



# Template Games, Simple Games, and Day Convolution

Clovis Eberhart, Tom Hirschowitz, Alexis Laouar

## ► To cite this version:

Clovis Eberhart, Tom Hirschowitz, Alexis Laouar. Template Games, Simple Games, and Day Convolution. 4th International Conference on Formal Structures for Computation and Deduction (FSCD 2019), 2019, Dortmund, Germany. pp.16:1–16:19, 10.4230/LIPIcs.FSCD.2019.16 . hal-01897309v2

**HAL Id: hal-01897309**

**<https://hal.science/hal-01897309v2>**

Submitted on 14 Feb 2019 (v2), last revised 29 Aug 2019 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Template games, simple games, and Day convolution

Clovis Eberhart 

National Institute of Informatics, Tokyo, Japan

Tom Hirschowitz 

Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000, Chambéry, France

Alexis Laouar

Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000, Chambéry, France

## Abstract

*Template games* [13] unify various approaches to game semantics, by exhibiting them as instances of a double-categorical variant of the slice construction. However, in the particular case of simple games [8, 11], template games do not quite yield the standard (bi)category. We refine the construction using factorisation systems, obtaining as an instance a slight generalisation of simple games and strategies. Another instance is Day’s *convolution* monoidal structure on the category of presheaves over a strict monoidal category [2], which answers a question raised in [3].

**2012 ACM Subject Classification** Theory of computation → Denotational semantics

**Keywords and phrases** Game semantics, Day convolution, Categorical semantics

**Digital Object Identifier** 10.4230/LIPIcs...??

**Funding** Clovis Eberhart: ERATO HASUO Metamathematics for Systems Design Project (No. JP-MJER1603, <http://dx.doi.org/10.13039/501100009024>), JST.

## 1 Introduction

Game semantics has provided adequate models for a variety of programming languages [10], in which types are interpreted as two-player games and programs as strategies. Most game models follow a common pattern. Typically, a function  $A \rightarrow B$  is interpreted as a strategy on a compound game made of  $A$  and  $B$ , where the program plays as *Proponent* (P) on  $B$  and as *Opponent* (O) on  $A$ . Another common feature is composition of strategies, which takes strategies  $\sigma: A \rightarrow B$  and  $\tau: B \rightarrow C$ , and returns a strategy  $\tau \circ \sigma: A \rightarrow C$  by letting  $\sigma$  and  $\tau$  interact on  $B$  until one of them produces a move in  $A$  or  $C$ .

Although widely acknowledged, this strong commonality is also recognised as poorly understood, particularly in the presence of *innocence*, a constraint on strategies that restricts them to purely functional behaviour. This has prompted a number of attempts at clarifying the situation [4, 8, 9]. Recently, Melliès [13] proposed a novel explanation, of unprecedented simplicity, named *template games*. It is based upon a purely categorical construction, essentially taking the slice of a weak double category over an internal monad, the *template*. This produces a new weak double category, in which composition of strategies occurs as so-called horizontal composition. In order to illustrate the construction, Melliès applies it to three different templates to obtain three models, respectively related to simple games [11], concurrent games [14], and the relational model. However, the first two differ significantly from their more standard counterparts.

This raises the question of whether template games only produce new game models, or whether they also cover standard models. In this paper, we show that, up to a slight refinement, they do cover simple games. More precisely, we (1) modify the original template for simple games (Lemma 62), and (2) enrich the general construction with a factorisation



© Clovis Eberhart, Tom Hirschowitz, and Alexis Laouar;  
licensed under Creative Commons License CC-BY

Leibniz International Proceedings in Informatics

**LIPICs** Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

system [1] (Theorem 59), obtaining a slight generalisation of standard simple games as an instance (Corollary 66). More precisely, we obtain a variant in which games may have several ‘initial positions’, before the game even starts, and similarly strategies may have several ‘initial states’ over each of these initial positions. We thus easily characterise standard simple games and strategies as so-called *definite* template games and strategies.

One motivation for simple and abstract constructions like template games is to find new connections with other settings. Our refined construction yields one such connection: we show that the Day *convolution* product [2] arises as an instance of our refined framework, though only in the restricted case of *strict* monoidal categories (Theorem 67). The convolution product, which arose in algebraic topology, extends the monoidal structure of a given category  $\mathbb{C}$  to the category  $\widehat{\mathbb{C}}$  of *presheaves* on  $\mathbb{C}$ , i.e., contravariant functors  $\mathbb{C}^{op} \rightarrow \mathbf{Set}$ . This makes formal the similarity, noted in [3, §6.5], between convolution and composition of strategies, by showing that both are instances of the same construction.

► **Related work.** Beyond Melliès’s obviously related work and the related [9], Garner and Shulman [7] prove results related to our Theorems 50 and 59. The common ground for comparison is the restriction of Theorem 50 to weak double categories with trivial vertical category, i.e., monoidal categories. Their Theorem 14.2 is a generalisation in another direction, namely that of monoidal bicategories, and their Theorem 14.5 could in particular accomodate various sorts of bicategorical factorisation systems.

► **Terminology.** Although template games and strategies are an abstract construction, we often abuse the term to denote the particular instance on the template for simple games.

► **Plan.** We follow the standard construction layers of game models: games, strategies, and composition of strategies. In Section 2, we analyse the differences between template and simple games, and describe our refinement of the former, which allows us to bridge the gap. In Section 3, we do the same at the level of strategies  $A \rightarrow B$ , for fixed games  $A$  and  $B$ . In Section 4, we recall the abstract construction of template games. In Section 5, we introduce our refined construction, and illustrate it on the promised instances.

## 2      Template games vs. simple games

In this section, we first recall template games, and then analyse the discrepancies with simple games. Finally, we introduce our solution.

Template games are based on the following simple category.

► **Definition 1.** Let  $\mathfrak{z}_v$  denote the category freely generated by the graph  $O \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} P$ .

Thus, objects are just  $O$  and  $P$ , which stand for *Opponent* and *Proponent*, as in most game models, and morphisms just count the number of (alternating) moves between them.

► **Definition 2.** A template game is a category  $A$ , equipped with a functor  $p: A \rightarrow \mathfrak{z}_v$ .

Intuitively, objects of  $A$  are positions, or plays, in the game, and  $p$  gives their polarity: by convention,  $O$  is to play in positions mapped to  $O$ , while  $P$  is to play in others.

Simple games clearly fit into this framework.

► **Definition 3.** A simple game  $A$  is a rooted tree.

The intuition is just as for template games, and indeed:

► **Proposition 4.** Any simple game  $A$ , viewed as a poset, hence as a category, forms a template game  $p_A: A \rightarrow \mathfrak{z}_v$ , where  $p_A$  maps the root to  $O$ , its children to  $P$ , and so on.

86 ▶ **Example 5.** The simple game  $\mathbb{B}$  for booleans is the tree



88 where we have labelled edges with moves, and each node with the corresponding play, i.e.,  
 89 the sequence of moves needed to reach it from the root. The first move,  $\mathbf{q}$ , represents  $O$   
 90 asking the value of the boolean, and the other moves represent the possible answers: true  
 91 and false. The functor  $p_{\mathbb{B}}: \mathbb{B} \rightarrow \mathfrak{A}_v$  maps  $\mathbf{q}$  to  $P$  and all other objects to  $O$ .

## 92 Discrepancies between template and simple games

93 Template games are significantly more general than simple games. To start with, simple  
 94 games have an empty, initial play, which template games need not. Furthermore, this initial  
 95 play is mapped to  $O$  by the functor to  $\mathfrak{A}_v$ .

96 ▶ **Lemma 6.** *For any simple game  $A$ , the category  $A$  has an initial object, mapped to  $O$  by*  
 97 *the functor  $A \rightarrow \mathfrak{A}_v$ .*

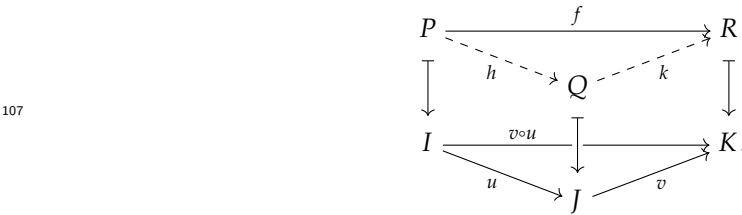
98 Let us exhibit some counterexamples among general template games.

