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### REPRESENTATION THEORY OF DISCONNECTED REDUCTIVE GROUPS

#### PRAMOD N. ACHAR, WILLIAM HARDESTY, AND SIMON RICHE

ABSTRACT. We study three fundamental topics in the representation theory of disconnected algebraic groups whose identity component is reductive: (i) the classification of irreducible representations; (ii) the existence and properties of Weyl and dual Weyl modules; and (iii) the decomposition map relating representations in characteristic 0 and those in characteristic p (for groups defined over discrete valuation rings). For each of these topics, we obtain natural generalizations of the well-known results for connected reductive groups.

#### 1. INTRODUCTION

Let G be a (possibly disconnected) affine algebraic group over an algebraically closed field  $\mathbb{k}$ , and let  $G^{\circ}$  be its identity component. We call G a *(possibly) disconnected reductive group* if  $G^{\circ}$  is reductive. The goal of this paper is to extend a number of well-known foundational facts about connected reductive groups to the disconnected case.

Such groups occur naturally, even when one is primarily interested in *connected* reductive groups. Namely, for a connected reductive group H, the stabilizer  $H^x$  of a nilpotent element in the Lie algebra of H may be disconnected. Let  $H^x_{\text{unip}}$  be its unipotent radical; then  $H^x/H^x_{\text{unip}}$  is a disconnected reductive group. The study of (the derived category of) coherent sheaves on the nilpotent cone  $\mathcal{N}$  of H, and in particular of *perverse-coherent sheaves* on  $\mathcal{N}$ , leads naturally to questions about representations of  $H^x/H^x_{\text{unip}}$ . See [AHR] for some questions of this form, and for some applications of the results of this paper.

The present paper contains three main results:

- (1) We classify the irreducible representations of G in terms of those of  $G^{\circ}$ , via an adaptation of Clifford theory (Theorem 2.16).
- (2) Assuming that the characteristic of k does not divide  $|G/G^{\circ}|$ , we prove that the category of finite-dimensional *G*-modules has a natural structure of a highest-weight category (Theorem 3.7).
- (3) Starting from a disconnected reductive group scheme over an extension of the *p*-adic integers (with algebraically closed residue field), one obtains a "decomposition map" relating the Grothendieck groups of representations in characteristic 0 and in characteristic *p*. We prove that this map is an isomorphism.

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These results are certainly not surprising, and some of them may be known to experts, but we are not aware of a reference that treats them in the detail and generality needed for the applications in [AHR].

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#### 2. Classification of simple representations

In this section we consider (affine) algebraic groups over an arbitrary algebraically closed field k. Our goal is to describe the representation theory of a disconnected algebraic group G whose neutral connected component  $G^{\circ}$  is reductive in terms of the representation theory of  $G^{\circ}$ , via a kind of Clifford theory.

2.1. Twist of a representation by an automorphism. Let G be an algebraic group,  $\varphi : G \xrightarrow{\sim} G$  an automorphism, and let  $\pi = (V, \varrho)$  be a representation of G. Then we define the representation  $\varphi \pi$  as the pair  $(V, \varrho \circ \varphi^{-1})$ . (Below, we will most of the time write V for  $\pi$ , and  $\varphi V$  for  $\varphi \pi$ .) It is straightforward to check that if  $\psi : G \xrightarrow{\sim} G$  is a second automorphism, then we have

(2.1) 
$${}^{\psi(\varphi\pi)} = {}^{\psi\circ\varphi\pi}.$$

If  $f: \pi \to \pi'$  is a morphism of *G*-representations, then the same linear map defines a morphism of *G*-representations  ${}^{\varphi}\pi \to {}^{\varphi}\pi'$ , which will sometimes be denoted  ${}^{\varphi}f$ .

**Lemma 2.1.** Let  $H \subset G$  be a subgroup, and  $(V, \varrho)$  be a representation of H. Then there exists a canonical isomorphism of G-modules

$${}^{\varphi}\operatorname{Ind}_{H}^{G}(V,\varrho)\cong\operatorname{Ind}_{\varphi(H)}^{G}(V,\varrho\circ\varphi^{-1}).$$

*Proof.* By definition, we have

$$\operatorname{Ind}_{\varphi(H)}^{G}(V,\varrho) = \{ f: G \to V \mid \forall h \in H, \ f(gh) = \varrho(h^{-1})(f(g)) \},$$
$$\operatorname{Ind}_{\varphi(H)}^{G}(V,\varrho \circ \varphi^{-1}) = \{ f: G \to V \mid \forall h \in \varphi(H), \ f(gh) = \varrho \circ \varphi^{-1}(h^{-1})(f(g)) \}.$$

Here, in both cases the functions are assumed to be algebraic, and the G-action is defined by  $(g \cdot f)(h) = f(g^{-1}h)$ . We have a natural isomorphism of vector spaces

$$\operatorname{Ind}_{H}^{G}(V,\varrho) \xrightarrow{\sim} \operatorname{Ind}_{\varphi(H)}^{G}(V,\varrho \circ \varphi^{-1})$$

sending f to  $f \circ \varphi^{-1}$ . It is straightforward to check that this morphism is an isomorphism of G-modules from  ${}^{\varphi}\operatorname{Ind}_{H}^{G}(V,\varrho)$  to  $\operatorname{Ind}_{\varphi(H)}^{G}(V,\varrho \circ \varphi^{-1})$ .

Remark 2.2. More generally, if G' is another algebraic group and  $\varphi : G \xrightarrow{\sim} G'$  is an isomorphism, for any G-module  $\pi$  we can consider the G' module  ${}^{\varphi}\pi$  defined as above. Then the same arguments as for Lemma 2.1 show that we have  ${}^{\varphi}\mathrm{Ind}_{H}^{G}(\pi) \cong$  $\mathrm{Ind}_{\omega(H)}^{G'}({}^{\varphi}\pi).$ 

In particular, assume that we are given an algebraic group G' and an embedding of G as a normal subgroup of G'. Then for any  $g \in G'$ , we have an automorphism  $\operatorname{ad}(g)$  of G sending h to  $ghg^{-1}$ . In this setting, we will write  ${}^{g}V$  for  ${}^{\operatorname{ad}(g)}V$ , and  ${}^{g}f$  for  ${}^{\operatorname{ad}(g)}f$ . Then for  $g, h \in G'$ , since  $\operatorname{ad}(g) \circ \operatorname{ad}(h) = \operatorname{ad}(gh)$ , (2.1) translates to  ${}^{g}({}^{h}V) = {}^{gh}V$ .

The verification of the following lemma is straightforward.

**Lemma 2.3.** Let  $(V, \varrho)$  be a representation of G. Then if  $g \in G$ ,  $\varrho(g^{-1})$  induces an isomorphism  $V \xrightarrow{\sim} {}^{g}V$ .

2.2. Disconnected reductive groups. From now on we fix an algebraic group G whose identity component  $G^{\circ}$  is reductive. We set  $A := G/G^{\circ}$  (a finite group). The canonical quotient morphism  $G \to A$  will be denoted  $\varpi$ .

Let T be the "universal maximal torus" of  $G^{\circ}$ , i.e., the quotient B/(B, B) for any Borel subgroup  $B \subset G^{\circ}$ . (Since all Borel subgroups in  $G^{\circ}$  are  $G^{\circ}$ -conjugate, and since  $B = N_{G^{\circ}}(B)$  acts trivially on B/(B, B), the quotient B/(B, B) does not depend on B, up to canonical isomorphism.) Let  $\mathbf{X} = X^*(T)$  be its weight lattice. If  $T' \subset B$  is any maximal torus, then the composition  $T' \hookrightarrow B \to T$  is an isomorphism, and this lets us identify  $\mathbf{X}$  with  $X^*(T')$ . The image in  $\mathbf{X}$  under this identification of the roots of (G, T'), and of the subset of positive roots (chosen as the opposite of the T'-weights on the Lie algebra of B), do not depend on the choice of T'; so they define the canonical root system  $\Phi \subset \mathbf{X}$  and the subset  $\Phi^+ \subset \Phi$  of positive roots. Similar comments apply to coroots, so that we can define the dominant weights  $\mathbf{X}^+ \subset \mathbf{X}$ . We denote by W the Weyl group of T. (This group is well defined because  $N_B(T') = T'$  for a maximal torus T' contained in a Borel subgroup B.)

Given a weight  $\lambda \in \mathbf{X}^+$ , we denote by

$$L(\lambda), \quad \Delta(\lambda), \quad \nabla(\lambda)$$

the irreducible, Weyl, and dual Weyl  $G^{\circ}$ -modules, respectively, corresponding to  $\lambda$ . Here  $\nabla(\lambda)$  is defined as the induced module  $\operatorname{Ind}_B^{G^{\circ}}(\Bbbk_B(\lambda))$  for some choice of Borel subgroup  $B \subset G^{\circ}$ ,  $L(\lambda)$  is the unique simple submodule of  $\nabla(\lambda)$ , and  $\Delta(\lambda)$  is defined as  $(\nabla(-w_0\lambda))^*$ , where  $w_0 \in W$  is the longest element. (These modules do not depend on the choice of B up to isomorphism thanks to Lemma 2.1 and Lemma 2.3.)

For any  $g \in G$  and any Borel subgroup  $B \subset G^{\circ}$ ,  $\operatorname{ad}(g)$  induces an isomorphism  $B/(B,B) \xrightarrow{\sim} gBg^{-1}/(gBg^{-1}, gBg^{-1})$ . Since  $gBg^{-1}$  is also a Borel subgroup of  $G^{\circ}$ , this defines an automorphism  $\operatorname{\underline{ad}}(g)$  of T. The fact that T is well defined translates to the property that  $\operatorname{\underline{ad}}(g) = \operatorname{id}$  if  $g \in G^{\circ}$ , so that  $\operatorname{\underline{ad}}$  factors through a morphism  $A \to \operatorname{Aut}(T)$ , which we will also denote by  $\operatorname{\underline{ad}}$ .

For  $a \in A$  and  $\lambda \in \mathbf{X}$ , we set

(2.2) 
$${}^{a}\lambda := \lambda \circ \underline{\mathrm{ad}}(a^{-1}).$$

This operation defines an action of A on  $\mathbf{X}$ , which preserves  $\Phi$ ,  $\Phi^+$  and  $\mathbf{X}^+$ . Moreover, Lemma 2.1 implies that for any  $\lambda \in \mathbf{X}^+$  and  $g \in G$ , we have canonical isomorphisms

(2.3) 
$${}^{g}\Delta(\lambda) \cong \Delta({}^{\varpi(g)}\lambda), {}^{g}L(\lambda) \cong L({}^{\varpi(g)}\lambda), {}^{g}\nabla(\lambda) \cong \nabla({}^{\varpi(g)}\lambda).$$

We will denote by  $\operatorname{Irr}(G^{\circ})$  the set of isomorphism classes of simple  $G^{\circ}$ -modules. This set admits an action of G, where g acts via  $[V] \mapsto [{}^{g}V]$ . (Of course, this action factors through an action of A.) The constructions above provide a natural bijection  $\mathbf{X}^{+} \xrightarrow{\sim} \operatorname{Irr}(G^{\circ})$  (sending  $\lambda$  to the isomorphism class of  $L(\lambda)$ ), which is A-equivariant in view of (2.3).

**Lemma 2.4.** Let V be an irreducible G-module. Then V is semisimple as a  $G^{\circ}$ -module. All of its irreducible  $G^{\circ}$ -submodules lie in a single G-orbit in  $Irr(G^{\circ})$ .

