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# Controlling frequency dispersion in electromagnetic invisibility cloaks

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## Abstract

Electromagnetic cloaking, as challenging as it may be to the physicist and the engineer has become a topical subject over the past decade. Thanks to the transformations optics (TO) invisibility devices are in sight even though quite drastic limitations remain yet to be lifted. The extreme material properties which are deduced from TO can be achieved in practice using dispersive metamaterials. However, the bandwidth over which a metamaterial cloak is efficient is drastically limited. We design and simulate a spherical cloak which takes into account the dispersive nature of relative permittivity and permeability tensors realized by plasma-like metamaterials. This spherical cloak works over a broad frequency-band even though these materials are of a highly dispersive nature. We establish two equations of state that link the eigenvalues of the permittivity and permeability tensors in every spherical cloak regardless of the geometrical transformation. Frequency dispersive properties do not disrupt cloaking as long as the equations of states are satisfied in the metamaterial cloak.

## 1 Introduction

Transformation optics (TO) is a powerful tool allowing us to imagine complex structured media that control the propagation of electromagnetic waves as if space was distorted. Twelve years ago, research groupings of Pendry [1] and Leonhardt [2] have shown how a special distribution of matter in a real physical space can mimic a distorted virtual electromagnetic space. Numerical simulations have validated the TO theory beyond doubt, notably for an electromagnetic source in the far field [3], and in the intense near field [4]. However, experimental realizations remain challenging : the design of the metamaterial coating must exhibit highly anisotropic effective tensors of relative permittivity and permeability that correspond to the spatially varying tensors given by the TO theory. A non-magnetic cylindrical cloak operating at optical wavelengths has been proposed with a reduced set of parameters, but the cloak's outer boundary is not perfectly matched with vacuum and thus some reflection persists [5]. In fact, effective tensors of relative permittivity and permeability within the mantle cloak are characterized by eigenvalues ranging from 0 to 1 if one wants to realize a mantle in free space. Therefore, we would like to address here three main issues : extreme anisotropy, inhomogeneity and values of relative permittivity and relative permeability bellow unity.

In practice, values of relative permittivity and permeability bellow unity are achievable using resonant metallic inclusions in the coating. Schurig et al. made the first experimental demonstration in 2006 [6] with concentric rings of split ring resonators. With the use of resonant devices comes one of the challenges of cloaking : the dispersion of materials makes it impossible for the coating to

operate over a range of frequencies. Many strategies have been tested in order to enlarge the frequency band [7] [8] [9] [10], some of them sacrifice the cloaking efficiency to broaden the bandwidth [11]. Limitations of cloaking with dispersive materials have been further analyzed in [12] [13] [14]. Kildishev published an interesting study ten years ago [15], about the engineering of dispersion : he considered that the inner radius of the T0 map could be frequency dependent. Considering a first order Taylor approximation for the dispersion of permittivity and permeability, he obtained for a cylindrical cloak with the electric field polarized along the (out-of-plane) z-axis (TE polarization), an elegant formula that exhibits a link between the space transformation and frequency. Besides, Rajput and Srivastava performed simulations of a cylindrical cloaking with an extended bandwidth thanks to a similar perturbative approach [16].

We present here a more general approach of dispersion engineering for spherical cloaking. We propose a convenient and versatile method to design a cloak with dispersive materials.

## 2 Control of dispersion

Let us consider a transformation that allows a cloaking of a sphere  $S_1$  (of radius  $R_1$ ) by a sphere  $S_2$  (of radius  $R_2$ ) with  $R_1 < R_2$ . We consider a bijective and smooth transformation between  $\mathbb{R}^3$  and  $\{\mathbb{R}^3/S_1\}$  of the form :

$$(r', \theta', \phi') = \begin{cases} (r, \theta, \phi) & \text{if } r > R_2 \\ (f^{-1}(r), \theta, \phi) & \text{if } r \leq R_2 \end{cases} \quad (1)$$

TO equations provide us with the formula for the permittivity and permeability tensors. We deduce the tensor  $\bar{\epsilon}_{coating}$  in the spherical coordinates as first established by Dolin in 1961 [17] :

$$\bar{\epsilon}_{coating}(r) = \bar{\mu}_{coating}(r) = \begin{pmatrix} \frac{f(r)^2}{r^2} \frac{1}{\frac{\partial f(r)}{\partial r}} & 0 & 0 \\ 0 & \frac{\partial f(r)}{\partial r} & 0 \\ 0 & 0 & \frac{\partial f(r)}{\partial r} \end{pmatrix}_{(r, \theta, \phi)} \quad (2)$$

We have seen in the introduction that for a metamaterial cloak, radial permittivity and permeability are necessarily dispersive. We asked ourselves whether it would be possible to engineer a cloaking device that fits the dispersion of materials, by using a frequency dependent geometric transformation.

