subgroup. Indeed those subgroups will not be a source of trouble.

If  $\bar{G}$  has no even torsion, one can prove using only *geometrical* arguments the following dichotomy – see for instance the proof of [[10](#), Proposition 5.28]. Given any chain  $\bar{\mathcal{C}} = (\bar{g}_0, \dots, \bar{g}_m)$  generating an elementary subgroup of  $\bar{G}$  and  $\bar{h} \in \bar{G}$  a loxodromic conjugating element of  $\bar{\mathcal{C}}$ ,

- (i) either  $\langle \bar{g}_0, \bar{h} \rangle$  is elementary,
- (ii) or the chain  $\bar{\mathcal{C}}$  can be lifted to a chain  $\mathcal{C} = (g_0, \dots, g_m)$  in  $G$  which generate an elementary subgroup of  $G$  and where (one of) the conjugating element  $h$  of  $\mathcal{C}$  is a pre-image of  $\bar{h}$ .

In the latter case,  $h$  is necessarily loxodromic. If in addition  $m \geq \nu(G, X)$ , we can conclude that  $\langle g_0, h \rangle$  and thus  $\langle \bar{g}_0, \bar{h} \rangle$  is elementary. It follows then that  $\nu(\bar{G}, \bar{X}) \leq \nu(G, X)$ .

Unfortunately this strategy fails in the presence of even torsion. In [Proposition 4.36](#) we observed the following phenomenon. Let  $F$  be an elliptic subgroup of  $G$  and  $\bar{F}$  its image in  $\bar{G}$ . If  $\bar{h}$  is an element of  $\bar{G}$  normalizing  $\bar{F}$ , then there exists an element  $h_0 \in G$  normalizing  $F$  whose action by conjugation on  $F$  coincide with the one of  $\bar{h}$  on  $\bar{F}$ . However  $h_0$  is not necessarily a pre-image of  $\bar{h}$ . In particular if  $\bar{h}$  is loxodromic, there is no reason that  $h_0$  should be loxodromic as well. The same issue arises when lifting chain. If a chain in  $\bar{G}$  admits a loxodromic conjugating element, there is not reason that its lift in  $G$  (provided it exists) has a loxodromic conjugating element. This motivates the definition of the strong  $\nu$ -invariant (see [Definition 3.10](#)). However this is not the only obstruction. As illustrated by the next example, the above dichotomy may fail as well.

**Example 4.47.** Start with the hyperbolic group

$$G = \langle a, b, c \mid a^2, b^2, c^2, [a, c] = 1 \rangle = \mathbf{D}_\infty *_{\mathbf{Z}_2} (\mathbf{Z}_2 \times \mathbf{Z}_2),$$

acting on its Cayley graph  $X$ . In this description  $a$  and  $b$  (respectively  $a$  and  $c$ ) generate the factor  $\mathbf{D}_\infty$  (respectively  $\mathbf{Z}_2 \times \mathbf{Z}_2$ ). The amalgamated group is  $\mathbf{Z}_2 = \langle a \rangle$ . Set  $r = ab$ . Let  $n \in \mathbf{N}$  be large integer divisible by 4, so that the group

$$\bar{G} = G / \langle\langle r^n \rangle\rangle.$$

is a small cancellation quotient of  $G$ . It follows from the additional relations that  $\bar{a}$  and  $\bar{b}$  generate a copy of  $\mathbf{D}_n$ . In particular,  $\bar{r}^{-n/4} \bar{a} \bar{r}^{n/4}$  is an involution which commutes with  $\bar{a}$ , so that the subgroup

$$\bar{E} = \langle \bar{a}, \bar{r}^{-n/4} \bar{a} \bar{r}^{n/4}, \bar{c} \rangle = \mathbf{Z}_2 \times \mathbf{D}_\infty$$

is loxodromic. We denote by  $\bar{t} = (\bar{r}^{-n/4} \bar{a} \bar{r}^{n/4}) \bar{c}$  the translation in the dihedral factor of  $\bar{E}$ . Finally set

$$\bar{g}_0 = \bar{r}^{-n/4} \bar{a} \bar{r}^{n/4}, \quad \bar{g}_1 = \bar{a}, \quad \text{and} \quad \bar{g}_2 = \bar{t} \bar{g}_0 \bar{t}^{-1}$$

Note that  $\langle \bar{g}_0, \bar{g}_1, \bar{g}_2 \rangle$  is a subgroup of  $\bar{E}$  which is also isomorphic to  $\mathbf{Z}_2 \times \mathbf{D}_\infty$  (see [Figure 9](#)). As  $\bar{t}$  commutes with  $\bar{g}_1$ , we observe that

$$\bar{g}_1 = \bar{h} \bar{g}_0 \bar{h}^{-1} \quad \text{and} \quad \bar{g}_2 = \bar{h} \bar{g}_1 \bar{h}^{-1}, \quad \text{where} \quad \bar{h} = \bar{t} \bar{r}^{n/4}.$$

Hence  $(\bar{g}_0, \bar{g}_1, \bar{g}_2)$  is a chain which generates an elementary subgroup of  $\bar{G}$ . The reader can check that  $\langle \bar{g}_0, \bar{h} \rangle$  is not an elementary subgroup of  $\bar{G}$ . It cannot either be lifted to a *chain* which generates an elementary subgroup of  $G$ .

In the previous example, the difficulty comes from the fact that the subgroup  $\langle \bar{g}_0, \bar{g}_1, \bar{g}_2 \rangle$  generated by the chain is a loxodromic subgroup of  $\bar{G}$  that cannot be lifted. Note that  $\langle \bar{g}_0, \bar{g}_1 \rangle$  is an elliptic subgroup that cannot be lifted either. As described in [Proposition 4.44](#),  $\bar{g}_1$  is obtained from  $\bar{g}_0$  by conjugation by a quarter turn.

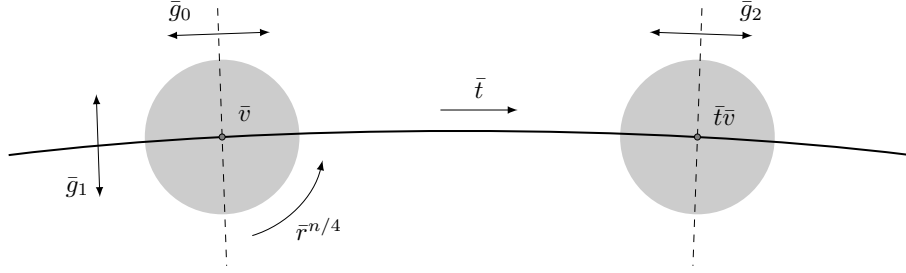


Figure 9: The action of  $\bar{E}$  and  $\bar{r}^{n/4}$  on the space  $\bar{X}$ . The shaded discs respectively represent  $B(\bar{v}, \rho)$  and  $B(\bar{t}\bar{v}, \rho)$  where  $\bar{v}$  is the cone point associated to the relation  $\bar{r}^n = 1$ .

This discussion suggests that the conjugation by a quarter turn rotation plays an important role. This is the place where *algebra* enters the stage. Take  $\bar{v} \in \bar{\mathcal{V}}$ . Let  $\bar{F}$  be the kernel of the geometric realization map  $q_{\bar{v}} \rightarrow \mathbf{D}_n$ . Assume as in our example that  $n$  is divisible by 4. Let  $\bar{r}$  be strict rotation at  $\bar{v}$ . From a geometric point of view, conjugating  $\bar{F}$  by  $\bar{r}$  is not a trackable operation. Indeed this will send an element  $\bar{g}$  which is almost trivial at  $\bar{v}$  to  $\bar{r}\bar{g}\bar{r}^{-1}$  which is still almost trivial at  $\bar{v}$ . Hence one cannot distinguish  $\bar{g}$  and  $\bar{r}\bar{g}\bar{r}^{-1}$  from their action on  $B(\bar{v}, \rho)$ . This does not mean that  $\bar{g}$  and  $\bar{r}$  commutes though. Nevertheless if we had a better understanding of the *algebraic structure* of  $\text{Stab}(\bar{v})$ , more precisely if we knew that  $\text{Stab}(\bar{v})$  is essentially a subproduct of dihedral groups, it could be possible to find a suitable quarter turn at  $\bar{v}$  which truly commutes with a prescribed subset of  $\bar{F}$  (see [Lemma 3.14](#)).

With additional algebraic hypotheses we are actually able to prove that if  $m$  is sufficiently large, any chain  $\bar{C} = (\bar{g}_0, \dots, \bar{g}_m)$  will satisfy (a variation of) the above dichotomy, which salvages the original strategy.

**Additional assumption.** We now begin a systematic study of the mixed invariants. As we pointed out above, we require some additional hypothesis on the algebraic structure of elementary subgroup.

We fix once for all an integer  $n$  and write  $n_2$  for the largest power of 2 dividing  $n$ . We also choose a model collection  $\mathcal{E}$ , i.e. a family of (abstract) torsion groups and assume that its exponent  $\mu = \mu(\mathcal{E})$  divides  $n$ . We suppose in addition that for every  $\mathbf{E} \in \mathcal{E}$ , the exponent of  $\mathbf{E}/Z(\mathbf{E})$  divides  $n/2$ , where  $Z(\mathbf{E})$  stands for the center of  $\mathbf{E}$ . From now on, we substitute [Assumption 4.11](#) on vertex stabilizer for the following stronger hypotheses. Recall that a dihedral pair  $(E, C)$  has type  $(\mathcal{E}, n_2)$  if there exist  $k \in \mathbf{N}$ , and a morphism  $\varphi: E \rightarrow \mathbf{E}$ , where  $\mathbf{E} \in \mathcal{E}$  such that  $\varphi$  extends to an embedding from  $E$  into  $E/C \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  (see [Definition 3.12](#)).

**Assumption 4.48** (Structure of elementary subgroups). *Every dihedral pair of  $G$  has type  $(\mathcal{E}, n_2)$ .*

**Assumption 4.49** (Relations). *For every  $(H, Y) \in \mathcal{Q}$ , there exists a primitive element  $g \in G$  such that  $H = \langle g^n \rangle$ .*

Let us now mention a few consequences of these new assumptions.

**Lemma 4.50.** *Every elliptic subgroup of a loxodromic subgroup of  $G$  is a dihedral germ.*

*Proof.* Let  $E$  be a loxodromic subgroup of  $G$  and  $F$  its maximal elliptic normal subgroup. Let  $g \in E$  be a loxodromic element. In particular the target of any marking of  $(E, F)$  is  $\mathbf{D}_\infty$ . Moreover such a marking maps  $g$  to an element of the rotation group  $\mathbf{Z} \subset \mathbf{D}_\infty$ . It follows from [Assumption 4.48](#) that some large power of  $g$  centralizes  $F$ . We conclude as in [Lemma 4.31](#).  $\square$

**Lemma 4.51.** *Let  $\bar{v} \in \bar{\mathcal{V}}$ . The cone point  $\bar{v}$  has order  $n$ . The group  $\text{Stab}(\bar{v})$  contains a central half-turn at  $\bar{v}$ . There exist  $k \in \mathbf{N}$  and a morphism  $\bar{\varphi}: \text{Stab}(\bar{v}) \rightarrow \mathbf{E}$ , where  $\mathbf{E} \in \mathcal{E}$  such that the geometric realization  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  together with  $\bar{\varphi}$  extend to an embedding from  $\text{Stab}(\bar{v})$  into  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ .*

*Remark.* In particular [Assumption 4.11](#) holds.

*Proof.* Fix  $(H, Y) \in \mathcal{Q}$  such that  $\pi: G \rightarrow \bar{G}$  maps  $E = \text{Stab}(Y)$  onto  $\text{Stab}(\bar{v})$ . Let  $F$  be the maximal elliptic normal subgroup of  $E$ . According to [Assumption 4.48](#), there exist an integer  $k \in \mathbf{N}$ , and a morphism  $\varphi: E \rightarrow \mathbf{E}$ , where  $\mathbf{E} \in \mathcal{E}$ , such that  $\varphi$  extends to an embedding from  $E$  into  $E/F \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ .

We write  $\alpha: E \rightarrow \mathbf{D}_{n_2}^k \times \mathbf{E}$  for the map obtained by composing the embedding  $E \hookrightarrow \mathbf{D}_\infty \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  with the natural projection onto  $\mathbf{D}_{n_2}^k \times \mathbf{E}$ . Note that  $\alpha$  is one-to-one when restricted to  $F$ . Recall that  $E/F$  is either  $\mathbf{Z}$  or  $\mathbf{D}_\infty$ . We denote by  $\mathfrak{t}$  a generator of the maximal infinite cyclic subgroup of  $E/F$ . By [Assumption 4.49](#), there exists a pre-image  $g \in E$  of  $\mathfrak{t}$  such that  $H = \langle g^n \rangle$ . In particular,  $n$  is the order of  $\bar{v}$ . By assumption, the exponent of  $\mathbf{D}_{n_2}^k \times \mathbf{E}$  divides  $n$ . Hence  $\alpha(H)$  is trivial. It follows that  $\alpha$  induces a map  $\bar{\alpha}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_{n_2}^k \times \mathbf{E}$  whose restriction to  $\bar{F}$  (the image of  $F$  in  $\bar{G}$ ) is an embedding. Hence the morphism  $\text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  given by  $q_{\bar{v}}$  and  $\bar{\alpha}$  is an embedding.

