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An Overview on Skew Constacyclic Codes and their Subclass of LCD Codes.

Ranya D.Boulanouar, Aicha Batoul *and Delphine Boucher †

Abstract

This text is about the derivation of some necessary and sufficient conditions for the equivalency between skew constacyclic codes, skew cyclic codes and skew negacyclic codes defined over finite fields. A first characterization of LCD skew constacyclic codes is also given and some constructions of LCD MDS skew cyclic and skew negacyclic codes are derived over $\mathbb{F}_{p^2}$.

Keywords Skew polynomial rings, Skew constacyclic codes,Equivalence,LCD codes,MDS codes

1 Introduction

One of the most active and important research areas in noncommutative algebra is the investigation of skew polynomial rings. Recently they have been successfully applied in many areas and specially in coding theory. The principal motivation for studying codes in this setting is that polynomials in skew polynomials rings exhibit many factorizations and hence there are many more ideals in a skew polynomial ring than in the commutative case. The research on codes in this setting has resulted in the discovery of many new codes with better Hamming distances than any previously linear code with the same parameters [3, 4, 5, 6, 7, 8].

On the other hand, constacyclic codes over finite fields is an important class of linear codes as it includes the well-known family of cyclic codes. They also have many practical applications as they can be efficiently encoded using simple shift registers. Further, they have a rich algebraic structure which can be used for efficient error detection and correction. As a consequence, these codes are preferred in practical applications. One of the contribution of this work is to generalize the results given in the commutative case by Batoul and al in [2] to the noncommutative case.

Linear code with complementary dual (LCD) codes, were introduced by Massey [13]. They provide an optimum linear coding solution for the two-user binary adder channel, and in [14] it was shown that asymptotically good LCD codes exist, since then several authors have studied these codes. But until now just a few works have been done on LCD codes in the non commutative case. In our work we give a new definition of θ-reversible codes. One of our results show that LCD codes are not θ-reversible codes which is true in the commutative case [15].

This article is organized as follows.

In Section 2, some preliminaries are given about skew constacyclic codes over finite fields and skew polynomial rings. In Section 3, the equivalency between skew constacyclic codes and skew cyclic codes is studied. In Section 4, conditions for the equivalency between skew constacyclic codes and skew negacyclic codes are provided. We give examples of skew cyclic codes which are not equivalent to any skew negacyclic codes in the case when the length is even. In Section 5, the notion of LCD skew codes is introduced and we give the best Hamming distances of LCD

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θ-negacyclic codes of length ≤ 10 and the best Hamming distances of LCD θ-cyclic codes of length ≤ 14. In Section 6, some constructions of LCD skew cyclic and negacyclic codes are given over $\mathbb{F}_{p^2}$. If $p \neq 2$ the Euclidean LCD θ-cyclic codes of length $2p^s$ and dimension p^s over $\mathbb{F}_{p^2}$ are all Hermitian LCD codes. All MDS LCD skew codes of length ≤ 16 over $\mathbb{F}_{p^2}$ for $p \in \{3, 5, 7\}$ are obtained and the results are presented in tables.

2 Preliminaries

Let q be a prime power, $\mathbb{F}_q$ a finite field and θ an automorphism of $\mathbb{F}_q$. We define the skew polynomial ring $R = \mathbb{F}_q[x; \theta]$ as

$$R = \mathbb{F}_q[x; \theta] = \{a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} \mid a_i \in \mathbb{F}_q \text{ and } n \in \mathbb{N}\}$$

under usual addition of polynomials and where multiplication is defined using the rule $(ax^i)(bx^j) = a\theta^i(b)x^{i+j}$. The ring R is noncommutative unless θ is the identity automorphism on $\mathbb{F}_q$. According to [16], an element $f(x) \in R$ is central if and only if $f(x) = \Sigma_{i=0}^{n} a_i x^{im}$ is in $\mathbb{F}_q^{\theta}[x^m]$ where $m =\mid \langle\theta\rangle \mid$ is the order of the automorphism θ and $\mathbb{F}_q^{\theta}$ is the fixed field by θ. The two-sided ideals of R are generated by elements having the form $(c_0 + c_1 x^m + \ldots + c_n x^{nm})x^l$, where l is an integer. Central elements of R are the generators of two-sided ideals in R [6]. Let $f(x)$ and $g(x)$ be in R with $f(x) \neq 0$. Then there exist unique skew polynomials $q(x)$ and $r(x)$ such that

$$g(x) = q(x)f(x) + r(x) \quad deg(r(x)) < deg(f(x)).$$

If $r(x) = 0$ then $f(x)$ is a right divisor of $g(x)$ in R, denoted by $f(x) \mid_r g(x)$ [16].

The **skew reciprocal polynomial** of $g(x) = \Sigma_{i=0}^{k} g_i x^i \in R$ of degree k is $g^*(x) = \Sigma_{i=0}^{k} \theta^i(g_{k-i})x^i$. The left monic skew reciprocal polynomial of $g(x)$ is $g^\sharp(x) = (1/\theta^k(g_0))g^*(x)$. If a skew polynomial is equal to its left monic skew reciprocal polynomial, then it is called self-reciprocal. In what follows, we consider a positive integer and a constant λ in $\mathbb{F}_q^*$. We assume that n is a multiple of the order of θ and λ fixed by θ.

According to [[6], [9]], A linear code C of length n over $\mathbb{F}_q$ is said to be (θ, λ)**-constacyclic** or **skew** λ**-constacyclic** if it satisfies

$$\forall c \in C, c = (c_0, c_1, \ldots, c_{n-1}) \in C \Rightarrow (\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C.$$

Any element of the quotient algebra $R/\langle x^n - \lambda\rangle$ is uniquely represented by a polynomial $\beta_0 + \beta_1 x + \ldots + \beta_{n-1}x^{n-1}$ of degree less than n, hence is identified with a word $(\beta_0, \beta_1, \ldots, \beta_{n-1})$ of length n over $\mathbb{F}_q$.

In this way, any skew λ-constacyclic code C of length n over $\mathbb{F}_q$ is identified with exactly one left ideal of the quotient algebra $R/\langle x^n - \lambda\rangle$, which is generated by a right divisor $g(x)$ of $x^n - \lambda$. In that case, $g(x)$ is called a **skew generator polynomial** of C.

Note that the skew 1-constacyclic codes are skew cyclic codes and the skew -1-constacyclic codes are skew negacyclic codes.

Lemma 1. *[5, Lemma 4] Let $h(x)$ and $g(x)$ be skew polynomials in R. Then the following assertions hold:*

1. $(h(x)g(x))^* = \Theta^t(g^*(x))h^*(x)$, *where* $t = deg(h(x))$ *and* $\Theta : R \mapsto R$ *given by* $\Sigma_{i=0}^{k} g_i x^i \mapsto \Sigma_{i=0}^{k} \theta(g_i)x^i$.
2. $(g^*)^* = \Theta^k(g)$ *where* $k = deg(g)$.

The **Hamming weight** $wt(y)$ of an n-tuple $y = (y_l, y_2 \ldots y_n) \in \mathbb{F}_q^n$ is the number of nonzero entries in y, that is, $wt(y) =\mid \{i : y_i \neq 0\} \mid$.

A $\mathbb{F}_q$-linear transformation $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is a monomial transformation if there exists a permutation σ of $\{1, 2 \ldots n\}$ and nonzero elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $\mathbb{F}_q^n$ such that

$$T(y_1, y_2, \ldots, y_n) = (\alpha_1 y_{\sigma(1)}, \alpha_2 y_{\sigma(2)}, \ldots, \alpha_n y_{\sigma(n)})$$

for all $(y_1, y_2 \dots y_n) \in \mathbb{F}_q^n$.

Two linear codes C_1, C_2 in $\mathbb{F}_q^n$ are **equivalent** if there exists a monomial transformation $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ taking C_1 to C_2 (i.e. there exists a linear Hamming isometry [12]).

3 The equivalency between skew λ-constacyclic codes and skew cyclic codes

Let $\mathbb{F}_q$ be a finite field where $q = p^{mr}$ and r, m are two positive integers, θ the automorphism of $\mathbb{F}_q$ given by $a \mapsto a^{p^r}$ and λ a unit in $\mathbb{F}_q$ which is fixed by θ. We assume that n is a multiple of the order of θ. For i in $\mathbb{N}^*$, we write $[i] = \dfrac{p^{ri} - 1}{p^r - 1}$.

In this section, we provide conditions on the existence of an isomorphism between skew λ-constacyclic codes and skew cyclic codes. We start with the following useful lemma.

Lemma 2. *Consider an element α of $\mathbb{F}_q^*$. The application*

$$\begin{aligned} \phi \ : \ R &\longrightarrow R \\ f(x) &\longmapsto f(\alpha x) \end{aligned}$$

is a morphism.

Proof.

- The application ϕ is well-defined.

  Consider $f(x) = \Sigma_{i=0}^{r} a_i x^i$ and $g(x) = \Sigma_{j=0}^{m} b_j x^j$ in R. If $f(x) = g(x)$ then $r = m$ and $a_i = b_i$ for $i = 0, \dots, r$.

  We have that $\phi(f(x)) = \Sigma_{i=0}^{r} a_i \Pi_{d=0}^{i-1} \theta^d(\alpha) x^i$ and $\phi(g(x)) = \Sigma_{j=0}^{r} b_j \Pi_{d=0}^{j-1} \theta^d(\alpha) x^j$ then $\phi(f(x)) = \phi(g(x))$.

- The application ϕ is a ring morphism.

1 By construction, $\phi(f(x) + g(x)) = \phi(f(x)) + \phi(g(x))$

2 Consider b in $\mathbb{F}_q$, then $\phi(x^i b) = \phi(\theta^i(b) x^i) = \theta^i(b) \alpha^{[i]} x^i = (\alpha^{[i]} x^i) b = \phi(x^i)\phi(b)$.

□

Theorem 1. *If $\mathbb{F}_q^*$ contains an element δ where $\lambda = \delta^{[n]}$ then the skew λ-constacyclic codes of length n over $\mathbb{F}_q$ are equivalent to the skew cyclic codes of length n over $\mathbb{F}_q$.*

Proof. Let δ be an element in $\mathbb{F}_q^*$ such that $\lambda = \delta^{[n]}$ and define

$$\begin{aligned} \varphi \ : \ R/\langle x^n - 1 \rangle &\longrightarrow R/\langle x^n - \lambda \rangle \\ f(x) &\longmapsto f(\delta^{-1} x) \end{aligned}$$

Let us prove that the application φ is an isomorphism who preserves the Hamming weight:

- The application φ is well-defined and is bijective :

$$f(x) \equiv g(x)(x^n - 1) \Leftrightarrow f(x) - g(x) \equiv 0(x^n - 1)$$

$$x^n - 1 \mid_r f(x) - g(x) \Rightarrow f(x) - g(x) = h(x)(x^n - 1)$$

By Lemma 2,

$$\phi(f(x) - g(x)) = \phi(h(x)(x^n - 1))$$

$$f(\delta^{-1} x) - g(\delta^{-1} x) = h(\delta^{-1} x)((\delta^{-1} x)^n - 1)$$

$$\begin{aligned} h(\delta^{-1}x)((\delta^{-1}x)^n-1) &= h(\delta^{-1}x)((\delta^{-1})^{[n]}x^n-1)\\ &= h(\delta^{-1}x)(\delta^{-1})^{[n]}(x^n-\delta^{[n]})\\ &= h(\delta^{-1}x)(\delta^{-1})^{[n]}(x^n-\lambda) \end{aligned}$$

$$f(\delta^{-1}x) \equiv g(\delta^{-1}x)(x^n-\lambda)$$

Therefore $\varphi(f(x)) = \varphi(g(x))$. Conversely, $\varphi(f(x)) = \varphi(g(x)) \Rightarrow f(x) = g(x)$. Therefore φ is an injective application between two sets with the same cardinality, so it is bijective.

- The application φ is a morphism: consider $f(x) = a_0 + a_1x + \ldots + a_ix^i + \ldots + a_{n-1}x^{n-1}$ and $g(x) = b_0 + b_1x + \ldots + b_ix^i + \ldots + b_{n-1}x^{n-1}$ in $R/\langle x^n-1\rangle$

  $f(x)g(x) = \Sigma_{j=0}^{n-1}(\Sigma_{i=0}^{j}a_i\theta^i(b_{j-i}) + \Sigma_{i=j+1}^{n-1}a_i\theta^i(b_{n-i+j}))x^j$ because $x^{j+n} = x^j$ in $R/\langle x^n-1\rangle$.

  As $\varphi(x^j) = (\delta^{-1})^{[j]}x^j$, one gets

  $\varphi(f(x)g(x)) = \Sigma_{j=0}^{n-1}(\Sigma_{i=0}^{j}a_i\theta^i(b_{j-i}) + \Sigma_{i=j+1}^{n-1}a_i\theta^i(b_{n-i+j}))(\delta^{-1})^{[j]}x^j$

  Furthermore,

  $\varphi(f(x)) = f(\delta^{-1}x) = a_0 + a_1\delta^{-1}x + \ldots + a_i(\delta^{-1})^{[i]}x^i + \ldots + a_{n-1}(\delta^{-1})^{[n-1]}x^{n-1}$

  $\varphi(g(x)) = g(\delta^{-1}x) = b_0 + b_1\delta^{-1}x + \ldots + b_i(\delta^{-1})^{[i]}x^i + \ldots + b_{n-1}(\delta^{-1})^{[n-1]}x^{n-1}$

  therefore

  $\varphi(f(x))\varphi(g(x)) = \Sigma_{j=0}^{n-1}(\Sigma_{i=0}^{j}a_i(\delta^{-1})^{[i]}\theta^i(b_{j-i}(\delta^{-1})^{[j-i]}))x^j +$
  $\Sigma_{j=0}^{n-1}(\Sigma_{i=j+1}^{n-1}a_i(\delta^{-1})^{[i]}\theta^i(b_{n-i+j}(\delta^{-1})^{[n-i+j]}))x^{j+n}$

  One can notice that $x^{j+n} = x^j(x^n-\lambda) + x^j\lambda = x^j\lambda$ in $R/\langle x^n-\lambda\rangle$. As $\lambda = (\delta)^{[n]}$ is fixed by θ, one gets $\lambda x^j = (\delta)^{[n]}x^j$ in $R/\langle x^n-\lambda\rangle$.

  Therefore $\varphi(f(x))\varphi(g(x)) = \Sigma_{j=0}^{n-1}(\Sigma_{i=0}^{j}a_i(\delta^{-1})^{[i]}\theta^i(b_{j-i}(\delta^{-1})^{[j-i]}) +$

  $\Sigma_{i=j+1}^{n-1}a_i(\delta^{-1})^{[i]}\theta^i(b_{n-i+j}(\delta^{-1})^{[n-i+j]})\delta^{[n]})x^j$

  $\varphi(f(x))\varphi(g(x)) =$

  $\Sigma_{j=0}^{n-1}(\Sigma_{i=0}^{j}a_i\theta^i(b_{j-i})(\delta^{-1})^{[j]} + \Sigma_{i=j+1}^{n-1}a_i\theta^i(b_{n-i+j})(\delta^{-1})^{[n+j]}(\delta)^{[n]})x^j$

  As the order of θ divides n, one has : $(\delta^{-1})^{[n+j]}(\delta)^{[n]} = \theta^n((\delta^{-1})^{[j]}) = (\delta^{-1})^{[j]}$ therefore one gets that $\varphi(f(x))\varphi(g(x)) = \varphi(f(x)g(x))$ and φ is a morphism.

