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Simple game semantics and Day convolution

CLOVIS EBERHART, National Institute of Informatics, Japan

TOM HIRSCHOWITZ, Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, France

ALEXIS LAOUAR, Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, France

Game semantics has provided adequate models for a variety of programming languages [13], in which types are interpreted as two-player games and programs as strategies. Melliès [14] suggested that such categories of games and strategies may be obtained as instances of a simple abstract construction on weak double categories. However, in the particular case of simple games [10], his construction slightly differs from the standard category. We refine the abstract construction using factorisation systems, and show that the new construction yields the standard category of simple games and strategies. Another perhaps surprising instance is Day’s *convolution* monoidal structure on the category of presheaves over a monoidal category.

1 INTRODUCTION

The construction of most game models follows a common pattern. Typically, a function $A \rightarrow B$ is interpreted as a strategy on a compound game made of A and B , where the program plays as *Proponent* (P) on B and as *Opponent* (O) on A . A crucial feature of game models is composition of strategies, by which two strategies $\sigma: A \rightarrow B$ and $\tau: B \rightarrow C$ yield a strategy $\tau \circ \sigma: A \rightarrow C$. Intuitively, $\tau \circ \sigma$ lets σ and τ interact on B until one of them produces a move in A or C . However, in order to obtain a strategy $A \rightarrow C$, everything occurring on B should be hidden.

Although widely acknowledged, this strong commonality is also recognised as poorly understood, particularly in the presence of *innocence*, a constraint on strategies that restricts them to purely functional behaviour. This has prompted a number of attempts at clarifying the situation [5, 10, 11]. In particular, Melliès [14] recently proposed a novel explanation, of unprecedented simplicity. Indeed, it rests upon a purely categorical construction, essentially taking the slice of a weak double category over an internal monad. This suggests that the approach may encompass a wide variety of game models, which is currently not the case as it is restricted to linear languages. More generally, it may prompt new connections between game semantics and other settings.

In this paper, we focus on a peculiar feature of Melliès’s approach, namely the slight discrepancy between his category of games and strategies and the standard one. Indeed, while standard strategies only play one move at a time, Melliès’s may play several moves simultaneously. This raises the question of whether the standard setting may be recovered by refining his approach. We answer this question positively by enriching the setting with a factorisation system [2]: using a double categorical variant of the so-called comprehensive factorisation system [15], we obtain the standard setting as an instance.

As a bonus, one of the connections alluded to above is established. Namely, we show that the Day *convolution* product [3] arises as an instance of our refined framework, though only in the restricted case of *strict* monoidal categories. The convolution product, which arose in algebraic topology, extends the monoidal structure of a given category \mathbb{C} to the category $\widehat{\mathbb{C}}$ of *presheaves* on \mathbb{C} , i.e., contravariant functors $\mathbb{C}^{op} \rightarrow \mathbf{Set}$. This makes formal the similarity, noted in [4, §6.5], between convolution and composition of strategies, by showing that both are instances of the same construction.

Authors’ addresses: Clovis Eberhart, National Institute of Informatics, Tokyo, Japan, clovis.eberhart@gmail.com; Tom Hirschowitz, Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000, Chambéry, France, tom.hirschowitz@univ-smb.fr; Alexis Laouar, Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000, Chambéry, France, alexis.laouar@ens-cachan.fr.

Related work

Beyond Melliès’s obviously related work and [11], Garner and Shulman [8] prove results related to Theorems 2.6 and 3.7. The common ground for comparison is the restriction of our Theorem 2.6 to weak double categories with trivial vertical category, i.e., monoidal categories. Their Theorem 14.2 is a generalisation in another direction, namely that of monoidal bicategories, and their Theorem 14.5 could in particular accomodate various sorts of bicategorical factorisation systems. What is needed for dealing with Day convolution in full generality (as mentioned in Section 4) is a common generalisation of their Theorem 14.5 and our Theorem 3.7.