We are left to prove existence of a central half turn. To that end we claim that for every  $u \in E$ ,

$$g^{n/2} u g^{-n/2} \in uH. \quad (21)$$

Let  $u$  in  $E$ . Recall that the exponents of  $\mathbf{E}$  and  $\mathbf{E}/Z(\mathbf{E})$  respectively divide  $n$  and  $n/2$ , hence

$$\mathbf{g}^{n/2} \mathbf{u} \mathbf{g}^{-n/2} = \mathbf{u} = \mathbf{u} \mathbf{g}^{-n}, \quad \text{where } \mathbf{g} = \varphi(g) \text{ and } \mathbf{u} = \varphi(u).$$

The same identities are also satisfied by the images of  $g$  and  $u$  in any factor  $\mathbf{D}_{n_2}$ . Assume now that  $u$  belongs to  $E^+$  (the maximal subgroup of  $E$  preserving the orientation). Then  $g^{n/2}$  commutes with  $u$  (one checks indeed that in each factor of  $E/F \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ , we have  $g^{n/2} u g^{-n/2} = u$ ). Assume now that  $u \in E \setminus E^+$ . In other words the image of  $u$  in  $E/F$  is a reflection. We similarly check that  $g^{n/2} u g^{-n/2} = u g^n$ , which completes the proof of our claim. As we already

observed  $\alpha(g^n) = 1$ , hence the image of  $g^{n/2}$  in  $\text{Stab}(\bar{v})$  is a non trivial rotation of order 2 at  $\bar{v}$ . It follows from (21) that it is also central in  $\text{Stab}(\bar{v})$ , thus it is a central half-turn at  $\bar{v}$ .  $\square$

**Model collection for  $\bar{G}$ .** We now prove that the elementary subgroups of  $\bar{G}$  satisfy an condition similar to [Assumption 4.48](#).

**Proposition 4.52.** *Dihedral pairs of  $\bar{G}$  (for its action on  $\bar{X}$ ) have type  $(\mathcal{E}, n_2)$ .*

*Remark.* Note that the model collection  $\mathcal{E}$  is the same as the one of [Assumption 4.48](#).

*Proof.* Let  $(\bar{E}, \bar{C})$  be a dihedral pair. We distinguish several cases. Assume first that  $\bar{E}$  is either an elliptic subgroup that *can* be lifted in  $G$  or a parabolic subgroup. Note that in the latter case  $\bar{E}$  can still be lifted in  $G$ . Indeed  $\bar{E}$  is the extension of an elliptic subgroup, namely  $\bar{C}$ , by a finitely generated subgroup. Thus there exist  $d \in \mathbf{R}_+$  and a finite subset  $\bar{S}$  generated  $\bar{E}$  such that  $\text{Mov}(\bar{S}, d)$  is non-empty. It follows then from [Proposition 4.39](#) that  $\bar{E}$  can be lifted. Let  $E$  be a lift of  $\bar{E}$  and  $C$  the pre-image of  $\bar{C}$  in  $E$ . According to [Lemma 4.33](#),  $C$  is a dihedral germ. It follows that  $(E, C)$  is a dihedral pair. Hence the result follows from [Assumption 4.48](#) applied to  $(E, C)$ .

Assume now that  $\bar{E}$  is a loxodromic subgroup. In particular  $\bar{C}$  is its maximal elliptic normal subgroup ([Lemma 3.9](#)). According to [Proposition 4.41](#) there exists a dihedral pair  $(E', C)$  in  $G$  where  $C$  is an elliptic subgroup lifting  $\bar{C}$  and an epimorphism  $\theta: \bar{E} \twoheadrightarrow E'$  with the following properties.

- (i) The morphism  $\pi \circ \theta$  is the identity when restricted to  $\bar{C}$ .
- (ii) The map  $\theta$  induces an embedding from  $\bar{E}$  into  $\bar{E}/\bar{C} \times E'$ .

Our assumption applied to  $(E', C)$  says that there exist  $k \in \mathbf{N}$ , and a morphism  $\varphi: E' \rightarrow \mathbf{E}$  where  $\mathbf{E} \in \mathcal{E}$ , which extends to an embedding  $E' \hookrightarrow E'/C \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ . We claim that  $\varphi \circ \theta: \bar{E} \rightarrow \mathbf{E}$  extends to a embedding  $\bar{E} \rightarrow \bar{E}/\bar{C} \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ . For simplicity we write  $\psi: E' \rightarrow \mathbf{D}_{n_2}^k \times \mathbf{E}$  for the composition of  $E' \rightarrow E'/C \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  with the canonical projection onto  $\mathbf{D}_{n_2}^k \times \mathbf{E}$ . It suffices to show that  $\psi \circ \theta: \bar{E} \rightarrow \mathbf{D}_{n_2}^k \times \mathbf{E}$  induces an embedding from  $\bar{E}$  into  $\bar{E}/\bar{C} \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ . Consider an element  $\bar{g} \in \bar{E}$  which is trivial in  $\bar{E}/\bar{C} \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ . In particular  $\bar{g}$  belongs to  $\bar{C}$ , hence  $\theta(\bar{g}) \in C$ . Moreover  $\psi \circ \theta(\bar{g})$  is trivial. Since  $\psi$  extends to an embedding from  $E' \hookrightarrow E'/C \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ , the element  $\theta(\bar{g})$  is trivial. On the other hand  $\theta$  induces an embedding from  $\bar{E}$  into  $\bar{E}/\bar{C} \times E'$ , thus  $\bar{g} = 1$ , which completes the proof our claim.

We finally assume that  $\bar{E}$  is an elliptic subgroup that *cannot* be lifted in  $G$ . In particular there exists an apex  $\bar{v} \in \bar{V}$  such that  $\bar{E}$  is contained in  $\text{Stab}(\bar{v})$  ([Proposition 4.29](#)). According to [Lemma 4.51](#) there exists  $k \in \mathbf{N}$ , and a morphism  $\bar{\varphi}: \text{Stab}(\bar{v}) \rightarrow \mathbf{E}$ , where  $\mathbf{E} \in \mathcal{E}$ , which combined with the geometric realization  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  provides an embedding from  $\text{Stab}(\bar{v})$  into  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ . For simplicity we write  $\bar{\psi}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_{n_2}^k \times \mathbf{E}$  for the composition of  $\text{Stab}(\bar{v}) \hookrightarrow \mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  with the natural projection onto  $\mathbf{D}_{n_2}^k \times \mathbf{E}$ .



Composing the geometric realization  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  with the canonical projection  $\mathbf{D}_n \rightarrow \mathbf{D}_{n_2}$  leads to a morphism that we denote  $q'_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_{n_2}$ . We are going to prove that  $q'_{\bar{v}}$  and  $\bar{\psi}$  extend to an embedding from  $\bar{E}$  into  $\bar{E}/\bar{C} \times \mathbf{D}_{n_2} \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ .

Let  $\bar{F}$  be the kernel of  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$ . We first claim that  $[\bar{C} : \bar{C} \cap \bar{F}]$  is a power of 2. Since  $\bar{C}$  is a dihedral germ, it contains a subgroup  $\bar{C}_0$  which is normalized by a loxodromic element and such that  $[\bar{C} : \bar{C}_0] = 2^m$  for some  $m \in \mathbf{N}$ . Observe that  $\bar{C}_0$  is contained in a reflection group at  $\bar{v}$ . Indeed otherwise, any element normalizing  $\bar{C}_0$  would belong to  $\text{Stab}(\bar{v})$  thus they would be no *loxodromic* element centralizing  $\bar{C}_0$ . In particular  $[\bar{C}_0 : \bar{C}_0 \cap \bar{F}]$  is at most 2. On the other hand

$$[\bar{C} : \bar{C} \cap \bar{F}][\bar{C} \cap \bar{F} : \bar{C}_0 \cap \bar{F}] = [\bar{C} : \bar{C}_0 \cap \bar{F}] = [\bar{C} : \bar{C}_0][\bar{C}_0 : \bar{C}_0 \cap \bar{F}]$$

Consequently  $[\bar{C} : \bar{C} \cap \bar{F}]$  divides  $[\bar{C} : \bar{C}_0 \cap \bar{F}]$ . In particular it is a power of 2 which completes the proof of our claim.

Consider now  $\bar{g} \in \bar{E}$  whose image in  $\bar{E}/\bar{C} \times \mathbf{D}_{n_2} \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  is trivial. First observe that  $\bar{g}$  belongs to  $\bar{C}$ . It follows from the previous claim that the order of  $q_{\bar{v}}(\bar{g})$  is a power of 2. Nevertheless the kernel of the projection  $\mathbf{D}_n \rightarrow \mathbf{D}_{n_2}$ , which contains  $q_{\bar{v}}(\bar{g})$ , consists only of element with odd order. Therefore  $q_{\bar{v}}(\bar{g})$  is trivial, i.e.  $\bar{g}$  belongs to  $\bar{F}$ . Observe that the map  $\bar{\psi}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_{n_2}^k \times \mathbf{E}$  is an embedding when restricted to  $\bar{F}$ . Since  $\bar{\psi}(\bar{g}) = 1$ , the element  $\bar{g}$  is trivial. This shows that  $\bar{E}$  embeds in  $\bar{E}/\bar{C} \times \mathbf{D}_{n_2} \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ .  $\square$

**The strong  $\nu$ -invariant.** We now start our study of the strong  $\nu$ -invariant. The ultimate goal is to prove the following statement.

**Proposition 4.53.** *Assume that  $2^{\nu+2}\mu$  divides  $n$  where  $\nu = \nu_{\text{stg}}(G, X)$ . Then*

$$\nu_{\text{stg}}(\bar{G}, \bar{X}) \leq \max \{ \nu_{\text{stg}}(G, X), \mu + 4 \}$$

For simplicity we adopt the following terminology.

**Definition 4.54.** A chain  $\mathcal{C} = (g_0, \dots, g_m)$  of  $G$  is a *strong chain* if it satisfies the following holds

- (i)  $g_0, \dots, g_m$  generate an elementary subgroup of  $G$  (for its action on  $X$ ).
- (ii) either  $\mathcal{C}$  admits a loxodromic conjugating element or  $\langle g_0, \dots, g_{m-1} \rangle$  is a dihedral germ.

We define *strong chains* of  $\bar{G}$  in the exact same way. A strong chain  $\bar{\mathcal{C}} = (\bar{g}_0, \dots, \bar{g}_m)$  can be *lifted* if there exists a strong chain  $\mathcal{C} = (g_0, \dots, g_m)$  of  $G$  such that the quotient map  $\pi: G \rightarrow \bar{G}$  sends  $g_k$  to  $\bar{g}_k$  for every  $k \in \llbracket 0, m \rrbracket$ . In this situation we also says that  $\mathcal{C}$  *lifts*  $\bar{\mathcal{C}}$ . As suggested at the beginning of this section we first prove the following dichotomy.

**Proposition 4.55.** *Let  $m \geq 3$  such that  $2^{m+2}\mu$  divides  $n$ . Let  $\bar{\mathcal{C}} = (\bar{g}_0, \dots, \bar{g}_m)$  be a strong chain of  $\bar{G}$  and  $\bar{h}$  a conjugating element of  $\bar{\mathcal{C}}$ . Then one of the following holds.*

- (i) There exists  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\langle \bar{g}_0, \bar{h} \rangle$  is contained in  $\text{Stab}(\bar{v})$ .
- (ii) The subgroup  $\langle \bar{g}_0, \bar{h} \rangle$  is loxodromic.
- (iii) There exists a strong chain  $\bar{\mathcal{C}}' = (\bar{g}'_0, \dots, \bar{g}'_m)$  of  $\bar{G}$  which can be lifted and such that  $(\bar{g}'_1, \dots, \bar{g}'_{m-1}) = (\bar{g}_1, \dots, \bar{g}_{m-1})$ .

We split the proof into several lemmas depending on the nature of the group generated by  $\bar{\mathcal{C}}$ .

**Lemma 4.56.** *Let  $\bar{\mathcal{C}} = (\bar{g}_0, \dots, \bar{g}_m)$  be a strong chain of  $\bar{G}$ . If the subgroup  $\bar{E}$  of  $\bar{G}$  generated by  $\bar{\mathcal{C}}$  is either elliptic and can be lifted or parabolic, then  $\bar{\mathcal{C}}$  can be lifted.*

*Proof.* We first claim that  $\bar{E}$  can always be lifted even if  $\bar{E}$  is parabolic. Indeed since  $\bar{E}$  is finitely generated [Proposition 4.39](#) applies. Let  $E$  be a lift of  $\bar{E}$ . For every  $k \in \llbracket 0, m \rrbracket$ , we denote by  $g_k$  the pre-image of  $\bar{g}_k$  in  $E$ . Note that, contrary to  $(\bar{g}_0, \dots, \bar{g}_m)$ , the tuple  $(g_0, \dots, g_m)$  could not be a chain. Nevertheless, according to [Proposition 4.26](#) applied with  $S_1 = \{g_0, \dots, g_{m-1}\}$  and  $S_2 = \{g_1, \dots, g_m\}$ , there exists  $h_0 \in G$  with the following properties

- (i) for every  $k \in \llbracket 0, m-1 \rrbracket$ , we have  $g_{k+1} = h_0 g_k h_0^{-1}$ .
- (ii) if  $\bar{h}$  is loxodromic, then either  $h_0$  is loxodromic or  $\langle \bar{g}_0, \dots, \bar{g}_{\nu-1} \rangle$  is contained in a reflection group.