- φ preserves the Hamming weight:

  consider $c(x) = a_0 + \ldots + a_{n-1}x^{n-1}$, then $\varphi(c(x)) = c(\delta^{-1}x) = a_0 + \ldots + a_{n-1}\delta^{-1}\ldots\theta^{n-2}(\delta^{-1})x^{n-1}$.

  The coefficients of $\varphi(c(x))$ are of the form $a_i(\delta^{-1})^r$ and $\forall i \in \{0,\ldots,n-1\}, \forall r \in \mathbb{N}\ \ a_i(\delta^{-1})^r = 0 \Leftrightarrow a_i = 0$ (because $\mathbb{F}_q$ integral domain), therefore $wt(c(x)) = wt(\varphi(c(x)))$.

Then φ is a ring isomorphism and a Hamming isometry. As the skew λ-constacyclic codes and skew cyclic codes are ideals, the result follows from [12] □

Corollary 1. *If $\mathbb{F}_q^*$ contains an element δ where $\lambda = \delta^{[n]}$ then the number of skew λ-constacyclic codes of given length n is equal to the number of skew cyclic codes of length n.*

Example 1. *For $\mathbb{F}_{2^4} = \mathbb{F}_2(w)$ where $w^4 = w+1$, θ is the automorphism of $\mathbb{F}_{2^4}$ given by $a \mapsto a^{2^2}$ and $\mathbb{F}_{2^4}^{\theta} = \mathbb{F}_4 = \{0, 1, w^5, w^{10}\}$.*

We have 33 skew cyclic codes of length 4 over $\mathbb{F}_{16}$,we give some of them:

$$\begin{aligned} x^4-1 &= (x^2+w^{13}x+w^9)(x^2+w^{13}x+w^6)\\ &= (x+w^{12})(x^3+w^3x^2+x+w^3) \end{aligned}$$

δ	$g(x)$	d
w^2	$x^2 + w^6x + w$	3
w^5	$x^2 + w^3x + w$	3
w^8	$x^2 + x + w$	3
w^{11}	$x^2 + w^{12}x + w$	3
w^{14}	$x^2 + w^9x + w$	3

Table 1: Some skew w^5-constacyclic codes of length 4 over $\mathbb{F}_{16}$

δ	g(x)	d
w	$x^3 + w^4x^2 + w^5x + w^9$	4
w^4	$x^3 + w^7x^2 + w^5x + w^{12}$	4
w^7	$x^3 + w^{10}x^2 + w^5x + 1$	4
w^{10}	$x^3 + w^{13}x^2 + w^5x + w^3$	4
w^{13}	$x^3 + wx^2 + w^5x + w^6$	4

Table 2: Some skew w^{10}-constacyclic code of length 4 over $\mathbb{F}_{16}$

All of the [4]*-th roots of* w^5 *are* $\{w^2, w^5, w^8, w^{11}, w^{14}\}$.

All of the [4]*-th roots of* w^{10} *are* $\{w, w^4, w^7, w^{10}, w^{13}\}$.

The table 1 below gives examples of skew w^5*-constacyclic codes of length* 4 *over* $\mathbb{F}_{16}$ *who are equivalent to the skew cyclic code* $C = \langle x^2 + w^{13}x + w^6\rangle$ *of length* 4 *over* $\mathbb{F}_{16}$.

The table 2 below gives examples of skew w^{10}*-constacyclic codes of length* 4 *over* $\mathbb{F}_{16}$ *who are equivalent to the skew cyclic code* $C = \langle x^3 + w^3x^2 + x + w^3\rangle$ *of length* 4 *over* $\mathbb{F}_{16}$.

In the following, using the ring isomorphism φ in Theorem 1, we give a factorization of the polynomial $x^n - \lambda$ in R.

Corollary 2. *If* $\mathbb{F}_q^*$ *contains an element* δ *where* $\lambda = \delta^{[n]}$*, and* $x^n - 1 = \prod_{i=1}^{k} f_i(x)$ *where the* $f_i(x)$ $1 \leq i \leq k$ *are monic irreducible in* R*, then* $x^n - \lambda = \lambda \prod_{i=1}^{k} f_i(\delta^{-1}x)$ *is a factorization of* $x^n - \lambda$ *into irreducible factors in* R.

Proof.

$$x^n - 1 = \prod_{i=1}^{k} f_i(x)$$

$$\phi(x^n - 1) = \phi(\prod_{i=1}^{k} f_i(x)) = \prod_{i=1}^{k} \phi(f_i(x))$$

(because ϕ is a morphism)

$$\lambda^{-1}(x^n - \lambda) = \prod_{i=1}^{k} (f_i(\delta^{-1}x))$$

$$x^n - \lambda = \lambda \prod_{i=1}^{k} f_i(\delta^{-1}x).$$

Furthermore, as ϕ is a morphism, the irreducibility of $f_i(x)$ implies the irreducibility of $f_i(\delta^{-1}x)$. □

Example 2. *Consider* $\mathbb{F}_{7^2} = \mathbb{F}_7(w)$ *where* $w^2 - w + 3 = 0$ *and* θ *the automorphism of* $\mathbb{F}_{49}$ *given by* $a \mapsto a^7$ *with fixed field* $\mathbb{F}_{49}^{\theta} = \mathbb{F}_7$*. All of the* [8]*-th roots of* $w^{16} = 2$ *are*

$\{w^2, w^5, 3, w^{11}, w^{17}, w^{14}, w^{20}, w^{26}, w^{29}, 4, w^{35}, w^{38}, w^{41}, w^{44}, w^{23}, w^{47}\}$.

$$x^8 - 1 = (x^4 + w^2x^2 + w^{36})(x^4 + w^{26}x^2 + w^{36}) \ .$$

Then by Theorem 1 the skew cyclic code $C = \langle x^4 + w^{26}x^2 + w^{36} \rangle$ *over* $\mathbb{F}_{49}$ *with parameter* $[8, 4, 3]$ *is equivalent to the skew* w^{16}*-constacyclic code of length* 8 *over* $\mathbb{F}_{49}$ *with skew generator polynomial* $g(x) = w^{-16}((wx)^4 + w^{26}(wx)^2 + w^{36}) = x^4 + w^{18}x^2 + w^{20}$.

Example 3. *For* $\mathbb{F}_{16} = \mathbb{F}_2(w)$ *where* $w^4 = w + 1$, θ *is the automorphism of* $\mathbb{F}_{16}$ *given by* $a \mapsto a^4$ *and* $\mathbb{F}_{16}^{\theta} = \mathbb{F}_4 = \{0, 1, w^5, w^{10}\}$.

A factorization of the skew polynomial $x^8 - 1$ *as a product of monic irreducible skew polynomials is:*

$$x^8 - 1 = (x + 1)(x + 1)(x + w^9)(x + w^6)(x + w^3)(x + w^{12})(x + w^3)(x + w^{12})$$

As w^5 *is a* [8]*-th root of* w^{13}*, by Corollary 2, a factorization of the skew polynomial* $x^8 - w^5$ *as a product of irreducible skew polynomials is:*

$$x^8 - w^5 = w^5(w^2x+1)(w^2x+1)(w^2x+w^9)(w^2x+w^6)(w^2x+w^3)(w^2x+w^{12})(w^2x+w^3)(w^2x+w^{12})$$
.

In the following we give a case where the skew constacyclic codes are equivalent to the skew cyclic codes using only a relation between the length n, the characteristic of $\mathbb{F}_q$ and the cardinality of $\mathbb{F}_q$. We start with the following useful lemma.

Lemma 3. *[2, Lemma 3.1] Let* α *be a primitive element of* $\mathbb{F}_q$ *and* $\lambda = \alpha^i$ *for* $i \leqslant q - 1$*. Then the equation* $x^s = \lambda$ *has a solution in* $\mathbb{F}_q$ *if and only if* $(s, q - 1) \mid i$.

In the following, we give a similar result of [2, Theorem 3.4] but in the noncommutative case.

Theorem 2. *Assume that* $([n], q - 1) = 1$*. Then all skew* λ*-constacyclic codes of length* n *over* $\mathbb{F}_q$ *are equivalent to skew cyclic codes of length* n *over* $\mathbb{F}_q$.

Proof. Let λ be a unit which is fixed by θ then there exist i such that $\lambda = \alpha^i$. As $([n], q-1) = 1 \mid i$, according to Lemma 3, there exist δ such that $\lambda = \delta^{[n]}$. By Theorem 1 skew λ-constacyclic codes of length n over $\mathbb{F}_q$ are equivalent to skew cyclic codes of length n over $\mathbb{F}_q$. □

In the following, we provide conditions on the existence of an isomorphism between skew λ-constacyclic codes and skew cyclic codes of length βp^s, where β an odd integer.

Theorem 3. *Let* β *be an odd integer, for* λ *such that there exists an element* $\delta \in \mathbb{F}_{p^r}^*$ *where* $\delta^{\beta} = \lambda$ *and* $s \equiv 0 \pmod{r}$ *then skew* $\pm\lambda$*-constacyclic codes of length* βp^s *over* $\mathbb{F}_q$ *are equivalent to skew cyclic codes of length* βp^s *over* $\mathbb{F}_q$.

Proof. Let λ be a unit which is fixed by θ and such that there exists $\delta \in \mathbb{F}_{p^r}^*$ and $\lambda = \delta^{\beta} = (\delta^{p^s})^{\beta}$, so $\lambda = \delta^{\beta p^s} = \delta^{[\beta p^s]}$ (because $\delta \in \mathbb{F}_{p^r}^*$) then by Theorem 1 skew λ-constacyclic codes of length βp^s over $\mathbb{F}_q$ are equivalent to skew cyclic codes of length βp^s over $\mathbb{F}_q$.
Since βp^s is odd, we obtain $-\lambda = (-\delta)^{\beta p^s} = (-\delta)^{[\beta p^s]}$ (because $\delta \in \mathbb{F}_{p^r}^*$ and $\theta(-\delta) = -\delta$) then by Theorem 1 skew $-\lambda$-constacyclic codes of length βp^s over $\mathbb{F}_q$ are equivalent to skew cyclic codes of length βp^s over $\mathbb{F}_q$. □

Remark 1. *When* β *is odd we have* $(-1)^{\beta} = -1$*, and we have* $-1 \in \mathbb{F}_{p^r}^*$*. Then from Theorem 3 the skew negacyclic codes of length* βp^s *over* $\mathbb{F}_q$ *are equivalent to the skew cyclic codes of length* βp^s *over* $\mathbb{F}_q$.

Theorem 3 yields the following corollary.

Corollary 3. *Let* β *be an odd integer and* $s \equiv 0 \pmod{r}$*. If* $(\beta, p^r - 1) = 1$ *then all skew* λ*-constacyclic codes of length* βp^s *over* $\mathbb{F}_q$ *are equivalent to skew cyclic codes of length* βp^s *over* $\mathbb{F}_q$.

Remark 2. *As a special case of Corollary 3 when $\beta = 1$ we obtain that all skew λ-constacyclic codes of length p^s over $\mathbb{F}_q$ are equivalent to skew cyclic codes of length p^s over $\mathbb{F}_q$.*

Proposition 1. *[3] Consider p a prime number, $\theta : a \mapsto a^p$ the Frobenius automorphism over $\mathbb{F}_{p^2}$ and α in $\mathbb{F}_{p^2}$. Then there are $p+1$ skew cyclic codes of length 2 over $\mathbb{F}_{p^2}$; their skew generator skew polynomials are the skew polynomials $x + \alpha$ where $\alpha^{p+1} = 1$.*

Proposition 1 yields the following corollary.

Corollary 4. *If $\mathbb{F}_q^*$ contains an element δ where $\lambda = \delta^{p+1}$ and λ in $\mathbb{F}_p$, then by Corollary 1, we have $p+1$ λ-constacyclic codes of length 2 over $\mathbb{F}_{p^2}$; their skew generator polynomials are the skew polynomials $\varphi(x + \alpha)$ where $\alpha^{p+1} = 1$.*

Example 4. *Consider $\mathbb{F}_{3^2} = \mathbb{F}_3(w)$ with $w^2 - w - 1 = 0$ and $\theta : a \mapsto a^3$. The skew polynomial $x + w^2 \in \mathbb{F}_{3^2}[x, \theta]$ generates a θ-cyclic code C of length 3 (because $(w^2)^{3+1} = w^8 = 1$).*

We have $2 = w^4$ then $x^2 + 1 = 2(w^7x + w^2)(w^7x + w^2)$. The skew negacyclic code of length 2 over $\mathbb{F}_9$ generated by $\varphi(x + w^2) = w^7x + w^2$ is equivalent to a skew cyclic code C.

4 The equivalency between skew λ-constacyclic codes and skew negacyclic codes

In this section, we provide conditions on the existence of an isomorphism between skew λ-constacyclic codes and skew negacyclic codes. Like in previous section we assume that $\mathbb{F}_q$ is a finite field, θ is an automorphism of $\mathbb{F}_q$ and n is a multiple of the order of θ and an odd intger.

Theorem 4. *If $\mathbb{F}_q^*$ contains an element δ where $\lambda = \delta^{[n]}$ then skew λ-constacyclic codes of length n over $\mathbb{F}_q$ are equivalent to skew negacyclic codes of length n over $\mathbb{F}_q$.*

Proof. Let δ an element in $\mathbb{F}_q^*$ sush that $\lambda = \delta^{[n]}$ and define

$$\begin{aligned} \psi \ : \ R/\langle x^n + 1\rangle &\longrightarrow R/\langle x^n - \lambda\rangle \\ f(x) &\longmapsto f(-\delta^{-1}x) \end{aligned}$$

as for the proof of Theorem 1, we prove that ψ is a ring isomorphism and a Hamming isometry. As the skew λ-constacyclic codes and skew negacyclic codes are ideals, the result follows from [12]. □

Corollary 5. *If $\mathbb{F}_q^*$ contains an element δ where $\lambda = \delta^{[n]}$ then the number of skew λ-constacyclic codes of length n over $\mathbb{F}_q$ is equal to the number of skew negacyclic codes of length n over $\mathbb{F}_q$.*

Example 5. *For $\mathbb{F}_{5^3} = \mathbb{F}_5(w)$ where $w^3 + 3w + 3 = 0$, θ is the automorphism of $\mathbb{F}_{5^3}$ given by $a \mapsto a^5$ and $\mathbb{F}_{5^3}^{\theta} = \mathbb{F}_5$.*

$$x^3 + 1 = (x + w^{52})(x^2 + w^{122}x + w^{72}).$$

The table 3 below gives examples of skew 2-constacyclic code of length 3 over $\mathbb{F}_{5^3}$ who are equivalent to skew negacyclic code $C = \langle x^2 + w^{122}x + w^{72}\rangle$ of length 3 over $\mathbb{F}_{5^3}$.

In the following, we give a relationship between skew cyclic codes and skew negacyclic codes.

Corollary 6. *the skew cyclic codes of length n over $\mathbb{F}_q$ are equivalent to the skew negacyclic codes of length n over $\mathbb{F}_q$.*

In the following example, we show that not all a skew cyclic codes of length n over $\mathbb{F}_q$ are equivalent to a skew negacyclic code of length n over $\mathbb{F}_q$, when n is even.

