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# An original geometrical contribution to special relativity.

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**Background:** Minkowski metric, geometrodynamics, special relativity, physics education.

**Purpose:** Overview of a deeper understanding of special relativity.

**Method:** From Minkowski metric, one can usually draw hyperbolic figures. By changing the point of view, it is possible to turn these hyperbolic figures into trigonometric figures, where velocity is represented by a sine, and gamma factor by the inverse of a cosine.

**Results:** One can study kinematics and dynamics in a new different way, which seems simpler for those who are at ease with geometry. This first article will essentially focus on kinematics.

**Conclusions:** This contribution could be helpful for physics education.

## I. INTRODUCTION

Nowadays, science seems to be more and more complicated. Indeed, this is a quite natural thing : high level researchers don't have always time to simplify their works. They essentially focus on their main "philosopher's stone" : new discoveries, which can change our minds and our lives forever. Consequently, it is important to go back in time, to read old researches of the two or three last centuries, in order to simplify science, for a better understanding of the laws of the Universe. To go further, one has to take stock of the situation. So has this article a role to play in this story. The geometry presented here is based on Minkowski metric, and is developed in two books that I have written on the subject. This new way of thinking could not be only helpful for young students, but also for researchers, to make new discoveries, hopefully.

## II. MINKOWSKI METRIC

### II.1. Background

First of all, a brief recall of Minkowski metric is necessary. Let us consider two dimensions instead of four, one of time and one of space. The convention adopted in this article is the  $(+ - - -)$  signature, which becomes  $(+ -)$  in two dimensions:

$$\begin{aligned} (\Delta s)^2 &= (c\Delta\tau_1)^2 = (c\Delta t_{0\rightarrow 1})^2 - (\Delta x_{0\rightarrow 1})^2 \\ (c\Delta\tau_1)^2 &= (c\Delta\tau_0)^2 - (\Delta x_{0\rightarrow 1})^2 \end{aligned} \quad (1)$$

Where:

- $(c\Delta\tau_1)$  is the proper time (or proper "temporal distance") of a particle  $1$ , i.e. measured in its proper Galilean system of reference  $1$  ;
- $(c\Delta t_{0\rightarrow 1}) = (c\Delta\tau_0)$  is the improper time (or improper "temporal distance") of this particle  $1$ , i.e. being observed by any other Galilean system of reference  $0$ . But this time is proper for  $0$  itself ;
- $(\Delta x_{0\rightarrow 1})$  is the "spatial distance" between the particle  $1$  and any other Galilean system of reference  $0$ , which is observing the particle  $1$ .

It is important to be accurate : as time is not absolute in relativity, it is important to define the observer. This is why "being observed by" and "which is observing" are underlined. The arrows between  $0$  and  $1$  try to make it clearer.

### II.2. An original geometrical view

From equation (1), one can obtain the following equation, which reminds us the Pythagorean theorem:

$$(c\Delta\tau_0)^2 = (c\Delta\tau_1)^2 + (c\Delta x_{0\rightarrow 1})^2 \quad (2)$$

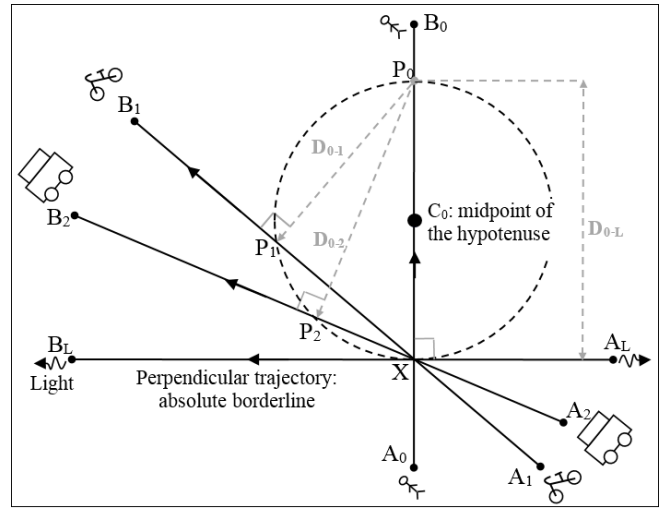


FIG. 1. An original space-time representation, [1], [2], where the pedestrian  $0$  is observing from his present  $P_0$  the cyclist  $1$  situated at his own present  $P_1$ , the car driver  $2$  situated at  $P_2$  and light  $L$  situated at  $X$ . The circle is the circumscribed circle to the infinite right triangles. The pedestrian is the observer.

From equation (2), by setting  $(c\Delta\tau_0)$  constant for any system of reference  $i \in \{0; 1; 2; 3; L\}$  :  $(c\Delta\tau_0) = (c\Delta t_{0\rightarrow i}) = D_{0\rightarrow L} - L$  associated to the extreme system

of reference : light —, it is possible to draw an infinity of right triangles — corresponding to the infinity of different velocities — which are included in a circle. This circle, centered in the middle of their common constant hypotenuse  $D_{0 \rightarrow L}$ , is circumscribed to this infinity of triangles, as shown in figure 1.

