Template Games, Simple Games, and Day Convolution

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Abstract

Template games [14] unify various approaches to game semantics, by exhibiting them as instances of a double-categorical variant of the slice construction. However, in the particular case of simple games [9, 12], template games do not quite yield the standard (bi)category. We refine the construction using factorisation systems, obtaining as an instance a slight generalisation of simple games and strategies. This proves that template games have the descriptive power to capture combinatorial constraints defining well-known classes of games. Another instance is Day's *convolution* monoidal structure on the category of presheaves over a strict monoidal category [2], which answers a question raised in [3].

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1 Introduction

Game semantics has provided adequate models for a variety of programming languages [11], in which types are interpreted as two-player games and programs as strategies. Most game models follow a common pattern. Typically, a function $A \to B$ is interpreted as a strategy on a compound game made of A and B, where the program plays as *Proponent* (P) on B and as *Opponent* (O) on A. Another common feature is composition of strategies, which takes strategies $\sigma\colon A \to B$ and $\tau\colon B \to C$, and returns a strategy $\tau \circ \sigma\colon A \to C$ by letting σ and τ interact on B until one of them produces a move in A or C.

Although widely acknowledged, this strong commonality is also recognised as poorly understood, particularly in the presence of *innocence*, a constraint on strategies that restricts them to purely functional behaviour. This has prompted a number of attempts at clarifying the situation [10, 9, 4]. Recently, Melliès [14] proposed a novel explanation, of unprecedented simplicity, named *template games*. It is based upon a purely categorical construction, essentially taking the slice of a weak double category over an internal monad, the *template*. This produces a new weak double category, in which composition of strategies occurs as so-called horizontal composition. In order to illustrate the construction, Melliès applies it to

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three different templates to obtain three models, respectively related to simple games [12], concurrent games [15], and the relational model. However, the first two differ significantly from their more standard counterparts.

This raises the question of whether template games only produce new game models, or whether they also cover standard models. In this paper, we show that, up to a slight refinement, they do cover simple games. More precisely, we (1) modify the original template for simple games (Lemma 64), and (2) enrich the general construction with a factorisation system [1] (Theorem 61), obtaining a slight generalisation of standard simple games as an instance (Corollary 68). More precisely, we obtain a variant in which games may have several 'initial positions', before the game even starts, and similarly strategies may have several 'initial states' over each of these initial positions. We thus easily characterise standard simple games and strategies as so-called *definite* template games and strategies.

One motivation for simple and abstract constructions like template games is to find new connections with other settings. Our refined construction yields one such connection: we show that the Day *convolution* product [2] arises as an instance of our refined framework, though only in the restricted case of *strict* monoidal categories (Theorem 73). The convolution product, which arose in algebraic topology, extends the monoidal structure of a given category $\mathbb{C}$ to the category $\widehat{\mathbb{C}}$ of *presheaves* on $\mathbb{C}$, i.e., contravariant functors $\mathbb{C}^{op} \to \mathsf{Set}$. This makes formal the similarity, noted in [3, §6.5], between convolution and composition of strategies, by showing that both are instances of the same construction.

Related work

Beyond [14] and the related [10], Garner and Shulman [7] prove results related to our Theorems 52 and 61. The common ground for comparison is the restriction of Theorem 52 to weak double categories with a trivial vertical category, i.e., monoidal categories. Their Theorem 14.2 is a generalisation in another direction, namely that of monoidal bicategories, and their Theorem 14.5 could in particular accomodate various sorts of bicategorical factorisation systems.

Terminology

Although template games and strategies are an abstract construction, we often abuse the term to denote the particular instance on the template for simple games.

Plan

We follow the standard construction layers of game models: games, strategies, and composition of strategies. In Section 2, we analyse the differences between template and simple games, and describe our refinement of the former, which allows us to bridge the gap. In Section 3, we do the same at the level of strategies $A \to B$, for fixed games A and B. In Section 4, we recall the abstract construction of template games. In Section 5, we introduce our refined construction, and illustrate it on the promised instances.

2 Template games vs. simple games

In this section, we first recall template games, and then analyse the discrepancies with simple games. Finally, we introduce our solution to bridge the gap between them.

Template games are based on the following simple category.

▶ **Definition 1.** *Let* $\pm_v$ *denote the category freely generated by the graph* $O \rightleftarrows P$.

Thus, objects are just O and P, which stand for *Opponent* and *Proponent*, as in most game models, and morphisms just count the number of (alternating) moves between them.

▶ **Definition 2.** *A* template game *is a category* A*, equipped with a functor* $p\colon A \to \pm_v$*.*

Intuitively, objects of A are positions, or plays, in the game, and p gives their polarity: by convention, O is to play in positions mapped to O, while P is to play in others.

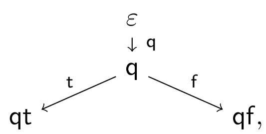
Simple games clearly fit into this framework.

▶ **Definition 3.** *A* simple game A *is a rooted tree.*

The intuition is the same as for template games, and indeed:

▶ **Proposition 4.** *Any simple game* A*, viewed as a poset, hence as a category, forms a template game* $p_A\colon A \to \pm_v$*, where* p_A *maps the root to* O*, its children to* P*, and so on.*

▶ **Example 5.** The simple game $\mathbb{B}$ for booleans is the tree



where we have labelled edges with moves, and each node with the corresponding play, i.e., the sequence of moves needed to reach it from the root. The first move, q, represents O asking the value of the boolean, and the other moves represent the possible answers: true and false. The functor $p_{\mathbb{B}}\colon \mathbb{B} \to \pm_v$ maps q to P and all other objects to O.

Discrepancies between template and simple games

Template games are significantly more general than simple games. To start with, simple games have an empty, initial play, which template games need not. Furthermore, this initial play is mapped to O by the functor to $\pm_v$.

▶ **Lemma 6.** *For any simple game* A*, the category* A *has an initial object, mapped to* O *by the functor* $A \to \pm_v$*.*

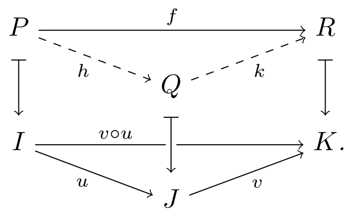
Let us exhibit some counterexamples among general template games.

▶ **Example 7.** The template game $\mathbb{1} \to \pm_v$ picking up P is not equivalent to any simple game in the slice 2-category $\mathsf{Cat}/\pm_v$.

▶ **Example 8.** Consider the template game consisting of $\mathbb{Z}$, the poset of integers, and the functor $\mathbb{Z} \to \pm_v$ that maps $2n$ to O and $2n+1$ to P. The category $\mathbb{Z}$ has no initial object, hence this template game is not equivalent to any simple game in $\mathsf{Cat}/\pm_v$.

Another discrepancy has to do with decomposing plays into moves.

▶ **Definition 9.** *A functor $F\colon \mathbb{E} \to \mathbb{B}$ is a* discrete Conduché fibration *iff for any $I \xrightarrow{u} J \xrightarrow{v} K$ in $\mathbb{B}$ and $P \xrightarrow{f} R$ in $\mathbb{E}$ mapped to $v \circ u$, f uniquely factors as $k \circ h$ with $F(h) = u$ and $F(k) = v$:*



▶ **Lemma 10.** *For any simple game A, the functor $p_A\colon A \to \pm_v$ is a discrete Conduché fibration.*

Proof. By construction, a morphism in A over a path of length n has the form $p \to pm_1 \dots m_n$ for some play p and moves $m_1, \dots, m_n$, hence decomposes as needed. ◀

Unlike simple games, not all template games are Conduché fibrations. Indeed, they may feature *atomic* sequences of moves, i.e., morphisms that are mapped to non-basic morphisms in $\pm_v$ and yet are 'indecomposable'.