Example 6. *Let $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w + 1$, θ the Frobenius automorphism. Let the skew cyclic code $C = \langle x^3 + x^2 + x + 1\rangle$ over $\mathbb{F}_9$ with parameter $[4, 1, 4]$. There is no skew negacyclic code of length 4 equivalent to C (because there is no skew negacyclic code of length 4 with minimum distance 4).*

δ	$g(x)$	d
w	$x^2 + w^{65}x + w^{78}$	3
w^5	$x^2 + w^{85}x + w^{102}$	3
w^{17}	$x^2 + w^{21}x + w^{50}$	3
w^{29}	$x^2 + w^{81}x + w^{122}$	3
w^{37}	$x^2 + w^{121}x + w^{46}$	3
w^{41}	$x^2 + w^{17}x + w^{70}$	3
w^{53}	$x^2 + w^{77}x + w^{18}$	3
w^{57}	$x^2 + w^{97}x + w^{42}$	3
w^{61}	$x^2 + w^{117}x + w^{66}$	3
w^{89}	$x^2 + w^{9}x + w^{110}$	3
w^{97}	$x^2 + w^{49}x + w^{34}$	3
w^{109}	$x^2 + w^{109}x + w^{106}$	3
w^{113}	$x^2 + w^{5}x + w^{6}$	3
w^{121}	$x^2 + w^{45}x + w^{54}$	3

Table 3: Some skew 2-constacyclic codes of length 3 over $\mathbb{F}_{5^3}$

Corollary 7. *the number of skew cyclic codes of length n over $\mathbb{F}_q$ is equal to the number of skew negacyclic codes of length n over $\mathbb{F}_q$.*

Example 7. *Using the same automorphism from Example 5 one gets:*

$$x^3 + 1 = (x + w^{12})(x^2 + w^{114}x + w^{112}).$$

Then by Theorem 4

$$x^3 - 1 = -(-x - w^{12})(x^2 - w^{114}x + w^{112}).$$

By Corollary 6 $C = \langle x^2 - w^{114}x + w^{112} \rangle$ is the skew cyclic code of length 3 over $\mathbb{F}_{5^3}$ equivalent to the skew negacyclic code $\langle x^2 + w^{114}x + w^{112} \rangle$ of length 3 over $\mathbb{F}_{5^3}$.

In the following, using the ring isomorphism ψ in Theorem 4, we give a factorization of the polynomial $x^n - \lambda$ in R.

Corollary 8. *If $\mathbb{F}_q^*$ contains an element δ where $\lambda = \delta^{[n]}$, and $x^n + 1 = \prod_{i=1}^{k} f_i(x)$ where $f_i(x)$ $1 \leq i \leq k$ are monic irreducible in R, then $x^n - \lambda = -\lambda \prod_{i=1}^{k} f_i(-\delta^{-1}x)$ is a factorization of $x^n - \lambda$ into irreducible factors in R.*

Proof.

$$x^n + 1 = \prod_{i=1}^{k} f_i(x)$$

$$\psi(x^n + 1) = \psi(\prod_{i=1}^{k} f_i(x)) = \prod_{i=1}^{k} \psi(f_i(x))$$

(because ψ is a morphism)

$$(-1)^n \lambda^{-1} x^n + 1 = -\lambda^{-1}(x^n - \lambda) = \prod_{i=1}^{k} (f_i(-\delta^{-1}x))$$

$$x^n - \lambda = -\lambda \prod_{i=1}^{k} f_i(-\delta^{-1}x).$$

□

Theorem 5. *If $([n], q-1) = 1$ Then all skew λ-constacyclic codes of length n over $\mathbb{F}_q$ are equivalent to skew negacyclic codes of length n over $\mathbb{F}_q$.*

Example 8. *Consider $\mathbb{F}_{5^3} = \mathbb{F}_5(w)$ where $w^3 + 3w + 3 = 0$, θ is the automorphism of $\mathbb{F}_{5^3}$ given by $a \mapsto a^5$ and $\mathbb{F}_{5^3}^{\theta} = \mathbb{F}_5$.*

Here is a factorization of the skew polynomial $x^9 + 1$ as a product of monic irreducible skew polynomials:

$x^9 + 1 = (x^2 + w^{121}x + w^{12})(x^2 + w^{65}x + w^{96})(x^2 + w^{91}x + w^{16})(x + w^{64})(x + w^{12})(x + w^{48})$.

As w^{123} is a $[9]$-th root of w^{31}, by Corollary 8 a factorization of the skew polynomial $x^9 - w^{31}$ as a product of irreducible skew polynomials is:

$$x^9 - w^{31} = -w^{31}(((-wx)^2 + w^{121}(-wx) + w^{12})((-wx)^2 + w^{65}(-wx) + w^{96})((-wx)^2 + w^{91}(-wx) + w^{16})((-wx) + w^{64})((-wx) + w^{12})((-wx) + w^{48}))$$

5 LCD skew cyclic and negacyclic codes

Like previously, in this section, we assume that $\mathbb{F}_q$ is a finite field, θ is an automorphism of $\mathbb{F}_q$ and n is a multiple of the order of θ.

The **(Euclidean) dual** of a linear code C of length n over $\mathbb{F}_q$ is defined as $C^{\perp} = \{x \in \mathbb{F}^n \mid \forall y \in C, < x, y >= 0\}$ where for x, y in $\mathbb{F}_q^n$, $< x, y >:= \sum_{i=1}^{n} x_i y_i$ is the (Euclidean) scalar product of x and y. A linear code is called an **Eulidean LCD code** (linear code with complementary dual) if $C \oplus C^{\perp} = \mathbb{F}_q^n$, which is equivalent to $C \cap C^{\perp} = \{0\}$.

Assume that $q = r^2$ is an even power of an arbitrary prime and denote for a in $\mathbb{F}_q$, $\overline{a} = a^r$. The **(Hermitian) dual** of a linear code C of length n over $\mathbb{F}_q$ is defined as $C^{\perp_H} = \{x \in \mathbb{F}^n \mid \forall y \in C, < x, y >_H = 0\}$ where for x, y in $\mathbb{F}_q^n$, $< x, y >_H := \sum_{i=1}^{n} x_i \overline{y_i}$ is the (Hermitian) scalar product of x and y. The code C is **Hermitian LCD** if $C \cap C^{\perp_H} = \{0\}$.

5.1 Skew generator polynomials of LCD skew cyclic and negacyclic codes

The Greatest Common Right Divisor (GCRD) $d(x)$ of $f(x)$ and $g(x)$ is a right divisor of $f(x)$ and $g(x)$, and if $e(x)$ is another right divisor of $f(x)$ and $g(x)$ then $d(x) = k(x)e(x)$ for some skew polynomial $k(x)$.

The Greatest Common Left Divisor (GCLD) of $f(x)$ and $g(x)$ is a monic skew polynomial defined in a similar way.

The Least Common Left Multiple (LCLM) $m(x)$ of $f(x)$ and $g(x)$ is the unique monic skew polynomial in R such that if there exists $h(x) \in R$ such that $f(x) \mid_r h(x)$ *and* $g(x) \mid_r h(x)$ then $h(x) = k(x)m(x)$ for some skew polynomial $k(x)$ and $f(x)$ is a right divisor of $m(x)$, $g(x)$ is a right divisor of $m(x)$.

The Least Common Right Multiple (LCRM) of $f(x)$ and $g(x)$ is a monic skew polynomial defined in a similar way.

In the non-commutative ring $R/(x^n - \lambda)$ we identify the image of $P \in R$ under the canonical morphism $\phi : R \longmapsto R/(x^n - \lambda)$ with the remainder of P by the right division with $x^n - \lambda$ in R.

Before we prove our main results in this section, we recall some results.

Lemma 4. *Let C_i,$1 \leq i \leq 2$ be a skew λ-constacyclic code of length n over $\mathbb{F}_q$ with skew generator polynomial $g_i(x)$, then:*

1. *$C_1 \cap C_2$ is a skew λ-constacyclic code of length n generated by $lclm(g_1(x), g_2(x))$.*
2. *$C_1 + C_2$ is a skew λ-constacyclic code of length n generated by $gcrd(g_1(x), g_2(x))$.*

Proof.

1. Let $c = (c_0, c_1, \ldots, c_{n-1}) \in C_1 \cap C_2 \Longrightarrow c \in C_1$ *and* $c \in C_2$

   $(\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C_1$ *and* $(\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C_2$ (because C_1 *and* C_2 skew constacyclic codes).

   Then $(\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C_1 \cap C_2$

   $C_1 \cap C_2$ **is generated by** $\phi(lclm(g_1(x), g_2(x)))$ **:**

   we have $g_1(x) \mid_r m(x) = lclm(g_1(x), g_2(x))$ *and* $g_2(x) \mid_r m(x) = lclm(g_1(x), g_2(x)) \Longrightarrow \phi(m(x)) \in \langle g_1(x)\rangle_l$ *and* $\phi(m(x)) \in \langle g_2(x)\rangle_l$ Then $\langle\phi(m(x))\rangle \subseteq C_1 \cap C_2$.
   conversely,
   Let $\phi(f(x)) \in C_1 \cap C_2$ use Right Division Algorithm

$$f(x) = q(x)m(x) + r(x) \quad deg(r(x)) < deg(m(x)).$$

   since $\phi(f(x)) \in C_1 \cap C_2$ and $m(x) = lclm(g_1(x), g_2(x))$, we have $g_1(x)$; $g_2(x)$ divides (on the right) both $f(x)$ and $m(x)$ then $g_1(x)$, $g_2(x)$ divides $r(x) = f(x) - q(x)m(x)$.
   Since $m(x)$ is the Least Common Left Multiple of $g_1(x)$ and $g_2(x)$ we must have $r(x) = 0$.

$$f(x) = q(x)m(x) \Rightarrow \phi(f(x)) = \phi(q(x))\phi(m(x)) \Rightarrow C_1 \cap C_2 \subseteq \langle\phi(m(x))\rangle$$

2. Let $c = a + b = (a_0 + b_0, a_1 + b_1, \ldots, a_{n-1} + b_{n-1}) \in C_1 + C_2$

   $(\lambda\theta(a_{n-1}), \theta(a_0), \ldots, \theta(a_{n-2})) \in C_1$ *and* $(\lambda\theta(b_{n-1}), \theta(b_0), \ldots, \theta(b_{n-2})) \in C_2$ (because C_1 *and* C_2 skew constacyclic codes).

   $(\lambda\theta(a_{n-1}) + \lambda\theta(b_{n-1}), \theta(a_0) + \theta(b_0), \ldots, \theta(a_{n-2}) + \theta(b_{n-2})) \in C_1 + C_2$ Then $(\lambda\theta(a_{n-1} + b_{n-1}), \theta(a_0 + b_0), \ldots, \theta(a_{n-2} + b_{n-2})) \in C_1 + C_2$

   Let $g(x) = gcrd(g_1(x), g_2(x))$. It follows by [1, Theorem 7] that

$$g(x) = a(x)g_1(x) + b(x)g_2(x) \text{ for some } a(x) \text{ and } b(x) \text{ in } \mathbb{F}_q[x;\theta].$$

$$\Rightarrow \phi(g(x)) = \phi(a(x))g_1(x) + \phi(b(x))g_2(x) \text{ for some } a(x) \text{ and } b(x) \text{ in } \mathbb{F}_q[x;\theta].$$

   So $\phi(g(x)) \in C_1 + C_2$. Since $C_1 + C_2$ is left ideal, $\langle\phi(g(x))\rangle \subseteq C_1 + C_2$.

   conversely, $g(x) \mid_r g_1(x)$, which shows that $C_1 \subseteq \langle\phi(g(x))\rangle$ similarly $C_2 \subseteq \langle\phi(g(x))\rangle$ implying $C_1 + C_2 = \langle\phi(g(x))\rangle$.

□

After we have proof the Lemma 4, we found it in [9], as a Theorem [Theorem 6.3.7]. We consider that our proof is different and it is more clair and simple as the one given in [9] because we have just using the Right Division Algorithm.

Example 9. *For* $\mathbb{F}_4 = \mathbb{F}_2(w)$ *where* $w^2 = w + 1$ *and* θ *the Frobenius automorphism* $\theta : a \mapsto a^2$ *we have :*

$$\begin{aligned} x^8 + 1 &= (x^3 + wx^2 + x + w)(x^5 + w^2x^4 + x + w^2) \\ &= (x^3 + wx^2 + w^2)(x^5 + w^2x^4 + x^3 + +x^2 + w) \end{aligned}$$

Let C_1 *be the skew cyclic code of length* 8 *and minimum distance* 4 *over* $\mathbb{F}_4$ *with skew generator polynomial* $g_1(x) = x^5 + w^2x^4 + x + w^2$ *and let* C_2 *be the skew cyclic code of length* 8 *and distance* 5 *over* $\mathbb{F}_4$ *with skew generator polynomial* $g_2(x) = x^5 + w^2x^4 + x^3 + x^2 + w$. *Then by lemma 4 :*
$C_1 + C_2 = \langle gcrd(g_1(x), g_2(x))\rangle = \langle x^3 + x^2 + x + 1\rangle$ *with parameter* $[8, 5, 2]$.
$C_1 \cap C_2 = \langle lclm(g_1(x), g_2(x))\rangle = \langle x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\rangle$ *with parameter* $[8, 1, 8]$.

Consider λ in $(\mathbb{F}_q^\theta)^*$ and C a skew λ-constacyclic code of length n and skew generator polynomial g. According to Theorem 1 of [8], the Euclidean dual $C^\perp$ of C is a skew λ^{-1}-constacyclic code generated by $h^\natural$ where $gh = hg = x^n - \lambda$. In order to consider $C \cap C^\perp$, we will assume that $\lambda^2 = 1$. Similarly, if $q = r^2$, the Hermitian dual $C^{\perp_H}$ of C is generated by $\overline{h^\natural}$.

In the following, we give a necessary and sufficient condition for a skew λ-constacyclic code to be a LCD code when $\lambda^2 = 1$.

Theorem 6. *Assume that $\lambda^2 = 1$. Consider a skew λ-constacyclic code C with skew generator polynomial g and length n. Consider h such that $hg = gh = x^n - \lambda$.*

1. *C is an Euclidean LCD if and only if* $\mathrm{gcrd}(g, h^\natural) = 1$.
2. *If q is an even power of a prime number, $q = r^2$, C is an Hermitian LCD code if and only if* $\mathrm{gcrd}(g, \overline{h^\natural}) = 1$.

Proof.

By Lemma 4 the skew polynomial $m(x) = lclm(g(x), h^*(x))$ is skew generator polynomial of the skew constacyclic code $C \cap C^\perp$.

$C \cap C^\perp = \{0\} \iff m(x) = x^n - \lambda$

$\iff deg(gcrd(g(x), h^*(x))) = 0$ (because by [18] we have $deg(gcrd(g(x), h^*(x))) + deg(lclm(g(x), h^*(x))) = deg(g(x)) + deg(h^*(x))$ and we have $deg(g(x)) = n - k$, $deg(h^*(x)) = k$)

$\iff gcrd(g(x), h^*(x)) = 1$

The same proof holds for Hermitian LCD codes. □

Example 10. *Consider $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w + 1$ and θ the Frobenius automorphism $\theta : a \mapsto a^3$. One has :*

$$x^4 + 1 = (x^2 + w^3x + 1)(x^2 + w^7x + 1).$$

The skew reciprocal polynomial of x^2+w^3x+1 is x^2+wx+1 and $gcrd(x^2+w^7x+1, x^2+wx+1) = 1$.