On this figure, where the past is on the bottom and the future on the top, one can see four objects belonging to four Galilean systems ( $i \in \{0; 1; 2; 3; L\}$ ) of reference, which cross each other at point X:

- the pedestrian  $\theta$ , with a straight line ( $A_0B_0$ ) which represents the passage of his proper time. He is the observer. At point X, he starts his chronometer. His present is at point  $P_0$ . The “temporal distance” ( $c\Delta t_{0 \rightarrow 1}$ ) =  $XP_0 = D_{0 \rightarrow L} = (c\Delta\tau_0)$  is his proper “temporal distance” — if he is observing himself —, but improper to measure time for the bicycle  $1$  or the car  $2$  ;

- the cyclist  $1$ , with a temporal straight line ( $A_1B_1$ ). He has a non-zero constant velocity relative to the pedestrian  $\theta$ . The cyclist  $1$  starts his chronometer at point X as well. His present  $P_1$  is the present observed from the pedestrian  $\theta$  at  $P_0$ .  $XP_1 = (c\Delta\tau_1)$  is his proper “temporal distance”. The distance  $P_0P_1 = D_{0 \rightarrow 1} = (\Delta x_{0 \rightarrow 1})$  is the “spatial distance” between  $\theta$  and  $1$ , from  $\theta$  point of view ;

- the car driver  $2$ , with a temporal straight line ( $A_2B_2$ ). He is driving faster than the cyclist  $1$ , with a constant velocity as well. The car driver  $2$  starts his chronometer at point X as well. His present  $P_2$  is the present observed from the pedestrian  $\theta$  at  $P_0$ .  $XP_2 = (c\Delta\tau_2)$  is his proper “temporal distance”. The distance  $P_0P_2 = D_{0 \rightarrow 2} = (\Delta x_{0 \rightarrow 2})$  is the “spatial distance” between  $\theta$  and  $2$ , from  $\theta$  point of view ;

- light  $L$ , which makes a right angle — which corresponds to maximum velocity, see further — with observer  $\theta$  time flow : ( $A_LB_L$ )  $\perp$  ( $A_0B_0$ ). The pedestrian  $\theta$  can only observe light  $L$  at point X, which seems frozen in time : ( $c\Delta\tau_L$ ) = 0. And the distance  $D_{0 \rightarrow L} = XP_0$  is a “spatial” and a “temporal distance” at the same time :  $(c\Delta\tau_0)^2 = (c\Delta\tau_L)^2 + (\Delta x_{0 \rightarrow L})^2 = 0 + (\Delta x_{0 \rightarrow L})^2$ . A “spatial distance” ( $\Delta x_{0 \rightarrow L}$ ) between light  $L$  and the pedestrian  $\theta$ . And the proper “temporal distance” ( $c\Delta t_{0 \rightarrow L}$ ) = ( $c\Delta\tau_0$ ) of the observer  $\theta$ .

As a partial conclusion, one can find in this first figure a high density of information. In particular, the faster the Galilean system of reference  $i$  relative to the observer  $\theta$ , the bigger the angle between their temporal trajectories. Angles have the advantage to be more intrinsic than velocities, and an angle of  $\frac{\pi}{2}$  corresponds to the maximum velocity  $c$ , the light vacuum velocity. Consequently, in the next section will be discussed the conversion from intrinsic angles to extrinsic velocities. Before going on, one can say in fact that this is time which is going at the maximum velocity  $c$ . As time is passing, and as time of light is frozen from the observer’s point of view, whose present  $P_0$  is attached to the time flow, this is light which is moving at the speed of time. It is quite important to mention it, because light does not decide its velocity. Light is subject to the laws of

Universe. And this is time flow which gives speed to light.

### III. VELOCITY CALCULATION

From these basics, one can now simply calculate velocity, which could be defined without any unit : spatial distance divided by temporal distance. As temporal distance is the maximum distance possible, velocity  $V_{0 \rightarrow 1}$  is a ratio between:

- the distance between the two systems of reference  $0$  and  $1$ , from  $\theta$ ’s point of view :  $D_{0 \rightarrow 1}$  ;

- the distance traveled by light  $L$  itself :  $D_{0 \rightarrow L}$ .