▶ **Example 11.** Consider the ordinal $\mathbb{2} = \{0 \leq 1\}$ viewed as a category, and the functor $\mathbb{2} \to \pm_v$ mapping $0 \leq 1$ to the path $O \to P \to O$. This is clearly not a Conduché fibration, hence is not equivalent to any simple game.

Refining template games

As announced in §1, our solution to bridge the gap between template and simple games is twofold: (1) we use a different template, $\mathbb{T}$, and (2) we introduce a factorisation system into the picture – concretely, we restrict attention to functors $A \to \mathbb{T}$ which are discrete fibrations. Let us start with the new template.

▶ **Definition 12.** *Let $\mathbb{T}_v = \omega$ denote the poset of natural numbers, seen as a category.*

▶ Remark 13. $\mathbb{T}_v$ is isomorphic to the coslice category $O/\pm_v$.

Defining games to be functors to $\mathbb{T}_v$ intuitively goes in the right direction, but does not quite solve any of our two problems.

▶ **Example 14.** Consider the functor $D\colon \omega \to \omega = \mathbb{T}_v$ defined by $D(n) = 2n + 1$. It exhibits both problems at once: it does not preserve the initial object, and, e.g., $0 \leq 1$ does not admit any decomposition along the decomposition $1 \leq 2 \leq 3$, a.k.a. $P \to O \to P$, of its image.

However, if we restrict attention to a certain kind of functors $A \to \mathbb{T}_v$, we solve both problems at once – almost. The relevant constraint on functors is generally stronger than being a Conduché fibration, but becomes equivalent when both categories have an initial object which is preserved by the functor.

▶ **Definition 15.** *A functor $F\colon \mathbb{E} \to \mathbb{B}$ is a* discrete fibration *when for any $E \in \mathbb{E}$ and $u\colon B \to F(E)$, there is a unique $f\colon E' \to E$ such that $F(f) = u$, as in*

$$\begin{array}{ccc} E' & \overset{f}{\dashrightarrow} & E \\ \downarrow & & \downarrow \\ B & \underset{u}{\longrightarrow} & F(E). \end{array}$$

Let $\mathsf{DFib}(\mathbb{C})$ *denote the full subcategory of* $\mathsf{Cat}/\mathbb{C}$ *spanning discrete fibrations.*

▶ **Definition 16.** *A* refined template game *is a discrete fibration to* $\mathbb{T}_v$*.*

Of course, any simple game A yields a discrete fibration $p_A \colon A \to \mathbb{T}_v$, and we have:

▶ **Proposition 17.** *Any refined template game in* $\mathsf{DFib}(\mathbb{T}_v)$ *is isomorphic to* p_A*, for some simple game* A*, iff it is* definite*, i.e., its fibre over* 0 *is a singleton.*

3 Template strategies vs. simple strategies

Let us now consider strategies. As for games, we start with Melliès's notion, to emphasise its simplicity. Template strategies are based on the following simple category:

▶ **Definition 18.** *Let* $\pm_V$ *denote the category freely generated by the graph*

$$OO \rightleftarrows OP \rightleftarrows PP.$$

This is essentially the well-known state diagram for strategies in a simple arrow game $A \to B$, extended to a category. We need a little lemma before defining strategies.

▶ **Lemma 19.** *The left and right projections give rise to functors* $s, t \colon \pm_V \to \pm_v$*, with* $s(XY) = X$ *and* $t(XY) = Y$*.*

▶ **Definition 20.** *A* template strategy *from* $p \colon A \to \pm_v$ *to* $q \colon B \to \pm_v$ *is a tuple* (S, s', t', r) *making the following diagram commute.*

$$\begin{array}{ccccc} A & \overset{s'}{\longleftarrow} & S & \overset{t'}{\longrightarrow} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\ \pm_v & \underset{s}{\longleftarrow} & \pm_V & \underset{t}{\longrightarrow} & \pm_v \end{array} \tag{1}$$

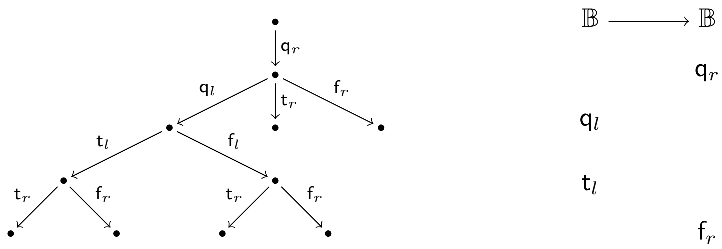
Let us now show that simple strategies give rise to template strategies. Simple strategies are rather subtle – which was one of the main motivations for template games in the first place! – so this is a bit technical. We first consider *boolean* simple strategies, which are easier, and then move on to general ones.

Boolean simple strategies

First, we need to define the *arrow* game $A \to B$, for any two simple games A and B. The following definition is slightly vague: we refer to [12] for a fully rigorous one.

▶ **Definition 21.** *Given two simple games* A *and* B*,* $A \to B$ *interleaves moves from* A *and* B *according to the following rules: (1) the polarity of moves in* A *is inverted, (2)* O *starts in* B*, and (3) only* P *gets to switch sides.*

▶ **Example 22.** The game $\mathbb{B} \to \mathbb{B}$ is depicted below left, with an example play on the right.



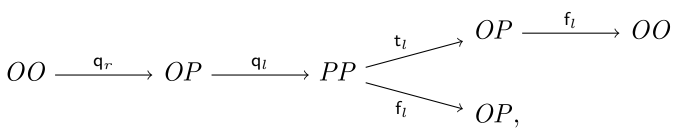
▶ **Definition 23.** *A* boolean simple strategy *$A \to B$ is a prefix-closed set of non-empty, even-length plays, called* accepted*, in $A \to B$.*

▶ **Example 24.** The set of non-empty, even-length prefixes of the play in Example 22 (i.e., the set $\{\mathsf{q}_r\mathsf{q}_l, \mathsf{q}_r\mathsf{q}_l\mathsf{t}_l\mathsf{f}_r\}$) forms a boolean simple strategy.

There are in fact several ways in which boolean simple strategies give rise to template strategies. We here present the relevant one in terms of semantics.

▶ **Definition 25.** *The template strategy associated to a boolean simple strategy $\sigma\colon A \to B$ is $(\mathbb{E}(\sigma), s', t', r)$, where $\mathbb{E}(\sigma)$ is the poset of all prefixes of accepted plays, plus all extensions sm of prefixes s of even length, and r, s', and t' are the obvious projections.*

▶ **Example 26.** Consider the strategy of Example 24. The functor $\mathbb{E}(\sigma) \to \pm_V$ can be represented as



(2)

with $s', t' \colon \mathbb{E}(\sigma) \to \mathbb{B}$ displayed on the left and right in Figure 1 (with polarity indications).

