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Quantum master equation on cyclic cochains and categorical Gromov-Witten invariants.

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(A, m) is a $d\text{-}\mathbb{Z}/2\mathbb{Z}$ graded associative algebra with odd scalar product β , $\dim_k A < \infty$.

H is an odd selfadjoint operator $H : A \rightarrow \Pi A$, $H^\vee = H$, such that $Id - [d, H] = P$, $dP = 0$, $P^2 = P$. B - the image of P .

Let Γ be a 3-valent ribbon graph with legs, then put:

- on every vertex $v \rightarrow$ the 3-tensors of the cyclic product on A

$$m^v \in ((\Pi A)^{\otimes 3})^\vee$$

- on every interior edge $e = (ff') \rightarrow$ the two tensor

$$\beta^\vee(H^\vee u_f, v_{f'}), \beta_H^{\vee, e} \in (\Pi A)^{\otimes 2},$$

- on every leg $l \in \text{Leg}(\Gamma) \rightarrow$ element $a_l \in \Pi B$
- make the contraction

$$W_\Gamma \left(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left(\bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{\vee, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle$$

The summation over 3-valent ribbon graphs with legs
(cont'd).

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$$W_{\Gamma} \left(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left(\bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{v,e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle$$

$$S = \sum_{\{\Gamma\}} \hbar^{1-\chi(\Gamma)} W_{\Gamma}$$

- $S \in \text{Sym} \left(\bigoplus_{j=1}^{\infty} \text{Hom} \left((\Pi B^{\otimes j}), k \right)^{\mathbb{Z}/j\mathbb{Z}} \right) [[\hbar]]$

Theorem (S.B.[2009a]) The sum over 3-valent ribbon graphs S satisfy the QMECC/noncommutative Batalin-Vilkovisky equation:

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0$$

- Let $V = V_0 \oplus V_1$ be $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, $\dim_k V < \infty$, with scalar product $\beta : V^{\otimes 2} \rightarrow k[p]$ of degree $p \bmod 2$
- $C^\lambda = (\bigoplus_{j=1}^{\infty} \text{Hom}((\Pi V)^{\otimes j}, k)^{\mathbb{Z}/j\mathbb{Z}})$,
 $S \in \text{Sym}(C^\lambda [1-p])[[\hbar]]$ (symmetric for the odd product β , antisymmetric for the even β)
- The noncommutative BV differential on $F = \text{Sym}(C^\lambda [1-p])$

$$\begin{aligned} \Delta_{NC}(a_{\rho_1} \dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon \beta_{\rho_p \tau_q}^\vee (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_{q-1}} a_{\rho_{p+1}} \dots a_{\rho_r})^\lambda + \\ &\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\rho_p \rho_q}^\vee (\dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^\lambda (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda \\ &\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\tau_p \tau_q}^\vee (\dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^\lambda (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^\lambda \end{aligned}$$

- signs are the standard Koszul signs taking into account that $(a_{\rho_1} \dots a_{\rho_r})^\lambda = 1 - p + \sum \bar{a}_{\rho_i}$, $a_i \in \text{Hom}(\Pi V)$.

Theorem (S.B.[2006a]) $\Delta_{NC}^2 = 0$

- The Quantum Master equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.[2006a])

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0 \quad \text{QMCC}$$

$$S = \sum_{g \geq 0, i > 0} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(\cdot).$$

-

$$\{S_{0,1}, S_{0,1}\} = 0,$$

$S_{0,1}$ - A_∞ - algebra with (even/odd) scalar product, so S -multiloop, higher genus generalization of A_∞ - algebra.

- (QMCC) $\iff \Delta_{NC}(\exp \frac{1}{\hbar} S) = 0$
- It is natural to call it the “noncommutative BV” differential because if the previous summation is over usual graphs with no cyclic orders at $v \in \text{Vert}(G)$ then it gives solutions to the standard BV.

A *stable ribbon graph* is a connected graph G (recall: a graph G is a triple $(Flag(G), \lambda, \sigma)$, where $Flag(G)$ is a finite set, whose elements are called flags, λ is a partition of $Flag(G) \leftrightarrow$ vertices, σ is an involution acting on $Flag(G) \leftrightarrow$ edges/legs) together with:

- partitions of the set of flags adjacent to every vertex into $i(v)$ subsets

$$Leg(v) = Leg(v)^{(1)} \sqcup \dots \sqcup Leg(v)^{(i(v))}, v \in Vert(G)$$

- fixed cyclic order on every subset $Leg(v)^{(k)}$,
- a map $g : Vert(G) \rightarrow \mathbb{Z}_{\geq 0}$ such that for any vertex $2(2g(v) + i(v) - 2) + n(v) > 0$.
- $\bigoplus_{\{G\}} k(G, or(G))$,
 $or(G) \in Det(\bigotimes_{v \in Vert(G)} (k^{Flag(v)} \oplus k^{Cycle(v)}))$ has natural
 “generalized contraction of edges” differential δ_{graph}

