

Quantum master equation on cyclic cochains and categorical Gromov-Witten invariants

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Quantum master equation on cyclic cochains and categorical Gromov-Witten invariants.

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(A,m) is a d- $\mathbb{Z}/2\mathbb{Z}$ graded associative algebra with odd scalar product $\beta,\ \dim_k A<\infty$.

H is an odd selfadjoint operator $H:A\to \Pi A,\ H^\vee=H$, such that $Id-[d,H]=P,\ dP=0$, $P^2=P.\ B$ - the image of P. Let Γ be a 3-valent *ribbon* graph with legs, then put:

• on every vertex $v \rightarrow$ the 3-tensors of the cyclic product on A

$$m^{\mathsf{v}} \in ((\Pi A)^{\otimes 3})^{\vee}$$

• on every interieur edge $e = (ff') \rightarrow$ the two tensor

$$\beta^{\vee}(H^{\vee}u_f, v_{f'}), \ \beta_H^{\vee,e} \in (\Pi A)^{\otimes 2},$$

- on every leg $I \in Leg(\Gamma) \to \text{element } a_I \in \Pi B$
- make the contraction

$$W_{\Gamma}(\bigotimes_{I \in Leg(\Gamma)} a_I) = \langle \bigotimes_{v \in Vert(\Gamma)} m^v, \left(\bigotimes_{e \in Edge(\Gamma)} \beta_H^{\lor,e}\right) \bigotimes_{I \in Leg(\Gamma)} a_I \rangle$$

The summation over 3-valent ribbon graphs with legs (cont'd).

$$W_{\Gamma}(igotimes_{I \in Leg(\Gamma)} a_I) = \langle igotimes_{v \in Vert(\Gamma)} m^v, \left(igotimes_{e \in Edge(\Gamma)} eta_H^{\lor,e}
ight) igotimes_{I \in Leg(\Gamma)} a_I
angle \ S = \sum_{\{\Gamma\}} \hbar^{1-\chi(\Gamma)} W_{\Gamma}$$

• $S \in \operatorname{Sym}(\bigoplus_{j=1}^{\infty} \operatorname{Hom}((\Pi B^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})[[\mathcal{T}]]$

Theorem (S.B.[2009a]) The sum over 3-valent ribbon graphs *S* satisfy the QMECC/noncommutative Batalin-Vilkovisky equation:

$$\hbar\Delta_{NC}S + \frac{1}{2}\{S, S\} = 0$$

 $\dim_k V < \infty$, with scalar product $\beta: V^{\otimes 2} o k[p]$ of degree p mod2 • $C^{\lambda} = (\bigoplus_{i=1}^{\infty} Hom((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})$,

• Let $V = V_0 \oplus V_1$ be $\mathbb{Z}/2\mathbb{Z}$ -graded vector space,

 $S \in \operatorname{Sym}(C^{\lambda}[1-p])[[h]]$ (symmetric for the odd product β , antisymmetric for the even β) • The noncommutative BV differential on $F = \operatorname{Sym}(C^{\lambda}[1-p])$

• The noncommutative BV differential on
$$F=\operatorname{Sym}(C^{\Lambda}[1-p])$$

$$\Delta_{NC}(a_{\rho_1}\dots a_{\rho_r})^{\lambda}(a_{\tau_1}\dots a_{\tau_t})^{\lambda}=$$

$$egin{aligned} egin{aligned} egin{aligned} eta_{\mathcal{N}C}(m{a}_{
ho_1} \dots m{a}_{
ho_r}) & (m{a}_{ au_1} \dots m{a}_{ au_t}) & - \ &= \sum_{p,q} (-1)^arepsilon eta_{
ho_p au_q}^ee (m{a}_{
ho_1} \dots m{a}_{
ho_{p-1}} m{a}_{ au_{q+1}} \dots m{a}_{ au_{q-1}} m{a}_{
ho_{p+1}} \dots m{a}_{
ho_r})^\lambda + \end{aligned}$$

$$\sum_{\rho\pm 1\neq q} (-1)^{\tilde{\varepsilon}} \beta_{\rho_\rho \rho_q}^\vee (\dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^\lambda (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda$$