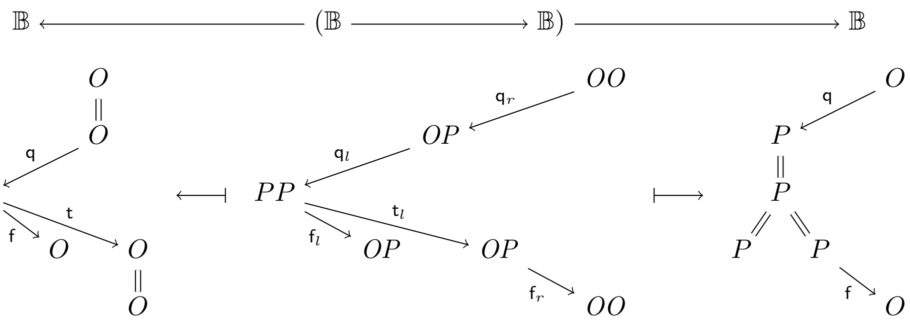


Figure 1 Template strategy associated to the simple strategy of Example 24.

General simple strategies

Let us now consider the more general, non-deterministic (i.e., non-boolean), but still standard, notion of simple strategy [12]. Let us start from an alternative presentation of boolean strategies.

▶ **Definition 27.** *Let* $(A \to B)^{P*}$ *denote the full subcategory of* $(A \to B)$ *spanning non-empty, even-length plays, and* $\mathsf{i}_{P*} \colon (A \to B)^{P*} \hookrightarrow (A \to B)$ *the inclusion functor.*

▶ **Proposition 28.** *Boolean simple strategies* $A \to B$ *are equivalent to functors* $((A \to B)^{P*})^{op} \to \mathbb{2}$*.*

The idea is that a strategy σ accepts a given play p iff the corresponding functor maps it to $1 \in \mathbb{2}$, observing that functoriality ensures prefix-closedness.

General simple strategies are obtained by generalising from $\mathbb{2}$ to Set:

▶ **Definition 29.** *Let the category of simple strategies* $A \to B$ *be* $\mathcal{S}(A, B) := \widehat{(A \to B)^{P*}}$*.*

▶ Remark 30. As noted in [12], this is equivalent to the standard definition as a slice category. Indeed, letting ω^{P*} denote the full subcategory of ω on positive, even ordinals, $p_{A \to B}$ restricts to a discrete fibration $(A \to B)^{P*} \to \omega^{P*}$. By the well-known equivalence $\partial^\star \colon \mathsf{DFib}(\mathbb{C}) \to \widehat{\mathbb{C}} : \mathrm{el}$ between presheaves and discrete fibrations (recalled as Lemma 32 below), and, for any $U \in \widehat{\mathbb{C}}$, the further equivalence $\widehat{\mathbb{C}}/U \simeq \widehat{\mathrm{el}(U)}$, we obtain

$$\mathcal{S}(A, B) = \widehat{(A \to B)^{P*}} \simeq \widehat{\omega^{P*}}/\partial^\star((A \to B)^{P*}).$$

The latter slice category is precisely the standard definition.

Of course, boolean simple strategies embed into general ones by postcomposition with the embedding $\mathbb{2} \hookrightarrow \mathsf{Set}$ determined by $0 \mapsto \varnothing$ and $1 \mapsto 1$.

From simple strategies to template strategies

Let us now informally describe how simple strategies $\sigma \in \widehat{(A \to B)^{P*}}$ give rise to template strategies, for which the following well-known result is the basis.

▶ **Definition 31.** *For any small category* $\mathbb{C}$*, let* $X \in \widehat{\mathbb{C}}$*. The* category of elements $\mathrm{el}(X)$ *of* X *has as objects pairs* (c, x) *with* $x \in X(c)$*, and as morphisms* $(c, x) \to (c', x')$ *all morphisms* $f \colon c \to c'$ *such that* $X(f)(x') = x$*.*

▶ **Lemma 32.** *For any small category* $\mathbb{C}$ *and* $X \in \widehat{\mathbb{C}}$*, the projection functor* $p_X \colon \mathrm{el}(X) \to \mathbb{C}$ *is a dicrete fibration, and the category of elements construction extends to an adjoint equivalence of categories* $\mathrm{el} \colon \widehat{\mathbb{C}} \to \mathsf{DFib}(\mathbb{C})$*. Let* $\partial^\star$ *denote the weak inverse to* el*.*

▶ **Example 33.** One possible definition of forests is as presheaves over ω. Indeed, any $T \in \widehat{\omega}$ models the forest whose nodes of depth n are $T(n)$, and whose parent relation is given by $T(n \leq n+1) \colon T(n+1) \to T(n)$. The category $\mathrm{el}(T)$ is just the same forest viewed as a poset, or rather a category, and the projection functor $p_T \colon \mathrm{el}(T) \to \omega$ maps vertices to their depths. Trees are those $T \in \widehat{\omega}$ such that $T(0)$ is a singleton.

Returning to simple and template strategies, we have a commuting diagram of functors

$$\begin{array}{ccccc} A & \longleftarrow & (A \to B) & \longrightarrow & B \\ {\scriptstyle p_A}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_B} \\ \curlywedge_v & \xleftarrow[s]{} & \curlywedge_V & \xrightarrow[t]{} & \curlywedge_v, \end{array} \tag{3}$$

so a first candidate follows by composition with the functor

$$\mathrm{el}(\sigma) \to (A \to B)^{P*} \xhookrightarrow{\mathsf{i}_{P*}} (A \to B).$$

However, the fibres of this functor over plays of odd length are all empty. We thus need to insert additional objects over odd-length plays whose immediate prefix is accepted, which will at last induce the desired template strategy by composition with (3). We do this using right Kan extension:

▶ **Definition 34.** *Let* $\overline{\sigma} \in \widehat{A \to B} := \prod_{\mathfrak{i}_{P*}} \sigma$*, i.e., the right Kan extension of* σ *along* $\mathfrak{i}^{op}_{P*}$*.*

This does yield the desired behaviour, by a direct application of the well-known end formula for right Kan extension:

▶ **Lemma 35.** *We have:*

- $\overline{\sigma}(\varepsilon) = 1$*;*
- *for all odd-length plays* sm*,* $\overline{\sigma}(sm) = \sigma(s)$*;*
- *for all non-empty, even-length plays* s*,* $\overline{\sigma}(s) = \sigma(s)$*.*

The discrete fibration corresponding to $\overline{\sigma}$, $\mathrm{el}(\overline{\sigma})$, thus yields the desired template strategy.

However, we obtain the following result, which shows that even the more general simple strategies do not cover all template strategies.

▶ **Lemma 36.** *Both functors* $\mathrm{el}(\overline{\sigma}) \to (A \to B) \to \pm_V$ *are discrete Conduché fibrations. Furthermore, the former functor* $\mathrm{el}(\overline{\sigma}) \to (A \to B)$ *is* receptive*, in the sense that for all even-length plays* s *with immediate extensions* sm*, and all* $(s, x) \in \mathrm{el}(\overline{\sigma})$*, there exists a unique* $(sm, y) \in \mathrm{el}(\overline{\sigma})$ *and morphism* $(s, x) \to (sm, y)$ *mapped to* $s \to sm$ *by the projection.*

Proof. An easy computation using the characterisation of right Kan extensions as ends. ◀

Refining template strategies

As we did for games, let us now analyse and resolve the discrepancies. For compatibility with what we did for games, we take the coslice $\pm_V$ under OO and restrict to discrete fibrations $S \to OO/\pm_V$. Analogously to Lemma 19, we have functors $O/\pm_v \xleftarrow{s} OO/\pm_V \xrightarrow{t} O/\pm_v$.