Let $S \in \text{Sym}(C^\lambda [1 - \rho])[[\hbar]]$ be a solution to the QMCC, with $S_{0,1,2} = 0$ ($d_V = 0$), let Γ be a *stable* ribbon graph, then put:

- on every vertex $v \rightarrow$ the multi-cyclic tensors

$$S_{g,i}^v \in \text{Sym}(\bigoplus_{j=1}^i \text{Hom}((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}} [1 - \rho])$$

- on every edge $e = (ff') \rightarrow$ the two tensor

$$\beta^v(u_f, v_{f'}), \beta^{v,e} \in (\Pi V)^{\otimes 2},$$

- take the contraction $W_\Gamma = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} S_{g,i}^v, \left(\bigotimes_{e \in \text{Edge}(\Gamma)} \beta^{v,e} \right) \right\rangle$

Theorem (S.B.[2006a]) For any S - a solution to the QMCC equation the following chain is a cycle in the *stable ribbon* graph complex

$$W(S) = \sum_{\{\Gamma\} \in \text{SRG}} \hbar^{1-\chi(\Sigma_\Gamma)} W_\Gamma [S]$$

$$\delta_{\text{graph}} W(S) = 0,$$

therefore $[W(S)] \in H_* (\overline{\mathcal{M}}_{*,*})$

A metric on the stable ribbon graph is a function $l : \text{Edge}(G) \rightarrow \mathbb{R}_{>0}$. Given a stable ribbon graph G w/out legs and a metric on G one can construct by standard procedure a punctured Riemann surface $\Sigma(G)$, which will have double points in general.

- replace every edge by oriented open strip $[0, l] \times] - i\infty, +i\infty[$ and glue them for each cyclically ordered subset of each vertex according to the cyclic order.
- identify points corresponding to 2 subsets at vertices with $g(v) = 0, i(v) = 2$, (double points); for points with $2g(v) + i(v) > 2$ remplace the vertex by the Riemann surface of genus $g(v)$, which does not contain any marked point, connected to the rest via $i(v)$ double points.
- This gives an isomorphism of complexes

$$\left(\bigoplus_{\{G\} \in \text{SRG}} k[(G, \text{or}(G)), \delta_{\text{graph}}] \right) \rightarrow \bigoplus_{g,v} C_*(\overline{\mathcal{M}}'_{g,v} \times \mathbb{R}_{>0}^v / \mathbb{S}_v)$$

More constructions of solutions to the QMCC equation

- **Conjecture** (S.B.[2006a]). Counting of holomorphic curves $(\Sigma, \partial\Sigma, p_i) \rightarrow (M, \coprod L_i, \oplus H_*(L_i \cap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the QMCC equation.
- **Theorem** (S.B.[2013]) If A is an A_∞ -infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC can be constructed step by step starting from $\{S_{0,1}, S_{0,1}\} = 0$

Homotopy theory of the QMCC eq

- **Theorem** (S.B.[2006a]) Solutions to the QMCC eq are in one-to-one correspondence with the structure of algebra over modular operad, which is the *Feynman transform* of $\widehat{k[\mathbb{S}_n]}$
- **Theorem** (S.B.[2006a]) $H_*(\text{The Feynman transform of } \widehat{k[\mathbb{S}_n]}(0, \gamma, \nu)) \simeq H^*(\overline{\mathcal{M}}'_{g,\nu}/\mathbb{S}_\nu)$

Construction of cohomology classes in $H^*(\overline{\mathcal{M}}'_{g,v}/\mathbf{S}_v)$

- A - associative algebra , with odd scalar product
- Assume: l - an *odd derivation* acting on A , preserving the scalar product: , in general $l^2 \neq 0$ (!), $\exists \tilde{l}, [l, \tilde{l}] = 1$, $str([a, \cdot]) = 0$ for any $a \in A$.
- **Theorem** (S.B.[2006b, 2009b]) This data \rightarrow Cohomology classes in $H^*(\overline{\mathcal{M}}'_{g,n})$
- Example $q(N)$, $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$, where π -odd involution, $q(N)$ has *odd trace* otr , $l = [\Xi, \cdot]$, Ξ -odd element $\Xi = (0 \quad | \quad diag(\lambda_1, \dots, \lambda_n))$, ($l^2 \neq 0$ (!))
- **Theorem** (S.B.[2009b]) This is the generating function for products of tautological classes $c_1(T_i)$.

- **Conjecture** (S.B.[2009b]) This construction, applied to A_∞ -algebra $A = \text{End}(C)$, C is a generating object of the $D^b(\text{Coh}(Y))$, Y is the mirror dual Calabi-Yau manifold to X
 $\rightarrow \dots$
- \rightarrow all genus Gromov-Witten invariants of X

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