Then by Theorem 6 the skew negacyclic code $C = \langle x^2 + w^7x + 1 \rangle$ of length 4 and distance 3 over $\mathbb{F}_9$ is an Euclidean LCD code.

Example 11. *For $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w + 1$ and θ the Frobenius automorphism $\theta : a \mapsto a^3$, one has :*

$$x^6 - 1 = (x^3 + wx^2 + x + 1)(x^3 + w^7x^2 + x + 2).$$

The skew reciprocal polynomial of $x^3 + wx^2 + x + 1$ is $x^3 + x^2 + w^3x + 1$ and $gcrd(x^3 + w^7x^2 + x + 2, x^3 + x^2 + wx + 1) = 1$.

Then by Theorem 6 the skew cyclic code $C = \langle x^3 + w^7x^2 + x + 2 \rangle$ of length 6 and distance 4 over $\mathbb{F}_9$ is Hermitian LCD code.

5.2 LCD and θ-reversible skew constacyclic codes

Over a finite field $\mathbb{F}_q$, there is a strong link between LCD cyclic codes and reversible codes ([13], [15]).

Definition 1. *Let $\mathbb{F}_q$ be the finite field where q is a prime power. A code C is called reversible if $(c_0, c_1, \ldots, c_{n-1}) \in C$ implies that $(c_{n-1}, c_{n-2}, \ldots, c_0) \in C$.*

The cyclic code generated by the monic polynomial g is reversible if and only if $g(x)$ is self-reciprocal (i.e $g(x) = g^\natural(x)$) [15, Theorem 1]. Furthermore, if q is coprime with n, a cyclic code of length n is LCD if and only if C is reversible [13]. The example below shows that it is not necessarily the case for skew cyclic codes when θ is not the identity.

Example 12. *Let $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w+1$, θ the Frobenius automorphism and $R = \mathbb{F}_9[x;\theta]$. We have :*

$$x^2 - 1 = (x + w^2)(x + w^2)$$

The skew polynomial $g = x + w^2$ is such that $g(x) = g^\sharp(x)$. The greatest common right divisor of $g(x)$ and $h^*(x)$ is $x + w^2$ (i.e $gcrd(g(x), h^*(x)) \neq 1$) therefore, by Theorem 6 C is not an LCD code.

Definition 2. *Let $\mathbb{F}_q$ be the finite field where q is a prime power and θ be an automorphism of $\mathbb{F}_q$, C be a code of length n over $\mathbb{F}_q$.*

1. *The code C is called a θ-reversible code if*

$$\forall c \in C \quad c = (c_0, c_1, \ldots, c_{n-1}) \in C \Longrightarrow \big(c_{n-1}, \theta(c_{n-2}), \ldots, \theta^{n-1}(c_0)\big) \in C$$

2. *If q is an even power of a prime number, $q = p^2$, C is a conjugate-skew reversible code if*

$$\forall c \in C \quad c = (c_0, c_1, \ldots, c_{n-1}) \in C \Longrightarrow \big(\overline{c_{n-1}}, \theta(\overline{c_{n-2}}), \ldots, \theta^{n-1}(\overline{c_0})\big) \in C$$

, where for a in $\mathbb{F}_q$ notice that $\overline{a} = a^p$

We can notice that if C is θ-reversible(resp. conjugate-skew reversible), then $g = g^\natural$(resp. $g = \overline{g^\natural}$). The converse is true when g is a proper central divisor of $x^n - \lambda$. In this case the skew constacyclic code is in fact a cyclic code.

Theorem 7. *Let $g(x)$ be a proper central divisor of $x^n - \lambda$. Then a skew constacyclic code C is a θ-reversible code (resp. conjugate-skew reversible code) if and only if $g(x)$ is self-reciprocal (resp. $g = \overline{g^\natural}$).*

Proof. If $g(x)$ is self-reciprocal then $g(x) = g^\sharp(x)$.

Let $c(x) = c_0 + c_1x + \ldots + c_{n-1}x^{n-1} \in C = \langle g(x)\rangle$.. Consider $f(x)$ such that $c(x) = f(x)g(x)$. As $g(x)$ is central, $c(x) = g(x)f(x)$, therefore

$$\begin{aligned} c^*(x) = (g(x)f(x))^* &= \Theta^{n-k}(f^*(x))g^*(x) \\ &= \Theta^{n-k}(f^*(x))\theta^{n-k}(g_0)((1/\theta^{n-k}(g_0))g^*(x)) \\ &= \Theta^{n-k}(f^*(x))\theta^{n-k}(g_0)g^\sharp(x) \\ &= \Theta^{n-k}(f^*(x))\theta^{n-k}(g_0)g(x) \end{aligned}$$

Then $c^*(x) \in C$ which means that $\big(c_{n-1}, \theta(c_{n-2}), \ldots, \theta^{n-1}(c_0)\big) \in C$.

Conversely if,

$$\begin{aligned} g(x) \in C &\Longrightarrow (g_0, g_1, \ldots, g_r, 0, \ldots, 0) \in C \\ &\Longrightarrow (0, \ldots, 0, \theta^{n-r-1}(g_r), \ldots, \theta^{n-1}(g_0)) \in C \\ &\Longrightarrow (\lambda\theta^n(g_r), \ldots, \lambda\theta^{r+1+n-1}(g_0), 0, \ldots, 0) \in C \\ &\text{(because } C \text{ is a skew } \lambda\text{-constacyclic code)} \\ &\Longrightarrow (g_r, \ldots, \theta^r(g_0), 0, \ldots, 0) \in C \\ &\text{(because } C \text{ is linear and the order of } \theta \text{ divides } n) \end{aligned}$$

we have that $g(x) \in C \Longrightarrow g^*(x) \in C \Longrightarrow g(x) \mid_r g^*(x)$

so $g(x) = g^\sharp(x)$

The same proof holds for conjugate-skew reversible code. □

Example 13. *For $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w + 1$ and θ the Frobenius automorphism $\theta : a \mapsto a^3$. In $\mathbb{F}_9[x;\theta]$ the polynomial $x^6 - 1$ has two θ-reversible codes generated by a proper central : $g_1(x) = x^2 + 2$ and $g_2(x) = x^4 + x^2 + 1$.*

Example 14. *Let $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w + 1$, θ the Frobenius automorphism.*

$$x^4 - 1 = (x^3 + w^7x^2 + 2x + w^3)(x + w)$$

The skew polynomial $g = x + w$ generates a skew cyclic code C of length 4. The greatest common right divisor of $g(x)$ and $h^*(x)$ is 1 therefore by Theorem 6 C is LCD. We have $(1, w^6, w^6, 0)$ in C but $(0, w^6, w^6, 1)$ is not in C then C is not a reversible code. Furthermore We have $(0, 0, 2, w^3)$ in C but $(w^3, 2, 0, 0)$ is not in C therefore C is not θ-reversible either.

In the following, we give a necessary and sufficient condition for a skew λ-constacyclic code generated by a skew polynomial $g(x)$ (not necessarily central) to be a LCD code under some assumptions (Theorem 8 and Corollary 9). We make some link with θ-reversible codes and conjugate-skew reversible codes. Before we prove our main results in this section, we define the following set: let f, g in R such that $gcrd(f(x), g(x)) = 1$,

$$A_{(f,g)} := \left\{(a(x), b(x)) \in R^2 \mid a(x)f(x) + b(x)g(x) = 1 \text{ and } b(x)g(x) = g(x)b(x)\right\}.$$

Lemma 5. *Let $\mathbb{F}_q$ be the finite field where q is a prime power and θ be an automorphism of $\mathbb{F}_q$, if there are two polynomials $a(x)$ and $b(x)$ in R such that*

$$a(x)f(x) + b(x)g(x) = 1,$$

then the greatest common right divisor (gcrd) of $f(x)$ and $g(x)$ is equal to 1 ($gcrd(f(x), g(x)) = 1$).

Proof. Assume that $h(x)$ is right divisor of $f(x)$ and $g(x)$ then there exists two skew polynomials $t(x)$ and $m(x)$ such that $f(x) = t(x)h(x)$,$g(x) = m(x)h(x)$

$$a(x)t(x)h(x) + b(x)m(x)h(x) = ((a(x)t(x) + b(x)m(x))h(x) = 1$$

$h(x)$ is right divisor of $((a(x)t(x) + b(x)m(x))h(x)$ then $h(x) \mid_r 1$. □

In the following, we give properties of skew polynomial ring.

Theorem 8. *Consider g, h in R and $\lambda \in \{-1, 1\}$ such that $x^n - \lambda = gh = hg$ with $\deg(h) = k$.*

1. *Assume that $A_{(g,\Theta^b(h^*))}$ is nonempty. Then $g = \Theta^{k+b}(g^\natural)$ for all b in $\{0, 1\}$.*
2. *If the greatest common right divisor of $h(x)$ and $g(x)$ is equal to 1, g_0 in $\mathbb{F}_q^\theta$ and $g = \Theta^{k+b}(g^\natural)$ then $gcrd(g(x), \Theta^b(h^\natural(x))) = 1$ for all b in $\{0, 1\}$.*
3. *If the greatest common left divisor of g and h is equal to 1 and if $g = \Theta^b(g^\natural)$, then $gcrd(g(x), \Theta^b(h^\natural(x))) = 1$ for all b in $\{0, 1\}$.*

Proof. By Lemma 1 we have $(x^n - \lambda)^* = (hg)^* = \Theta^k(g^*)h^*$
$\iff x^n - \lambda = -\lambda\Theta^k(g^*)h^* = hg$
(because $\lambda^2 = 1$).

1. Consider $a(x)$, $b(x)$ in R such that $a(x)g(x) + b(x)\Theta^b(h^*(x)) = 1$ and $b(x)\Theta^b(h^*(x)) = \Theta^b(h^*(x))b(x)$.
   $a(x)g(x) + b(x)\Theta^b(h^*(x)) = 1$
   $\iff \Theta^{k+b}(g^*)a(x)g(x) + \Theta^{k+b}(g^*)b(x)\Theta^b(h^*(x)) = \Theta^{k+b}(g^*)$
   $\iff \Theta^{k+b}(g^*)a(x)g(x) + \Theta^{k+b}(g^*)\Theta^b(h^*(x))b(x) = \Theta^{k+b}(g^*)$
   $\iff \Theta^{k+b}(g^*)a(x)g(x) + \Theta^b(\Theta^k(g^*)h^*(x))b(x) = \Theta^{k+b}(g^*)$
   $\iff \Theta^{k+b}(g^*)a(x)g(x) - \lambda\Theta^b(-\lambda\Theta^k(g^*)h^*(x))b(x) = \Theta^{k+b}(g^*)$
   $\iff \Theta^{k+b}(g^*)a(x)g(x) - \lambda b(x)(x^n - \lambda) = \Theta^{k+b}(g^*)$
   Then $g \mid_r \Theta^{k+b}(g^*) \Rightarrow \Theta^{k+b}(g^*) = \alpha g \Rightarrow \alpha = \theta^{k+b}(\theta^{n-k}(g_0)) \Rightarrow \alpha = \theta^b(g_0)$, therefore
   $$g = \Theta^b(g_0^{-1}\Theta^k(g^*)) = \Theta^b(g_0^{-1}\Theta^k(\Theta^{n-k}(g_0)g^\natural)) = \Theta^{k+b}(g^\natural).$$
2. We have

$$
\begin{aligned}
-\lambda\Theta^k(g^*)h^* = hg \quad &\iff -\lambda\Theta^k(g^*)h^* - hg = 0 \\
&\iff -\lambda\Theta^k(g^*)h^* - gh = 0 \\
&\text{(because } x^n - \lambda \in Z(R)\text{)} \\
&\iff -\lambda\Theta^{k+b}(g^*)\Theta^b(h^*) - gh = 0 \\
&\iff -\lambda\Theta^{k+b}(g^*)\Theta^b(h^*) - \Theta^b(g_0^{-1}\Theta^k(g^*))h = 0 \\
&\iff \Theta^{k+b}(g^*)(\Theta^b(h^*) + \lambda g_0^{-1}h) = 0 \\
&\text{(because } g_0 \in \mathbb{F}_q^\theta\text{)}
\end{aligned}
$$

$$h = -\lambda g_0\Theta^b(h^*)$$

$gcrd(h, g) = 1 \iff gcrd(-\lambda g_0\Theta^b(h^*), g) = 1 \iff gcrd(\Theta^b(h^*), g) = 1$

3. Assume that the greatest common left divisor of g and h is equal to 1 and consider a right divisor u of degree d of $\Theta^b(g^\natural)$ and $\Theta^b(h^\natural)$. Then u divides on the right both $\Theta^b(g^*)$ and $\Theta^b(h^*)$. Consider G such that $\Theta^b(g^*) = G \cdot u$, and denote $r = \deg(g)$, then $\Theta^{r+e}(g) = \Theta^b((\Theta^b(g^*))^*) = \Theta^{r-d+b}(u^*) \cdot \Theta^b(G^*)$ where $e = 0$ if $b = 0$ and $e = 2$ if $b = 1$, therefore $\Theta^{-d-b}(u^*)$ divides g on the left. Similarly, $\Theta^{-d-b}(u^*)$ divides $\Theta^b(h)$ on the left, therefore u is a constant and the greatest common right divisor of $\Theta^b(g^\natural)$ and $\Theta^b(h^\natural)$ is equal to 1. Lastly, if $g = \Theta^b(g^\natural)$, then $gcrd(g, \Theta^b(h^\natural)) = 1$.

□

In the following, using the Theorem 8 we give a necessary and sufficient condition for a skew constacyclic code to be an Euclidean LCD code and Hermitian LCD code.

Corollary 9. *Consider C a skew λ-constacyclic code with skew generator g and length n.*

1. *If $A_{(g,\Theta^b(h^*))}$ is nonempty and if C is an Euclidean LCD skew constacyclic code or C is an Hermitian LCD skew constacyclic code then $g = \Theta^{k+b}(g^\natural)$ for all b in $\{0, 1\}$.*

2. *If the greatest common right divisor of $h(x)$ and $g(x)$ is equal to 1, g_0 in $\mathbb{F}_q^\theta$ and $g(x) = \Theta^{k+b}(g^\natural(x))$ then C is an Euclidean LCD skew constacyclic code when $b = 0$ and C is an Hermitian LCD skew constacyclic code when $b = 1$.*

3. *If the greatest common left divisor of $h(x)$ and $g(x)$ is equal to 1 and $g = \Theta^b(g^\natural)$ then C is an Euclidean LCD skew constacyclic code when $b = 0$ and C is an Hermitian LCD skew constacyclic code when $b = 1$.*

4. *If the greatest common left divisor of $h(x)$ and $g(x)$ is equal to 1 and C is a θ-reversible code (resp.conjugate-skew reversible code) then C is an Euclidean LCD skew constacyclic code (resp. C is an Hermitian LCD skew constacyclic code).*

Remark 3. *Suppose $gcrd(g(x), \Theta^b(h^*))) = 1$, there are polynomials $(a(x), b(x)) \in A_{(g,\Theta^b(h^*))}$, as a special case of Corollary 9 when $|\langle\theta\rangle| \mid k$ we obtain C is an Euclidean LCD skew constacyclic code or C is an Hermitian LCD skew constacyclic code then $g = \Theta^b(g^\natural)$.*

In the following, we give necessary and sufficient condition for existence LCD skew cyclic code and LCD skew negacyclic code when n an odd integer, The following proposition is inspired from Theorem 4 and Corollary 6.