Here is the simple equation:

$$|V_{0 \rightarrow 1}| = |\beta_{0 \rightarrow 1}| = \frac{|v_{0 \rightarrow 1}|}{c} = \frac{D_{0 \rightarrow 1}}{D_{0 \rightarrow L}} \quad (3)$$

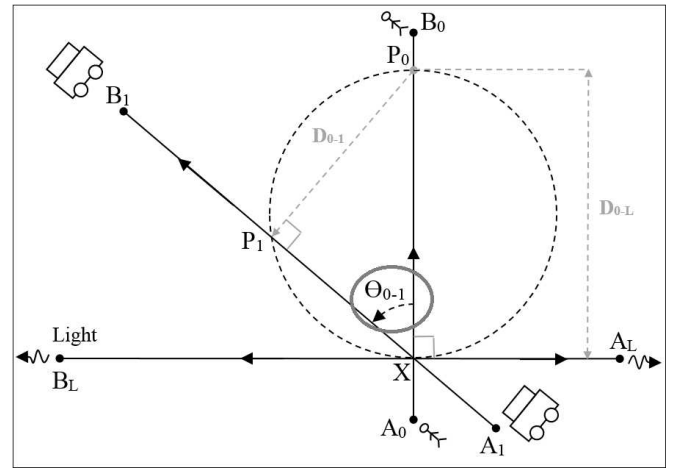


FIG. 2. Angles are equivalent to velocities, [1], [2].

From equation (3) and figure 2, one can define the sine of the angle  $\theta_{0 \rightarrow 1}$  between  $\theta$  and  $1$ , from  $\theta$ ’s point of view:

$$|\sin(\theta_{0 \rightarrow 1})| = \frac{D_{0 \rightarrow 1}}{D_{0 \rightarrow L}} = |V_{0 \rightarrow 1}| = |\beta_{0 \rightarrow 1}| \quad (4)$$

From this equation (4), one can represent velocity as a percentage of speedlight in function of the spatio-temporal angle, see figure 3. To be more accurate with the definition of velocity, a clockwise convention of angles will be defined in this article:

$$V_{0 \rightarrow 1} = \beta_{0 \rightarrow 1} = \sin(\theta_{0 \rightarrow 1}) \leq 0$$

Indeed, according to figure 2,  $V_{0 \rightarrow 1}$  is the velocity of the car (1) going to the left, hence negative, and been observed by the pedestrian (0). Finally, with two Galilean systems of reference  $i$  and  $j$ , with clockwise convention, instead of the usual anti-clockwise convention:

$$V_{i \rightarrow j} = \beta_{i \rightarrow j} = \sin(\theta_{i \rightarrow j}) \quad (5)$$



FIG. 3. Speed function of spatio-temporal angle, [1], [2].

#### IV. TIME DILATION – LENGTH CONTRACTION

##### IV.1. Time dilation

If one consider figure 2 again, when the pedestrian (0) observes the driver (1), this is improper time ( $D_{0 \rightarrow L} = XP_0$ ), which is measured from pedestrian point of view, is always longer than the proper time ( $XP_1$ ) of the driver :  $XP_0 \geq XP_1$ . Hence, the observer (0) thinks there is a dilation of time : from pedestrian (0) point of view, if the driver (1) is speaking for instance, it seems to be in slow motion. The dilation ratio  $\rho_{0 \rightarrow 1}$  is easy to calculate:

$$\rho_{0 \rightarrow 1} = \frac{XP_0}{XP_1} = \frac{1}{\cos(\theta_{0 \rightarrow 1})} \geq 1$$

This dilation ratio is equal to the Lorentz factor  $\gamma_{0 \rightarrow 1}$ :

$$\gamma_{0 \rightarrow 1} = \frac{1}{\sqrt{1 - (\beta_{0 \rightarrow 1})^2}} = \frac{1}{\sqrt{1 - \sin^2(\theta_{0 \rightarrow 1})}}$$

$$\gamma_{0 \rightarrow 1} = \frac{1}{\sqrt{\cos^2(\theta_{0 \rightarrow 1})}} = \frac{1}{|\cos(\theta_{0 \rightarrow 1})|}$$

Finally, as  $\theta_{0 \rightarrow 1} \in [-\frac{\pi}{2}; +\frac{\pi}{2}]$ :

$$\gamma_{0 \rightarrow 1} = \frac{1}{\cos(\theta_{0 \rightarrow 1})} = \rho_{0 \rightarrow 1} \quad (6)$$

Note that if the driver (1) becomes the observer :

- the Lorentz factor remains the same :  $\gamma_{1 \rightarrow 0} = \gamma_{0 \rightarrow 1}$  ;
- the beta factor becomes the opposite :  $\beta_{1 \rightarrow 0} = -\beta_{0 \rightarrow 1}$ .

