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# Mean-field limit of a spatially-extended FitzHugh-Nagumo neural network

Joachim Crevat<sup>\*1</sup>

<sup>1</sup>Institut de Mathématiques de Toulouse ; UMR5219, Université de Toulouse ; UPS IMT, F-31062  
Toulouse Cedex 9 France

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## Abstract

Motivated by the consideration of the behavior of large assemblies of neurons, we introduce a spatially-extended model of a neural network of interacting neurons of FitzHugh-Nagumo type without noise, and we establish the mean-field limit of this system towards a nonlocal kinetic equation as the number of neurons in the network goes to infinity. We also state a well-posedness result of this kinetic equation. We work in a space of measures equipped with the Wasserstein distance of order 2 in order to use optimal transport theory. Our approach is based on deterministic methods, and on an argument of stability of solutions of the kinetic equation in their initial data. The difficulty, as compared to other mean-field models, lies in the spatially-extended aspect of this model, and in the fact that the interaction kernel is not globally Lipschitz continuous. This result rigorously provides a link between the microscopic and mesoscopic scales of observation of the network, which can be used as an intermediary step to derive macroscopic models.

*keywords* : Mean-field limit, neural network, FitzHugh-Nagumo, Wasserstein distance.

## 1 Introduction

The FitzHugh-Nagumo (FHN) model [28, 36] focuses on the evolution of the membrane electrical potential  $v$  of a nerve cell depending on the input it receives. Such variations depend on the ion exchanges between the neuron and its environment through synapses, which were precisely described by the Hodgkin-Huxley model [33]. The FHN model was then developed as a simplification of the Hodgkin-Huxley model, introducing an adaptation variable  $w$  which approximates the effect of the concentrations of ions on the membrane potential of the nerve cell. It can be written as follows:

$$\begin{cases} \frac{dv}{dt} = N(v) - w + I_{\text{ext}}, \\ \frac{dw}{dt} = \tau(v + a - bw), \end{cases} \quad t > 0, \quad (1.1)$$

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\*joachim.crevat@math.univ-toulouse.fr

where  $I_{\text{ext}}$  stands for the input current the neuron receives from its environment,  $\tau \geq 0$ ,  $a \in \mathbb{R}$  and  $b \geq 0$  are some given constants, and  $N(v)$  is a nonlinearity which models the cell excitability. A typical choice for the nonlinearity  $N$  (see for instance [3, 6, 35, 36]) is the bistable function

$$N : v \mapsto v(\alpha - \beta v^2) \quad (1.2)$$

for any  $\alpha, \beta > 0$ . Since the input  $I_{\text{ext}}$  is the result of the interactions between the nerve cell and its neighbours, we can replace it with an interaction term with other similar neurons, to introduce a model of a network of  $n$  interacting neurons, where  $n \in \mathbb{N}$ . Biological observations seem to exclude the case of homogeneous interactions [38, 47], and show that the interactions inside a network of neurons are spatially structured [10]. Hence, as in [10, 34], we choose to modulate the neural connectivity with a spatial weight. Thus, we consider the following spatially-extended FitzHugh-Nagumo network for  $n$  interacting neurons:

$$\begin{cases} \frac{d\mathbf{x}_n^i}{dt} = 0, \\ \frac{dv_n^i}{dt} = N(v_n^i) - w_n^i - \frac{1}{n} \sum_{j=1}^n \Psi(\mathbf{x}_n^i, \mathbf{x}_n^j)(v_n^i - v_n^j), & t > 0, \\ \frac{dw_n^i}{dt} = \tau(v_n^i + a - b w_n^i). \end{cases} \quad (1.3)$$

where each neuron within the network is labelled by its index  $i \in \{1, \dots, n\}$ ,  $\mathbf{x}_i \in \mathbb{R}^d$  is a parameter which stands for the position of the neuron  $i$ , and  $(v_n^i, w_n^i) \in \mathbb{R}^2$  is the couple membrane potential and adaptation variable of the neuron  $i$ . Here, the connectivity weight  $\Psi : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  models the effect of spatial dependence on the strength of neuronal interactions which we assume to be of electric type. More precisely, for every neural cell  $i$ , each other neuron  $j$  contributes in the input current received by neuron  $i$ , and we write this interaction using Ohm's law *via*  $\Psi(\mathbf{x}_n^i, \mathbf{x}_n^j)(v_n^i - v_n^j)$ , which is then summed over the network. In doing so, we assumed that conductances from the neuron  $j$  to the neuron neuron  $i$ , denoted  $\Psi(\mathbf{x}_n^i, \mathbf{x}_n^j)$ , are modulated by their relative positions. Moreover, the scaling factor  $\frac{1}{n}$  is introduced so that each neuron in the network contributes equally to the dynamics of a given neuron.

The main purpose of this paper is to rigorously derive the behaviour of the solutions to (1.3) as the number of neurons  $n$  tends to infinity. We use a similar approach as what has been done in [29] for general large particle systems. That is, instead of focusing on the solutions of (1.3), whose dimension tends to infinity as  $n$  tends to infinity, we use the following notion of empirical measure.

**Definition 1.1** (Empirical measure). *To each  $n$ -tuple  $\mathbf{X}_n = (\mathbf{x}^1, \dots, \mathbf{x}^n) \in (\mathbb{R}^d)^n$ ,  $V_n = (v^1, \dots, v^n) \in \mathbb{R}^n$  and  $W_n = (w^1, \dots, w^n) \in \mathbb{R}^n$ , one associates its empirical measure:*

$$\mu_{(\mathbf{X}_n, V_n, W_n)} := \frac{1}{n} \sum_{j=1}^n \delta_{(\mathbf{x}^j, v^j, w^j)}. \quad (1.4)$$

Let us highlight that an empirical measure is a probability measure on  $\mathbb{R}^{d+2}$ . As the number of neurons  $n$  goes to infinity, the interaction term in (1.3) is expected to satisfy formally for all  $i$ :

$$\frac{1}{n} \sum_{j=1}^n \Psi(\mathbf{x}, \mathbf{x}_n^j)(v - v_n^j) \rightarrow \int \Psi(\mathbf{x}, \mathbf{x}') (v - v') f(t, d\mathbf{x}', dv', dw'),$$

where  $f(t, d\mathbf{x}, dv, dw)$  is the probability measure of finding neurons in an elementary volume of side  $d\mathbf{x}$  with a potential membrane and an adaptation variable in an elementary interval respectively of length  $dv$  and  $dw$  at time  $t \geq 0$  within the cortex. Our purpose is thus to prove the convergence of the empirical measures associated with the solutions of (1.3) towards such a probability function  $f$  satisfying the following nonlocal kinetic equation:

$$\partial_t f + \partial_v (f [N(v) - w - \mathcal{K}[f]]) + \partial_w (f A(v, w)) = 0, \quad t > 0, \quad (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}, \quad (1.5)$$

where

$$\begin{cases} \mathcal{K}[f](t, \mathbf{x}, v) & := \int_{\mathbb{R}^{d+2}} \Psi(\mathbf{x}, \mathbf{x}') (v - v') f(t, d\mathbf{x}', dv', dw'), \\ A(v, w) & := \tau (v + a - bw). \end{cases}$$

The term  $-\partial_v (f\mathcal{K}[f])$  describes nonlocal interactions through the whole field, and  $\partial_v (f(N(v) - w)) + \partial_w (f A(v, w))$  accounts for the local reaction due to the excitability of nerve cells. This kinetic transport equation (1.5) provides a mesoscopic description of the network. This is particularly valuable if we seek for a macroscopic model, that is a system involving macroscopic quantities of the network such as the average membrane potential. Notice that if we study monokinetic solutions of (1.5), that are solutions of the form

$$f(t, \mathbf{x}, v, w) = \rho(t, \mathbf{x}) \otimes \delta_{V(t, \mathbf{x})}(v) \otimes \delta_{W(t, \mathbf{x})}(w),$$

where  $\delta_z$  stands for the Dirac distribution centered in  $z$ ,  $V$  is the average membrane potential in the network,  $W$  is the average adaptation potential, and  $\rho$  is the average density of neurons in the network, then the kinetic equation (1.5) reduces to the following nonlocal reaction-diffusion FitzHugh-Nagumo system:

$$\begin{cases} \partial_t \rho(t, \mathbf{x}) = 0, \\ \partial_t (\rho V)(t, \mathbf{x}) + \int \Psi(\mathbf{x}, \mathbf{x}') (V(t, \mathbf{x}) - V(t, \mathbf{x}')) \rho(t, \mathbf{x}) \rho(t, \mathbf{x}') d\mathbf{x}' = \rho(t, \mathbf{x}) N(V(t, \mathbf{x})) - (\rho W)(t, \mathbf{x}), \\ \partial_t (\rho W)(t, \mathbf{x}) = \tau \rho(t, \mathbf{x}) (V(t, \mathbf{x}) + a - bW(t, \mathbf{x})). \end{cases} \quad t > 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad (1.6)$$

Consequently, (1.6) describes the evolution of macroscopic quantities in the network. The rigorous link between the kinetic equation (1.5) in the regime of strong local interactions and the reaction-diffusion system (1.6) is proved in [24].

In the following, we denote by  $\mathcal{P}_2(\mathbb{R}^{d+2})$  the set of probability measures on  $\mathbb{R}^{d+2}$  with finite moments of order 2. Since this PDE can be written in a divergence form, we directly have the conservation of mass, which leads us to complement (1.5) with an initial condition:

$$f(0, \cdot) = f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2}). \quad (1.7)$$

The secondary purpose of this paper is to prove the well-posedness of the nonlocal kinetic equation (1.5). Since (1.5) is a transport equation, we chose a similar approach as what was done in [29] for general mean-field equations, which consists in proving the well-posedness of the characteristic system of (1.5) constructing an adequate Cauchy sequence.

The problem of characterizing mesoscopic and macroscopic scale in large neural network has been much investigated for other models. We have to mention the work of Amari, Wilson and Cowan in the 1970s [1, 48, 49] who heuristically derived some macroscopic models focusing on the variations of the spatial average membrane potential in the network. These models were successful in reproducing some pattern formations in the cortex, for example associated to visual hallucinations. More recent works focus on this kind of macroscopic models and their applications in neuroscience [8, 9, 23]. Then, the problem of deriving macroscopic models from microscopic models has been undertaken, for example with integrate-and-fire neurons as in [11, 13, 18]. Cai *et al.* [14] also introduced a kinetic framework for a noisy integrate-and-fire neurons network, completed in a more realistic model by Perthame and Salort [42]. In [25, 45], the authors focused on the mean-field limit of a network of neurons modeled by their firing rates, divided into a fixed number of populations, with random synaptic weights whose probability distributions only depend upon the population indexes. They also studied the influence of the noise on collective behaviors in the network. Another approach is to focus on the probability that each neuron releases an action potential. This kind of model is called time-elapsd neuron (see for instance [19–21, 39–41]). Here, in this article, we use another approach, with the derivation of a kinetic equation from the FHN model, which does not abstract the emission of action potentials, following the example of the Hodgkin-Huxley model.

The mean-field limit of the stochastic spatially-extended FHN model for interacting neurons has already been studied in the specific case of a compactly supported connectivity kernel in [34], and in the case when each cell interacts only with its  $p$  nearest neighbours in [34, 37]. Other approaches are considered in the literature to adjust the interactions with other quantities than space. For example, in [3, 6], the authors focused on the mean-field limit of a stochastic FHN model or Hodgkin-Huxley model of  $n$  neurons interacting through electro-chemical synapses towards a kinetic PDE similar to (1.5), and in [7], the authors proved the synchronization of a similar stochastic mean-field model of a neural network of Hodgkin-Huxley type. Similarly, in [35], the authors proved the mean-field limit of a stochastic FHN model of a network of neurons interacting with a constant conductance, towards a kinetic model, and in [43], the authors showed the synchronization of a similar stochastic mean-field model of a neural network of FHN type. The specificity of our present work is that we only consider spatially-weighted neural networks without noise.

The main difficulty here is to get a stability result of the characteristic flow associated to the transport equation (1.5). Indeed, in our framework, the interaction kernel  $(\mathbf{x}, \mathbf{y}, v) \mapsto \Psi(\mathbf{x}, \mathbf{y}) v$  is not globally Lipschitz continuous, unlike in [29] for example. In [16], the authors tackled a similar problem for some kinetic models of collective motion, and they chose to work with compactly supported solutions specifically to overcome this problem. Here, similarly as in [5] for stochastic individual-based models, we deal with this issue with the estimate of exponential moments of the solution of the kinetic equation (1.5). The most convenient distance to describe the convergence of an empirical measure in such problems is the Wasserstein distance [5, 16, 29, 30]. Here, we work with the Wasserstein distance of order 2, defined as follows:

**Definition 1.2** (Wasserstein distance). *Let  $\mu$  and  $\nu$  be in  $\mathcal{P}_2(\mathbb{R}^{d+2})$ . The Wasserstein distance of order 2 between  $\mu$  and  $\nu$  is defined by*

$$d_2(\mu, \nu) = \inf_{\pi \in \Lambda(\mu, \nu)} \left\{ \left( \iint_{\mathbb{R}^{d+2} \times \mathbb{R}^{d+2}} |(\mathbf{x} - \mathbf{x}', v - v', w - w')|^2 \pi(d\mathbf{x}, dv, dw, d\mathbf{x}', dv', dw') \right)^{\frac{1}{2}} \right\},$$

where  $\Lambda(\mu, \nu)$  is the set of couplings of  $\mu$  and  $\nu$ , that is to say for all  $\pi \in \Lambda(\mu, \nu)$ , for all function  $\phi \in \mathcal{C}(\mathbb{R}^{d+2})$  such that  $\phi(\mathbf{z}) = \mathcal{O}(|\mathbf{z}|^2)$  as  $|\mathbf{z}|$  goes to infinity,

$$\iint \phi(\mathbf{x}, v, w) \pi(d\mathbf{x}, dv, dw, d\mathbf{x}', dv', dw') = \int \phi(\mathbf{x}, v, w) \mu(d\mathbf{x}, dv, dw),$$

and

$$\iint \phi(\mathbf{x}', v', w') \pi(d\mathbf{x}, dv, dw, d\mathbf{x}', dv', dw') = \int \phi(\mathbf{x}', v', w') \nu(d\mathbf{x}', dv', dw').$$

We choose to work with the Wasserstein distance of order 2 instead of 1 to deal with the nonlinearity  $N$ , which naturally makes appear some moments of order 2 in  $v$  and  $w$  in the computation of the stability estimate.

## 2 Main results

In this section, we present our main result on the mean-field limit from the solution of the microscopic model (1.3) towards the measure solution of the kinetic equation (1.5) as the number of neurons  $n$  goes to infinity. Before establishing this result, we need to prove the well-posedness of these models.