Plan

In Section 2, after briefly reviewing *double categories* [6], a 2-dimensional generalisation of categories, we recall the cornerstone of Melliès’s variant of simple games, the double category \mathcal{A} . We then explain how Melliès’s bicategory of simple games may be obtained by a double categorical generalisation of the slice construction. In Section 3, we then refine this construction. We introduce *double factorisation systems* and show that restricting a slice weak double category to members of the right class of a double factorisation system again yields a weak double category. Finally, we observe that this construction has both standard simple games and Day convolution as instances. We conclude and give some perspective on future work in Section 4.

2 MELLIÈS’S SIMPLE GAMES

The cornerstone of Melliès’s account of simple games is a double category \mathcal{A} (called the *clock*) which embodies the essence of scheduling. So let us briefly recall what a double category is, and then describe \mathcal{A} .

2.1 Recap on double categories

A double category \mathcal{C} essentially consists of two categories \mathcal{C}_h and \mathcal{C}_v sharing the same object set, together with a set of *cells*

$$\begin{array}{ccc} A & \xrightarrow{f \bullet} & B \\ u \downarrow & \Downarrow \alpha & \downarrow v \\ C & \xrightarrow{g \bullet} & D, \end{array}$$

where A, B, C , and D are objects, f and g are morphisms in \mathcal{C}_h , and u and v are morphisms in \mathcal{C}_v . In order to distinguish notationally between horizontal and vertical morphisms, we mark horizontal ones with a bullet. Cells are furthermore equipped with composition and identities in both directions. E.g., to any given cells with compatible vertical border as below left is assigned a composite cell as below right

$$\begin{array}{ccc} A \xrightarrow{S \bullet} B \xrightarrow{S' \bullet} E & & A \xrightarrow{S' \bullet S \bullet} E \\ p \downarrow \quad \Downarrow \alpha \quad \downarrow q & & p \downarrow \quad \Downarrow \beta \bullet \alpha \quad \downarrow r \\ C \xrightarrow{T \bullet} D \xrightarrow{T' \bullet} F & \mapsto & C \xrightarrow{T' \bullet T \bullet} F, \end{array} \quad (1)$$

Similarly, we have horizontal identities

$$\begin{array}{ccc} A & \xrightarrow{id \bullet_A} & A \\ p \downarrow & \Downarrow id \bullet_p & \downarrow p \\ B & \xrightarrow{id \bullet_B} & B, \end{array}$$

Both notions of composition are required to be associative and the corresponding identities unital. Finally, the *interchange law* requires the two different ways of parsing any compatible pasting

$$\begin{array}{ccccc}
 A & \xrightarrow{\bullet} & B & \xrightarrow{\bullet} & C \\
 \downarrow & & \Downarrow \alpha & & \downarrow \\
 D & \xrightarrow{\bullet} & E & \xrightarrow{\bullet} & F \\
 \downarrow & & \Downarrow \beta & & \downarrow \\
 G & \xrightarrow{\bullet} & H & \xrightarrow{\bullet} & I
 \end{array}$$

to agree, i.e.,

$$(\delta \circ \gamma) \bullet (\beta \circ \alpha) = (\delta \bullet \beta) \circ (\gamma \bullet \alpha).$$

There is an alternative point of view on double categories which will be crucial to us. The previous presentation has emphasised \mathcal{E}_v and \mathcal{E}_h , and added the set of cells. But we could also put forward \mathcal{E}_V , the category whose objects are horizontal morphisms, and whose morphisms are cells. Indeed, double categories may be axiomatised based on a pair of functors

$$l, r: \mathcal{E}_V \rightrightarrows \mathcal{E}_v. \quad (2)$$

What is missing from this is just horizontal composition and identities (for horizontal arrows and cells), which may be postulated by requiring that l and r form a *category object* in **Cat**.

Equivalently, still, we can consider the following structure, which is almost a (large) double category.

Definition 2.1. Let **Span(Cat)** have

- as objects all small categories,
- as vertical morphisms all functors,
- as horizontal morphisms $A \rightarrow B$ all spans $A \leftarrow C \rightarrow B$ of functors, and
- as cells

$$\begin{array}{ccc}
 A \xrightarrow{\bullet} B & & A \longleftarrow C \longrightarrow B \\
 \downarrow \quad \Downarrow \quad \downarrow & \text{all commuting diagrams} & \downarrow \quad \downarrow \quad \downarrow \\
 U \xrightarrow{\bullet} V & & U \longleftarrow W \longrightarrow V
 \end{array}$$

in **Cat**.