▶ **Definition 37.** *Recalling Remark 13,* naively refined template strategies *are just as template strategies in Definition 20, but over* $O/\pm_v \xleftarrow{s} OO/\pm_V \xrightarrow{t} O/\pm_v$*, and with* r *a discrete fibration.*

We obtain a first improvement:

▶ **Lemma 38.** *The game* $A \to B$ *is a limit*

$$\begin{array}{ccccc} A & \longleftarrow & (A \to B) & \longrightarrow & B \\ {\scriptstyle p_A}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_B} \\ O/\pm_v & \xleftarrow[s]{} & OO/\pm_V & \xrightarrow[t]{} & O/\pm_v, \end{array} \tag{4}$$

and naively refined template strategies are equivalent to presheaves over $A \to B$*.*

Proof. The first statement essentially says that plays in $A \to B$ are uniquely determined by their projections and the interleaving schedule, which is clear. For the second statement, $(A \to B) \to OO/\pm_V$ is a discrete fibration, so naively refined template strategies are in one-to-one correspondence with discrete fibrations over $A \to B$, hence with presheaves. ◀

In order to precisely capture simple strategies, it remains to account for receptiveness. For this, our solution is to further restrict the template:

▶ **Definition 39.** *Let* $\mathbb{T}_V$ *denote the full subcategory of* $OO/\pm_V$ *spanning even-length schedules.*

▶ Remark 40. It is tempting to further restrict to non-empty, even-length schedules. However, the obtained span does not form a monad, hence the general template games construction does not apply.

▶ **Definition 41.** Refined template strategies *are defined just as template strategies in Definition 20, but over* $\mathbb{T}_v \xleftarrow{s} \mathbb{T}_V \xrightarrow{t} \mathbb{T}_v$*, and with* r *a discrete fibration.*

We obtain a restriction of Lemma 38 to the relevant plays:

▶ **Lemma 42.** *The full subcategory* $(A \to B)^P \hookrightarrow (A \to B)$ *spanning even-length plays is a limit*

$$\begin{array}{ccccc} A & \longleftarrow & (A \to B)^P & \longrightarrow & B \\ {\scriptstyle p_A}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_B} \\ \mathbb{T}_v & \xleftarrow[s]{} & \mathbb{T}_V & \xrightarrow[t]{} & \mathbb{T}_v, \end{array} \qquad (5)$$

and refined template strategies $A \to B$ *are equivalent to presheaves over* $(A \to B)^P$*.*

Proof. By repeated application of stability of discrete fibrations under pullback, $(A \to B)^P \to \mathbb{T}_V$ is a discrete fibration. Hence, in a diagram like (1) (with $\mathbb{T}$ instead of $\pm$), assuming $p, q \in \mathsf{DFib}$, r is a discrete fibration iff the induced morphism to $(A \to B)^P$ is. The result then follows from Lemma 32. ◀

▶ **Corollary 43.** *There is a full, reflective embedding from simple strategies* $\mathcal{S}(A \to B)$ *to refined template strategies, whose essential image consists of* definite *refined template strategies, i.e., those whose associated presheaf* $X \in \widehat{(A \to B)^P}$ *is such that* $X(\varepsilon) = 1$*.*

Proof. The embedding $(A \to B)^{P*} \hookrightarrow (A \to B)^P$ being full, right Kan extension along its opposite defines a full, reflective embedding $\widehat{(A \to B)^{P*}} \hookrightarrow \widehat{(A \to B)^P}$, which returns definite refined template strategies by the standard end formula. ◀

Summary

Until now, we have refined template games and strategies, first by replacing the original template $\pm$ by our $\mathbb{T}$, and second by restricting template games and strategies to be discrete fibrations. We have then identified simple games as definite template games, and constructed a full, reflective embedding from simple strategies $A \to B$ to refined template strategies $p_A \to p_B$, with essential image the definite ones. What remains to be seen is whether we can refine Melliès's double-categorical construction accordingly.

4 Template games

In this section, we review Melliès's double-categorical variant of the slice construction. We then apply it to deduce that the new template $\mathbb{T}$ yields a weak double category as desired. In the next section, we refine the construction using a factorisation system, which allows us to account for restriction to discrete fibrations.

4.1 Double categories

The key point is that the template $\pm$ forms a *double category* [5], in a way that describes the scheduling of composition of strategies. So let us first briefly review double categories.

A double category $\mathcal{C}$ essentially consists of a *horizontal* category $\mathcal{C}_h$ and a *vertical* one $\mathcal{C}_v$ sharing the same object set, together with a set of *cells* as in

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle u}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle v} \\ C & \xrightarrow[\quad g \quad]{} & D, \end{array}$$

where A, B, C, and D are objects, f and g are morphisms in $\mathcal{C}_h$, and u and v are morphisms in $\mathcal{C}_v$. In order to distinghish notationally between horizontal and vertical morphisms, we mark horizontal ones with a bullet. Cells are furthermore equipped with composition and identities in both directions. E.g., to any given cells α and β with compatible vertical border is assigned a composite cell $\beta \bullet \alpha$, as below left. Similarly, we have horizontal identities $id^{\bullet}_p$ as below right.

$$\begin{array}{ccccc} A & \xrightarrow{S} & B & \xrightarrow{S'} & E \\ {\scriptstyle p}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle q} & \Downarrow{\scriptstyle \beta} & \downarrow{\scriptstyle r} \\ C & \xrightarrow[T]{} & D & \xrightarrow[T']{} & F \end{array} \mapsto \begin{array}{ccc} A & \xrightarrow{S' \bullet S} & E \\ {\scriptstyle p}\downarrow & \Downarrow{\scriptstyle \beta \bullet \alpha} & \downarrow{\scriptstyle r} \\ C & \xrightarrow[T' \bullet T]{} & F \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{id^{\bullet}_A} & A \\ {\scriptstyle p}\downarrow & \Downarrow{\scriptstyle id^{\bullet}_p} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[id^{\bullet}_B]{} & B \end{array} \tag{6}$$

Both notions of composition are required to be associative and the corresponding identities unital. Thus, e.g., (6) defines a *horizontal cell category* $\mathcal{C}_H$. Similarly, there is a *vertical cell category* $\mathcal{C}_V$. Finally, the *interchange law* requires the two different ways of parsing any compatible pasting as below to agree, i.e., $(\delta \circ \gamma) \bullet (\beta \circ \alpha) = (\delta \bullet \beta) \circ (\gamma \bullet \alpha)$:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow & \Downarrow{\scriptstyle \gamma} & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \\ \downarrow & \Downarrow{\scriptstyle \beta} & \downarrow & \Downarrow{\scriptstyle \delta} & \downarrow \\ G & \longrightarrow & H & \longrightarrow & I. \end{array}$$

4.2 The template $\pm$ as a double category

Let us now show that the template $\pm$ forms a double category. As suggested by the notation, its vertical category and vertical cell category are $\pm_v$ and $\pm_V$. Its horizontal category $\pm_h$ is generated by the graph $O \to P$. It is thus isomorphic to the ordinal $\mathbb{2}$: there are exactly three horizontal morphisms: OO, PP, and OP. To complete the definition, it remains to define composition and identities in $\pm_H$. One way is to depict basic cells as triangles

$$\begin{array}{ccc} O & \longrightarrow & O \\ & \searrow & \downarrow \\ & & P \end{array} \qquad \begin{array}{ccc} O & \longrightarrow & P \\ \downarrow & \nearrow & \\ P & & \end{array} \qquad \begin{array}{ccc} P & \longrightarrow & P \\ \downarrow & \nearrow & \\ O & & \end{array} \qquad \begin{array}{ccc} O & \longrightarrow & P \\ & \searrow & \downarrow \\ & & O, \end{array} \tag{7}$$

respectively denoting $OO \to OP$, $OP \to PP$, $PP \to OP$, and $OP \to OO$. General cells are obtained by stacking up such basic triangles. Depicting cells as stacks of triangles yields the following inductive definition of composition of cells α and β as in (6):

- If there is an 'outwards' bottom triangle, i.e., the bottom of α and β look like either of

with $X \in \{O, P\}$ and $X^\perp$ denoting the other player, then the composite is obtained by composing the rest of α and β, and appending the obvious triangle ($(OX, OX^\perp)$, resp. $(XP, X^\perp P)$).
- Otherwise, there is a pair of interacting bottom triangles, as below left, in which case the composite is simply the composite of the rest of α and β – which is precisely where game semantical *hiding* is encoded in $\pm$.