99 ▶ **Example 7.** The template game  $\mathbb{1} \rightarrow \mathfrak{A}_v$  picking up  $P$  is not equivalent to any simple game  
 100 in the 2-category  $\mathbf{Cat}/\mathfrak{A}_v$ .

101 ▶ **Example 8.** Consider the template game consisting of  $\mathbb{Z}$ , the poset of integers, and the  
 102 functor  $\mathbb{Z} \rightarrow \mathfrak{A}_v$  that maps  $2n$  to  $O$  and  $2n + 1$  to  $P$ . The category  $\mathbb{Z}$  has no initial object,  
 103 hence this template game is not equivalent to any simple game in  $\mathbf{Cat}/\mathfrak{A}_v$ .

104 Another discrepancy has to do with decomposing plays into moves.

105 ▶ **Definition 9.** *A functor  $F: \mathbb{E} \rightarrow \mathbb{B}$  is a discrete Conduché fibration iff for any  $I \xrightarrow{u} J \xrightarrow{v} K$*   
 106 *in  $\mathbb{B}$  and  $P \xrightarrow{f} R$  in  $\mathbb{E}$  mapped to  $v \circ u$ ,  $f$  uniquely factors as  $k \circ h$  with  $F(h) = u$  and  $F(k) = v$ :*



108 ▶ **Lemma 10.** *For any simple game  $A$ , the functor  $p_A: A \rightarrow \mathfrak{A}_v$  is a discrete Conduché*  
 109 *fibration.*

110 **Proof.** By construction, a morphism in  $A$  over a path of length  $n$  has the form  $p \rightarrow pm_1 \dots m_n$   
 111 for some play  $p$  and moves  $m_1, \dots, m_n$ , hence decomposes as needed. ◀

112 Unlike simple games, not all template games are Conduché fibrations. Indeed, they may  
 113 feature *atomic* sequences of moves, i.e., morphisms that are mapped to non-basic morphisms  
 114 in  $\mathfrak{A}_v$  and yet are ‘indecomposable’.

115 ▶ **Example 11.** Consider the ordinal  $2 = \{0 \leq 1\}$  viewed as a category, and the functor  
 116  $2 \rightarrow \mathfrak{A}_v$  mapping  $0 \leq 1$  to the path  $O \rightarrow P \rightarrow O$ . This is clearly not a Conduché fibration,  
 117 hence is not equivalent to any simple game.

### 118    Refining template games

119    As announced in §1, our solution to bridge the gap between template and simple games is  
 120    twofold: (1) we use a different template,  $\mathbb{T}$ , and (2) we restrict attention to functors  $A \rightarrow \mathbb{T}$   
 121    which are discrete fibrations. Let us start with the new template.

122    ▶ **Definition 12.** Let  $\mathbb{T}_v = \omega$  denote the poset of natural numbers, viewed as a category.

123    ▶ **Remark 13.**  $\mathbb{T}_v$  is isomorphic to the slice category  $O/\mathbb{T}_v$ .

124    Defining games to be functors to  $\mathbb{T}_v$  intuitively goes in the right direction, but does not  
 125    quite solve any of our two problems.

126    ▶ **Example 14.** Consider the functor  $D: \omega \rightarrow \omega = \mathbb{T}_v$  defined by  $D(n) = 2n + 1$ . It exhibits  
 127    both problems at once: it does not preserve the initial object, and, e.g.,  $0 \leq 1$  does not admit  
 128    any decomposition along the decomposition  $1 \leq 2 \leq 3$ , a.k.a.  $P \rightarrow O \rightarrow P$ , of its image.

129    However, if we restrict attention to a certain kind of functors  $A \rightarrow \mathbb{T}_v$ , we do solve both  
 130    problems at once. The relevant constraint on functors is generally stronger than being a  
 131    Conduché fibration, but becomes equivalent when both categories have an initial object  
 132    which is preserved by the functor.

133    ▶ **Definition 15.** A functor  $F: \mathbb{E} \rightarrow \mathbb{B}$  is a discrete fibration when for any  $E \in \mathbb{E}$  and  
 134     $u: B \rightarrow F(E)$ , there is a unique  $f: E' \rightarrow E$  such that  $F(f) = u$ , as in

$$\begin{array}{ccc} E' & \xrightarrow{\quad f \quad} & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{\quad u \quad} & F(E). \end{array}$$

136    Let  $\mathbf{DFib}(\mathbb{C})$  denote the full subcategory of  $\mathbf{Cat}/\mathbb{C}$  spanning discrete fibrations.

137    ▶ **Definition 16.** A refined template game is a discrete fibration to  $\mathbb{T}_v$ .

138    Of course, any simple game  $A$  yields a discrete fibration  $p_A: A \rightarrow \mathbb{T}_v$ , and we have:

139    ▶ **Proposition 17.** Any refined template game in  $\mathbf{DFib}(\mathbb{T}_v)$  is isomorphic to  $p_A$ , for some  
 140    simple game  $A$ , iff it is definite, i.e., its fibre over 0 is a singleton.

## 141    3    Template strategies vs. simple strategies

142    Let us now consider strategies. As for games, we start with Melliès's notion, to emphasise  
 143    its simplicity. Template strategies are based on the following simple category:

144    ▶ **Definition 18.** Let  $\mathbb{T}_V$  denote the category freely generated by the graph

$$\begin{array}{ccccc} OO & \xrightarrow{\quad} & OP & \xrightarrow{\quad} & PP. \\ & \xleftarrow{\quad} & & \xleftarrow{\quad} & \\ & & & & \end{array}$$

146    This is essentially the well-known state diagram for strategies in a simple arrow game  $A \rightarrow B$ ,  
 147    extended to a category. We need a little lemma before defining strategies.

148    ▶ **Lemma 19.** The left and right projections give rise to functors  $s, t: \mathbb{T}_V \rightarrow \mathbb{T}_v$ , with  
 149     $s(XY) = X$  and  $t(XY) = Y$ .

150 ▶ **Definition 20.** A template strategy from  $p: A \rightarrow \mathfrak{A}_v$  to  $q: B \rightarrow \mathfrak{A}_v$  is a tuple  $(S, s', t', r)$   
 151 making the following diagram commute.

$$\begin{array}{ccccc}
 A & \xleftarrow{s'} & S & \xrightarrow{t'} & B \\
 p \downarrow & & \downarrow r & & \downarrow q \\
 \mathfrak{A}_v & \xleftarrow{s} & \mathfrak{A}_V & \xrightarrow{t} & \mathfrak{A}_v
 \end{array} \tag{1}$$

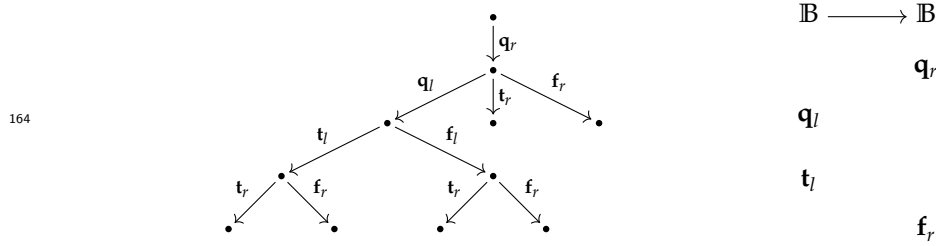
153 Let us now show that simple strategies give rise to template strategies. Simple strategies are  
 154 rather subtle – which was one of the main motivations for template games in the first place!  
 155 – so this is a bit technical. We first consider *boolean* simple strategies, which are easier, and  
 156 then move on to general ones.

### 157 Boolean simple strategies

158 First, we need to define the *arrow game*  $A \rightarrow B$ , for any two simple games  $A$  and  $B$ . The  
 159 following definition is slightly vague: we refer to [11] for a fully rigorous one.

160 ▶ **Definition 21.** Given two simple games  $A$  and  $B$ ,  $A \rightarrow B$  interleaves moves from  $A$  and  $B$   
 161 according to the following rules: (1) the polarity of moves in  $A$  is inverted, (2)  $O$  starts in  
 162  $B$ , and (3) only  $P$  gets to switch sides.

163 ▶ **Example 22.** The game  $\mathbb{B} \rightarrow \mathbb{B}$  is depicted below left, with an example play on the right.