Thus  $\mathcal{C} = (g_0, \dots, g_\nu)$  is actually a chain and  $h_0$  a conjugating element of  $\mathcal{C}$ . Obviously  $\mathcal{C}$  generates an elementary subgroup of  $G$ . Note that either  $h_0$  is loxodromic or  $\langle g_0, \dots, g_{\nu-1} \rangle$  is a dihedral germ ([Lemmas 4.33](#) and [4.32](#)) hence the proof is complete.  $\square$

**Lemma 4.57.** *Let  $\bar{\mathcal{C}} = (\bar{g}_0, \dots, \bar{g}_m)$  be a strong chain of  $\bar{G}$  which generates an elliptic subgroup  $\bar{E}$  of  $\bar{G}$  which cannot be lifted. Let  $\bar{h}$  be a conjugating element of  $\bar{\mathcal{C}}$ . Then either  $\bar{E}_0 = \langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$  contains a strict rotation and  $\langle \bar{g}_0, \bar{h} \rangle$  is contained in  $\text{Stab}(\bar{v})$  for some  $\bar{v} \in \bar{\mathcal{V}}$ , or  $\bar{\mathcal{C}}$  can be lifted.*

*Proof.* There exists a (unique) apex  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\bar{E}$  is a subgroup of  $\text{Stab}(\bar{v})$  ([Proposition 4.29](#)). Let  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  be the canonical geometric realization map. Assume first that  $\bar{E}_0 = \langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$  contains a strict rotation at  $\bar{v}$ . Consequently  $\bar{h}\bar{E}_0\bar{h}^{-1}$  contains a strict rotation at  $\bar{h}\bar{v}$ . Since strict rotations fix a unique cone point we get  $\bar{h}\bar{v} = \bar{v}$ . Thus  $\langle \bar{g}_0, \bar{h} \rangle$  is contained in  $\text{Stab}(\bar{v})$ , hence is elementary.

Assume now that  $\bar{E}_0$  does not contain a strict rotation at  $\bar{v}$ . Since  $\bar{E}$  cannot be lifted,  $q_{\bar{v}}$  maps  $\bar{g}_0$  and  $\bar{g}_m$  to two distinct reflections, and  $\bar{g}_1, \dots, \bar{g}_{m-1}$  to the identity. We denote by  $\bar{C}$  the intersection of  $\bar{E}$  with the kernel of  $q_{\bar{v}}$  and let  $\bar{A} = \langle \bar{g}_0, \bar{C} \rangle$  and  $\bar{B} = \langle \bar{g}_m, \bar{C} \rangle$ . Note that  $\bar{C}$  is a subgroup of index 2 in both  $\bar{A}$  and  $\bar{B}$ . Let  $(H, Y) \in \mathcal{Q}$  such that the projection  $\pi: G \rightarrow \bar{G}$  maps  $\text{Stab}(Y)$  onto  $\text{Stab}(\bar{v})$ . We choose a lift  $A$  (respectively  $B$ ) of  $\bar{A}$  (respectively  $\bar{B}$ ) contained in  $\text{Stab}(Y)$  so that  $A \cap B$  is a lift of  $\bar{C}$  that we denote by  $C$

(see [Corollary 4.30](#)). Let  $a_0, \dots, a_{m-1}$  (respectively  $b_1, \dots, b_m$ ) the pre-images of  $\bar{g}_0, \dots, \bar{g}_{m-1}$  (respectively  $\bar{g}_1, \dots, \bar{g}_m$ ) in  $A$  (respectively  $B$ ). As we already observed  $\bar{g}_1, \dots, \bar{g}_{m-1}$  belong to  $\bar{C}$ , thus  $a_k = b_k$ , for all  $k \in \llbracket 1, m-1 \rrbracket$ . Applying [Proposition 4.26](#) with  $S_1 = A$  and  $S_2 = B$ , we get that there exists  $h_0 \in G$  such that for every  $k \in \llbracket 0, m-1 \rrbracket$  we have  $b_{k+1} = h_0 a_k h_0^{-1}$ . It follows that  $\mathcal{C} = (a_0, a_1, \dots, a_{\nu-1}, b_\nu)$  is a chain and  $h_0$  a conjugating element of  $\mathcal{C}$ . In addition this chain generates a subgroup of  $\text{Stab}(Y)$ , which is therefore elementary. Recall that  $\langle a_0, \dots, a_{\nu-1} \rangle$  is an elliptic subgroup of  $\text{Stab}(Y)$ . Hence it is a dihedral germ ([Lemma 4.31](#)). Consequently  $\mathcal{C}$  is a strong chain of  $G$  lifting  $\bar{\mathcal{C}}$ .  $\square$

**Lemma 4.58.** *Let  $\bar{\mathcal{C}} = (\bar{g}_0, \dots, \bar{g}_m)$  be a strong chain generating a loxodromic subgroup  $\bar{E}$  of  $\bar{G}$ . Let  $\bar{h}$  be a conjugating element of  $\bar{\mathcal{C}}$ . If  $2^{m+2}\mu$  divides  $n$  then one of the following holds*

- (i) *The subgroup  $\langle \bar{g}_0, \bar{h} \rangle$  is either loxodromic or contained in  $\text{Stab}(\bar{v})$  for some  $\bar{v} \in \bar{\mathcal{V}}$ .*
- (ii) *There exists a strong chain  $\bar{\mathcal{C}}' = (\bar{g}'_0, \dots, \bar{g}'_m)$  which can be lifted and such that  $(\bar{g}'_1, \dots, \bar{g}'_{m-1}) = (\bar{g}_1, \dots, \bar{g}_{m-1})$ .*

*Proof.* Assume first that  $\bar{E}_0 = \langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$  contains a loxodromic element say  $\bar{t}$ . Since  $\bar{E}_0$  and  $\bar{h}\bar{E}_0\bar{h}^{-1}$  generate an elementary subgroup, namely  $\bar{E}$ , both  $\bar{g}_0$  and  $\bar{h}$  are contained in the maximal elementary subgroup containing  $\bar{t}$ . In particular,  $\langle \bar{g}_0, \bar{h} \rangle$  is loxodromic.

Assume now that  $\bar{E}_0$  is elliptic (a subgroup of a loxodromic subgroup cannot be parabolic). Let  $\bar{C}$  be the maximal normal elliptic subgroup of  $\bar{E}$ . Since  $\bar{E}$  is generated by two elliptic subgroup (namely  $\bar{E}_0$  and  $\bar{h}\bar{E}_0\bar{h}^{-1}$ ) it does not preserve the orientation. Hence the quotient  $\bar{E}/\bar{C}$  is isomorphic to  $\mathbf{D}_\infty$ . We write

$$1 \rightarrow \bar{C} \rightarrow \bar{E} \xrightarrow{q} \mathbf{D}_\infty \rightarrow 1$$

for the corresponding short exact sequence. One observes that  $q$  maps  $\bar{g}_0$  and  $\bar{g}_m$  to two distinct reflections, while  $\bar{C}$  is the normal subgroup of  $\bar{E}$  generated by  $\bar{g}_0^2, \bar{g}_1, \dots, \bar{g}_{m-1}, \bar{g}_m^2$ . We let  $\bar{A} = \langle \bar{g}_0, \bar{C} \rangle$  and  $\bar{B} = \langle \bar{g}_m, \bar{C} \rangle$  so that  $\bar{E}$  is isomorphic to  $\bar{A} *_{\bar{C}} \bar{B}$ . We now distinguish two cases.

**Case 1.** *Assume first that both  $\bar{A}$  and  $\bar{B}$  can be lifted in  $G$ . We denote by  $A$  and  $B$  a lift of  $\bar{A}$  and  $\bar{B}$  respectively. According to [Proposition 4.42](#) there exists  $u \in G$  whose image  $\bar{u}$  in  $\bar{G}$  centralizes  $\bar{C}$  such that  $E_u = \langle A, uBu^{-1} \rangle$  is elementary. We let  $\bar{g}'_k = \bar{g}_k$ , for every  $k \in \llbracket 0, m-1 \rrbracket$  and  $\bar{g}'_m = \bar{u}\bar{g}_m\bar{u}^{-1}$ . Since  $\bar{u}$  centralizes  $\bar{C}$  we observe that  $\bar{\mathcal{C}}' = (\bar{g}'_0, \dots, \bar{g}'_m)$  is a chain and  $\bar{h}_0 = \bar{u}\bar{h}$  a conjugating element of  $\bar{\mathcal{C}}'$ . Note also that  $\bar{C}$  and  $\bar{\mathcal{C}}'$  only differ on the last element. We now focus on this new chain. Let  $a'_0, \dots, a'_{m-1}$  be the lift of  $\bar{g}'_0, \dots, \bar{g}'_{m-1}$  in  $A$  and  $b'_1, \dots, b'_m$  the lifts of  $\bar{g}'_1, \dots, \bar{g}'_m$  in  $uBu^{-1}$ . We now proceed exactly as in the proof of [Lemma 4.57](#). We first observe that  $\mathcal{C}' = (a'_0, a'_1, \dots, a'_{m-1}, b'_m)$  is a chain for some conjugating element  $h'_0 \in G$ . Moreover it generate a subgroup*

of  $E_u$  which is therefore elementary. Recall that  $\bar{E}_0 = \langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$  is an elliptic subgroup of the loxodromic subgroup  $\bar{E}$ , therefore it is a dihedral germ (Lemma 4.50). As a lift of  $\langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$ , the subgroup  $\langle a'_0, \dots, a'_{m-1} \rangle$  is a dihedral germ as well (Lemma 4.33). Hence  $\mathcal{C}'$  is a strong chain of  $G$  lifting  $\bar{\mathcal{C}}'$ .

**Case 2.** Assume that either  $\bar{A}$  or  $\bar{B}$  cannot be lifted in  $G$ . Up to replacing  $(\bar{g}_0, \dots, \bar{g}_m)$  by  $(\bar{g}_m, \dots, \bar{g}_0)$ , we can assume that  $\bar{A}$  cannot be lifted in  $G$ . In particular there exists a (unique) apex  $\bar{v} \in \bar{\mathcal{V}}$  such that  $\bar{A}$  contains a strict rotation at  $\bar{v}$  (Proposition 4.29). We write  $q_{\bar{v}}: \text{Stab}(\bar{v}) \rightarrow \mathbf{D}_n$  for the geometric realization map associated to  $\bar{v}$  and  $\mathbf{r} \in \mathbf{D}_n$  for a generator of the rotation group. Let  $\bar{E}_1 = \langle \bar{g}_1, \dots, \bar{g}_{m-2} \rangle$ . Observe that  $\bar{A}$  is generated by  $\bar{h}^{-1}\bar{E}_1\bar{h}$  and  $\bar{C}$ . Moreover  $\bar{E}_1$  is contained in  $\bar{A}$ . It follows from Proposition 4.44 that either  $\bar{E}_1$  contains a strict rotation, in which case  $\langle \bar{g}_0, \bar{h} \rangle$  is a subgroup of  $\text{Stab}(\bar{v})$ , or there exists a reflection  $\mathbf{x} \in \mathbf{D}_n$  such that

$$q_{\bar{v}}(\bar{C}) = \langle \mathbf{x} \rangle \quad \text{and} \quad q_{\bar{v}}(\bar{E}_1) = \langle \mathbf{x}' \rangle, \quad \text{where} \quad \mathbf{x}' = \mathbf{r}^{n/4}\mathbf{x}\mathbf{r}^{-n/4},$$

See Figure 10. Let us explore further this second configuration. Recall also

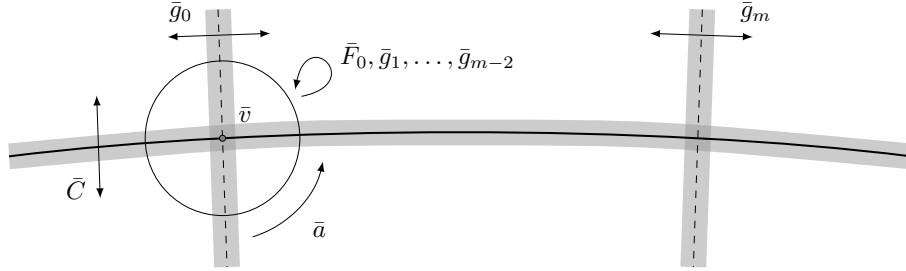


Figure 10: Lifting chains. The shared areas represent  $\text{Mov}(\bar{A}, 11\bar{\delta})$ ,  $\text{Mov}(\bar{B}, 11\bar{\delta})$ , and  $\text{Mov}(\bar{C}, 11\bar{\delta})$  respectively. The isometry  $\bar{g}_{m-1}$  is not on the picture. It is not clear a priori whether it is a reflection or trivial at  $\bar{v}$ .

that there exists  $k \in \mathbf{N}$ , and an abstract group  $\mathbf{E} \in \mathcal{E}$ , such that  $\text{Stab}(\bar{v})$  embeds in  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  (Lemma 4.51). Let  $\bar{F}_0$  be the normal subgroup of  $\text{Stab}(\bar{v})$  generated by  $\bar{g}_0^2, \bar{g}_1, \dots, \bar{g}_{m-1}$ . Seen as a subgroup of  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ , the reflection rank of  $\bar{F}_0$  is at most  $m-1$ . Recall that  $2^{m+2}\mu$  divides  $n$ . According to Lemma 3.14 there exists a pre-image  $\bar{a} \in \text{Stab}(\bar{v})$  of  $\mathbf{r}^{n/4}$  such that

- (i)  $\bar{a}$  centralizes  $\bar{F}_0$
- (ii)  $[[\bar{a}, \bar{u}_1], \bar{u}_2] = 1$ , for every  $\bar{u}_1, \bar{u}_2 \in \text{Stab}(\bar{v})$ .