Vertical composition is given by (componentwise) composition of functors, while horizontal composition is given by pullbacks and their universal property.

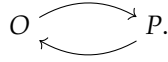
The structure formed by **Span(Cat)** is a *weak double category* [7], a weak form of double category where horizontal composition is only associative and unital up to coherent isomorphism, in a suitable sense. In particular, the horizontal arrows and *special* cells of a weak double category form a bicategory, where special means that the left and right borders are identities. This entails that we may define monads in weak double categories just like one usually does in bicategories, and we have:

PROPOSITION 2.2. *A double category is the same as a monad in Span(Cat).*

Indeed, composing a span (2) with itself in **Span(Cat)** amounts to constructing the category of pairs of compatible horizontal morphisms and cells (1), so requiring a monad multiplication is requiring horizontal composition.

2.2 The clock

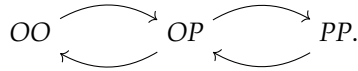
The vertical category of Melliès's \pm is the free category on the graph



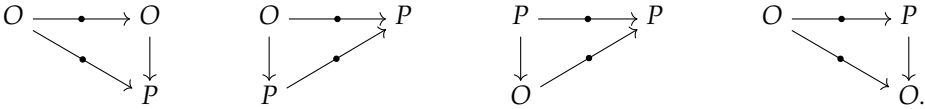
So its objects are just O and P , which stand for *Opponent* and *Proponent*, as in game models, and morphisms just count the number of (alternating) moves between them. One way to denote such morphisms is just as alternating strings of O s and P s.

The horizontal category \pm_h is simply the ordinal 2, viewed as a category, except that 0 is here renamed to O and 1 to P , so we have $O \leq P$, and there are only three horizontal morphisms: OO , PP , and OP .

Finally, cells describe the allowed schedulings in an arrow game: they are simply arrows in the free category \pm_V over the famous state diagram for simple games



Again, a way to write such morphisms is by valid sequences in $\{OO, OP, PP\}$. Vertical composition of cells is simply composition in \pm_V , i.e., concatenation. The most intuitive way to introduce horizontal composition is to depict basic cells as triangles



General cells are obtained by stacking up such basic triangles.

Example 2.3. To see the connection with standard game models, consider a typical play like below left, whose scheduling is the cell below right in \pm . Written as a sequence, this morphism is (OO, OP, PP, OP, OO) – the sequence of involved horizontal morphisms.



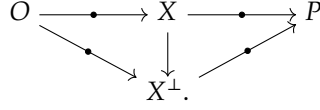
Depicting cells as stacks of triangles yields the following inductive definition of horizontal composition. Given two cells as in (1), if the middle object D is O then the last move in α , if any, cannot be on the left. Symmetrically, if the middle object is P then the last move in β , if any, cannot be on the right. This simple case analysis reveals that

- either there is an ‘outwards’ bottom triangle, i.e., the bottom of α and β look like either of



with $X \in \{O, P\}$ and X^\perp denoting the other player,

- or there is a pair of interacting bottom triangles, as in



In the first case, the composite is the obvious triangle $(OX, OX^\perp, \text{ resp. } XP, X^\perp P)$, while in the last case, the composite is the identity – which is precisely where game semantical *hiding* is encoded in \multimap . Melliès obtains:

PROPOSITION 2.4. \multimap forms a double category.

2.3 Simple games and strategies

This is where Melliès’s approach is novel. He simply puts:

Definition 2.5. A *game* is a functor to \multimap_v , and a *strategy* from $p: A \rightarrow \multimap_v$ to $q: B \rightarrow \multimap_v$ is a commuting diagram

$$\begin{array}{ccccc}
 A & \longleftarrow & S & \longrightarrow & B \\
 p \downarrow & & \downarrow & & \downarrow q \\
 \multimap_v & \longleftarrow & \multimap_V & \longrightarrow & \multimap_v
 \end{array} \tag{3}$$

in **Cat**.