##### IV.2. Length contraction

Until now, every studied object was a point, i.e. in zero dimension. To show length contraction, it is necessary to consider at least an object in one dimension, like a stick in our next example. This stick (1) is a special one, to actually show simultaneity issues in relativity : every second, its color alternatively changes, from light

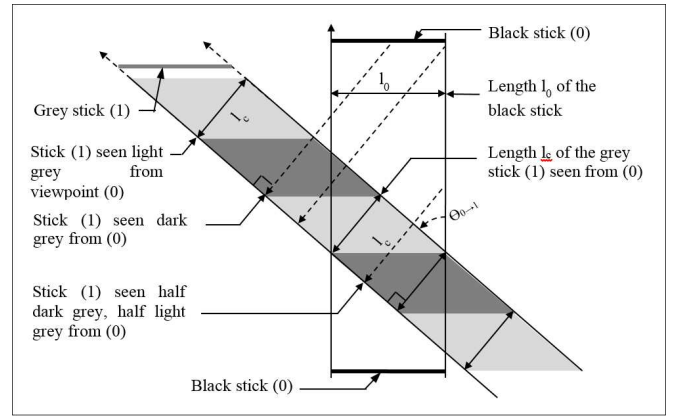


FIG. 4. Length contraction phenomenon, [1], [2].

grey to dark grey. To compare this stick with a reference, let us consider another one with exactly the same length ( $l_0$ ) but only colored in black (0). The special stick called “1” is in movement relative to the stick called “0”, which is the “observer”. The figure 4 represents the results of the observation, which shows, from 0’s point of view, the grey stick with a “contracted” length equal to  $l_c$ . This “contraction” is obviously an optical illusion in space-time. And the contraction ratio  $\kappa_{0 \rightarrow 1}$  is simply the inverse of the dilation ratio:

$$\kappa_{0 \rightarrow 1} = \frac{l_c}{l_0} = \cos \theta_{0 \rightarrow 1} = \frac{1}{\rho_{0 \rightarrow 1}} = \frac{1}{\gamma_{0 \rightarrow 1}} \leq 1 \quad (7)$$

Moreover, it is interesting to consider the problem of simultaneity. Indeed, from the black stick (0) point of view, the special stick (1) will appear with different colors, light and dark grey, at the same time. Which means that the left extremity of the special stick appears delayed in comparison to its right extremity.

To give more credit to the whole section, let us use in the next subsection the Lorentz transformation, which is equivalent but complementary to the Minkowski metric.

##### IV.3. Use of Lorentz transformation

Let us consider figure 5 : in comparison to figure 4, only the right extremity of the black stick (0) is kept, to focus on the special stick (1).

The objective of this subsection is to prove that the analytic calculations of length contraction and time delay between the right and the left extremities of the special stick (1) exactly match with the graphic constructions of figures 4 and 5.

Before going on, here is a brief recap of a 2 dimensional Lorentz transformation, one in space and one in time. If the origin  $O_j$  of a Galilean system of reference  $j$  is observing, from its present  $O_j \equiv P_j$  a random point  $M$  belonging to any system of reference, Galilean or not (with

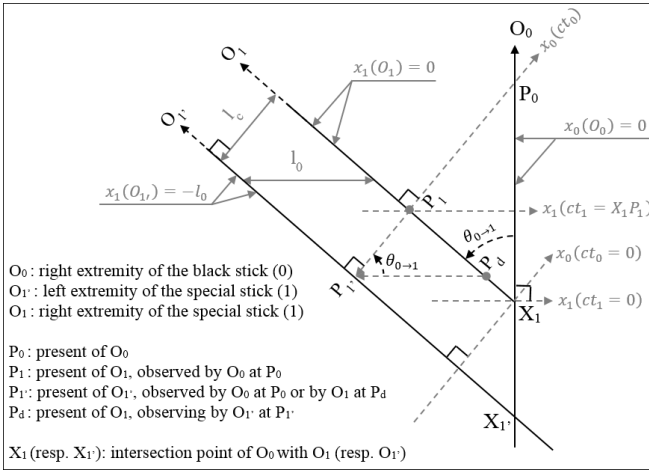


FIG. 5. Analytic calculations on the special stick.

X, the origin of time):

$$\mathbf{O}_j\mathbf{M} = \begin{pmatrix} \overline{X_j P_j} \\ P_j M \end{pmatrix} = \begin{pmatrix} ct_{j \rightarrow M} \\ x_{j \rightarrow M} \end{pmatrix}$$

The origin  $O_i$  of another Galilean system of reference  $i$  will be observing  $M$  with the relation below:

$$\begin{pmatrix} ct_{i \rightarrow M} \\ x_{i \rightarrow M} \end{pmatrix} = \gamma_{i \rightarrow j} \begin{pmatrix} 1 & \beta_{i \rightarrow j} \\ \beta_{i \rightarrow j} & 1 \end{pmatrix} \begin{pmatrix} ct_{j \rightarrow M} \\ x_{j \rightarrow M} \end{pmatrix}$$