First of all, we present the hypotheses we make for the study of the microscopic model (1.3) and the kinetic model (1.5). To prove our mean-field result, we choose a connectivity kernel  $\Psi : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  which satisfies

$$\Psi \in \text{Lip}_b(\mathbb{R}^{2d}), \quad (2.1)$$

where  $\text{Lip}_b(\mathbb{R}^{2d})$  is the set of bounded and globally Lipschitz continuous functions from  $\mathbb{R}^{2d}$  to  $\mathbb{R}$ . In the following, we note  $L > 0$  the Lipschitz constant of the connectivity kernel  $\Psi$ , and

$$\|\Psi\|_\infty := \sup_{(\mathbf{x}, \mathbf{y})} |\Psi(\mathbf{x}, \mathbf{y})|.$$

We also choose a locally Lipschitz continuous nonlinearity  $N : \mathbb{R} \rightarrow \mathbb{R}$  such that there exist two constants  $\kappa, \tilde{\kappa} > 0$  satisfying:

$$\begin{cases} v N(v) \leq \kappa |v|^2 \quad \forall v \in \mathbb{R}, \\ (v - u)(N(v) - N(u)) \leq \tilde{\kappa} |v - u|^2 \quad \forall (v, u) \in \mathbb{R}^2. \end{cases} \quad (2.2)$$

In order to define the measure solutions of (1.5), we first have to study the system of characteristic equations. Consider  $T > 0$  and an initial data  $f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2})$  for the kinetic equation (1.5). Let us consider the system of characteristic equations for all  $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$  associated to (1.5):

$$\begin{cases} \frac{\partial}{\partial t} \mathcal{V}(t, \mathbf{z}) = N(\mathcal{V}(t, \mathbf{z})) - \mathcal{W}(t, \mathbf{z}) - \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(t, \mathbf{z}')) f_0(d\mathbf{x}', dv', dw'), \\ \frac{\partial}{\partial t} \mathcal{W}(t, \mathbf{z}) = A(\mathcal{V}(t, \mathbf{z}), \mathcal{W}(t, \mathbf{z})), \\ \mathcal{W}|_{t=0} = w, \mathcal{V}|_{t=0} = v. \end{cases} \quad t > 0, \quad (2.3)$$

We note  $(\mathcal{V}_{f_0}, \mathcal{W}_{f_0})$  the solution of the characteristic system (2.3), and we define for all  $t \in [0, T]$ ,  $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ :

$$\mathcal{Z}_{f_0}(t, \mathbf{z}) := (\mathbf{x}, \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z})) \quad , \quad \mathcal{Z}_{f_0}(0, \mathbf{z}) = \mathbf{z}. \quad (2.4)$$

**Remark 2.1.** In the rest of this paper, for the sake of clarity, we will use the notation  $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$  and  $\mathbf{z}' = (\mathbf{x}', v', w') \in \mathbb{R}^{d+2}$ .

Now, we have all the tools we need to define the notion of measure solution of the kinetic equation (1.5).

**Definition 2.2** (Measure solution of (1.5)). Consider a connectivity kernel  $\Psi$  satisfying (2.1), a locally Lipschitz continuous nonlinearity  $N$  satisfying (2.2) and an initial data  $f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2})$ . Let  $T > 0$ . Then,  $f$  is said to be a measure solution of (1.5) with initial data  $f_0$  if  $f \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$  and for all  $t \in [0, T]$ ,

$$f(t) = \mathcal{Z}_{f_0}(t) \# f_0,$$

where  $\#$  is our notation for the push-forward<sup>1</sup>.

**Remark 2.3.** A measure solution of (1.5) as defined previously is also a solution in the sense of distribution of (1.5).

Let  $n \in \mathbb{N}$ . First of all, we establish the well-posedness of the system of  $2n$  ODEs (1.3), and its link with the characteristic system (2.3).

**Proposition 2.4** (Well-posedness of (1.3)). Let  $n \in \mathbb{N}$ . We choose a connectivity kernel  $\Psi$  satisfying (2.1), and a locally Lipschitz continuous nonlinearity  $N$  satisfying (2.2). Consider the parameters  $\mathbf{X}_n := (\mathbf{x}_n^i)_{1 \leq i \leq n} \in (\mathbb{R}^d)^n$  and the initial data  $(V_{0,n}, W_{0,n}) = (v_{0,n}^i, w_{0,n}^i)_{1 \leq i \leq n} \in (\mathbb{R}^2)^n$ . We note  $Z_0 := (\mathbf{X}_n, V_{0,n}, W_{0,n}) \in (\mathbb{R}^{d+2})^n$ .

- (i) For all  $T > 0$ , there exists a unique solution  $(V_n(\cdot, Z_0), W_n(\cdot, Z_0)) := (v_i, w_i)_{1 \leq i \leq n} \in \mathcal{C}^1([0, T], (\mathbb{R}^2)^n)$  of (1.3) with initial data  $(V_{0,n}, W_{0,n})$ .
- (ii) We define  $f_{0,n}$  the empirical measure  $\mu(\mathbf{x}_n, V_n(t, Z_0), W_n(t, Z_0))$  defined as in (1.4). Let  $T > 0$ . Assume that there exists a solution  $(\mathcal{V}_{f_{0,n}}, \mathcal{W}_{f_{0,n}}) \in \mathcal{C}([0, T] \times \mathbb{R}^{d+2})$  of (2.3) with the initial distribution  $f_{0,n}$  noted. Then for all  $1 \leq i \leq n$  and all  $t \in [0, T]$ ,

$$(\mathcal{V}_{f_{0,n}}(t, \mathbf{x}_n^i, v_{0,n}^i, w_{0,n}^i), \mathcal{W}_{f_{0,n}}(t, \mathbf{x}_n^i, v_n^i, w_n^i)) = (v_n^i(t), w_n^i(t)).$$

**Proof.**

- (i) The Cauchy-Lipschitz Theorem yields the local existence and uniqueness of a solution of class  $\mathcal{C}^1$  of (1.3) with initial conditions  $(v_{0,n}^i, w_{0,n}^i)_{1 \leq i \leq n}$  noted:

$$\begin{pmatrix} V_n(\cdot, Z_0) \\ W_n(\cdot, Z_0) \end{pmatrix} : t \mapsto \begin{pmatrix} V_n(t, Z_0) \\ W_n(t, Z_0) \end{pmatrix} := \begin{pmatrix} (v_n^i(t))_{1 \leq i \leq n} \\ (w_n^i(t))_{1 \leq i \leq n} \end{pmatrix}.$$

---

<sup>1</sup>For all  $h$ , map from  $\mathbb{R}^{d+2}$  to itself, for all probability measure  $\mu$ , the notation  $\nu = h\#\mu$  is equivalent to

$$\int 1_B(z) \nu(dz) = \int 1_B(h(y)) \mu(dy)$$

for all  $B$  measurable subset of  $\mathbb{R}^{d+2}$ .

Let  $t \in [0, T]$  be small enough so that  $V_n(t, Z_0)$  and  $W_n(t, Z_0)$  exist. We get global solutions from the following energy estimate:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|(V_n(t, Z_0), W_n(t, Z_0))\|^2 \\
&= \sum_{i=1}^n v_n^i (N(v_n^i) - w_n^i) + \tau \sum_{i=1}^n w_n^i (v_n^i + a - b w_n^i) - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \Psi(\mathbf{x}_n^i, \mathbf{x}_n^j) v_n^i (v_n^i - v_n^j) \\
&\leq \left( \frac{1}{2} + \kappa + \tau \right) \|(V_n(t, Z_0), W_n(t, Z_0))\|^2 + \frac{\tau a^2}{2} + \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n |\Psi(\mathbf{x}_n^i, \mathbf{x}_n^j)| \left( \frac{3}{2} |v_n^i|^2 + \frac{1}{2} |v_n^j|^2 \right) \\
&\leq \left( \frac{1}{2} + \kappa + \tau + 2 \|\Psi\|_\infty \right) \|(V_n(t, Z_0), W_n(t, Z_0))\|^2 + \frac{\tau a^2}{2}.
\end{aligned}$$

(ii) Let  $1 \leq i \leq n$ . Thus,  $(\mathcal{V}_{f_{0,n}}(\cdot, \mathbf{x}_n^i, v_{0,n}^i, w_{0,n}^i), \mathcal{W}_{f_{0,n}}(\cdot, \mathbf{x}_n^i, v_n^i, w_n^i))$  is solution of the following system

$$\begin{cases} \frac{d}{dt} \mathcal{V}_{f_{0,n}}(t, \mathbf{x}_n^i, v_{0,n}^i, w_{0,n}^i) &= N(\mathcal{V}_{f_{0,n}}) - \mathcal{W}_{f_{0,n}} - \mathcal{K}[f_{0,n}](t, \mathbf{x}_n^i, \mathcal{V}_{f_{0,n}}(t, \mathbf{x}_n^i, v_{0,n}^i, w_{0,n}^i)) \\ \frac{d}{dt} \mathcal{W}_{f_{0,n}}(t, \mathbf{x}_n^i, v_{0,n}^i, w_{0,n}^i) &= A(\mathcal{V}_{f_{0,n}}, \mathcal{W}_{f_{0,n}}), \end{cases} \quad (2.5)$$

which is exactly (1.3) according to the definition of  $f_{0,n}$ . We conclude with the uniqueness of the solution of (1.3). ■

Then, we focus on the existence of a measure solution to the kinetic equation (1.5). We have the following result.

**Proposition 2.5** (Well-posedness of (1.5)). *We choose a connectivity kernel  $\Psi$  satisfying (2.1), and a locally Lipschitz continuous nonlinearity  $N$  satisfying (2.2). Consider an initial data  $f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2})$ . Then, for all  $T > 0$ , there exists a unique measure solution  $f \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$  of (1.5) with initial data  $f_0$ .*

The proof is postponed to Section 4, and it relies on the well-posedness of the characteristic system (2.3) using an adequate Cauchy sequence, following ideas from [29]. Let us mention that in [16], the authors proved an analogous result for a kinetic equation derived from the Cucker-Smale model using a fixed point method. The main difference between our work and [16] is the presence of the drift term  $\partial_w(f(A(v, w)))$  from the adaption variable.

**Remark 2.6.** *We can extend the result of Proposition 2.5 and the definition 2.2 of measure solutions to the case where  $\Psi$  is only bounded and continuous with respect to its first variable uniformly relative to its second variable.*

Now, we can state our main result on the mean-field limit from the system of ODEs (1.3) towards the kinetic equation (1.5).



**Theorem 1** (Mean-field limit). *We choose a connectivity kernel  $\Psi$  satisfying (2.1), and a locally Lipschitz continuous nonlinearity  $N$  satisfying (2.2). Let  $T > 0$ . For all  $n \in \mathbb{N}$ , consider the initial data  $(\mathbf{x}_n^i, v_{0,n}^i, w_{0,n}^i)_{1 \leq i \leq n} \in (\mathbb{R}^{d+2})^n$  and their associated empirical measure*

$$f_{0,n} := \frac{1}{n} \sum_{j=1}^n \delta_{(\mathbf{x}_n^j, v_{0,n}^j, w_{0,n}^j)}. \quad (2.6)$$

Consider an initial data  $f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2})$  satisfying

$$\int e^{\alpha_0(1+|v|^2+|w|^2)^{\frac{1}{2}}} f_0(d\mathbf{x}, dv, dw) < \infty. \quad (2.7)$$

Assume that the initial data satisfies

$$\lim_{n \rightarrow +\infty} d_2(f_{0,n}, f_0) = 0. \quad (2.8)$$

Consider for all  $n \in \mathbb{N}$  the solution  $(\mathbf{x}_n^i, v_n^i, w_n^i)_{1 \leq i \leq n} \in (\mathbb{R}^{d+2})^n$  of the FitzHugh-Nagumo system (1.3) with initial data  $(\mathbf{x}_n^i, v_{0,n}^i, w_{0,n}^i)_{1 \leq i \leq n}$ , and their associated empirical measures for all  $t \in [0; T]$

$$f_n(t) := \frac{1}{n} \sum_{j=1}^n \delta_{(\mathbf{x}_n^j, v_n^j(t), w_n^j(t))}. \quad (2.9)$$

Also consider  $f \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$  the solution to (1.5) with initial data  $f_0$  provided by Proposition 2.5. Then, there exist two constants  $C_T > 0$  and  $K_T > 0$  such that for all  $t \in [0; T]$  and all  $n \in \mathbb{N}$ ,

$$d_2(f_n(t), f(t)) \leq K_T d_2(f_{0,n}, f_0) e^{-C_T t}.$$

Therefore, we have:

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, T]} d_2(f_n(t), f(t)) = 0. \quad (2.10)$$

**Sketch of the proof of Theorem 1 :** Our approach is based on a stability estimate of the solutions of the kinetic equation (1.5), similarly as [5, 16, 29]. This stability estimate will be proved in Section 5. Nevertheless, in [16], the authors worked with compactly supported solutions of kinetic models for collective motions. Our approach is closer to what was done in [5] for general mean-field equations with suitable assumptions on the interaction kernel. Indeed, we overcome the issues arising from the fact that the solutions of (1.5) are not compactly supported in general with an estimate of the exponential moments of the solutions of (1.5), detailed in Section 3. More precisely, the difficulties come from the fact that the solutions of the characteristic system (2.3) are not bounded in general.

**Outline.** The rest of this paper is organized as follows. In Section 3, we prove an *a priori* estimate of exponential moments of the solution of the kinetic equation (1.5) which will be crucial for the proof of Theorem 1. Then, in Section 4, we present the proof of Proposition 2.5 on the existence and uniqueness of the measure solution of the kinetic equation (1.5). Furthermore, we prove our main theorem of mean-field limit in Section 5 and present as an application a stability result regarding monokinetic solutions in the following Section 6. Finally, in Section 7, we provide a numerical scheme for the kinetic equation (1.5) and we display some numerical simulations.

### 3 *A priori* estimate

In this section, we prove an *a priori* estimate of the exponential moments of a solution of the kinetic equation (1.5) that will be useful for the proof of Theorem 1. We will need to work with the characteristic system (2.3). In Section 4, we will prove that for all  $T > 0$ , the characteristics are well-defined in  $\mathcal{C}([0, T], \mathcal{E})$ , where

$$\mathcal{E} := \left\{ \mathcal{U} \in \mathcal{C}(\mathbb{R}^{d+2}, \mathbb{R}) \mid \|\mathcal{U}\|_{\mathcal{E}} < \infty \right\}, \quad (3.1)$$

with

$$\|\mathcal{U}\|_{\mathcal{E}} := \sup_{(\mathbf{x}, v, w) \in \mathbb{R}^{d+2}} \frac{|\mathcal{U}(\mathbf{x}, v, w)|}{1 + |v| + |w|}. \quad (3.2)$$

This choice of Banach space is justified since in general, the characteristics are not bounded, so we need to control some moments in  $v$  and  $w$ . In the following, for all  $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ , we will use the notation:

$$\gamma(\mathbf{z}) := 1 + |v| + |w|. \quad (3.3)$$

**Lemma 3.1.** *We choose a connectivity kernel  $\Psi$  satisfying (2.1) and a locally Lipschitz continuous non-linearity  $N$  satisfying (2.2). For all  $(v, w) \in \mathbb{R}^2$ , we note*

$$\langle v, w \rangle := (1 + |v|^2 + |w|^2)^{\frac{1}{2}} \geq 1.$$

Let  $p \geq 1$ . Consider an initial data  $f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2})$  such that

$$\int e^{\alpha_0 \langle v, w \rangle^p} f_0(d\mathbf{x}, dv, dw) < \infty, \quad (3.4)$$

where  $\alpha_0 > 0$  is a positive constant. Let  $T > 0$ . Assume that there exists  $f \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$  a measure solution of (1.5) and a couple  $(\mathcal{V}_{f_0}, \mathcal{W}_{f_0}) \in \mathcal{C}([0, T], \mathcal{E})^2$  solution of the characteristic system (2.3), such that if we define  $\mathcal{Z}_{f_0} := (id_{\mathbb{R}^d}, \mathcal{V}_{f_0}, \mathcal{W}_{f_0})$ , then for all  $t \in [0, T]$ ,

$$f(t) = \mathcal{Z}_{f_0}(t) \# f_0.$$

Then, there exists a constant  $C_{f_0}^T > 0$  which depends only on the parameters of the equation (1.5), on  $T$  and on the moments of  $f_0$ , and  $\alpha_T := \alpha_0 e^{-p C_{f_0}^T T}$  such that for all  $t \in [0, T]$ ,

$$\int e^{\alpha_T \langle v, w \rangle^p} f(t, d\mathbf{x}, dv, dw) \leq \int e^{\alpha_0 \langle v, w \rangle^p} f_0(d\mathbf{x}, dv, dw). \quad (3.5)$$