Intuitively, in a game $p: A \rightarrow \multimap_v$, the objects of A are plays, and p indicates which player should play next. Morphisms are sequences of moves, whose number and polarity is again indicated by p . The notion of strategy may be understood as follows: the limit

$$\begin{array}{ccccc}
 & & \mathbb{P}_{A,B} & & \\
 & \swarrow & \downarrow & \searrow & \\
 A & & & & B \\
 p \downarrow & & \downarrow & & \downarrow q \\
 \multimap_v & \longleftarrow & \multimap_V & \longrightarrow & \multimap_v
 \end{array}$$

may be thought of as a category of plays on the arrow game $A \rightarrow B$, and the induced map $S \rightarrow \mathbb{P}_{A,B}$ describes which plays are accepted by the strategy.

Recalling the weak double category **Span(Cat)** from Definition 2.1, we observe that a game is a morphism to \multimap_v in **Span(Cat)**_v, while a strategy $A \rightarrow B$ is merely a cell

$$\begin{array}{ccc}
 A & \xrightarrow{S} & B \\
 p \downarrow & \Downarrow & \downarrow q \\
 \multimap_v & \xrightarrow{\bullet} & \multimap_v
 \end{array}$$

This provides a simple way of composing strategies, using the monad structure of \multimap (given by Propositions 2.2 and 2.4): the composite of $S: A \rightarrow B$ and $T: B \rightarrow C$ is simply the pasting

$$\begin{array}{ccccccc}
 A & \xrightarrow{S} & B & \xrightarrow{T} & C \\
 p \downarrow & \Downarrow & \downarrow q & \Downarrow & \downarrow r \\
 \multimap_v & \xrightarrow{\bullet} & \multimap_v & \xrightarrow{\bullet} & \multimap_v \\
 & \searrow & \downarrow \mu & \swarrow & \\
 & & \multimap_v & &
 \end{array} \tag{4}$$

where μ denotes the monad multiplication for \multimap , i.e., horizontal composition of cells.

Strategies are thus equipped with weak double category structure (although Melliès only considers the underlying bicategory) by applying the following general result to the monad \pm in $\mathbf{Span}(\mathbf{Cat})$:

THEOREM 2.6. *Given any monad $M_V : M_v \rightarrow M_v$ in a weak double category \mathcal{C} , there is a slice weak double category \mathcal{C}/M whose*

- vertical category is $(\mathcal{C}/M)_v = \mathcal{C}_v/M_v$;
- vertical category of cells is $(\mathcal{C}/M)_V = \mathcal{C}_V/M_V$;
- horizontal composition of cells is given by pasting with monad multiplication, as in (4); and
- horizontal identity on $p : A \rightarrow M_v$ is the pasting

$$\begin{array}{ccc}
 A & \xrightarrow{\quad \bullet \quad} & A \\
 p \downarrow & \Downarrow \text{id}_p & \downarrow p \\
 M_v & \xrightarrow{\quad \bullet \quad} & M_v \\
 & \Downarrow \eta & \\
 & M_V &
 \end{array}
 \tag{5}$$

PROPOSITION 2.7. *Melliès’s bicategory of simple games is the underlying bicategory of $\mathbf{Span}(\mathbf{Cat})/\pm$.*

A weak double category with trivial vertical category is nothing but a monoidal category. In that case, the theorem reduces to the following well-known result used, e.g., in Weber [17]:

COROLLARY 2.8. *The slice of a monoidal category over a monoid is again monoidal.*

3 RECOVERING SIMPLE GAMES

3.1 Restricting to discrete fibrations

The present work was prompted by Melliès’s observation that $\mathbf{Span}(\mathbf{Cat})/\pm$ is not quite equivalent to the standard category of simple games and strategies. Indeed, one might have expected that, restricting to strategies whose underlying functor $S \rightarrow \mathbb{P}_{A,B}$ is an inclusion, we would obtain an equivalent category. But this is not the case: Melliès’s games and strategies are intrinsically more general. This is even emphasised as a feature, as it has the advantage of smoothing things up in the context of asynchronous games, where a different clock is used. In our sequential setting, the extra generality is twofold. First, the considered games may have no time origin – concretely there is no empty play.