165 ▶ **Definition 23.** A boolean simple strategy  $A \rightarrow B$  is a prefix-closed set of non-empty,  
 166 even-length plays, called *accepted*, in  $A \rightarrow B$ .

167 ▶ **Example 24.** The set of non-empty, even-length prefixes of the play in Example 22 (i.e.,  
 168 the set  $\{q_r q_l, q_r q_l t_l f_r\}$ ) forms a boolean simple strategy.

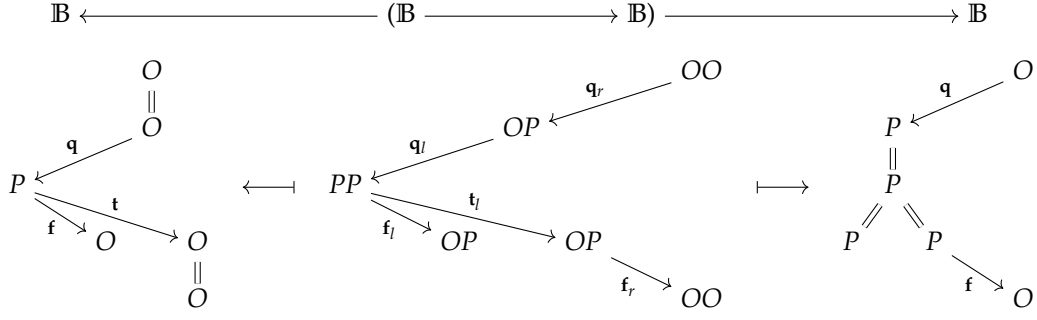
169 There are in fact several ways in which boolean simple strategies give rise to template  
 170 strategies. We here present the relevant one in terms of semantics.

171 ▶ **Definition 25.** The template strategy associated to a boolean simple strategy  $\sigma: A \rightarrow B$  is  
 172  $(\mathbb{E}(\sigma), s', t', r)$ , where  $\mathbb{E}(\sigma)$  is the poset of all prefixes of accepted plays, plus all extensions *sm*  
 173 of prefixes *s* of even length, and  $r$ ,  $s'$ , and  $t'$  are the obvious projections.

174 ▶ **Example 26.** Consider the strategy of Example 24. The functor  $\mathbb{E}(\sigma) \rightarrow \mathfrak{A}_V$  can be  
 175 represented as

$$\begin{array}{ccccccc}
 OO & \xrightarrow{q_r} & OP & \xrightarrow{q_l} & PP & \begin{array}{l} \xrightarrow{t_l} OP \\ \xrightarrow{f_l} OP \end{array} & \xrightarrow{f_l} OO
 \end{array} \tag{2}$$

177 with  $s', t': \mathbb{B} \rightarrow \mathbb{B}$  displayed on the left and right in Figure 1 (with polarity indications).



■ **Figure 1** Template strategy associated to the simple strategy of Example 24

### 178 General simple strategies

179 Let us now consider the more general, non-deterministic (= non-boolean) but still standard,  
 180 notion of simple strategy [11]. Let us start from an alternative presentation of boolean  
 181 strategies.

182 ▶ **Definition 27.** Let  $(A \rightarrow B)^{P^*} \hookrightarrow (A \rightarrow B)$  denote the full subcategory spanning non-empty,  
 183 even-length plays.

184 ▶ **Proposition 28.** Boolean simple strategies  $A \rightarrow B$  are equivalent to functors  $((A \rightarrow$   
 185  $B)^{P^*})^{op} \rightarrow 2$ .

186 The idea is that a strategy  $\sigma$  accepts a given play  $p$  iff the corresponding functor maps it to  
 187  $1 \in 2$ , observing that functoriality ensures prefix-closedness.

188 General simple strategies are obtained by generalising from **2** to **Set**:

189 ▶ **Definition 29.** Let the category of simple strategies  $A \rightarrow B$  be  $\mathcal{S}(A, B) := \widehat{(A \rightarrow B)^{P^*}}$ .

▶ **Remark 30.** This is equivalent to the standard definition [11] as a slice category. Indeed,  
 letting  $\omega^{P^*}$  denote the full subcategory of  $\omega$  on positive, even ordinals,  $p_{A \rightarrow B}$  restricts to a  
 discrete fibration  $(A \rightarrow B)^{P^*} \rightarrow \omega^{P^*}$ . By the well-known equivalence  $\partial^*: \mathbf{DFib}(\mathbb{C}) \rightarrow \widehat{\mathbb{C}} : el$   
 between presheaves and discrete fibrations (recalled as Lemma 32 below), and, for any  $U \in \widehat{\mathbb{C}}$ ,  
 the further equivalence  $\widehat{\mathbb{C}}/U \simeq el(\widehat{U})$ , we obtain

$$\mathcal{S}(A, B) = \widehat{(A \rightarrow B)^{P^*}} \simeq \widehat{\omega^{P^*}} / \partial^* ((A \rightarrow B)^{P^*}).$$

190 The latter slice category is precisely the standard definition.

191 Of course, boolean simple strategies embed into general ones by postcomposition with the  
 192 embedding  $2 \hookrightarrow \mathbf{Set}$  determined by  $0 \mapsto \emptyset$  and  $1 \mapsto 1$ .

193 Let us now informally describe how simple strategies  $\sigma \in \widehat{(A \rightarrow B)^{P^*}}$  give rise to template  
 194 strategies, for which the following well-known result is the basis.

195 ▶ **Definition 31.** For any small category  $\mathbb{C}$ , let  $X \in \widehat{\mathbb{C}}$ . The category of elements  $el(X)$  of  
 196  $X$  has as objects pairs  $(c, x)$  with  $x \in X(c)$ , and as morphisms  $(c, x) \rightarrow (c', x')$  all morphisms  
 197  $f: c \rightarrow c'$  such that  $X(f)(x') = x$ .

198 ▶ **Lemma 32.** For any small category  $\mathbb{C}$  and  $X \in \widehat{\mathbb{C}}$ , the projection functor  $p_X: el(X) \rightarrow \mathbb{C}$   
 199 is a discrete fibration, and the category of elements construction extends to an equivalence of  
 200 categories  $el: \widehat{\mathbb{C}} \rightarrow \mathbf{DFib}(\mathbb{C})$ . Let  $\partial^*$  denote the weak inverse to  $el$ .

201 ▶ **Example 33.** One possible definition of forests is as presheaves over  $\omega$ . Indeed, any  $T \in \widehat{\omega}$   
 202 models the forest whose nodes of depth  $n$  are  $T(n)$ , and whose parent relation is given by  
 203  $T(n \leq n+1): T(n+1) \rightarrow T(n)$ . The category  $el(T)$  is just the same forest viewed as a poset,  
 204 or rather a category, and the projection functor  $p_T: el(T) \rightarrow \omega$  maps vertices to their depths.  
 205 Trees are those  $T \in \widehat{\omega}$  such that  $T(0)$  is a singleton.

206 Returning to simple and template strategies, we have a commuting diagram of functors

$$\begin{array}{ccccc}
 A & \xleftarrow{\quad} & (A \rightarrow B) & \xrightarrow{\quad} & B \\
 p_A \downarrow & & \downarrow & & \downarrow p_B \\
 \mathbb{A}_v & \xleftarrow{s} & \mathbb{A}_V & \xrightarrow{t} & \mathbb{A}_v,
 \end{array} \tag{3}$$

so a first candidate follows by composition with the functor

$$el(\sigma) \rightarrow (A \rightarrow B)^{P^*} \hookrightarrow (A \rightarrow B).$$

208 However, the fibres of this functor over plays of odd length are all empty. We thus need to  
 209 insert additional objects over plays of odd length whose immediate prefix is accepted, thus  
 210 yielding a functor  $\overline{el(\sigma)} \rightarrow (A \rightarrow B)$  which at last induces the desired template strategy by  
 211 composition with (3).

212 We obtain the following result, which shows that even the more general simple strategies  
 213 do not cover all template strategies.