**Proof.** Consider  $\alpha \in \mathcal{C}^1([0, T], \mathbb{R})$  a positive function to be determined later. Let  $t \in [0, T]$ . We have the following estimate:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int e^{\alpha(t) \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p} f_0(d\mathbf{z}) &\leq \int \left[ \frac{1}{2} \alpha'(t) \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p \right. \\ &\quad \left. + \frac{p}{2} \alpha(t) \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^{p-2} (I_1 + I_2 + I_3) \right] e^{\alpha(t) \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p} f_0(d\mathbf{z}), \end{aligned}$$

where

$$\begin{cases} I_1 := \tau \mathcal{W}_{f_0}(t, \mathbf{z}) (\mathcal{V}_{f_0}(t, \mathbf{z}) + a - b \mathcal{W}_{f_0}(t, \mathbf{z})), \\ I_2 := \mathcal{V}_{f_0}(t, \mathbf{z}) (N(\mathcal{V}_{f_0}(t, \mathbf{z})) - \mathcal{W}_{f_0}(t, \mathbf{z})), \\ I_3 := -\mathcal{V}_{f_0}(t, \mathbf{z}) \int \Psi(\mathbf{x}, \mathbf{x}') [\mathcal{V}_{f_0}(t, \mathbf{z}) - \mathcal{V}_{f_0}(t, \mathbf{z}')] f_0(d\mathbf{z}'). \end{cases}$$

First of all, we can easily compute the first term  $I_1$  with Young's inequality, which yields:

$$\begin{aligned} I_1 &\leq \frac{\tau}{2} |\mathcal{V}_{f_0}(t, \mathbf{z})|^2 + \tau |\mathcal{W}_{f_0}(t, \mathbf{z})|^2 + \frac{\tau a^2}{2} - \tau b |\mathcal{W}_{f_0}(t, \mathbf{z})|^2 \\ &\leq \tau \left(1 + \frac{a^2}{2}\right) |\langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle|^2. \end{aligned}$$

Furthermore, we deal with the second term  $I_2$  using the assumption on  $N$  (2.2) and Young's inequality:

$$\begin{aligned} I_2 &\leq \kappa |\mathcal{V}_{f_0}(t, \mathbf{z})|^2 + \frac{1}{2} (|\mathcal{V}_{f_0}(t, \mathbf{z})|^2 + |\mathcal{W}_{f_0}(t, \mathbf{z})|^2) \\ &\leq \left(\kappa + \frac{1}{2}\right) |\langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle|^2. \end{aligned}$$

Finally, we treat the third term  $I_3$  using Young's inequality and then factorizing by  $|\langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle|^2$  as follows:

$$\begin{aligned} I_3 &\leq \frac{3}{2} |\mathcal{V}_{f_0}(t, \mathbf{z})|^2 \int |\Psi(\mathbf{x}, \mathbf{x}')| f_0(d\mathbf{z}') + \frac{1}{2} \int |\Psi(\mathbf{x}, \mathbf{x}')| |\mathcal{V}_{f_0}(t, \mathbf{z}')|^2 f_0(d\mathbf{z}') \\ &\leq \frac{1}{2} \|\Psi\|_\infty \left(3 |\mathcal{V}_{f_0}(t, \mathbf{z})|^2 + \int |\mathcal{V}_{f_0}(t, \mathbf{z}')|^2 f_0(d\mathbf{z}')\right) \\ &\leq \frac{1}{2} \|\Psi\|_\infty \left(3 + \sup_{s \in [0, T]} \|\mathcal{V}_{f_0}(s)\|_{\mathcal{E}}^2 \int |\gamma(\mathbf{z}')|^2 f_0(d\mathbf{z}')\right) |\langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle|^2. \end{aligned}$$

Finally, we get that there exists a positive constant  $C_{f_0}^T > 0$  such that:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int e^{\alpha(t) \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p} f_0(d\mathbf{z}) \\ &\leq \int \left[ \frac{1}{2} \alpha'(t) + \frac{p}{2} C_{f_0}^T \alpha(t) \right] \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p e^{\alpha(t) \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p} f_0(d\mathbf{z}). \end{aligned}$$

We choose for all  $s \in [0, T]$ ,  $\alpha(s) := \alpha_0 e^{-p C_{f_0}^T s}$ , so that for all  $s \in [0, T]$ ,

$$\alpha'(s) + p C_{f_0}^T \alpha(s) = 0, \quad \alpha(0) = \alpha_0.$$

Hence,

$$\frac{d}{dt} \int e^{\alpha(t) \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p} f_0(d\mathbf{z}) \leq 0.$$

Finally, we define:

$$\alpha_T := \alpha(T),$$

and we deduce that for all  $t \in [0, T]$ ,

$$\begin{aligned} \int e^{\alpha T} \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p f_0(d\mathbf{z}) &\leq \int e^{\alpha(t)} \langle \mathcal{V}_{f_0}(t, \mathbf{z}), \mathcal{W}_{f_0}(t, \mathbf{z}) \rangle^p f_0(d\mathbf{z}) \\ &\leq \int e^{\alpha_0 \langle v, w \rangle^p} f_0(d\mathbf{z}). \end{aligned}$$

■

## 4 Proof of Proposition 2.5

This section is devoted to the proof of the well-posedness of the equation (1.5). First, we focus on the well-posedness of the characteristic system (2.3), and then we will conclude by defining the measure solution to (1.5) as the push-forward of the initial distribution by the solution of the characteristic system. Let  $T > 0$  be fixed. To prove the Proposition 2.5, we start by proving that the characteristic system (2.3) is well-defined. Then, we define the solution of (1.5) as the push-forward of the initial data with the flow of the characteristic equation.

### 4.1 Proof of the well-posedness of the characteristic system (2.3)

First, we establish the existence and uniqueness of the solution of the characteristic system (2.3) in  $\mathcal{C}([0, T], \mathcal{E})$ , where  $\mathcal{E}$  is defined with (3.1)-(3.2), with an approach based on the construction of a Cauchy sequence.

**Proposition 4.1.** *Under the assumptions of Proposition 2.5, there exists a unique couple  $(\mathcal{V}_{f_0}, \mathcal{W}_{f_0})$  solution of (2.3) on  $[0, T]$  such that*

$$\mathcal{V}_{f_0}, \mathcal{W}_{f_0} \in \mathcal{C}([0, T], \mathcal{E}).$$

**Proof.** We want to construct a Cauchy sequence in  $\mathcal{E}$  with functions  $(\mathcal{V}_p, \mathcal{W}_p)_{p \in \mathbb{N}}$  initialized with the couple  $(\mathcal{V}_0, \mathcal{W}_0) := (0, 0)$  and satisfying for all  $p \in \mathbb{N}$ :

$$\left\{ \begin{array}{l} \frac{d}{dt} \mathcal{V}_{p+1}(t, \mathbf{z}) = N(\mathcal{V}_{p+1}(t, \mathbf{z})) - \mathcal{W}_{p+1}(t, \mathbf{z}) - \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{V}_p(t, \mathbf{z}) - \mathcal{V}_p(t, \mathbf{z}')) f_0(d\mathbf{z}'), \\ \hspace{15em} \mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}, \quad t > 0, \\ \frac{d}{dt} \mathcal{W}_{p+1}(t, \mathbf{z}) = A(\mathcal{V}_{p+1}(t, \mathbf{z}), \mathcal{W}_{p+1}(t, \mathbf{z})), \\ \mathcal{V}_{p+1}(0, \mathbf{z}) = v, \quad \mathcal{W}_{p+1}(0, \mathbf{z}) = w. \end{array} \right. \quad (4.1)$$

Then, we will define the couple  $(\mathcal{V}_{f_0}, \mathcal{W}_{f_0})$  as the its limit as  $p$  tends to infinity. Taking the limit  $p \rightarrow +\infty$  in the system (4.1), we will show that  $(\mathcal{V}_{f_0}, \mathcal{W}_{f_0})$  is the unique solution of the characteristic system (2.3).

#### Step 1: construction of the sequences

The existence and uniqueness of the sequences  $\{\mathcal{V}_p\}_{p \in \mathbb{N}}$  and  $\{\mathcal{W}_p\}_{p \in \mathbb{N}}$  is a consequence of the following lemma.

**Lemma 4.2.** *Let  $T > 0$ . Under the assumptions of Proposition 4.1, assume that there exists  $\mathcal{U} \in \mathcal{C}([0, T], \mathcal{E})$  of class  $\mathcal{C}^1$  in time. Then, there exists unique couple  $(\mathcal{V}, \mathcal{W}) \in \mathcal{C}([0, T], \mathcal{E})^2$  solution of class  $\mathcal{C}^1$  in time of the system*

$$\begin{cases} \frac{d}{dt} \mathcal{V}(t, \mathbf{z}) = N(\mathcal{V}(t, \mathbf{z})) - \mathcal{W}(t, \mathbf{z}) - \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{U}(t, \mathbf{z}) - \mathcal{U}(t, \mathbf{z}')) f_0(d\mathbf{z}'), \\ \frac{d}{dt} \mathcal{W}(t, \mathbf{z}) = A(\mathcal{V}(t, \mathbf{z}), \mathcal{W}(t, \mathbf{z})), \\ \mathcal{V}(0, \mathbf{z}) = v, \quad \mathcal{W}(0, \mathbf{z}) = w, \end{cases} \quad \mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}, \quad t > 0, \quad (4.2)$$

The proof of Lemma 4.2 is postponed to the Appendix A. Then, by induction, Lemma 4.2 implies the existence and uniqueness of  $(\mathcal{V}_p, \mathcal{W}_p)$  for all  $p \in \mathbb{N}$ .

### Step 2: Cauchy sequences

Now, we want to prove that for all  $t \in [0, T]$ ,  $\{\mathcal{V}_p(t)\}_{p \in \mathbb{N}}$  and  $\{\mathcal{W}_p(t)\}_{p \in \mathbb{N}}$  are two Cauchy sequences in  $\mathcal{E}$ . For all  $p \in \mathbb{N}$ , we define for all  $\mathbf{z} \in \mathbb{R}^{d+2}$ :

$$G_{p+1}(t) := \left( \|\mathcal{V}_{p+1}(t) - \mathcal{V}_p(t)\|_{\mathcal{E}}^2 + \|\mathcal{W}_{p+1}(t) - \mathcal{W}_p(t)\|_{\mathcal{E}}^2 \right)^{\frac{1}{2}},$$

and we want to prove by induction that this quantity is summable. Let  $p \in \mathbb{N}$ ,  $t \in [0, T]$  and  $\mathbf{z} = (\mathbf{x}, v, w)$ . Thus, we have:

$$\frac{1}{2} \left( |\mathcal{V}_{p+2}(t, \mathbf{z}) - \mathcal{V}_{p+1}(t, \mathbf{z})|^2 + |\mathcal{W}_{p+2}(t, \mathbf{z}) - \mathcal{W}_{p+1}(t, \mathbf{z})|^2 \right) \leq \int [\mathcal{T}_1(s) + \mathcal{T}_2(s) + \mathcal{T}_3(s) + \mathcal{T}_4(s)] ds,$$

where for all  $s \in [0, t]$ ,

$$\begin{cases} \mathcal{T}_1(s) & := \tau (\mathcal{W}_{p+2}(s, \mathbf{z}) - \mathcal{W}_{p+1}(s, \mathbf{z})) [(\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})) - b (\mathcal{W}_{p+2}(s, \mathbf{z}) - \mathcal{W}_{p+1}(s, \mathbf{z}))], \\ \mathcal{T}_2(s) & := (\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})) [N(\mathcal{V}_{p+2}(s, \mathbf{z})) - N(\mathcal{V}_{p+1}(s, \mathbf{z}))], \\ \mathcal{T}_3(s) & := - (\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})) (\mathcal{W}_{p+2}(s, \mathbf{z}) - \mathcal{W}_{p+1}(s, \mathbf{z})), \\ \mathcal{T}_4(s) & := - (\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})) \\ & \quad \times \int \Psi(\mathbf{x}, \mathbf{x}') [(\mathcal{V}_{p+1}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z}')) - (\mathcal{V}_p(s, \mathbf{z}) - \mathcal{V}_p(s, \mathbf{z}'))] f_0(d\mathbf{z}'). \end{cases}$$

Let  $s \in [0, t]$ . The first term  $\mathcal{T}_1(s)$  is easily controled using Young's inequality:

$$\begin{aligned} \mathcal{T}_1(s) & \leq \frac{\tau}{2} \left( |\mathcal{V}_{p+2}(t, \mathbf{z}) - \mathcal{V}_{p+1}(t, \mathbf{z})|^2 + |\mathcal{V}_{p+2}(t, \mathbf{z}) - \mathcal{V}_{p+1}(t, \mathbf{z})|^2 \right) \\ & \leq \frac{\tau}{2} |\gamma(\mathbf{z})|^2 \int_0^t |G_{p+2}(s)|^2 ds \end{aligned} \quad (4.3)$$

Then, we treat the second term  $\mathcal{T}_2(s)$  using assumption (2.2) satisfied by  $N$  :

$$\mathcal{T}_2(s) \leq \tilde{\kappa} |\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})|^2 \leq \tilde{\kappa} |\gamma(\mathbf{z})|^2 |G_{p+2}(s)|^2. \quad (4.4)$$

Furthermore, we can estimate the third term  $\mathcal{T}_3(s)$  with Young's inequality:

$$\begin{aligned}\mathcal{T}_3(s) &\leq \frac{1}{2} \left( |\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})|^2 + |\mathcal{W}_{p+2}(s, \mathbf{z}) - \mathcal{W}_{p+1}(s, \mathbf{z})|^2 \right) \\ &\leq \frac{1}{2} |\gamma(\mathbf{z})|^2 |G_{p+2}(s)|^2.\end{aligned}\tag{4.5}$$

Finally, to deal with the nonlocal term  $\mathcal{T}_4(s)$ , we start by defining the constant

$$K := \int |\gamma(\mathbf{z}')|^2 f_0(d\mathbf{z}').$$

Then, using the assumption (2.1) and Young's inequality, we can compute:

$$\begin{aligned}\mathcal{T}_4(s) &= (\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})) \\ &\quad \times \int \Psi(\mathbf{x}, \mathbf{x}') [(\mathcal{V}_{p+1}(s, \mathbf{z}') - \mathcal{V}_p(s, \mathbf{z}')) - (\mathcal{V}_{p+1}(s, \mathbf{z}) - \mathcal{V}_p(s, \mathbf{z}))] f_0(d\mathbf{z}') \\ &\leq \frac{1}{2} |\mathcal{V}_{p+2}(s, \mathbf{z}) - \mathcal{V}_{p+1}(s, \mathbf{z})|^2 \\ &\quad + \frac{1}{2} \|\Psi\|_\infty^2 \int \left( |\mathcal{V}_{p+1}(s, \mathbf{z}') - \mathcal{V}_p(s, \mathbf{z}')|^2 + |\mathcal{V}_{p+1}(s, \mathbf{z}) - \mathcal{V}_p(s, \mathbf{z})|^2 \right) f_0(d\mathbf{z}')\end{aligned}$$

Moreover, factorizing each term with  $|\gamma(\mathbf{z})|^2$ , we get:

$$\begin{aligned}\mathcal{T}_4(s) &\leq \frac{1}{2} |\gamma(\mathbf{z})|^2 \|\mathcal{V}_{p+2}(s) - \mathcal{V}_{p+1}(s)\|_{\mathcal{E}}^2 + \frac{1}{2} \|\Psi\|_\infty^2 \left( \int |\gamma(\mathbf{z}')|^2 f_0(d\mathbf{z}') + |\gamma(\mathbf{z})|^2 \right) \|\mathcal{V}_{p+1}(s) - \mathcal{V}_p(s)\|_{\mathcal{E}}^2 \\ &\leq \frac{1}{2} |\gamma(\mathbf{z})|^2 \left( |G_{p+2}(s)|^2 + \|\Psi\|_\infty^2 (K + 1) |G_{p+1}(s)|^2 \right).\end{aligned}\tag{4.6}$$