*Proof.* Choose an irreducible  $G^{\circ}$ -submodule  $M \subset V$ , and choose a set of coset representatives  $g_1, \ldots, g_r$  for  $G^{\circ}$  in G. The subspace

$$\sum_{i=1}^r g_i M \subset V$$

is stable under the action of G, so it must be all of V. Each summand  $g_i M$  is stable under  $G^\circ$ , so there is a surjective map of  $G^\circ$ -representations

$$\bigoplus_{i=1}^{r} g_i M \to \sum_{i=1}^{r} g_i M = V.$$

Now,  $g_i M$  is isomorphic as a  $G^{\circ}$ -module to  $g_i M$ ; in particular, each  $g_i M$  is an irreducible  $G^{\circ}$ -module, and  $\bigoplus_i g_i M$  is semisimple. Thus, as a  $G^{\circ}$ -module, V is a quotient of a semisimple module, all of whose summands lie in a single G-orbit of  $\operatorname{Irr}(G^{\circ})$ , so the same holds for V itself.  $\Box$ 

2.3. The component group and induced representations. For each  $a \in A = G/G^{\circ}$ , let us choose, once and for all, a representative  $\iota(a) \in G$ . In the special case  $a = 1_A$ , we require that

$$\iota(1_A) = 1_G.$$

Given  $a, b \in A$ , the representative  $\iota(ab)$  need not be equal to  $\iota(a)\iota(b)$ ; but these elements lie in the same coset of  $G^{\circ}$ . Explicitly, there is a unique element  $\gamma(a, b) \in G^{\circ}$  such that

$$\iota(a)\iota(b) = \iota(ab)\gamma(a,b).$$

Our assumption on  $\iota(1_A)$  implies that for any  $a \in A$ , we have

$$\gamma(1_A, a) = \gamma(a, 1_A) = 1_G.$$

By expanding  $\iota(abc)$  in two ways, one finds that

(2.4) 
$$\gamma(ab,c) \cdot \operatorname{ad}(\iota(c)^{-1})(\gamma(a,b)) = \gamma(a,bc)\gamma(b,c).$$

Now let V be a G°-module. By Lemma 2.3, for any  $a, b \in A$  the action of  $\gamma(a, b)$  defines an isomorphism of G°-modules

$$\gamma^{(a,b)}V \xrightarrow{\sim} V.$$

Twisting by  $\iota(ab)$  we deduce an isomorphism

$$\phi_{a,b}: {}^{\iota(a)\iota(b)}V \xrightarrow{\sim} {}^{\iota(ab)}V.$$

We can use the maps  $\iota$  and  $\gamma$  to explicitly describe representations of G that are induced from  $G^{\circ}$ , as follows. Let us denote by  $\Bbbk[A]$  the group algebra of A over  $\Bbbk$ . Let V be a  $G^{\circ}$ -module, and consider the vector space

(2.5) 
$$\tilde{V} = \Bbbk[A] \otimes V = \bigoplus_{f \in A} \Bbbk f \otimes V.$$

We now explain how to make  $\tilde{V}$  into a *G*-module. Note that every element of *G* can be written uniquely as  $\iota(a)g$  for some  $a \in A$  and  $g \in G^{\circ}$ . We put

(2.6) 
$$\iota(a)g \cdot (f \otimes v) = af \otimes \gamma(a, f) \cdot \mathrm{ad}(\iota(f)^{-1})(g) \cdot v.$$

Using (2.4) one can check that this does indeed define an action of G on V.

Lemma 2.5. The map

$$f\mapsto \sum_{a\in A}a\otimes f(\iota(a))$$

defines an isomorphism of G-modules  $\operatorname{Ind}_{G^{\circ}}^{G}(V) \xrightarrow{\sim} \tilde{V}$ .

*Proof.* It is clear that our map is an isomorphism of vector spaces, and that its inverse sends  $a \otimes v$  to the function  $f : G \to V$  such that  $f(\iota(a)g) = g^{-1} \cdot v$  for  $g \in G^{\circ}$  and  $f(\iota(b)g) = 0$  for  $g \in G^{\circ}$  and  $b \in A \setminus \{a\}$ . It is not difficult to check that this inverse map respects the G-actions, proving the proposition.  $\Box$ 

In view of Lemma 2.5, it is clear that as  $G^{\circ}$ -modules, we have

(2.7) 
$$\operatorname{Ind}_{G^{\circ}}^{G}(V) \cong \bigoplus_{f \in A} {}^{\iota(f)}V,$$

as expected.

2.4. A twisted group algebra of a stabilizer. Let  $\lambda \in \mathbf{X}^+$ , and let  $A^{\lambda} = \{a \in A \mid a\lambda = \lambda\}$  be its stabilizer. We also set  $G^{\lambda} := \varpi^{-1}(A^{\lambda})$ . In view of (2.3), we have

(2.8) 
$$G^{\lambda} = \{g \in G \mid {}^{g}L(\lambda) \cong L(\lambda)\}$$

We fix a representative for the simple  $G^{\circ}$ -module  $L(\lambda)$  and, for each  $a \in A^{\lambda}$ , an isomorphism of  $G^{\circ}$ -modules

$$\theta_a: L(\lambda) \xrightarrow{\sim} {}^{\iota(a)}L(\lambda).$$

In the special case that  $a = 1_A$ , we require that

$$\theta_{1_A} = \mathrm{id}_{L(\lambda)}.$$

Explicitly, these maps have the property that for any  $g \in G^{\circ}$  and  $v \in L(\lambda)$ , we have

(2.9) 
$$\theta_a(g \cdot v) = \operatorname{ad}(\iota(a)^{-1})(g) \cdot \theta_a(v),$$

where on the right-hand side we consider the given action of  $G^{\circ}$  on  $L(\lambda)$ .

Now let  $a, b \in A^{\lambda}$ , and consider the diagram



This is *not* a commutative diagram. Rather, both  $\theta_{ab}$  and  $\phi_{a,b} \circ {}^{\iota(a)}\theta_b \circ \theta_a$  are isomorphisms of simple  $G^{\circ}$ -modules, so they must be scalar multiples of one another. Let  $\alpha(a,b) \in \mathbb{k}^{\times}$  be the scalar such that

$$\phi_{a,b}{}^{\iota(a)}\theta_b\theta_a = \alpha(a,b)\cdot\theta_{ab}$$

Our assumptions on  $\iota(1_A)$  and  $\theta_{1_A}$  imply that for all  $a \in A$ , we have

$$\alpha(1_A, a) = \alpha(a, 1_A) = 1.$$

Given three elements  $a, b, c \in A^{\lambda}$ , we can form the diagram shown in Figure 1. The subdiagram consisting of straight arrows is commutative (by (2.4), (2.9) and the definitions), whereas each curved arrow introduces a scalar factor. Comparing the different scalars shows that

$$\alpha(a,b)\alpha(ab,c) = \alpha(a,bc)\alpha(b,c).$$



FIGURE 1. Isomorphisms of  $L(\lambda)$  with  ${}^{\iota(abc)}L(\lambda)$ 

In other words,  $\alpha: A^{\lambda} \times A^{\lambda} \to \mathbb{k}^{\times}$  is a 2-cocycle.

Let  $\mathscr{A}^{\lambda}$  be the twisted group algebra of  $A^{\lambda}$  determined by this cocycle. Explicitly, we define  $\mathscr{A}^{\lambda}$  to be the k-vector space spanned by symbols  $\{\rho_a : a \in A^{\lambda}\}$  with multiplication given by

$$\rho_a \rho_b = \alpha(a, b) \rho_{ab}.$$

This is a unital k-algebra, with unit  $\rho_{1_A}$ .

The algebra  $\mathscr{A}^{\lambda}$  can be described in more canonical terms as follows.

**Proposition 2.6.** There exists a canonical isomorphism of k-algebras

$$\operatorname{End}_{G^{\lambda}}\left(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))\right) \cong (\mathscr{A}^{\lambda})^{\operatorname{op}}.$$

*Proof.* We will work with the description of  $\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))$  from Lemma 2.5 (applied to the group  $G^{\lambda}$ ): we identify it with  $\Bbbk[A^{\lambda}] \otimes L(\lambda)$ , where the action of  $G^{\lambda}$  is given by (2.6).

We begin by equipping  $\Bbbk[A^{\lambda}] \otimes L(\lambda)$  with the structure of a right  $\mathscr{A}^{\lambda}$ -module as follows: given  $a, f \in A^{\lambda}$  and  $v \in L(\lambda)$ , we put

(2.10) 
$$(f \otimes v) \cdot \rho_a := (fa) \otimes \gamma(f, a) \cdot \theta_a(v).$$

Let us check that this is indeed a right  $\mathscr{A}^{\lambda}$ -module structure:

$$\begin{split} ((f \otimes v) \cdot \rho_a) \cdot \rho_b &= ((fa) \otimes \gamma(f, a) \cdot \theta_a(v)) \cdot \rho_b \\ &= (fab) \otimes \gamma(fa, b) \cdot \theta_b(\gamma(f, a) \cdot \theta_a(v)) \\ &= (fab) \otimes \gamma(fa, b) \mathrm{ad}(\iota(b)^{-1})(\gamma(f, a)) \cdot \theta_b(\theta_a(v)) \\ &= (fab) \otimes \gamma(f, ab)\gamma(a, b) \cdot \theta_b(\theta_a(v)) \\ &= (fab) \otimes \alpha(a, b)(\gamma(f, ab) \cdot \theta_{ab}(v)) \\ &= (f \otimes v) \cdot (\alpha(a, b)\rho_{ab}). \end{split}$$

(Here, the third equality relies on (2.9), and the fourth one on (2.4).) Next, we check that the right action of  $\mathscr{A}^{\lambda}$  commutes with the left action of G:

$$\begin{split} \iota(a)g \cdot ((f \otimes v) \cdot \rho_b) &= \iota(a)g \cdot ((fb) \otimes \gamma(f, b) \cdot \theta_b(v)) \\ &= (afb) \otimes \gamma(a, fb) \mathrm{ad}(\iota(fb)^{-1})(g)\gamma(f, b) \cdot \theta_b(v) \\ &= (afb) \otimes \gamma(a, fb)\gamma(f, b) \mathrm{ad}((\iota(fb)\gamma(f, b))^{-1})(g) \cdot \theta_b(v) \\ &= (afb) \otimes \gamma(af, b) \mathrm{ad}(\iota(b)^{-1})(\gamma(a, f)) \mathrm{ad}((\iota(f)\iota(b))^{-1})(g) \cdot \theta_b(v) \\ &= (afb) \otimes \gamma(af, b)\theta_b(\gamma(a, f) \mathrm{ad}(\iota(f)^{-1})(g) \cdot v) \\ &= ((af) \otimes \gamma(a, f) \mathrm{ad}(\iota(f)^{-1})(g) \cdot v) \cdot \rho_b \\ &= (\iota(a)g \cdot (f \otimes v)) \cdot \rho_b. \end{split}$$

As a consequence, the right  $\mathscr{A}^{\lambda}$ -action gives rise to an algebra homomorphism

$$\varphi: (\mathscr{A}^{\lambda})^{\mathrm{op}} \to \mathrm{End}_{G^{\lambda}}(\Bbbk[A^{\lambda}] \otimes L(\lambda)).$$

For each  $a \in A^{\lambda}$ , the operator  $\varphi(\rho_a)$  permutes the direct summands  $\Bbbk f \otimes L(\lambda) \subset \Bbbk[A^{\lambda}] \otimes L(\lambda)$ , as f runs over elements of  $A^{\lambda}$ . Moreover, distinct a's give rise to distinct permutations. It follows from this that the collection of linear operators  $\{\varphi(\rho_a) : a \in A^{\lambda}\}$  is linearly independent. In other words,  $\varphi$  is injective.

On the other hand, by adjunction, we have

(2.11) 
$$\dim \operatorname{End}_{G^{\lambda}}\left(\operatorname{Ind}_{G^{\circ}}^{G^{\wedge}}(L(\lambda))\right) = \dim \operatorname{Hom}_{G^{\circ}}\left(\operatorname{Ind}_{G^{\circ}}^{G^{\wedge}}(L(\lambda)), L(\lambda)\right).$$

Now, (2.7) implies that as a  $G^{\circ}$ -module,  $\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))$  is isomorphic to a direct sum of  $|A^{\lambda}|$  copies of  $L(\lambda)$ . So (2.11) shows that

$$\dim \operatorname{End}_{G^{\lambda}}\left(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))\right) = |A^{\lambda}| = \dim \mathscr{A}^{\lambda}.$$

Since  $\varphi$  is an injective map between k-vector spaces of the same dimension, it is also surjective, and hence an isomorphism.