Example 3.1. A symptom is that the categories A may not be well founded, i.e., they may contain infinite chains $\dots \rightarrow a_n \rightarrow \dots \rightarrow a_0$.

The second source of extra generality is that games may feature ‘compound moves’, i.e., indecomposable morphisms whose image in \pm_v has length > 1 .

Example 3.2. Take for A , e.g., the ordinal $\mathbb{1}$ viewed as a category, and map its unique morphism to $OPOP$ in \pm_v .

This thus raises the question: can standard simple games be recovered by refining the abstract Theorem 2.6?

The first step towards this is to characterise the games $A \rightarrow \pm_v$ that correspond to standard simple games. This is easy: by definition, standard simple games are trees, which may be defined as presheaves over the ordinal ω . But presheaves are equivalent to discrete fibrations, hence the idea of restricting $(\mathbf{Span}(\mathbf{Cat})/\pm)_v$ to discrete fibrations. However, this does not quite work, as \pm_v lacks a ‘time origin’: presheaves on \pm_v describe games in which there is no first move – all

moves have predecessors. But if we slice \pm_v under OO , then we get it right, as we have $OO/\pm_v \cong \omega$. Similarly, OO/\pm_V describes scheduling in the arrow category starting from OO .

Funnily enough, $OO: O \rightarrow O$, viewed as a horizontal endomorphism in \pm , is a comonad (the identity comonad on O), hence we may apply (the dual of) Theorem 2.6 to obtain:

COROLLARY 3.3. *The slice $\mathbb{H} := OO/\pm$ forms a double category.*

Taking \mathbb{H} as a replacement for \pm , we get the desired property that discrete fibrations over \mathbb{H}_v are equivalent to $\widehat{\omega}$.

In a similar vein, in recent work on game semantics [5, 12, 16], a concurrent notion of strategy was defined as presheaves on plays. It thus would seem natural to also restrict strategies (3) to ensure that the induced functor $S \rightarrow \mathbb{P}_{A,B}$ is a discrete fibration. This may be enforced directly:

LEMMA 3.4. *Given a commuting diagram of functors*

$$\begin{array}{ccccc} A & \longleftarrow & S & \longrightarrow & B \\ p \downarrow & & \downarrow m & & \downarrow q \\ X & \xleftarrow{l} & T & \xrightarrow{r} & Y \end{array}$$

where p and q are discrete fibrations, letting P denote the limit of the subdiagram

$$A \rightarrow X \leftarrow T \rightarrow Y \leftarrow B,$$

the induced functor $S \rightarrow P$ is a discrete fibration iff the middle functor $m: S \rightarrow T$ is.

PROOF. Discrete fibrations are the right class of a (strong) factorisation system (see Lemma 3.6 below), hence are stable under composition and pullback, and furthermore enjoy left cancellation: if $g \circ f$ and g are discrete fibrations, then so is f .

Now, the limit P may be computed by taking pullbacks of p and q , respectively along l and r , and then taking the pullback of the obtained cospan. Thus, if p and q are discrete fibrations, then by stability under pullback and composition, so is the projection functor $P \rightarrow T$.

Thus, if the induced functor $S \rightarrow P$ is a discrete fibration, then so is the middle functor $S \rightarrow T$ by stability under composition.

Conversely, if the middle functor $S \rightarrow T$ is a discrete fibration, then so is the induced functor $S \rightarrow P$ by left cancellation. \square

3.2 Simple games

We thus hope to recover standard simple games by slicing \pm under OO , and restricting the slice construction to discrete fibrations (for vertical morphisms and cells). This may be carried over to the abstract setting using the observation, recalled in the above proof, that discrete fibrations are the right class of a factorisation system.