As  $\gamma_{i \rightarrow j} = \frac{1}{\cos \theta_{i \rightarrow j}}$  and  $\beta_{i \rightarrow j} = \sin \theta_{i \rightarrow j}$ , because of clock-wise convention:

$$\cos \theta_{i \rightarrow j} \begin{pmatrix} ct_{i \rightarrow M} \\ x_{i \rightarrow M} \end{pmatrix} = \begin{pmatrix} 1 & \sin \theta_{i \rightarrow j} \\ \sin \theta_{i \rightarrow j} & 1 \end{pmatrix} \begin{pmatrix} ct_{j \rightarrow M} \\ x_{j \rightarrow M} \end{pmatrix}$$

Let us first study  $O_1$ . For practical reasons, the arrows between the subscripts will be retrieved. From its own viewpoint, its spatio-temporal coordinate are:

$$\mathbf{O}_1\mathbf{O}_1 = \begin{pmatrix} \overline{X_1 P_1} \\ P_1 P_1 \end{pmatrix} = \begin{pmatrix} ct_{11} \\ 0 \end{pmatrix}$$

With the convention  $ct_{11} = 0$  if  $P_1 \equiv X_1$ , see figure 5. From  $O_0$ 's point of view, the coordinate of  $O_1$  is the following:

$$\mathbf{O}_0\mathbf{O}_1 = \begin{pmatrix} \overline{X_1 P_0} \\ P_0 P_1 \end{pmatrix} = \begin{pmatrix} ct_{01} \\ x_{01} \end{pmatrix}$$

And according to Lorentz transformation from  $\mathbf{O}_1\mathbf{O}_1$  to  $\mathbf{O}_0\mathbf{O}_1$ :

$$\cos \theta_{01} \begin{pmatrix} ct_{01} \\ x_{01} \end{pmatrix} = \begin{pmatrix} 1 & \sin \theta_{01} \\ \sin \theta_{01} & 1 \end{pmatrix} \begin{pmatrix} ct_{11} \\ 0 \end{pmatrix} \quad (8)$$

Equation (8) will be used later. Let us now study the "length contraction" on figure 6, which can be explained by non simultaneity between the right ( $O_1'$ ) and the left ( $O_1$ ) extremities of the special stick (1). Indeed, if  $O_1'$  and  $O_1$  were simultaneous from  $O_0$  viewpoint, there would be no illusion of contraction phenomenon.

So let us now study  $O_1'$ . From  $O_0$ 's viewpoint, its spatio-temporal coordinates are:

$$\mathbf{O}_0\mathbf{O}_1' = \begin{pmatrix} \overline{X_1 P_0} \\ P_0 P_1' \end{pmatrix} = \begin{pmatrix} ct_{01} \\ x_{01} + (-l_c) \end{pmatrix}$$

With  $x_{01} \leq 0$  on this figure, because the special stick is going to the left. From  $O_1$ 's viewpoint, its spatio-temporal coordinates are:

$$\mathbf{O}_1\mathbf{O}_1' = \begin{pmatrix} \overline{X_1 P_d} \\ P_d P_1' \end{pmatrix} = \begin{pmatrix} ct_{11} + c\Delta t_d \\ -l_0 \end{pmatrix}$$

With the subscript  $d$  for *delay*. Indeed, as the special stick system of reference (1) is considered, the simultaneity line, which has to be perpendicular to the system of reference  $\theta$ , is horizontal on this figure. It gives a delay between  $P_1$  and  $P_d$  which is constant and equal to  $c\Delta t_d$ . An analytic calculation will be done on this constant later, but figure 6 gives us the result :  $c\Delta t_d = l_0 \sin \theta_{01}$ . In any case,  $P_d$  and  $P_1'$  are simultaneous, considering a clock synchronization. It means an event in  $P_1'$  is happening before an event in  $P_1$ .

Let us use the Lorentz transformation from  $\mathbf{O}_1\mathbf{O}_1'$  to  $\mathbf{O}_0\mathbf{O}_1'$ :

$$\cos \theta_{01} \begin{pmatrix} ct_{01} \\ x_{01} - l_c \end{pmatrix} = \begin{pmatrix} 1 & \sin \theta_{01} \\ \sin \theta_{01} & 1 \end{pmatrix} \begin{pmatrix} ct_{11} + c\Delta t_d \\ -l_0 \end{pmatrix} \quad (9)$$