- Remark 2.7. (1) The  $G^{\circ}$ -module  $L(\lambda)$  is defined only up to isomorphism. But if  $L'(\lambda)$  is another choice for this module, then an isomorphism  $L(\lambda) \xrightarrow{\sim} L'(\lambda)$  is unique up to scalar (and exists). Hence the induced isomorphism  $\operatorname{End}_{G^{\lambda}}(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))) \xrightarrow{\sim} \operatorname{End}_{G^{\lambda}}(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L'(\lambda)))$  does not depend on the choice of isomorphism. In other words, the algebra  $\operatorname{End}_{G^{\lambda}}(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda)))$  is completely canonical, i.e. does not depend on any choice.
  - (2) Once the  $G^{\circ}$ -module  $L(\lambda)$  is fixed, our description of the k-algebra  $\mathscr{A}^{\lambda}$ , and of its identification with  $\operatorname{End}_{G^{\lambda}}(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda)))^{\operatorname{op}}$  in Proposition 2.6, depend on the choice of the isomorphisms  $\theta_a$  for  $a \in A \setminus \{1\}$ . However, if  $\{\theta'_a : a \in A \setminus \{1\}\}$  is another choice of such isomorphisms, and  $\{\rho'_a : a \in A\}$ is the basis of the corresponding algebra  $(\mathscr{A}')^{\lambda}$ , then for any  $a \in A$  there exists a unique  $t_a \in \mathbb{k}^{\times}$  such that  $\theta'_a = t_a \theta_a$ . It is easy to check that the assignment  $\rho'_a \mapsto t_a \rho_a$  defines an algebra isomorphism  $(\mathscr{A}')^{\lambda} \xrightarrow{\sim} \mathscr{A}^{\lambda}$  which commutes with the identifications provided by Proposition 2.6.
  - (3) If, instead of using Lemma 2.5 to describe the  $G^{\lambda}$ -module  $\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))$ , we describe it in terms of algebraic functions  $\phi : G^{\lambda} \to L(\lambda)$  satisfying  $\phi(gh) = h^{-1} \cdot \phi(g)$  for  $h \in G^{\circ}$ , then the right action of  $\mathscr{A}^{\lambda}$  on this module satisfies  $(\phi \cdot \rho_a)(g) = \theta_a \circ \phi(g\iota(a)^{-1})$ .

2.5. Simple  $G^{\lambda}$ -modules whose restriction to  $G^{\circ}$  is a direct sum of copies of  $L(\lambda)$ . We continue with the setting of §2.4, and in particular with our fixed  $\lambda \in \mathbf{X}^+$ . If E is a finite-dimensional left  $\mathscr{A}^{\lambda}$ -module, we define a  $G^{\lambda}$ -action on the  $\Bbbk$ -vector space  $E \otimes L(\lambda)$  by

(2.12) 
$$\iota(a)g \cdot (u \otimes v) = \rho_a u \otimes \theta_a^{-1}(gv) \quad \text{for } a \in A^{\lambda} \text{ and } g \in G^{\circ}$$

**Lemma 2.8.** The rule (2.12) defines a structure of  $G^{\lambda}$ -module on  $E \otimes L(\lambda)$ .

Proof. Note that

$$\iota(a)g\iota(b)h = \iota(a)\iota(b)\mathrm{ad}(\iota(b)^{-1})(g)h = \iota(ab)(\gamma(a,b)\mathrm{ad}(\iota(b)^{-1})(g)h).$$

We now have

$$\iota(a)g \cdot (\iota(b)h \cdot (u \otimes v)) = \iota(a)g \cdot (\rho_b u \otimes \theta_b^{-1}(hv))$$
  
=  $\rho_a \rho_b u \otimes \theta_a^{-1}(g\theta_b^{-1}(hv))$   
=  $\alpha(a,b)\rho_{ab}u \otimes (\theta_b \circ \theta_a)^{-1}(\mathrm{ad}(\iota(b)^{-1})(g)hv)$   
=  $\rho_{ab}u \otimes \theta_{ab}^{-1}(\gamma(a,b)\mathrm{ad}(\iota(b)^{-1})(g)hv)$   
=  $(\iota(a)g\iota(b)h) \cdot (u \otimes v),$ 

proving the desired formula.

**Proposition 2.9.** The assignment  $E \mapsto E \otimes L(\lambda)$  defines a bijection between the set of isomorphism classes of simple  $\mathscr{A}^{\lambda}$ -modules and the set of isomorphism classes of simple  $G^{\lambda}$ -modules whose restriction to  $G^{\circ}$  is a direct sum of copies of  $L(\lambda)$ .

*Proof.* We will show that if V is a finite dimensional  $G^{\lambda}$ -module whose restriction to  $G^{\circ}$  is a direct sum of copies of  $L(\lambda)$ , and if we set  $E := \operatorname{Hom}_{G^{\circ}}(L(\lambda), V)$ , then E has a natural structure of a left  $\mathscr{A}^{\lambda}$ -module, and there exists an isomorphism of  $G^{\lambda}$ -modules

$$\eta_{\lambda,E}: E \otimes L(\lambda) \xrightarrow{\sim} V.$$

We define the  $\mathscr{A}^{\lambda}$ -action on E by

$$(\rho_a \cdot f)(x) = \iota(a) \cdot f(\theta_a(x))$$

for  $f \in E = \text{Hom}_{G^{\circ}}(L(\lambda), V)$  and  $x \in L(\lambda)$ . (We leave it to the reader to check that  $\rho_a \cdot f$  is a morphism of  $G^{\circ}$ -modules.) To justify that this defines an  $\mathscr{A}^{\lambda}$ -module structure, we simply compute:

$$\begin{aligned} (\rho_a \cdot (\rho_b \cdot f))(x) &= \iota(a) \cdot (\rho_b \cdot f)(\theta_a(x)) \\ &= \iota(a) \cdot \iota(b) \cdot f(\theta_b \circ \theta_a(x)) \\ &= \iota(ab) \cdot \gamma(a,b) \cdot f(\theta_b \circ \theta_a(x)) \\ &= \iota(ab) \cdot f(\gamma(a,b) \cdot \theta_b \circ \theta_a(x)) \\ &= \alpha(a,b) \cdot \iota(ab) \cdot f(\theta_{ab}(x)) \\ &= ((\alpha(a,b)\rho_{ab}) \cdot f)(x). \end{aligned}$$

Now there exists a canonical isomorphism of  $G^{\circ}$ -modules

$$\eta_{\lambda,E}: E \otimes L(\lambda) = \operatorname{Hom}_{G^{\circ}}(L(\lambda), V) \otimes L(\lambda) \xrightarrow{\sim} V$$

defined by  $\eta_{\lambda,E}(f \otimes v) = f(v)$ . Let us check that this morphism also commutes with the action of  $\iota(A)$ . By definition we have

$$\iota(a) \cdot (f \otimes v) = (\rho_a \cdot f) \otimes \theta_a^{-1}(v) = \sigma(\iota(a)) \circ f \circ \theta_a \otimes \theta_a^{-1}(v),$$

where  $\sigma: G^{\lambda} \to \operatorname{GL}(V)$  is the morphism defining the  $G^{\lambda}$ -action. Hence

$$\eta_{\lambda,E}(\iota(a)\cdot(f\otimes v)) = \iota(a)\cdot f(v) = \iota(a)\cdot\eta_{\lambda,E}(f\otimes v),$$

proving that  $\eta_{\lambda,E}$  is an isomorphism of  $G^{\lambda}$ -modules.

It is clear that the assignments

$$-\otimes L(\lambda): E \mapsto E \otimes L(\lambda)$$
 and  $\operatorname{Hom}_{G^{\circ}}(L(\lambda), -): V \mapsto \operatorname{Hom}_{G^{\circ}}(L(\lambda), V)$ 

define functors from the category of finite-dimensional  $\mathscr{A}^{\lambda}$ -modules to the category of finite-dimensional  $G^{\lambda}$ -modules whose restriction to  $G^{\circ}$  are isomorphic to a direct sum of copies of  $L(\lambda)$ , and from the category of finite-dimensional  $G^{\lambda}$ modules whose restriction to  $G^{\circ}$  are isomorphic to a direct sum of copies of  $L(\lambda)$  to the category of finite-dimensional  $\mathscr{A}^{\lambda}$ -modules respectively. It is straightforward to construct an isomorphism of functors  $\operatorname{Hom}_{G^{\circ}}(L(\lambda), -) \circ (- \otimes L(\lambda)) \xrightarrow{\sim} \operatorname{id}$ , as well as an isomorphism  $(- \otimes L(\lambda)) \circ \operatorname{Hom}_{G^{\circ}}(L(\lambda), -) \xrightarrow{\sim} \operatorname{id}$  defined by  $\eta_{\lambda, -}$ . Our functors are thus equivalences of categories, quasi-inverse to each other; hence they define bijections between the sets of isomorphism classes of simple objects in these categories.  $\Box$ 

Remark 2.10. As in Remark 2.7, it can be easily checked that the assignment  $E \mapsto E \otimes L(\lambda)$  does not depend on the choice of the isomorphisms  $\{\theta_a : a \in A\}$ , in the sense that if  $\{\theta'_a : a \in A\}$  is another choice of such isomorphisms, and if  $(\mathscr{A}')^{\lambda}$  is the corresponding algebra, then the identification  $(\mathscr{A}')^{\lambda} \xrightarrow{\sim} \mathscr{A}^{\lambda}$  considered in Remark 2.7 defines a bijection between isomorphism classes of simple  $(\mathscr{A}')^{\lambda}$ -modules and  $\mathscr{A}^{\lambda}$ -modules, which commutes with the operations  $- \otimes L(\lambda)$ . Of course, these constructions do not depend on the choice of  $L(\lambda)$  in its isomorphism class either.

2.6. Induction from  $G^{\lambda}$  to G. We continue with the setting of §§2.4–2.5. If E is a finite-dimensional  $\mathscr{A}^{\lambda}$ -module, we now consider the G-module

$$L(\lambda, E) := \operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda)).$$

**Lemma 2.11.** If E is a simple  $\mathscr{A}^{\lambda}$ -module, then  $L(\lambda, E)$  is a simple G-module.

*Proof.* Let  $V \subset L(\lambda, E)$  be a simple G-submodule. For any simple  $G^{\circ}$ -module L, let  $[V:L]_{G^{\circ}}$  denote the multiplicity of L as a composition factor of V, regarded as a  $G^{\circ}$ -module. The image of the embedding  $V \hookrightarrow L(\lambda, E)$  under the isomorphism

 $\operatorname{Hom}_{G}(V, L(\lambda, E)) = \operatorname{Hom}_{G}(V, \operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda))) \cong \operatorname{Hom}_{G^{\lambda}}(V, E \otimes L(\lambda))$ 

given by Frobenius reciprocity provides a nonzero morphism of  $G^{\lambda}$ -modules  $V \to E \otimes L(\lambda)$ , which must be surjective since  $E \otimes L(\lambda)$  is simple by Proposition 2.9. It follows that  $[V : L(\lambda)]_{G^{\circ}} \geq \dim(E)$ . Now, as in (2.7), if  $g_1, \ldots, g_r$  are representatives in G of the cosets in  $G/G^{\lambda}$ , then as  $G^{\circ}$ -modules we have

$$L(\lambda, E) = \operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda)) \cong \bigoplus_{i=1}^{r} {}^{g_i}L(\lambda)^{\oplus \dim(E)}.$$

Since V is stable under the G-action, we have  $[V : L(\lambda)]_{G^{\circ}} = [V : g_i L(\lambda)]_{G^{\circ}}$  for all *i* (see Lemma 2.3), and hence  $[V : g_i L(\lambda)]_{G^{\circ}} \ge \dim(E)$  for all *i*. This implies that  $\dim(V) \ge \dim(\operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda)))$ , so in fact  $V = \operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda))$ , as desired.  $\Box$  2.7. Simple *G*-modules. We come back to the general setting of §2.2. (In particular, the dominant weight  $\lambda$  is not fixed anymore.) We can now prove that the procedure explained in §§2.4–2.6 allows us to construct *all* simple *G*-modules (up to isomorphism).

**Lemma 2.12.** Let V be a simple G-module. Then there exists  $\lambda \in \mathbf{X}^+$ , a simple  $\mathscr{A}^{\lambda}$ -module E, and an isomorphism of G-modules

$$V \xrightarrow{\sim} L(\lambda, E).$$

*Proof.* Certainly there exists  $\lambda \in \mathbf{X}^+$  and a surjection of  $G^\circ$ -modules  $V \to L(\lambda)$ . By Frobenius reciprocity we deduce a nonzero (hence injective) morphism of G-modules  $V \to \operatorname{Ind}_{G^\circ}^G(L(\lambda))$ . So to conclude, it suffices to prove that all composition factors of  $\operatorname{Ind}_{G^\circ}^G(L(\lambda))$  are of the form  $L(\lambda, E)$  (with E a simple  $\mathscr{A}^\lambda$ -module). However, we have

$$\operatorname{Ind}_{G^{\diamond}}^{G}(L(\lambda)) \cong \operatorname{Ind}_{G^{\lambda}}^{G}(\operatorname{Ind}_{G^{\diamond}}^{G^{\lambda}}(L(\lambda))).$$

The restriction of  $\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))$  to  $G^{\circ}$  is a direct sum of copies of  $L(\lambda)$  by (2.7) applied to  $G^{\lambda}$ . Therefore, all of its composition factors are of the form  $E \otimes L(\lambda)$  with E a simple  $\mathscr{A}^{\lambda}$ -module by Proposition 2.9. Since the functor  $\operatorname{Ind}_{G^{\lambda}}^{G}$  is exact (by Lemma 2.5, or by [Ja, Corollary I.5.13]) and sends simple  $G^{\lambda}$ -modules of the form  $E \otimes L(\lambda)$  to simple G-modules by Lemma 2.11, the claim follows.