Definition 3.5. A double factorisation system on a weak double category \mathcal{C} consists of factorisation systems $(\mathcal{L}_v, \mathcal{R}_v)$ and $(\mathcal{L}_V, \mathcal{R}_V)$ on \mathcal{C}_v and \mathcal{C}_V , respectively, such that

- (A) \mathcal{L}_V is preserved under horizontal composition and contains horizontal identities, and
- (B) all cells α as below left with $\ell, \ell' \in \mathcal{L}_v$ and $r, r' \in \mathcal{R}_v$ factor as below right, with $\lambda \in \mathcal{L}_V$ and $\rho \in \mathcal{R}_V$.

$$\begin{array}{ccc}
A & \xrightarrow{\dot{S}} & A' \\
\ell \downarrow & \Downarrow \alpha & \downarrow \ell' \\
B & & B' \\
r \downarrow & \Downarrow & \downarrow r' \\
C & \xrightarrow{\dot{U}} & C'
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{\dot{S}} & A' \\
\ell \downarrow & \Downarrow \lambda & \downarrow \ell' \\
B & \xrightarrow{\bullet} T & B' \\
r \downarrow & \Downarrow \rho & \downarrow r' \\
C & \xrightarrow{\dot{U}} & C'
\end{array}$$

LEMMA 3.6. *Discrete fibrations and componentwise discrete fibrations are the right classes of factorisation systems which together form a double factorisation system for $\mathbf{Span}(\mathbf{Cat})$.*

PROOF. It is well-known that discrete fibrations may be defined by unique lifting w.r.t. the injection $\mathbf{1} \hookrightarrow \mathbf{2}$ mapping 0 to 1. The only non-obvious point is then that componentwise discrete fibrations are stable under horizontal composition, which follows from the general fact that the right class of any factorisation system is stable under pullback in the arrow category. \square

This leads us to the following generalisation of Theorem 2.6:

THEOREM 3.7. *Given any monad $M_V : M_v \rightarrow M_v$ in a weak double category \mathcal{C} with double factorisation system $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$, there is a slice weak double category $\mathcal{C}/_{\mathcal{R}}M$ whose*

- vertical category $(\mathcal{C}/_{\mathcal{R}}M)_v$ is $\mathcal{C}_v/\mathcal{R}_v M_v$, the full subcategory of \mathcal{C}_v/M_v on maps in \mathcal{R}_v ;
- vertical category of cells is $(\mathcal{C}/_{\mathcal{R}}M)_V = \mathcal{C}_V/\mathcal{R}_V M_V$;
- horizontal composition of cells is given by factoring the pasting (4) as $\rho \circ \lambda$ and returning ρ ;
- and whose horizontal identity on any $p : A \rightarrow M_v$ is given by factoring (5) as $\rho \circ \lambda$ and returning ρ .

Remark 3.8. The fact that identities and horizontal composites have the right perimeter follows from Condition (B) in Definition 3.5.

Remark 3.9. When the monad multiplication is in \mathcal{R}_V , and \mathcal{R}_V is stable under horizontal composition and contains horizontal identities, then so are (4) and (5), hence $\mathcal{C}/_{\mathcal{R}}M$ is a sub weak double category of \mathcal{C}/M .

In the general case, we may picture composition in $(\mathcal{C}/_{\mathcal{R}}M)_h$ as follows:

$$\begin{array}{ccccc}
A & \xrightarrow{\dot{S}} & B & \xrightarrow{\dot{T}} & C \\
\downarrow p & \Downarrow & \downarrow q & \Downarrow \lambda & \downarrow r \\
M_v & \xrightarrow{\dot{M}_V} & M_v & \xrightarrow{\dot{M}_V} & M_v \\
& \Downarrow \mu & & \Downarrow \rho & \\
& M_v & & M_v &
\end{array}$$

PROOF OF THEOREM 3.7. By coherence for weak double categories [9, Theorem 7.5], and assuming a higher universe in which \mathcal{C} is small, we may assume that \mathcal{C} is in fact a (strict) double category.