$$\sum_{
ho\pm1
eq q}(-1)^{ ilde{arepsilon}}eta_{
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ho_{oldsymbol{q}}}^{arphi}(\ldots a_{
ho_{
ho-1}}a_{
ho_{q+1}}\ldots a_{
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ho_{q-1}})^{lpha}(a_{ au_{1}}\ldots a_{ au_{r-1}}a_{ au_{q+1}}\ldots a_{ au_{r}})^{lpha}(a_{ au_{r+1}}\ldots a_{ au_{r+1}})^{lpha}(a_{ au_{r+1}}\ldots a_{ au_{r+1}}\ldots a_{ au_{r+1}})^{lpha}(a_{ au_{r+1}}\ldots a_{ au_{r+1}})^{lpha}(a_{ au_{r+1}}\ldots a_{ au_{r+1}})^{lpha}(a_{ au_{r+1}}\ldots a_{ au_{r+1}}\ldots a_{ au_{r+$$

 $\sum (-1)^{\tilde{\varepsilon}} \beta_{\tau_n \tau_n}^{\vee} (\dots a_{\rho_r})^{\lambda} (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^{\lambda} (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^{\lambda}$

 $p\pm 1\neq a$

signs are the standard Koszul signs taking into account that

 $(a_{\varrho_1}\dots a_{\varrho_r})^{\lambda}=1-p+\sum \bar{a_{\varrho_i}}$, $a_i\in {\sf Hom}\,(\Pi V)$. **Theorem** (S.B.[2006a]) $\Delta_{NC}^2 = 0$

 The Quantum Master equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.[2006a])

$$\hbar \Delta_{NC} S + \frac{1}{2} \{ S, S \} = 0$$
 QMCC

$$S = \sum_{g \geq 0, i > 0} \mathcal{T}^{2g-1+i} S_{g,i}$$
 , $S_{g,i} \in \mathsf{Sym}^i(\cdot)$.

$${S_{0,1}, S_{0,1}} = 0,$$

 $S_{0,1}$ - A_{∞} — algebra with (even/odd) scalar product, so S — multiloop, higher genus generalization of A_{∞} — algebra.

- (QMCC) $\iff \Delta_{NC}(\exp \frac{1}{\hbar}S) = 0$
- It is natural to call it the "noncommutative BV" differential because if the previous summation is over usual graphs with no cyclic orders at $v \in Vert(G)$ then it gives solutions to the standard BV.

A stable ribbon graph is a connected graph G (recall: a graph G is a triple $(Flag(G), \lambda, \sigma)$, where Flag(G) is a finite set, whose elements are called flags, λ is a partition of $Flag(G) \leftrightarrow$ vertices, σ is an involution acting on $Flag(G) \leftrightarrow$ edges/legs) together with:

• partitions of the set of flags adjacent to every vertex into i(v) subsets

$$Leg(v) = Leg(v)^{(1)} \sqcup \ldots \sqcup Leg(v)^{(i(v))}, v \in Vert(G)$$

- fixed cyclic order on every subset $Leg(v)^{(k)}$,
- a map $g: Vert(G) \to \mathbb{Z}_{\geq 0}$ such that for any vertex 2(2g(v)+i(v)-2)+n(v)>0 .
- $\bigoplus_{\{G\}} k(G, or(G))$, $or(G) \in Det(\bigotimes_{v \in Vert(G)} (k^{Flag(v)} \bigoplus k^{Cycle(v)})$ has natural "generalized contraction of edges" differential δ_{graph}

Let $S \in \operatorname{Sym}(C^{\lambda}[1-p])[[\hbar]]$ be a solution to the QMCC, with $S_{0,1,2}=0$ ($d_V=0$), let Γ be a *stable* ribbon graph, then put:

ullet on every vertex v o the multi-cyclic tensors

$$S_{g,i}^{v} \in \operatorname{Sym}^{i}(\oplus_{j=1}^{\infty} \operatorname{Hom}((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}} [1-p])$$