2.8. Conjugation. It now remains to understand when two modules of the form  $L(\lambda, E)$  are isomorphic. For this, we need to analyze the relation between this construction applied to a dominant weight, and to a twist of this dominant weight by an element of A.

So, let  $\lambda \in \mathbf{X}^+$ , and  $a \in A$ . Then we have

$$A^{a_{\lambda}} = aA^{\lambda}a^{-1}, \quad G^{a_{\lambda}} = \iota(a)G^{\lambda}\iota(a)^{-1},$$

and we can choose as  $L({}^{a}\lambda)$  the module  ${}^{\iota(a)}L(\lambda)$ , cf. (2.3).

Let us choose isomorphisms  $\theta_b : L(\lambda) \xrightarrow{\sim} \iota^{(b)}L(\lambda)$  for all  $b \in A^{\lambda}$ . Again for  $b \in A^{\lambda}$ , we can consider the isomorphism

$$\begin{split} \tilde{\theta}_{aba^{-1}} &: L({}^{a}\lambda) = {}^{\iota(a)}L(\lambda) \xrightarrow{\theta_{b}} {}^{\iota(a)\iota(b)}L(\lambda) = {}^{\iota(a)\iota(b)\iota(a)^{-1}} \big({}^{\iota(a)}L(\lambda)\big) \\ &= {}^{\iota(a)\iota(b)\iota(a)^{-1}} \big(L({}^{a}\lambda)\big) \xrightarrow{\iota(aba^{-1})^{-1}\iota(a)\iota(b)\iota(a)^{-1}\cdot(-)} {}^{\iota(aba^{-1})}L({}^{a}\lambda). \end{split}$$

(Here, the last isomorphism means the action of  $\iota(aba^{-1})^{-1}\iota(a)\iota(b)\iota(a)^{-1}$  on  $L(^a\lambda)$ , or in other words the action of  $\iota(a)^{-1}\iota(aba^{-1})^{-1}\iota(a)\iota(b)$  on  $L(\lambda)$ .)

The following claim can be checked directly from the definitions.

**Lemma 2.13.** For any  $b, c \in A^{\lambda}$ , we have

$$\gamma(aca^{-1}, aba^{-1}) \circ \tilde{\theta}_{aba^{-1}} \circ \tilde{\theta}_{aca^{-1}} = \alpha(c, b) \cdot \tilde{\theta}_{acba^{-1}}$$

(where here  $\gamma(aca^{-1}, aba^{-1})$  means the action of this element on  $L(a\lambda)$ ).

If  $\mathscr{A}^{\lambda}$  and its basis  $\{\rho_b : b \in A^{\lambda}\}$  are defined in terms of the isomorphisms  $\{\theta_b : b \in A^{\lambda}\}$  and if  $\mathscr{A}^{a_{\lambda}}$  and its basis  $\{\tilde{\rho}_b : b \in A^{a_{\lambda}}\}$  are defined in terms of the isomorphisms  $\{\tilde{\theta}_a : a \in A^{a_{\lambda}}\}$ , then Lemma 2.13 allows us to compare the cocycles that arise in the definitions of  $\mathscr{A}^{\lambda}$  and  $\mathscr{A}^{a_{\lambda}}$ . More precisely, this lemma shows that the assignment  $\rho_b \mapsto \tilde{\rho}_{aba^{-1}}$  defines an algebra isomorphism  $\xi^a_{\lambda} : \mathscr{A}^{\lambda} \xrightarrow{\sim} \mathscr{A}^{a_{\lambda}}$ .

The isomorphism  $\xi^a_{\lambda}$  can be described more canonically as follows. Recall that Proposition 2.6 provides canonical identifications

 $(\mathscr{A}^{\lambda})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{G^{\lambda}} (\mathrm{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))), \quad (\mathscr{A}^{a_{\lambda}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{G^{\lambda}} (\mathrm{Ind}_{G^{\circ}}^{G^{\lambda}}(L(a_{\lambda}))).$ 

One can check that under these identifications, the automorphism  $\xi^a_\lambda$  is given by the isomorphism

$$\operatorname{End}_{G^{\lambda}}\left(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))\right) = \operatorname{End}_{G^{a_{\lambda}}}\left({}^{\iota(a)}\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))\right) \xrightarrow{\sim} \operatorname{End}_{G^{a_{\lambda}}}\left(\operatorname{Ind}_{G^{\circ}}^{G^{*\lambda}}(L({}^{a}\lambda))\right)$$

(where we use the notation of Remark 2.2).

The properties of these isomorphisms that we will need below are summarized in the following lemma.

#### Lemma 2.14. Let $\lambda \in \mathbf{X}^+$ .

- If a, b ∈ A, then we have ξ<sup>ab</sup><sub>λ</sub> = ξ<sup>a</sup><sub>bλ</sub> ∘ ξ<sup>b</sup><sub>λ</sub>.
   If a ∈ A<sup>λ</sup>, then ξ<sup>a</sup><sub>λ</sub> is an inner automorphism of 𝔄<sup>λ</sup>.

*Proof.* (1) To simplify notation, we set  $\mu := {}^{ab}\lambda$ . Note that the simple  $G^{\circ}$ -modules of highest weight  $\mu$  used in the definitions of  $\xi_{\lambda}^{ab}$  and  $\xi_{b_{\lambda}}^{a} \circ \xi_{\lambda}^{b}$  are different: for the former we use the module  $L_1(\mu) := {}^{\iota(ab)}L(\lambda)$ , while for the latter we use the module  $L_2(\mu) := {}^{\iota(a)\iota(b)}L(\lambda)$ . There exists a canonical isomorphism

$$(2.13) L_1(\mu) \xrightarrow{\sim} L_2(\mu).$$

given by the action of  $\gamma(a, b)^{-1}$  on  $L(\lambda)$  (i.e. the inverse of the isomorphism denoted  $\phi_{a,b}$  in §2.3).

Our algebras are all defined as endomorphisms of some induced module, which can be described in terms of functions with values in the vector space underlying the representation  $L(\lambda)$ . From this point of view,  $\xi^a_{b\lambda} \circ \xi^b_{\lambda}$  is conjugation by the isomorphism of vector spaces  $\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda)) \xrightarrow{\sim} \operatorname{Ind}_{G^{\circ}}^{G^{\mu}}(L_{2}(\mu))$  sending functions  $G^{\lambda} \to L(\lambda)$  to functions  $G^{\mu} \to L(\lambda)$  and given by  $\phi \mapsto \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b))$ , while  $\xi_{\lambda}^{ab}$  is conjugation by the isomorphism  $\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda)) \xrightarrow{\sim} \operatorname{Ind}_{G^{\circ}}^{G^{\mu}}(L_{1}(\mu))$  given by  $\phi \mapsto \phi(\iota(ab)(-)\iota(ab)^{-1})$ . Taking into account the isomorphism (2.13), we have to check that conjugation by the isomorphism given by

(2.14) 
$$\phi \mapsto \gamma(a,b) \cdot \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b))$$

(where  $\gamma(a,b) \cdot (-)$  means the action of  $\gamma(a,b) \in G^{\circ}$  on  $L(\lambda)$ ) coincides with conjugation by the isomorphism given by

(2.15) 
$$\phi \mapsto \phi(\iota(ab)(-)\iota(ab)^{-1}).$$

However, since  $\gamma(a, b)$  belongs to  $G^{\circ}$ , the functions  $\phi$  we consider satisfy

$$\begin{split} \gamma(a,b) \cdot \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b)) &= \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b)\gamma(a,b)^{-1}) \\ &= \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(ab)) = \phi(\gamma(a,b)^{-1}\iota(ab)^{-1}(-)\iota(ab)). \end{split}$$

Thus, the isomorphisms (2.14) and (2.15) do not coincide, but they differ only by the action of an element of  $G^{\lambda}$  (which, in fact, even belongs to  $G^{\circ}$ ) on  $\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))$ . Therefore, conjugation by either (2.14) or (2.15) induces the *same* isomorphism of algebras  $\operatorname{End}_{G^{\lambda}}(\operatorname{Ind}_{G^{\circ}}^{G^{\lambda}}(L(\lambda))) \xrightarrow{\sim} \operatorname{End}_{G^{\mu}}(\operatorname{Ind}_{G^{\circ}}^{G^{\mu}}(L_{1}(\mu))).$ (2) By the comments preceding the statement,  $\xi^{a}_{\lambda}$  is conjugation by an isomor-

phism  $\operatorname{Ind}_{G^{\diamond}}^{G^{\lambda}}(L(\lambda)) \xrightarrow{\sim} \operatorname{Ind}_{G^{\diamond}}^{G^{a_{\lambda}}}(L(^{a_{\lambda}}))$ . If  $a \in A^{\lambda}$  then this isomorphism defines an invertible element of  $\mathscr{A}^{\lambda}$ , so that  $\xi^{a}_{\lambda}$  is indeed an inner automorphism.  $\Box$ 

Given  $a \in A$  and  $\lambda \in \mathbf{X}^+$ , the isomorphism  $\xi_{\lambda}^a$  defines a bijection between the set of isomorphism classes of simple  $\mathscr{A}^{\lambda}$ -modules and the set of isomorphism classes of simple  $\mathscr{A}^{a\lambda}$ -modules. From Lemma 2.14(1) we see that this operation defines an action of the group A on the set of pairs  $(\lambda, E)$  where  $\lambda \in \mathbf{X}^+$  and E is a simple  $\mathscr{A}^{\lambda}$ -module. Moreover, it follows from Lemma 2.14(2) that the induced action of  $A^{\lambda}$  on the set of isomorphism classes of simple  $\mathscr{A}^{\lambda}$ -modules is trivial.

**Lemma 2.15.** Let  $\lambda \in \mathbf{X}^+$ , and let E be a simple  $\mathscr{A}^{\lambda}$ -module. Let  $a \in A$ , and let E' be the simple  $\mathscr{A}^{a_{\lambda}}$ -module deduced from E via the isomorphism  $\xi^a_{\lambda} : \mathscr{A}^{\lambda} \xrightarrow{\sim} \mathscr{A}^{a_{\lambda}}$ . Then there exists an isomorphism of G-modules

$$L(\lambda, E) \xrightarrow{\sim} L({}^a\lambda, E').$$

*Proof.* As above we choose for our simple  $G^{\circ}$ -module of highest weight  ${}^{a}\lambda$  the module  ${}^{\iota(a)}L(\lambda)$ . Then conjugation by  $\iota(a)$  induces an isomorphism  $G^{\lambda} \xrightarrow{\sim} G^{a\lambda}$ , and using the notation of Remark 2.2 we have as  $G^{a\lambda}$ -modules

$${}^{\iota(a)}(E \otimes L(\lambda)) = E' \otimes L({}^{a}\lambda)$$

In view of Lemma 2.1 we deduce an isomorphism of G-modules

$${}^{\iota(a)}\mathrm{Ind}_{G^{\lambda}}^{G}(E\otimes L(\lambda)) \xrightarrow{\sim} \mathrm{Ind}_{G^{a_{\lambda}}}^{G}(E'\otimes L({}^{a}\lambda)).$$

Now by Lemma 2.3 the left-hand side is isomorphic to  $L(\lambda, E)$ , and the claim follows.

2.9. Classification of simple *G*-modules. We denote by Irr(G) the set of isomorphism classes of simple *G*-modules. Now we can finally state the main result of this section.

**Theorem 2.16.** The assignment  $(\lambda, E) \mapsto L(\lambda, E)$  induces a bijection

$$\left\{ (\lambda, E) \middle| \begin{array}{c} \lambda \in \mathbf{X}^+ \text{ and } E \text{ an isom. class} \\ of simple \ left \ \mathscr{A}^{\lambda} \text{-modules} \end{array} \right\} \middle/ A \longleftrightarrow \operatorname{Irr}(G).$$

*Proof.* From Lemma 2.11, we see that the assignment  $(\lambda, E) \mapsto L(\lambda, E)$  defines a map from the set of pairs  $(\lambda, E)$  as in the statement to the set Irr(G). By Lemma 2.15 this map factors through a map

$$\left\{ (\lambda, E) \left| \begin{array}{c} \lambda \in \mathbf{X}^+ \text{ and } E \text{ an isom. class} \\ \text{of simple left } \mathscr{A}^{\lambda}\text{-modules} \end{array} \right\} \middle/ A \to \operatorname{Irr}(G) \right.$$

By Lemma 2.12, this latter map is surjective. Hence, all that remains is to prove that it is injective.