▶ Remark 44. Ambiguous configurations as above right, where we would not know which triangle to put last in the composite, cannot occur. Indeed, existence of the left-hand triangle forces $M = P$, while existence of the right-hand one forces $M = O$.

Horizontal identities are the so-called *copycat* schedules. The copycat on the vertical morphism $O \to P$ is obtained by composing $OO \to OP$ and $OP \to PP$, and dually for $P \to O$. The copycat schedule of a general morphism is the obvious composite of these basic copycats.

We obtain as promised:

▶ **Proposition 45** (Melliès [14]). *The template $\pm$ forms a double category.*

4.3 Template games as a double slice

There is an alternative point of view on double categories which will be crucial to us: they may be axiomatised based on a span of functors $\mathcal{C}_v \xleftarrow{s} \mathcal{C}_V \xrightarrow{t} \mathcal{C}_v$. For this, let us consider the following structure, which is almost a (large) double category.

▶ **Definition 46.** *Let* $\mathsf{Span}(\mathsf{Cat})$ *have as objects all small categories, as vertical morphisms all functors, as horizontal morphisms* $A \to B$ *all spans* $A \leftarrow C \to B$ *of functors, and as cells below left all commuting diagrams as below right in* Cat.

Vertical composition is given by (componentwise) composition of functors, while horizontal composition is given by pullbacks and their universal property.

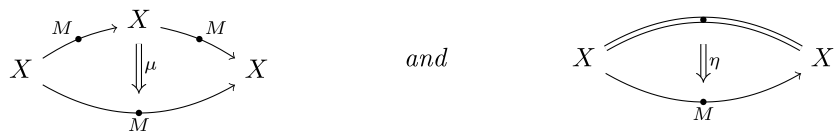
The structure formed by $\mathsf{Span}(\mathsf{Cat})$ is a *weak double category* [6], a weak form of double category where horizontal composition is only associative and unital up to coherent isomorphism, in a suitable sense.

▶ Remark 47. The horizontal arrows and *special* cells of a weak double category $\mathcal{C}$ form a bicategory $\mathcal{H}(\mathcal{C})$, where special means that the left and right borders are identities.

▶ Remark 48. The reason Span(Cat) is weak is that one cannot hope to make a strictly associative choice of pullbacks.

Just like one usually does in bicategories, we may define monads internally to weak double categories.

▶ **Definition 49.** *A* monad *in a weak double category $\mathcal{C}$ is a horizontal morphism $M\colon X \multimap X$, equipped with special cells*



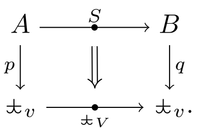
satisfying the obvious generalisation of the usual monad laws.

▶ **Proposition 50.** *A double category is the same as a monad in* Span(Cat)*.*

Proof. Composing an endo-span $\mathcal{C}_v \xleftarrow{s} \mathcal{C}_V \xrightarrow{t} \mathcal{C}_v$ with itself in Span(Cat) amounts to constructing the category of pairs of compatible horizontal morphisms and cells (6), so requiring a monad multiplication is requiring horizontal composition. Similarly, requiring a monad unit amounts to requiring horizontal identities. ◀

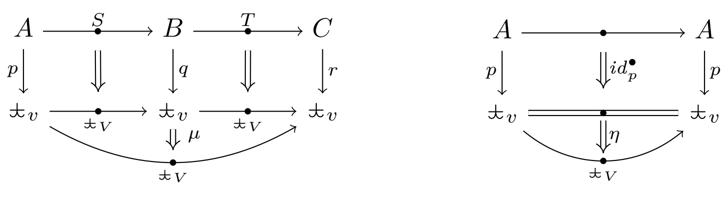
Explicitly, if $\mathcal{C}$ is a double category, then it can be seen as a monad $\mathcal{C}_V : \mathcal{C}_v \multimap \mathcal{C}_v$ in Span(Cat) with μ and η given by horizontal composition and identities.

It should now be clear that a template game is a vertical morphism to $\pm_v$ in Span(Cat), while a template strategy $A \to B$ is merely a cell



This allows us to define composition of template strategies, using the monad structure of $\pm$ given by Propositions 50 and 45.

▶ **Proposition 51.** *The composite of $S\colon A \to B$ and $T\colon B \to C$ in the sense of [14] is the pasting below left, while the identity on any $p\colon A \to \pm_v$ is the one below right.*



(8)

From this, it easily follows that strategies in fact form a weak double category, and clearly this construction works for any monad in any weak double category. Namely, Melliès's construction may be obtained by applying the following theorem.

▶ **Theorem 52.** *Given any monad $M_V\colon M_v \multimap M_v$ in a weak double category $\mathcal{C}$, there is a* slice *weak double category $\mathcal{C}/M$ whose*

- *vertical category is* $(\mathcal{C}/M)_v = \mathcal{C}_v/M_v$*;*
- *vertical cell category is* $(\mathcal{C}/M)_V = \mathcal{C}_V/M_V$*;*
- *horizontal composition is given by pasting with* μ*, as on the left in* (8)*;*
- *horizontal identity on* $p\colon A \to M_v$ *is by pasting with* η *as on the right in* (8)*.*
- *and all operations on cells are given by their counterparts in* $\mathcal{C}$*.*

▶ **Proposition 53.** *The weak double category* $\underline{\mathbf{Games}}(\pm)$ *of template games [14] is equal to* $\mathsf{Span}(\mathsf{Cat})/\pm$*.*

A weak double category with trivial category $\mathcal{C}_v$ is nothing but a monoidal category. In that case, the theorem reduces to the following well-known result used, e.g., in Weber [17]:

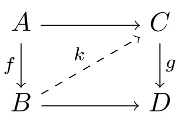
▶ **Corollary 54.** *The slice of a monoidal category over a monoid is again monoidal.*

5 Refined template games, simple games, and Day convolution

5.1 Refined template games

We now want to recover simple games by replacing $\pm$ with $\mathbb{T}$, and restricting the slice construction in $\mathsf{Span}(\mathsf{Cat})$ to discrete fibrations (for vertical morphisms and cells). For this, we appeal to factorisation systems.