We now let  $\bar{g}'_0 = \bar{a}\bar{g}_0\bar{a}^{-1}$  and  $\bar{g}'_k = \bar{g}_k$  for every  $k \in \llbracket 1, m \rrbracket$ . By construction  $\bar{a}$  commutes with  $\bar{g}_1, \dots, \bar{g}_{m-1}$ . Hence  $\bar{\mathcal{C}}' = (\bar{g}'_0, \dots, \bar{g}'_m)$  is a chain with  $\bar{h}_0 = \bar{h}\bar{a}^{-1}$

as conjugating element. Identity (ii) also tells us that given  $\bar{c} \in \bar{C}$  and  $\varepsilon \in \{\pm 1\}$ , we have  $[[\bar{a}, \bar{g}_0^\varepsilon], \bar{c}] = 1$ , which can be reformulated as

$$(\bar{g}'_0)^{-\varepsilon} \bar{c} (\bar{g}'_0)^\varepsilon = (\bar{a} \bar{g}_0^{-\varepsilon} \bar{a}^{-1}) \bar{c} (\bar{a} \bar{g}_0^\varepsilon \bar{a}^{-1}) = \bar{g}_0^{-\varepsilon} \bar{c} \bar{g}_0^\varepsilon.$$

As  $\bar{g}_0$  normalizes  $\bar{C}$ , so does  $\bar{g}'_0$ . Since  $\bar{a}$  commutes with  $\bar{g}_0^2$ , we have  $(\bar{g}'_0)^2 = \bar{g}_0^2$ , hence  $(\bar{g}'_0)^2$  belongs to  $\bar{C}$ . Consequently  $\bar{C}$  is a subgroup of index 2 of  $\bar{A}' = \langle \bar{g}'_0, \bar{C} \rangle$ . In particular  $\bar{E}' = \langle \bar{A}', \bar{B} \rangle$  is elementary. Hence  $\bar{C}'$  generates an elementary subgroup. Recall that  $\bar{a}$  is a preimage of  $\mathbf{r}^{n/4}$ . Hence

$$q_{\bar{v}}(\bar{g}'_0) = \mathbf{r}^{n/4} \mathbf{x}' \mathbf{r}^{-n/4} = \mathbf{x}$$

On the other hand  $q_{\bar{v}}$  maps  $\bar{C}$  to  $\langle \mathbf{x} \rangle$ . Consequently  $q_{\bar{v}}(\bar{A}') = \langle \mathbf{x} \rangle$  and  $\bar{A}'$  can be lifted in  $G$ . As observed above  $\bar{C}'$  generates a subgroup  $\bar{E}'$ , which is either elliptic, parabolic or loxodromic. We let  $\bar{E}'_0 = \langle \bar{g}'_0, \dots, \bar{g}'_{m-1} \rangle$  and distinguish again two cases.

**Case 2.1** *Assume that  $\bar{E}'$  is not loxodromic.* Note that  $\bar{E}'_0$  is contained in  $\bar{A}'$ . Since  $\bar{A}'$  can be lifted  $\bar{E}'_0$  does not contain a strict rotation. If  $\bar{E}'$  is elliptic or loxodromic, then Lemmas 4.56 and 4.57 tell us that there exists a strong chain  $C'$  lifting  $\bar{C}'$ .

**Case 2.2** *Assume that  $\bar{E}'$  is loxodromic.* Observe that  $\bar{E}'_0$  does not contain a loxodromic element (it lies in the elliptic subgroup  $\bar{A}'$ ). We rerun the previous discussion replacing  $\bar{C}$  and  $\bar{E} = \bar{A} *_{\bar{C}} \bar{B}$  by  $\bar{C}'$  and  $\bar{E}' = \bar{A}' *_{\bar{C}} \bar{B}$ . In particular, if  $\bar{B}$  can be lifted, we are back to Case 1. This means that there exist a strong chain  $C'$  lifting  $\bar{C}'$ . Assume now that  $\bar{B}$  cannot be lifted. Note that  $\bar{E}'_1 = \langle \bar{g}'_1, \dots, \bar{g}'_{m-1} \rangle$  coincides with  $\bar{E}_1$ , and therefore does not contain a strict rotation. We permute in Case 2 the role of  $\bar{A}$  and  $\bar{B}$  and produce a new strong chain  $\bar{C}'' = (\bar{g}''_0, \dots, \bar{g}''_m)$  such that  $(\bar{g}''_1, \dots, \bar{g}''_{m-1}) = (\bar{g}_1, \dots, \bar{g}_{m-1})$  which generates an elementary subgroup of the form  $\bar{E}'' = \bar{A}' *_{\bar{C}} \bar{B}'$  where both  $\bar{A}'$  and  $\bar{B}'$  are elliptic subgroups which can be lifted. Following Case 1. we observe that there exists a strong chain  $C''$  of  $G$  which lifts  $\bar{C}''$ .  $\square$

Observe that Proposition 4.55 is combination of Lemmas 4.56, 4.57 and 4.58

*Proof of Proposition 4.53.* Let  $m \geq \max\{\nu, \mu + 4\}$ . Let  $\bar{C} = (\bar{g}_0, \dots, \bar{g}_m)$  be a strong chain of  $\bar{G}$ . Let  $\bar{h}$  be a conjugating element of  $\bar{C}$ . According to Proposition 4.55 one of the following holds.

- (i) There exists  $\bar{v} \in \bar{V}$  such that  $\langle \bar{g}_0, \bar{h} \rangle$  is contained in  $\text{Stab}(\bar{v})$ .
- (ii) The subgroup  $\langle \bar{g}_0, \bar{h} \rangle$  is loxodromic,
- (iii) There exists a strong chain  $\bar{C}' = (\bar{g}'_0, \dots, \bar{g}'_m)$  of  $\bar{G}$  which can be lifted such that  $(\bar{g}'_1, \dots, \bar{g}'_{m-1}) = (\bar{g}_1, \dots, \bar{g}_{m-1})$ .

We study each case separately. Assume first that  $\langle \bar{g}_0, \bar{h} \rangle$  is contained in  $\text{Stab}(\bar{v})$  for some  $\bar{v} \in \bar{\mathcal{V}}$ . Note that  $\bar{h}$  cannot be loxodromic (it fixes  $\bar{v}$ ). By the very definition of strong chain  $\langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$  is a dihedral germ. According to [Lemma 4.51](#) there exist  $k \in \mathbf{N}$  and  $\mathbf{E} \in \mathcal{E}$  such that  $\text{Stab}(\bar{v})$  embeds in  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ . Since  $m \geq \mu + 2$ , it follows from [Proposition 3.13](#) that  $\bar{g}_0$  and  $\bar{g}_m$  respectively belong to  $\langle \bar{g}_1, \dots, \bar{g}_m \rangle$  and  $\langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$ . In other words  $\bar{h}$  normalizes  $\langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$ . Hence  $\langle \bar{g}_0, \bar{h} \rangle$  is a cyclic extension of the dihedral germ  $\langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$ , therefore it has dihedral shape.

Assume now that  $\langle \bar{g}_0, \bar{h} \rangle$  is loxodromic. Then it automatically has dihedral shape ([Lemma 3.9](#)).

We are left with the last case. Let  $\mathcal{C}' = (g'_0, \dots, g'_m)$  be a strong chain of  $G$  lifting the chain  $\bar{\mathcal{C}}'$  given by Point (iii). Let  $h_0$  be a conjugating element of  $\mathcal{C}'$ . Recall that  $m \geq \nu$ . Thus  $\langle g'_0, h_0 \rangle$  is an elementary subgroup with dihedral shape. Recall that by [Assumption 4.48](#) every dihedral pair of  $G$  has type  $(\mathcal{E}, n_2)$ . Since  $m - 2 \geq \mu + 2$ , it follows from [Proposition 3.13](#) that  $g'_1$  and  $g'_{m-1}$  respectively belong to  $\langle g'_2, \dots, g'_{m-1} \rangle$  and  $\langle g'_1, \dots, g'_{m-2} \rangle$ . Recall that  $\bar{\mathcal{C}}$  and  $\bar{\mathcal{C}}'$  coincide everywhere but except maybe on the first and the last element. Pushing the previous observation in  $G$  we get that  $\bar{g}_1$  and  $\bar{g}_{m-1}$  respectively belong to  $\langle \bar{g}_2, \dots, \bar{g}_{m-1} \rangle$  and  $\langle \bar{g}_1, \dots, \bar{g}_{m-2} \rangle$ . Hence  $\bar{h}$  normalizes  $\langle \bar{g}_1, \dots, \bar{g}_{m-2} \rangle$  and a fortiori  $\langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$ . In particular  $\langle \bar{g}_0, \bar{h} \rangle$  is elementary. If  $\bar{h}$  is loxodromic, then  $\langle \bar{g}_0, \bar{h} \rangle$  has automatically dihedral shape ([Lemma 3.9](#)) Otherwise  $\langle \bar{g}_0, \bar{h} \rangle$  is a cyclic extension of the dihedral germ  $\langle \bar{g}_0, \dots, \bar{g}_{m-1} \rangle$ , and thus it has dihedral shape.  $\square$

## 5 Periodic groups

### 5.1 Induction step

**Proposition 5.1.** *There exist positive constants  $\delta_1$ ,  $C_0$  and  $C_1$  such that for every positive integer  $\nu_0$ , there exists a critical exponent  $N_0 \in \mathbf{N}$  with the following properties. Let  $\mathcal{E}$  be a model collection of abstract groups whose exponent  $\mu = \mu(\mathcal{E})$  is finite. Let  $N_1 \geq N_0$  and  $n \geq N_1$  a multiple of  $2^{\nu_0+2}\mu$ .*

*Let  $G$  be a group acting by isometries on a  $\delta_1$ -hyperbolic length space  $X$  and satisfying the following assumptions.*

(i) *The action of  $G$  on  $X$  is gentle and non-elementary.*

(ii) *Dihedral pairs of  $G$  have type  $(\mathcal{E}, n_2)$ .*

(iii)  $A(G, X) \leq (\nu_0 + 5)C_0$ ,  
 $\text{inj}(G, X) \geq 1/C_0\sqrt{N_1}$ ,  
 $\max\{\nu_{\text{stg}}(G, X), \mu + 4\} \leq \nu_0$ .

*We denote by  $P$  the set of all primitive loxodromic elements  $h \in G$  such that  $\|h\| < C_0$ . Let  $K$  be the (normal) subgroup of  $G$  generated by  $\{h^n : h \in P\}$  and  $\bar{G}$  the quotient of  $G$  by  $K$ . We write  $\pi : G \rightarrow \bar{G}$  for the corresponding quotient map.*

Then there exists a  $\delta_1$ -hyperbolic length space  $\bar{X}$  on which  $\bar{G}$  acts by isometries satisfying (i)-(iii). In addition there exists a  $\pi$ -equivariant map  $X \rightarrow \bar{X}$  with the following properties.

- The map  $X \rightarrow \bar{X}$  is  $C_1/\sqrt{N_1}$ -Lipschitz.
- If  $F$  is an elliptic (respectively parabolic) subgroup of  $G$ , then  $\pi$  induces an isomorphism from  $F$  onto its image  $\bar{F}$  which is also elliptic (respectively elliptic or parabolic).
- Any elliptic subgroup of  $\bar{G}$ , is either the isomorphic image of an elliptic subgroup of  $G$ , or has finite exponent dividing  $n$ .
- Any finitely generated parabolic subgroup of  $\bar{G}$  is the isomorphic image of a parabolic subgroup of  $G$ .
- Let  $F_1$  and  $F_2$  be two subgroups of  $G$ . Assume that  $F_1$  is elliptic and  $F_2$  is generated by a set  $S_2$  such that  $\text{Mov}(S_2, C_0)$  is non-empty. If the images of  $F_1$  and  $F_2$  are conjugated in  $\bar{G}$ , then  $F_1$  and  $F_2$  are conjugated in  $G$ .

*Vocabulary.* Assume that  $\nu_0$ ,  $n$  and the model collection  $\mathcal{E}$  have been already fixed. If  $G$  is a group acting on a metric space  $X$  satisfying the assumptions of the proposition, including Points (i)-(iii), we say that  $(G, X)$  satisfies the induction hypotheses relative to  $(\nu_0, n, \mathcal{E})$ . The proposition says among others that if  $(G, X)$  satisfies the induction hypotheses relative to  $(\nu_0, n, \mathcal{E})$ , then so does  $(G, \bar{X})$ .

*Proof.* We start by defining the several constants appearing in the statement. Recall that  $L_0$  stands for the constant given by the stability of local quasi-geodesics (Proposition 2.3). Let  $\delta_0$ ,  $\delta_1$ ,  $\Delta_0$ , and  $\rho_0$  be the parameter given by the small cancellation theorem (Theorem 4.7). We define  $\kappa = \delta_1/\pi \text{sh}(25\delta_1)$  (so that we can apply Proposition 4.45). We choose  $\rho \geq \rho_0$  such that

$$\frac{1}{\sqrt[3]{5\kappa\pi \text{sh } \rho}} < \delta_1.$$

We now fix  $C_0$  and  $C_1$  as follows.

$$C_0 = \pi \text{sh}(4L_0\delta_1) \quad \text{and} \quad C_1 = 10C_0\pi \text{sh } \rho.$$

Observe that  $\delta_1 \ll C_0 \ll \rho \ll C_1$ . For every integer  $N \in \mathbf{N}$  we define a rescaling parameter  $\lambda_N$  as follows

$$\lambda_N = \frac{C_1}{\sqrt{N}} = \frac{10C_0\pi \text{sh } \rho}{\sqrt{N}}$$

Let  $\nu_0 \in \mathbf{N}$ . The sequence  $(\lambda_N)$  converges to 0 as  $N$  tends to infinity. Therefore there exists a critical exponent  $N_0 \in \mathbf{N}$ , such that for every integer  $N \geq N_0$  we

have

$$\lambda_N \delta_1 \leq \delta_0, \quad (22)$$

$$\lambda_N \lambda [(2\nu_0 + 8)C_0 + (95\nu_0 + 494)\delta_1] \leq \Delta_0, \quad (23)$$

$$\lambda_N (\nu_0 + 5)C_0 \leq C_0, \quad (24)$$

$$\lambda_N \kappa C_0 \leq 2\delta_1. \quad (25)$$

Let  $\mathcal{E}$  be a collection of (abstract) groups and  $\mu$  its exponent. We now fix  $N_1 \geq N_0$ . Let  $n \geq N_1$  be a multiple of  $2^{\nu_0+2}\mu$ . In particular  $\mu$  divides  $n/2$ . Consequently for every  $\mathbf{E} \in \mathcal{E}$ , then exponent of  $\mathbf{E}$  and  $\mathbf{E}/Z(\mathbf{E})$  respectively divide  $n$  and  $n/2$ , which means that the model collection  $\mathcal{E}$  satisfies the assumptions stated in [Section 4.7.2](#). For simplicity we write  $\lambda$  instead of  $\lambda_{N_1}$ .