214 ▶ **Lemma 34.** *Both functors  $\overline{el(\sigma)} \rightarrow (A \rightarrow B) \rightarrow \mathbb{A}_v$  are discrete Conduché fibrations.*  
 215 *Furthermore, the former functor  $\overline{el(\sigma)} \rightarrow (A \rightarrow B)$  is receptive, in the sense that for all*  
 216 *even-length plays  $s$  with immediate extensions  $sm$ , and all  $(s, x) \in \overline{el(\sigma)}$ , there exists a unique*  
 217  *$(sm, y) \in \overline{el(\sigma)}$  and morphism  $(s, x) \rightarrow (sm, y)$  mapped to  $s \rightarrow sm$  by the projection.*

## 218 Refining template strategies

219 As we did for games, let us now analyse and resolve the discrepancies. Clearly, the root and  
 220 atomicity problems arise again. So a natural idea is to slice  $\mathbb{A}_v$  under  $OO$  and restrict to  
 221 discrete fibrations  $S \rightarrow OO/\mathbb{A}_v$ .

222 ▶ **Definition 35.** *Recalling Remark 13, naively refined template strategies are just as template*  
 223 *strategies in Definition 20, but over  $O/\mathbb{A}_v \xleftarrow{s} OO/\mathbb{A}_v \xrightarrow{t} O/\mathbb{A}_v$ , and with  $r$  a discrete fibration.*

224 We obtain a first improvement:

225 ▶ **Lemma 36.** *With  $s$  and  $t$  as in Lemma 19 but in slices,  $A \rightarrow B$  is a limit*

$$\begin{array}{ccccc}
 A & \xleftarrow{\quad} & (A \rightarrow B) & \xrightarrow{\quad} & B \\
 p_A \downarrow & & \downarrow & & \downarrow p_B \\
 O/\mathbb{A}_v & \xleftarrow{s} & OO/\mathbb{A}_v & \xrightarrow{t} & O/\mathbb{A}_v,
 \end{array} \tag{4}$$

227 and naively refined template strategies are equivalent to presheaves over  $A \rightarrow B$ .

228 In order to precisely capture simple strategies, it remains to account for receptiveness. For  
 229 this, our solution is to further restrict the template:

230 ▶ **Definition 37.** *Let  $\mathbb{T}_V$  denote the full subcategory of  $OO/\mathbb{A}_v$  spanning even-length schedules.*



231 ▶ **Remark 38.** It is tempting to further restrict to non-empty, even-length schedules. However,  
 232 the obtained span does not form a monad, hence the general template games construction  
 233 does not apply.

234 ▶ **Definition 39.** Refined template strategies are defined just as template strategies in Defini-  
 235 tion 20, but over  $\mathbb{T}_v \xleftarrow{s} \mathbb{T}_V \xrightarrow{t} \mathbb{T}_{v'}$ , and with  $r$  a discrete fibration.

236 We obtain a restriction of Lemma 36 to the relevant plays:

237 ▶ **Lemma 40.** The full subcategory  $(A \rightarrow B)^P \hookrightarrow (A \rightarrow B)$  spanning even-length plays is a  
 238 limit

$$\begin{array}{ccccc}
 A & \xleftarrow{\quad} & (A \rightarrow B)^P & \xrightarrow{\quad} & B \\
 p_A \downarrow & & \downarrow & & \downarrow p_B \\
 \mathbb{T}_v & \xleftarrow{s} & \mathbb{T}_V & \xrightarrow{t} & \mathbb{T}_{v'}
 \end{array} \tag{5}$$

240 and refined template strategies  $A \rightarrow B$  are equivalent to discrete fibrations over  $(A \rightarrow B)^P$ .

241 **Proof.** By repeated application of stability of discrete fibrations under pullback,  $(A \rightarrow$   
 242  $B)^P \rightarrow \mathbb{T}_V$  is a discrete fibration. Hence, in a diagram like (1) (with  $\mathbb{T}$  instead of  $\mathbb{A}$ ),  
 243 assuming  $p, q \in \mathbf{DFib}$ ,  $r$  is a discrete fibration iff the induced morphism to  $(A \rightarrow B)^P$  is. The  
 244 result then follows from Lemma 32. ◀

245 ▶ **Corollary 41.** There is a full, reflective embedding from simple strategies  $\mathcal{S}(A \rightarrow B)$   
 246 to refined template strategies, whose essential image consists of definite refined template  
 247 strategies, i.e., those whose associated presheaf  $X \in (A \rightarrow B)^P$  is such that  $X(\varepsilon) = 1$ .

248 **Proof.** The embedding  $(A \rightarrow B)^{P*} \hookrightarrow (A \rightarrow B)^P$  being full, right Kan extension along its  
 249 opposite defines a full, reflective embedding  $(A \rightarrow B)^{P*} \hookrightarrow (A \rightarrow B)^P$ , which returns definite  
 250 refined template strategies by the standard end formula. ◀

## 251 Summary

252 Until now, we have refined template games and strategies, first by replacing the original  
 253 template  $\mathbb{A}$  by our  $\mathbb{T}$ , and second by restricting template games and strategies to be discrete  
 254 fibrations. We have then identified simple games as definite template games, and constructed  
 255 a full, reflective embedding from simple strategies  $A \rightarrow B$  to refined template strategies  
 256  $p_A \rightarrow p_B$ , with essential image the definite ones. What remains to be seen is whether we can  
 257 refine Melliès's double-categorical construction accordingly.

## 258 4 Template games

259 In this section, we review Melliès's double-categorical variant of the slice construction. We  
 260 then apply it to deduce that the new template  $\mathbb{T}$  yields a weak double category as desired.  
 261 In the next section, we refine the construction using a factorisation system, which allows us  
 262 to account for restriction to discrete fibrations.

### 263 4.1 Double categories

264 The key point is that the template  $\mathbb{A}$  forms a *double category* [5], in a way that describes  
 265 the scheduling of composition of strategies. So let us first briefly review double categories.

266 A double category  $\mathcal{C}$  essentially consists of a *horizontal* category  $\mathcal{C}_h$  and a *vertical* one  $\mathcal{C}_v$  sharing the same object set, together with a set of  
 267 *cells* as on the right where  $A, B, C$ , and  $D$  are objects,  $f$  and  $g$  are morphisms in  $\mathcal{C}_h$ , and  $u$  and  $v$  are morphisms in  $\mathcal{C}_v$ . In order to distinguish  
 268 notationally between horizontal and vertical morphisms, we mark horizontal ones with a  
 269 bullet. Cells are furthermore equipped with composition and identities in both directions.  
 270 E.g., to any given cells  $\alpha$  and  $\beta$  with compatible vertical border is assigned a composite cell  
 271  $\beta \bullet \alpha$ , as below left. Similarly, we have horizontal identities  $id_p^\bullet$  as below right.

$$274 \quad \begin{array}{ccccc} A & \xrightarrow{\bullet s} & B & \xrightarrow{\bullet s'} & E \\ p \downarrow & \Downarrow \alpha & \downarrow q & \Downarrow \beta & \downarrow r \\ C & \xrightarrow{\bullet t} & D & \xrightarrow{\bullet t'} & F \end{array} \mapsto \begin{array}{ccc} A & \xrightarrow{\bullet s' \bullet s} & E \\ p \downarrow & \Downarrow \beta \bullet \alpha & \downarrow r \\ C & \xrightarrow{\bullet t' \bullet t} & F \end{array}, \quad \begin{array}{ccc} A & \xrightarrow{id_A^\bullet} & A \\ p \downarrow & \Downarrow id_p^\bullet & \downarrow p \\ B & \xrightarrow{id_B^\bullet} & B \end{array} \quad (6)$$

275 Both notions of composition are required to be associative and the corresponding identities  
 276 unital, giving two categories  $\mathcal{C}_H$  and  $\mathcal{C}_V$  of cells. Finally, the *interchange law* requires the  
 277 two different ways of parsing any compatible pasting as below to agree, i.e.,  $(\delta \circ \gamma) \bullet (\beta \circ \alpha) =$   
 278  $(\delta \bullet \beta) \circ (\gamma \bullet \alpha)$ .

$$\begin{array}{ccccc} A & \xrightarrow{\bullet} & B & \xrightarrow{\bullet} & C \\ \downarrow & \Downarrow \alpha & \downarrow & \Downarrow \gamma & \downarrow \\ D & \xrightarrow{\bullet} & E & \xrightarrow{\bullet} & F \\ \downarrow & \Downarrow \beta & \downarrow & \Downarrow \delta & \downarrow \\ G & \xrightarrow{\bullet} & H & \xrightarrow{\bullet} & I \end{array}$$