Composition of cells in $\mathcal{C}/_{\mathcal{R}}M$ is just as in \mathcal{C} , so the only non-trivial point to check is weak associativity and unitality of horizontal composition (of morphisms). For weak associativity, we observe that both cells

$$\begin{array}{c}
A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D \\
p \downarrow \quad \alpha \Downarrow \quad q \downarrow \quad \beta \Downarrow \quad r \downarrow \quad \gamma \Downarrow \quad s \downarrow \\
M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \\
\downarrow \mu \quad \Downarrow \mu \\
M_v
\end{array}
\qquad
\begin{array}{c}
A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D \\
p \downarrow \quad \alpha \Downarrow \quad q \downarrow \quad \beta \Downarrow \quad r \downarrow \quad \gamma \Downarrow \quad s \downarrow \\
M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \\
\downarrow \mu \quad \Downarrow \mu \\
M_v
\end{array}$$

are equal. Now, denoting composition in $(\mathcal{E}/\mathcal{R}M)_h$ by $\tilde{\bullet}$, $\gamma \tilde{\bullet} (\beta \tilde{\bullet} \alpha)$ and $(\gamma \tilde{\bullet} \beta) \tilde{\bullet} \alpha$ are obtained by factoring them as follows. For the former, we factor

$$T \bullet S \xrightarrow{\beta \bullet \alpha} M_V \bullet M_V \xrightarrow{\mu} M_V \qquad \text{as} \qquad T \bullet S \xrightarrow{\lambda_{\beta,\alpha}} K_{\beta,\alpha} \xrightarrow{\rho_{\beta,\alpha}} M_V,$$

in which, by Condition (B), $\lambda_{\beta,\alpha}$ has identity left and right borders, i.e., is *special*. We then factor

$$U \bullet K_{\beta,\alpha} \xrightarrow{\gamma \bullet \rho_{\beta,\alpha}} M_V \bullet M_V \xrightarrow{\mu} M_V \qquad \text{as} \qquad U \bullet K_{\beta,\alpha} \xrightarrow{\lambda_{\gamma,(\beta,\alpha)}} K_{\gamma,(\beta,\alpha)} \xrightarrow{\rho_{\gamma,(\beta,\alpha)}} M_V.$$

The other composite may be computed symmetrically, so that we obtain factorisations:

$$\begin{array}{c}
A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D \\
p \downarrow \quad \lambda_{\beta,\alpha} \Downarrow \quad \lambda_{\gamma,(\beta,\alpha)} \Downarrow \quad s \downarrow \\
M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \\
\downarrow \mu \quad \Downarrow \mu \\
M_v
\end{array}
\qquad
\begin{array}{c}
A \xrightarrow{S} B \xrightarrow{T} C \xrightarrow{U} D \\
p \downarrow \quad \lambda_{(\gamma,\beta),\alpha} \Downarrow \quad \lambda_{\beta,\gamma} \Downarrow \quad s \downarrow \\
M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \xrightarrow{\quad} M_v \\
\downarrow \mu \quad \Downarrow \mu \\
M_v
\end{array}$$

By Condition (A), both are in fact factorisations for $(\mathcal{L}_V, \mathcal{R}_V)$, so that by lifting, we obtain a special cell $a_{\alpha,\beta,\gamma}: K_{\gamma,(\beta,\alpha)} \Rightarrow K_{(\gamma,\beta),\alpha}$ such that $\rho_{(\gamma,\beta),\alpha} \circ a_{\alpha,\beta,\gamma} = \rho_{\gamma,(\beta,\alpha)}$, which is our candidate associator for $\mathcal{E}/\mathcal{R}M$. It satisfies the MacLane pentagon by uniqueness of lifting.

Weak unitality follows similarly. □

We finally obtain:

COROLLARY 3.10. *Consider the weak double category $\mathbf{Span}(\mathbf{Cat})/_{DFib} \mathbf{H}$. Restricting horizontal morphisms to discrete fibrations $S \rightarrow \mathbb{P}_{A,B}$ that are subcategory inclusions, we obtain a category which is isomorphic to the standard category of simple games.*

3.3 Day convolution

We finally reach the surprising application mentioned in the introduction, Day convolution. The purpose of this operation is to show that the Yoneda embedding $\mathbf{y}: \mathbf{C} \rightarrow \widehat{\mathbf{C}}$ is monoidal when \mathbf{C} is. This means that $\widehat{\mathbf{C}}$ may be equipped with monoidal structure preserved by \mathbf{y} up to coherent isomorphism. The tensor is given as follows:

Definition 3.11. For any small monoidal category \mathbf{C} and $X, Y \in \widehat{\mathbf{C}}$, let

$$(X \otimes Y)(c) = \int_{(c_1, c_2) \in \mathbf{C}^2} X(c_1) \times Y(c_2) \times \mathbf{C}(c, c_1 \otimes c_2).$$

Let us now recover this structure from Theorem 3.7, in the particular case where \mathbb{C} is strictly monoidal. The starting point is the sub weak double category, say \mathscr{W} , of $\mathbf{Span}(\mathbf{Cat})$, obtained by restricting attention to just one object and one vertical morphism, namely the terminal category $\mathbb{1}$ and the identity thereon. Thus, \mathscr{W}_V consists of categories and functors, and horizontal composition is given by cartesian product. Furthermore, a monad in \mathscr{W} is nothing but a monoid in \mathbf{Cat} for the cartesian product, i.e., a strict monoidal category \mathbb{C} . The double factorisation system induced by discrete fibrations on $\mathbf{Span}(\mathbf{Cat})$ restricts to one on \mathscr{W} , and obviously the weak double category $\mathscr{W}/_{DFib} \mathbb{C}$ is vertically trivial, hence underlies a monoidal category, say \mathbb{C}' .

THEOREM 3.12. *For any strictly monoidal category \mathbb{C} , the monoidal category \mathbb{C}' is equivalent to $\widehat{\mathbb{C}}$ equipped with the convolution tensor product.*

In order to prove this, let us first show:

LEMMA 3.13. *Let $f : A \rightarrow B$ be a functor. The discrete fibration ρ_f associated to f is determined up to isomorphism by*

$$\partial^\star(\rho_f)(b) \cong \int^{a \in A} B(b, f(a)),$$

where $\partial^\star : DFib_B \rightarrow \widehat{B}$ is the standard equivalence between discrete fibrations and presheaves.

PROOF. This is actually obvious by construction. In [15], the dual case is actually treated, initial functors and discrete opfibrations. But up to this discrepancy, $\partial^\star(f)$ is precisely k in the proof of [15, Theorem 3], which would in our case be defined as the left Kan extension of $A^{op} \xrightarrow{!} 1 \xrightarrow{\square} \mathbf{Set}$ along f^{op} . By the well-known characterisation of left Kan extensions by coends, we readily obtain the desired formula. \square

PROOF OF THEOREM 3.12. By construction, given two presheaves $X, Y \in \widehat{\mathbb{C}}$ and transporting them to their corresponding discrete fibrations, say $S : el(X) \rightarrow \mathbb{C}$ and $T : el(Y) \rightarrow \mathbb{C}$, their tensor product $S \bullet T$ in \mathbb{C}' is the right factor of the composite

$$el(X) \times el(Y) \xrightarrow{S \times T} \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}.$$

By Lemma 3.13, the result has its corresponding presheaf defined up to isomorphism by

$$\begin{aligned} \partial^\star(S \bullet T)(c) &\cong \int^{(a,b) \in el(X) \times el(Y)} \mathbb{C}(c, \otimes((S \times T)(a, b))) \\ &= \int^{(a,b) \in el(X) \times el(Y)} \mathbb{C}(c, S(a) \otimes T(b)) \\ &\cong \int^{((c_1, x), (c_2, y)) \in el(X) \times el(Y)} \mathbb{C}(c, c_1 \otimes c_2) \\ &\cong \int^{c_1, c_2} X(c_1) \times Y(c_2) \times \mathbb{C}(c, c_1 \otimes c_2), \text{ as desired.} \end{aligned} \quad \square$$

4 CONCLUSION AND PERSPECTIVES

We have designed an abstract slice construction over monads in weak double categories, which has as instances

- a weak double category of simple games and concurrent strategies,
- and the monoidal category of presheaves over any strict monoidal category.

We see at least two directions for future work. First, we should try to accomodate not only the weak double category structure of Melliès's construction, but also symmetric monoidal closedness. Melliès is also currently working on the construction of a linear exponential comonad [1] on his category of simple games and concurrent strategies. This will of course be a useful feature

to incorporate to our framework. The second direction for future work is to generalise our construction to encompass Day convolution for non-strict monoidal categories. This will involve a 3-dimensional refinement of weak double categories.

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