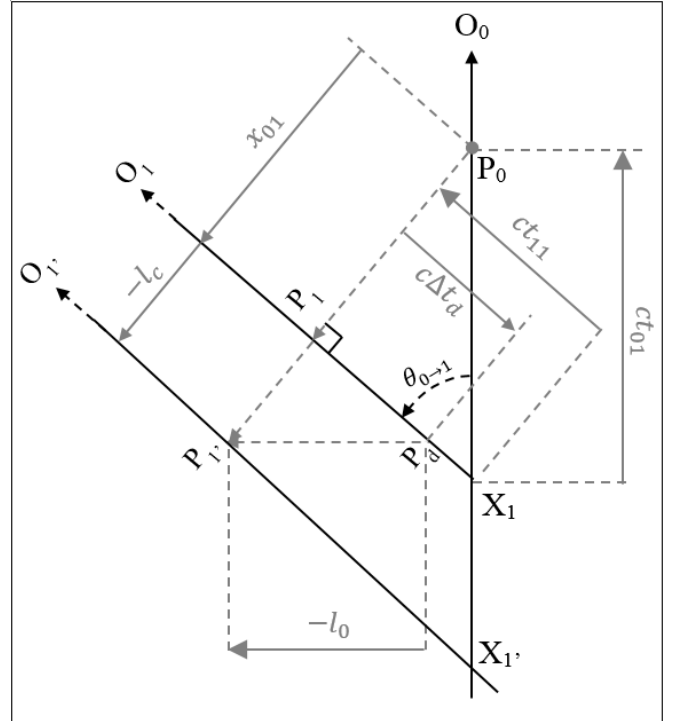


FIG. 6. A zoom of figure 5.

Let us now subtract equation (9) to equation (8):

$$\cos \theta_{01} \begin{pmatrix} 0 \\ l_c \end{pmatrix} = \begin{pmatrix} 1 & \sin \theta_{01} \\ \sin \theta_{01} & 1 \end{pmatrix} \begin{pmatrix} -c\Delta t_d \\ l_0 \end{pmatrix}$$

Hence:

$$\begin{aligned} \begin{pmatrix} c\Delta t_d \\ l_c \cos \theta_{01} \end{pmatrix} &= \begin{pmatrix} l_0 \sin \theta_{01} \\ c\Delta t_d \sin \theta_{01} + l_0 \end{pmatrix} \\ &= \begin{pmatrix} l_0 \sin \theta_{01} \\ l_0(1 - \sin^2 \theta_{01}) \end{pmatrix} \end{aligned}$$

Finally, one can get simultaneously time delay and length contraction in equation (10):

$$\begin{pmatrix} c\Delta t_d \\ l_c \end{pmatrix} = l_0 \begin{pmatrix} \sin \theta_{01} \\ \cos \theta_{01} \end{pmatrix} \quad (10)$$

## V. THE EYE OF RELATIVITY

Let us take a step back in this section. From figure 1, a new figure called *the eye of relativity* will be built, which takes into account human perception of space-time, which is obviously different from what is really happening in the corridors of the Universe (figure 1). The gap between human perception and hidden real world explains why relativity is not an intuitive science.

### V.1. Plato's cave : the *Platosphere*

Let us first remember Plato's cave. In Plato's work entitled *the Republic*, book VII, Socrates mentions the *Allegory of the Cave*. He describes a group of prisoners stuck in a cave, facing a blank wall and unable to turn their heads. They are interpreting shadows on this wall like real objects, whereas these shadows are coming from walking puppets lit by a burning fire behind them. And these puppets are actually real objects. In a certain way, concerning space-time, one could say that Albert Einstein is one of the prisoners who managed to turn his head back and understand the system in its globality. So let us draw a figure which represents both the *puppets* and the *shadows*:

- the *puppets* will be called *corrisphere*, with "corri-" for "corridors", i.e. "behind the scenes" of Universe, real space-time, an undistorted reality ;
- the *shadows* will be called *platosphere*, with "plato-" for "Plato's cave", i.e. space-time being observed by our senses, a distorted reality.

This representation can be found on figure 7:

- the thick grey horizontal segment  $[P_{L-}P_{L+}]$  on the top is the *platosphere*, which represents all the objects which have crossed the pedestrian at X.  $P_{L-}$  (respectively  $P_{L+}$ ) is the left (respectively right) border of the *platosphere* : it represents a photon coming from the right (respectively left), which has crossed the pedestrian at X ;
- the circle centered on  $C_0$  is the circle presented in figure 1. It belongs to the *corrisphere*, and the *platosphere* is the projection of this circle.

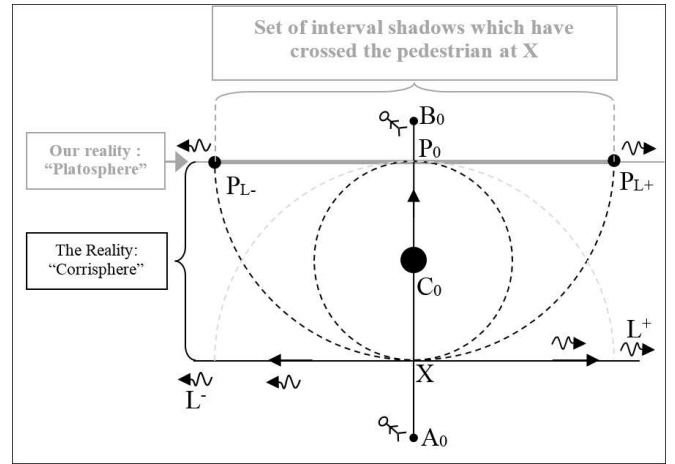


FIG. 7. The eye of relativity, [1], [2].