Finally, (4.3), (4.4), (4.5) and (4.6) together yield that there exist two positive constants  $C_1 > 0$  and  $C_2 > 0$  independent of  $p$  such that:

$$\begin{aligned}|\mathcal{V}_{p+2}(t, \mathbf{z}) - \mathcal{V}_{p+1}(t, \mathbf{z})|^2 + |\mathcal{W}_{p+2}(t, \mathbf{z}) - \mathcal{W}_{p+1}(t, \mathbf{z})|^2 \\ \leq |\gamma(\mathbf{z})|^2 \left[ C_1 \int_0^t |G_{p+1}(s)|^2 ds + C_2 \int_0^t |G_{p+2}(s)|^2 ds \right],\end{aligned}$$

which implies, by dividing this inequality by  $|\gamma(\mathbf{z})|^2$  and then taking the supremum on  $\mathbb{R}^{d+2}$ :

$$|G_{p+2}(t)|^2 \leq C_1 \int_0^t |G_{p+1}(s)|^2 ds + C_2 \int_0^t |G_{p+2}(s)|^2 ds.\tag{4.7}$$

On the one hand, using Grönwall's lemma, we have for all  $t \in [0, T]$ :

$$|G_{p+2}(t)|^2 \leq C_1 e^{C_2 T} \int_0^t |G_{p+1}(s)|^2 ds.\tag{4.8}$$

On the other hand, for all  $t \in [0, T]$ ,

$$|G_1(t)|^2 = \|\mathcal{V}_1(t)\|_{\mathcal{E}}^2 + \|\mathcal{W}_1(t)\|_{\mathcal{E}}^2 \leq C_T,\tag{4.9}$$

for some constant  $C_T > 0$ , since  $\mathcal{V}_1$  and  $\mathcal{W}_1 \in \mathcal{C}([0, T], \mathcal{E})$  according to Lemma 4.2. Hence, by induction, we can deduce from (4.8) and (4.9) that for all  $p \in \mathbb{N}$  and all  $t \in [0, T]$ :

$$|G_{p+1}(t)|^2 \leq C_T \frac{(C_1 e^{C_2 T} t)^p}{p!},\tag{4.10}$$

which is summable. Consequently, for all  $t \in [0, T]$ ,  $\{\mathcal{V}_p(t)\}_{p \in \mathbb{N}}$  and  $\{\mathcal{W}_p(t)\}_{p \in \mathbb{N}}$  are Cauchy sequences in  $\mathcal{E}$ . Since  $\mathcal{E}$  is a Banach space, and since for all  $p \in \mathbb{N}$ ,  $\mathcal{V}_p$  and  $\mathcal{W}_p \in \mathcal{C}([0, T], \mathcal{E})$ , there exist  $\mathcal{V}_{f_0}$  and  $\mathcal{W}_{f_0} \in \mathcal{C}([0, T], \mathcal{E})$  such that for all  $t \in [0, T]$ ,  $\mathcal{V}_p(t, \cdot)$  (respectively  $\mathcal{W}_p$ ) converges towards  $\mathcal{V}_{f_0}(t, \cdot)$  (respectively  $\mathcal{W}_{f_0}$ ) uniformly in  $\mathcal{E}$ . Thus, passing to the limit  $p \rightarrow +\infty$  in

$$\begin{cases} \mathcal{V}_{p+1}(t, \mathbf{z}) = v + \int_0^t \left[ N(\mathcal{V}_{p+1}(s, \mathbf{z})) - \mathcal{W}_{p+1}(s, \mathbf{z}) - \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{V}_p(s, \mathbf{z}) - \mathcal{V}_p(s, \mathbf{z}')) f_0(d\mathbf{z}') \right] ds, \\ \mathcal{W}_{p+1}(s, \mathbf{z}) = w + \int_0^s A(\mathcal{V}_{p+1}(s, \mathbf{z}), \mathcal{W}_{p+1}(s, \mathbf{z})) ds, \end{cases} \quad \mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}, \quad t \in (0, T], \quad (4.11)$$

we get that for all  $\mathbf{z} \in \mathbb{R}^{d+2}$ ,  $(\mathcal{V}_{f_0}(\cdot, \mathbf{z}), \mathcal{W}_{f_0}(\cdot, \mathbf{z}))$  is a solution of (2.3) of class  $\mathcal{C}^1$  in time.

### Step 3: Uniqueness

Now, we want to check that the solution of (2.3) is unique. Suppose that there exist  $\mathcal{Z}_1 := (\text{id}_{\mathbb{R}^d}, \mathcal{V}_1, \mathcal{W}_1)$  and  $\mathcal{Z}_2 := (\text{id}_{\mathbb{R}^d}, \mathcal{V}_2, \mathcal{W}_2)$  such that  $(\mathcal{V}_1, \mathcal{W}_1)$  and  $(\mathcal{V}_2, \mathcal{W}_2)$  are two solutions of (2.3) in  $\mathcal{C}([0, T], \mathcal{E})^2$ . We define for all  $t \in [0, T]$ :

$$G(t) := \left( \|\mathcal{V}_1(t) - \mathcal{V}_2(t)\|_{\mathcal{E}}^2 + \|\mathcal{W}_1(t) - \mathcal{W}_2(t)\|_{\mathcal{E}}^2 \right)^{\frac{1}{2}}. \quad (4.12)$$

Thus, using similar computations as previously, we get that there exists a positive constant  $C$  such that for all  $t \in [0, T]$ :

$$|G(t)|^2 \leq C \int_0^t |G(s)|^2 ds.$$

Using Grönwall's inequality we get that  $\mathcal{Z}_1 = \mathcal{Z}_2$ . ■

## 4.2 Conclusion

We have proved so far that there exists a unique map  $\mathcal{Z}_{f_0} := (\text{id}_{\mathbb{R}^d}, \mathcal{V}_{f_0}, \mathcal{W}_{f_0})$  such that  $(\mathcal{V}_{f_0}, \mathcal{W}_{f_0})$  is a solution of the characteristic system (2.3) in  $\mathcal{C}([0, T], \mathcal{E})$ . We define:

$$f : t \mapsto \mathcal{Z}_{f_0}(t) \# f_0. \quad (4.13)$$

Thus, for all  $t \in [0, T]$ :

$$\begin{aligned} \int (|\mathbf{x}|^2 + |v|^2 + |w|^2) f(t, d\mathbf{z}) &= \int (|\mathbf{x}|^2 + |\mathcal{V}_{f_0}(t, \mathbf{z})|^2 + |\mathcal{W}_{f_0}(t, \mathbf{z})|^2) f_0(d\mathbf{z}) \\ &\leq \int |\mathbf{x}|^2 f_0(d\mathbf{z}) + (\|\mathcal{V}_{f_0}(t)\|_{\mathcal{E}}^2 + \|\mathcal{W}_{f_0}(t)\|_{\mathcal{E}}^2) \int |\gamma(\mathbf{z})|^2 f_0(d\mathbf{z}), \end{aligned}$$

which is uniformly bounded. Hence, for all  $t \in [0, T]$ :

$$f(t) \in \mathcal{P}_2(\mathbb{R}^{d+2}).$$

Then, we want to prove that  $f \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$ , where  $\mathcal{P}_2(\mathbb{R}^{d+2})$  is equipped with the Wasserstein distance  $d_2$ . Let  $t$  and  $t' \in [0, T]$ . Notice that the measure  $(\mathcal{Z}_{f_0}(t) \times \mathcal{Z}_{f_0}(t')) \# f_0 \in \Lambda(f(t), f(t'))$ . Therefore,

we have:

$$\begin{aligned}
d_2^2(f(t), f(t')) &= \inf_{\pi \in \Lambda(f(t), f(t'))} \iint |\mathbf{z}_1 - \mathbf{z}_2|^2 \pi(d\mathbf{z}_1 d\mathbf{z}_2) \\
&\leq \int |\mathcal{Z}_{f_0}(t, \mathbf{z}) - \mathcal{Z}_{f_0}(t', \mathbf{z})|^2 f_0(d\mathbf{z}) \\
&\leq (\|\mathcal{V}_{f_0}(t) - \mathcal{V}_{f_0}(t')\|_{\mathcal{E}}^2 + \|\mathcal{W}_{f_0}(t) - \mathcal{W}_{f_0}(t')\|_{\mathcal{E}}^2) \int |\gamma(\mathbf{z})|^2 f_0(d\mathbf{z}).
\end{aligned}$$

Since  $\mathcal{V}_{f_0}$  and  $\mathcal{W}_{f_0} \in \mathcal{C}([0, T], \mathcal{E})$ , and  $f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2})$ , we have that:

$$f \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2})).$$

Hence,  $f$  is a measure solution of (1.5) in the sense of definition 2.2.

## 5 A stability result and proof of Theorem 1

This section is devoted to the proof of the following stability result of measure solutions of the kinetic equation (1.5) with respect to their initial condition which is crucial to the proof of Theorem 1.

**Proposition 5.1.** *We choose a connectivity kernel  $\Psi$  satisfying (2.1), and a locally Lipschitz continuous nonlinearity  $N$  satisfying (2.2). Let  $T > 0$ . Consider two initial conditions  $f_{0,1}$  and  $f_{0,2} \in \mathcal{P}_2(\mathbb{R}^{d+2})$  such that  $d_2(f_{0,1}, f_{0,2}) \leq 1$ . Assume that there exists  $\alpha_0 > 0$  such that*

$$\int e^{\alpha_0 \langle v, w \rangle} f_{0,2}(d\mathbf{z}) < \infty.$$

Let  $f_1$  and  $f_2 \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$  be the solutions of the kinetic equation (1.5) respectively with initial conditions  $f_{0,1}$  and  $f_{0,2}$  provided by Proposition 2.5. Then there exist two positive constants  $C_T > 0$  and  $K_T > 0$  such that for all  $t \in [0, T]$ ,

$$d_2(f_1(t), f_2(t)) \leq K_T d_2(f_{0,1}, f_{0,2})^{\beta(t)}, \quad (5.1)$$

where

$$\beta(t) := e^{-C_T t}. \quad (5.2)$$

Let us postpone the proof of this proposition and conclude the proof of Theorem 1 first. We notice that for all  $n \in \mathbb{N}$ , the function  $f_n$  defined with (2.9) is the measure solution of the kinetic equation (1.5) with initial condition  $f_{0,n}$ , the empirical measure defined in (2.6). Then Proposition 5.1 yields that there exist two positive constants  $C_T$  and  $K_T$  independent of  $n$  such that for all  $t \in [0, T]$  and for all  $n \in \mathbb{N}$ ,

$$d_2(f_n(t), f(t)) \leq K_T d_2(f_{0,n}, f_0)^{\beta(t)},$$

where

$$\beta(t) := e^{-C_T t}.$$

Finally, using the assumption (2.8), we get that

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, T]} d_2(f_n(t), f(t)) = 0,$$

which concludes the proof of Theorem 1.



**Proof.** [of Proposition 5.1.]

For  $j \in \{1, 2\}$ , we define the map  $\mathcal{Z}_j$  such that for all  $t \in [0, T]$  and all  $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ :

$$\mathcal{Z}_j(t, \mathbf{z}) := (\mathbf{x}, \mathcal{V}_j(t, \mathbf{z}), \mathcal{W}_j(t, \mathbf{z})),$$

where  $(\mathcal{V}_j, \mathcal{W}_j)$  is the solution of the characteristic system (2.3) with initial distribution  $f_{0,j}$ .  $\mathcal{V}_j$  and  $\mathcal{W}_j$  are in  $\mathcal{C}([0, T], \mathcal{E})$  according to Proposition 4.1. Let  $\pi \in \Lambda(f_{0,1}, f_{0,2})$  be the optimal measure to compute  $d_2(f_{0,1}, f_{0,2})$ , that is

$$d_2(f_{0,1}, f_{0,2}) = \left( \iint \|\mathbf{z}_1 - \mathbf{z}_2\|^2 \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \right)^{\frac{1}{2}}.$$

The existence of such a minimizing measure is proved in [46]. First, we want to estimate the function  $D[\pi]$  defined for all  $t \in [0, T]$  with

$$D[\pi](t) := \left( \iint \|\mathcal{Z}_1(t, \mathbf{z}'_1) - \mathcal{Z}_2(t, \mathbf{z}'_2)\|^2 \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \right)^{\frac{1}{2}}.$$

Then, we will be able to conclude using the fact that for all  $t \in [0, T]$ ,

$$d_2(f_1(t), f_2(t)) \leq D[\pi](t).$$

### Step 1: Estimate of $D[\pi]$

Let  $t \in [0, T]$ ,  $\mathbf{z}_1 = (\mathbf{x}_1, v_1, w_1)$  and  $\mathbf{z}_2 = (\mathbf{x}_2, v_2, w_2) \in \mathbb{R}^{d+2}$ . We start by estimating  $\|\mathcal{Z}_1(t, \mathbf{z}_1) - \mathcal{Z}_2(t, \mathbf{z}_2)\|^2$ . Then, we will integrate with respect to the measure  $\pi(d\mathbf{z}_1, d\mathbf{z}_2)$  in order to estimate  $D[\pi]^2(t)$ . In the following, we use the shorthand notations  $\mathcal{V}_j := \mathcal{V}_j(t, \mathbf{z}_j)$  and  $\mathcal{V}'_j := \mathcal{V}_j(t, \mathbf{z}'_j)$ , and the same for  $\mathcal{W}_j$ . First, we have:

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{Z}_1(t, \mathbf{z}_1) - \mathcal{Z}_2(t, \mathbf{z}_2)\|^2 \leq \mathcal{T}_{l,1} + \mathcal{T}_{l,2} + \mathcal{T}_{nl}, \quad (5.3)$$

where

$$\left\{ \begin{array}{l} \mathcal{T}_{l,1} := \tau (\mathcal{W}_1 - \mathcal{W}_2) [(\mathcal{V}_1 - \mathcal{V}_2) - b(\mathcal{W}_1 - \mathcal{W}_2)], \\ \mathcal{T}_{l,2} := (\mathcal{V}_1 - \mathcal{V}_2) [(N(\mathcal{V}_1) - N(\mathcal{V}_2)) - (\mathcal{W}_1 - \mathcal{W}_2)], \\ \mathcal{T}_{nl} := (\mathcal{V}_1 - \mathcal{V}_2) \left[ \int \Psi(\mathbf{x}_1, \mathbf{x}'_1) (\mathcal{V}_1 - \mathcal{V}'_1) f_{0,1}(d\mathbf{z}'_1) - \int \Psi(\mathbf{x}_2, \mathbf{x}'_2) (\mathcal{V}_2 - \mathcal{V}'_2) f_{0,2}(d\mathbf{z}'_2) \right]. \end{array} \right.$$

By integrating (5.3) with respect to the measure  $\pi(d\mathbf{z}_1, d\mathbf{z}_2)$ , we get:

$$\frac{1}{2} \frac{d}{dt} D[\pi](t)^2 \leq \iint [\mathcal{T}_{l,1} + \mathcal{T}_{l,2} + \mathcal{T}_{nl}] \pi(d\mathbf{z}_1, d\mathbf{z}_2). \quad (5.4)$$