Proposition 2. *Let $\mathbb{F}_q$ be a finite field where q is a prime power and θ be an automorphism over $\mathbb{F}_q$ of order m,n an odd integer then there exist LCD skew cyclic code if and only if there exist LCD skew negacyclic code.*

Proof. $g(x)$ is a generated polynomial of the LCD skew negacyclic code C then

$$x^n + 1 = h(x)g(x)$$

by Theorem 4 we have

$$\psi(x^n+1) = \psi(h(x))\psi(g(x)) \Rightarrow x^n - 1 = -\psi(h(x))\psi(g(x))$$

because we have n odd and $\lambda = 1 = 1^{[n]}$ in this case we have $\delta = 1$
$\psi(g(x))$ is a generated polynomial of the LCD skew cyclic code.

C is LCD skew negacyclic code therefore $gcrd(g, h^*) = 1$ Furthermore, $g(x)$ coprime with $h^*(x)$ Then by [1, Theorem 7] there are polynomials polynomials $a(x)$, and $b(x)$ such that

$$a(x)g(x) + b(x)h^*(x) = 1$$

by Theorem 4 we have

$$\psi(a(x))\psi(g(x)) + \psi(b(x))\psi(h^*(x)) = 1$$

Then

$$(\psi(g(x)), \psi(h(x))^*) = 1.$$

Note that

$$j = \begin{cases} 1 & \text{if} \quad \text{k odd} \\ 0 & otherwise \end{cases}$$

where $k = deg(h(x))$.
$\psi(h^*(x) = (-1)^j \psi(h(x))^*$ because
$\psi(h^*(x)) = \sum_{i=0}^{k} (-1)^i (\delta^{-1})^{[i]} \theta^i(h_{k-i}) x^i$;
$\psi(h(x))^* = (\delta^{-1})^{[k]} \sum_{i=0}^{k} (-1)^{i+j} (\delta)^{[i]} \theta^i(h_{k-i}) x^i$
and we have $\delta = 1$.
Similarly, if there exist LCD skew cyclic code Then there exist LCD skew negacyclic code. □

Remark 4. *from Proposition 2 if there exist LCD skew negacyclic code Then there exist LCD skew cyclic code and by Theorem 4 these last two are equivalent.*

In table 4, one lists the best Hamming distances of LCD θ-negacyclic codes with length $\leqslant 10$. For each n, we give a skew generator polynomial g we list parameters of LCD θ-negacyclic codes over $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w + 1$, θ the Frobenius automorphism.

In table 5, one lists the best Hamming distances of LCD θ-cyclic codes with length $\leqslant 14$. For each n, we give a skew generator polynomial $g(x)$ we list parameters of LCD θ-cyclic codes over $\mathbb{F}_4 = \mathbb{F}_2(w) =$ where $w^2 = w + 1$, θ the Frobenius automorphism.

The values d_{opt} correspond to the best distances for linear codes from [11], the distances does not attaining the optimal value for linear codes are lines on them.

We focus now on Euclidean and Hermitian LCD θ-cyclic and θ-negacyclic codes.

6 LCD skew cyclic and negacyclic codes over $\mathbb{F}_{p^2}$

In this section, we describe LCD skew cyclic and negacyclic codes in the particular case when $q = p^2$ is the square of a prime number p and $\theta : a \mapsto a^p$ is the Frobenius automorphism over $\mathbb{F}_q$. We first give a characterization of LCD skew cyclic and negacyclic codes using a lclm factorization of their skew generator polynomials (Theorem 9). This point of view is useful for the construction and the enumeration of the codes (Propositions 5, 6, 7).

Like previously, we assume that the order of θ divides n, therefore n is even. Euclidean LCD skew cyclic and negacyclic codes of length $n = 2k$ are generated by monic skew polynomials g satisfying :

$$g \cdot h = h \cdot g = x^{2k} \pm 1 \text{ and } \operatorname{gcrd}(g, h^{\natural}) = 1 \tag{1}$$

$[n,k,d]$	d_{opt}	$g(x)$
$[2,1,2]$	2	$x+w^3$
$[2,1,2]$	2	$x+w^7$
$[4,2,3]$	3	x^2+w^2x+2
$[4,2,3]$	3	x^2+w^5x+1
$[6,3,4]$	4	$x^3+x^2+wx+w^7$
$[6,3,4]$	4	$x^3+w^3x^2+w^7x+w^7$
$[8,4,5]$	5	$x^4+x^3+wx^2+w^5x+w^2$
$[8,4,5]$	5	$x^4+wx^3+x^2+w^3x+1$
$[8,4,5]$	5	$x^4+wx^3+w^5x^2+w^2x+w^2$
$[10,9,2]$	2	$x+w$
$[10,8,\underline{2}]$	$\underline{3}$	x^2+1
$[10,6,\underline{4}]$	$\underline{5}$	$x^4+x^2+w^6x+w^6$
$[10,6,4]$	4	$x^4+x^3+w^6x+2$
$[10,5,6]$	6	$x^5+x^4+w^3x^3+w^6x^2+wx+w^5$
$[10,5,6]$	6	$x^5+x^4+wx^3+w^2x^2+w^3x+w^7$
$[10,4,\underline{6}]$	$\underline{7}$	$x^6+w^6x^5+x^4+w^5x^3+2x^2+x+2$
$[10,4,\underline{6}]$	$\underline{7}$	$x^6+w^7x^5+w^6x^4+w^6x^3+w^6x^2+wx+1$
$[10,2,\underline{5}]$	$\underline{9}$	$x^8+2x^6+x^4+2x^2+1$
$[10,1,10]$	10	$x^9+wx^8+2x^7+w^5x^6+x^5+wx^4+2x^3+w^5x^2+x+w$

Table 4: Skew generator polynomials of Euclidean LCD skew-negacyclic codes over $\mathbb{F}_9$ of lengths $\leqslant 10$

while Hermitian LCD skew cyclic and negacyclic codes are characterized by :

$$g \cdot h = h \cdot g = x^{2k} \pm 1 \text{ and } \operatorname{gcrd}(g, \Theta(h^{\natural})) = 1 \tag{2}$$

where $\Theta : \sum a_i x^i \mapsto \sum \theta(a_i) x^i$.

Let us introduce a first notation. For $F(x^2) \in \mathbb{F}_p[x^2]$ and b in $\{0,1\}$, consider the following set :

$$\mathcal{L}^{(b)}_{F(x^2)} := \{g \in R \mid g \text{ monic},\ g \cdot h = F(x^2) \text{ and } \operatorname{gcrd}(\Theta^b(h^{\natural}), g) = 1\}.$$

The following proposition is inspired from Proposition 28 of [5] and Proposition 2 of [3]. It will enable to construct LCD skew cyclic and negacyclic codes over $\mathbb{F}_{p^2}$.

Proposition 3. *Consider $\mathbb{F}_q$ a finite field with $q = p^2$ elements where p is a prime number, $\theta : a \mapsto a^p$ the Frobenius automorphism over $\mathbb{F}_{p^2}$, $R = \mathbb{F}_q[x;\theta]$. Consider $F(x^2) = f_1(x^2) \cdots f_r(x^2)$ where $f_1(x^2), \ldots, f_r(x^2)$ are polynomials of $\mathbb{F}_p[x^2]$ such that f_i is coprime with f_j and $f_j^{\natural}$ for all $i \neq j$. The application*

$$\phi : \left\{ \begin{array}{ccc} \mathcal{L}^{(b)}_{f_1(x^2)} \times \cdots \times \mathcal{L}^{(b)}_{f_r(x^2)} & \to & \mathcal{L}^{(b)}_{F(x^2)} \\ (g_1, \ldots, g_r) & \mapsto & \operatorname{lclm}(g_1, \ldots, g_r) \end{array} \right.$$

is bijective.

Proof. • The application ϕ is well-defined.

Consider $(g_1, \ldots, g_r)$ in $\mathcal{L}^{(b)}_{f_1(x^2)} \times \cdots \times \mathcal{L}^{(b)}_{f_r(x^2)}$ and $g = \operatorname{lclm}(g_1, \ldots, g_r)$. Consider $h_1, \ldots, h_r$ such that $g_i h_i = h_i g_i = f_i(x^2)$ and $\operatorname{gcrd}(g_i, \Theta^b(h_i^{\natural})) = 1$. Consider $h = \operatorname{lcrm}(h_1, \ldots, h_r)$

First of all, as $h_1, \ldots, h_r$ divide respectively $f_1(x^2), \ldots, f_r(x^2)$, and as $f_1(x^2), \ldots, f_r(x^2)$ are pairwise coprime central polynomials, the degree of $\operatorname{lcrm}(h_1, \ldots, h_r)$ is equal to $\sum_{i=1}^{r} \deg(h_i)$.

In the same way, the degree of $\text{lclm}(g_1, \ldots, g_r)$ is equal to $\sum_{i=1}^{r} \deg(g_i)$. Furthermore, as $g_i \cdot h_i = f_i(x^2)$, the degree of $g_i h_i$ is equal to the degree of $f_i(x^2)$ in x, therefore the degree of gh is equal to the degree of $F(x^2)$ in x.

Consider, for i in $\{1, \ldots, r\}$, A_i in R such that $g = A_i \cdot g_i$ and B_i in R such that $h = h_i \cdot B_i$. One gets $g \cdot h = A_i \cdot g_i \cdot h_i \cdot B_i = A_i \cdot f_i(x^2) \cdot B_i$. As $f_i(x^2)$ is central, it divides $g \cdot h$. The polynomials $f_i(x^2)$ are pairwise coprime in $\mathbb{F}_p[x^2]$, therefore their least common right multiple is equal to their product $F(x^2)$, and $F(x^2)$ divides $g \cdot h$. Considerations on the degrees of the involved polynomials imply the equality $g \cdot h = F(x^2)$. Now $\text{gcrd}(g, \Theta^b(h^\natural)) = \text{gcrd}(\text{lclm}(g_1, \ldots, g_r), \text{lclm}(\Theta^b(h_1^\natural), \ldots, \Theta^b(h_r^\natural))$. One can note that the skew polynomials g_i and $\Theta^b(h_j^\natural)$ are right coprime if $i = j$ ($\text{gcrd}(g_i, \Theta^b(h_i^\natural)) = 1$). If $i \neq j$ consider a right divisor u of g_i and $\Theta^b(h_j^\natural)$, then u divides $f_i(x^2)$ and $f_j^\natural(x^2)$, as $f_i(x^2)$ and $f_j^\natural(x^2)$ are coprime one gets that $u = 1$. One deduces that $\text{gcrd}(g, \Theta^b(h^\natural)) = 1$.

The skew polynomial g belongs to $\mathcal{L}^{(b)}_{F(x^2)}$ therefore ϕ is well defined.

- The application ϕ is bijective.

  Consider g in $\mathcal{L}^{(b)}_{F(x^2)}$, then g divides $F(x^2)$, therefore, according to Theorem 4.1 of [10], $g = \text{lclm}(g_1, \ldots, g_r)$ where $g_i = \text{gcrd}(f_i(x^2), g)$ and this lclm-decomposition into skew polynomials dividing $f_1(x^2), \ldots, f_r(x^2)$ is unique. Furthermore $\deg(g) = \sum_{i=1}^{r} \deg(g_i)$ bevause $f_i(x^2)$ and $f_j(x^2)$ are coprime. Let us prove that g_i belongs to $\mathcal{L}^{(b)}_{f_i(x^2)}$. Consider h in R such that $gh = hg = F(x^2)$ and $\text{gcrd}(g, \Theta^b(h^\natural)) = 1$. As h divides $F(x^2)$, according to Theorem 4.1 of [10], $h = \text{lcrm}(h_1, \ldots, h_r)$ where $h_i = \text{gcld}(f_i(x^2), h)$. This lcrm-decomposition into skew polynomials dividing $f_1(x^2), \ldots, f_r(x^2)$ is unique and $\deg(h) = \sum_{i=1}^{r} \deg(h_i)$.

  Consider, for i in $\{1, \ldots, r\}$, A_i in R such that $g = A_i \cdot g_i$ and B_i in R such that $h = h_i \cdot B_i$. As $g \cdot h = F(x^2)$ and as $F(x^2)$ is central, the skew polynomial $g_i \cdot h_i$ divides $F(x^2)$ on the right. Therefore, $g_i \cdot h_i = \text{lclm}(\text{gcrd}(g_i \cdot h_i, f_j(x^2)), j = 1, \ldots, r) = \text{gcrd}(g_i \cdot h_i, f_i(x^2))$. Therefore $g_i \cdot h_i$ divides $f_i(x^2)$. As $\sum_{i=1}^{r} \deg(g_i h_i) = \deg(g) + \deg(h) = \deg(F(x^2)) = \sum_{i=1}^{r} \deg(f_i(x^2))$, one gets $g_i \cdot h_i = f_i(x^2)$. Lastly, consider u in R such that u divides on the right g_i and $\Theta^b(h_i^\natural)$. As h_i divides on the left h, $\Theta^b(h_i^\natural)$ divides on the right $\Theta^b(h^\natural)$, therefore u divides on the right both g and $\Theta^b(h^\natural)$, and $u = 1$.

□

We know introduce some notations that will be useful later :

$$\mathcal{D}_{F(x^2)} := \{f \in \mathbb{F}_p[x^2] \mid f \text{ monic and divides } F(x^2) \text{ in } \mathbb{F}_p[x^2]\}$$

$$\mathcal{F}_{ir} := \{f = f(x^2) \in \mathbb{F}_p[x^2] \mid f = f^\natural \text{ irreducible in } \mathbb{F}_p[x^2], \deg_{x^2} f > 1\}$$

$$\mathcal{F}_{red} := \{f = f(x^2) \in \mathbb{F}_p[x^2] \mid f = f_{ir} f_{ir}^\natural,\ f_{ir} \neq f_{ir}^\natural \text{ irreducible in } \mathbb{F}_p[x^2]\}.$$

Theorem 9. *Consider $\mathbb{F}_q$ a finite field with $q = p^2$ elements where p is a prime number, $\theta : a \mapsto a^p$ the Frobenius automorphism over $\mathbb{F}_{p^2}$, $R = \mathbb{F}_q[x; \theta]$, $\lambda \in \{-1, 1\}, b \in \{0, 1\}$. Consider $x^n - \lambda = f_1(x^2)^{p^s} \cdots f_r(x^2)^{p^s}$ where $f_1(x^2), \ldots, f_r(x^2)$ are polynomials of $\mathbb{F}_p[x^2]$ belonging to $\{x^2 \pm 1\} \cup \mathcal{F}_{ir} \cup \mathcal{F}_{red}$. Consider a (θ, λ)-constacyclic code C of length n and skew generator polynomial g.*

1. *C is an Euclidean LCD code if and only if $g = \text{lclm}(g_1, \ldots, g_r)$ where for all i in $\{1, \ldots, r\}$, $g_i \in \mathcal{L}^{(0)}_{f_i(x^2)}$.*
2. *C is an Hermitian LCD code if and only if $g = \text{lclm}(g_1, \ldots, g_r)$ where for all i in $\{1, \ldots, r\}$, $g_i \in \mathcal{L}^{(1)}_{f_i(x^2)}$.*

Proof. According to Theorem 6, the Euclidean (resp. Hermitian) LCD θ-cyclic codes of length $2k$ are generated by the elements of the set $\mathcal{L}^{(b)}_{x^{2k}-1}$ where $b=0$ (resp. $b=1$). The Euclidean (resp. Hermitian) LCD θ-negacyclic codes of length $2k$ are associated to the set $\mathcal{L}^{(b)}_{x^{2k}+1}$ where $b=0$ (resp. $b=1$). As $x^{2k}\pm 1$ is self-reciprocal, one has $x^{2k}\pm 1 = f_1(x^2)^{p^s}\cdots f_r(x^2)^{p^s}$ where $f_1(x^2),\ldots,f_r(x^2)$ are self-reciprocal polynomials of $\mathbb{F}_p[x^2]$ who are either irreducible or products of an irreducible polynomial and its reciprocal polynomial. Therefore for $i\neq j$, f_i is coprime with f_j and $f_j^{\natural}$ and Theorem 3 can be applied to $F(x^2)=x^{2k}\pm 1$.