And the figure 8 is the superposition of figures 1 and 7. The half concentric circles represent the distances travelled by the cyclist ( $P_0P_1 = P_0P_1'$ ), the car driver ( $P_0P_2 = P_0P_2'$ ) and light ( $P_0X = P_0P_{L+} = P_0P_{L-}$ ) from the pedestrian viewpoint  $P_0$ . And the intersections of these circles with the *platosphere* give us every object aligned. That is what we could see in our distorted reality, where the notion of proper time for each object is hard to represent. With these two figures, one can see that it is impossible to be faster than time, to travel a distance bigger than  $P_0X$ . Indeed, the structure of space-time itself makes impossible to reach speedlight, or rather *speed of time*.

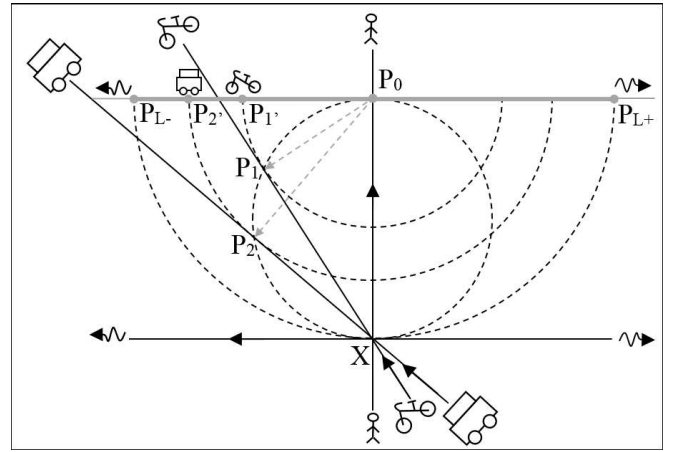


FIG. 8. The eye of relativity more detailed, [1], [2].

### V.2. Application : longitudinal Doppler effect

Before starting, only longitudinal Doppler-Fizeau effect will be presented in this subsection. Indeed, in special relativity, a quite negligible transversal Doppler ef-

fect exists [5], but 2 dimensional figures do not allow to show it. At least, 3 dimensional figures are needed — 2 of space and 1 of time — to show it. Such figures will be presented in section VIII.

Let us consider the relativistic longitudinal Doppler-Fizeau effect in figure 9, which will be more visual, thanks to the *the eye of relativity*. In this figure, the bicycle is removed, the car consequently becomes the system of reference number 1. The measures in figure 9 are correct, which allows us to make right calculations from it:

- one electromagnetic wave, with a wavelength equal to  $\lambda_0$ , is emitted from the front of the car, in the direction of motion. The first top is at X in the *corrisphere* or at  $P_{L-}$  in the *platosphere* ; the second top is at  $P_1$  in the *corrisphere* or at  $P_{1'}$  in the in the *platosphere* ;

- the other one, with the same wavelength, is emitted from the back of it, in the opposite direction of motion : the first top is at X in the *corrisphere* or at  $P_{L+}$  in the *platosphere* ; the second top is at  $P_1$  in the *corrisphere* or at  $P_{1'}$  in the in the *platosphere*.

The perfect match between the wavelength  $\lambda_0$  and

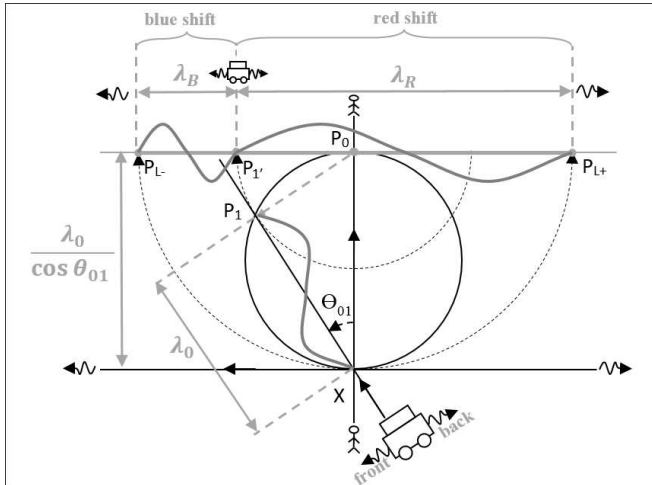


FIG. 9. Relativistic longitudinal Doppler-Fizeau effect geometrically explained, [3].

proper time  $XP_1$  is made on purpose on this figure, to make things easier. As a result, one can immediately measure the red  $\lambda_R$  and the blue  $\lambda_B$  shifts with a simple ruler.