We easily treat the first local term  $\mathcal{T}_{l,1}$  using Young's inequality:

$$\begin{aligned} \iint \mathcal{T}_{l,1} \pi(d\mathbf{z}_1, d\mathbf{z}_2) &\leq \frac{\tau}{2} \iint (|\mathcal{V}_1 - \mathcal{V}_2|^2 + |\mathcal{W}_1 - \mathcal{W}_2|^2) \pi(d\mathbf{z}_1, d\mathbf{z}_2) - \tau b \iint |\mathcal{W}_1 - \mathcal{W}_2|^2 \pi(d\mathbf{z}_1, d\mathbf{z}_2) \\ &\leq \frac{\tau}{2} D[\pi](t)^2. \end{aligned} \quad (5.5)$$

Then, we can estimate the second local term  $\mathcal{T}_{l,2}$  using Young's inequality and the assumption on  $N$  (2.2) as follows:

$$\begin{aligned}
\iint \mathcal{T}_{l,2} \pi(d\mathbf{z}_1, d\mathbf{z}_2) &\leq \iint \left[ \tilde{\kappa} |\mathcal{V}_1 - \mathcal{V}_2|^2 + \frac{1}{2} |\mathcal{V}_1 - \mathcal{V}_2|^2 + \frac{1}{2} |\mathcal{W}_1 - \mathcal{W}_2|^2 \right] \pi(d\mathbf{z}_1, d\mathbf{z}_2) \\
&\leq \left( \frac{1}{2} + \tilde{\kappa} \right) \iint \|\mathcal{Z}_1(t, \mathbf{z}_1) - \mathcal{Z}_2(t, \mathbf{z}_2)\|^2 \pi(d\mathbf{z}_1, d\mathbf{z}_2) \\
&\leq \left( \frac{1}{2} + \tilde{\kappa} \right) D[\pi](t)^2.
\end{aligned} \tag{5.6}$$

Now, we deal with the nonlocal terms  $\mathcal{T}_{nl}$ . Since we have:

$$\begin{aligned}
\Psi(\mathbf{x}_1, \mathbf{x}'_1) (\mathcal{V}_1 - \mathcal{V}'_1) + \Psi(\mathbf{x}_2, \mathbf{x}'_2) (\mathcal{V}_2 - \mathcal{V}'_2) \\
= \Psi(\mathbf{x}_1, \mathbf{x}'_1) [(\mathcal{V}_1 - \mathcal{V}_2) - (\mathcal{V}'_1 - \mathcal{V}'_2)] + (\Psi(\mathbf{x}_1, \mathbf{x}'_1) - \Psi(\mathbf{x}_2, \mathbf{x}'_2)) (\mathcal{V}_2 - \mathcal{V}'_2),
\end{aligned}$$

we get:

$$\iint \mathcal{T}_{nl} \pi(d\mathbf{z}_1, d\mathbf{z}_2) \leq J_1 + J_2 + J_3,$$

where

$$\begin{cases} J_1 := \iint \iint |\Psi(\mathbf{x}_1, \mathbf{x}'_1)| |\mathcal{V}_1 - \mathcal{V}_2|^2 \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2), \\ J_2 := \iint \iint |\Psi(\mathbf{x}_1, \mathbf{x}'_1)| |\mathcal{V}_1 - \mathcal{V}_2| |\mathcal{V}'_1 - \mathcal{V}'_2| \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2), \\ J_3 := \iint \iint |\Psi(\mathbf{x}_1, \mathbf{x}'_1) - \Psi(\mathbf{x}_2, \mathbf{x}'_2)| |\mathcal{V}_1 - \mathcal{V}_2| |\mathcal{V}_2 - \mathcal{V}'_2| \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2). \end{cases}$$

The two first terms  $J_1$  and  $J_2$  are easy to estimate, using the fact that  $\Psi$  is bounded and Young's inequality. We find:

$$\begin{cases} J_1 \leq \|\Psi\|_\infty D[\pi](t)^2, \\ J_2 \leq \|\Psi\|_\infty D[\pi](t)^2. \end{cases}$$

Furthermore, since  $\Psi$  satisfies the assumption (2.1), we have for all  $(\mathbf{x}_1, \mathbf{x}'_1)$  and  $(\mathbf{x}_2, \mathbf{x}'_2) \in \mathbb{R}^{2d}$ :

$$|\Psi(\mathbf{x}_1, \mathbf{x}'_1) - \Psi(\mathbf{x}_2, \mathbf{x}'_2)| \leq \min \{ L (|\mathbf{x}_1 - \mathbf{x}_2| + |\mathbf{x}'_1 - \mathbf{x}'_2|), 2\|\Psi\|_\infty \}. \tag{5.7}$$

Let  $R(t)$  to be determined later. We define the sets

$$\begin{cases} \varepsilon_{R(t)} := \{ \mathbf{z} \in \mathbb{R}^{d+2}, |V_2(t, \mathbf{z})| \leq R(t) \}, \\ \Theta_{R(t)} := (\mathbb{R}^{d+2} \times \varepsilon_{R(t)})^2. \end{cases} \tag{5.8}$$

Then, we can deal with the third term  $J_3$  using (5.7) and then splitting the set of integration into  $\Theta_{R(t)}$  and its complementary  $\Theta_{R(t)}^C$ :

$$\begin{aligned}
J_3 &\leq \iint \iint \min \{ L (|\mathbf{x}_1 - \mathbf{x}_2| + |\mathbf{x}'_1 - \mathbf{x}'_2|), 2\|\Psi\|_\infty \} |\mathcal{V}_1 - \mathcal{V}_2| |\mathcal{V}_2 - \mathcal{V}'_2| \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2) \\
&\leq J_{31} + J_{32},
\end{aligned}$$

where

$$\begin{cases} J_{31} := L \iint \iint_{\Theta_{R(t)}} |\mathcal{V}_1 - \mathcal{V}_2| (|\mathbf{x}_1 - \mathbf{x}_2| + |\mathbf{x}'_1 - \mathbf{x}'_2|) |\mathcal{V}_2 - \mathcal{V}'_2| \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2), \\ J_{32} := 2 \|\Psi\|_\infty \iint \iint_{\Theta_{R(t)}^C} |\mathcal{V}_1 - \mathcal{V}_2| |\mathcal{V}_2 - \mathcal{V}'_2| \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2). \end{cases}$$

We can treat the term  $J_{31}$  with the Cauchy-Schwarz inequality:

$$\begin{aligned} J_{31} &\leq 2LR(t) \iint \iint_{\Theta_{R(t)}} |\mathcal{V}_1 - \mathcal{V}_2| (|\mathbf{x}_1 - \mathbf{x}_2| + |\mathbf{x}'_1 - \mathbf{x}'_2|) \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2) \\ &\leq 2LR(t) D[\pi](t)^2 + 2LR(t) \left( \iint |\mathcal{V}_1 - \mathcal{V}_2|^2 \pi(d\mathbf{z}_1, d\mathbf{z}_2) \right)^{\frac{1}{2}} \left( \iint |\mathbf{x}'_1 - \mathbf{x}'_2|^2 \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \right)^{\frac{1}{2}} \\ &\leq 4LR(t) D[\pi](t)^2. \end{aligned}$$

Then, using the Young and Cauchy-Schwarz inequalities, we can estimate the second term  $J_{32}(s)$  as follows:

$$\begin{aligned} J_{32}(s) &\leq \|\Psi\|_\infty D[\pi](t)^2 + \|\Psi\|_\infty \iint \iint_{\Theta_{R(t)}^C} |\mathcal{V}_2 - \mathcal{V}'_2|^2 \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2) \\ &\leq \|\Psi\|_\infty D[\pi](t)^2 \\ &\quad + \|\Psi\|_\infty \left( \iint \iint |\mathcal{V}_2 - \mathcal{V}'_2|^4 \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2) \right)^{\frac{1}{2}} \left( \iint \iint_{\Theta_{R(t)}^C} \pi(d\mathbf{z}'_1, d\mathbf{z}'_2) \pi(d\mathbf{z}_1, d\mathbf{z}_2) \right)^{\frac{1}{2}} \\ &\leq \|\Psi\|_\infty D[\pi](t)^2 + C \left( \int |\mathcal{V}_2|^4 f_{0,2}(d\mathbf{z}_2) \right)^{\frac{1}{2}} \left( \int_{\varepsilon_{R(t)}^C} f_{0,2}(d\mathbf{z}_2) \right)^{\frac{1}{2}}. \end{aligned}$$

On the one hand, according to Lemma 3.1, there exists a constant  $\alpha_T > 0$  such that we have:

$$\int_{\varepsilon_{R(t)}^C} f_{0,2}(d\mathbf{z}_2) \leq \int_{\varepsilon_{R(t)}^C} \frac{e^{\alpha_T |\mathcal{V}_2|}}{e^{\alpha_T R(t)}} f_{0,2}(d\mathbf{z}_2) \leq e^{-\alpha_T R(t)} \int e^{\alpha_0 \langle v_2, w_2 \rangle} f_{0,2}(d\mathbf{z}_2).$$

On the other hand, there exists a constant  $C > 0$  such that

$$\int |\mathcal{V}_2|^4 f_{0,2}(d\mathbf{z}_2) \leq C \int \left(1 + e^{\alpha_T |\mathcal{V}_2|}\right) f_{0,2}(d\mathbf{z}_2) < \infty,$$

according to Lemma 3.1. Finally, there exists a constant  $C > 0$  such that

$$J_{32} \leq C(1 + R(t)) D[\pi](t)^2 + C e^{-\frac{\alpha_T}{2} R(t)}.$$

and consequently, there exists a constant  $\tilde{C} > 0$  such that

$$\iint \mathcal{T}_{nl} \pi(d\mathbf{z}_1 d\mathbf{z}_2) \leq \tilde{C}(1 + R(t)) D[\pi](t)^2 + \tilde{C} e^{-\frac{\alpha_T}{2} R(t)}. \quad (5.9)$$

Now, we use (5.5), (5.6) and (5.9) to estimate respectively the local terms  $\mathcal{T}_{l,1}$ ,  $\mathcal{T}_{l,2}$ , and the nonlocal terms  $\mathcal{T}_{nl}$  in (5.4). Finally, we get that there exists a constant  $C_T > 0$  such that for all  $t \in [0, T]$  and all  $R(t) > 0$ :

$$\frac{d}{dt} D[\pi](t)^2 \leq C_T (1 + R(t)) D[\pi](t)^2 + C_T e^{-\frac{\alpha_T}{2} R(t)}. \quad (5.10)$$

First, if we choose  $R(t) = 1$  for all  $t \in [0, T]$  in the inequality (5.10), Grönwall's lemma yields that there exists a constant  $K_T > 0$  such that for all  $t \in [0, T]$ ,  $D[\pi](t)^2 < K_T$ . Then, we define the function

$$u : t \mapsto \frac{D[\pi](t)^2}{e K_T},$$

so that for all  $t \in [0, T]$  such that  $u(t) > 0$ , we have  $1 \leq -\ln(u(t))$ . Let  $t \in [0, T]$  such that  $u(t) > 0$ . Then, we choose the quantity  $R(t)$  in (5.10) as follows:

$$R(t) := -\frac{2}{\alpha_T} \ln(u(t)).$$

Hence, (5.10) becomes

$$\begin{aligned} u'(t) &\leq C_T \left(1 - \frac{2}{\alpha_T} \ln(u(t))\right) u(t) + \frac{C_T}{e K_T} u(t) \\ &\leq -C_T \left(1 + \frac{2}{\alpha_T}\right) u(t) \ln(u(t)) - \frac{C_T}{e K_T} u(t) \ln(u(t)) \\ &\leq -\tilde{C}_T u(t) \ln(u(t)), \end{aligned}$$

for some constant  $\tilde{C}_T > 0$ . With the convention  $v \ln(v) = 0$  if  $v = 0$ , according to Osgood's lemma, we have for all  $t \in [0, T]$ :

$$\int_{u(0)}^{u(t)} \frac{-dy}{\tilde{C}_T y \ln(y)} \leq t.$$

This implies that

$$\ln(-\ln(u(t))) - \ln(-\ln(u(0))) \geq -\tilde{C}_T t.$$

Consequently, since  $\ln(u(0)) < 0$ , if we define the function

$$\beta : t \mapsto e^{-\tilde{C}_T t}, \quad (5.11)$$

we get that for all  $t \in [0, T]$ :

$$u(t) \leq u(0)^{\beta(t)},$$

and this remains true if  $u(t) = 0$ . Finally, we can conclude that for all  $t \in [0, T]$

$$\begin{aligned} D[\pi](t) &\leq (e K_T)^{\frac{1-\beta(t)}{2}} D[\pi](0)^{\beta(t)} \\ &\leq \tilde{K}_T D[\pi](0)^{\beta(t)}, \end{aligned} \quad (5.12)$$

where  $\tilde{K}_T > 0$  is a positive constant.

## Step 2: Conclusion

We note for all  $j \in \{1, 2\}$  and all  $t \in [0, T]$

$$f_j(t) := \mathcal{Z}_j(t) \# f_{0,j}.$$

Thus, for all  $t \in [0, T]$  and for all  $\pi \in \Lambda(f_{0,1}, f_{0,2})$ ,

$$(\mathcal{Z}_1(t) \times \mathcal{Z}_2(t)) \# \pi \in \Lambda(f_1(t), f_2(t)),$$

and therefore,

$$d_2(f_1(t), f_2(t)) \leq \left( \iint \|\mathcal{Z}_1(t, \mathbf{z}_1) - \mathcal{Z}_2(t, \mathbf{z}_2)\|^2 \pi(d\mathbf{z}_1 d\mathbf{z}_2) \right)^{\frac{1}{2}} = D[\pi](t).$$

Finally, we can conclude using the estimate (5):

$$\begin{aligned} d_2(f_1(t), f_2(t)) &\leq \tilde{K}_T D[\pi](0)^{\beta(t)} \\ &= \tilde{K}_T d_2(f_{0,1}, f_{0,2})^{\beta(t)}. \end{aligned}$$

■

**Remark 5.2.** *If we make the additional assumption that there exists  $p > 1$  such that*

$$\int e^{\langle v, w \rangle^p} f_{0,2}(d\mathbf{z}) < \infty,$$

*then instead of the estimate (5.9), we can conclude using a similar argument that there exists a constant  $C_T > 0$  such that for all  $t \in [0, T]$ ,*

$$\frac{d}{dt} D[\pi](t)^2 \leq C_T (1 + R) D[\pi](t)^2 + C_T e^{-\frac{\alpha T}{2} R^p}.$$

*Hence, Grönwall's lemma yields that for all  $R > 0$  and all  $t \in [0, T]$ ,*

$$D[\pi](t)^2 \leq \left( D[\pi](0)^2 + \frac{e^{-\frac{\alpha T}{2} R^p}}{1 + R} \right) e^{C_T (1+R)t}.$$

*Therefore, this implies that for all  $t \in [0, T]$  and all  $R > 0$ ,*

$$d_2^2(f_1(t), f_2(t)) \leq D[\pi](t)^2 \leq \left( d_2^2(f_{0,1}, f_{0,2}) + \frac{e^{-\frac{\alpha T}{2} R^p}}{1 + R} \right) e^{C_T (1+R)t}. \quad (5.13)$$

*Choosing for instance*

$$R = -\frac{1}{2C_T T} \ln(d_2^2(f_{0,1}, f_{0,2})),$$

*the estimate (5.13) is enough to prove Theorem 1. Indeed, since  $(f_{0,n})_{n \in \mathbb{N}}$  and  $f_0$  satisfy assumption (2.8), we get*

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, T]} d_2(f_n(t), f(t)) = 0.$$

## 6 Application: stability of monokinetic solutions

As explained in the introduction, the analysis of the mean-field equation can be seen as an intermediary step for the derivation of macroscopic models, which describe the dynamics of the average electrical activity in large assemblies of neurons. A way to study macroscopic quantities is to look for monokinetic solutions of (1.5), that is solutions  $f$  of the form

$$f(t, \mathbf{x}, v, w) = \rho(t, \mathbf{x}) \otimes \delta_{V(t, \mathbf{x})}(v) \otimes \delta_{W(t, \mathbf{x})}(w), \quad (6.1)$$

where  $\rho$  is the average density of neurons, and  $(V, W)$  is the average couple membrane potential-adaptation variable in the network. Then, we formally get that the triplet  $(\rho, \rho V, \rho W)$  satisfies the nonlocal reaction-diffusion FitzHugh-Nagumo system (1.6). We are all the more interested in monokinetic solutions of the kinetic model (1.5), as this is the kind of solutions we are going to numerically display in the next section. In this subsection, we consider a connectivity kernel  $\Psi$  of the form:

$$\Psi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}).$$

Note that if  $\rho_0$  is a positive function, the system (1.6) reduces to the following nonlocal reaction-diffusion system:

$$\begin{cases} \partial_t \rho(t, \mathbf{x}) = 0, \\ \partial_t V(t, \mathbf{x}) - \int \Phi(\mathbf{x} - \mathbf{x}') (V(t, \mathbf{x}) - V(t, \mathbf{x}')) \rho_0(\mathbf{x}') d\mathbf{x}' = N(V(t, \mathbf{x})) - W(t, \mathbf{x}), \\ \partial_t W(t, \mathbf{x}) = \tau (V(t, \mathbf{x}) + a - b W(t, \mathbf{x})). \end{cases} \quad t > 0, \quad \mathbf{x} \in \mathbb{R}^d \quad (6.2)$$

We want to use the previous stability result provided by Proposition 5.1 to prove that monokinetic solutions of the kinetic equation (1.5) are stable in the Wasserstein distance  $d_2$ . Under some additional assumptions, we can state the following well-posedness result.