Tedious but straightforward calculations lead to (see supplemental material) system (3) , in which we consider that the geometric transformation depends on the pulsation  $\omega$  :

$$\begin{cases} \epsilon_{radial}(r, \omega) \epsilon_{ortho}(r, \omega) = \frac{f(r, \omega)^2}{r^2} \\ \epsilon_{ortho}(r, \omega) = \frac{\partial f(r, \omega)}{\partial r} \end{cases} \quad (3)$$

System (3) is written here for the permittivity components, but takes the same form with permeability components. We assume in the sequel that  $\epsilon_{radial} = \mu_{radial}$  and  $\epsilon_{ortho} = \mu_{ortho}$  . As soon as there is a function  $f$  that allows the one to one correspondence (bijection) between  $\mathbb{R}^3$  and  $\{\mathbb{R}^3/S_1(\omega)\}$  and such that (3) is fulfilled, cloaking works.

Let us recast (3) to get more convenient criteria that  $\epsilon_{ortho}$  and  $\epsilon_{radial}$  have to satisfy in order to achieve cloaking. We note that in (3) equations link  $\epsilon_{radial}$  ,  $\epsilon_{ortho}$  , and  $f$  : it is thus possible to manipulate equations in order to make  $f$  disappear. We obtain two equivalent systems that are equivalent to (3) as long as some classical hypotheses on the integrability and derivability of  $\epsilon_{radial}$  and  $\epsilon_{ortho}$  (and likewise for the permeability) are verified :

$$\left\{ \begin{array}{l} \exists R_1(\omega) / \int_{R_1(\omega)}^{R_2} \epsilon_{ortho}(r, \omega) dr = R_2 \\ \epsilon_{radial}(r, \omega) = \frac{1}{\epsilon_{ortho}(r, \omega)} \left( \frac{R_2 - \int_r^{R_2} \epsilon_{ortho}(s, \omega) ds}{r} \right)^2 \end{array} \right. \quad (4)$$

$$\left\{ \begin{array}{l} \exists R_1(\omega) / \lim_{r \rightarrow R_1(\omega)} \left( \int_r^{R_2} \frac{dx}{x^2 \epsilon_{radial}(x, \omega)} \right) = +\infty \\ \epsilon_{ortho}(r, \omega) = \frac{1}{\epsilon_{radial}(r, \omega) r^2 \left( \frac{1}{R_2} + \int_{R_2}^r \frac{-1}{s^2 \epsilon_{radial}(s, \omega)} ds \right)^2} \end{array} \right. \quad (5)$$

In (4) and (5), the geometric transformation is implicit. We have here two equations in each system :

- First equation of each system gives a criterion on  $\epsilon_{ortho}$  or  $\epsilon_{radial}$  that has to be verified if we want a bijection between  $\mathbb{R}^3$  and  $\{\mathbb{R}^3/S_1(\omega)\}$
- Second equation of each system gives a constitutive relation between the components of the permittivity (and also permeability) tensor(s). They ensure the required optical path (in the light ray point of view) in the coating.

The main idea that emerges from these equations is that as soon as radial components of permittivity and permeability are small enough, there is a function that gives the orthoradial permittivity and permeability in order to realize a cloak. Conversely, as soon as we have large enough orthoradial components, there exists a function that describes the radial components.

### 3 Application

We now consider as an illustrative example a cloak for which the radial components of permittivity and permeability behave as an ideal lossless Drude metal near its plasma pulsation. Let us set the plasma pulsation as a function of radius,  $\omega_p = \frac{R_2 - r}{R_2 - R_1} \omega_0$ , then we deduce the radial permittivity and permeability :

$$\epsilon_{radial}(r, \omega) = \mu_{radial}(r, \omega) = 1 - \frac{\omega_0^2}{\omega^2} \left( \frac{R_2 - r}{R_2 - R_1} \right)^2 \quad (6)$$

Pulsation  $\omega_0$  is the maximum pulsation of operation of the cloak. The components of the permittivity and permeability tensors are known, so it is possible to perform numerical simulations. Full wave simulations have been realized with *COMSOL Multiphysics*<sup>®</sup> and validate the theory : in order to simulate a frequency dependent cloaking, we chose to design a multilayered cloak. Every layer of that cloak is composed of an adapted theoretical material : layers behave as plasmons in the radial direction, and their orthoradial properties are deduced using the constitutive equation of system (5). Radial and orthoradial properties of a representative layer can be seen on Fig.2. The choice of a multilayer design limits the number of different plasma pulsations in the cloak, and makes it more realistic : instead of using function (6) as written here, the variable "r" has been discretized,