□

Remark 5. $\{1, F(x^2)\}\subset \mathcal{L}^{(b)}_{F(x^2)}$. *That means that the trivial codes* $\{0\}$ *and* $\mathbb{F}_q^n$ *are Hermitian and Euclidean LCD* θ*-cyclic and* θ*-negacyclic codes.*

Remark 6. *The Euclidean LCD skew cyclic (resp. negacyclic) codes of length* n *over* $\mathbb{F}_{p^2}$ *are the skew cyclic (resp. negacyclic) codes* $C=\bigcap_{i=1}^r C_i$ *where* C_i *is a skew cyclic (resp. negacyclic) code generated by* $g_i\in\mathcal{L}^{(0)}(f_i(x^2)^{p^s})$ *and where the polynomial* $F(x^2)=x^n-1$ *(resp.* x^n+1*) factorizes in* $\mathbb{F}[x^2]$ *as* $F(x^2)=\prod_{i=1}^r f_i(x^2)^{p^s}$, $f_i(x^2)\in\mathcal{F}_{ir}\bigcup\mathcal{F}_{red}\bigcup\{x^2\pm 1\}$. *The same remark holds for Hermitian LCD skew cyclic and negacyclic codes where* $g_i\in\mathcal{L}^{(0)}(f_i(x^2)^{p^s})$ *is replaced with* $g_i\in\mathcal{L}^{(1)}(f_i(x^2)^{p^s})$.

Therefore, LCD θ-cyclic and θ-negacyclic codes of length $2k$ over $\mathbb{F}_{p^2}$ are generated by skew polynomials which are least common left multiples of skew polynomials $g_i\in\mathcal{L}^{(b)}_{f_i(x^2)^{p^s}}$ where $f=f_i$ satisfies one of the following cases :

- $f(x^2)=x^2-\epsilon$ where $\epsilon=\pm 1$ (see Lemma 6)
- $f(x^2)\in\mathcal{F}_{ir}$ irreducible in $\mathbb{F}_p[x^2]$ with degree $d>1$ in x^2 (Lemma 6);
- $f(x^2)\in\mathcal{F}_{red}$ is the product of two irreducible distinct polynomials in $\mathbb{F}_p[x^2]$ (Lemma 7).

The following proposition enables to characterize those skew polynomials having a unique factorization into the product of monic irreducible skew polynomials. It will be useful later.

Proposition 4 (Proposition 16 of [5])**.** *Consider* $R=\mathbb{F}_q[x;\theta]$ *with* $q=p^2$ *and* $\theta : a\mapsto a^p$ *of order two,* $f\in\mathbb{F}_p[x^2]$ *irreducible in* $\mathbb{F}_p[x^2]$ *and* $h=h_1\cdots h_m$ *a product of irreducible monic polynomials dividing* f*. The following assertions are equivalent*

(i) h *has a unique factorization into irreducible monic polynomials;*

(ii) f *does not divide* h *in* R*;*

(iii) for all i *in* $\{1,\ldots,m-1\}$, $f\neq h_{i+1}h_i$.

6.1 Length $2p^s$ over $\mathbb{F}_{p^2}$

Lemma 6 describes the set $\mathcal{L}^{(b)}_{f(x^2)^m}$ where $f(x^2)\in\mathcal{F}_{ir}\bigcup\{x^2\pm 1\}$. It will be useful to characterize all nontrivial LCD skew cyclic and negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^2}$ (Proposition 5).

Lemma 6. *Consider* $m\in\mathbb{N}$, $f(x^2)\in\mathcal{F}_{ir}\bigcup\{x^2\pm 1\}$ *with degree* d *in* x^2 *and* g *in* R*. The skew polynomial* g *belongs to the set* $\mathcal{L}^{(b)}_{f(x^2)^m}$ *if, and only if,* $g=1$ *or* $g=f(x^2)^m$ *or* g *has a unique factorization into the product of* m *monic skew polynomials* $g=g_m\cdots g_1$ *where*

$$\begin{cases} \forall i\in\{1,\ldots,m\}, \deg(g_i)=d \\ g_i \text{ divides on the right } f(x^2) \\ \forall i\in\{1,\ldots,m-1\}, g_{i+1}g_i\neq f(x^2) \\ g_1\neq\Theta^b(h_1^{\natural}) \text{ where } \Theta^b(h_1^{\natural})h_1=f(x^2). \end{cases} \tag{3}$$

Proof. Consider g in $\mathcal{L}^{(b)}_{f(x^2)^m} \setminus \{f(x^2)^m, 1\}$. Consider h in R such that $gh = hg = f(x^2)^m$. As $f(x^2)$ is central and irreducible in $\mathbb{F}_p[x^2]$, the skew polynomials g and h are products of irreducible monic factors dividing $f(x^2)$. As the degree of g belongs to $\{1, \ldots, 2m-1\}$, there exists $r \in \{1, \ldots, 2m-1\}$, $g_1, \ldots, g_{2m-r}, h_1, \ldots, h_r$ monic of degree d dividing $f(x^2)$ such that

$$g = g_{2m-r} \cdots g_1$$

and

$$h = h_1 \cdots h_r.$$

The skew polynomial $\Theta^b(h_1^\natural)$ is an irreducible right factor of $\Theta^b(h^\natural)$ which divides $\Theta^b(f^\natural(x^2)) = f(x^2)$ and does not divide g on the right because $\text{gcrd}(\Theta^b(h^\natural), g) = 1$. Therefore $f(x^2)$ does not divide g. Similarly, one gets that $f(x^2)$ does not divide $\Theta^b(h^\natural)$ and h. Therefore, according to Proposition 4, the above factorizations of g and h into the products of monic irreducible factors are unique and for all i in $\{1, \ldots, m-1\}$, $g_{i+1}g_i \neq f(x^2)$.
As $gh = hg = f(x^2)^m$ one gets that for all i, $g_i h_i = f(x^2)$, therefore, $r = m$.
Laslty, as g and $\Theta^b(h^\natural)$ are right coprime, necessarily, $g_1 \neq \Theta^b(h_1^\natural)$.

Conversely, consider $g = g_m \cdots g_1$ where $g_1, \ldots, g_m$ are monic skew polynomials satisfying (3). Consider $h = h_1 \cdots h_m$ with $g_i h_i = h_i g_i = f(x^2)$ then $gh = hg = f(x^2)^m$. Furthermore as $g_{i+1}g_i \neq f(x^2)$, according to Proposition 4, the above factorization of g into the product of monic irreducible factors is unique. Similarly, the factorizations of h and $\Theta^b(h^\natural)$ into the products of monic irreducible factors are unique.
Consider u a right factor of g and $\Theta^b(h^\natural)$ with degree > 1. Necessarily, u has a unique factorization into the product of monic skew polynomials. The unique monic linear right factor of u is also the unique monic right factor of g and $\Theta^b(h^\natural)$, therefore $u = g_1 = \Theta^b(h_1^\natural)$, which is impossible according to (3). Therefore $\text{gcrd}(g, \Theta^b(h^\natural)) = 1$ and g belong to $\mathcal{L}^{(b)}_{f(x^2)^m}$. □

Lemma 6 enables to characterize all nontrivial LCD skew cyclic and negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^2}$. Their dimensions are necessarily equal to p^s :

Proposition 5. *Consider $\theta : a \mapsto a^p$ the Frobenius automorphism over $\mathbb{F}_{p^2}$ and s in $\mathbb{N}$. There is no (Euclidean and Hermitian) LCD skew (θ-cyclic and θ-negacyclic) codes of length $n = 2p^s$ and dimension k if $k \notin \{0, p^s, 2p^s\}$.*

1. *The number of Euclidean LCD θ-cyclic of length $n = 2p^s$ and dimension $k = p^s$ over $\mathbb{F}_{p^2}$ is*

$$\begin{cases} 2^k & \text{if} \quad p = 2 \\ p^{k-1}(p - (-1)^{(p+1)/2}) & \text{if} \quad p \neq 2. \end{cases}$$

2. *The number of Hermitian LCD θ-cyclic of length $n = 2p^s$ and dimension $k = p^s$ over $\mathbb{F}_{p^2}$ is*

$$\begin{cases} 0 & \text{if} \quad p = 2 \\ p^{k-1}(p+1) & \text{if} \quad p \neq 2. \end{cases}$$

3. *For p odd prime, the number of Euclidean LCD θ-negacyclic of length $n = 2p^s$ and dimension $k = p^s$ over $\mathbb{F}_{p^2}$ is $p^{k-1}(p - (-1)^{(p-1)/2})$.*

4. *For p odd prime, there is no Hermitian LCD θ-negacyclic of length $n = 2p^s$ and dimension $k = p^s$ over $\mathbb{F}_{p^2}$.*

Proof.

- The Euclidean LCD θ-cyclic (resp. θ-negacyclic) codes of length $n = 2p^s$ over $\mathbb{F}_{p^2}$ are generated by the monic skew polynomials belonging to $\mathcal{L}^{(0)}_{(x^2-\epsilon)^{p^s}}$ where $\epsilon = 1$ (resp. $\epsilon = -1$). According to Lemma 6, a monic skew polynomial g belongs to the set $\mathcal{L}^{(0)}_{(x^2-\epsilon)^{p^s}}$ if, and only if, $g = 1$ or $g = (x^2 - \epsilon)^{p^s}$ or $g = g_{p^s} \cdots g_1$ where $g_1, \ldots, g_{p^s}$ are monic skew polynomials satisfying :

$$\begin{cases} \deg(g_i) = 1 \\ g_i \text{ divides on the right } x^2 - \epsilon \\ g_{i+1} g_i \neq x^2 - \epsilon \\ g_1 \neq h_1^\natural \text{ where } h_1^\natural \cdot h_1 = x^2 - \epsilon. \end{cases}$$

Therefore $g \neq 1, (x^2-\epsilon)^{p^s}$ belongs to the set $\mathcal{L}^{(0)}_{(x^2-\epsilon)^{p^s}}$ if, and only if, $g = (x+\alpha_{p^s}) \cdots (x+\alpha_1)$ where

$$\begin{cases} \alpha_i^{p+1} = \epsilon \\ \alpha_{i+1} \neq -\epsilon/\alpha_i \\ \epsilon \neq -\theta(\alpha_1)/\alpha_1. \end{cases} \tag{4}$$

Furthermore the above factorization of g is unique, therefore the number of skew polynomials of degree $k = p^s$ in $\mathcal{L}^{(0)}_{(x^2-\epsilon)^k}$ is

$$\begin{cases} p^{k-1}(p-1) & \text{if} \quad p \text{ odd and } \epsilon = (-1)^{(p+1)/2} \\ p^{k-1}(p+1) & \text{if} \quad p \text{ odd and } \epsilon \neq (-1)^{(p+1)/2} \\ 2^k & \text{if} \quad p = 2. \end{cases}$$

- The Hermitian LCD θ-cyclic (resp. θ-negacyclic) codes of length $n = 2p^s$ over $\mathbb{F}_{p^2}$ are generated by the monic skew polynomials belonging to $\mathcal{L}^{(1)}_{(x^2-\epsilon)^{p^s}}$ where $\epsilon = 1$ (resp. $\epsilon = -1$). According to Lemma 6, a monic skew polynomial g belongs to the set $\mathcal{L}^{(1)}_{(x^2-\epsilon)^{p^s}}$ if, and only if, $g = 1$ or $g = (x^2-\epsilon)^{p^s}$ or $g = (x+\alpha_{p^s}) \cdots (x+\alpha_1)$ where

$$\begin{cases} x + \alpha_i \text{ divides on the right } x^2 - \epsilon \\ (x+\alpha_{i+1})(x+\alpha_i) \neq x^2 - \epsilon \\ x + \alpha_1 \neq \Theta^b(h_1^\natural) \text{ where } \Theta^b(h_1^\natural) \cdot h_1 = x^2 - \epsilon. \end{cases}$$

If $\epsilon = -1$ or $p = 2$, $\mathcal{L}^{(1)}_{(x^2-\epsilon)^{p^s}} = \{1, (x^2-\epsilon)^{p^s}\}$. If $\epsilon = 1$ and $p \neq 2$, the skew polynomial $g \neq 1, (x^2-\epsilon)^{p^s}$ belongs to the set $\mathcal{L}^{(1)}_{(x^2-\epsilon)^{p^s}} = \mathcal{L}^{(1)}_{(x^2-1)^{p^s}}$ if, and only if, $g = (x+\alpha_{p^s}) \cdots (x+\alpha_1)$ where

$$\begin{cases} \alpha_i^{p+1} = 1 \\ \alpha_{i+1} \neq -1/\alpha_i. \end{cases} \tag{5}$$

In this case there are $(p+1)p^{k-1}$ skew polynomials of degree $k = p^s$ in $\mathcal{L}^{(1)}_{(x^2-\epsilon)^k}$.