• on every edge $e = (ff') \rightarrow$ the two tensor

$$eta^ee(u_f, v_{f'}), \ eta^{ee, e} \in (\Pi V)^{\otimes 2},$$

• take the contraction $W_{\Gamma} = \langle \bigotimes_{v \in Vert(\Gamma)} S_{g,i}^{v}, \begin{pmatrix} \bigotimes_{e \in Edge(\Gamma)} \beta^{\vee,e} \end{pmatrix} \rangle$ **Theorem** (S.B.[2006a]) For any *S*- a solution to the QMCC equation the following chain is a cycle in the stable ribbon graph

 $\delta_{graph}W(S)=0.$

Theorem (S.B.[2006a]) For any *S*- a solution to the QMCC equation the following chain is a cycle in the *stable ribbon* graph complex

complex
$$W(S) = \sum_{\{\Gamma\} \in \mathit{SRG}}
olimits_h^{1-\chi(\Sigma_\Gamma)} W_\Gamma\left[\Gamma
ight]$$

therefore
$$[W(S)] \in H_*(\overline{\mathcal{M}}_{*,*})$$

A metric on the stable ribbon graph is a function $I: Edge(G) \to \mathbb{R}_{>0}$. Given a stable ribbon graph G w/out legs and a metric on G one can construct by standard procedure a punctured Riemann surface $\Sigma(G)$, which will have double points in general.

- replace every edge by oriented open strip $[0, l] \times] i \infty, + i \infty [$ and glue them for each cyclically ordered subset of each vertex according to the cyclic order.
- identify points corresponding to 2 subsets at vertices with g(v)=0, i(v)=2, (double points); for points with 2g(v)+i(v)>2 remplace the vertex by the Riemann surface of genus g(v), which does not contain any marked point, connected to the rest via i(v) double points.
- This gives an isomorphism of complexes $(\bigoplus_{\{G\}\in SRG} k\left[(G,or(G)],\delta_{graph})
 ightarrow \oplus_{g,\nu} C_*(\overline{\mathcal{M}}'_{g,\nu} imes \mathbb{R}^{\nu}_{>0}/\mathbb{S}_{\nu})$

More constructions of solutions to the QMCC equation

- Conjecture (S.B.[2006a]). Counting of holomorphic curves $(\Sigma, \partial \Sigma, p_i) \to (M, \coprod L_i, \oplus H_*(L_i \cap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the QMCC equation.
- Theorem (S.B.[2013]) If A is an A_{∞} —infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC can be constructed step by step starting from $\{S_{0,1}, S_{0,1}\} = 0$

Homotopy theory of the QMCC eq

- **Theorem** (S.B.[2006a]) Solutions to the QMCC eq are in one-to-one correspondence with the structure of algebra over modular operad, which is the *Feynman transform* of $\widehat{k[S_n]}$
- **Theorem** (S.B.[2006a]) H_* (The Feynman transform of $\widehat{k[S_n]}(0, \gamma, \nu)$) $\simeq H^*(\overline{\mathcal{M}}'_{\sigma, \nu}/S_{\nu})$

Construction of cohomology classes in $H^*(\overline{\mathcal{M}}'_{g,\nu}/\mathbb{S}_{\nu})$

- A- associative algebra, with odd scalar product
- Assume: I an odd derivation acting on A, preserving the scalar product: , in general $I^2 \neq 0$ (!), $\exists \widetilde{I}$, $[I, \widetilde{I}] = 1$, $str([a, \cdot]) = 0$ for any $a \in A$.
- Theorem (S.B.[2006b, 2009b])This data →Cohomology classes in H*(M'_{g,n})
- Example q(N), $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$, where π -odd involution, q(N) has odd trace otr, $I = [\Xi, \cdot]$, Ξ -odd element $\Xi = \begin{pmatrix} 0 & | diag(\lambda_1, \dots, \lambda_n) \end{pmatrix}$, $\begin{pmatrix} I^2 \neq 0 & (!) \end{pmatrix}$
- Theorem (S.B.[2009b]) This is the generating function for products of tautological classes c₁(T_i).

• **Conjecture** (S.B.[2009b]) This construction, applied to A_{∞} —algebra A = End(C), C is a generating object of the

• \rightarrow all genus Gromov-Witten invariants of X

 \rightarrow ...

 $D^b(Coh(Y))$, Y is the mirror dual Calabi-Yau manifold to X

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