**Lemma 6.1.** *We choose a non-negative symmetric connectivity kernel  $\Phi \in L^1(\mathbb{R}^d)$ , and a nonlinearity*

$$N : v \mapsto v - v^3.$$

*Consider an initial data  $(\rho_0, V_0, W_0)$  satisfying*

$$\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d), \quad \rho_0 \geq 0, \quad V_0, W_0 \in L^\infty(\mathbb{R}^d). \quad (6.3)$$

*Then for any  $T > 0$ , there exists a unique couple  $(V, W)$  which is a classical solution to the nonlocal reaction-diffusion system (6.2) with initial data  $(V_0, W_0)$ , where*

$$V, W \in \mathcal{C}^1([0, T], L^\infty(\mathbb{R}^d)).$$

The proof of Lemma 6.1 is based on a classical fixed point argument, and we refer to [24] for the details. Now, as a direct consequence of Proposition 5.1, we get the following result of stability of monokinetic solution of the equation (1.5).

**Proposition 6.2** (Stability of monokinetic solutions). *Let  $T > 0$ . We choose a non-negative symmetric connectivity kernel  $\Phi \in L^1(\mathbb{R}^d)$  satisfying (2.1), and the nonlinearity*

$$N : v \mapsto v - v^3.$$

*Consider the initial data  $f_0 \in \mathcal{P}_2(\mathbb{R}^{d+2})$ , and  $(\rho_0, V_0, W_0)$  satisfying (6.3) and*

$$\int_{\mathbb{R}^d} \rho_0(\mathbf{x}) \, d\mathbf{x} = 1, \quad \int_{\mathbb{R}^d} \left( |\mathbf{x}|^2 + e^{\alpha_0 \langle V_0(\mathbf{x}), W_0(\mathbf{x}) \rangle} \right) \rho_0(\mathbf{x}) \, d\mathbf{x} < \infty, \quad (6.4)$$

*for some positive constant  $\alpha_0 > 0$ . Let  $(V, W)$  be the solution of (6.2) with initial data  $(V_0, W_0)$  provided by Lemma 6.1, and let  $f \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$  be the measure solution of (1.5) with initial data  $f_0$  provided by Proposition 2.5. Then there exist two positive constants  $C_T > 0$  and  $K_T > 0$  such that for all  $t \in [0, T]$ ,*

$$d_2(f(t), \rho(t, \cdot) \otimes \delta_{V(t, \cdot)} \otimes \delta_{W(t, \cdot)}) \leq K_T d_2(f_0, \rho_0 \otimes \delta_{V_0} \otimes \delta_{W_0})^{\beta(t)}, \quad (6.5)$$

*where  $\beta(t) := e^{-C_T t}$ .*

**Proof.** According to the assumption (6.4), we have:

$$\int \left( |\mathbf{x}|^2 + e^{\alpha_0 \langle v, w \rangle} \right) \rho_0(\mathbf{x}) \delta_{V_0}(v) \delta_{W_0}(w) \, d\mathbf{x} \, dv \, dw < \infty,$$

so  $\rho_0 \otimes \delta_{V_0} \otimes \delta_{W_0} \in \mathcal{P}_2(\mathbb{R}^{d+2})$ . Moreover,  $\rho_0 \otimes \delta_V \otimes \delta_W \in \mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^{d+2}))$  is a measure solution of (1.5) since  $(\rho_0, \rho_0 V, \rho_0 W)$  is a classical solution of (1.6). Thus, we can apply Proposition 5.1, which yields (6.5). ■

## 7 Numerical simulations

In this section, we approximate the solution  $f$  of the kinetic equation (1.5) for a one-dimensional network (*i.e.*  $d = 1$ ), normalized to  $[0, 1]$ . There are few numerical methods specifically adapted to kinetic theory. In [2], the authors numerically approximate a mean-field model of neural network of FitzHugh-Nagumo type using finite differences without considering any space dependence. On the contrary, we are particularly interested in the influence of space in the kinetic model (1.5). In order to approximate (1.5), we use a particle method. This kind of numerical scheme was first introduced by Harlow [31] for the numerical computation of specific problems in fluid dynamics, and precisely mathematically studied later [44]. Then, a large diversity of particle methods were introduced for simulations in fluid mechanics and plasma physics (see for instance [15, 27, 32] and references therein). Throughout this section, we fix  $N(v)$  to be the following cubic nonlinearity:

$$N : v \mapsto v(1 - v)(v - 0.25). \quad (7.1)$$

### 7.1 Principle of the particle method

For  $T > 0$ , the standard particle method consists in approximating the solution  $f$  of the kinetic equation (1.5) on  $[0, T]$  by a finite sum of Dirac masses

$$f_M(t, \mathbf{x}, v, w) := \frac{1}{M} \sum_{i=1}^M \delta(\mathbf{x} - \mathbf{X}_i) \delta(v - \mathcal{V}_i(t)) \delta(w - \mathcal{W}_i(t)),$$

where  $M$  is the number of particles considered in the network, and for all  $1 \leq i \leq M$ ,  $(\mathcal{V}_i, \mathcal{W}_i)$  is the solution of the characteristic system (2.3) provided by Proposition 4.1 with initial condition  $(\mathbf{X}_i, V_i^0, W_i^0) \in \mathbb{R}^3$  and initial distribution  $f_M(0, \cdot)$ . In order to provide an approximation of the macroscopic quantities necessary to solve (2.3), we approach the Dirac masses by  $\varphi_h := h^{-1} \varphi(\cdot/h)$ , where  $h > 0$  is a small fixed parameter, and

$$\varphi : z \mapsto \begin{cases} 0 & \text{if } |z| \geq 1, \\ 1 + z & \text{if } |z| \in [-1, 0], \\ 1 - z & \text{if } |z| \in [0, 1]. \end{cases}$$

Therefore, we can define the discrete densities on a mesh  $(\mathbf{y}_i)_{1 \leq i \leq n_{\mathbf{y}}}$  of step size  $h > 0$  of the considered interval  $[0, 1]$ , where  $n_{\mathbf{y}} \in \mathbb{N}$  and  $h > 0$  satisfy  $h = \frac{1}{n_{\mathbf{y}}}$ , with:

$$\begin{cases} \rho_{M,h}(\mathbf{y}_i) & := \frac{1}{M} \sum_{j=1}^M \varphi_h(\mathbf{y}_i - \mathbf{X}_j), \\ j_{M,h}(t, \mathbf{y}_i) & := \frac{1}{M} \sum_{j=1}^M \varphi_h(\mathbf{y}_i - \mathbf{X}_j) \mathcal{V}_j(t). \end{cases} \quad \forall 1 \leq i \leq M, \quad (7.2)$$

The whole point of the particle method is that  $n_{\mathbf{y}} \ll M$ . Now, for all  $1 \leq i \leq M$ , the couple  $(\mathcal{V}_i, \mathcal{W}_i)$  is approximated by the solution of the system

$$\begin{cases} \frac{d}{dt} V_i(t) = N(V_i(t)) - W_i(t) - \int \Psi(\mathbf{X}_i, \mathbf{x}') (\rho_{M,h}(\mathbf{x}') V_i(t) - j_{M,h}(t, \mathbf{x}')) d\mathbf{x}', \\ \frac{d}{dt} W_i(t) = A(V_i(t), W_i(t)), \end{cases} \quad t > 0, \quad (7.3)$$

and we approximate the integral terms in (7.3) with the rectangle method using the mesh  $(\mathbf{y}_i)_{1 \leq i \leq n_{\mathbf{y}}}$ . Finally, for the time discretization of the system (7.3), we use a Runge-Kutta of order 2 scheme, with a time grid of step 0.01. In the following, we work in the case where the neurons in the network are homogeneously distributed in the interval  $[0, 1]$ . Consequently, we choose the parameters  $(\mathbf{X}_i)_{1 \leq i \leq M}$  forming a regular mesh of the interval  $[0, 1]$ , and we fix  $M = 5001$  and  $n_{\mathbf{y}} = 501$ .

## 7.2 Numerical investigation of three different regimes

The dynamics of the microscopic system (1.3) when the number of neurons is large and of the kinetic model (1.5) are both rich, but not well known. Nevertheless, the model (1.1) for one isolated neuron has been extensively studied, and its asymptotic behaviours are perfectly predictable. Thus, we consider three different sets of parameters corresponding to three different regimes of the FitzHugh-Nagumo model (1.1) with  $I_{\text{ext}} = 0$ .

- (i) First, we investigate the bistable regime, without the adaptation variable  $w$ , that is  $\tau = 0$ . In this case, the equation (1.1) admits exactly two stable fixed points at  $v = 0$  and  $v = 1$  and one unstable at  $v = 0.25$ . Thus, the solution  $v(t)$  of the equation (1.1) converges towards 0 as  $t \rightarrow +\infty$  if  $v(0) < 0.25$ , or towards 1 if  $v(0) > 0.25$ .



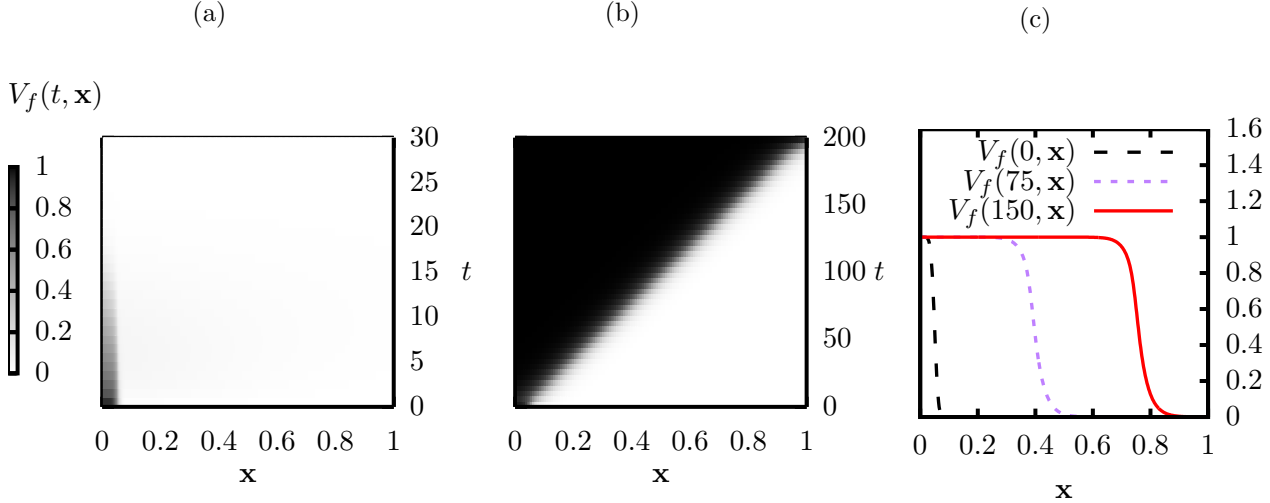


Figure 7.1: **Bistable regime.** (a)-(b) Spatio-temporal evolution of the macroscopic function  $V_f$  computed from the solution  $f$  of the kinetic equation (1.5) with  $\tau = 0$ , and different values of the parameter  $\varepsilon$ , fixed at (a)  $10^{-1}$ , (b)  $10^{-3}$ . (c) Profile of the macroscopic function  $V_f(t, \cdot)$  at different fixed times, computed with  $\varepsilon = 10^{-3}$  and with  $\tau = 0$ .

- (ii) Then, we focus on the oscillatory regime. We choose  $a = -0.25$  and  $b = 3$ , so that the system (1.1) admits a unique fixed point  $(0.25, 0)$ , which is unstable. We also consider  $\tau = 0.02$ . Consequently, in this setting, the solution of the system (1.1) converges towards a stable limit cycle if it is not initialized at the fixed point.
- (iii) Finally, we study the excitable regime. We choose  $a = 0$  and  $b = 7$ , so that the system (1.1) admits a unique fixed point  $(0, 0)$ , which is stable. All the solutions of (1.1) converge towards  $(0, 0)$  as  $t \rightarrow +\infty$ . Moreover, we fix  $\tau = 0.002$ , so that (1.1) exhibits a slow/fast dynamics.

From now on, we define the connectivity kernel to be

$$\Psi(\mathbf{x}, \mathbf{y}) = G_\varepsilon(\mathbf{x} - \mathbf{y}) \quad \forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \quad \text{with} \quad G_\varepsilon(\mathbf{x}) := \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{|\mathbf{x}|^2}{2\varepsilon}\right), \quad (7.4)$$

where  $\varepsilon > 0$  is a parameter.

We are interested in the dynamics of the average macroscopic couple  $(V_f, W_f)$  computed from the solution  $f$  of the kinetic model (1.5). In the previous section, we have proved the stability of monokinetic solutions (6.1) from well-prepared initial data (see Proposition 6.2). In the following, we present in the three regimes discussed above some numerical evidences that monokinetic solutions have a larger basin of attraction in the sense that the dynamics of  $(V_f, W_f)$  is found to be close to the dynamics of the solutions of the nonlocal reaction-diffusion system (1.6) which now reads

$$\begin{cases} \partial_t V(t, \mathbf{x}) - \int_{[0,1]} G_\varepsilon(\mathbf{x} - \mathbf{x}') (V(t, \mathbf{x}) - V(t, \mathbf{x}')) d\mathbf{x}' = N(V(t, \mathbf{x})) - W(t, \mathbf{x}), \\ \partial_t W(t, \mathbf{x}) = \tau (V(t, \mathbf{x}) + a - b W(t, \mathbf{x})), \end{cases} \quad (7.5)$$

for  $t > 0$  and  $\mathbf{x} \in [0, 1]$ .