## 279 4.2 The template $\mathfrak{A}$ as a double category

280 Let us now show that the template  $\mathfrak{A}$  forms a double category. As suggested by the notation,  
 281 its vertical category and vertical category of cells are  $\mathfrak{A}_v$  and  $\mathfrak{A}_V$ . Its horizontal category  $\mathfrak{A}_h$   
 282 is generated by the graph  $O \rightarrow P$ . It is thus isomorphic to the ordinal  $\mathbf{2}$ : there are exactly  
 283 three horizontal morphisms:  $OO$ ,  $PP$ , and  $OP$ . To complete the definition, it remains to  
 284 define composition and identities in  $\mathfrak{A}_H$ . One way is to depict basic cells as triangles

$$285 \quad \begin{array}{c} O \xrightarrow{\bullet} O \\ \searrow \bullet \downarrow \\ P \end{array} \quad \begin{array}{c} O \xrightarrow{\bullet} P \\ \downarrow \bullet \nearrow \\ P \end{array} \quad \begin{array}{c} P \xrightarrow{\bullet} P \\ \downarrow \bullet \nearrow \\ O \end{array} \quad \begin{array}{c} O \xrightarrow{\bullet} P \\ \searrow \bullet \downarrow \\ O \end{array}, \quad (7)$$

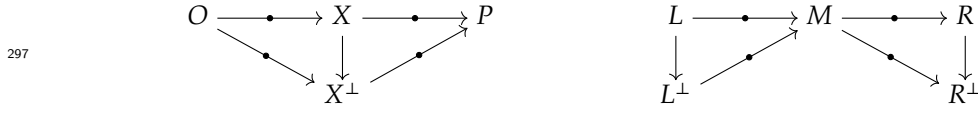
286 respectively denoting  $OO \rightarrow OP$ ,  $OP \rightarrow PP$ ,  $PP \rightarrow OP$ , and  $OP \rightarrow OO$ . General cells are  
 287 obtained by stacking up such basic triangles. Depicting cells as stacks of triangles yields the  
 288 following inductive definition of composition of cells  $\alpha$  and  $\beta$  as in (6):

289 ■ If there is an ‘outwards’ bottom triangle, i.e., the bottom of  $\alpha$  and  $\beta$  look like either of

$$290 \quad \begin{array}{ccc} O & \xrightarrow{\bullet} & O \\ & \searrow \bullet & \downarrow \\ & & X^\perp \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\bullet} & P \\ \downarrow & \bullet \nearrow & \\ X^\perp & & \end{array}$$

291 with  $X \in \{O, P\}$  and  $X^\perp$  denoting the other player, then the composite is obtained by  
 292 composing the rest of  $\alpha$  and  $\beta$ , and appending the obvious triangle  $((OX, OX^\perp)$ , resp.  
 293  $(XP, X^\perp P)$ .

294 ■ Otherwise, there is a pair of interacting bottom triangles, as below left, in which case  
 295 the composite is simply the composite of the rest of  $\alpha$  and  $\beta$  – which is precisely where  
 296 game semantical *hiding* is encoded in  $\mathfrak{A}$ .



► Remark 42. Ambiguous configurations as above right, where we would not know which triangle to put last in the composite, cannot occur. Indeed, existence of the left-hand triangle forces  $M = P$ , while existence of the right-hand one forces  $M = O$ . Horizontal identities are the so-called *copycat* schedules. The copycat on the vertical morphism  $O \rightarrow P$  is obtained by composing  $OO \rightarrow OP$  and  $OP \rightarrow PP$ , and dually for  $P \rightarrow O$ . The copycat schedule of a general morphism is the obvious composite of these basic copycats.

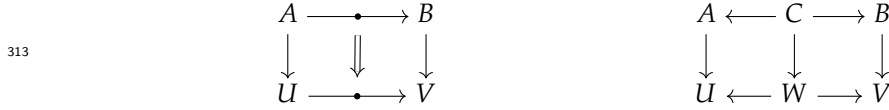
We obtain as promised:

► Proposition 43 (Melliès [13]). *The template  $\pm$  forms a double category.*

### 4.3 Template games as a double slice

There is an alternative point of view on double categories which will be crucial to us: they may be axiomatised based on a span of functors  $\mathcal{C}_v \xleftarrow{s} \mathcal{C}_V \xrightarrow{t} \mathcal{C}_v$ . For this, let us consider the following structure, which is almost a (large) double category.

► Definition 44. Let **Span**(Cat) have as objects all small categories, as vertical morphisms all functors, as horizontal morphisms  $A \multimap B$  all spans  $A \leftarrow C \rightarrow B$  of functors, and as cells below left all commuting diagrams as below right in Cat.



Vertical composition is given by (componentwise) composition of functors, while horizontal composition is given by pullbacks and their universal property.

The structure formed by **Span**(Cat) is a *weak double category* [6], a weak form of double category where horizontal composition is only associative and unital up to coherent isomorphism, in a suitable sense.

► Remark 45. The horizontal arrows and *special* cells of a weak double category  $\mathcal{C}$  form a bicategory  $\mathcal{H}(\mathcal{C})$ , where special means that the left and right borders are identities.

► Remark 46. The reason **Span**(Cat) is weak is that one cannot hope to make a strictly associative choice of pullbacks.

Just like one usually does in bicategories, we may define monads internally to weak double categories.

► Definition 47. A monad in a weak double category  $\mathcal{C}$  is a horizontal morphism  $M: X \multimap X$ , equipped with special cells



satisfying the obvious generalisation of the usual monad laws.

329 ▶ **Proposition 48.** *A double category is the same as a monad in  $\mathbf{Span}(\mathbf{Cat})$ .*

330 **Proof.** Composing an endo-span  $\mathcal{C}_v \xleftarrow{s} \mathcal{C}_V \xrightarrow{t} \mathcal{C}_v$  with itself in  $\mathbf{Span}(\mathbf{Cat})$  amounts to  
 331 constructing the category of pairs of compatible horizontal morphisms and cells (6), so  
 332 requiring a monad multiplication is requiring horizontal composition. ◀

333 Explicitly, if  $\mathcal{C}$  is a double category, then it can be seen as a monad  $\mathcal{C}_V: \mathcal{C}_v \rightarrow \mathcal{C}_v$  in  
 334  $\mathbf{Span}(\mathbf{Cat})$  with  $\mu$  and  $\eta$  given by horizontal composition and identities.

335 It should now be clear that a template game is a vertical morphism to  $\mathfrak{x}_v$  in  $\mathbf{Span}(\mathbf{Cat})$ ,  
 336 while a template strategy  $A \rightarrow B$  is merely a cell

$$\begin{array}{ccc}
 A & \xrightarrow{\quad \dot{S} \quad} & B \\
 p \downarrow & \Downarrow & \downarrow q \\
 \mathfrak{x}_v & \xrightarrow{\quad \dot{\mathfrak{x}}_V \quad} & \mathfrak{x}_v.
 \end{array}$$

338 This allows us to define composition of template strategies, using the monad structure of  $\mathfrak{x}$   
 339 given by Propositions 48 and 43.

340 ▶ **Proposition 49.** *The composite of  $S: A \rightarrow B$  and  $T: B \rightarrow C$  is the pasting below left, while  
 341 the identity on any  $p: A \rightarrow \mathfrak{x}_v$  is the one below right.*

$$\begin{array}{ccc}
 A & \xrightarrow{\quad \dot{S} \quad} & B & \xrightarrow{\quad \dot{T} \quad} & C \\
 p \downarrow & \Downarrow & \downarrow q & \Downarrow & \downarrow r \\
 \mathfrak{x}_v & \xrightarrow{\quad \dot{\mathfrak{x}}_V \quad} & \mathfrak{x}_v & \xrightarrow{\quad \dot{\mathfrak{x}}_V \quad} & \mathfrak{x}_v \\
 & \searrow \mu & & \nearrow \mu & \\
 & \mathfrak{x}_V & & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\quad \bullet \quad} & A \\
 p \downarrow & \Downarrow id_p & \downarrow p \\
 \mathfrak{x}_v & \xrightarrow{\quad \bullet \quad} & \mathfrak{x}_v \\
 & \searrow \eta & \nearrow \eta \\
 & \mathfrak{x}_V & 
 \end{array}
 \tag{8}$$

343 From this, it easily follows that strategies in fact form a weak double category, and clearly  
 344 this construction works for any monad in any weak double category. Namely, Mellies's  
 345 construction may be obtained by applying the following theorem.