▶ **Definition 55.** *For all morphisms* f *and* g *in a category* $\mathcal{C}$*, let* $f \perp g$ *iff for all commuting squares as in the solid part of*



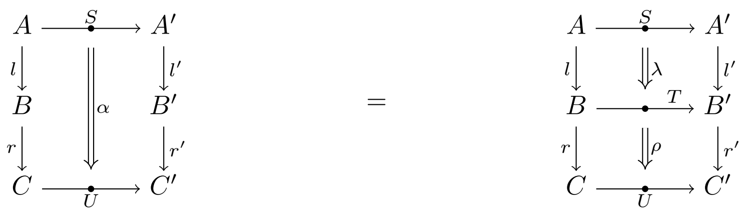
there exists a unique k *as shown making both triangles commute. For any class* $\mathcal{M}$ *of morphisms, let* $\mathcal{M}^{\perp} = \{g \mid \forall f \in \mathcal{M}, f \perp g\}$*. We define* ${}^{\perp}\mathcal{M}$ *symmetrically.*

A (strong) factorisation system *[1] on a category* $\mathcal{C}$ *consists in classes* $\mathcal{L}$ *and* $\mathcal{R}$ *of arrows such that* $\mathcal{L}^{\perp} = \mathcal{R}$*,* $\mathcal{L} = {}^{\perp}\mathcal{R}$*, and every arrow factors as* $r \circ l$ *with* $l \in \mathcal{L}$ *and* $r \in \mathcal{R}$*.*

▶ **Example 56.** Discrete fibrations form the right class of the *comprehensive* factorisation system $(\mathsf{Fin}, \mathsf{DFib})$ on Cat, whose left class is that of *final* functors [16].

Our refined construction is based on the following generalisation of factorisation systems.

▶ **Definition 57.** *A* double factorisation system *on a weak double category* $\mathcal{C}$ *consists of factorisation systems* $(\mathcal{L}_v, \mathcal{R}_v)$ *and* $(\mathcal{L}_V, \mathcal{R}_V)$ *on* $\mathcal{C}_v$ *and* $\mathcal{C}_V$*, respectively, such that the source and target functors* $\mathcal{C}_V \to \mathcal{C}_v$ *both map* $\mathcal{L}_V$ *to* $\mathcal{L}_v$ *and* $\mathcal{R}_V$ *to* $\mathcal{R}_v$*, and all cells* α *as below left with* $l, l' \in \mathcal{L}_v$ *and* $r, r' \in \mathcal{R}_v$ *factor as below right, with* $\lambda \in \mathcal{L}_V$ *and* $\rho \in \mathcal{R}_V$*.*



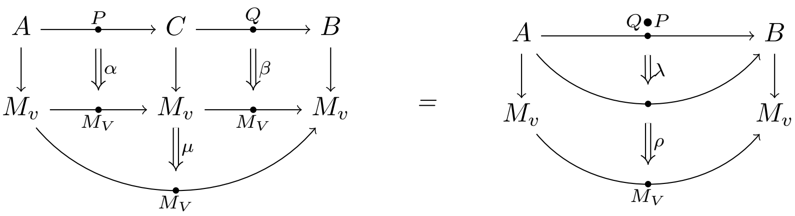
▶ **Lemma 58.** *Discrete fibrations and componentwise discrete fibrations are the right classes of a double factorisation system* $((\mathsf{Fin}, \mathsf{DFib}), (\mathsf{Fin}_V, \mathsf{DFib}_V))$ *on* $\mathsf{Span}(\mathsf{Cat})$*. Furthermore,* DFib_V *is stable under horizontal composition and identities.*

Proof. As is well-known, discrete fibrations may be defined by unique lifting w.r.t. the injection $\mathbb{1} \hookrightarrow \mathbb{2}$ mapping 0 to 1. Componentwise discrete fibrations may be defined similarly, in $\mathsf{Span}(\mathsf{Cat})_V$. The result then follows from componentwise discrete fibrations being stable under horizontal composition, which holds because they are stable under pullback in the arrow category. ◀

In order to state the promised generalisation of Theorem 52 in full generality, we need the following, somewhat awkward notion of stability of left residuals. Indeed, our two applications work for rather different reasons, as emphasised by Corollaries 62 and 63 below.

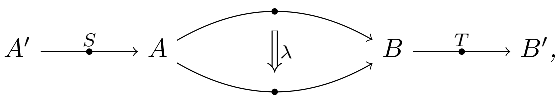
▶ **Definition 59.** *A monad* $M_V : M_v \to M_v$ *in a weak double category* $\mathcal{C}$ *has* stable left residuals *w.r.t. a double factorisation system* $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$*, iff*

(a) *for all* $\alpha, \beta \in \mathcal{R}_V$ *such that the composite below left factors as on the right,*

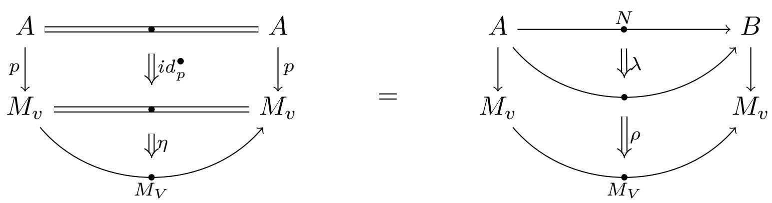


(9)

for any $S: A' \to A$ *and* $T: B \to B'$ *in* $\mathcal{C}_h$*, the composite* $id_T \bullet \lambda \bullet id_S$ *below is in* $\mathcal{L}_V$*;*



(b) *for all* $p \in \mathcal{R}_v$ *such that the composite below left factors as on the right, for any* $S: A' \to A$ *and* $T: B \to B'$ *in* $\mathcal{C}_h$*,* $id_T \bullet \lambda \bullet id_S$ *is in* $\mathcal{L}_V$*.*



(10)

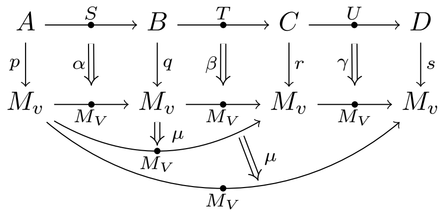
▶ Remark 60. In both cases, existence of a special λ follows by Definition 57.

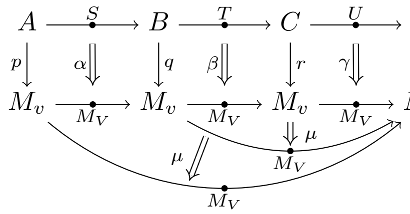
▶ **Theorem 61.** *Consider any monad* $M_V : M_v \to M_v$ *in a weak double category* $\mathcal{C}$ *with stable left residuals w.r.t. a double factorisation system* $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$*. Then there is a* slice *weak double category* $\mathcal{C}/_{\mathcal{R}_V} M$ *whose*

- *vertical category* $(\mathcal{C}/_{\mathcal{R}_V} M)_v$ *is* $\mathcal{C}_v/_{\mathcal{R}_v} M_v$*, the full subcategory of* $\mathcal{C}_v/M_v$ *on maps in* $\mathcal{R}_v$*;*
- *vertical category of cells is* $(\mathcal{C}/_{\mathcal{R}_V} M)_V = \mathcal{C}_V/_{\mathcal{R}_V} M_V$*;*
- *horizontal composition is given by* ρ *in* (9)*;*
- *horizontal identity on any* $p: A \to M_v$ *is given by* ρ *in* (10)*;*
- *and all operations on cells are given by their counterparts in* $\mathcal{C}$*.*

Proof. By coherence for weak double categories [8, Theorem 7.5], and assuming a higher universe in which $\mathcal{C}$ is small, we may assume that $\mathcal{C}$ is in fact a (strict) double category.