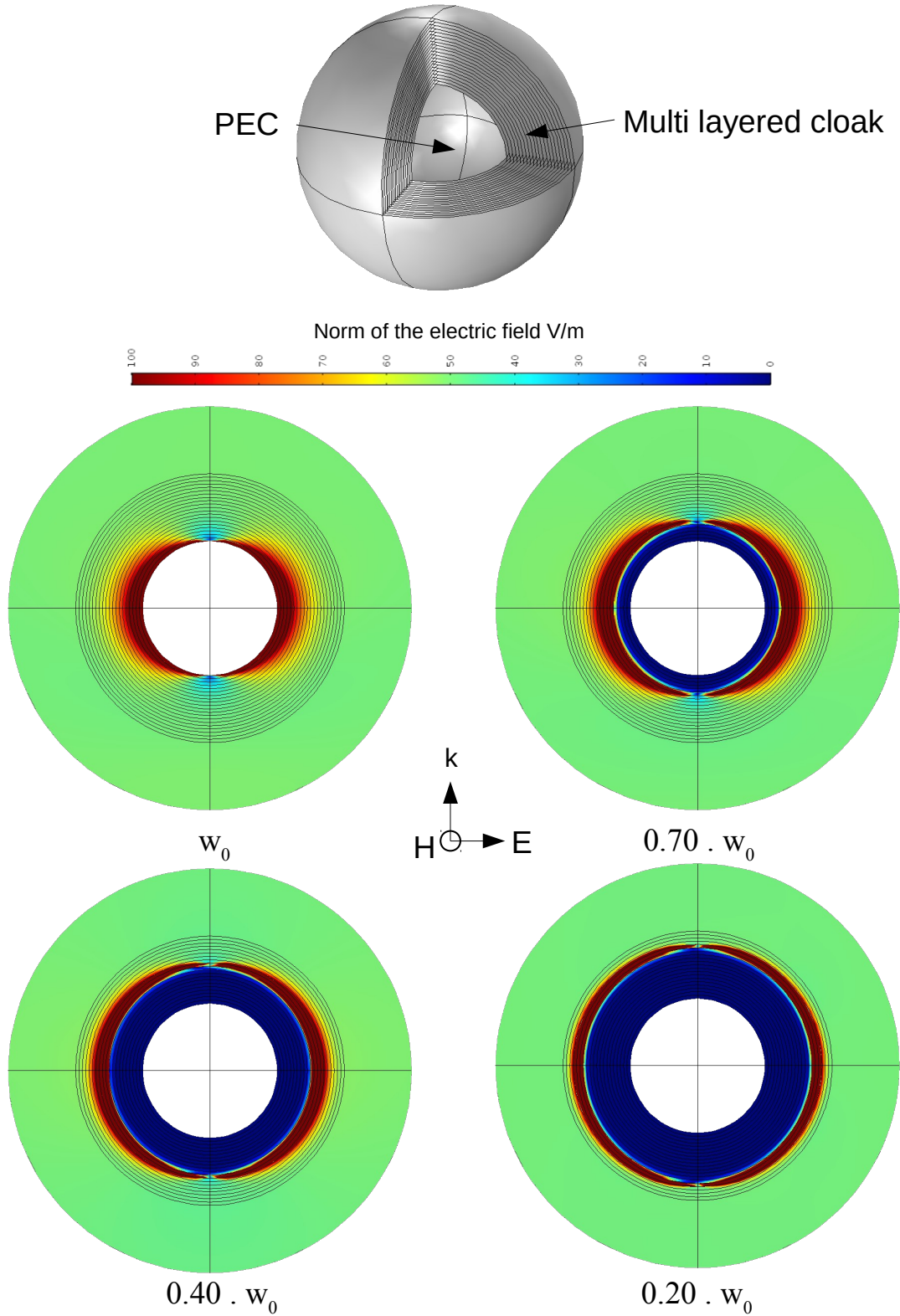


Figure 1: Time-averaged total electric field for different pulsations : simulation of the dispersion engineered cloak. The magnetic field of the source is oriented perpendicularly to the cut plane. For this simulations,  $R_1 = 0.5R_2$ ,  $\lambda_0 = 3R_2$  with  $\lambda_0$  the wavelength in the vacuum at the pulsation  $\omega_0$ .