346 ▶ **Theorem 50.** *Given any monad  $M_V: M_v \rightarrow M_v$  in a weak double category  $\mathcal{C}$ , there is a  
 347 slice weak double category  $\mathcal{C}/M$  whose*

- 348 ■ *vertical category is  $(\mathcal{C}/M)_v = \mathcal{C}_v/M_v$ ;*
- 349 ■ *vertical category of cells is  $(\mathcal{C}/M)_V = \mathcal{C}_V/M_V$ ;*
- 350 ■ *horizontal composition is given by pasting with  $\mu$ , as on the left in (8);*
- 351 ■ *horizontal identity on  $p: A \rightarrow M_v$  is by pasting with  $\eta$  as on the right in (8).*
- 352 ■ *and all operations on cells are given by their counterparts in  $\mathcal{C}$ .*

353 ▶ **Proposition 51.** *The weak double category  $\mathbf{Games}(\mathfrak{x})$  of template games [13] is equal to  
 354  $\mathbf{Span}(\mathbf{Cat})/\mathfrak{x}$ .*

355 A weak double category with trivial category  $\mathcal{C}_v$  is nothing but a monoidal category. In  
 356 that case, the theorem reduces to the following well-known result used, e.g., in Weber [16]:

357 ▶ **Corollary 52.** *The slice of a monoidal category over a monoid is again monoidal.*

## 358 5 Refined template games, simple games, and Day convolution

### 359 5.1 Refined template games

We now want to recover simple games by replacing  $\pm$  with  $\mathbb{T}$ , and restricting the slice construction in  $\mathbf{Span}(\mathbf{Cat})$  to discrete fibrations (for vertical morphisms and cells). For this, we appeal to factorisation systems.

$$\begin{array}{ccc} A & \xrightarrow{\quad} & C \\ f \downarrow & \dashrightarrow k & \downarrow g \\ B & \xrightarrow{\quad} & D \end{array}$$

► **Definition 53.** For all morphisms  $f$  and  $g$  in a category  $\mathcal{C}$ , let  $f \perp g$  iff for all commuting squares as above right, there exists a unique  $k$  as shown making both triangles commute. For any class  $\mathcal{M}$  of morphisms, let  $\mathcal{M}^\perp = \{g \mid \forall f \in \mathcal{M}, f \perp g\}$ . We define  ${}^\perp \mathcal{M}$  symmetrically.

A (strong) factorisation system on a category  $\mathcal{C}$  consists in classes  $\mathcal{L}$  and  $\mathcal{R}$  of arrows such that  $\mathcal{L}^\perp = \mathcal{R}$ ,  $\mathcal{L} = {}^\perp \mathcal{R}$ , and every arrow factors as  $r \circ l$  with  $l \in \mathcal{L}$  and  $r \in \mathcal{R}$ .

► **Example 54.** Discrete fibrations form the right class of the *comprehensive* factorisation system  $(\mathbf{Fin}, \mathbf{DFib})$  on  $\mathbf{Cat}$ , whose left class is that of *final* functors [15].

Our refined construction is based on the following generalisation of factorisation systems.

► **Definition 55.** A double factorisation system on a weak double category  $\mathcal{C}$  consists of factorisation systems  $(\mathcal{L}_v, \mathcal{R}_v)$  and  $(\mathcal{L}_V, \mathcal{R}_V)$  on  $\mathcal{C}_v$  and  $\mathcal{C}_V$ , respectively, such that the source and target functors  $\mathcal{C}_V \rightarrow \mathcal{C}_v$  both map  $\mathcal{L}_V$  to  $\mathcal{L}_v$  and  $\mathcal{R}_V$  to  $\mathcal{R}_v$ , and all cells  $\alpha$  as below left with  $\ell, \ell' \in \mathcal{L}_v$  and  $r, r' \in \mathcal{R}_v$  factor as below right, with  $\lambda \in \mathcal{L}_V$  and  $\rho \in \mathcal{R}_V$ .

$$\begin{array}{ccc} A & \xrightarrow{\quad \bullet \quad} & A' \\ \ell \downarrow & \Downarrow \alpha & \downarrow \ell' \\ B & & B' \\ r \downarrow & & \downarrow r' \\ C & \xrightarrow{\quad \bullet \quad} & C' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\quad \bullet \quad} & A' \\ \ell \downarrow & \Downarrow \lambda & \downarrow \ell' \\ B & \xrightarrow{\quad \bullet \quad} & B' \\ r \downarrow & \Downarrow \rho & \downarrow r' \\ C & \xrightarrow{\quad \bullet \quad} & C' \end{array}$$

► **Lemma 56.** Discrete fibrations and componentwise discrete fibrations are the right classes of a double factorisation system  $((\mathbf{Fin}, \mathbf{DFib}), (\mathbf{Fin}_V, \mathbf{DFib}_V))$  on  $\mathbf{Span}(\mathbf{Cat})$ . Furthermore,  $\mathbf{DFib}_V$  is stable under horizontal composition and identities.

**Proof.** As is well-known, discrete fibrations may be defined by unique lifting w.r.t. the injection  $\mathbf{1} \hookrightarrow \mathbf{2}$  mapping 0 to 1. Componentwise discrete fibrations may be defined similarly, in  $\mathbf{Span}(\mathbf{Cat})_V$ . The result then follows from componentwise discrete fibrations being stable under horizontal composition, which holds because they are stable under pullback in the arrow category. ◀

In order to state the promised generalisation of Theorem 50 in full generality, we need the following, somewhat awkward notion of stability of left residuals. Indeed, our two applications work for rather different reasons, as emphasised by Corollaries 60 and 61 below.

► **Definition 57.** A monad  $M_V: M_v \rightarrow M_v$  in a weak double category  $\mathcal{C}$  has stable left residuals w.r.t. a double factorisation system  $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$ , iff

(a) for all  $\alpha, \beta \in \mathcal{R}_V$  such that the composite below left factors as on the right,

$$\begin{array}{ccc} A & \xrightarrow{\quad \bullet \quad} & C & \xrightarrow{\quad \bullet \quad} & B \\ \downarrow & \Downarrow \alpha & \downarrow & \Downarrow \beta & \downarrow \\ M_v & \xrightarrow{\quad \bullet \quad} & M_v & \xrightarrow{\quad \bullet \quad} & M_v \\ & & \Downarrow \mu & & \\ & & M_v & & \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\quad \bullet \quad} & B \\ \downarrow & \Downarrow \lambda & \downarrow \\ M_v & \xrightarrow{\quad \bullet \quad} & M_v \\ & \Downarrow \rho & \\ & M_v & \end{array} \quad (9)$$

for any  $S: A' \rightarrow A$  and  $T: B \rightarrow B'$  in  $\mathcal{C}_h$ , the composite  $\text{id}_T \bullet \lambda \bullet \text{id}_S$  below is in  $\mathcal{L}_V$ ;

$$A' \xrightarrow{S} A \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \lambda \\ \xrightarrow{\quad} \end{array} B \xrightarrow{T} B',$$

(b) for all  $p \in \mathcal{R}_v$  such that the composite below left factors as on the right, for any  $S: A' \rightarrow A$  and  $T: B \rightarrow B'$  in  $\mathcal{C}_h$ ,  $\text{id}_T \bullet \lambda \bullet \text{id}_S$  is in  $\mathcal{L}_V$ .

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A \\ p \downarrow & \Downarrow \text{id}_p & \downarrow p \\ M_v & \xrightarrow{\quad} & M_v \\ & \Downarrow \eta & \\ & M_V & \end{array} = \begin{array}{ccc} A & \xrightarrow{N} & B \\ \downarrow & \Downarrow \lambda & \downarrow \\ M_v & \xrightarrow{\quad} & M_v \\ & \Downarrow \rho & \\ & M_V & \end{array} \quad (10)$$

► Remark 58. In both cases, existence of a special  $\lambda$  follows by Definition 55.