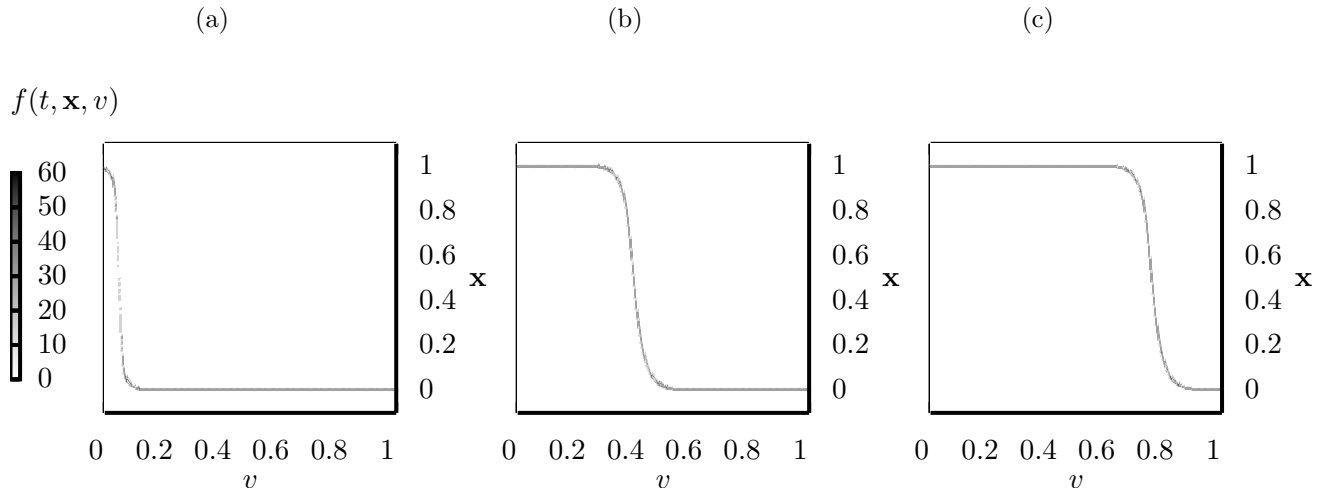


Figure 7.2: **Bistable regime.** Numerical approximation of the density function  $f$  solution of the kinetic equation (1.5) at fixed time (a)  $t = 0$ , (b)  $t = 75$  and (c)  $t = 150$ , computed with the parameters  $\varepsilon = 10^{-3}$  and  $\tau = 0$ .

### 7.2.1 Case (i) – Bistable regime

We study the kinetic model (1.5) in the bistable regime with  $\tau = 0$ . As initial condition for our numerical scheme, we choose for all  $1 \leq i \leq M$  and  $\mathbf{x} \in [0, 1]$ ,

$$\begin{cases} W_i^0(\mathbf{x}) = 0, \\ V_i^0(\mathbf{x}) = \frac{1}{2} \operatorname{erfc}(\sqrt{5000}(\mathbf{x} - 0.05)), \end{cases}$$

where  $\operatorname{erfc}$  is the complementary error function.

In Figure 7.1, we show the spatio-temporal evolution of the macroscopic quantity  $V_f$  for different values of the parameter  $\varepsilon$ . First, in the case (a),  $\varepsilon = 10^{-1}$  is large compared to the width of the considered interval  $[0, 1]$ . Thus, the space influence is almost homogenized, and the interactions between the particles of the neural network are expected to vanish after a few time. Consequently, for  $t$  large enough and for all fixed position  $\mathbf{x} \in [0, 1]$ , the macroscopic quantity  $V_f(\cdot, \mathbf{x})$  is expected to behave as a solution of the one-neuron equation (1.1) in the same framework, that is to converge towards one of the two stable fixed points. Indeed, we can observe that it converges towards 0. Then, in the case (b),  $\varepsilon = 10^{-3}$  is sufficiently small to observe another dynamic. Here, the behavior of the function  $V_f$  qualitatively looks like an invasion front, connecting the steady state 0 to 1, propagating at constant speed. Moreover, as shown in (c), after an initial transition phase, the shape of the front seems to be invariant and smooth.

We note that the qualitative behavior of the macroscopic function  $V_f$  when  $\varepsilon$  is small enough corresponds well to the dynamics of the nonlocal reaction-diffusion system (7.5) for which traveling front solutions are known to exist [4] when considered on the real line. Regarding the density function  $f$ , we show in Figure 7.2 its temporal evolution for  $\varepsilon = 10^{-3}$ . We observe that the density  $f$  remains concentrated around the states  $v = 1$  in an interval  $[0, \mathbf{x}_0(t)]$  and then around  $v = 0$  in the complementary interval  $[\mathbf{x}_0(t), 1]$  for some  $\mathbf{x}_0(t) \in (0, 1)$  propagating at a constant speed. This shows that  $f$  remains close to a monokinetic

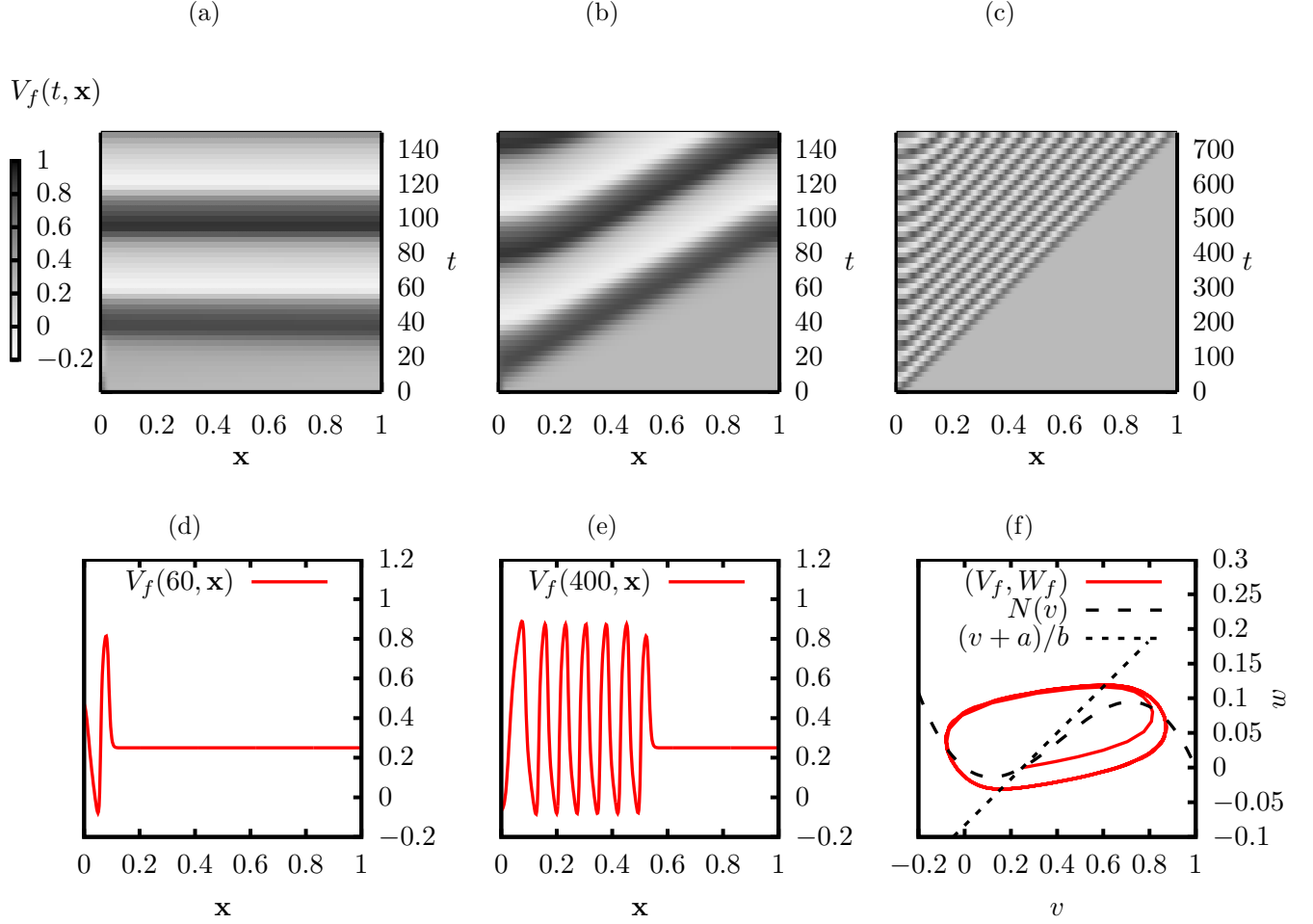


Figure 7.3: **Oscillatory regime.** (a)-(b)-(c) Spatio-temporal evolution of the macroscopic function  $V_f$  computed from the solution  $f$  of the kinetic equation (1.5) with three different values of the parameter  $\varepsilon$ , fixed at (a)  $\varepsilon = 10^{-1}$ , (b)  $\varepsilon = 10^{-3}$  and (c)  $\varepsilon = 10^{-5}$ . (d)-(e) Profile of the macroscopic function  $V_f(t, \cdot)$  computed with  $\varepsilon = 10^{-5}$  at time  $t = 60$  and  $t = 400$  respectively. (f) Trajectory in the phase space  $(v, w)$  of the couple  $(V_f, W_f)$  at fixed time  $t = 400$  computed with  $\varepsilon = 10^{-5}$ . The other parameters are fixed at  $a = -0.25$ ,  $b = 3$ , and  $\tau = 0.02$ .

solution of the form  $\rho_0(\mathbf{x}) \otimes \delta_{V_f(t, \mathbf{x})}(v)$ , where the qualitative behavior of  $V_f$  is that of a traveling front, as previously detailed. This validates the fact monokinetic solutions seem to have a large basin of attraction.

### 7.2.2 Case (ii) – Oscillatory regime

In the oscillatory regime, we choose as initial data a perturbation of the steady state  $(0.25, 0)$  concentrated around the position  $\mathbf{x} = 0$ :

$$\begin{cases} V_i^0(\mathbf{x}) = 0.25 + 0.5 \exp(-5000 \mathbf{x}^2), \\ W_i^0(\mathbf{x}) = 0. \end{cases}$$

In Figure 7.3, we display the spatio-temporal evolution of the macroscopic function  $V_f$  computed from the solution  $f$  of the kinetic equation (1.5) with three different values of the variance of  $\varepsilon$ . In the first

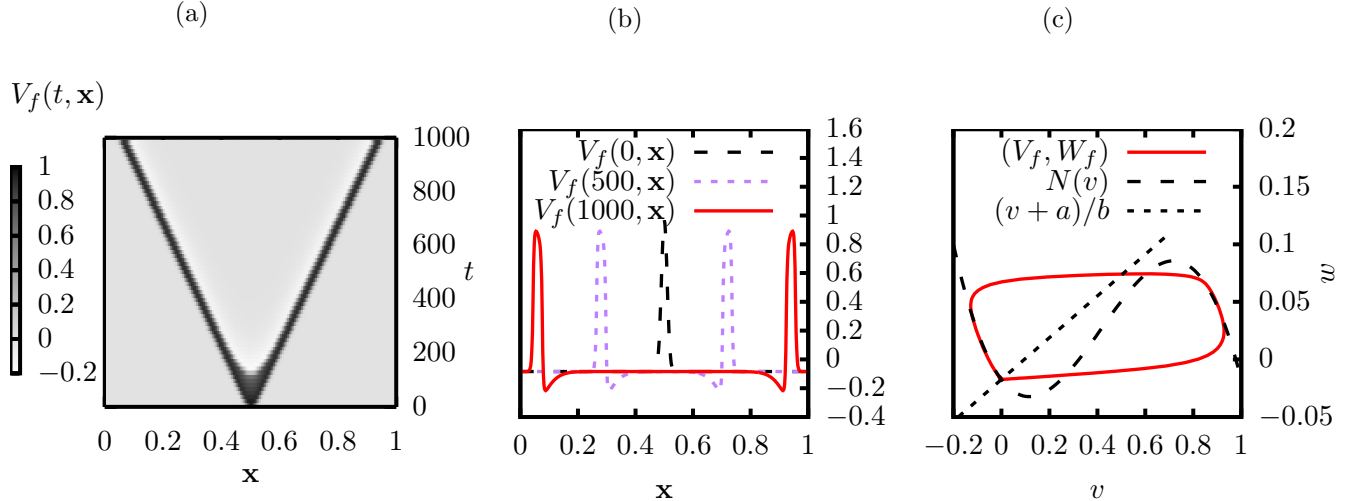


Figure 7.4: **Excitable regime.** (a) Spatio-temporal evolution of the macroscopic function  $V_f$  computed from the solution  $f$  of the kinetic equation (1.5) with  $\varepsilon = 10^{-5}$ . (b) Corresponding profile of the macroscopic function  $V_f(t, \cdot)$  computed at different times. (c) Trajectory in the phase space  $(v, w)$  of the couple  $(V_f, W_f)$  at fixed time  $t = 500$  computed with  $\varepsilon = 10^{-5}$ . The other parameters are fixed at  $a = 0$ ,  $b = 7$ , and  $\tau = 0.002$ .

case (a), we fix the parameter  $\varepsilon = 10^{-1}$ . In that case, we expect that the space dependence of  $V_f$  to be suppressed and we indeed observe synchronized homogeneous oscillations. Then, we reduce the value of  $\varepsilon$  to localize the interactions and enforce the spatial dependence. For both  $\varepsilon = 10^{-3}$  and  $\varepsilon = 10^{-5}$ , we observe temporal oscillations whose phase is modulated spatially, similar to what is usually found for the local FitzHugh-Nagumo reaction-diffusion system in the oscillatory regime [17]. More precisely, the dynamics is that of a modulated traveling wave propagating at constant speed from 0 towards the right, and leaving in the wake an oscillatory pattern with a constant frequency and amplitude. This is illustrated in Figure 7.3 panel (f) where the trajectory followed by the average couple  $(V_f, W_f)$  in the phase space at a given time converges towards a limit cycle.

### 7.2.3 Case (iii) – Excitable regime

We consider an initial condition that is a perturbation of the steady state  $(0, 0)$  concentrated at the middle of the interval  $[0, 1]$ , that is for all  $1 \leq i \leq M$  and all  $\mathbf{x} \in [0, 1]$ :

$$\begin{cases} V_i^0(\mathbf{x}) = \exp(-5000 |\mathbf{x} - 0.5|^2), \\ W_i^0(\mathbf{x}) = 0. \end{cases}$$

We report in Figure 7.4, panel (a), the space-time representation of the macroscopic function  $V_f$  for  $\varepsilon = 10^{-5}$ . In that case, we see that the dynamics generates two counter-propagating traveling pulses. Once again, the behavior of the function  $V_f$  is qualitatively the same as the expected dynamics of the corresponding macroscopic model (7.5), where it is well known that the nonlocal reaction-diffusion FitzHugh-Nagumo system supports traveling pulse solution [26]. We have drawn in Figure 7.4 panel (b) the profile of the  $v$ -component of the traveling pulse at different times and the corresponding trajectory in the phase plane

in Figure 7.4 panel (c) where we recover that the profile of the traveling pulse is a homoclinic orbit to the stable fixed point  $(0, 0)$ . The study of traveling pulses in excitable media has received lots of interests in the past decades, especially for the local reaction-diffusion FitzHugh-Nagumo system, and to our best knowledge, it is the first time that traveling pulses are reported for the FitzHugh-Nagumo kinetic model.