Let  $(\lambda, E)$  and  $(\lambda', E')$  be pairs as above. Let  $V = L(\lambda, E)$  and  $V' = L(\lambda', E')$ , and assume that  $V \cong V'$ . As a  $G^{\circ}$ -representation, V is isomorphic to a direct sum of twists of  $L(\lambda)$ , and V' is isomorphic to a direct sum of twists of  $L(\lambda')$  (see the proof of Lemma 2.11). Hence  $L(\lambda)$  and  $L(\lambda')$  are twists of each other, which implies that  $\lambda$  and  $\lambda'$  are in the same A-orbit. Therefore, we can (and shall) assume that  $\lambda = \lambda'$ . Fix some isomorphism  $V \xrightarrow{\sim} V'$ , and consider the morphism of  $G^{\lambda}$ modules  $f : V \to E' \otimes L(\lambda)$  deduced by Frobenius reciprocity. If  $g_1, \ldots, g_r$  are representatives of the cosets in  $G/G^{\lambda}$ , with  $g_1 = 1_G$ , then we have an isomorphism of  $G^{\circ}$ -modules

$$\operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda)) \cong \bigoplus_{i=1}^{\prime} {}^{g_{i}}L(\lambda) \otimes E.$$

If  $i \neq 1$ , then  $g_i L(\lambda)$  is not isomorphic to  $L(\lambda)$ . Hence f is zero on the corresponding summand of  $\operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda))$ . We deduce that the composition

$$E \otimes L(\lambda) \hookrightarrow \operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes L(\lambda)) \xrightarrow{J} E' \otimes L(\lambda),$$

where the first morphism is again deduced from Frobenius reciprocity, is nonzero. But this morphism is a morphism of  $G^{\lambda}$ -modules. Since  $L(\lambda, E)$  and  $L(\lambda, E')$  are simple, it must be an isomorphism, and by Proposition 2.9 this implies that  $E \cong E'$ as  $\mathscr{A}^{\lambda}$ -modules.

- Remark 2.17. (1) As explained above Lemma 2.15, for any  $\lambda \in \mathbf{X}^+$  the action of  $A^{\lambda}$  on the set of isomorphism classes of irreducible  $\mathscr{A}^{\lambda}$ -modules is trivial. Hence if  $\Lambda \subset \mathbf{X}^+$  is a set of representatives of the *A*-orbits in  $\mathbf{X}^+$ , the quotient considered in the statement of Theorem 2.16 can be described more explicitly as the set of pairs  $(\lambda, E)$  where  $\lambda \in \Lambda$  and *E* is an isomorphism class of simple  $\mathscr{A}^{\lambda}$ -modules.
  - (2) Assume that  $\iota$  is a group morphism (so that G is isomorphic to the semidirect product  $A \ltimes G^{\circ}$ ) and that moreover there exists a Borel subgroup  $B \subset G^{\circ}$  such that  $\iota(a)B\iota(a)^{-1} = B$  for any  $a \in A$ . Then if we define the standard and costandard  $G^{\circ}$ -modules using this Borel subgroup, the isomorphisms

$${}^{\iota(a)}\Delta(\lambda) \cong \Delta({}^{a}\lambda), \quad {}^{\iota(a)}\nabla(\lambda) \cong \nabla({}^{a}\lambda)$$

(see (2.3)) can be chosen in a canonical way. In fact, our assumptions imply that there exist unique *B*-stable lines in  ${}^{\iota(a)}\Delta(\lambda)$  and  $\Delta({}^{a}\lambda)$ , and moreover that these lines coincide. Hence there exists a unique isomorphism of *G*modules  ${}^{\iota(a)}\Delta(\lambda) \xrightarrow{\sim} \Delta({}^{a}\lambda)$  which restricts to the identity on these *B*-stable lines. Similar comments apply to  ${}^{\iota(a)}\nabla(\lambda)$  and  $\nabla({}^{a}\lambda)$ .

In particular, the isomorphisms  $\theta_a$  of §2.4 can be chosen in a canonical way. Then the cocycle  $\alpha$  will be trivial, so that in this case  $\mathscr{A}^{\lambda}$  is canonically isomorphic to the group algebra  $\Bbbk[A^{\lambda}]$ .

2.10. Semisimplicity. We finish this section with a criterion ensuring that the algebra  $\mathscr{A}^{\lambda}$  is semisimple unless p is small.

**Lemma 2.18.** Assume that  $p \nmid |A|$ . If V is a simple  $G^{\circ}$ -module, then  $\operatorname{Ind}_{G^{\circ}}^{G}(V)$  is a semisimple G-module.

*Proof.* Let M be a G-submodule of  $\operatorname{Ind}_{G^{\circ}}^{G}(V)$ , and let  $N = \operatorname{Ind}_{G^{\circ}}^{G}(V)/M$ . We will show that the image c of the exact sequence  $M \hookrightarrow \operatorname{Ind}_{G^{\circ}}^{G}(V) \twoheadrightarrow N$  in  $\operatorname{Ext}_{G}^{1}(N, M)$  vanishes.

First we remark that for any two algebraic G-modules X, Y, the forgetful functor from Rep(G) to  $\text{Rep}(G^{\circ})$  induces an isomorphism

$$\operatorname{Hom}_{G}(X,Y) \xrightarrow{\sim} (\operatorname{Hom}_{G^{\circ}}(X,Y))^{A}.$$

Under our assumptions the functor  $(-)^A$  is exact. On the other hand, it is easily checked that the restriction of any injective *G*-module to  $G^{\circ}$  is injective. Hence this isomorphism induces an isomorphism

$$\operatorname{Ext}_{G}^{n}(X,Y) \xrightarrow{\sim} \left(\operatorname{Ext}_{G^{\circ}}^{n}(X,Y)\right)^{A}$$

for any  $n \ge 0$ . We deduce in particular that the forgetful functor induces an injection

$$\operatorname{Ext}^1_G(N, M) \hookrightarrow \operatorname{Ext}^1_{G^\circ}(N, M).$$

Hence to prove that c = 0 it suffices to prove that the sequence  $M \hookrightarrow \operatorname{Ind}_{G^{\circ}}^{G}(V) \twoheadrightarrow N$ , considered as an exact sequence of  $G^{\circ}$ -modules, splits. This fact is clear since  $\operatorname{Ind}_{G^{\circ}}^{G}(V)$  is semisimple as a  $G^{\circ}$ -module, see (2.7).

From this lemma (applied to the group  $A^{\lambda}$ ) and Proposition 2.6 we deduce the following.

**Lemma 2.19.** If  $p \nmid |A^{\lambda}|$ , then the algebra  $\mathscr{A}^{\lambda}$  is semisimple (and in fact isomorphic to a product of matrix algebras).

#### 3. Highest weight structure

Our goal in this section is to prove that if  $p \nmid |A|$ , then the category  $\operatorname{Rep}(G)$  of finite-dimensional G-modules admits a natural structure of a highest weight category.

For the beginning of the section, we continue with the setting of  $\S2.2$  (not imposing any further assumption).

3.1. The order. If  $(\lambda, E)$  is a pair as in Theorem 2.16, we denote by  $[\lambda, E]$  the corresponding A-orbit. We define a relation < on the set of such orbits as follows:

(3.1)  $[\lambda, E] < [\lambda', E']$  if for some  $a \in A$ , we have  ${}^a\lambda < \lambda'$ .

(Here, the order on **X** is the standard one, where  $\lambda \leq \mu$  iff  $\mu - \lambda$  is a sum of positive roots.)

**Lemma 3.1.** The relation < is a partial order.

*Proof.* Using the fact that for  $a \in A$  and  $\lambda, \mu \in \mathbf{X}$  such that  $\lambda \leq \mu$  we have  ${}^{a}\lambda \leq {}^{a}\mu$  (because the A-action is linear and preserves positive roots), one can easily check that this relation is transitive. What remains to be seen is that there cannot exist classes  $[\lambda, E], [\lambda', E']$  such that

$$[\lambda, E] < [\lambda', E'] < [\lambda, E].$$

However, in this case we have  ${}^{a}\lambda < \lambda$  for some  $a \in A$ . Since a permutes the positive coroots of  $G^{\circ}$ , then if we denote by  $2\rho^{\vee}$  the sum of these coroots we must have  $\langle {}^{a}\lambda, 2\rho^{\vee}\rangle = \langle \lambda, 2\rho^{\vee}\rangle$ , hence  $\langle \lambda - {}^{a}\lambda, 2\rho^{\vee}\rangle = 0$ . On the other hand, by assumption  $\lambda - {}^{a}\lambda$  is a nonzero sum of positive roots, so that its pairing with  $2\rho^{\vee}$  cannot vanish. This provides the desired contradiction.

3.2. Standard *G*-modules. Let  $\lambda \in \mathbf{X}^+$ . We will work in the setting of §§2.3–2.4, including, in particular, fixing a  $G^\circ$ -module  $L(\lambda)$ , and notation such as  $\iota$ ,  $\gamma$ ,  $\theta$ , and  $\alpha$ . We also fix a representative  $\Delta(\lambda)$  for the Weyl module surjecting to  $L(\lambda)$ , and a surjection  $\pi^{\lambda} : \Delta(\lambda) \to L(\lambda)$ .

Since  $\operatorname{End}_{G^{\circ}}(\Delta(\lambda)) = \mathbb{k} \cdot \operatorname{id}$ , from (2.3) we see that for each  $a \in A^{\lambda}$ , there exists a unique isomorphism  $\theta_a^{\Delta} : \Delta(\lambda) \xrightarrow{\sim} \iota^{(a)} \Delta(\lambda)$  such that the following diagram

commutes:

$$\Delta(\lambda) \xrightarrow{\theta_a^{\Delta}} {}^{\iota(a)}\Delta(\lambda)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$L(\lambda) \xrightarrow{\theta_a} {}^{\iota(a)}L(\lambda).$$

Moreover, this uniqueness implies that for any  $a, b \in A^{\lambda}$ , if we define  $\phi_{a,b}^{\Delta} : \Delta(\lambda) \to \Delta(\lambda)$  as the action of  $\gamma(a, b)$ , then we have

(3.2) 
$$\phi_{a,b}^{\Delta}\theta_{b}^{\Delta}\theta_{a}^{\Delta} = \alpha(a,b)\theta_{ab}^{\Delta}$$

Remark 3.2. These considerations show that the subgroup  $A^{\lambda} \subset A$  can be equivalently defined as consisting of the elements  $a \in A$  such that  ${}^{\iota(a)}\Delta(\lambda) \cong \Delta(\lambda)$ . The twisted group algebra  $\mathscr{A}^{\lambda}$  can also be defined in terms of a choice of isomorphisms  $(\theta_a^{\Delta} : a \in A^{\lambda})$  instead of isomorphisms  $(\theta_a : a \in A^{\lambda})$ .

**Lemma 3.3.** Let E be a simple left  $\mathscr{A}^{\lambda}$ -module. The following rule defines the structure of a  $G^{\lambda}$ -module on the vector space  $E \otimes \Delta(\lambda)$ :

$$\iota(a)g \cdot (u \otimes v) = \rho_a u \otimes (\theta_a^{\Delta})^{-1}(gv) \quad \text{for any } a \in A^{\lambda} \text{ and } g \in G^{\circ}.$$

This  $G^{\lambda}$ -module has  $E \otimes L(\lambda)$  as its unique irreducible quotient. Moreover, all the  $G^{\circ}$ -composition factors of the kernel of the quotient map  $E \otimes \Delta(\lambda) \to E \otimes L(\lambda)$  are of the form  $L(\mu)$  with  $\mu < \lambda$ .

*Proof.* We begin by noting that thanks to (3.2), the calculation from Lemma 2.8 can be repeated to show that the formula above does, indeed, define the structure of a  $G^{\lambda}$ -module on  $E \otimes \Delta(\lambda)$ . Moreover, the quotient map  $\pi^{\lambda} : \Delta(\lambda) \to L(\lambda)$  induces a surjective map of  $G^{\lambda}$ -modules

$$\pi_E^{\lambda} := \mathrm{id}_E \otimes \pi : E \otimes \Delta(\lambda) \to E \otimes L(\lambda).$$

If we forget the  $G^{\lambda}$ -module structure and regard  $E \otimes \Delta(\lambda)$  as just a  $G^{\circ}$ -module, then it is clear that its unique maximal semisimple quotient can be identified with  $E \otimes L(\lambda)$ , and that the highest weights of the kernel of  $\pi_E^{\lambda}$  are  $< \lambda$ . Now, let Mbe the head of  $E \otimes \Delta(\lambda)$  as a  $G^{\lambda}$ -module. Since M must remain semisimple as a  $G^{\circ}$ -module (by Lemma 2.4), it cannot be larger than  $E \otimes L(\lambda)$ . In other words,  $E \otimes L(\lambda)$  is the unique simple quotient of  $E \otimes \Delta(\lambda)$ .