Composition of cells in $\mathcal{C}/_{\mathcal{R}_V}M$ is just as in $\mathcal{C}$, so the only non-trivial point to check is weak associativity and unitality of horizontal composition (of morphisms). For weak associativity, we observe that both cells





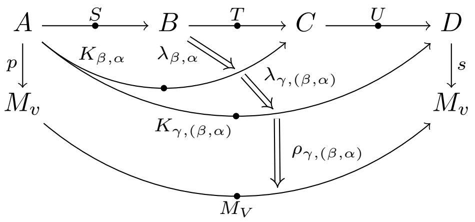
are equal. Now, denoting composition in $(\mathcal{C}/_{\mathcal{R}_V}M)_h$ by $\tilde{\bullet}$, $\gamma\tilde{\bullet}(\beta\tilde{\bullet}\alpha)$ and $(\gamma\tilde{\bullet}\beta)\tilde{\bullet}\alpha$ are obtained by factoring them as follows. For the former, we factor

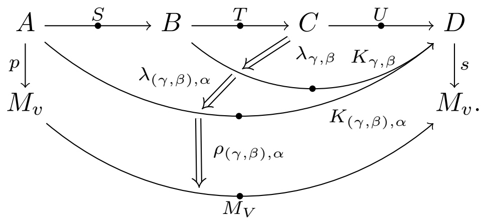
$$T \bullet S \xrightarrow{\beta\bullet\alpha} M_V \bullet M_V \xrightarrow{\mu} M_V \qquad \text{as} \qquad T \bullet S \xrightarrow{\lambda_{\beta,\alpha}} K_{\beta,\alpha} \xrightarrow{\rho_{\beta,\alpha}} M_V,$$

in which $\lambda_{\beta,\alpha}$ special by definition. We then factor

$$U \bullet K_{\beta,\alpha} \xrightarrow{\gamma\bullet\rho_{\beta,\alpha}} M_V \bullet M_V \xrightarrow{\mu} M_V \qquad \text{as} \qquad U \bullet K_{\beta,\alpha} \xrightarrow{\lambda_{\gamma,(\beta,\alpha)}} K_{\gamma,(\beta,\alpha)} \xrightarrow{\rho_{\gamma,(\beta,\alpha)}} M_V.$$

The other composite may be computed symmetrically, so that we obtain factorisations:





By stability of left residuals, both are in fact factorisations for $(\mathcal{L}_V, \mathcal{R}_V)$, so that by lifting, we obtain a special cell $a_{\alpha,\beta,\gamma} : K_{\gamma,(\beta,\alpha)} \xrightarrow{\simeq} K_{(\gamma,\beta),\alpha}$ such that $\rho_{(\gamma,\beta),\alpha} \circ a_{\alpha,\beta,\gamma} = \rho_{\gamma,(\beta,\alpha)}$, which is our candidate associator for $\mathcal{C}/_{\mathcal{R}_V}M$. It satisfies the MacLane pentagon by uniqueness of lifting.

Weak unitality follows similarly. ◀

▶ **Corollary 62.** *Consider any monad $M_V : M_v \to M_v$ in a weak double category $\mathcal{C}$ with double factorisation system $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$. If $\eta, \mu \in \mathcal{R}_V$ and $\mathcal{R}_V$ is stable under horizontal composition and identities, then $\mathcal{C}/_{\mathcal{R}_V}M$ exists and is a sub weak double category of $\mathcal{C}/M$.*

Proof. Both pastings on the left of (9) and (10) are already in $\mathcal{R}_V$. ◀

▶ **Corollary 63.** *Consider any monad $M_V : M_v \to M_v$ in a weak double category $\mathcal{C}$ with double factorisation system $((\mathcal{L}_v, \mathcal{R}_v), (\mathcal{L}_V, \mathcal{R}_V))$. If $\mathcal{L}_V$ is stable under horizontal composition, then $\mathcal{C}/_{\mathcal{R}_V}M$ is a weak double category.*

Proof. Stability under horizontal composition entails stability under whiskering. ◀

5.2 Simple games

Let us at last return to simple games. We have:

▶ **Lemma 64.** $\mathbb{T}$ *is a monad whose multiplication and unit are in* DFib_V*.*

Proof. In $\mathbb{T}$, multiplication is composition of schedules (through parallel composition and hiding) and the unit is given by copycat schedules. The crucial point to prove that multiplication is in DFib_V is that, for any pair (p, q) of schedules in $\mathbb{T}_V \bullet \mathbb{T}_V$, the last move in $s(q)$ $(= t(p))$ cannot be last in both p and q for polarity reasons. For the unit, the crucial point is that copycat schedules are closed under restrictions. ◀

By Corollary 62, we have:

▶ **Corollary 65.** $\mathsf{Span}(\mathsf{Cat})/_{\mathsf{DFib}_V}\mathbb{T}$ *is a sub weak double category of* $\mathsf{Span}(\mathsf{Cat})$*.*

Finally, let us relate to simple strategies.

▶ **Definition 66** ([12, Definition 9])**.** *Simple games, strategies, and natural transformations form a bicategory* $\mathcal{S}$*.*

▶ **Theorem 67.** *The full, reflective embeddings* $F_{A,B} \colon \mathcal{S}(A, B) \rightsquigarrow \mathsf{DFib}((A \to B)^P)$ *of Corollary 43 determine a locally reflective and fully-faithful weak 2-functor*

$$\mathcal{S} \to \mathcal{H}(\mathsf{Span}(\mathsf{Cat})/_{\mathsf{DFib}_V}\mathbb{T})$$

(where $\mathcal{H}$ *is the bicategory of special cells, as in Remark 47), whose essential image consists of definite refined template games and strategies.*

Proof. The essential image part of the result is clear. By [13, §2.2], we need to organise the full, reflective embeddings $F_{A,B}$ into a weak 2-functor, which here means that F commutes with composition of strategies up to coherent isomorphism. ◀

▶ **Corollary 68.** *The bicategory* $\mathcal{S}$ *of simple games and strategies is biequivalent to the locally full sub-bicategory of* $\mathcal{H}(\mathsf{Span}(\mathsf{Cat})/_{\mathsf{DFib}_V}\mathbb{T})$ *spanning definite refined template games and strategies.*

5.3 Day convolution

We finally reach the application mentioned in the introduction: Day convolution. The purpose of this operation is to show that $\widehat{\mathbb{C}}$ is monoidal when $\mathbb{C}$ is. Let us now recover this structure from Theorem 61, in the particular case where $\mathbb{C}$ is strictly monoidal. The starting point is the following weak double category:

▶ **Definition 69.** *Let* $\mathcal{W}$ *be the sub weak double category of* $\mathsf{Span}(\mathsf{Cat})$ *obtained by restricting objects to the terminal object* $\mathbb{1}$ *and vertical morphisms to the identity thereon.*

Thus, $\mathcal{W}_V$ consists of categories and functors, and horizontal composition is given by cartesian product and its universal property.

▶ **Lemma 70.** *A monad in* $\mathcal{W}$ *is a strict monoidal category* $\mathbb{C}$*.*

Proof. The monad multiplication $\mu\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ gives the tensor product while the unit $\eta\colon \mathbb{1} \to \mathbb{C}$ gives the tensor unit, and coherence equations for the monad those of the monoidal category. ◀

Clearly, final functors and discrete fibrations form a double factorisation system $(\mathcal{L}_V, \mathcal{R}_V) = (\mathsf{Fin}, \mathsf{DFib})$ on $\mathcal{W}$. Final functors being closed under binary products, $\mathcal{L}_V$ is closed under horizontal composition and identities, so Corollary 63 applies and we obtain a weak double category $\mathcal{W}/_{\mathsf{DFib}}\mathbb{C}$. This weak double category is vertically trivial, hence underlies a monoidal category, say $\mathbb{C}'$.