## 8 Discussion

In this paper, we have proved the mean-field limit of the spatially-extended FitzHugh-Nagumo model for neural networks towards a nonlocal kinetic equation as the number of neurons goes to infinity. Our approach is based on a stability estimate of solutions of the kinetic equation (1.5) in their initial distribution. We have also proved the well-posedness of the kinetic equation in the space of probability measures with finite second moments, equipped with the Wasserstein distance of order 2. This mean-field limit provides a link between the microscopic scale of the neural network and a mesoscopic scale. Our microscopic model was obtained coupling the FitzHugh-Nagumo model for a finite number of neurons, whose interactions with their neighbours is modulated only with their spatial position in the network with a connectivity kernel  $\Psi$ . Moreover, we have ignored the noise in the interactions, so that we only used deterministic tools. Finally, we numerically proved that the kinetic model (1.5), with a sufficiently localized connectivity kernel, is robust enough to display some qualitative behaviours expected for macroscopic quantities in some specific frameworks, while retaining more information than macroscopic models. The main motivation was to derive some macroscopic descriptions of the neural network, in order to rigorously establish a link between the FitzHugh-Nagumo system and some nonlocal reaction-diffusion systems. Such macroscopic models provide a better understanding of dynamics of large assemblies of neurons.

Several extensions to this work seem natural. For example, taking inspiration from [12], a possibility could be to randomly choose the connectivity weights  $\Psi_{i,j}$  in (1.3) with a probability law which depends only on the distance between the neurons  $i$  and  $j$ . A first idea could be to try the approach introduced in the article by Bolley *et al.* [5] for a stochastic model of collective motion with alignment. A direct consequence of a mean-field limit result could be the propagation of chaos in the network as the number of neurons goes to infinity, that is the neurons become less and less correlated as their number gets large. Furthermore, another interesting extension comes from the recent work of Chiba and Medvedev [22] for the Kuramoto model, and to study the mean-field limit of the FitzHugh-Nagumo model on various types of random graphs.

## 9 Acknowledgements

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## A Proof of Lemma 4.2

We start by proving that for all fixed  $\mathbf{z} \in \mathbb{R}^{d+2}$ , there exists a unique solution  $(\mathcal{V}(\cdot, \mathbf{z}), \mathcal{W}(\cdot, \mathbf{z})) \in \mathcal{C}^1([0, T])^2$  of (4.2) such that for all  $t \in [0, T]$ ,  $(\mathcal{V}(t, \cdot), \mathcal{W}(t, \cdot)) \in \mathcal{C}(\mathbb{R}^{d+2}, \mathbb{R})^2$ . Then, we prove that  $\mathcal{V}$  and  $\mathcal{W} \in$

$\mathcal{C}([0, T], \mathcal{E})$ , where  $\mathcal{E}$  is defined with (3.1)-(3.2).

**Step 1: Existence and uniqueness of the solution of (4.2)**

First of all, the function

$$\begin{pmatrix} v \\ w \end{pmatrix} \mapsto \begin{pmatrix} N(v) - w \\ A(v, w) \end{pmatrix}$$

is locally Lipschitz continuous in  $\mathbb{R}^2$ . Moreover, since  $\mathcal{U} \in \mathcal{C}([0, T], \mathcal{E})$ , then for all  $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ , the function

$$t \mapsto \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{U}(t, \mathbf{z}) - \mathcal{U}(t, \mathbf{z}')) f_0(d\mathbf{z}')$$

is continuous on  $[0, T]$ . Therefore, for all  $\mathbf{z} \in \mathbb{R}^{d+2}$ , the Cauchy-Lipschitz Theorem gives us the local existence and uniqueness of the solution  $(\mathcal{V}(\cdot, \mathbf{z}), \mathcal{W}(\cdot, \mathbf{z}))$  of class  $\mathcal{C}^1$ . Then, using an energy estimate, we want to prove that for all  $\mathbf{z} \in \mathbb{R}^{d+2}$ , the couple  $(\mathcal{V}(\cdot, \mathbf{z}), \mathcal{W}(\cdot, \mathbf{z}))$  is well-defined on  $[0, T]$ . According to the definition of  $\mathcal{E}$ , we know that for all  $(t, \mathbf{z}) \in [0, T] \times \mathbb{R}^{d+2}$ :

$$|\mathcal{U}(t, \mathbf{z})| \leq \gamma(\mathbf{z}) \sup_{s \in [0, T]} \|\mathcal{U}(s)\|_{\mathcal{E}}. \quad (\text{A.1})$$

Let  $\mathbf{z} \in \mathbb{R}^{d+2}$  and  $t \in [0, T]$  small enough so that  $(\mathcal{V}, \mathcal{W})$  exists in  $(t, \mathbf{z})$ . We have the following energy estimate:

$$\frac{1}{2} \frac{d}{dt} \left( |\mathcal{V}(t, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z})|^2 \right) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where

$$\begin{cases} \mathcal{I}_1 := \tau \mathcal{W}(t, \mathbf{z}) (\mathcal{V}(t, \mathbf{z}) + a - b \mathcal{W}(t, \mathbf{z})), \\ \mathcal{I}_2 := \mathcal{V}(t, \mathbf{z}) (N(\mathcal{V}(t, \mathbf{z})) - \mathcal{W}(t, \mathbf{z})), \\ \mathcal{I}_3 := -\mathcal{V}(t, \mathbf{z}) \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{U}(t, \mathbf{z}) - \mathcal{U}(t, \mathbf{z}')) f_0(d\mathbf{z}'). \end{cases}$$

First of all, we treat the first term  $\mathcal{I}_1$  using Young's inequality

$$\begin{aligned} \mathcal{I}_1 &\leq \frac{\tau}{2} |\mathcal{V}(t, \mathbf{z})|^2 + \tau |\mathcal{W}(t, \mathbf{z})|^2 + \frac{\tau a^2}{2} \\ &\leq \tau \left( |\mathcal{V}(t, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z})|^2 \right) + \frac{\tau a^2}{2}. \end{aligned}$$

Then, we can estimate the second term  $\mathcal{I}_2$  using Young's inequality and the assumption (2.2) satisfied by  $N$ :

$$\begin{aligned} \mathcal{I}_2 &\leq \kappa |\mathcal{V}(t, \mathbf{z})|^2 + \frac{1}{2} \left( |\mathcal{V}(t, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z})|^2 \right) \\ &\leq \left( \kappa + \frac{1}{2} \right) \left( |\mathcal{V}(t, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z})|^2 \right). \end{aligned}$$

Furthermore, we deal with the second term  $\mathcal{I}_3$  using Young's inequality, the assumption that  $\Psi$  is bounded

and the estimate (A.1):

$$\begin{aligned}
\mathcal{I}_3 &\leq \frac{1}{2} |\mathcal{V}(t, \mathbf{z})|^2 + \frac{1}{2} \int |\Psi(\mathbf{x}, \mathbf{x}')|^2 (|\mathcal{U}(t, \mathbf{z})|^2 + |\mathcal{U}(t, \mathbf{z}')|^2) f_0(d\mathbf{z}') \\
&\leq \frac{1}{2} |\mathcal{V}(t, \mathbf{z})|^2 + \frac{1}{2} \|\Psi\|_\infty^2 \left( |\mathcal{U}(t, \mathbf{z})|^2 + \int |\mathcal{U}(t, \mathbf{z}')|^2 f_0(d\mathbf{z}') \right) \\
&\leq \frac{1}{2} |\mathcal{V}(t, \mathbf{z})|^2 + \frac{1}{2} \|\Psi\|_\infty^2 \left( |\gamma(\mathbf{z})|^2 + \int |\gamma(\mathbf{z}')|^2 f_0(d\mathbf{z}') \right) \sup_{s \in [0, T]} \|\mathcal{U}(s)\|_{\mathcal{E}}^2 \\
&\leq \frac{1}{2} |\mathcal{V}(t, \mathbf{z})|^2 + \frac{1}{2} \|\Psi\|_\infty^2 \left( 1 + \int |\gamma(\mathbf{z}')|^2 f_0(d\mathbf{z}') \right) \sup_{s \in [0, T]} \|\mathcal{U}(s)\|_{\mathcal{E}}^2 |\gamma(\mathbf{z})|^2.
\end{aligned}$$

Hence, there exist two constants  $C_1 > 0$  and  $C_2 > 0$  such that for all  $t \in [0, T]$  and  $\mathbf{z} \in \mathbb{R}^{d+2}$ ,

$$\frac{d}{dt} \left( |\mathcal{V}(t, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z})|^2 \right) \leq C_1 \left( |\mathcal{V}(t, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z})|^2 \right) + C_2 |\gamma(\mathbf{z})|^2.$$

Therefore, Grönwall's inequality yields that for all  $t \in [0, T]$ ,

$$\left( |\mathcal{V}(t, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z})|^2 \right) \leq (|v|^2 + C_2 t |\gamma(\mathbf{z})|^2) e^{C_1 t}. \quad (\text{A.2})$$

This implies that  $\mathcal{V}(\cdot, \mathbf{z})$  and  $\mathcal{W}(\cdot, \mathbf{z})$  are defined on  $[0, T]$ , and thus  $(\mathcal{V}(\cdot, \mathbf{z}), \mathcal{W}(\cdot, \mathbf{z})) \in \mathcal{C}^1([0, T])$ . Moreover, if we divide (A.2) by  $|\gamma(\mathbf{z})|^2$  and if we take the supremum on  $\mathbb{R}^{d+2}$ , we conclude that for all  $t \in [0, T]$ ,

$$\|\mathcal{V}(t)\|_{\mathcal{E}}^2 \leq (1 + C_2 t) e^{C_1 t}. \quad (\text{A.3})$$

Moreover, since  $\Psi$  is continuous with respect to its first variable uniformly relative to its second variable, and since  $\mathcal{U} \in \mathcal{C}([0, T], \mathcal{E})$ , we get that the function

$$(s, \mathbf{z}) = (s, \mathbf{x}, v, w) \mapsto \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{U}(t, \mathbf{z}) - \mathcal{U}(t, \mathbf{z}')) f_0(d\mathbf{z}')$$

is continuous on  $[0, T] \times \mathbb{R}^{d+2}$ . Therefore, one can check that for all  $t \in [0, T]$ ,

$$(\mathcal{V}(t, \cdot), \mathcal{W}(t, \cdot)) \in \mathcal{C}(\mathbb{R}^{d+2}, \mathbb{R}^2). \quad (\text{A.4})$$

Furthermore, (A.3) together with (A.4) yield that for all  $t \in [0, T]$ ,

$$(\mathcal{V}(t, \cdot), \mathcal{W}(t, \cdot)) \in \mathcal{E}^2. \quad (\text{A.5})$$

**Step 2:  $\mathcal{V}$  and  $\mathcal{W}$  are in  $\mathcal{C}([0, T], \mathcal{E})$**

It remains to prove that  $(\mathcal{V}, \mathcal{W}) \in \mathcal{C}([0, T], \mathcal{E})^2$ . Let us fix  $t \in [0, T]$ . Consider  $\tilde{t} \in [0, T]$  and  $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ . Thus, we have:

$$\frac{1}{2} \frac{d}{dt} \left( |\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(\tilde{t}, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z}) - \mathcal{W}(\tilde{t}, \mathbf{z})|^2 \right) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,$$

where

$$\left\{ \begin{array}{l}
\mathcal{J}_1 := (\mathcal{W}(t, \mathbf{z}) - \mathcal{W}(\tilde{t}, \mathbf{z})) (A(\mathcal{V}(t, \mathbf{z}), \mathcal{W}(t, \mathbf{z})) - A(\mathcal{V}(\tilde{t}, \mathbf{z}), \mathcal{W}(\tilde{t}, \mathbf{z}))), \\
\mathcal{J}_2 := (\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(\tilde{t}, \mathbf{z})) [(N(\mathcal{V}(t, \mathbf{z})) - N(\mathcal{V}(\tilde{t}, \mathbf{z}))) - (\mathcal{W}(t, \mathbf{z}) - \mathcal{W}(\tilde{t}, \mathbf{z}))], \\
\mathcal{J}_3 := -(\mathcal{V}(t, \mathbf{z}) - \mathcal{V}_{p+1}(\tilde{t}, \mathbf{z})) \\
\quad \times \int [\Psi(\mathbf{x}, \mathbf{x}') (\mathcal{U}(t, \mathbf{z}) - \mathcal{U}(t, \mathbf{z}')) + \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{U}(\tilde{t}, \mathbf{z}) - \mathcal{U}(\tilde{t}, \mathbf{z}'))] f_0(d\mathbf{z}').
\end{array} \right.$$

We deal with the first term  $\mathcal{J}_1$  using Young's inequality

$$\mathcal{J}_1 \leq \frac{\tau}{2} \left( |\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(\tilde{t}, \mathbf{z})|^2 + |\mathcal{W}(t, \mathbf{z}) - \mathcal{W}(\tilde{t}, \mathbf{z})|^2 \right).$$

Then, we can estimate the second term  $\mathcal{J}_2$  using Young's inequality and the assumption (2.2) satisfied by  $N$ :

$$\mathcal{J}_2 \leq \left( \tilde{\kappa} + \frac{1}{2} \right) |\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(\tilde{t}, \mathbf{z})|^2 + \frac{1}{2} |\mathcal{W}(t, \mathbf{z}) - \mathcal{W}(\tilde{t}, \mathbf{z})|^2.$$

Then, to treat the last term  $\mathcal{J}_3$ , we use Young's inequality and the assumption that  $\Psi$  is bounded:

$$\begin{aligned} \mathcal{J}_3 &= -(\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(\tilde{t}, \mathbf{z})) \\ &\quad \times \left( (\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(\tilde{t}, \mathbf{z})) \int \Psi(\mathbf{x}, \mathbf{x}') f_0(d\mathbf{z}') + \int \Psi(\mathbf{x}, \mathbf{x}') (\mathcal{U}(t, \mathbf{z}') - \mathcal{U}(\tilde{t}, \mathbf{z}')) f_0(d\mathbf{z}') \right) \\ &\leq \frac{1}{2} |\mathcal{V}(t, \mathbf{z}) - \mathcal{V}(\tilde{t}, \mathbf{z})|^2 \\ &\quad + \frac{1}{2} \|\Psi\|_\infty^2 |\mathcal{U}(t, \mathbf{z}) - \mathcal{U}(\tilde{t}, \mathbf{z})|^2 + \frac{1}{2} \|\Psi\|_\infty^2 \|\mathcal{U}(t) - \mathcal{U}(\tilde{t})\|_{\mathcal{E}}^2 \int |\gamma(\mathbf{z}')|^2 f_0(d\mathbf{z}'). \end{aligned}$$

Then, dividing  $\mathcal{J}_1$ ,  $\mathcal{J}_2$  and  $\mathcal{J}_3$  by  $|\gamma(\mathbf{z})|^2$  and taking the supremum on  $\mathbb{R}^{d+2}$ , we get that there exist two constants  $C_1$  and  $C_2$  such that:

$$\begin{aligned} \frac{d}{dt} \left( \|\mathcal{V}(t) - \mathcal{V}(\tilde{t})\|_{\mathcal{E}}^2 + \|\mathcal{W}(t) - \mathcal{W}(\tilde{t})\|_{\mathcal{E}}^2 \right) \\ \leq C_1 \left( \|\mathcal{V}(t) - \mathcal{V}(\tilde{t})\|_{\mathcal{E}}^2 + \|\mathcal{W}(t) - \mathcal{W}(\tilde{t})\|_{\mathcal{E}}^2 \right) + C_2 \|\mathcal{U}(t) - \mathcal{U}(\tilde{t})\|_{\mathcal{E}}^2. \end{aligned}$$

Using Grönwall's lemma the assumption  $\mathcal{U} \in \mathcal{C}([0, T], \mathcal{E})$ , we conclude that

$$\lim_{\tilde{t} \rightarrow t} \left( \|\mathcal{V}(t) - \mathcal{V}(\tilde{t})\|_{\mathcal{E}}^2 + \|\mathcal{W}(t) - \mathcal{W}(\tilde{t})\|_{\mathcal{E}}^2 \right) = 0,$$

and consequently,

$$(\mathcal{V}, \mathcal{W}) \in \mathcal{C}([0, T], \mathcal{E})^2.$$

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