□

Remark 7. *Over $\mathbb{F}_4$, one can notice that when $k = 2^s$, the number of Euclidean LCD θ-cyclic codes $[2k, k]$ growths exponentially with k (Proposition 5, 1. above), while the number of Euclidean self-dual θ-cyclic codes $[2k, k]$ is constant (Corollary 26 of [5]).*

Remark 8. *As (4) $\Rightarrow$ (5), if $p \neq 2$, the Euclidean LCD θ-cyclic codes of length $2p^s$ and dimension p^s over $\mathbb{F}_{p^2}$ are all Hermitian LCD.*

Example 15. *There are $16 = 2^{2^2}$ nontrivial Euclidean LCD θ-cyclic codes of length 8 over $\mathbb{F}_4 = \mathbb{F}_2(w)$where θ is the Frobenius automorphism . Their dimensions are all equal to 4. Consider $g = (x+1)(x+w^2)(x+w^2)(x+w^2) = x^4+wx^3+wx^2+x+1$ and $h = (x+w)(x+w)(x+w)(x+1)$. Then $g \cdot h = h \cdot g = x^8-1$, $h^\natural = (x+1)(x+w^2)(x+1)(x+w)$ and $\Theta(h^\natural) = (x+1)(x+w)(x+1)(x+w^2)$. These factorizations of g, h and $\Theta(h^\natural)$ into the products of monic linear skew polynomials are unique. As $x + w^2 \neq x + w$, $h^\natural$ is right coprime with g, therefore g generates an Euclidean LCD θ-cyclic code. However, as $x+w^2$ divides on the right both $\Theta(h^\natural)$ and g, this code is not Hermitian LCD.*

Example 16. *Consider $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w+1$. There are $18 = 3^2 \times (3-(-1)^2)$ nontrivial Euclidean LCD θ-cyclic codes and $36 = 3^2 \times (3+1)$ nontrivial Hermitian LCD θ-cyclic codes of length 6 over $\mathbb{F}_9$. Their dimensions are equal to 3.*

- *Consider $g = (x+a^2)(x+a^6)(x+1) = x^3+a^5x^2+a^5x+1$ and $h = (x+2)(x+w^6)(x+w^2)$. Then $g \cdot h = h \cdot g = x^6-1$, $h^\natural = (x+w^6)(x+w^2)(x+2)$ and $\Theta(h^\natural) = (x+w^2)(x+w^6)(x+2)$. These factorizations of g, h and $\Theta(h^\natural)$ into the products of monic linear skew polynomials are unique. As $x+2 \neq x+1$, $h^\natural$ is right coprime with g and $\Theta(h^\natural)$ is also rigth coprime to g. Therefore g generates a θ-cyclic code of length 6 which is both Euclidean LCD and Hermitian LCD.*
- *Consider $g = (x-1)^2(x-w^6)$ and $h = (x-w^6)(x+1)^2$. Then $g \cdot h = h \cdot g = x^6-1$. Consider the θ-cyclic code C of length 6 and dimension 3 generated by g. This code is MDS. Furthermore $h^\natural = g$, therefore C is not Euclidean LCD and $\Theta(h^\natural) = (x-1)^2(x-a^2)$ is right coprime with g, therefore C is a MDS Hermitian LCD code.*

Example 17. *Consider $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w+1$. There are $36 = 3^2 \times (3-(-1))$ nontrivial Euclidean LCD θ-negacyclic codes of length 6 over $\mathbb{F}_9$. Their dimensions are equal to 3. Consider $g = (x+w^5)(x+w^7)(x+w^3) = x^3+x^2+wx+w^7$ and $h = (x+w^5)(x+w)(x+w^3)$. Then $g \cdot h = h \cdot g = x^6+1$, $h^\natural = (x+w^3)(x+w)(x+w)$ and $\Theta(h^\natural) = (x+w^2)(x+w^3)(x+w^3)$. These factorizations of g, h and $\Theta(h^\natural)$ into the products of monic linear skew polynomials are unique. As $x+w^3 \neq x+w$, $h^\natural$ is right coprime with g. Therefore g generates a θ-negacyclic code of length 6 which is Euclidean LCD. However g and $\Theta(h^\natural)$ are not right coprime because $x+w^3$ divides both g and $\Theta(h^\natural)$ and the code is not Hermitian LCD.*

Example 18. *Consider $\mathbb{F}_{25} = \mathbb{F}_5(w)$ where $w^2 = w-2$. The skew polynomials $g = (x+1)^3(x+w)(x+1) = x^5+w^{16}x^4+w^5x^3+2x^2+wx+w^4$ and $h = (x-1)(x+w^8)(x-1)^3$ satisfy $g \cdot h = h \cdot g = x^{10}-1$. Furthermore $h^\natural = (x+w^4)(x+w^{20})(x+w^4)(x+w^{16})(x-1)$, $\Theta(h^\natural) = (x+w^{20})(x+w^4)(x+w^{20})(x+w^8)(x-1)$. The factorizations of g, h and $\Theta(h^\natural)$ into the products of monic linear skew polynomials are unique. As $x-1 \neq x+1$, $h^\natural$ and $\Theta(h^\natural)$ are right coprime with g. Therefore g generates a θ-cyclic code $[10,5]$ which is Euclidean LCD and Hermitian LCD. Furthermore, this is an MDS code.*

6.2 Any length over $\mathbb{F}_{p^2}$

In order to consider LCD skew cyclic and negacyclic codes of any even length over $\mathbb{F}_{p^2}$, we will characterize $\mathcal{L}^{(b)}_{f(x^2)^m}$ for $f(x^2) \in \mathcal{F}_{red}$ (Lemma 7 below) and then use Theorem 9 and Lemma 6 to conclude (Proposition 6).

Lemma 7. *Consider $f(x^2) = f_{ir}(x^2)f^\natural_{ir}(x^2)$ in $\mathcal{F}_{red}$ with degree $d = 2\delta$ in x^2. The monic skew polynomial $g \in R$ belongs to the set $\mathcal{L}^{(b)}_{f(x^2)^m}$ if, and only if, $g = 1$ or $g = f(x^2)^m$ or $g = \mathrm{lclm}(g_1, g_2)$ where $g_1 = g_{1,m} \cdots g_{1,1}$ and $g_2 = g_{2,m} \cdots g_{2,1}$ have unique factorizations into the products of m monic skew polynomials satisfying :*

$$\begin{cases} \deg(g_{i,j}) = \delta \\ g_{1,j} \text{ divides } f_{ir}(x^2) \\ g_{2,j} \text{ divides } f^\natural_{ir}(x^2) \\ g_{1,j}g_{1,j-1} \neq f_{ir}(x^2) \\ g_{2,j}g_{2,j-1} \neq f^\natural_{ir}(x^2) \\ g_{1,1} \neq \Theta^b(h^\natural_{2,1}), h_{2,1}g_{2,1} = f^\natural_{ir}(x^2) \end{cases} \tag{6}$$

Proof. Consider g in $\mathcal{L}^{(b)}_{f(x^2)^m} \backslash \{1, f_{ir}(x^2)^m f^\natural_{ir}(x^2)^m\}$ and h in R such that $gh = hg = f_{ir}(x^2)^m f^\natural_{ir}(x^2)^m$ and $\mathrm{gcrd}(g, \Theta^b(h^\natural)) = 1$. As g divides $f_{ir}(x^2)^m f^\natural_{ir}(x^2)^m$, according to Theorem 4.1 of [10], $g = \mathrm{lclm}(g_1, g_2)$ where $g_1 = \mathrm{gcrd}(f_{ir}(x^2)^m, g)$ and $g_2 = \mathrm{gcrd}(f^\natural_{ir}(x^2)^m, g)$. Similarly, $h = \mathrm{lcrm}(h_1, h_2)$, where $h_1 = \mathrm{gcld}(f_{ir}(x^2), h)$ and $h_2 = \mathrm{gcld}(f^\natural_{ir}(x^2), h)$.

As $gh = hg = f_{ir}(x^2)^m f_{ir}^\natural(x^2)^m$, one has $g_1h_1 = f_{ir}(x^2)^m$ and $g_2h_2 = f_{ir}^\natural(x^2)^m$, therefore g_1 and h_1 (resp. g_2 and h_2) are products of irreducible skew polynomials dividing $f_{ir}(x^2)$ (resp. $f_{ir}^\natural(x^2)$).

If $f_{ir}(x^2)$ divides g_1, then, as $\Theta^b(h_2^\natural)$ divides $f_{ir}(x^2)^m$, g_1 and $\Theta^b(h_2^\natural))$ have a common right divisor (dividing $f_{ir}(x^2)$), therefore g and $\Theta^b(h^\natural)$ also have a common nontrivial right divisor, which is impossible as g and $\Theta^b(h^\natural)$ are right coprime. Therefore $f_{ir}(x^2)$ does not divide g_1. In the same way, $f_{ir}(x^2)$ does not divide h_1, $f_{ir}^\natural(x^2)$ does not divide g_2 and h_2, therefore using Proposition 4, one gets that the factorizations of g_1, g_2, h_1 and h_2 into the products of monic irreducible skew polynomials are unique and

$$\begin{cases} g_1 = g_{1,m} \cdots g_{1,1} \\ g_2 = g_{2,m} \cdots g_{2,1} \\ h_1 = h_{1,1} \cdots h_{1,m} \\ h_2 = h_{2,1} \cdots h_{2,m} \\ \deg(g_{i,j}) = \delta \\ g_{1,j}g_{1,j-1} \neq f_{ir}(x^2) \\ g_{2,j}g_{2,j-1} \neq f_{ir}^\natural(x^2) \\ g_{1,i}h_{1,i} = f(x^2) \\ g_{2,i}h_{2,i} = f^\natural(x^2) \end{cases}$$

As g and $\Theta^b(h^\natural)$ are right coprime, g_1 and $\Theta^b(h_2^\natural)$ are right coprime, therefore $g_{1,1} \neq \Theta^b(h_{2,1}^\natural)$. Conversely, assume that $g = \mathrm{lclm}(g_1, g_2)$ where

$$\begin{cases} g_1 = g_{1,m} \cdots g_{1,1} \\ g_2 = g_{2,m} \cdots g_{2,1} \\ \deg(g_{i,j}) = \delta \\ g_{1,j} \text{ divides } f_{ir}(x^2) \\ g_{2,j} \text{ divides } f_{ir}^\natural(x^2) \\ g_{1,j}g_{1,j-1} \neq f_{ir}(x^2) \\ g_{2,j}g_{2,j-1} \neq f_{ir}^\natural(x^2) \\ g_{1,1} \neq \Theta^b(h_{2,1}^\natural), h_{2,1}g_{2,1} = f_{ir}^\natural(x^2) \end{cases}$$

Consider $h_{i,j}$ such that $g_{1,i}h_{1,i} = f_{ir}(x^2)$ and $g_{2,i}h_{2,i} = f_{ir}^\natural(x^2)$. Consider $h_1 = h_{1,1} \cdots h_{1,m}$, $h_2 = h_{2,1} \cdots h_{2,m}$ and $h = \mathrm{lcrm}(h_1, h_2)$. Then $gh = hg = f_{ir}(x^2)^m f_{ir}^\natural(x^2)^m$ and g and $\Theta^b(h^\natural)$ are right coprime.

□

Theorem 9, Lemma 6 and Lemma 7 enable to characterize the skew generators of LCD θ-cyclic and θ-negacyclic codes over $\mathbb{F}_{p^2}$ as follows :

Proposition 6. *Consider $\mathbb{F}_q$ a finite field with $q = p^2$ elements where p is a prime number, $\theta : a \mapsto a^p$ the Frobenius automorphism over $\mathbb{F}_{p^2}$, $R = \mathbb{F}_q[x;\theta]$, $\lambda \in \{-1, 1\}, b \in \{0, 1\}$. Consider $x^n - \lambda = f_1(x^2)^{p^s} \cdots f_r(x^2)^{p^s}$ where $f_1(x^2), \ldots, f_r(x^2)$ are polynomials of $\mathbb{F}_p[x^2]$ belonging to $\{x^2 \pm 1\} \cup \mathcal{F}_{ir} \cup \mathcal{F}_{red}$. Consider a (θ, λ)-constacyclic code C of skew generator g dividing $x^{2k} - \lambda$. C is Euclidean (resp Hermitian) LCD if and only if*

$$g = \prod_{i \in I} f_i(x^2)^{p^s} \ \mathrm{lclm}_{j \in J}(g_j)$$

where

$$\begin{cases} I, J \subset \{1, \ldots, r\} \\ I \cap J = \emptyset \\ \forall j \in J, g_j \in \mathcal{L}^{(b)}_{f_j(x^2)^{p^s}} \setminus \{1, f_j(x^2)^{p^s}\} \text{ with } b = 0 \text{ (resp } b = 1). \end{cases}$$

Proof. Proposition 3, Lemma 6 and Lemma 7. □

Remark 9. *In Proposition 6, if J is empty then g is central; if I is empty then g is not divisible by any central polynomial. If $J = \{1, \ldots, r\}$ (and $I = \emptyset$), then g is not divisible by any central polynomial and* $\deg(g) = \sum_{i=1}^{r} \deg(g_i) = k$.

The following proposition characterizes LCD skew cyclic and negacyclic codes of even length $n = 2k$ and of dimension k whose generator polynomials are not divisible by any central polynomial ($J = \{1, \ldots, r\}$ and $I = \emptyset$ in Proposition 6).

Proposition 7. *1. The number of Euclidean LCD θ-cyclic codes of length $2k$ and dimension k whose skew generator polynomial is not divisible by any central polynomial is*

$$N \times \prod_{\substack{f \in \mathcal{F}_{ir} \cap \mathcal{D}_{x^n-1} \\ d=\deg(f)}} (p^d - p^{d/2}) p^{d(p^s-1)} \times \prod_{\substack{f \in \mathcal{F}_{red} \cap \mathcal{D}_{x^n-1} \\ d=\deg(f)}} (1 + p^{d/2}) p^{(2p^s-1)d/2}$$

$$\text{where } N = \begin{cases} 2^{2^s} & \text{if } p=2 \\ (p^{p^s-1})^2(p^2-1) & \text{if } k \text{ is even and } p \text{ is odd} \\ p^{p^s-1}(p-(-1)^{(p+1)/2}) & \text{if } k \text{ is odd and } p \text{ is odd} \end{cases}$$

2. The number of Hermitian LCD θ-cyclic codes of length $2k$ and dimension k whose skew generator polynomial is not divisible by any central polynomial is

$$N \times \prod_{\substack{f \in \mathcal{F}_{ir} \cap \mathcal{D}_{x^n-1} \\ d=\deg(f)}} (p^d - p^{d/2}) p^{d(p^s-1)} \times \prod_{\substack{f \in \mathcal{F}_{red} \cap \mathcal{D}_{x^n-1} \\ d=\deg(f)}} (1 + p^{d/2}) p^{(2p^s-1)d/2}$$

$$\text{where } N = \begin{cases} 0 & \text{if } p=2 \\ 0 & \text{if } k \text{ is even and } p \text{ is odd} \\ p^{p^s-1}(p+1) & \text{if } k \text{ is odd and } p \text{ is odd} \end{cases}$$

3. The number of Euclidean LCD θ-negacyclic codes of length $2k$ and dimension k whose skew generator polynomial is not divisible by any central polynomial is

$$N \times \prod_{\substack{f \in \mathcal{F}_{ir} \cap \mathcal{D}_{x^n+1} \\ d=\deg(f)}} (p^d - p^{d/2}) p^{d(p^s-1)} \times \prod_{\substack{f \in \mathcal{F}_{red} \cap \mathcal{D}_{x^n+1} \\ d=\deg(f)}} (1 + p^{d/2}) p^{(2p^s-1)d/2}$$

$$\text{where } N = \begin{cases} 1 & \text{if } k \text{ is even} \\ p^{p^s-1}(p-(-1)^{(p-1)/2}) & \text{if } k \text{ is odd} \end{cases}$$

4. The number of Hermitian LCD θ-negacyclic codes of length $2k$ and dimension k whose skew generator polynomial is not divisible by any central polynomial is

$$N \times \prod_{\substack{f \in \mathcal{F}_{ir} \cap \mathcal{D}_{x^n+1} \\ d=\deg(f)}} (p^d - p^{d/2}) p^{d(p^s-1)} \times \prod_{\substack{f \in \mathcal{F}_{red} \cap \mathcal{D}_{x^n+1} \\ d=\deg(f)}} (1 + p^{d/2}) p^{(2p^s-1)d/2}$$

$$\text{where } N = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. If $f(x^2) \in \mathbb{F}_p[x^2]$ is irreducible in $\mathbb{F}_p[x^2]$ with degree d, according to [17], the number of monic irreducible right factors of $f(x^2)$ is equal to $1 + p^d$, and according to Lemma 3.4 of [7], the number of irreducible monic right factors h_1 of $f(x^2)$ such that $\Theta^b(h_1^\natural)h_1 \neq f(x^2)$ is equal to $1 + p^{d/2}$. Therefore according to Lemma 6, if $f(x^2)$ belongs to $\mathcal{F}_{ir}$, the number of monic skew polynomials of degree md in $\mathcal{L}^{(b)}_{f(x^2)^m}$ is $((p^d+1)-(1+p^{d/2}))(1+p^d-1)^{m-1} = (p^d - p^{d/2})p^{d(m-1)}$.