Let  $G$  be a group acting on a  $\delta_1$ -hyperbolic space  $X$  such that  $(G, X)$  satisfies the induction hypotheses relative to  $(\nu_0, n, \mathcal{E})$ . We denote by  $P$  the set of all primitive loxodromic elements  $h \in G$  such that  $\|h\| < C_0$ . Let  $K$  be the (normal) subgroup of  $G$  generated by  $\{h^n : h \in P\}$  and  $\bar{G}$  the quotient of  $G$  by  $K$ . If  $P$  is empty, then  $\bar{G} = G$ . Thus  $\bar{X} = \lambda X$  obviously satisfies the conclusion of the proposition. Otherwise, we are going to prove that  $\bar{G}$  is a small cancellation quotient of  $G$ . To that end we consider the action of  $G$  on the rescaled space  $\lambda X$ . According to [\(22\)](#) this space is  $\delta$ -hyperbolic with  $\delta \leq \lambda\delta_1 \leq \delta_0$ . We define the family  $\mathcal{Q}$  by

$$\mathcal{Q} = \left\{ \left( \langle h^n \rangle, Y_h \right) : h \in P \right\}.$$

**Lemma 5.2.** *The family  $\mathcal{Q}$  satisfies the small cancellation hypotheses, i.e.  $\Delta(\mathcal{Q}, \lambda X) \leq \Delta_0$  and  $\text{inj}(\mathcal{Q}, \lambda X) \geq 10\pi \text{sh } \rho$ .*

*Proof.* We start with the upper bound of  $\Delta(\mathcal{Q}, \lambda X)$ . Let  $h_1$  and  $h_2$  be two elements of  $P$  such that  $(\langle h_1^n \rangle, Y_{h_1})$  and  $(\langle h_2^n \rangle, Y_{h_2})$  are distinct. We first claim that  $h_1$  and  $h_2$  generate a non-elementary subgroup. Assume on the contrary that it is not the case. Let  $E$  be the maximal elementary subgroup containing  $h_1$  and  $h_2$ . This subgroup is necessarily loxodromic. We denote by  $F$  its maximal elliptic normal subgroup, so that  $(E, F)$  is a dihedral pair. According to our assumption there exists  $k \in \mathbf{N}$  and  $\mathbf{E} \in \mathcal{E}$  such that  $E$  embeds in  $E/F \times \mathbf{D}_{n_2}^k \times \mathbf{E}$ , where  $n_2$  is the largest power of 2 dividing  $n$ . Recall that  $h_1$  and  $h_2$  are primitive. Hence up to replacing  $h_2$  by its inverse, we may assume that  $h_1$  and  $h_2$  have the same image in  $E/F$ . Since the exponents of  $\mathbf{D}_{n_2}$  and  $\mathbf{E}$  divides  $n$ , the images of  $h_1^n$  and  $h_2^n$  are trivial in  $\mathbf{D}_{n_2}^k \times \mathbf{E}$ . Consequently  $h_1^n = h_2^n$  and thus  $Y_{h_1} = Y_{h_2}$ . This contradicts the fact that  $(\langle h_1^n \rangle, Y_{h_1})$  and  $(\langle h_2^n \rangle, Y_{h_2})$  are distinct and completes the proof of our claim. Recall that  $h_i$  moves the points of  $Y_{h_i}$  by at most  $\|h_i\|_{\lambda X} + 84\delta$  ([Lemma 2.10](#)), while  $\|h_i\|_{\lambda X} \leq \lambda C_0$ . Consequently

$$Y_{h_1}^{+5\delta} \cap Y_{h_2}^{+5\delta} \subset \text{Mov}(\{h_1, h_2\}, \lambda C_0 + 95\delta).$$

Since  $h_1$  and  $h_2$  generate a non-elementary subgroup we get from [Corollary 3.5](#) that

$$\text{diam}(Y_{h_1}^{+5\delta} \cap Y_{h_2}^{+5\delta}) \leq A(G, \lambda X) + [\nu(G, X) + 3](\lambda C_0 + 95\delta) + 209\delta.$$



Recall that  $\nu(G, X)$  is bounded above by  $\nu_{\text{stg}}(G, X)$ , hence by  $\nu_0$ . Moreover by assumption  $A(G, \lambda X) \leq \lambda(\nu_0 + 5)C_0$ , hence

$$\text{diam}(Y_{h_1}^{+5\delta} \cap Y_{h_2}^{+5\delta}) \leq \lambda[(2\nu_0 + 8)C_0 + (95\nu_0 + 494)\delta_1].$$

Using (23) we get  $\Delta(\mathcal{Q}, \lambda X) \leq \Delta_0$ . Let us now focus on  $\text{inj}(\mathcal{Q}, \lambda X)$ . It follows from our assumption on  $\text{inj}(\mathcal{Q}, X)$  that

$$\text{inj}(G, \lambda X) \geq \lambda \text{inj}(G, X) \geq \frac{C_1}{\sqrt{N_1}} \frac{1}{C_0 \sqrt{N_1}} \geq \frac{10\pi \text{sh } \rho}{N_1} \geq \frac{10\pi \text{sh } \rho}{n}.$$

Let  $(H, Y) \in \mathcal{Q}$ . By construction, any element  $g \in H$  is the  $n$ -th power of a loxodromic element of  $G$ . Consequently

$$\|g\|_{\lambda X} \geq n \text{inj}(G, \lambda X) \geq 10\pi \text{sh } \rho.$$

It follows that  $\text{inj}(\mathcal{Q}, \lambda X) \geq 10\pi \text{sh } \rho$ .  $\square$

On account of the previous lemma, we can now apply the small cancellation theorem (Theorem 4.7) to the action of  $G$  on the rescaled space  $\lambda X$  and the family  $\mathcal{Q}$ . We denote by  $\dot{X}$  the space obtained by attaching on  $\lambda X$  for every  $(H, Y) \in \mathcal{Q}$ , a cone of radius  $\rho$  over the set  $Y$ . The space  $\bar{X}$  is the quotient of  $\dot{X}$  by  $K$ . According to Theorem 4.7,  $\bar{X}$  is a  $\delta_1$ -hyperbolic length space and  $\bar{G}$  acts by isometries on it. As usual we write  $\mathcal{V}$  for the set of apices in  $\dot{X}$  and  $\bar{\mathcal{V}}$  for its image in  $\bar{X}$ . We now prove that the action of  $\bar{G}$  on  $\bar{X}$  satisfies the induction hypotheses relative to  $(\nu_0, n, \mathcal{E})$ . This action is gentle (Proposition 4.21) and non-elementary (Proposition 4.23), which provides (i). In addition dihedral pairs of  $\bar{G}$  have type  $(\mathcal{E}, n_2)$  (Proposition 4.52). Thus (ii) holds. Point (iii) is a consequence of the following lemma.

**Lemma 5.3.** *The parameters  $A(\bar{G}, \bar{X})$ ,  $\text{inj}(\bar{G}, \bar{X})$  and  $\nu_{\text{stg}}(\bar{G}, \bar{X})$  satisfy*

- (i)  $A(\bar{G}, \bar{X}) \leq (\nu_0 + 5)C_0$ ;
- (ii)  $\text{inj}(\bar{G}, \bar{X}) \geq 1/C_0 \sqrt{N_1}$ ;
- (iii)  $\max\{\nu_{\text{stg}}(\bar{G}, \bar{X}), \mu + 4\} \leq \nu_0$ .

*Proof.* We start with the upper bound of  $A(\bar{G}, \bar{X})$ . Recall that  $C_0$  is bounded below by  $\pi \text{sh}(4L_0\delta_1)$ . Hence Proposition 4.46 yields

$$A(\bar{G}, \bar{X}) \leq A(G, \lambda X) + [\nu(G, X) + 4]\pi \text{sh}(4L_0\delta_1) \leq \lambda(\nu_0 + 5)C_0 + [\nu(G, X) + 4]C_0.$$

Since  $\nu(G, X)$  is bounded above by  $\nu_{\text{stg}}(G, X)$ , hence by  $\nu_0$  we get that

$$A(\bar{G}, \bar{X}) \leq \lambda(\nu_0 + 5)C_0 + (\nu_0 + 4)C_0.$$

Using (24) we obtain  $A(\bar{G}, \bar{X}) \leq (\nu_0 + 5)C_0$ . We now focus on the injectivity radius of  $\bar{G}$ . Let  $g$  be a loxodromic isometry of  $G$ . Since dihedral pairs have type  $(\mathcal{E}, n_2)$  we can write  $g = g_0^k u$  where  $k$  is a positive integer,  $g_0$  a primitive

element and  $u$  an elliptic element centralized by some large power of by  $g_0$ . In particular,  $\|g\|_{\lambda X}^\infty \geq \|g_0\|_{\lambda X}^\infty$ . Assume now that  $g$  does not stabilize any cylinder  $Y_h$ , where  $h \in P$ . It follows that  $g_0$  does not belong to  $P$ . Thus by (10)

$$\|g_0\|_{\lambda X}^\infty \geq \|g_0\|_{\lambda X} - 16\delta > \lambda C_0 - 16\lambda\delta_1$$

Recall that  $C_0 \geq 32\delta_1$ , hence  $\|g\|_{\lambda X}^\infty \geq \lambda C_0/2$ . [Proposition 4.45](#) applied with  $N = G$  yields

$$\text{inj}(\bar{G}, \bar{X}) \geq \min\left\{\frac{\lambda\kappa C_0}{2}, \delta_1\right\}$$

Combined with (25) we obtain

$$\text{inj}(\bar{G}, \bar{X}) \geq \frac{\lambda\kappa C_0}{2} \geq \frac{5\kappa C_0^2 \pi \text{sh } \rho}{\sqrt{N_1}}.$$

However  $C_0^3 \geq 1/5\pi\kappa \text{sh } \rho$ , hence  $\text{inj}(\bar{G}, \bar{X}) \geq 1/C_0\sqrt{N_1}$ . The upper bound for  $\nu_{\text{stg}}(\bar{G}, \bar{X})$  directly follows from [Proposition 4.53](#).  $\square$

We now study the properties of the projection  $\pi: G \rightarrow \bar{G}$ . Recall that the map  $\zeta: \lambda X \rightarrow \bar{X}$  is 1-Lipschitz. Hence  $X \rightarrow \bar{X}$  is  $\lambda$ -Lipschitz (and  $\pi$ -equivariant by construction). If  $F$  is an elliptic (respectively parabolic) subgroup of  $G$ , then it follows from [Lemma 4.25](#) that  $\pi: G \rightarrow \bar{G}$  induces an isomorphism from  $F$  onto its image  $\bar{F}$ . Moreover,  $\bar{F}$  is elliptic (respectively elliptic or parabolic). Let  $\bar{F}$  be an elliptic subgroup of  $\bar{G}$ . If  $\bar{F}$  is not the isomorphic image of an elliptic subgroup of  $G$ , then there exists an apex  $\bar{v} \in \bar{V}$  such that  $\bar{F}$  is contained in  $\text{Stab}(\bar{v})$  ([Proposition 4.29](#)). On the other hand there exists  $k \in \mathbf{N}$  and  $\mathbf{E} \in \mathcal{E}$  such that  $\text{Stab}(\bar{v})$  embeds in  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k \times \mathbf{E}$  ([Lemma 4.51](#)). Since the exponent of  $\mathbf{E}$  divides  $n$ , the exponent of  $\bar{F}$  is finite and divides  $n$  as well. By [Proposition 4.39](#) any finitely generated parabolic subgroup of  $\bar{G}$  is the isomorphic image of a parabolic subgroup of  $G$ . Let  $F_1$  and  $F_2$  be two subgroups of  $G$ . Assume that  $F_1$  is elliptic and  $F_2$  is generated by a finite set  $S_2$  such that  $\text{Mov}(S_2, C_0)$  is non-empty. It follows from our choice of  $C_0$  and  $\rho$ , that  $C_0 \leq \rho/100$ . Thus, if the respective images  $\bar{F}_1$  and  $\bar{F}_2$  are conjugated in  $\bar{G}$ , then so are  $F_1$  and  $F_2$  ([Corollary 4.28](#)). We have checked all the announced properties of the projection  $\pi: G \rightarrow \bar{G}$ , and the proof of the proposition is completed.  $\square$

## 5.2 Construction of periodic groups

The number of variables in the next statement can be confusing at first sight. Basically we are stating the fact that the critical exponent  $N_1$  does not depend on the group  $G$ , but only on certain parameters related to its action on a hyperbolic space  $X$ . More precisely  $N_1$  is a function of

- the hyperbolicity constant  $\delta$  of  $X$ ;
- the invariants  $A(G, X)$ ,  $\text{inj}(G, X)$  and  $\nu_{\text{stg}}(G, X)$ ;
- the structure of subgroup of  $G$  with dihedral shape.