**Proposition 3.4.** Let E be a simple  $\mathscr{A}^{\lambda}$ -module. The G-module

$$\Delta(\lambda, E) := \operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes \Delta(\lambda))$$

admits  $L(\lambda, E)$  as its unique irreducible quotient. Moreover, all the composition factors of the kernel of the quotient map  $\Delta(\lambda, E) \rightarrow L(\lambda, E)$  are of the form  $L(\mu, E')$  with  $[\mu, E'] < [\lambda, E]$ .

*Proof.* The surjection  $E \otimes \Delta(\lambda) \to E \otimes L(\lambda)$  from Lemma 3.3 induces a surjection  $\Delta(\lambda, E) \to L(\lambda, E)$  since the functor  $\operatorname{Ind}_{G^{\lambda}}^{G_{\lambda}}$  is exact (see the proof of Lemma 2.12). If  $g_1, \dots, g_r$  are representatives of the cosets in  $G/G^{\lambda}$ , then as  $G^{\circ}$ -modules we have

(3.3) 
$$\Delta(\lambda, E) \cong \bigoplus_{i=1}^{r} E \otimes {}^{g_i} \Delta(\lambda), \quad L(\lambda, E) \cong \bigoplus_{i=1}^{r} E \otimes {}^{g_i} L(\lambda)$$

Therefore, as in the proof of Lemma 2.12,  $L(\lambda, E)$  is the head of  $\Delta(\lambda, E)$  as a  $G^{\circ}$ -module, hence also as a G-module.

If  $L(\mu, E')$  is a *G*-composition factor of the kernel of the surjection  $\Delta(\lambda, E) \rightarrow L(\lambda, E)$ , then some twist of  $L(\mu)$  must be a *G*°-composition factor of the surjection  $g_i \Delta(\lambda) \rightarrow g_i L(\lambda)$  for some *i*. Therefore  $\mu$  is smaller than some twist of  $\lambda$ , and we deduce that  $[\mu, E'] < [\lambda, E]$ .

3.3. Ext<sup>1</sup>-vanishing. The same proof as for Lemma 2.15 shows that, up to isomorphism,  $\Delta(\lambda, E)$  only depends on the orbit  $[\lambda, E]$ . The following lemma shows that this module is a "partial projective cover" of  $L(\lambda, E)$  (under the assumption that  $p \nmid |A|$ ).

**Lemma 3.5.** Assume that  $p \nmid |A|$ . For any two pairs  $(\lambda, E)$  and  $(\mu, E')$ , we have  $\operatorname{Ext}^1_G(\Delta(\lambda, E), L(\mu, E')) \neq 0 \implies [\mu, E'] > [\lambda, E].$ 

*Proof.* As in the proof of Lemma 2.18, we have a canonical isomorphism

$$\operatorname{Ext}^{1}_{G}(\Delta(\lambda, E), L(\mu, E')) \cong \left(\operatorname{Ext}^{1}_{G^{\circ}}(\Delta(\lambda, E), L(\mu, E'))\right)^{A}.$$

If we assume that  $\operatorname{Ext}_{G}^{1}(\Delta(\lambda, E), L(\mu, E')) \neq 0$ , then this isomorphism shows that we must also have  $\operatorname{Ext}_{G^{\circ}}^{1}(\Delta(\lambda, E), L(\mu, E')) \neq 0$ . Using (3.3), we deduce that for some  $g, h \in G$  we have

$$\operatorname{Ext}_{G^{\circ}}^{1}({}^{g}\Delta(\lambda), {}^{h}L(\mu)) \neq 0.$$
  
This implies that  ${}^{\varpi(h)}\mu > {}^{\varpi(g)}\lambda$ , hence that  $[\mu, E'] > [\lambda, E].$ 

3.4. Costandard *G*-modules. Fix again  $\lambda \in \mathbf{X}^+$  and a simple  $\mathscr{A}^{\lambda}$ -module *E*. Then after fixing a costandard module  $\nabla(\lambda)$  with socle  $L(\lambda)$  and an embedding  $L(\lambda) \hookrightarrow \nabla(\lambda)$ , as in §3.2 the isomorphisms  $\theta_a$  can be "lifted" to isomorphisms  $\theta_a^{\nabla} : \nabla(\lambda) \xrightarrow{\rightarrow} \iota^{(a)} \nabla(\lambda)$ , which satisfy the appropriate analogue of (3.2). Using these isomorphisms one can define a  $G^{\lambda}$ -module structure on  $E \otimes \nabla(\lambda)$  by the same procedure as in Lemma 3.3. Then the same arguments as for Proposition 3.4 show that  $\nabla(\lambda, E) := \operatorname{Ind}_{G^{\lambda}}^{G}(E \otimes \nabla(\lambda))$  admits  $L(\lambda, E)$  as its unique simple submodule, and that all the composition factors of the injection  $L(\lambda, E) \hookrightarrow \nabla(\lambda, E)$  are of the form  $L(\mu, E')$  with  $[\mu, E'] < [\lambda, E]$ . Moreover, as in Lemma 3.5, if  $p \nmid |A|$  we have

$$\operatorname{Ext}^{1}_{G}(L(\mu, E'), \nabla(\lambda, E)) \neq 0 \quad \Rightarrow \quad [\mu, E'] > [\lambda, E].$$

**Lemma 3.6.** Assume that  $p \nmid |A|$ , and let  $(\lambda, E)$  and  $(\mu, E')$  be pairs as above. Then for any i > 0 we have

$$\operatorname{Ext}_{G}^{i}(\Delta(\lambda, E), \nabla(\mu, E')) = 0.$$

Moreover

$$\operatorname{Hom}_{G}(\Delta(\lambda, E), \nabla(\mu, E')) = 0$$

unless  $[\lambda, E] = [\mu, E']$ , in which case this space is 1-dimensional.

*Proof.* As in the proof of Lemma 2.18, for any i > 0 we have

$$\operatorname{Ext}_{G}^{i}(\Delta(\lambda, E), \nabla(\mu, E')) \cong \left(\operatorname{Ext}_{G^{\circ}}^{i}(\Delta(\lambda, E), \nabla(\mu, E'))\right)^{A}.$$

As  $G^{\circ}$ -modules  $\Delta(\lambda, E)$  is isomorphic to a direct sum of Weyl modules, and  $\nabla(\mu, E')$  is isomorphic to a direct sum of induced modules. Hence, the right-hand side vanishes unless i = 0, which proves the first claim.

For the second claim we remark that if  $\operatorname{Hom}_G(\Delta(\lambda, E), \nabla(\mu, E')) \neq 0$ , then  $L(\lambda, E)$  is a composition factor of  $\nabla(\mu, E')$ , so that  $[\lambda, E] \leq [\mu, E']$ , and  $L(\mu, E')$  is a composition factor of  $\Delta(\lambda, E)$ , so that  $[\mu, E'] \leq [\lambda, E]$ . We deduce that  $[\mu, E'] =$ 

 $[\lambda, E]$ . Moreover, in this case any nonzero morphism in this space must be a multiple of the composition

$$\Delta(\lambda, E) \twoheadrightarrow L(\lambda, E) \hookrightarrow \nabla(\lambda, E),$$

which concludes the proof.

3.5. Highest weight structure. Let  $\mathscr{C}$  be a finite-length k-linear abelian category such that  $\operatorname{Hom}_{\mathscr{C}}(M,N)$  is finite-dimensional for any M, N in  $\mathscr{C}$ . Let  $\mathscr{S}$  be the set of isomorphism classes of irreducible objects of  $\mathscr{C}$ . Assume that  $\mathscr{S}$  is equipped with a partial order  $\leq$ , and that for each  $s \in \mathscr{S}$  we have a fixed representative of the simple object  $L_s$ . Assume also we are given, for any  $s \in \mathscr{S}$ , objects  $\Delta_s$  and  $\nabla_s$ , and morphisms  $\Delta_s \to L_s$  and  $L_s \to \nabla_s$ . For  $\mathscr{T} \subset \mathscr{S}$ , we denote by  $\mathscr{C}_{\mathscr{T}}$  the Serre subcategory of  $\mathscr{C}$  generated by the objects  $L_t$  for  $t \in \mathscr{T}$ . We write  $\mathscr{C}_{\leq s}$  for  $\mathscr{C}_{\{t\in\mathscr{S}\mid t< s\}}$ , and similarly for  $\mathscr{C}_{< s}$ . Finally, recall that an *ideal* of  $\mathscr{S}$  is a subset  $\mathscr{T} \subset \mathscr{S}$  such that if  $t \in \mathscr{T}$  and  $s \in \mathscr{S}$  are such that  $s \leq t$ , then  $s \in \mathscr{T}$ .

Recall that the category  $\mathscr{C}$  (together with the above data) is said to be a *highest* weight category if the following conditions hold:

- (1) for any  $s \in \mathscr{S}$ , the set  $\{t \in \mathscr{S} \mid t \leq s\}$  is finite;
- (2) for each  $s \in \mathscr{S}$ , we have  $\operatorname{End}_{\mathscr{C}}(L_s) = \Bbbk$ ;
- (3) for any  $s \in \mathscr{S}$  and any ideal  $\mathscr{T} \subset \mathscr{S}$  such that  $s \in \mathscr{T}$  is maximal,  $\Delta_s \to L_s$ is a projective cover in  $\mathscr{C}_{\mathscr{T}}$  and  $L_s \to \nabla_s$  is an injective envelope in  $\mathscr{C}_{\mathscr{T}}$ ;
- (4) the kernel of  $\Delta_s \to L_s$  and the cokernel of  $L_s \to \nabla_s$  belong to  $\mathscr{C}_{\langle s \rangle}$ ; (5) we have  $\operatorname{Ext}^2_{\mathscr{C}}(\Delta_s, \nabla_t) = 0$  for all  $s, t \in \mathscr{S}$ .

In this case, the poset  $(\mathscr{S}, \leq)$  is called the *weight poset* of  $\mathscr{C}$ .

See [Ri, §7] for the basic properties of highest weight categories (following Cline-Parshall–Scott and Beilinson–Ginzburg–Soergel).

We can finally state the main result of this section.

**Theorem 3.7.** Assume that  $p \nmid |A|$ . The category  $\operatorname{Rep}(G)$ , equipped with the poset

$$\left\{ (\lambda, E) \left| \begin{array}{c} \lambda \in \mathbf{X}^+ \text{ and } E \text{ an isom. class} \\ of \text{ simple } \mathscr{A}^{\lambda}\text{-modules} \end{array} \right\} \right/ A$$

(with the order defined in (3.1)) and the objects  $\Delta(\lambda, E)$ ,  $L(\lambda, E)$ ,  $\nabla(\lambda, E)$ , is a highest weight category.

*Proof.* The desired properties are verified in Theorem 2.16, Proposition 3.4 and Lemma 3.5, their variants for costandard objects (see  $\S3.4$ ), and Lemma 3.6.  $\square$ 

#### 4. Grothendieck groups

Our goal in this section is to prove a generalization of a result of Serre [Se] providing a description of the Grothendieck group of any split connected reductive group over a complete discrete valuation ring. (In [Se], the author considers more general coefficients, but we will restrict to the setting we need for the application in [AHR].)

4.1. Setting. We will denote by  $\mathbb{O}$  a complete discrete valuation ring, with algebraically closed residue field  $\mathbb{F}$ . We also denote by  $\mathbb{K}$  the fraction field of  $\mathbb{O}$ , and by  $\overline{\mathbb{K}}$  an algebraic closure of  $\mathbb{K}$ . We will assume that  $\mathbb{K}$  has characteristic 0, and that  $\mathbb{F}$  has characteristic p > 0.

We will consider an affine  $\mathbb{O}$ -group scheme G, a closed normal subgroup  $G^{\circ} \subset G$ , and we will denote by A the factor group of G by  $G^{\circ}$  in the sense of [Ja, §I.6.1] (i.e. of [DG, III, §3, n. 3]). We will make the following assumptions:

- (1)  $G^{\circ}$  is a reductive group scheme over  $\mathbb{O}$  (in the sense of [SGA3.3]);
- (2) A is the constant group scheme associated with a finite group  $\mathbf{A}$ , and moreover char( $\mathbb{F}$ ) does not divide  $|\mathbf{A}|$ .

We will now note a few consequences of these assumptions. We begin with the following observation.