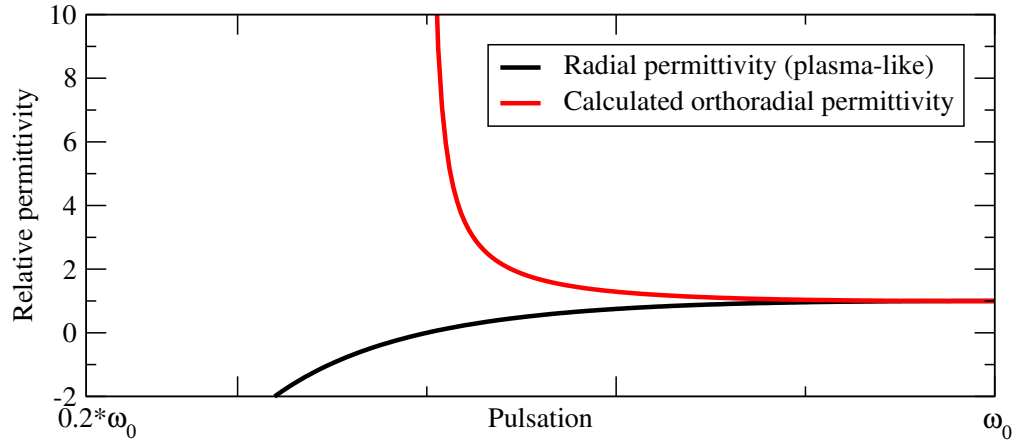


Figure 2: Typical radial and orthoradial permittivity and permeability of a material composing one of the layers in the cloak. The black curve gives the dispersive behaviour of the plasma-like radial permittivity, whereas the red curve gives the orthoradial permittivity that was calculated using the second equation of (5). The plasma pulsation is  $\omega_p = \omega_0/2$ , and it corresponds to the material in the 10<sup>th</sup> layer (counting from the outside of the cloak to the inside).

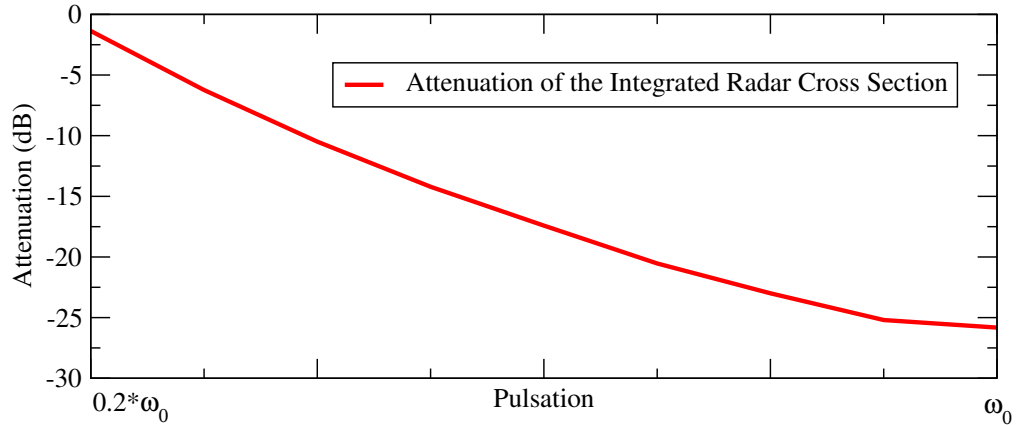


Figure 3: Attenuation (dB) of the Integrated Radar Cross Section of the cloaked metallic sphere. The closer we are to  $\omega_0$ , the more layers contribute to the cloaking : this explains why we have a better efficiency near  $\omega_0$ . These results have been obtained with a total of 20 layers, and would be improved if we consider more layers.

so that tensor components in each layer are constant.

The sphere of radius  $R_1(\omega_0)$  inside the coating is a perfect conductor that exhibits a large Radar Cross Section in the absence of cloaking. The maximum element size of the mesh we used for the calculation is equal to  $\lambda_0/10$ , with  $\lambda_0$  the wavelength in free space for the pulsation  $\omega_0$ . We used the scattered field module of COMSOL, with a background linear polarized electromagnetic wave. The wave vector is oriented vertically on Fig.1 so that we can note that the cloak is working well. Indeed, we neither observe a shadow behind the cloak nor wave interference in front of it.

We can observe on Fig.1 that the cloak is working approximatively over a frequency band from  $0.2 \omega_0$  to  $\omega_0$ . Over that interval of pulsations the norm of the scattered far field remains very low in comparison with the metallic sphere without the cloak (see Fig.3). The maximal theoretical band of operation is contained in the interval  $[\omega_{min}, \omega_0]$  where  $\omega_{min}$  is given by the plasma pulsation at the external boundary of the cloak :  $\omega_{min} = 0$  theoretically. In other words, we can cloak with this design over the entire pulsation band within which we are able to obtain a plasma-like resonance. As in [15] and [16] we obtain an internal radius that varies with the pulsation. Looking at the simulations, that phenomenon can be seen very clearly. We can also clearly identify that there is no electric field near the metallic sphere, inside the frequency-dependent cloaked region. The reduction of Radar Cross Section between the metallic sphere uncloaked and cloaked, reaches values of typical cloaking for a multilayered cloak. The attenuation of the Radar Cross Sections, which is proportional to the square of the norm of the scattered far field is represented in Fig.3. The reduced attenuation for low frequencies is caused by the reduction of the number of layers in the cloak, that has an impact on its efficiency. The attenuation can theoretically be infinite for continuously varying materials as soon as and the components of permeability and permittivity tensors satisfy the equations over the frequency band of interest. Indeed, no approximation has been done to derive the equations.

