Quantum master equation on cyclic cochains and categorical Gromov-Witten invariants

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(A, m) is a d-$\mathbb{Z}/2\mathbb{Z}$ graded *associative* algebra with odd scalar product β, $\dim_k A < \infty$.
H is an odd selfadjoint operator $H : A \to \Pi A$, $H^\vee = H$, such that $Id - [d, H] = P$, $dP = 0$, $P^2 = P$. B - the image of P.
Let Γ be a 3-valent *ribbon* graph with legs, then put:

- on every vertex $v \to$ the 3-tensors of the cyclic product on A

$$m^v \in ((\Pi A)^{\otimes 3})^\vee$$

- on every interieur edge $e = (ff') \to$ the two tensor

$$\beta^\vee(H^\vee u_f, v_{f'}),\ \beta_H^{\vee,e} \in (\Pi A)^{\otimes 2},$$

- on every leg $l \in Leg(\Gamma) \to$ element $a_l \in \Pi B$
- make the contraction

$$W_\Gamma(\bigotimes_{l \in Leg(\Gamma)} a_l) = \langle \bigotimes_{v \in Vert(\Gamma)} m^v, \left(\bigotimes_{e \in Edge(\Gamma)} \beta_H^{\vee,e} \right) \bigotimes_{l \in Leg(\Gamma)} a_l \rangle$$

The summation over 3-valent ribbon graphs with legs (cont'd).

- $$W_\Gamma(\bigotimes_{l\in Leg(\Gamma)} a_l) = \langle \bigotimes_{v\in Vert(\Gamma)} m^v, \left(\bigotimes_{e\in Edge(\Gamma)} \beta_H^{\vee,e}\right) \bigotimes_{l\in Leg(\Gamma)} a_l\rangle$$

$$S = \sum_{\{\Gamma\}} \hbar^{1-\chi(\Gamma)} W_\Gamma$$

- $S \in \mathrm{Sym}(\oplus_{j=1}^{\infty} Hom((\Pi B^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})[[\hbar]]$

Theorem (S.B.[2009a]) The sum over 3-valent ribbon graphs S satisfy the QMECC/noncommutative Batalin-Vilkovisky equation:

$$\hbar\Delta_{NC} S + \frac{1}{2}\{S, S\} = 0$$

- Let $V = V_0 \oplus V_1$ be $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, $\dim_k V < \infty$, with scalar product $\beta : V^{\otimes 2} \to k\,[p]$ of degree $p \bmod 2$
- $C^\lambda = (\oplus_{j=1}^{\infty} Hom((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})$, $S \in \mathrm{Sym}(C^\lambda\,[1-p])[[\hbar]]$ (symmetric for the odd product β , antisymmetric for the even β)
- The noncommutative BV differential on $F = \mathrm{Sym}(C^\lambda\,[1-p])$

$$\Delta_{NC}(a_{\rho_1}\dots a_{\rho_r})^\lambda(a_{\tau_1}\dots a_{\tau_t})^\lambda =$$
$$= \sum_{p,q}(-1)^{\varepsilon}\beta^\vee_{\rho_p\tau_q}(a_{\rho_1}\dots a_{\rho_{p-1}}a_{\tau_{q+1}}\dots a_{\tau_{q-1}}a_{\rho_{p+1}}\dots a_{\rho_r})^\lambda +$$
$$\sum_{p\pm 1\neq q}(-1)^{\tilde{\varepsilon}}\beta^\vee_{\rho_p\rho_q}(\dots a_{\rho_{p-1}}a_{\rho_{q+1}}\dots a_{\rho_r})^\lambda(a_{\rho_{p+1}}\dots a_{\rho_{q-1}})^\lambda(a_{\tau_1}\dots a_{\tau_t})^\lambda$$
$$\sum_{p\pm 1\neq q}(-1)^{\tilde{\tilde{\varepsilon}}}\beta^\vee_{\tau_p\tau_q}(\dots a_{\rho_r})^\lambda(a_{\tau_1}\dots a_{\tau_{p-1}}a_{\tau_{q+1}}\dots a_{\tau_t})^\lambda(a_{\tau_{p+1}}\dots a_{\tau_{q-1}})^\lambda$$

- signs are the standard Koszul signs taking into account that $(a_{\rho_1}\dots a_{\rho_r})^\lambda = 1-p+\sum \bar{a}_{\rho_i}$, $a_i \in \mathrm{Hom}\,(\Pi V)$.

Theorem (S.B.[2006a]) $\Delta_{NC}^2 = 0$

- The Quantum Master equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.[2006a])

$$\hbar\Delta_{NC}S + \frac{1}{2}\{S,S\} = 0 \qquad \text{QMCC}$$

$S = \sum_{g\geq 0, i>0} \hbar^{2g-1+i} S_{g,i}\,,\ S_{g,i} \in \mathrm{Sym}^i(\cdot)\ .$

- $$\{S_{0,1}, S_{0,1}\} = 0,$$

  $S_{0,1}$ - $A_\infty-$ algebra with (even/odd) scalar product, so $S-$ *multiloop, higher genus* generalization of $A_\infty-$ algebra.
- (QMCC) $\Longleftrightarrow \Delta_{NC}(\exp\frac{1}{\hbar}S) = 0$
- It is natural to call it the *"noncommutative BV" differential* because if the previous summation is over usual graphs with no cyclic orders at $v \in Vert(G)$ then it gives solutions to the standard BV.