#### **Lemma 4.1.** Any reductive group scheme over $\mathbb{O}$ (in particular, $G^{\circ}$ ) is split.

*Proof.* This result follows from the arguments in the proof of [McN, Corollary 2.1.2]. Namely, let H be a reductive group scheme over  $\mathbb{O}$ , and let K be a split  $\mathbb{O}$ -torus such that there exists a closed embedding  $\operatorname{Spec}(\mathbb{F}) \times_{\operatorname{Spec}(\mathbb{O})} K \to \operatorname{Spec}(\mathbb{F}) \times_{\operatorname{Spec}(\mathbb{O})} H$ . By [SGA3.2, Exp. XI, Corollaire 4.2] and [EGA4.4, Théorème 18.5.17], there exists an  $\mathbb{O}$ -group scheme morphism  $f: K \to H$  which lifts this embedding. If we let K act on itself by left multiplication, and on H by left multiplication via f, then f is K-equivariant. Hence so is the induced morphism

$$(4.1) \qquad \qquad \mathcal{O}(H) \to \mathcal{O}(K).$$

In other words (see [Ja, §I.2.11]), both  $\mathcal{O}(H)$  and  $\mathcal{O}(K)$  admit gradings by characters of K, and (4.1) is a graded morphism. Now each graded piece of  $\mathcal{O}(K)$  has finite rank over  $\mathbb{O}$  (in fact, these pieces are free of rank 1), and the image of this morphism under the functor  $\mathbb{F} \otimes_{\mathbb{O}} -$  is surjective by the construction of f. By the Nakayama lemma, we deduce that (4.1) itself is surjective, i.e. that f is a closed embedding.

Finally, it follows from [SGA3.3, Exp. XIX, Théorème 2.5] that the rank of the  $\overline{\mathbb{K}}$ -group  $\operatorname{Spec}(\overline{\mathbb{K}}) \times_{\operatorname{Spec}(\mathbb{O})} H$  coincides with that of  $\operatorname{Spec}(\mathbb{F}) \times_{\operatorname{Spec}(\mathbb{O})} H$ , so that  $\operatorname{Spec}(\overline{\mathbb{K}}) \times_{\operatorname{Spec}(\mathbb{O})} K$  is a maximal torus in  $\operatorname{Spec}(\overline{\mathbb{K}}) \times_{\operatorname{Spec}(\mathbb{O})} H$ . We have thus proved that K is a maximal torus in H, and hence that the latter group is split.  $\Box$ 

Next, from [Ja, §I.5.7] (or from [DG, III, §3, Proposition 2.5]) we deduce that the quotient morphism  $\varpi : G \to A$  is flat, and hence that G is a flat  $\mathbb{O}$ -group scheme (since A is flat by assumption). If  $\Bbbk$  is one of  $\mathbb{F}$ ,  $\mathbb{K}$  or  $\overline{\mathbb{K}}$ , we set

$$G_{\Bbbk} := \operatorname{Spec}(\Bbbk) \times_{\operatorname{Spec}(\mathbb{O})} G, \quad G^{\circ}_{\Bbbk} := \operatorname{Spec}(\Bbbk) \times_{\operatorname{Spec}(\mathbb{O})} G^{\circ}$$

Then  $G_{\Bbbk}$  is an extension of the constant (hence smooth)  $\Bbbk$ -group scheme associated with **A** by the smooth group scheme  $G_{\Bbbk}^{\circ}$ , so it is smooth by [Mi, Proposition 8.1]. From this we deduce that G itself is smooth (see [SP, Tag 01V8]).

In particular, the groups  $G_{\mathbb{F}}$  and  $G_{\overline{\mathbb{K}}}$  are algebraic groups (over  $\mathbb{F}$  and  $\mathbb{K}$ ) in the usual "naive" sense. Since  $G_{\mathbb{F}}^{\circ}$  is connected and  $\mathbf{A}$  is finite, the latter group identifies with the group of components of  $G_{\mathbb{F}}$ . Similarly,  $\mathbf{A}$  also identifies with the group of  $G_{\overline{\mathbb{K}}}$ .

**Lemma 4.2.** The morphism  $G(\mathbb{O}) \to \mathbf{A}$  induced by  $\varpi$  is surjective.

*Proof.* We consider the commutative diagram

$$\begin{array}{ccc} G(\mathbb{O}) & \longrightarrow & A(\mathbb{O}) \\ & & & & \\ & & & \\ & & & \\ G(\mathbb{F}) & \longrightarrow & A(\mathbb{F}) \end{array}$$

where the horizontal maps are induced by  $\varpi$  and the vertical ones by the quotient morphism  $\mathbb{O} \to \mathbb{F}$ . Here the lower arrow is surjective, and the right vertical arrow is an isomorphism. On the other hand the left vertical arrow is surjective by [EGA4.4, Théorème 18.5.17]. We deduce that the upper arrow is surjective, as desired.  $\Box$ 

Thanks to Lemma 4.2, we can (and will) choose a section  $\iota : \mathbf{A} \to G(\mathbb{O})$  of the projection induced by  $\varpi$ . (Of course, we do not assume that  $\iota$  is a group morphism.) We will assume that  $\iota(1) = 1$ . For simplicity, we will also denote by  $\iota : A \to G$  the morphism of  $\mathbb{O}$ -group schemes defined by  $\iota$ : for any  $\mathbb{O}$ -algebra R, the induced morphism on R-points is the composition  $A(R) = \mathbf{A} \stackrel{\iota}{\to} G(\mathbb{O}) \to G(R)$ .

Lemma 4.3. The morphism

$$A \times G^{\circ} \to G$$

defined by  $(a,g) \mapsto \iota(a) \cdot g$  is an isomorphism of  $\mathbb{O}$ -schemes.

*Proof.* Consider the algebra morphism  $\varphi : \mathcal{O}(G) \to \prod_{a \in \mathbf{A}} \mathcal{O}(G^{\circ})$  induced by our morphism. From the remarks above, we know that the algebra morphism  $\mathbb{F} \otimes_{\mathbb{O}} \varphi$  is an isomorphism. Here  $\prod_{a \in \mathbf{A}} \mathcal{O}(G^{\circ})$  is finitely generated as an  $\mathcal{O}(G)$ -module, since  $\mathcal{O}(G^{\circ})$  is. Hence, by (a variant for discrete valuation rings of) [BR, Lemma 1.4.1], we deduce that  $\varphi$  is an isomorphism.  $\Box$ 

4.2. Statement. Let us consider the Grothendieck groups

$$\mathsf{K}(G), \qquad \mathsf{K}(G_{\mathbb{K}}), \qquad \mathsf{K}(G_{\mathbb{F}})$$

of the categories of (algebraic) G-modules of finite type over  $\mathbb{O}$ , of finite-dimensional (algebraic)  $G_{\mathbb{K}}$ -modules, and of finite-dimensional (algebraic)  $G_{\mathbb{F}}$ -modules, respectively. We will also denote by  $\mathsf{K}_{\mathrm{pr}}(G)$  the Grothendieck group of the exact category of G-modules which are free of finite rank over  $\mathbb{O}$ . Following [Se] we consider the natural morphisms of abelian groups

(4.2) 
$$\begin{array}{c} \mathsf{K}_{\mathrm{pr}}(G) \xrightarrow{\sim} \mathsf{K}(G) \xrightarrow{} \mathsf{K}(G_{\mathbb{K}}) \\ & \swarrow \\ \mathsf{K}(G_{\mathbb{F}}) \end{array} \end{array}$$

Here, on the upper line, the left horizontal map (which is induced by the natural inclusion of categories) is an isomorphism by [Se, Proposition 4]. The right horizontal map (induced by the exact functor  $\mathbb{K} \otimes_{\mathbb{O}} (-)$ ) is surjective by [Se, Théorème 1]. The map from the top left-hand corner to the group on the bottom line is induced by the (exact) functor  $\mathbb{F} \otimes_{\mathbb{O}} (-)$ . Finally, the map  $d_G$  is the "decomposition" morphism from [Se, Théorème 2].

The main result of this section is the following.

**Theorem 4.4.** All the maps in (4.2) are isomorphisms.

According to [Se, Théorème 3], if  $d_G$  is surjective, then the right-hand morphism on the upper line is automatically an isomorphism. Thus, to prove Theorem 4.4, it is enough to prove that  $d_G$  is an isomorphism. This will be accomplished in §4.5 below. 4.3. Lattices. Our starting point will be the main result of [Se], which is applicable here thanks to Lemma 4.1. This result asserts that if we consider the diagram

(4.3) 
$$\begin{array}{c} \mathsf{K}_{\mathrm{pr}}(G^{\circ}) \xrightarrow{\sim} \mathsf{K}(G^{\circ}) \xrightarrow{\longrightarrow} \mathsf{K}(G^{\circ}_{\mathbb{K}}) \\ & \swarrow \\ \mathsf{K}(G^{\circ}_{\mathbb{F}}) \xrightarrow{d_{G^{\circ}}} \end{array}$$

similar to (4.2) but for the group  $G^{\circ}$ , then the decomposition morphism  $d_{G^{\circ}}$  is an isomorphism, so that all the maps in (4.3) are isomorphisms.

More precisely, let us a fix a split torus  $T \subset G^{\circ}$  and set  $\mathbf{X} := X^*(T)$ . Choosing a system of positive roots in the root system of  $(G^{\circ}, T)$ , we obtain a Borel subgroup  $B \subset G^{\circ}$  containing T (chosen such that B is the negative Borel subgroup), and a subset  $\mathbf{X}^+ \subset \mathbf{X}$  of dominant weights.

By the well known representation theory of connected reductive groups over algebraically closed fields, both the set of isomorphism classes of simple  $G^{\circ}_{\mathbb{K}}$ -modules and the set of isomorphism classes of simple  $G^{\circ}_{\mathbb{F}}$ -modules are in bijection with  $\mathbf{X}^+$ . More concretely, if  $\lambda \in \mathbf{X}^+$  and if  $L_{\mathbb{K}}(\lambda)$  is a simple  $G^{\circ}_{\mathbb{K}}$ -module of highest weight  $\lambda$ , then there exists a simple  $G^{\circ}_{\mathbb{K}}$ -module  $L_{\mathbb{K}}(\lambda)$  and an isomorphism  $\overline{\mathbb{K}} \otimes_{\mathbb{K}} V_{\mathbb{K}} \cong$  $V_{\overline{\mathbb{K}}}$  (see e.g. [Se, §4.6] or [Ja, Corollary II.2.9] for details). If  $V_{\mathbb{O}} \subset V_{\mathbb{K}}$  is a  $G^{\circ}$ stable  $\mathbb{O}$ -lattice, then the class of  $\mathbb{F} \otimes_{\mathbb{O}} V_{\mathbb{O}}$  in  $\mathsf{K}(G^{\circ}_{\mathbb{F}})$  coincides with the class of the Weyl module  $\Delta_{\mathbb{F}}(\lambda)$  of highest weight  $\lambda$  (because  $L_{\overline{\mathbb{K}}}(\lambda)$  and  $\Delta_{\mathbb{F}}(\lambda)$  have the same characters). In fact, it is well known that the lattice  $V_{\mathbb{O}}$  can be chosen in such a way that  $\mathbb{F} \otimes_{\mathbb{O}} V_{\mathbb{O}} \cong \Delta_{\mathbb{F}}(\lambda)$  as  $G^{\circ}_{\mathbb{F}}$ -modules. For each  $\lambda$  we will fix such a lattice, and denote it by  $L_{\mathbb{O}}(\lambda)$ .

In the present setting,  $\mathbf{A}$  is the group of components both of  $G_{\mathbb{F}}$  and of  $G_{\overline{\mathbb{K}}}$ . Identifying  $T_{\mathbb{F}}$  and  $T_{\overline{\mathbb{K}}}$  with the universal maximal tori of  $G_{\mathbb{F}}^{\circ}$  and  $G_{\overline{\mathbb{K}}}^{\circ}$  respectively (via the choice of Borel subgroups obtained from B by base change), we obtain two actions of  $\mathbf{A}$  on  $\mathbf{X} = X^*(T_{\mathbb{F}}) = X^*(T_{\overline{\mathbb{K}}})$ , see §2.2. The description of this action involves the property that Borel subgroups are conjugate, which is not true over  $\mathbb{O}$ ; so it is not clear from the definition that they must coincide. In the next lemma we will show that they do at least coincide on  $\mathbf{X}^+$ .

