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Quantum master equation on cyclic cochains and categorical Gromov-Witten invariants.
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\((A, m)\) is a \(d\mathbb{Z}/2\mathbb{Z}\) graded associative algebra with odd scalar product \(\beta\), \(\dim_k A < \infty\).

\(H\) is an odd selfadjoint operator \(H : A \rightarrow \Pi A, \ H^\dagger = H\), such that \(\text{Id} - [d, H] = P, \ dP = 0, \ P^2 = P\). \(B\) - the image of \(P\).

Let \(\Gamma\) be a 3-valent ribbon graph with legs, then put:

- on every vertex \(v \rightarrow\) the 3-tensors of the cyclic product on \(A\)

\[
m^v \in ((\Pi A)^ \otimes 3)^\dagger
\]

- on every interior edge \(e = (ff') \rightarrow\) the two tensor

\[
\beta^\dagger (H^\dagger u_f, v_{f'}), \ \beta^\dagger,^e_H \in (\Pi A)^ \otimes 2,
\]

- on every leg \(l \in \text{Leg}(\Gamma) \rightarrow\) element \(a_l \in \Pi B\)

- make the contraction

\[
\mathcal{W}_\Gamma \left( \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^\dagger,^e_H \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle
\]
The summation over 3-valent ribbon graphs with legs (cont’d).

\[ W_\Gamma(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l) = \langle \bigotimes_{\nu \in \text{Vert}(\Gamma)} m^{\nu}, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta_{\Gamma, e}^{\nu, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \rangle \]

\[ S = \sum_{\{\Gamma\}} \mathcal{h}^{1-\chi(\Gamma)} W_\Gamma \]

\[ S \in \text{Sym}\left( \bigoplus_{j=1}^{\infty} \text{Hom}(\Pi B^\otimes j), k^{\mathbb{Z}/j\mathbb{Z}} \right)[[\mathcal{h}]] \]

**Theorem** (S.B.[2009a]) The sum over 3-valent ribbon graphs \( S \) satisfy the QMECC/noncommutative Batalin-Vilkovisky equation:

\[ \hbar \Delta_{NC} S + \frac{1}{2} \{ S, S \} = 0 \]
• Let $V = V_0 \oplus V_1$ be $\mathbb{Z}/2\mathbb{Z}$-graded vector space, $\dim_k V < \infty$, with scalar product $\beta : V \otimes^2 \to k[p]$ of degree $p \mod 2$

• $C^\lambda = (\bigoplus_{j=1}^\infty \text{Hom}((\prod V \otimes^j), k)\mathbb{Z}/j\mathbb{Z})$, $S \in \text{Sym}(C^\lambda [1 - p])[\hbar]$ (symmetric for the odd product $\beta$, antisymmetric for the even $\beta$)

• The noncommutative BV differential on $F = \text{Sym}(C^\lambda [1 - p])$

$$\Delta_{NC}(a_{\rho_1} \ldots a_{\rho_r})^\lambda (a_{\tau_1} \ldots a_{\tau_t})^\lambda =$$

$$= \sum_{p,q} (-1)^{\epsilon} \beta^\lambda_{\rho_p \rho_q} (a_{\rho_1} \ldots a_{\rho_{p-1}} a_{\tau_{q+1}} \ldots a_{\tau_{q-1}} a_{\rho_{p+1}} \ldots a_{\rho_r})^\lambda +$$

$$\sum_{p \pm 1 \neq q} (-1)^{\epsilon} \beta^\lambda_{\rho_p \rho_q} (\ldots a_{\rho_{p-1}} a_{\rho_{q+1}} \ldots a_{\rho_r})^\lambda (a_{\rho_{p+1}} \ldots a_{\rho_{q-1}})^\lambda (a_{\tau_1} \ldots a_{\tau_t})^\lambda$$

$$\sum_{p \pm 1 \neq q} (-1)^{\epsilon} \beta^\lambda_{\tau_p \tau_q} (\ldots a_{\rho_r})^\lambda (a_{\tau_1} \ldots a_{\tau_{p-1}} a_{\tau_{q+1}} \ldots a_{\tau_t})^\lambda (a_{\tau_{p+1}} \ldots a_{\tau_{q-1}})^\lambda$$

• signs are the standard Koszul signs taking into account that $(a_{\rho_1} \ldots a_{\rho_r})^\lambda = 1 - p + \sum a_{\rho_i} \bar{a}_{\rho_i}$, $a_i \in \text{Hom}(\prod IV)$

**Theorem** (S.B.[2006a]) $\Delta_{NC}^2 = 0$
The Quantum Master equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.[2006a])

\[ \hbar \Delta_{NC} S + \frac{1}{2} \{ S, S \} = 0 \quad \text{QMCC} \]

\[ S = \sum_{g \geq 0, i > 0} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^{i}(\cdot) . \]

\{ S_{0,1}, S_{0,1} \} = 0,

\( S_{0,1} \) - \( A_{\infty} \) — algebra with (even/odd) scalar product, so \( S \) — multiloop, higher genus generalization of \( A_{\infty} \) — algebra.

(QMCC) \iff \( \Delta_{NC}(\exp \frac{1}{\hbar} S) = 0 \)

It is natural to call it the “noncommutative BV” differential because if the previous summation is over usual graphs with no cyclic orders at \( \nu \in \text{Vert}(G) \) then it gives solutions to the standard BV.
A stable ribbon graph is a connected graph $G$ (recall: a graph $G$ is a triple $(\text{Flag}(G), \lambda, \sigma)$, where $\text{Flag}(G)$ is a finite set, whose elements are called flags, $\lambda$ is a partition of $\text{Flag}(G) \leftrightarrow$ vertices, $\sigma$ is an involution acting on $\text{Flag}(G) \leftrightarrow$ edges/legs) together with:

- partitions of the set of flags adjacent to every vertex into $i(v)$ subsets

$$\text{Leg}(v) = \text{Leg}(v)^{(1)} \sqcup \ldots \sqcup \text{Leg}(v)^{(i(v))}, \, v \in \text{Vert}(G)$$

- fixed cyclic order on every subset $\text{Leg}(v)^{(k)}$,

- a map $g : \text{Vert}(G) \to \mathbb{Z}_{\geq 0}$ such that for any vertex $2(2g(v) + i(v) - 2) + n(v) > 0$.