**Theorem 5.4.** *Let  $\delta, r \in \mathbf{R}_+^*$ ,  $\nu, \mu \in \mathbf{N}$  and set  $\nu_1 = \max\{\nu + 2, \mu + 6\}$ . There exist  $N_1 \in \mathbf{N}$  such that for every integer  $n \geq N_1$  which is a multiple of  $2^{\nu_1}\mu$ , the following holds.*

*Let  $\mathcal{E}$  be a model collection of groups whose exponent divides  $\mu$ . Let  $G$  be a group acting on a  $\delta$ -hyperbolic length space  $X$  such that*

- *the action of  $G$  on  $X$  is gentle and non-elementary;*
- *for every dihedral pair  $(E, C)$  the group  $E$  embeds in  $E/C \times \mathbf{E}$  for some  $\mathbf{E} \in \mathcal{E}$ .*
- *$A(G, X) \leq r$ ,  $\nu_{\text{stg}}(G, X) \leq \nu$ , and  $\text{inj}(G, X) \geq 1/r$*

*Then there exists a quotient  $Q$  of  $G$  with the following properties.*

- (i) *For every elliptic (respectively parabolic) subgroup  $F$  of  $G$ , the projection  $G \twoheadrightarrow Q$  induces an embedding from  $F$  into  $Q$ .*
- (ii) *For every  $q \in Q$ , either  $q^n = 1$  or  $q$  is the image of an elliptic or parabolic element of  $G$ .*
- (iii) *The projection  $G \twoheadrightarrow G/G^n$  induces an epimorphism  $Q \twoheadrightarrow G/G^n$ . In particular, if  $G$  has no parabolic element, and every elliptic element of  $G$  has finite order dividing  $n$ , then  $Q = G/G^n$ .*
- (iv) *For every  $x \in X$ , the map  $G \rightarrow Q$  is one-to-one when restricted to*

$$\{g \in G : |gx - x| < r\}.$$

- (v) *There are infinitely many elements of  $Q$  which are not the image of an elliptic or a parabolic element of  $G$ .*
- (vi) *The kernel  $K$  of  $G \rightarrow Q$  is purely loxodromic (i.e. all its non-trivial elements are loxodromic). As a normal subgroup  $K$  is not finitely generated.*

*Proof.* The main ideas of the proof are the following. Using [Proposition 5.1](#) we construct by induction a sequence of groups  $G = G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$  where  $G_{k+1}$  is obtained from  $G_k$  by adding new relations of the form  $h^n$  where  $h$  is a primitive element of  $G$ . Then we chose for the quotient  $Q = G/K$  the direct limit of these groups.

**Critical exponent.** Let us define first all the parameters leading to the critical exponent. For simplicity we let  $\nu_0 = \max\{\nu, \mu + 4\}$  and  $\nu_1 = \nu_0 + 2$ . The parameters  $\delta_1, C_0, C_1$ , and  $N_0$  are the one given by [Proposition 5.1](#). We choose  $\varepsilon > 0$  and  $N_1 \in \mathbf{N}$  such that

$$\varepsilon\delta \leq \delta_1, \quad \varepsilon r \leq \min \left\{ (\nu_0 + 5)C_0, \frac{C_0}{2} \right\}, \quad \frac{\varepsilon}{r} \geq \frac{1}{C_0\sqrt{N_1}}, \quad \text{and} \quad \frac{C_1}{\sqrt{N_1}} < 1.$$

We now a fix an integer  $n \geq N_1$  which is divisible by  $2^{\nu_1}\mu$ .

**The initialization.** Let  $\mathcal{E}$  be a model collection of groups and  $G$  be a group acting on a  $\delta$ -hyperbolic length space  $X$  as in the theorem. Let  $X_0$  be the space  $X$  whose metric has been rescaled by  $\varepsilon$ . It follows from our choice of  $\varepsilon$  and  $N_1$  that  $X_0$  is  $\delta_1$ -hyperbolic,  $A(G, X_0) \leq (\nu_0 + 5)C_0$ , and  $\text{inj}(G, X_0) \geq 1/C_0\sqrt{N_1}$ . In addition  $\max\{\nu_{\text{stg}}(G, X_0), \mu + 4\} \leq \nu_0$ . In other words, if  $G_0 = G$ , then  $(G_0, X_0)$  satisfies the induction hypotheses relative to  $(\nu_0, n, \mathcal{E})$ .

**The induction step.** Let  $k \in \mathbf{N}$ . We assume that we already constructed the group  $G_k$  and the space  $X_k$  such that  $(G_k, X_k)$  satisfies the induction hypotheses relative to  $(\nu_0, n, \mathcal{E})$ . We denote by  $P_k$  the set of primitive loxodromic elements  $h \in G_k$  such that  $\|h\|_{X_k} < C_0$ . Let  $K_k$  be the normal subgroup of  $G_k$  generated by  $\{h^n, h \in P_k\}$ . We write  $G_{k+1}$  for the quotient of  $G_k$  by  $K_k$ . According to [Proposition 5.1](#), there exists a metric space  $X_{k+1}$  such that  $(G_{k+1}, X_{k+1})$  satisfies the induction hypotheses relative to  $(\nu_0, n, \mathcal{E})$ . Moreover  $X_{k+1}$  comes with a  $C_1/\sqrt{N_1}$ -Lipschitz map  $X_k \rightarrow X_{k+1}$  which is  $\pi_k$ -equivariant, where  $\pi_k: G_k \rightarrow G_{k+1}$  is the canonical projection, and fulfills the following properties.

- (P1) If  $F$  is an elliptic (respectively parabolic) subgroup of  $G_k$ , then  $\pi_k$  induces an isomorphism from  $F$  onto its image which is also elliptic (respectively elliptic or parabolic).
- (P2) Any elliptic subgroup of  $G_{k+1}$  is either isomorphic to an elliptic subgroup of  $G_k$  or a finite group whose exponent divides  $n$ .
- (P3) Any finitely generated parabolic subgroup of  $G_{k+1}$  is the isomorphic image of a parabolic subgroup of  $G_k$ .
- (P4) Let  $F_1$  and  $F_2$  be two subgroups of  $G_k$ . Assume that  $F_1$  is elliptic and  $F_2$  is generated by a finite set  $S_2$  such that  $\text{Mov}(S_2, C_0)$  is non-empty. If the images of  $F_1$  and  $F_2$  are conjugated in  $G_{k+1}$ , then  $F_1$  and  $F_2$  are conjugated in  $G_k$ .

**Direct limit.** The direct limit of the sequence  $(G_k)$  is a quotient  $Q = G/K$  of  $G$ . We claim that this group satisfies the announced properties. Let  $E$  be a subgroup of  $G$  which is either elliptic or parabolic. A proof by induction on  $k \in \mathbf{N}$  using (P1) shows that for every  $k \in \mathbf{N}$ , the map  $G \rightarrow G_k$  induces an isomorphism from  $E$  onto its image which is either elliptic or parabolic for the action of  $G_k$  on  $X_k$ . It follows that  $G \rightarrow Q$  induces an isomorphism from  $E$  onto its image, which proves (i).

A proof by induction on  $k \in \mathbf{N}$  using (P2) and (P3) shows that if  $g \in G_k$  is elliptic or parabolic (for its action on  $X_k$ ) then either  $g^n = 1$  or  $g$  is the image of an elliptic or a parabolic element of  $G$  (for its action on  $X$ ). Let  $q \in Q$  and  $g \in G$  be a pre-image of  $q$ . For simplicity we still write  $g$  for the image of  $g$  in  $G_k$ . Since the map  $X_k \rightarrow X_{k+1}$  is  $C_1/\sqrt{N_1}$ -Lipschitz, we get for every  $k \in \mathbf{N}$ ,

$$\|g\|_{X_k}^\infty \leq \left( \frac{C_1}{\sqrt{N_1}} \right)^k \|g\|_X^\infty.$$

As  $C_1/\sqrt{N_1} < 1$ , there exists  $k \in \mathbf{N}$  such that

$$\|g\|_{X_k}^\infty < \frac{1}{C_0\sqrt{N_1}} \leq \text{inj}(G_k, X_k).$$

Consequently  $g$  is elliptic or parabolic as an element of  $G_k$ . It follows from the previous observation that one of the following holds.

- The element  $g$  coincide in  $G_k$  with an elliptic or a parabolic element of  $G$ , hence  $q$  is the image of an elliptic or a parabolic element of  $G$ .
- We have  $g^n = 1$  (in  $G_k$ ), hence  $q^n = 1$ .

This completes the proof of (ii).

All the relation we added to build the sequence of groups  $(G_k)$  have the form  $h^n = 1$ . Hence the projection  $G \rightarrow Q$  induces an epimorphism  $Q \rightarrow G/G^n$ , which gives (iii).

Let  $g$  be an elliptic or a parabolic element of  $K$ . It follows from (i) that the map  $G \rightarrow G/K$  induces an isomorphism from  $\langle g \rangle$  onto its image. Hence  $g$  is trivial. Consequently  $K$  is purely loxodromic. For every  $k \in \mathbf{N}$ , the action of  $G_k$  on  $X_k$  is non-elementary. It follows that the sequence  $(G_k)$  does not ultimately stabilize. Indeed, otherwise (ii) would fail. Thus  $K$  is infinitely generated as a normal subgroup, which completes the proof of (vi).

Let  $x \in X$ . Let  $g_1, g_2 \in G$  such that  $|g_i x - x|_X < r$ . It follows from our choice of  $\varepsilon$  that  $\text{Mov}(g, C_0) \subset X_0$  is non empty, where  $g = g_1^{-1}g_2$ . Assume now that  $g_1$  and  $g_2$  have the same image in  $Q$ , i.e.  $g$  is trivial in  $Q$ . In particular there exists  $i \in \mathbf{N}$  such that the image of  $g$  in  $G_i$  is trivial. Recall that the map  $X_k \rightarrow X_{k+1}$  is 1-Lipschitz for every  $k \in \mathbf{N}$ . In particular  $\text{Mov}(g, C_0) \subset X_k$  is non-empty for every  $k \in \mathbf{N}$ . A proof by induction using (P4) show that  $g = 1$ . Hence the quotient map  $G \rightarrow Q$  is one-to-one when restricted to the set

$$\{g \in G : |gx - x| < r\},$$

whence (iv).

We are left to prove (v). Let  $S$  be the collection of all elements of  $Q$  which are not the image of an elliptic or a parabolic element of  $G$ . Assume contrary to our claim that  $S$  is finite. Let  $S_0$  be a finite pre-image of  $S$  in  $G_0$ . Using the same argument as above we observe that there exists  $i \in \mathbf{N}$ , such that the image of  $S_0$  in  $G_i$  only consists of elliptic and parabolic elements. As we already observed, the sequence  $(G_k)$  is not ultimately constant. Consequently there exists  $j \geq i$  such that  $P_j$  is non-empty (recall that  $P_j$  is a set of primitive elements of  $G_j$  such that  $\|h\|_{X_j} < C_0$ ). We fix  $g \in P_j$ . We claim that  $g$  does not coincide in  $Q$  with an elliptic or a parabolic element of  $G_j$ . Assume on the contrary that it is the case. There exists an elliptic or a parabolic element  $u \in G_j$  as well as an index  $k > j$  such that  $g$  and  $u$  coincide in  $G_k$ . Note that the set  $\text{Mov}(g, C_0) \subset X_\ell$  is non empty, for every  $\ell \geq j$ . A proof by induction using (P4) shows that  $g$  and  $u$  are conjugated as elements of  $G_j$ . It contradicts the fact that  $g$  is loxodromic and  $u$  is not, hence the claim is proved. Our claim

has two consequences for the image  $q$  of  $g$  in  $Q$ . First  $q$  is not the image of an elliptic or parabolic element of  $G$ , hence  $q \in S$ . Since every element of  $S_0$  is elliptic in  $G_j$ ,  $q$  does not belong to  $S$ , a contradiction.  $\square$

### 5.3 Examples

One source of examples comes from groups acting acylindrically on a  $\delta$ -hyperbolic length space  $X$ . Let us recall first the definition of acylindricity. For our purpose we need to keep in mind the parameters that appear in the definition.

**Definition 5.5** (Acylindrical action). Let  $N, L, d \in \mathbf{R}_+^*$ . The group  $G$  acts  $(d, L, N)$ -acylindrically on  $X$  if the following holds: for every  $x, y \in X$  with  $|x - y| \geq L$ , the number of elements  $u \in G$  satisfying  $|ux - x| \leq d$  and  $|uy - y| \leq d$  is bounded above by  $N$ . The group  $G$  acts *acylindrically* on  $X$  if for every  $d > 0$  there exist  $N, L > 0$  such that  $G$  acts  $(d, L, N)$ -acylindrically on  $X$ .

Since  $X$  is a hyperbolic space, one can decide whether an action is acylindrical by looking at a single value of  $d$ .

**Proposition 5.6** (Dahmani-Guirardel-Osin [13, Proposition 5.31]). *The action of  $G$  on  $X$  is acylindrical if and only if there exists  $N, L > 0$  such that the action is  $(100\delta, L, N)$ -acylindrical.*

*Remark.* Dahmani, Guirardel and Osin work in a class of geodesic spaces. Nevertheless, following the proof of [13, Proposition 5.31] one observes that the statement also holds for length spaces. Moreover one gets the following *quantitative* statement. Assume that the action of  $G$  on  $X$  is  $(100\delta, L, N)$ -acylindrical, then for every  $d > 0$  the action is  $(d, L(d), N(d))$ -acylindrical where

$$\begin{aligned} L(d) &= L + 4d + 100\delta, \\ N(d) &= \left( \frac{d}{5\delta} + 3 \right) N. \end{aligned}$$

Assume now that the action of  $G$  on  $X$  is  $(100\delta, L, N)$ -acylindrical. One can obtain the following estimates of the various invariants defined in Section 3. More precisely

- (i)  $\text{inj}(G, X) \geq \delta/N$ , [12, Lemma 3.9]
- (ii)  $\nu(G, X) \leq N(2 + L/\delta)$  [10, Lemmas 6.12]
- (iii)  $A(G, X) \leq 10L_3^2 N^3(L + 5\delta)$ , [10, Lemma 6.14].