**Lemma 4.5.** The two actions of  $\mathbf{A}$  on  $\mathbf{X}$  agree on  $\mathbf{X}^+$ .

*Proof.* Let us provisionally denote the two actions of  $\mathbf{A}$  on  $\mathbf{X}$  by  $\cdot_{\mathbb{F}}$  and  $\cdot_{\mathbb{K}}$ . Since  $\mathbf{A}$  acts by algebraic group automorphisms on  $G^{\circ}$ , this group acts on all the Grothendieck groups in (4.3), and all the maps in this diagram are obviously  $\mathbf{A}$ -equivariant. Now for  $\lambda \in \mathbf{X}^+$  we have  $a \cdot [L_{\mathbb{K}}(\lambda)] = [L_{\mathbb{K}}(a \cdot_{\mathbb{K}} \lambda)]$ . Hence

$$d_{G^{\circ}}(a \cdot [L_{\mathbb{K}}(\lambda)]) = [\Delta_{\mathbb{F}}(a \cdot \overline{\mathbb{K}} \lambda)].$$

On the other hand, since  $d_{G^{\circ}}$  is **A**-equivariant we have

$$d_{G^{\circ}}(a \cdot [L_{\mathbb{K}}(\lambda)]) = a \cdot [\Delta_{\mathbb{F}}(\lambda)] = [\Delta_{\mathbb{F}}(a \cdot \lambda)]$$

(see (2.3)). We deduce that  $[\Delta_{\mathbb{F}}(a \cdot_{\overline{\mathbb{K}}} \lambda)] = [\Delta_{\mathbb{F}}(a \cdot_{\mathbb{F}} \lambda)]$ , hence that  $a \cdot_{\overline{\mathbb{K}}} \lambda = a \cdot_{\mathbb{F}} \lambda$ .  $\Box$ 

From now on we fix  $\lambda \in \mathbf{X}^+$ . It follows in particular from Lemma 4.5 that the two possible definitions of the subgroup  $\mathbf{A}^{\lambda} \subset \mathbf{A}$  (see §2.4) coincide.

**Lemma 4.6.** (1) We have  $\operatorname{End}_{G^{\circ}}(L_{\mathbb{O}}(\lambda)) = \mathbb{O}$ .

(2) For any  $a \in \mathbf{A}^{\lambda}$ , there exists an isomorphism of  $G^{\circ}$ -modules

$${}^{\iota(a)}L_{\mathbb{O}}(\lambda) \cong L_{\mathbb{O}}(\lambda).$$

*Proof.* We only explain the proof of (2); the proof of (1) is similar. Consider the object

$$R \operatorname{Hom}_{G^{\circ}}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda))$$

of the derived category of  $\mathbb{O}$ -modules. By [Ja, Lemma II.B.5 and its proof], this complex has bounded cohomology, and each of its cohomology objects is finitely generated. This implies that it is isomorphic (in the derived category) to a finite direct sum of shifts of finitely generated  $\mathbb{O}$ -modules.

It follows from [MR, Proposition A.6 and Proposition A.8] that we have

$$\mathbb{F} \overset{L}{\otimes}_{\mathbb{O}} R \operatorname{Hom}_{G^{\circ}}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda)) \cong R \operatorname{Hom}_{G^{\circ}_{\mathbb{F}}}(\Delta_{\mathbb{F}}(\lambda), \Delta_{\mathbb{F}}(\lambda)),$$
$$\overline{\mathbb{K}} \overset{L}{\otimes}_{\mathbb{O}} R \operatorname{Hom}_{G^{\circ}}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda)) \cong R \operatorname{Hom}_{G^{\circ}_{\overline{\pi}}}(L_{\overline{\mathbb{K}}}(\lambda), L_{\overline{\mathbb{K}}}(\lambda)).$$

Now we have  $R \operatorname{Hom}_{G_{\overline{\mathbb{K}}}^{\circ}}(L_{\overline{\mathbb{K}}}(\lambda), L_{\overline{\mathbb{K}}}(\lambda)) \cong \overline{\mathbb{K}}$ , so that  $\operatorname{Hom}_{G^{\circ}}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda))$  is a sum of  $\mathbb{O}$  and a torsion module. But since  $\operatorname{Hom}_{G_{\mathbb{F}}^{\circ}}(\Delta_{\mathbb{F}}(\lambda), \Delta_{\mathbb{F}}(\lambda)) = \mathbb{F}$ , this torsion module is zero; in other words we have  $\operatorname{Hom}_{G^{\circ}}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda)) \cong \mathbb{O}$ . If  $f : L_{\mathbb{O}}(\lambda) \to {}^{\iota(a)}L_{\mathbb{O}}(\lambda)$  is a generator of this rank-1  $\mathbb{O}$ -module, the  $G_{\mathbb{F}}$ -module morphism  $\mathbb{F} \otimes_{\mathbb{O}} f$  is an isomorphism, so that f is also an isomorphism.  $\Box$ 

4.4. Comparison of twisted group algebras. We continue with the setting of §4.3 (and in particular with our fixed  $\lambda \in \mathbf{X}^+$ ).

By Lemma 4.6 we can choose, for any  $a \in \mathbf{A}^{\lambda}$ , an isomorphism  $\theta_a : L_{\mathbb{O}}(\lambda) \xrightarrow{\sim} \iota^{(a)}L_{\mathbb{O}}(\lambda)$ . Then  $\overline{\mathbb{K}} \otimes_{\mathbb{O}} \theta_a$  is an isomorphism from  $L_{\overline{\mathbb{K}}}(\lambda)$  to  $\iota^{(a)}L_{\overline{\mathbb{K}}}(\lambda)$ , and for  $a, b \in \mathbf{A}^{\lambda}$  the scalar  $\alpha(a, b) \in \overline{\mathbb{K}}$  defined in §2.4 using these isomorphisms in fact belongs to  $\mathbb{O}^{\times}$ . In particular, if  $\mathscr{A}_{\overline{\mathbb{K}}}^{\lambda}$  is the associated twisted group algebra (over  $\overline{\mathbb{K}}$ ), then the  $\mathbb{O}$ -lattice  $\mathscr{A}_{\mathbb{O}}^{\lambda} := \bigoplus_{a \in A^{\lambda}} \mathbb{O} \cdot \rho_a$  is an  $\mathbb{O}$ -subalgebra in  $\mathscr{A}_{\overline{\mathbb{K}}}^{\lambda}$ . On the other hand,  $\mathbb{F} \otimes_{\mathbb{O}} \theta_a$  is an isomorphism from  $\Delta_{\mathbb{F}}(\lambda)$  to  $\iota^{(a)}\Delta_{\mathbb{F}}(\lambda)$ , and by Remark 3.2 the algebra  $\mathscr{A}_{\mathbb{F}}^{\lambda}$  from §2.4 (now for the group  $G_{\mathbb{F}}$  and its simple module  $L_{\mathbb{F}}(\lambda)$ ) can be described as the twisted group algebra of  $\mathbf{A}^{\lambda}$  defined by the cocyle sending (a, b) to the image of  $\alpha(a, b)$  in  $\mathbb{F}$ .

Summarizing, we have obtained an  $\mathbb{O}\text{-algebra}\ \mathscr{A}^\lambda_{\mathbb{O}}$  which is free over  $\mathbb{O}$  and such that

$$\overline{\mathbb{K}} \otimes_{\mathbb{O}} \mathscr{A}^{\lambda}_{\mathbb{O}} \cong \mathscr{A}^{\lambda}_{\overline{\mathbb{K}}}, \quad \mathbb{F} \otimes_{\mathbb{O}} \mathscr{A}^{\lambda}_{\mathbb{O}} \cong \mathscr{A}^{\lambda}_{\mathbb{F}}.$$

From Lemma 2.19 we know that  $\mathscr{A}_{\mathbb{F}}^{\lambda}$  and  $\mathscr{A}_{\mathbb{K}}^{\lambda}$  are products of matrix algebras (over  $\mathbb{F}$  and  $\mathbb{K}$  respectively). In fact, the same arguments show that  $\mathscr{A}_{\mathbb{K}}^{\lambda} := \mathbb{K} \otimes_{\mathbb{O}} \mathscr{A}_{\mathbb{O}}^{\lambda}$  is also a product of matrix algebras (over  $\mathbb{K}$ ). Hence we are in the setting of Tits' deformation theorem (see e.g. [GP, Theorem 7.4.6]), and we deduce that we have a canonical bijection between the sets of isomorphism classes of simple  $\mathscr{A}_{\mathbb{K}}^{\lambda}$ -modules and isomorphism classes of simple  $\mathscr{A}_{\mathbb{F}}^{\lambda}$ -modules, which sends a simple module M to  $\mathbb{F} \otimes_{\mathbb{O}} M_{\mathbb{O}}$ , where  $M_{\mathbb{O}}$  is any  $\mathscr{A}_{\mathbb{O}}^{\lambda}$ -stable  $\mathbb{O}$ -lattice in M.

Let now E be a simple  $\mathscr{A}_{\mathbb{K}}^{\lambda}$ -module. The same procedure as in §2.5 allows us to define a  $G_{\mathbb{K}}^{\lambda}$ -module structure on  $E \otimes_{\mathbb{K}} L_{\mathbb{K}}(\lambda)$ , where  $G_{\mathbb{K}}^{\lambda}$  is the inverse image of  $\mathbf{A}^{\lambda}$  under the map  $G_{\mathbb{K}} \to A$  induced by  $\varpi$ .

**Lemma 4.7.** Let  $\tilde{E}$  be the simple  $\mathscr{A}_{\mathbb{F}}^{\lambda}$ -module corresponding to E under the bijection above. Then we have

$$d_G([\operatorname{Ind}_{G^{\lambda}_{\mathbb{K}}}^{G_{\mathbb{K}}}(E \otimes L_{\mathbb{K}}(\lambda))]) = [\Delta_{\mathbb{F}}(\lambda, \tilde{E})],$$

where  $G_{\mathbb{F}}^{\lambda}$  is the inverse image of  $\mathbf{A}^{\lambda}$  under the map  $G_{\mathbb{F}} \to A$  induced by  $\varpi$ .

Proof. If  $E_{\mathbb{O}} \subset E$  is an  $\mathscr{A}^{\lambda}_{\mathbb{O}}$ -stable lattice in E, then  $E_{\mathbb{O}} \otimes_{\mathbb{O}} L_{\mathbb{O}}(\lambda)$  has a natural structure of  $G^{\lambda}$ -module (where  $G^{\lambda} = \varpi^{-1}(\mathbf{A}^{\lambda})$ ), and is a  $G^{\lambda}$ -stable lattice in  $E \otimes L_{\mathbb{K}}(\lambda)$ . Inducing to G, we deduce that  $\operatorname{Ind}_{G^{\lambda}}^{G}(E_{\mathbb{O}} \otimes_{\mathbb{O}} L_{\mathbb{O}}(\lambda))$  is a G-stable lattice in  $\operatorname{Ind}_{G^{\lambda}_{\mathbb{V}}}^{G_{\mathbb{K}}}(E \otimes L_{\mathbb{K}}(\lambda))$ , whose modular reduction is  $\Delta_{\mathbb{F}}(\lambda, \tilde{E})$ .

4.5. Invertibility of  $d_G$ . We can now prove that  $d_G$  is an isomorphism, which will finish the proof of Theorem 4.4.

In fact, for any  $\lambda \in \mathbf{X}^+$ , since  $\mathscr{A}_{\mathbb{K}}^{\lambda}$  is a product of matrix algebras the assignment  $E \mapsto \overline{\mathbb{K}} \otimes_{\mathbb{K}} E$  induces induces a bijection between the sets of isomorphism classes of simple modules for the algebras  $\mathscr{A}_{\mathbb{K}}^{\lambda}$  and  $\mathscr{A}_{\overline{\mathbb{K}}}^{\lambda}$  from §4.4. Then, using Theorem 2.16 and arguing as in [Se, §3.6], we see that the similar operation induces a bijection between the sets of isomorphism classes of simple  $G_{\mathbb{K}}$ -modules and of simple  $G_{\overline{\mathbb{K}}}$ -modules.

Let us now fix a subset  $\Lambda \subset \mathbf{X}^+$  of representatives for the **A**-orbits on  $\mathbf{X}^+$ . By the remarks above, the classes of the modules  $\operatorname{Ind}_{G_{\mathbb{K}}^{\lambda}}^{G_{\mathbb{K}}}(E \otimes L_{\mathbb{K}}(\lambda))$ , where  $(\lambda, E)$  runs over the pairs consisting of an element  $\lambda \in \Lambda$  and a simple  $\mathscr{A}_{\mathbb{K}}^{\lambda}$ -modules, form a basis of  $\mathsf{K}(G_{\mathbb{K}})$  (see in particular Remark 2.17(1)). In view of Lemma 4.7 and Theorem 3.7, the image of this basis under  $d_G$  is a basis of  $\mathsf{K}(G_{\mathbb{F}})$ . Hence,  $d_G$  is indeed an isomorphism.

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