Let us now show that $\mathbb{C}'$ is equivalent to $\widehat{\mathbb{C}}$ equipped with the convolution tensor product. We first recall that the latter is given as follows:

▶ **Definition 71.** *For any small monoidal category $\mathbb{C}$ and $X, Y \in \widehat{\mathbb{C}}$, let*

$$(X \otimes Y)(c) = \int^{(c_1,c_2)\in\mathbb{C}^2} X(c_1) \times Y(c_2) \times \mathbb{C}(c, c_1 \otimes c_2).$$

▶ **Lemma 72.** *Let $f\colon A \to B$ be a functor. The discrete fibration ρ_f associated to f by the comprehensive factorisation system is determined up to isomorphism by*

$$\partial^\star(\rho_f)(b) \cong \int^{a\in A} B(b, f(a)),$$

where $\partial^\star\colon \mathsf{DFib}(B) \to \widehat{B}$ is the standard equivalence between discrete fibrations and presheaves.

Proof. This is actually obvious by construction. In [16], the dual case is actually treated, initial functors and discrete opfibrations. But up to this discrepancy, $\partial^\star(\rho_f)$ is precisely k in the proof of [16, Theorem 3], which would in our case be defined as the left Kan extension of $A^{op} \xrightarrow{!} 1 \xrightarrow{1} \mathsf{Set}$ along f^{op}. By the well-known characterisation of left Kan extensions by coends, we readily obtain the desired formula. ◀

▶ **Theorem 73.** *For any strictly monoidal category $\mathbb{C}$, the monoidal category $\mathbb{C}'$ is equivalent to $\widehat{\mathbb{C}}$ equipped with the convolution tensor product.*

Proof. By construction, given two presheaves $X, Y \in \widehat{\mathbb{C}}$ and transporting them to their corresponding discrete fibrations, say $S\colon \mathrm{el}(X) \to \mathbb{C}$ and $T\colon \mathrm{el}(Y) \to \mathbb{C}$, their tensor product $S\tilde{\bullet}T$ in $\mathbb{C}'$ is the right factor of the composite

$$\mathrm{el}(X) \times \mathrm{el}(Y) \xrightarrow{S\times T} \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}.$$

By Lemma 72, the result has its corresponding presheaf defined up to isomorphism by

$$\begin{array}{rcl} \partial^\star(S\tilde{\bullet}T)(c) & \cong & \int^{(a,b)\in\mathrm{el}(X)\times\mathrm{el}(Y)} \mathbb{C}(c, \otimes((S\times T)(a,b))) \\ & = & \int^{(a,b)\in\mathrm{el}(X)\times\mathrm{el}(Y)} \mathbb{C}(c, S(a)\otimes T(b)) \\ & \cong & \int^{((c_1,x),(c_2,y))\in\mathrm{el}(X)\times\mathrm{el}(Y)} \mathbb{C}(c, c_1\otimes c_2) \\ & \cong & \int^{c_1,c_2} X(c_1)\times Y(c_2)\times\mathbb{C}(c, c_1\otimes c_2), \end{array}$$

as desired. ◀

6 Conclusion and perspectives

We have designed an abstract slice construction over monads in weak double categories, which has as instances (1) a weak double category of simple games and non-deterministic strategies, whose underlying bicategory of definite games and strategies is biequivalent to the standard one, and (2) the monoidal category of presheaves over any small strict monoidal category.

For future work, we first could try to accomodate not only the weak double category structure of template games, but also symmetric monoidal closedness. Furthermore, Melliès is also currently working on a template game model of full linear logic. This will of course be a useful feature to incorporate to our framework. Another direction for future work is to generalise our construction to encompass Day convolution for non-strict monoidal categories. What is needed here is a common generalisation of [7, Theorem 14.5] and Theorem 61. Finally, it is slightly unsatisfactory to only get standard simple games up to a locally fully-faithful embedding. In order to obtain a biequivalence, preliminary investigation suggests that instead of restricting the slice construction w.r.t. some factorisation system, it would be more expressive to construct the factorisation system directly in the relevant slice, for which fibrant objects would be the desired strategies. Indeed, e.g., polarities, which cannot be used before slicing, become available in the slice. This technique also seems to apply for refining the correspondence to, e.g., deterministic strategies, and possibly even innocence.

References

1. A. K. Bousfield. Constructions of Factorization Systems in Categories. *Journal of Pure and Applied Algebra*, 9(2-3):287–329, 1977.
2. B. Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, volume 137 of *Lecture Notes in Mathematics*, pages 1–38. Springer, 1970.
3. C. Eberhart. *Catégories et diagrammes de cordes pour les jeux concurrents*. PhD thesis, Université Savoie Mont Blanc, 2018.
4. C. Eberhart and T. Hirschowitz. What's in a game?: A theory of game models. In *Proc. 33rd Symposium on Logic in Computer Science*, pages 374–383. ACM, 2018. doi:10.1145/3209108.3209114.
5. C. Ehresmann. Catégories structurées. *Annales scientifiques de l'Ecole Normale Supérieure*, 80(4):349–426, 1963.
6. R. Garner. *Polycategories*. PhD thesis, University of Cambridge, 2006.
7. R. Garner and M. Shulman. Enriched categories as a free cocompletion. *Advances in Mathematics*, 289:1–94, 2016.
8. M. Grandis and R. Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999.
9. R. Harmer, J. M. E. Hyland, and P.-A. Melliès. Categorical Combinatorics for Innocent Strategies. In *Proc. 22nd Symposium on Logic in Computer Science*, pages 379–388. IEEE, 2007.
10. M. Hirschowitz, A. Hirschowitz, and T. Hirschowitz. A theory for game theories. In *Proc. 27th Foundations of Software Technology and Theoretical Computer Science*, volume 4855 of *Lecture Notes in Computer Science*, pages 192–203. Springer, 2007.
11. J. M. E. Hyland. Game Semantics. In Andrew M. Pitts and Peter Dybjer, editors, *Semantics and Logics of Computation*, pages 131–184. Cambridge University Press, 1997.
12. C. Jacq and P.-A. Melliès. Categorical Combinatorics for Non Deterministic Strategies on Simple Games. In *Proc. 21st Foundations of Software Science and Computational Structures*, volume 10803 of *Lecture Notes in Computer Science*, pages 39–70. Springer, 2018. doi:10.1007/978-3-319-89366-2_3.

13 T. Leinster. Basic Bicategories. *arXiv Mathematics e-prints*, page math/9810017, October 1998. `arXiv:math/9810017`.

14 P.-A. Melliès. Categorical combinatorics of scheduling and synchronization in game semantics. *Proc. ACM Program. Lang.*, 3(POPL):23:1–23:30, January 2019. `doi:10.1145/3290336`.

15 S. Rideau and G. Winskel. Concurrent Strategies. In *Proc. 26th Symposium on Logic in Computer Science*, pages 409–418. IEEE, 2011.

16 R. Street and R. F. C. Walters. The Comprehensive Factorization of a Functor. *Bulletin of the American Mathematical Society*, 79(5), 1973.

17 M. Weber. Generic morphisms, parametric representations and weakly cartesian monads. *Theory and Applications of Categories*, 13:191–234, 2004.