- $\bigoplus_G k(G, \text{or}(G))$,

$$\text{or}(G) \in \text{Det}(\otimes_{v \in \text{Vert}(G)} (k^{\text{Flag}(v)} \oplus k^{\text{Cycle}(v)}))$$ has natural “generalized contraction of edges” differential $\delta_{\text{graph}}$
Let \( S \in \text{Sym}(C^1 [1 - p])[\hbar]\) be a solution to the QMCC, with \( S_{0,1,2} = 0 \) (\( d_V = 0 \)), let \( \Gamma \) be a stable ribbon graph, then put:

- on every vertex \( \nu \rightarrow \) the multi-cyclic tensors

\[
S^\nu_{g,i} \in \text{Sym}(\bigoplus_{j=1}^\infty \text{Hom}( (\Pi V^\otimes j), k)^{\mathbb{Z}/j\mathbb{Z}} [1 - p])
\]

- on every edge \( e = (ff') \rightarrow \) the two tensor

\[
\beta^\nu(u_f, v_{f'}), \ \beta^\nu,e \in (\Pi V)^\otimes 2,
\]

- take the contraction \( W_\Gamma = \langle \bigotimes_{\nu \in \text{Vert}(\Gamma)} S^\nu_{g,i}, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^\nu,e \right) \rangle \)

**Theorem** (S.B.[2006a]) For any \( S\)- a solution to the QMCC equation the following chain is a cycle in the stable ribbon graph complex

\[
W(S) = \sum_{\{\Gamma\} \in \text{SRG}} \hbar^{1-\chi(\Sigma \Gamma)} W_\Gamma [\Gamma]
\]

\[
\delta_{\text{graph}} W(S) = 0,
\]

therefore \([W(S)] \in H_* (\overline{M}_{\ast,\ast}) \)
A metric on the stable ribbon graph is a function $l : \text{Edge}(G) \to \mathbb{R}_{>0}$. Given a stable ribbon graph $G$ w/out legs and a metric on $G$ one can construct by standard procedure a punctured Riemann surface $\Sigma(G)$, which will have double points in general.

- replace every edge by oriented open strip $[0, l] \times ] - i\infty, + i\infty[$ and glue them for each cyclically ordered subset of each vertex according to the cyclic order.
- identify points corresponding to 2 subsets at vertices with $g(v) = 0, i(v) = 2$, (double points); for points with $2g(v) + i(v) > 2$ remplace the vertex by the Riemann surface of genus $g(v)$, which does not contain any marked point, connected to the rest via $i(v)$ double points.
- This gives an isomorphism of complexes

$$
\left( \bigoplus_{\{G\} \in \text{SRG}^k} \left[ (G, or(G)), \delta_{\text{graph}} \right] \right) \to \\
\bigoplus_{g,v} C_* \left( \overline{\mathcal{M}}_{g,v}^/ \times \mathbb{R}_{>0}^v / S_v \right)
$$
More constructions of solutions to the QMCC equation

- **Conjecture** (S.B. [2006a]). Counting of holomorphic curves $(\Sigma, \partial \Sigma, p_i) \rightarrow (M, \bigsqcup L_i, \bigoplus H_*(L_i \cap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$-graded local systems, gives solution to the QMCC equation.

- **Theorem** (S.B. [2013]) If $A$ is an $A_\infty$—infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC can be constructed step by step starting from $\{S_{0,1}, S_{0,1}\} = 0$
Homotopy theory of the QMCC eq

- **Theorem** (S.B.[2006a]) Solutions to the QMCC eq are in one-to-one correspondence with the structure of algebra over modular operad, which is the *Feynman transform* of $k[\mathcal{S}_n]$
- **Theorem** (S.B.[2006a]) $H_*(\text{The Feynman transform of } k[\mathcal{S}_n](0, \gamma, \nu)) \simeq H^*(\bar{\mathcal{M}}'_{g,\nu}/\mathcal{S}_\nu)$
Construction of cohomology classes in $H^*(\overline{\mathcal{M}}_{g,v}^I / \mathcal{S}_v)$

- $A$- associative algebra, with odd scalar product
- Assume: $I$ - an odd derivation acting on $A$, preserving the scalar product: $I^2 \neq 0 (!)$, $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, $str([a, \cdot]) = 0$ for any $a \in A$.
- **Theorem** (S.B.[2006b, 2009b]) This data $\rightarrow$ Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n}^I)$
- Example $q(N)$, $q(N) = \{ [X, \pi] = 0 | X \in gl(N|N) \}$, where $\pi$—odd involution, $q(N)$ has odd trace otr, $I = [\Xi, \cdot]$, $\Xi$—odd element $\Xi = (0 | diag(\lambda_1, \ldots, \lambda_n))$, ($I^2 \neq 0 (!)$)
- **Theorem** (S.B.[2009b]) This is the generating function for products of tautological classes $c_1(T_i)$. 
• **Conjecture** (S.B.[2009b]) This construction, applied to $A_\infty$—algebra $A = \text{End}(C)$, $C$ is a generating object of the $D^b(\text{Coh}(Y))$, $Y$ is the mirror dual Calabi-Yau manifold to $X$ →...

• → all genus Gromov-Witten invariants of $X$