A *stable ribbon graph* is a connected graph G (recall: a graph G is a triple $(Flag(G), \lambda, \sigma)$, where $Flag(G)$ is a finite set, whose elements are called flags, λ is a partition of $Flag(G) \leftrightarrow$ vertices , σ is an involution acting on $Flag(G) \leftrightarrow$ edges/legs) together with:

- partitions of the set of flags adjacent to every vertex into $i(v)$ subsets

$$Leg(v) = Leg(v)^{(1)} \sqcup \ldots \sqcup Leg(v)^{(i(v))}, v \in Vert(G)$$

- fixed cyclic order on every subset $Leg(v)^{(k)}$,
- a map $g : Vert(G) \to \mathbb{Z}_{\geq 0}$ such that for any vertex $2(2g(v) + i(v) - 2) + n(v) > 0$.
- $\oplus_{\{G\}} k(G, or(G))$,
  $or(G) \in Det(\otimes_{v \in Vert(G)}(k^{Flag(v)} \oplus k^{Cycle(v)})$ has natural "generalized contraction of edges" differential δ_{graph}

Let $S \in \mathrm{Sym}(C^{\lambda}[1-p])[[\hbar]]$ be a solution to the QMCC, with $S_{0,1,2} = 0$ ($d_V = 0$), let Γ be a *stable* ribbon graph, then put:

- on every vertex $v \rightarrow$ the multi-cyclic tensors

$$S^v_{g,i} \in \overset{i}{\mathrm{Sym}}(\oplus_{j=1}^{\infty} Hom((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}}[1-p])$$

- on every edge $e = (ff') \rightarrow$ the two tensor

$$\beta^{\vee}(u_f, v_{f'}),\ \beta^{\vee,e} \in (\Pi V)^{\otimes 2},$$

- take the contraction $W_{\Gamma} = \langle \bigotimes_{v \in Vert(\Gamma)} S^v_{g,i}, \left(\bigotimes_{e \in Edge(\Gamma)} \beta^{\vee,e} \right) \rangle$

Theorem (S.B.[2006a]) For any S- a solution to the QMCC equation the following chain is a cycle in the *stable ribbon* graph complex

$$W(S) = \sum_{\{\Gamma\} \in SRG} \hbar^{1-\chi(\Sigma_{\Gamma})} W_{\Gamma}[\Gamma]$$

$$\delta_{graph} W(S) = 0,$$

therefore $[W(S)] \in H_* \left(\overline{\mathcal{M}}_{*,*} \right)$

A metric on the stable ribbon graph is a function $l : Edge(G) \to \mathbb{R}_{>0}$. Given a stable ribbon graph G w/out legs and a metric on G one can construct by standard procedure a punctured Riemann surface $\Sigma(G)$, which will have double points in general.

- replace every edge by oriented open strip $[0, l] \times]-i\infty, +i\infty[$ and glue them for each cyclically ordered subset of each vertex according to the cyclic order.
- identify points corresponding to 2 subsets at vertices with $g(v) = 0, i(v) = 2$, (double points); for points with $2g(v) + i(v) > 2$ remplace the vertex by the Riemann surface of genus $g(v)$, which does not contain any marked point, connected to the rest via $i(v)$ double points.
- This gives an isomorphism of complexes $(\oplus_{\{G\} \in SRG} k\left[(G, or(G)\right], \delta_{graph}) \to$ $\oplus_{g,\nu} C_*(\overline{\mathcal{M}}'_{g,\nu} \times \mathbb{R}^{\nu}_{>0}/\mathbb{S}_{\nu})$

More constructions of solutions to the QMCC equation

- **Conjecture** (S.B.[2006a]). Counting of holomorphic curves $(\Sigma, \partial\Sigma, p_i) \to (M, \coprod L_i, \oplus H_*(L_i \bigcap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the QMCC equation.
- **Theorem** (S.B.[2013]) If A is an $A_\infty-$infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC can be constructed step by step starting from $\{S_{0,1}, S_{0,1}\} = 0$

Homotopy theory of the QMCC eq

- **Theorem** (S.B.[2006a]) Solutions to the QMCC eq are in one-to-one correspondence with the structure of algebra over modular operad, which is the *Feynman transform* of $\widehat{k[\mathbb{S}_n]}$
- **Theorem** (S.B.[2006a]) $H_*(\text{The Feynman transform of } \widehat{k[\mathbb{S}_n]}(0,\gamma,\nu)) \simeq H^*(\overline{\mathcal{M}}'_{g,\nu}/\mathbb{S}_\nu)$

Construction of cohomology classes in $H^*(\overline{\mathcal{M}}'_{g,\nu}/\mathbb{S}_\nu)$

- A- associative algebra , with odd scalar product
- Assume: I - an *odd derivation* acting on A, preserving the scalar product: , in general $I^2 \neq 0$ (!), $\exists \widetilde{I}$, $[I, \widetilde{I}] = 1$, $str([a, \cdot]) = 0$ for any $a \in A$.
- **Theorem** (S.B.[2006b, 2009b])This data $\rightarrow$Cohomology classes in $H^*(\overline{\mathcal{M}}'_{g,n})$
- Example $q(N)$, $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$,where $\pi-$odd involution, $q(N)$ has *odd trace otr*, $I = [\Xi, \cdot]$, Ξ-odd element $\Xi = \begin{pmatrix} 0 & | diag(\lambda_1, \ldots, \lambda_n) \end{pmatrix}$, ($I^2 \neq 0$ (!))
- **Theorem** (S.B.[2009b]) This is the generating function for products of tautological classes $c_1(T_i)$.

- **Conjecture** (S.B.[2009b]) This construction, applied to $A_\infty-$algebra $A = End(C)$, C is a generating object of the $D^b(Coh(Y))$, Y is the mirror dual Calabi-Yau manifold to X $\rightarrow$...
- $\rightarrow$all genus Gromov-Witten invariants of X

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