One of the limitations of that method is the exotic dispersion profile that we obtain for  $\epsilon_{ortho}$ . We cannot guarantee that a material exhibiting such properties exist. Fortunately it is possible to find a solution to that problem : considering a given collection of materials and metamaterials with their dispersion properties, an optimization algorithm could determine what materials should be used in every layer of the coating. The strange properties showed in Fig.2 could possibly be approximated with a combination of a high pass filter or even band pass filters and regular dielectrics. In the field of microwave frequencies, one can use for example high pass filters realized with microstrip lines imitating a T-circuit [18], whereas band pass filters are very satisfying using rectangular complementary split ring resonators [19]. Combinations of dispersive materials in the metamaterial of the coating could lead to a good approximation of the desired permittivity dispersion.

## 4 Conclusion

Transformation optics allows us to imagine many ways to cloak objects as soon as we are able to manufacture the coating. Nonetheless, one of the main limitations to cloaking is the bandwidth, that cannot easily be enlarged. The system of equations we set up allows us to design broadband cloaking using dispersive materials. We cannot force materials to be non-dispersive, and moreover we cannot impose a specific dependence between permittivity (or permeability) and frequency. However, thanks to existing materials and with metamaterials technology at hand, it may be possible to design a broadband cloak by following the proposed strategy.

## Aknowledgments

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## 5 Supplemental

### 5.1 Calculations

Let us consider a transformation that allows for a cloaking of a sphere  $S_1$  (of radius  $R_1$ ) by a sphere  $S_2$  (of radius  $R_2$ ) with  $R_1 < R_2$ . We consider a bijective and smooth transformation between  $\mathbb{R}^3$  and  $\{\mathbb{R}^3/S_1\}$  of the form :

$$(r', \theta', \phi') = \begin{cases} (r, \theta, \phi) & \text{if } r > R_2 \\ (f^{-1}(r), \theta, \phi) & \text{if } r \leq R_2 \end{cases} \quad (7)$$

Tools of transformation optics (TO) lead to permittivity and permability tensors as functions of the radius  $r$  :

$$\bar{\epsilon}_{coating}(r) = \bar{J}^{-1}(r) \bar{\epsilon}_{medium}(r) \bar{J}^{-T}(r) \det(\bar{J}(r)) \quad (8)$$

If the background medium is air we have  $\bar{\epsilon}_{medium} = I_3$ , the identity matrix.  $J$  stands for the jacobian matrix of the transformation. We deduce the tensor  $\bar{\epsilon}_{coating}$  in the spherical coordinates as established in a paper that predates TO [17] :

$$\bar{\epsilon}_{coating}(r) = \bar{\mu}_{coating}(r) = \begin{pmatrix} \frac{f(r)^2}{r^2} \frac{1}{\frac{\partial f(r)}{\partial r}} & 0 & 0 \\ 0 & \frac{\partial f(r)}{\partial r} & 0 \\ 0 & 0 & \frac{\partial f(r)}{\partial r} \end{pmatrix}_{(r,\theta,\phi)} \quad (9)$$

We deduce the following system :

$$\begin{cases} \epsilon_{radial}(r)\epsilon_{ortho}(r) = \frac{f(r)^2}{r^2} \\ \epsilon_{ortho}(r) = \frac{\partial f(r)}{\partial r} \end{cases} \quad (10)$$

We then consider that the whole system is frequency-dependent, in order to take into account the dispersion of materials :

$$\begin{cases} \epsilon_{radial}(r, \omega)\epsilon_{ortho}(r, \omega) = \frac{f(r, \omega)^2}{r^2} \\ \epsilon_{ortho}(r, \omega) = \frac{\partial f(r, \omega)}{\partial r} \end{cases} \quad (11)$$

For given permeability components, as soon as there is a function  $f$  that allows the bijection between  $\mathbb{R}^3$  and  $\{\mathbb{R}^3/S_1\}$  and that verifies (11) cloaking works. However such a system is difficult to manipulate : if the radial component of permittivity is imposed by materials and happens to be dispersive, we won't easily find function  $f$ .

We transform that system to obtain more convenient criteria that  $\epsilon_{ortho}$  and  $\epsilon_{radial}$  have to satisfy in order to achieve cloaking. We can first integrate the second equation between  $R_1$  and  $R_2$ , to obtain a criterion on  $\epsilon_{ortho}$  :

$$\int_{R_1(\omega)}^{R_2} \epsilon_{ortho}(r, \omega) dr = f(R_2) - f(R_1(\omega))$$

The continuity of the function  $f$  gives us  $f(R_2) = R_2$  and  $f(R_1(\omega)) = 0$ . We obtain the following criterion :

$$\exists R_1(\omega) / \int_{R_1(\omega)}^{R_2} \epsilon_{ortho}(r, \omega) dr = R_2 \quad (12)$$