If $f(x^2)$ belongs to $\mathcal{F}_{red}$, $\mathcal{L}^{(b)}_{f(x^2)^m} \setminus \{1, f(x^2)^m\}$ has $(1+p^{d/2})p^{(2m-1)d/2}$ elements. Namely, according to Lemma 7, the elements of $\mathcal{L}^{(b)}_{f(x^2)^m} \setminus \{1, f(x^2)^m\}$ are the skew polynomials g in bijection with the couples (g_1, g_2) satisfying (6). There are $1+p^\delta$ possibilities for $g_{1,1}$ and p^δ possibilities for each $g_{1,j}$, with $j = 2, \ldots, m$, therefore $(1+p^\delta)p^{\delta(m-1)}$ possibilities for g_1. For each j in $\{1, \ldots, m\}$ there are p^δ possibilities for $g_{2,j}$, therefore, one gets $p^{\delta m}$ possibilities for g_2.

Lastly the value of N is deduced from the cardinalities of $\mathcal{L}^{(b)}_{(x^2\pm 1)^{p^s}}$ (Proposition 5).

□

We give below some examples over $\mathbb{F}_4$, $\mathbb{F}_{25}$ and $\mathbb{F}_{49}$.

Example 19. *One has $x^6+1 = (x^2+1)(x^4+x^2+1)$ in $\mathbb{F}_2[x^2]$. For $\theta : a \mapsto a^2$, the Euclidean LCD θ-cyclic codes of length 6 over $\mathbb{F}_4 = \mathbb{F}_2(w)$ are generated by the monic skew polynomials $g = \mathrm{lclm}(g_1, g_2)$ where $g_1 \in \mathcal{L}^{(0)}_{x^2+1} = \{x+w, x+w^2, 1, x^2+1\}$ and $g_2 \in \mathcal{L}^{(0)}_{x^4+x^2+1} = \{x^2+wx+1, x^2+w^2x+1, 1, x^4+x^2+1\}$. There are four $[6,3]_4$ Euclidean LCD θ-cyclic codes. These codes are also Hermitian self-dual.*

Namely $x+w = \Theta(x+w^2) = \Theta((x+w^2)^\natural)$ and $\Theta((x+w^2)^\natural)(x+w^2) = x^2+1$ $x^2+wx+1 = \Theta(x^2+w^2x+1) = \Theta((x^2+wx+1)^\natural)$ and $\Theta((x^2+wx+1)^\natural)(x^2+wx+1) = x^4+x^2+1$ therefore the above skew polynomials g all satisfy $g = \Theta(h^\natural)$ where $\Theta(h^\natural)h = x^6-1$.

Example 20. *There are 32 Euclidean LCD θ-cyclic codes of length 12 and dimension 6 over $\mathbb{F}_4$ where $\theta : a \mapsto a^2$. They are generated by the skew polynomials $g = \mathrm{lclm}(g_1, g_2)$ where $g_1 \in \{(x+1)(x+w), (x+w)(x+w), (x+1)(x+w^2), (x+w^2)(x+w^2)\}$ and $g_2 \in \{(x^2+w)(x^2+wx+1), (x^2+w^2)(x^2+wx+1), (x^2+x+1)(x^2+wx+1), (x^2+w^2x+1)(x^2+wx+1), (x^2+w)(x^2+w^2x+1), (x^2+w^2)(x^2+w^2x+1), (x^2+x+1)(x^2+w^2x+1), (x^2+wx+1)(x^2+w^2x+1)\}$.*

Table 6 sums up the repartition of all the Euclidean LCD θ-cyclic codes of length 12 over $\mathbb{F}_4$ and gives one example for each dimension.

Example 21. *As $x^{14}+1 = (x^2+1)(x^6+x^4+1)(x^6+x^2+1)$ over $\mathbb{F}_2[x^2]$, there are $2\times(1+2^3)\times 2^3 = 144$ Euclidean LCD θ-cyclic codes of length 14 and dimension 7 over $\mathbb{F}_4$ where $\theta : a \mapsto a^2$. For example, consider $g_0 = x+w$, $g_1 = x^3+x^2+w^2x+1$ and $g_2 = x^3+w^2x^2+x+w$. g_1 is a right divisor of x^6+x^2+1 and g_2 is a right divisor of x^6+x^4+1. Furthermore $h_2 = x^3+wx^2+x+w^2$ is such that $g_2h_2 = x^6+x^4+1$ and $h_2^\natural = x^3+w^2x^2+wx+w^2 \neq g_1$ therefore $g = \mathrm{lclm}(g_0, g_1, g_2) = 1+w^2x+wx^2+w^2x^4+w^2x^5+wx^6+x^7$ generates an Euclidean LCD θ-cyclic code of length 14 over $\mathbb{F}_4$. The minimal distance of the code is equal to 6.*

Example 22. *Consider θ the Frobenius automorphism over $\mathbb{F}_{25} = \mathbb{F}_5(w)$ where $w^2+4w+2 = 0$. The skew polynomial $g = x^7+4x^6+w^4x^5+w^7x^4+w^7x^3+w^{16}x^2+w^8x+w^{20}$ generates a MDS $[14,7,8]_{25}$ θ-cyclic code over $\mathbb{F}_{25}$ which is both Euclidean LCD and Hermitian LCD. The skew polynomial g is obtained as $g = \mathrm{lclm}(g_1, g_2)$ where $g_1 = x+w^{16}$ belongs to $\mathcal{L}^{(0)}_{x^2-1} \cap \mathcal{L}^{(1)}_{x^2-1}$ and $g_2 = x^6+w^{11}x^5+2x^4+w^3x^3+2x^2+w^{11}x+1$ belongs to $\mathcal{L}^{(0)}_{x^{12}+x^{10}+x^8+x^6+x^4+x^2+1} \cap \mathcal{L}^{(1)}_{x^{12}+x^{10}+x^8+x^6+x^4+x^2+1}$.*

Example 23. *Consider θ the Frobenius automorphism over $\mathbb{F}_{49} = \mathbb{F}_7(w)$ where $w^2+6w+3 = 0$. The skew polynomial $g = x^8+6x^7+w^7x^6+w^{28}x^5+w^{11}x^4+w^{47}x^3+w^{10}x^2+w^{30}x+w^{12}$ generates a MDS $[16,8,9]_{49}$ θ-negacyclic code over $\mathbb{F}_{49}$ which is both Euclidean LCD and Hermitian LCD. The skew polynomial g is obtained as $g = \mathrm{lclm}(g_1, g_2)$ where $g_1 = x^4+w^{39}x^3+w^3x^2+w^{33}x+1$ belongs to $\mathcal{L}^{(0)}_{x^8+4x^4+1} \cap \mathcal{L}^{(1)}_{x^8+4x^4+1}$ and $g_2 = x^4+5x^3+w^{30}x^2+w^{35}x+w^6$ belongs to $\mathcal{L}^{(0)}_{x^8+3x^4+1} \cap \mathcal{L}^{(1)}_{x^8+3x^4+1}$.*

Lastly, tables 7 sums up the dimensions of MDS LCD skew codes of given length ≤ 20 over $\mathbb{F}_9$. Tables 8 and 9 sum up the dimensions of MDS LCD skew codes of length ≤ 18 over $\mathbb{F}_{25}$ and of length ≤ 16 over $\mathbb{F}_{49}$.

7 Conclusion

In this text, the notions of skew constacyclic codes, skew cyclic codes, skew negacyclic codes and skew LCD codes are studied. Equivalency between some of these classes are derived under some

conditions. Lastly LCD skew cyclic and negacyclic codes are constructed and enumerated over $\mathbb{F}_{p^2}$.

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$[n,k,d]$	d_{opt}	$g(x)$
$[2,1,2]$	2	$x+w$
$[2,1,2]$	2	$x+w^2$
$[4,2,3]$	3	x^2+x+w
$[4,2,3]$	3	$x^2+w^2x+w^2$
$[6,5,2]$	2	$x+w^2$
$[6,5,2]$	2	$x+w$
$[6,4,2]$	2	x^2+wx+1
$[6,4,2]$	2	x^2+w^2x+1
$[6,3,4]$	4	$x^3+wx^2+w^2x+1$
$[6,3,4]$	4	$x^3+w^2x^2+wx+1$
$[6,2,4]$	4	x^4+wx^3+wx+1
$[6,2,4]$	4	$x^4+w^2x^3+w^2x+1$
$[6,1,6]$	6	$x^5+w^2x^4+x^3+w^2x^2+x+w^2$
$[8,4,4]$	4	$x^4+x^3+w^2x^2+w^2x+1$
$[8,4,4]$	4	$x^4+wx^3+wx^2+x+1$
$[8,4,4]$	4	$x^4+wx^3+w^2x^2+w^2$
$[8,4,4]$	4	$x^4+w^2x^3+w^2x^2+w^2$
$[10,9,2]$	2	$x+w^2$
$[10,8,2]$	2	x^2+1
$[10,6,4]$	4	$x^4+w^2x^3+x+1$
$[10,6,4]$	4	x^4+wx^3+wx+w
$[10,5,5]$	5	$x^5+x^4+wx^3+wx^2+w$
$[10,5,5]$	5	$x^5+x^3+w^2x^2+w^2x+1$
$[10,4,6]$	6	$x^6+x^5+x^4+wx^2+x+w$
$[10,4,6]$	6	$x^6+x^5+x^4+w^2x^2+x+w^2$
$[10,2,\underline{5}]$	$\underline{8}$	$x^8+x^6+x^4+x^2+1$
$[10,1,10]$	10	$x^9+wx^8+x^7+wx^6+x^5+wx^4+x^3+wx^2+x+w$
$[10,1,10]$	10	$x^9+w^2*x^8+x^7+w^2x^6+x^5+w^2x^4+x^3+w^2x^2+x+w^2$
$[12,10,2]$	2	x^2+wx+w
$[12,8,\underline{3}]$	$\underline{4}$	$x^4+w^2x^3+w^2x^2+x+w$
$[12,8,\underline{3}]$	$\underline{4}$	$x^4+w^2x^3+w^2x^2+w^2x+1$
$[12,6,\underline{5}]$	$\underline{6}$	$x^6+w^2x^5+wx^4+wx^3+x^2+wx+w^2$
$[12,6,\underline{5}]$	$\underline{6}$	$x^6+w^2x^5+wx^4+w^2x^3+w^2x^2+x+w$
$[12,4,\underline{6}]$	$\underline{7}$	$x^8+w^2x^7+wx^6+wx^5+wx^4+w^2x^3+wx^2+wx+w^2$
$[12,4,\underline{6}]$	$\underline{7}$	$x^8+x^7+w^2x^6+x^5+x^3+w^2x^2+x+1$
$[12,2,9]$	9	$x^{10}+x^9+w^2x^8+x^6+x^5+w^2x^4+x^2+x+w^2$
$[12,2,9]$	9	$x^{10}+w^2x^9+w^2x^8+x^6+w^2x^5+w^2x^4+x^2+w^2x+w^2$
$[14,13,2]$	2	$x+w$
$[14,12,2]$	2	x^2+1
$[14,8,5]$	5	$x^6+w^2x^5+x^4+wx^3+w^2x^2+w^2x+w^2$
$[14,8,5]$	5	$x^6+wx^5+x^4+w^2x^3+wx^2+wx+w$
$[14,7,6]$	6	$x^7+w^2x^6+w^2x^5+w^2x^4+wx^3+wx^2+w^2x+w^2$
$[14,7,6]$	6	$x^7+wx^6+w^2x^5+w^2x^4+wx^2+w^2x+1$
$[14,6,7]$	7	$x^8+w^2x^7+x^6+w^2x^5+w^2x^4+w^2x^3+w^2x+w^2$
$[14,6,7]$	7	$x^8+wx^7+x^6+wx^5+w^2x^4+wx^3+wx+w^2$
$[14,2,\underline{7}]$	$\underline{11}$	$x^{12}+x^{10}+x^8+x^6+x^4+x^2+1$
$[14,1,14]$	14	$x^{13}+w^2x^{12}+x^{11}+w^2x^{10}+x^9+w^2x^8+x^7+w^2x^6+x^5+w^2x^4+x^3+w^2x^2+x+w^2$
$[14,1,14]$	14	$x^{13}+wx^{12}+x^{11}+wx^{10}+x^9+wx^8+x^7+wx^6+x^5+wx^4+x^3+wx^2+x+w$

Table 5: Skew generator polynomials of Euclidean LCD skew-cyclic codes over $\mathbb{F}_4$ of lengths $\leqslant 14$

Dimension	Number	Best distance	A skew generator polynomial
10	4	2	$g_1 = (x+1)(x+w^2)$
8	9	3	$g_2 = (x^2+ax+1)(x^2+w^2x+1)$
6	32	5	$\text{lclm}(g_1, g_2)$
4	9	6	$(x^4+1)g_2$
2	4	9	$(x^8+x^4+1)g_1$

Table 6: Euclidean LCD skew-cyclic codes over $\mathbb{F}_4$ of length 12 and dimension $\leqslant 12$

length	Euclidean LCD		Hermitian LCD	
	skew cyc	skew nega	skew cyc	skew nega
2	1	1	1	no
4	1,2,3	2	1,3	2
6	3	3	3	no
8	1,3,4,5,7	4	1,3,5,7	4
10	1,5,9	1,5,9	1,5,9	no
12	no	no	no	no
14	1,13	1, 13	1, 13	no
16	1, 15	no	1,15	no
18	no	no	no	no
20	1,19	no	1,19	no

Table 7: Dimensions of non trivial MDS LCD skew codes over $\mathbb{F}_9$ with length ≤ 20

length	Euclidean LCD		Hermitian LCD	
	skew cyc	skew nega	skew cyc	skew nega
2	1	1	1	no
4	1,2,3	2	1,3	2
6	1,2,3,4,5	1,2,3,4,5	1,2,3,4,5	2,4
8	1,3,4,5,7	4	1,3,5,7	4
10	5	5	5	no
12	1,3,5,6,7,9,11	6	1,3,5,7,9,11	6
14	1,7,13	1, 7,13	1, 7,13	no
16	1, 7,8,9,15	no	1,7,9,15	no
18	1,9,17	1,9,17	1,9,17	no

Table 8: Dimensions of non trivial MDS LCD skew codes over $\mathbb{F}_{25}$ with length ≤ 18

length	Euclidean LCD		Hermitian LCD	
	skew cyc	skew nega	skew cyc	skew nega
2	1	1	1	no
4	1,2,3	2	1,3	2
6	1,2,3,4,5	1,2,3,4,5	1,2,3,4,5	2,4
8	1,3,4,5,7	2,4,6	1,3,5,7	2,4,6
10	1,4,5,6,9	1,4,5,6,9	1,4,5,6,9	4,6
12	1,3,5,6,7,9,11	6	1,3,5,7,9,11	6
14	7	7	7	no
16	1,3,5,7,8,9,11,13,15	8	1,3,5,7,9,11,13,15	8

Table 9: Dimensions of non tirivial MDS LCD skew codes over $\mathbb{F}_{49}$ with length ≤ 16