The statements given in [10] do not mention an explicit upper bound for  $\nu(G, X)$  and  $A(G, X)$ . However following the proofs yields directly to the result. Our estimates are very generous. The important point to notice is that they only depend on  $\delta$ ,  $L$  and  $N$ .

Recall that given a group  $\mathbf{E}$ , its *holomorph* is the semi-direct product  $\text{Hol}(\mathbf{E}) = \text{Aut}(\mathbf{E}) \rtimes \mathbf{E}$ . If  $\mathcal{E}_0$  stands for a collection of groups, we let

$$\text{Hol}(\mathcal{E}_0) = \{\text{Hol}(\mathbf{E}) : \mathbf{E} \in \mathcal{E}_0\}.$$

**Theorem 5.7.** *Let  $\delta, L, r \in \mathbf{R}_+^*$  and  $N \in \mathbf{N}$ . Let  $\mathcal{E}_0$  be a finite collection of finite groups. We write  $\mu$  for the exponent of  $\text{Hol}(\mathcal{E}_0)$ . There exists  $\nu_1, N_1 \in \mathbf{N}$  such that for every integer  $n \geq N_1$  which is a multiple of  $2^{\nu_1}\mu$  the following holds.*

*Let  $G$  be a group acting by isometries on a  $\delta$ -hyperbolic length space  $X$ . We assume that this action is  $(100\delta, L, N)$ -acylindrical and non-elementary. In addition we suppose that every finite subgroups of  $G$  with dihedral shape is isomorphic to a group of  $\mathcal{E}_0$ . Then there exists a quotient  $Q$  of  $G$  with the following properties.*

- (i) *For every elliptic subgroup  $F$  of  $G$ , the projection  $G \twoheadrightarrow Q$  induces an embedding from  $F$  into  $Q$ .*
- (ii) *For every  $q \in Q$ , either  $q^n = 1$  or  $q$  is the image of an elliptic element of  $G$ .*
- (iii) *The projection  $G \twoheadrightarrow G/G^n$  induces an epimorphism  $Q \twoheadrightarrow G/G^n$ .*
- (iv) *For every  $x \in X$ , the map  $G \rightarrow Q$  is one-to-one when restricted to*

$$\{g \in G : |gx - x| < r\}.$$

- (v) *There are infinitely many elements of  $Q$  which are not the image of an elliptic element of  $G$ .*
- (vi) *The kernel  $K$  of  $G \twoheadrightarrow Q$  is purely loxodromic. As a normal subgroup  $K$  is not finitely generated.*

*Proof.* We are going to apply [Theorem 5.4](#). To that end we let we denote by  $M$  the cardinality of the biggest group in  $\mathcal{E}_0$ . Up to replacing  $r$  by a largest value, we can assume that

$$r \geq 10L_S^2 N^3 (L + 5\delta) \quad \text{and} \quad \frac{1}{r} \leq \frac{\delta}{N}.$$

Recall that  $\mu$  is the exponent of  $\text{Hol}(\mathcal{E}_0)$ . We now set

$$\nu = \max\{N(2 + L/\delta), M + 1\} \quad \text{and} \quad \nu_1 = \max\{\nu + 2, \mu + 6\}.$$

and denote by  $N_1$  the critical exponent given by [Theorem 5.4](#). Let  $n \geq N_1$  be an integer divisible by  $2^{\nu_1}\mu$ .

Let  $G$  be a group acting on a  $\delta$ -hyperbolic length space  $X$  as in the theorem. It follows from our previous discussion that  $A(G, X) \leq r$ ,  $\text{inj}(G, X) \geq 1/r$  and  $\nu(G, X) \leq \nu$ . Let us prove that  $\nu_{\text{stg}}(G, X) \leq \nu$ . Let  $g, h \in G$  and  $m \geq \nu$ . For every  $k \in \mathbf{N}$ , we write  $g_k = h^k g h^{-k}$ . Suppose that  $E = \langle g_0, \dots, g_m \rangle$  is elementary. Assume first that  $h$  is loxodromic. According to our choice of  $\nu$ , we have  $m \geq \nu(G, X)$ , thus the elements  $g$  and  $h$  generate an elementary subgroup of  $G$ . Note that this group is necessarily loxodromic, hence has dihedral shape. Assume now that  $E_0 = \langle g_0, \dots, g_{m-1} \rangle$  is a dihedral germ. Since the action of

$G$  is acylindrical, every loxodromic subgroup of  $G$  is virtually cyclic. Hence  $E_0$  is finite. As a dihedral germ it also has dihedral shape. Consequently  $E_0$  is isomorphic to a group in  $\mathcal{E}_0$  hence contains at most  $M$  element. Since  $m \geq M + 1$ , there exists  $i, j \in \llbracket 0, m - 1 \rrbracket$  with  $i < j$  such that  $g_i = g_j$ . In particular  $g_{j-i} = g_0$ . It follows that  $h$  normalizes  $\langle g_0, \dots, g_{m-1} \rangle$ . Hence the group generated by  $g$  and  $h$  is elementary. More precisely, it is a cyclic extension of the dihedral germ  $\langle g_0, \dots, g_{m-1} \rangle$ , thus it has dihedral shape. This proves that  $\nu \geq \nu_{\text{stg}(G, X)}$  and completes the proof of our claim.

We now build an appropriate model collection  $\mathcal{E}$ . Let  $(E, C)$  be a dihedral pair where  $E$  is infinite. Since  $C$  is finite, it fits into the following short exact sequence

$$1 \rightarrow C \rightarrow E \rightarrow \mathbf{L} \rightarrow 1,$$

where  $\mathbf{L}$  is either  $\mathbf{Z}$  or  $\mathbf{D}_\infty$ . We choose an element  $g \in E$  whose image in  $\mathbf{L}$  generates the maximal infinite cyclic subgroup of  $\mathbf{L}$ . Recall that  $\mu$  is the exponent of  $\text{Hol}(\mathcal{E}_0)$ . We claim that  $\langle g^\mu \rangle$  is a normal subgroup of  $E$ . Let  $u \in E$ . There exists  $\varepsilon \in \{\pm 1\}$  and  $c \in C$  such that  $ugu^{-1} = g^\varepsilon c$ . The value of  $\varepsilon$  depends whether the image of  $u$  in  $\mathbf{L}$  is a reflection or not. Since  $g$  normalizes  $C$  its action by conjugation on  $C$  induces an automorphism of  $C$  that we denote by  $\varphi$ . One checks that for every  $p \in \mathbf{N}$ ,

$$ug^p u^{-1} = (g^\varepsilon c)^p = g^{\varepsilon p} \varphi^{\varepsilon(p-1)}(c) \cdots \varphi^\varepsilon(c)c.$$

However in the holomorph  $\text{Hol}(C)$ , whose exponent divides  $\mu$ , we have

$$1 = (\varphi^\varepsilon, c)^\mu = (\varphi^{\varepsilon\mu}, \varphi^{\varepsilon(\mu-1)}(c) \cdots \varphi^\varepsilon(c)c).$$

Hence  $ug^\mu u^{-1} = g^{\varepsilon\mu}$ . Consequently  $\langle g^\mu \rangle$  is a normal subgroup of  $E$  as we announced. Note also that the exponent of  $E/\langle g^\mu \rangle$  divides  $\mu$ . Moreover  $E$  embeds in  $E/C \times E/\langle g^\mu \rangle$ .

We denote by  $\mathcal{E}_1$  the collection of quotients  $E/\langle g^\mu \rangle$  obtained as above, where  $(E, C)$  runs over all dihedral pairs with  $E$  infinite. In addition we let  $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$ . It follows from the construction that the exponent of  $\mathcal{E}$  divides  $\mu$ . Moreover, for every dihedral pair  $(E, C)$  there exists  $\mathbf{E} \in \mathcal{E}$  such that  $E$  embeds in  $E/C \times \mathbf{E}$ . All the assumptions of [Theorem 5.4](#) are satisfied and the result follows.  $\square$

### Free Burnside groups and periodic groups.

**Theorem 5.8.** *Let  $r \geq 2$ . There exists  $N_1 \in \mathbf{N}$  such that for every integer  $n \geq N_1$  that is a multiple of 128, the free Burnside group  $\mathbf{B}_r(n)$  is not finitely presented and therefore infinite. Moreover if  $n_2$  stands for the largest power of 2 dividing  $n$ , then every finite subgroup of  $\mathbf{B}_r(n)$  embeds in  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k$  for some  $k \in \mathbf{N}$ .*

*Proof.* Let  $X$  be the Cayley graph of the free group  $\mathbf{F}_r$  of rank  $r$ . It is  $\delta$ -hyperbolic with  $\delta \in \mathbf{R}_+^*$  such that  $L_0\delta < 1$ . It follows that  $A(\mathbf{F}_r, X) = 0$ . Moreover  $\text{inj}(\mathbf{F}_r, X) \geq 1$ . Every dihedral germ is trivial. Hence  $\nu_{\text{stg}(G, X)} = 1$ . We choose for  $\mathcal{E}$  the class of group reduced to the trivial one. Its exponent  $\mu =$



$\mu(\mathcal{E})$  is 1. Every subgroup with dihedral shape is trivial or infinite cyclic. Hence the assumption of [Theorem 5.4](#) are fulfilled. Note that  $\nu_1 = \max\{\nu_{\text{stg}}(G, X) + 2, \mu + 6\} = 7$ . Hence there exists  $N_1 \in \mathbf{N}$  such that for every  $n \geq N_1$  that is a multiple of 128, the group  $\mathbf{B}_r(n)$  is not finitely presented, hence infinite. We now prove the second assertion. Let

$$G_0 = \mathbf{F}_r \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \dots$$

be the sequence of groups produced in the proof of [Theorem 5.4](#), whose direct limit is  $\mathbf{B}_r(n)$ . Let  $F$  be a finite subgroup of  $\mathbf{B}_r(n)$ . By construction there exists  $\ell \in \mathbf{N}$  and an elliptic subgroup  $F_\ell$  of  $G_\ell$  such that  $G_\ell \twoheadrightarrow \mathbf{B}_r(n)$  maps  $F_\ell$  onto  $F$ . We assume that  $\ell$  is the smallest for this property. Recall that each map  $G_k \twoheadrightarrow G_{k+1}$  is one-to-one when restricted to an elliptic subgroup. Hence  $F_\ell$  is actually isomorphic to  $F$ . According to our choice of  $\ell$ , the subgroup  $F_\ell$  cannot be lifted. Therefore there exists  $v$  in the vertex set  $\mathcal{V}_\ell$  of  $X_\ell$  such that  $F_\ell$  is contained in  $\text{Stab}(v)$ . It follows from [Lemma 4.51](#) that  $F_\ell$  embeds in  $\mathbf{D}_n \times \mathbf{D}_{n_2}^k$  for some  $k \in \mathbf{N}$ .  $\square$

*Remark.* Although we have not written the details here, a careful reader can follow the induction in the proof of [Theorem 5.4](#) to show that the word and the conjugacy problems are solvable in  $\mathbf{B}_r(n)$ . With the previous notations, the solution of the word problem is based on the following observation. Given an element  $g \in \mathbf{F}_r$ , there exists  $k \in \mathbf{N}$ , which only depends on the words length of  $g$ , such that  $g$  is trivial in  $\mathbf{B}_r(n)$  is trivial if and only if so is it in  $G_k$ . Hence it suffices to run the solution of the word problem in the hyperbolic group  $G_k$ . The conjugacy problem can be solved as follows. Given  $g_1, g_2 \in \mathbf{F}_r$ , there exists  $k \in \mathbf{N}$ , which only depends on the word length of  $g_1$  and  $g_2$ , such that both  $g_1$  and  $g_2$  are elliptic in  $G_k$ . A proof by induction based on [Corollary 4.28](#) shows that  $g_1$  and  $g_2$  are conjugated in  $\mathbf{B}_r(n)$  if and only if so are they in  $G_k$ . Hence it suffices to run the solution of the conjugacy problem in  $G_k$ .

**Corollary 5.9.** *Let  $r \geq 2$ . There exists  $N_1 \in \mathbf{N}$ , such that for every integer  $n \geq N_1$ , the group  $\mathbf{B}_r(n)$  is infinite.*

*Proof.* Recall that free Burnside groups of sufficiently large odd exponents are infinite, see for instance [\[15, 9\]](#) for a geometric proof. Hence it suffices to observe that given two integers  $p, n \in \mathbf{N}$ , the group  $\mathbf{B}_r(pn)$  maps onto  $\mathbf{B}_r(n)$ .  $\square$

**Theorem 5.10.** *Let  $G$  be a non-elementary hyperbolic group. There exists  $p, N_1 \in \mathbf{N}$  such that for every integer  $n \geq N_1$  that is a multiple of  $p$ , the quotient  $G/G^n$  is infinite. Moreover*

$$\bigcap_{n \geq 1} G^n = \{1\}.$$

*Proof.* Let  $\mathcal{E}_0$  be the collection of all isomorphism classes of finite subgroups of  $G$ . Since  $G$  is hyperbolic,  $\mathcal{E}_0$  is finite. We write  $\mu$  for its exponent. Let  $X$  be the Cayley graph of  $G$  relative to some finite generating set. The action of  $G$

on this hyperbolic space is acylindrical hence the assumptions of [Theorem 5.7](#) holds. Let  $\nu_1, N_1 \in \mathbf{N}$  be the parameters given by [Theorem 5.7](#) and set  $p = 2^{\nu_1} \mu$ . Observe that the order of every elliptic element of  $G$  divides  $\mu$ , hence  $p$ . Assume now that  $n \geq N_1$  is a multiple of  $p$ . According to [Theorem 5.7](#) there exists an infinite quotient  $Q$  of  $G$  such that the projection  $G \twoheadrightarrow G/G^n$  induces a map  $Q \twoheadrightarrow G/G^n$ . Moreover every element  $q \in Q$  satisfies the following dichotomy. Either  $q^n = 1$ , or  $q$  is the image of an elliptic element of  $G$ . In the latter case we have  $q^p = 1$ , and thus  $q^n = 1$ . In other words  $Q$  is a quotient of  $G/G^n$ , hence  $Q$  is isomorphic to  $G/G^n$  which is therefore infinite.