Conversely, considering a given function of radius  $\epsilon_{ortho}$ , if there exists  $R_1$  for which the integral between  $R_1$  and  $R_2$  of  $\epsilon_{ortho}$  is equal to  $R_2$  and  $\epsilon_{ortho}$  is positive and integrable over that interval, then there is a function  $f$  for which  $\epsilon_{ortho}$  is adapted. In that case the function  $f$  is the primitive of  $\epsilon_{ortho}$  that vanishes in  $R_1$ . We can find a similar criterion for  $\epsilon_{radial}$  that is equivalent to (12) :

$$\exists R_1(\omega) / \lim_{r \rightarrow R_1(\omega)} \left( \int_r^{R_2} \frac{dx}{x^2 \epsilon_{radial}(x, \omega)} \right) \rightarrow +\infty \quad (13)$$

As soon as  $\epsilon_{ortho}$  is a function of radius that is positive and integrable, equation (12) and the second equation of the system (11) are equivalent. We will now consider the following system as equivalent to (11) :

$$\begin{cases} \epsilon_{radial}(r, \omega) \epsilon_{ortho}(r, \omega) = \frac{f(r, \omega)^2}{r^2} \\ \exists R_1(\omega) / \int_{R_1(\omega)}^{R_2} \epsilon_{ortho}(r, \omega) dr = R_2 \end{cases} \quad (14)$$

Using the boundary conditions  $f(R_2) = R_2$  and  $f(R_1) = 0$ , we can extract an equation that links  $\epsilon_{radial}$  and  $\epsilon_{ortho}$  without  $f$  :

$$\epsilon_{radial}(r, \omega) = \frac{1}{\epsilon_{ortho}(r, \omega)} \left( \frac{R_2 - \int_r^{R_2} \epsilon_{ortho}(s, \omega) ds}{r} \right)^2 \quad (15)$$

Equation (15) gives us the criterion that links the components of permittivity (and also permeability) tensor(s) without explicit dependence upon the geometric transformation. However it would be more convenient to have an equation that gives  $\epsilon_{ortho}$  as a function of  $\epsilon_{radial}$ . For this purpose, let us introduce the function  $\alpha$ , defined as :

$$\alpha(r, \omega) = \sqrt{\epsilon_{radial}(r, \omega) \epsilon_{ortho}(r, \omega)} \quad (16)$$

When we differentiate the first equation in (4) we obtain :

$$\begin{aligned} \frac{\partial f(r, \omega)}{\partial r} &= \frac{\partial \sqrt{r^2 \epsilon_{radial}(r, \omega) \epsilon_{ortho}(r, \omega)}}{\partial r} \\ \epsilon_{ortho}(r, \omega) &= \frac{\partial (r \alpha(r, \omega))}{\partial r} \end{aligned}$$

$$\frac{\alpha(r, \omega)^2}{\epsilon_{radial}(r, \omega)} = \frac{\partial (r\alpha(r, \omega))}{\partial r}$$

Calculations bring us then to the following differential equation for  $\alpha$  :

$$\frac{\partial \alpha(r, \omega)}{\partial r} + \frac{\alpha(r, \omega)}{r} = \frac{\alpha(r, \omega)^2}{\epsilon_{radial}(r, \omega)r} \quad (17)$$

We recognize a Bernoulli differential equation that can be solved with the boundary conditions we have on  $\alpha$  :

$$\alpha(r, \omega) = \frac{1}{r \left( \frac{1}{R_2} + \int_{R_2}^r \frac{-1}{s^2 \epsilon_{radial}(s, \omega)} ds \right)} \quad (18)$$

We finally have  $\epsilon_{ortho}$  as a function of  $\epsilon_{radial}$  :

$$\epsilon_{ortho}(r, \omega) = \frac{1}{\epsilon_{radial}(r, \omega)r^2 \left( \frac{1}{R_2} + \int_{R_2}^r \frac{-1}{s^2 \epsilon_{radial}(s, \omega)} ds \right)^2} \quad (19)$$

## 5.2 Simulations

Simulations have been performed with the RF module of *COMSOL Multiphysics*<sup>®</sup> 5.3 in the frequency domain. We created 21 concentric spheres, delimiting the 20 different layers composing the cloak. Two additional concentric spheres delimit the heterogeneous anisotropic and absorptive perfectly matched layer (spherical PML), outside the cloak.

The permittivity and permeability tensors in the layers are frequency-dependent, in order to match those calculated by our methodology. The mesh was built to make possible a simulation at the highest pulsation, that is the pulsation of reference of the cloak : we used the same mesh for every pulsation. In the same way, the PML thickness was dimensioned in order to work at  $\omega_0$ .