► Theorem 59. Consider any monad  $M_V: M_v \rightarrow M_v$  in a weak double category  $\mathcal{C}$  with stable left residuals w.r.t. a double factorisation system  $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$ . Then there is a slice weak double category  $\mathcal{C}/_{\mathcal{R}_V} M$  whose

- vertical category  $(\mathcal{C}/_{\mathcal{R}_V} M)_v$  is  $\mathcal{C}_v/\mathcal{R}_v M_v$ , the full subcategory of  $\mathcal{C}_v/M_v$  on maps in  $\mathcal{R}_v$ ;
- vertical category of cells is  $(\mathcal{C}/_{\mathcal{R}_V} M)_V = \mathcal{C}_V/\mathcal{R}_V M_V$ ;
- horizontal composition is given by  $\rho$  in (9);
- horizontal identity on any  $p: A \rightarrow M_v$  is given by  $\rho$  in (10);
- and all operations on cells are given by their counterparts in  $\mathcal{C}$ .

Proof. See Appendix A. ◀

► Corollary 60. Consider any monad  $M_V: M_v \rightarrow M_v$  in a weak double category  $\mathcal{C}$  with double factorisation system  $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$ . If  $\eta, \mu \in \mathcal{R}_V$  and  $\mathcal{R}_V$  is stable under horizontal composition and identities, then  $\mathcal{C}/_{\mathcal{R}_V} M$  exists and is a sub weak double category of  $\mathcal{C}/M$ .

Proof. Both pastings on the left of (9) and (10) are already in  $\mathcal{R}_V$ . ◀

► Corollary 61. Consider any monad  $M_V: M_v \rightarrow M_v$  in a weak double category  $\mathcal{C}$  with double factorisation system  $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$ . If  $\mathcal{L}_V$  is stable under horizontal composition, then  $\mathcal{C}/_{\mathcal{R}_V} M$  is a weak double category.

Proof. Stability under horizontal composition entails stability under whiskering. ◀

## 5.2 Simple games

Let us at last return to simple games. We have:

► Lemma 62.  $\mathbb{T}$  is a monad whose multiplication and unit are in  $\text{DFib}_V$ .

Proof. In  $\mathbb{T}$ , multiplication is composition of schedules (through parallel composition and hiding) and the unit is given by copycat schedules. The crucial point to prove that multiplication is in  $\text{DFib}_V$  is that, for any pair  $(p, q)$  of schedules in  $\mathbb{T}_V \bullet \mathbb{T}_V$ , the last move in  $s(q)$  ( $= t(p)$ ) cannot be last in both  $p$  and  $q$  for polarity reasons. For the unit, the crucial point is that the restriction of a copycat schedule is itself copycat. ◀

By Corollary 60, we have:

► Corollary 63.  $\text{Span}(\text{Cat})/\text{DFib}_V \mathbb{T}$  is a sub weak double category of  $\text{Span}(\text{Cat})$ .

Finally, let us relate to simple strategies.

425 ▶ **Definition 64** ([11, Definition 9]). *Simple games, strategies, and natural transformations*  
 426 *form a bicategory  $\mathcal{S}$ .*

▶ **Theorem 65.** *The full, reflective embeddings  $F_{A,B}: \mathcal{S}(A,B) \xrightarrow{\sim} \mathbf{DFib}((A \rightarrow B)^P)$  of Corollary 41 determine a locally reflective and fully-faithful weak 2-functor*

$$\mathcal{S} \rightarrow \mathcal{H}(\mathbf{Span}(\mathbf{Cat})/\mathbf{DFib}_V \mathbb{T})$$

427 (where  $\mathcal{H}$  is the bicategory of special cells, as in Remark 45), whose essential image consists  
 428 of definite refined template games and strategies.

429 **Proof.** The essential image part of the result is clear. By [12, §2.2], we need to organise the  
 430 full, reflective embeddings  $F_{A,B}$  into a weak 2-functor, which here means that  $F$  commutes  
 431 with composition of strategies up to coherent isomorphism. ◀

432 ▶ **Corollary 66.** *The bicategory  $\mathcal{S}$  of simple games and strategies is biequivalent to the*  
 433 *locally full sub-bicategory of  $\mathcal{H}(\mathbf{Span}(\mathbf{Cat})/\mathbf{DFib}_V \mathbb{T})$  spanning definite refined template games*  
 434 *and strategies.*

### 435 5.3 Day convolution

436 We finally reach the application mentioned in the introduction: Day convolution. The  
 437 purpose of this operation is to show that  $\widehat{\mathbf{C}}$  is monoidal when  $\mathbf{C}$  is. Let us now recover this  
 438 structure from Theorem 59, in the particular case where  $\mathbf{C}$  is strictly monoidal. The starting  
 439 point is the sub weak double category, say  $\mathcal{W}$ , of  $\mathbf{Span}(\mathbf{Cat})$ , obtained by restricting attention  
 440 to just one object and one vertical morphism, namely the terminal category  $\mathbf{1}$  and the  
 441 identity thereon. Thus,  $\mathcal{W}_V$  consists of categories and functors, and horizontal composition  
 442 is given by cartesian product. Furthermore, a monad in  $\mathcal{W}$  is nothing but a monoid in  
 443  $\mathbf{Cat}$  for the cartesian product, i.e., a strict monoidal category  $\mathbf{C}$ . Clearly, final functors and  
 444 discrete fibrations form a double factorisation system  $(\mathcal{L}_V, \mathcal{R}_V) = (\mathbf{Fin}, \mathbf{DFib})$  on  $\mathcal{W}$ . Final  
 445 functors being closed under binary products,  $\mathcal{L}_V$  is closed under horizontal composition  
 446 and identities, so Corollary 61 applies and we obtain a weak double category  $\mathcal{W}/\mathbf{DFib} \mathbf{C}$ . This  
 447 weak double category is vertically trivial, hence underlies a monoidal category, say  $\mathbf{C}'$ .

448 ▶ **Theorem 67.** *For any strictly monoidal category  $\mathbf{C}$ , the monoidal category  $\mathbf{C}'$  is equivalent*  
 449 *to  $\widehat{\mathbf{C}}$  equipped with the convolution tensor product.*

450 **Proof.** See Appendix B ◀

## 451 6 Conclusion and perspectives

452 We have designed an abstract slice construction over monads in weak double categories,  
 453 which has as instances (1) a weak double category of simple games and concurrent strategies,  
 454 whose underlying bicategory of definite games and strategies is biequivalent to the standard  
 455 one, and (2) the monoidal category of presheaves over any small strict monoidal category.

456 For future work, we first could try to accomodate not only the weak double category  
 457 structure of template games, but also symmetric monoidal closedness. Furthermore, Melliès  
 458 is also currently working on a template game model of full linear logic. This will of course  
 459 be a useful feature to incorporate to our framework. Another direction for future work is to  
 460 generalise our construction to encompass Day convolution for non-strict monoidal categories.  
 461 What is needed here is a common generalisation of [7, Theorem 14.5] and Theorem 59.

Finally, it is slightly unsatisfactory to only get standard simple games up to a locally fully-faithful embedding. In order to obtain a biequivalence, preliminary investigation suggests that instead of restricting the slice construction w.r.t. some factorisation system, it would be more expressive to construct the factorisation system directly in the relevant slice, for which fibrant objects would be the desired strategies. Indeed, e.g., polarities, which cannot be used before slicing, become available in the slice. This technique also seems to apply for refining the correspondence to, e.g., deterministic strategies, and possibly even innocence.