According to [Theorem 5.7](#) the quotient map  $G \rightarrow G/G^n$  can be made one-to-one on arbitrarily large balls by enlarging the value of  $n$ . Hence the intersection of all the subgroups  $G^n$  is trivial.  $\square$

*Remark.* As for free Burnside groups, one can give a precise description of the finite subgroups of  $G/G^n$ , provided  $n$  is sufficiently large and divisible by  $p$ . One can also prove that the word and the conjugacy problem are solvable in these periodic quotients.

**Relatively hyperbolic groups.** Since Gromov's original paper [\[19\]](#), several different definitions of relatively hyperbolic groups have emerged, see for instance [\[3, 16\]](#). These definitions have been shown to be almost equivalent [\[3, 34, 23\]](#). For our purpose we will use the following one.

**Definition 5.11** ([\[23, Definition 3.3\]](#)). Let  $G$  be a group and  $\{P_1, \dots, P_m\}$  be a collection of subgroups of  $G$ . We say that  $G$  is *hyperbolic relative to*  $\{P_1, \dots, P_m\}$  if there exists a proper geodesic hyperbolic space  $X$  and a collection  $\mathcal{Y}$  of disjoint open horoballs satisfying the following properties.

- (i)  $G$  acts properly by isometries on  $X$  and  $\mathcal{Y}$  is  $G$ -invariant.
- (ii) If  $U$  stands for the union of the horoballs of  $\mathcal{Y}$  then  $G$  acts co-compactly on  $X \setminus U$ .
- (iii)  $\{P_1, \dots, P_m\}$  is a set of representatives of the  $G$ -orbits of  $\{\text{Stab}(Y) : Y \in \mathcal{Y}\}$ .

The action of  $G$  on the space  $X$  given by [Definition 5.11](#) is not acylindrical. Indeed the subgroups  $P_j$  can be parabolic. This cannot happen with an acylindrical action [\[2, Lemma 2.2\]](#). More generally, the elementary subgroups of  $G$  are exactly the virtually cyclic subgroups of  $G$  and the ones which are conjugated to a subgroup of some  $P_j$ . As in the case of groups with an acylindrical action, one can prove that  $\text{inj}(G, X)$  is positive whereas  $\nu(G, X)$  and  $A(G, X)$  are finite. Proceeding as in [Theorem 5.7](#) we get the following result.

**Theorem 5.12.** *Let  $G$  be a group and  $\{P_1, \dots, P_m\}$  be a collection of subgroups of  $G$  such that  $G$  is hyperbolic relative to  $\{P_1, \dots, P_m\}$ . Assume that there are only finitely many isomorphism classes of finite subgroups with dihedral shape. There exist  $p, N_1 \in \mathbf{N}$  such that every integer  $n \geq N_1$  multiple of  $p$ , there exists a quotient  $Q$  of  $G$  with the following properties.*

- (i) if  $E$  is a finite subgroup of  $G$  or conjugated to some  $P_j$ , then the projection  $G \twoheadrightarrow Q$  induces an isomorphism from  $E$  onto its image;
- (ii) for every element  $g \in Q$ , either  $g^n = 1$  or  $g$  is the image a non-loxodromic element of  $G$ ;
- (iii) there are infinitely many elements in  $Q$  which do not belong to the image of an elementary non-loxodromic subgroup of  $G$ .

**Mapping class groups.** Let  $\Sigma$  be a compact surface of genus  $g$  with  $p$  boundary components. In the rest of this paragraph we assume that its complexity  $3g + p - 3$  is larger than 1. The *mapping class group*  $\text{MCG}(\Sigma)$  of  $\Sigma$  is the group of orientation preserving self homeomorphisms of  $\Sigma$  defined up to homotopy. A mapping class  $f \in \text{MCG}(\Sigma)$  is

- (i) *periodic*, if it has finite order;
- (ii) *reducible*, if it permutes a collection of essential non-peripheral curves (up to isotopy);
- (iii) *pseudo-Anosov*, if there exists an homotopy in the class of  $f$  that preserves a pair of transverse foliations and rescale these foliations in an appropriate way.

It follows from Thurston's work that any element of  $\text{MCG}(\Sigma)$  falls into one these three categories [35, Theorem 4]. The *complex of curves*  $X$  is a simplicial complex associated to  $\Sigma$ . It has been first introduced by Harvey [22]. A  $k$ -simplex of  $X$  is a collection of  $k + 1$  homotopy classes of curves of  $\Sigma$  that can be disjointly realized. Masur and Minsky proved that this new space is hyperbolic [28]. By construction,  $X$  is endowed with an action by isometries of  $\text{MCG}(\Sigma)$ . Moreover Bowditch showed that this action is acylindrical [2, Theorem 1.3]. This action provides an other characterization of the elements of  $\text{MCG}(\Sigma)$ . An element of  $\text{MCG}(\Sigma)$  is periodic or reducible (respectively pseudo-Anosov) if and only it is elliptic (respectively loxodromic) for the action on the complex of curves [28]. Recall that  $\text{MCG}(\Sigma)$  contains only finitely many conjugacy classes of finite subgroups [17, Theorem 7.14]. Hence the next statement is a direct application of Theorem 5.7.

**Theorem 5.13.** *Let  $\Sigma$  be a compact surface of genus  $g$  with  $p$  boundary components such that  $3g + p - 3 > 1$ . There exist  $p, N_1$  such that for every integer  $n \geq N_1$  which is a multiple of  $p$ , there exists a quotient  $Q$  of  $\text{MCG}(\Sigma)$  with the following properties.*

- (i) *If  $E$  is a subgroup of  $\text{MCG}(\Sigma)$  that does not contain a pseudo-Anosov element, then the projection  $\text{MCG}(\Sigma) \twoheadrightarrow Q$  induces an isomorphism from  $E$  onto its image.*
- (ii) *Let  $f$  be a pseudo-Anosov element of  $\text{MCG}(\Sigma)$ . Either  $f^n = 1$  in  $Q$  or  $f$  coincide in  $Q$  with a periodic or a reducible element.*

(iii) There are infinitely many elements in  $Q$  which are not the image of a periodic or reducible element of  $\text{MCG}(\Sigma)$ .

**Amalgamated product.** Let  $G$  be a group. A subgroup  $H$  of  $G$  is *malnormal* if for every  $g \in G$ ,  $gHg^{-1} \cap H = \{1\}$  unless  $g$  belongs to  $H$ .

**Theorem 5.14.** *Let  $A$  and  $B$  be two groups. Let  $C$  be a subgroup of  $A$  and  $B$  malnormal in  $A$  or  $B$ . Assume that there exists  $M \in \mathbf{N}$  such that every subgroup of  $A$  (respectively  $B$ ) that is isomorphic to the extension of a 2-group by finite cyclic or dihedral group contains at most  $M$  elements. There exist  $p, N_1$  such that for every integer  $n \geq N_1$  which is a multiple of  $p$ , there exists a quotient  $Q$  of  $A *_C B$  with the following properties.*

- (i) *The natural projection  $A *_C B \rightarrow Q$  induces an embedding of  $A$  and  $B$  into  $Q$ .*
- (ii) *For every  $g \in Q$ , if  $g$  is not a conjugate of an element of  $A$  or  $B$  then  $g^n = 1$ .*
- (iii) *There are infinitely many elements in  $Q$  which are not conjugates of elements of  $A$  or  $B$ .*

*Proof.* We denote by  $X$  the Bass-Serre tree associated to the amalgamated product  $G = A *_C B$  [33]. As  $C$  is malnormal in  $A$ , the action of  $G$  on  $X$  is acylindrical. Moreover every loxodromic subgroup is either  $\mathbf{Z}$  or  $\mathbf{D}_\infty$ . Hence every dihedral germ is necessarily a 2-group. Consequently, a finite group with dihedral shape is isomorphic to the extension of a 2-group by a finite cyclic or dihedral group. Being finite, such a group is contained in a conjugate of  $A$  or  $B$ . Thus it follows from our assumption that  $G$  admits only finitely many isomorphism classes of finite subgroup with dihedral shape. The conclusion follows from Theorem 5.7.  $\square$

## References

- [1] S. I. Adian. Periodic products of groups. *Akademiya Nauk Soyuz Sovetskikh Sotsialisticheskikh Respublik. Trudy Matematicheskogo Instituta imeni V. A. Steklova*, 142:3–21– 268, 1976.
- [2] B. H. Bowditch. Tight geodesics in the curve complex. *Inventiones Mathematicae*, 171(2):281–300, 2008.
- [3] B. H. Bowditch. Relatively hyperbolic groups. *International Journal of Algebra and Computation*, 22(3):1250016–1250066, 2012.
- [4] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

- [5] W. Burnside. On an unsettled question in the theory of discontinuous groups. *The Quarterly Journal of Pure and Applied Mathematics*, 33:230–238, 1902.
- [6] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990.
- [7] R. Coulon. Asphericity and small cancellation theory for rotation families of groups. *Groups, Geometry, and Dynamics*, 5(4):729–765, 2011.
- [8] R. Coulon. Outer automorphisms of free Burnside groups. *Commentarii Mathematici Helvetici*, 88(4):789–811, 2013.
- [9] R. Coulon. On the geometry of Burnside quotients of torsion free hyperbolic groups. *International Journal of Algebra and Computation*, 24(3):251–345, 2014.
- [10] R. Coulon. Partial periodic quotients of groups acting on a hyperbolic space. *Université de Grenoble. Annales de l’Institut Fourier*, 66(5):1773–1857, 2016.
- [11] R. Coulon. Théorie de la petite simplification: une approche géométrique [d’après F. Dahmani, V. Guirardel, D. Osin et S. Cantat, S. Lamy]. *Astérisque*, 380(Séminaire Bourbaki. Vol. 2014/2015):Exp. No. 1089, 1–33, 2016.
- [12] R. Coulon and D. Gruber. Small cancellation theory over Burnside groups. *arXiv.org* 1705.09651, May 2017.
- [13] F. Dahmani, V. Guirardel, and D. V. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Memoirs of the American Mathematical Society*, 245(1156):0–0, Jan. 2017.
- [14] T. Delzant. Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Mathematical Journal*, 83(3):661–682, 1996.
- [15] T. Delzant and M. Gromov. Courbure mésoscopique et théorie de la toute petite simplification. *Journal of Topology*, 1(4):804–836, 2008.
- [16] B. Farb. Relatively hyperbolic groups. *Geometric and Functional Analysis*, 8(5):810–840, 1998.
- [17] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [18] É. Ghys and P. de la Harpe. *Sur les groupes hyperboliques d’après Mikhael Gromov*, volume 83 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, Boston, MA, 1990.

- [19] M. Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, New York, New York, 1987.
- [20] M. Gromov. *Mesosopic curvature and hyperbolicity*, volume 288 of *Contemporary Mathematics*. American Mathematical Society, Providence, Rhode Island, 2001.
- [21] V. Guirardel. Geometric small cancellation. In *Geometric group theory*, pages 55–90. Amer. Math. Soc., Providence, RI, 2014.
- [22] W. J. Harvey. Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, pages 245–251. Princeton Univ. Press, Princeton, N.J., 1981.
- [23] G. C. Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. *Algebraic & Geometric Topology*, 10(3):1807–1856, 2010.
- [24] S. V. Ivanov. The free Burnside groups of sufficiently large exponents. *International Journal of Algebra and Computation*, 4(1-2):ii+308, 1994.
- [25] S. V. Ivanov and A. Y. Ol’shanskii. Hyperbolic groups and their quotients of bounded exponents. *Transactions of the American Mathematical Society*, 348(6):2091–2138, 1996.
- [26] I. G. Lysenok. Infinite Burnside groups of even period. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 60(3):3–224, 1996.
- [27] I. G. Lysenok. Burnside structures of finite subgroups. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 71(5):81–110, 2007.
- [28] H. Masur and Y. N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Inventiones Mathematicae*, 138(1):103–149, 1999.
- [29] P. S. Novikov and S. I. Adian. Infinite periodic groups. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 32:212–524 and 709–731, 1968.
- [30] A. Y. Ol’shanskii. The Novikov-Adyan theorem. *Matematicheskii Sbornik*, 118(160)(2):203–235, 287, 1982.
- [31] A. Y. Ol’shanskii. Periodic quotient groups of hyperbolic groups. *Matematicheskii Sbornik*, 182(4):543–567, 1991.
- [32] A. Y. Ol’shanskii. On residualing homomorphisms and G-subgroups of hyperbolic groups. *International Journal of Algebra and Computation*, 3(4):365–409, 1993.
- [33] J.-P. Serre. *Arbres, amalgames,  $SL_2$* . Société Mathématique de France, Paris, 1977.

- [34] A. Szczepański. Relatively hyperbolic groups. *Michigan Mathematical Journal*, 45(3):611–618, 1998.
- [35] W. P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *American Mathematical Society. Bulletin. New Series*, 19(2):417–431, 1988.

*Rémi Coulon*

Univ Rennes, CNRS

IRMAR - UMR 6625

F-35000 Rennes, France

`remi.coulon@univ-rennes1.fr`

`http://rcoulon.perso.math.cnrs.fr`