The PML may have caused some slight problems at lower pulsation, because it was too thin compared to the wavelength in free space. However, it was difficult to enlarge it, because even with a less accurate mesh the number of elements would have been too large for computational resources available. With the mesh of Fig.4, composed of 315 000 elements of domain, the simulation on a 8-Core processor (Xeon(R) CPU E5-2637 v2) lasted 4 hours for 9 different pulsations : 27 minutes per pulsation on average.

## 5.3 Calculation of the attenuation

In order to evaluate quantitatively the efficiency of our cloaking device in the simulation, we set up the following procedure :

- we realize the numerical simulation of the metallic sphere cloaked with a frequency sweep : the pulsation ration  $\omega/\omega_0$  starts at 0.2 and stop at 1 with a step of 0.1.
- we realize the same simulation without the cloak.
- we extract the norm of the electrical scattered far field along 1250 different directions for each frequency, with and without the cloak.
- we calculate the average far field for each frequency.
- we calculate for each frequency the following attenuation, that is equivalent the attenuation of the Integrated Radar Cross Section :  $A = 10 \log \left( \frac{\langle E_{far, cloaked} \rangle^2}{\langle E_{far, uncloaked} \rangle^2} \right)$

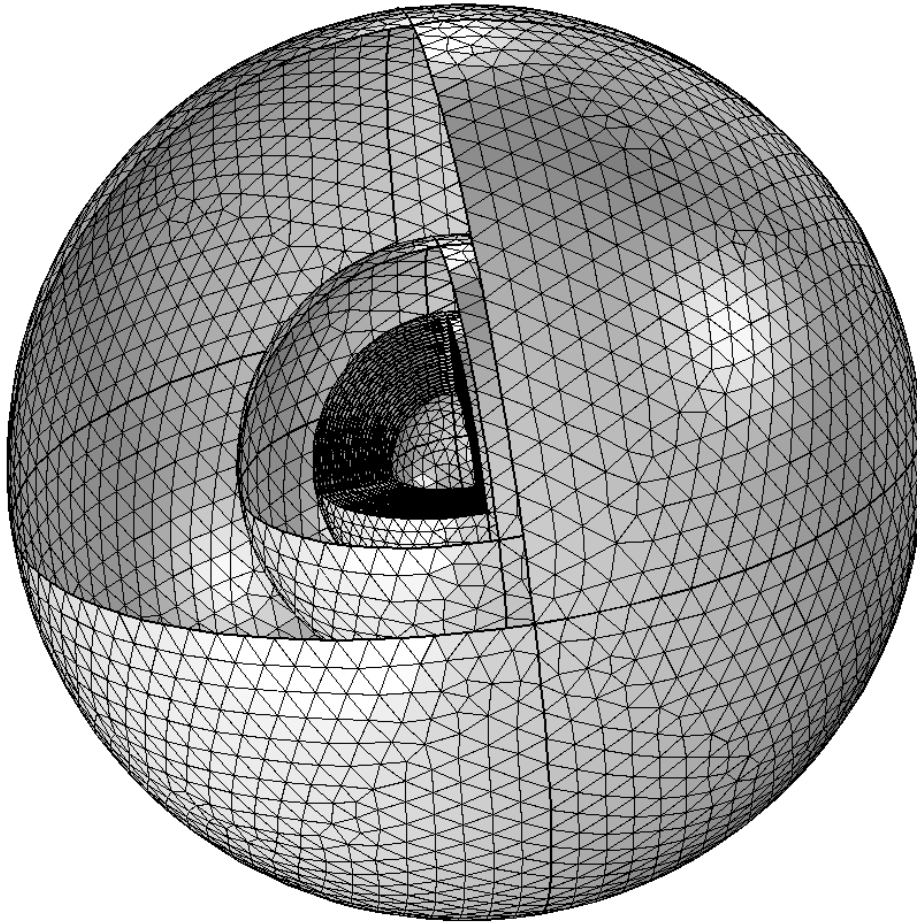


Figure 4: Mesh used for the simulations : the external large layer is the PML region. The maximum size of a mesh element is equal to  $\lambda_0/10$ , with  $\lambda_0$  the wavelength in free space for the highest pulsation. It is composed of 315 000 domain elements.