## References

- 1 A. K. Bousfield. Constructions of factorization systems in categories. *J. Pure and Appl. Alg.*, 9(2-3):287–329, 1977.
- 2 B. Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, volume 137 of *Lecture Notes in Math.*, pages 1–38. Springer, 1970.
- 3 C. Eberhart. *Catégories et diagrammes de cordes pour les jeux concurrents*. PhD thesis, Université Savoie Mont Blanc, 2018.
- 4 C. Eberhart and T. Hirschowitz. What’s in a game?: A theory of game models. In *Proc. 33rd Logic in Comp. Sci.*, pages 374–383. ACM, 2018.
- 5 C. Ehresmann. Catégories structurées. *Annales scientifiques de l’Ecole Normale Supérieure*, 80(4):349–426, 1963.
- 6 R. Garner. *Polycategories*. PhD thesis, University of Cambridge, 2006.
- 7 R. Garner and M. Shulman. Enriched categories as a free cocompletion. *Advances in Mathematics*, 289:1–94, 2016.
- 8 R. Harmer, J. M. E. Hyland, and P.-A. Melliès. Categorical combinatorics for innocent strategies. In *Proc. 22nd Logic in Comp. Sci.*, pages 379–388. IEEE, 2007.
- 9 M. Hirschowitz, A. Hirschowitz, and T. Hirschowitz. A theory for game theories. In *Proc. 27th Found. of Soft. Tech. and Theor. Comp. Sci.*, volume 4855 of *Lecture Notes in Comp. Sci.*, pages 192–203. Springer, 2007.
- 10 J. M. E. Hyland. Game semantics. In Andrew M. Pitts and Peter Dybjer, editors, *Semantics and Logics of Computation*, pages 131–184. Cambridge University Press, 1997.
- 11 C. Jacq and P.-A. Melliès. Categorical combinatorics for non deterministic strategies on simple games. In *Proc. 21st Foundations Soft. Science Comp. Structures*, volume 10803 of *Lecture Notes in Comp. Sci.*, pages 39–70. Springer, 2018.
- 12 T. Leinster. Basic Bicategories. *arXiv Mathematics e-prints*, page math/9810017, October 1998. [arXiv:math/9810017](https://arxiv.org/abs/math/9810017).
- 13 P.-A. Melliès. Categorical combinatorics of scheduling and synchronization in game semantics. In *Proc. 46th Int. Symp. Principles of Progr. Lang.* ACM, 2019.
- 14 S. Rideau and G. Winskel. Concurrent strategies. In *Proc. 26th Logic in Comp. Sci.*, pages 409–418. IEEE, 2011.
- 15 R. Street and R. F. C. Walters. The comprehensive factorization of a functor. *Bull. AMS*, 79(5), 1973.
- 16 M. Weber. Generic morphisms, parametric representations and weakly cartesian monads. *Theory and Applications of Categories*, 13:191–234, 2004.

## A Proof of Theorem 59

By coherence for weak double categories (M. Grandis and R. Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999,



Theorem 7.5), and assuming a higher universe in which  $\mathcal{C}$  is small, we may assume that  $\mathcal{C}$  is in fact a (strict) double category.

Composition of cells in  $\mathcal{C}/\mathcal{R}_V M$  is just as in  $\mathcal{C}$ , so the only non-trivial point to check is weak associativity and unitality of horizontal composition (of morphisms). For weak associativity, we observe that both cells

$$\begin{array}{ccc}
 A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D & & A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D \\
 p \downarrow \alpha \downarrow q \downarrow \beta \downarrow r \downarrow \gamma \downarrow s \downarrow & & p \downarrow \alpha \downarrow q \downarrow \beta \downarrow r \downarrow \gamma \downarrow s \downarrow \\
 M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v & & M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v \\
 \mu \searrow \mu \searrow \mu \searrow \mu & & \mu \searrow \mu \searrow \mu \searrow \mu \\
 & M_v & & M_v
 \end{array}$$

are equal. Now, denoting composition in  $(\mathcal{C}/\mathcal{R}_V M)_h$  by  $\tilde{\bullet}$ ,  $\gamma \tilde{\bullet} (\beta \tilde{\bullet} \alpha)$  and  $(\gamma \tilde{\bullet} \beta) \tilde{\bullet} \alpha$  are obtained by factoring them as follows. For the former, we factor

$$T \bullet S \xrightarrow{\beta \bullet \alpha} M_V \bullet M_V \xrightarrow{\mu} M_V \quad \text{as} \quad T \bullet S \xrightarrow{\lambda_{\beta, \alpha}} K_{\beta, \alpha} \xrightarrow{\rho_{\beta, \alpha}} M_V,$$

in which, by definition,  $\lambda_{\beta, \alpha}$  has identity left and right borders, i.e., is *special*. We then factor

$$U \bullet K_{\beta, \alpha} \xrightarrow{\gamma \bullet \rho_{\beta, \alpha}} M_V \bullet M_V \xrightarrow{\mu} M_V \quad \text{as} \quad U \bullet K_{\beta, \alpha} \xrightarrow{\lambda_{\gamma, (\beta, \alpha)}} K_{\gamma, (\beta, \alpha)} \xrightarrow{\rho_{\gamma, (\beta, \alpha)}} M_V.$$

The other composite may be computed symmetrically, so that we obtain factorisations:

$$\begin{array}{ccc}
 A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D & & A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D \\
 p \downarrow \alpha \downarrow q \downarrow \beta \downarrow r \downarrow \gamma \downarrow s \downarrow & & p \downarrow \alpha \downarrow q \downarrow \beta \downarrow r \downarrow \gamma \downarrow s \downarrow \\
 M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v & & M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v \xrightarrow{\mu} M_v \\
 \mu \searrow \mu \searrow \mu \searrow \mu & & \mu \searrow \mu \searrow \mu \searrow \mu \\
 & M_v & & M_v
 \end{array}$$

By stability of left residuals, both are in fact factorisations for  $(\mathcal{L}_V, \mathcal{R}_V)$ , so that by lifting, we obtain a special cell  $a_{\alpha, \beta, \gamma}: K_{\gamma, (\beta, \alpha)} \rightrightarrows K_{(\gamma, \beta), \alpha}$  such that  $\rho_{(\gamma, \beta), \alpha} \circ a_{\alpha, \beta, \gamma} = \rho_{\gamma, (\beta, \alpha)}$ , which is our candidate associator for  $\mathcal{C}/\mathcal{R}_V M$ . It satisfies the MacLane pentagon by uniqueness of lifting.

Weak unitality follows similarly.

## B Proof of Theorem 67

Let us first recall that the convolution tensor product is given as follows:

► **Definition 68.** For any small monoidal category  $\mathbb{C}$  and  $X, Y \in \widehat{\mathbb{C}}$ , let

$$(X \otimes Y)(c) = \int^{(c_1, c_2) \in \mathbb{C}^2} X(c_1) \times Y(c_2) \times \mathbb{C}(c, c_1 \otimes c_2).$$

In order to prove Theorem 67, let us first show:

► **Lemma 69.** Let  $f: A \rightarrow B$  be a functor. The discrete fibration  $\rho_f$  associated to  $f$  is determined up to isomorphism by

$$\partial^\star(\rho_f)(b) \cong \int^{a \in A} B(b, f(a)),$$

where  $\partial^\star: \text{DFib}(B) \rightarrow \widehat{B}$  is the standard equivalence between discrete fibrations and presheaves.

528 **Proof.** This is actually obvious by construction. In [15], the dual case is actually treated,  
 529 initial functors and discrete opfibrations. But up to this discrepancy,  $\partial^\star(f)$  is precisely  $k$  in  
 530 the proof of [15, Theorem 3], which would in our case be defined as the left Kan extension  
 531 of  $A^{op} \xrightarrow{!} 1 \xrightarrow{1} \mathbf{Set}$  along  $f^{op}$ . By the well-known characterisation of left Kan extensions by  
 532 coends, we readily obtain the desired formula. ◀

**Proof of Theorem 67.** By construction, given two presheaves  $X, Y \in \widehat{\mathbb{C}}$  and transporting them to their corresponding discrete fibrations, say  $S: el(X) \rightarrow \mathbb{C}$  and  $T: el(Y) \rightarrow \mathbb{C}$ , their tensor product  $S \bullet T$  in  $\mathbb{C}'$  is the right factor of the composite

$$el(X) \times el(Y) \xrightarrow{S \times T} \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}.$$

533 By Lemma 69, the result has its corresponding presheaf defined up to isomorphism by

$$\begin{aligned} \partial^\star(S \bullet T)(c) &\cong \int^{(a,b) \in el(X) \times el(Y)} \mathbb{C}(c, \otimes((S \times T)(a, b))) \\ &= \int^{(a,b) \in el(X) \times el(Y)} \mathbb{C}(c, S(a) \otimes T(b)) \\ &\cong \int^{((c_1, x), (c_2, y)) \in el(X) \times el(Y)} \mathbb{C}(c, c_1 \otimes c_2) \\ 534 &\cong \int^{c_1, c_2} X(c_1) \times Y(c_2) \times \mathbb{C}(c, c_1 \otimes c_2), \end{aligned}$$

535 as desired. ◀