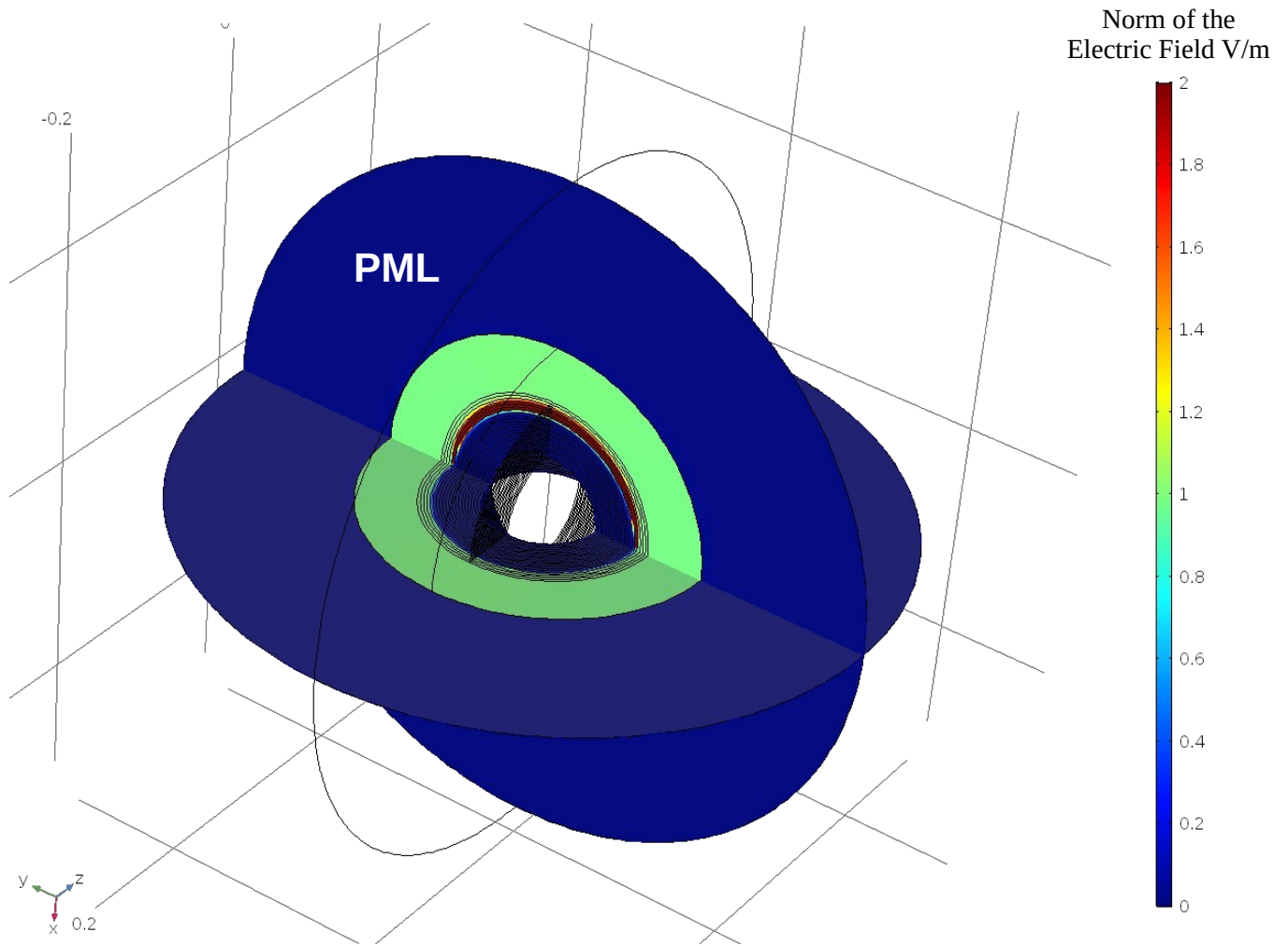


Figure 5: Time-averaged norm of the electric field for a simulation with  $\omega/\omega_0 = 0.2$ . The wave vector of the background wave is along the y direction, and the electric field of the background wave is along the x direction.

## 5.4 Layers of the cloak

Each layer of the cloak is composed of an anisotropic theoretical material whose permittivity and permeability tensors are diagonal in the spherical coordinates. The radial components of permeability and permittivity have a plasma-like dispersive behavior and the other two components are calculated using the equations we developed. In order to have constant tensor components in the spherical coordinates we assume that the radius is constant inside each layer : the thickness of each layer is supposed to be small compared to its radius of curvature. The plasma pulsation is then given by :

$$\omega_{p,layer} = \omega_p(r_{layer}) = \frac{R_2 - r_{layer}}{R_2 - R_1} \omega_0 \quad (20)$$

We deduce the permittivity and permeability tensors :

$$\bar{\bar{\epsilon}}_{layer}(\omega) = \bar{\bar{\mu}}_{layer}(\omega) = \left( 1 - \frac{\omega_{p,layer}^2}{\omega^2} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_{layer}(\omega) & 0 \\ 0 & 0 & \alpha_{layer}(\omega) \end{pmatrix}_{(r,\theta,\phi)} \quad (21)$$

with  $\alpha_{layer}$  a function of the pulsation that describes the anisotropy of the material :

$$\alpha_{layer}(\omega) = \left( 1 - \frac{\omega_{p,layer}^2}{\omega^2} \right)^{-2} \left( \frac{r_{layer}}{R_2} + \int_{R_2}^{r_{layer}} \frac{-r_{layer}}{s^2 \left( 1 - \frac{\omega_p(s)^2}{\omega^2} \right)} ds \right)^{-2} \quad (22)$$