Let us calculate  $\lambda_R$  and  $\lambda_B$  :

$$\begin{cases} \lambda_R = XP_0 + P_0P_1 = \frac{\lambda_0}{\cos \theta_{01}} (1 + |\sin \theta_{01}|) \\ \lambda_B = XP_0 - P_0P_1 = \frac{\lambda_0}{\cos \theta_{01}} (1 - |\sin \theta_{01}|) \end{cases}$$

Finally:

$$\lambda_R = \lambda_0 \sqrt{\frac{1 + |\sin \theta_{01}|}{1 - |\sin \theta_{01}|}} \quad (11)$$

$$\lambda_B = \lambda_0 \sqrt{\frac{1 - |\sin \theta_{01}|}{1 + |\sin \theta_{01}|}} \quad (12)$$

By the way :

$$\begin{aligned} \lambda_B \lambda_R &= (XP_0 - P_0P_1)(XP_0 + P_0P_1) \\ &= XP_0^2 - P_0P_1^2 = XP_1^2 \\ &= \lambda_0^2 \end{aligned}$$

Finally:

$$\lambda_B \lambda_R = \lambda_0^2$$

So,  $XP_1 = \lambda_0$  is the geometric mean of  $\lambda_R$  and  $\lambda_B$ , and this figure reminds us a little on Archimedes representations on harmonic, geometric, arithmetic and quadratic means [6], or on Mohr's circle — even though the circles drawn are slightly different:

$$XP_1 = \lambda_0 = \sqrt{\lambda_B \lambda_R} \quad (13)$$

This geometric mean remains constant, because of the constant proper wavelength  $\lambda_0$ . On the contrary, the arithmetic mean,  $XP_0$ , is indirectly dependent on relative velocity:

$$XP_0 = \frac{\lambda_0}{\cos \theta_{01}} = \frac{1}{2}(\lambda_B + \lambda_R) \quad (14)$$

Moreover, one can build a quadratic polynomial from the doublet  $(y_1, y_2) = (\lambda_B, \lambda_R)$  of solutions:

$$Q(y) = y^2 - Sy + P$$

With:

$$\begin{cases} S = \lambda_B + \lambda_R \\ P = \lambda_R \lambda_B \end{cases}$$

Finally:

$$Q(y) = y^2 - \frac{2\lambda_0}{\cos \theta_{01}}y + \lambda_0^2 \quad (15)$$

As geometric mean is always inferior or equal to arithmetic mean, the following condition is obviously respected:

$$\frac{\sqrt{P}}{\frac{1}{2}S} = \cos \theta_{01} \leq 1 \quad (16)$$

Which is equivalent to  $\Delta = b^2 - 4ac \geq 0$ , with  $a = 1$ ,  $b > 0$  and  $c \geq 0$ .

At last, one should notice that this polynomial (15) is the characteristic polynomial of two similar matrix. The first one is the following diagonal matrix:

$$D(\theta_{01}, \lambda_0) = \begin{pmatrix} \lambda_R & 0 \\ 0 & \lambda_B \end{pmatrix}$$

With:

$$\begin{cases} \text{tr}(D) = S = \lambda_B + \lambda_R \\ |D| = P = \lambda_R \lambda_B \end{cases}$$

And:

$$Q(y) = |y I_2 - D| = y^2 - \text{tr}(D) y + |D|$$

With  $I_2$  the identity matrix.

The second one is more interesting, because it is proportional to  $L(\theta_{01})$ , the Lorentz transformation matrix:

$$\begin{aligned} A(\theta_{01}, \lambda_0) &= \lambda_0 L(\theta_{01}) \\ &= \frac{\lambda_0}{\cos \theta_{01}} \begin{pmatrix} 1 & \sin \theta_{01} \\ \sin \theta_{01} & 1 \end{pmatrix} \end{aligned} \quad (17)$$

With:

$$\begin{cases} \text{tr}(A) = 2 \frac{\lambda_0}{\cos \theta_{01}} = \text{tr}(D) \\ |A| = \left( \frac{\lambda_0}{\cos \theta_{01}} \right)^2 (1 - \sin^2 \theta_{01}) = \lambda_0^2 = |D| \end{cases}$$

As a partial conclusion, one can consider the longitudinal Doppler wavelengths as geometric lengths from figure 9, or as eigenvalues of the Lorentz transformation matrix – with a  $\lambda_0$  factor.

Let us go a bit further with the eigenvectors. As  $A(\theta_{01}, \lambda_0)$  is a symmetric matrix of real coefficients, similar to  $D(\theta_{01}, \lambda_0)$ , one can find an orthogonal transfer matrix  $P$ , such as:

$$A(\theta_{01}, \lambda_0) = P \cdot D(\theta_{01}, \lambda_0) \cdot P^T$$

$P$  is a basis of orthogonal eigenvectors, which are obviously light vectors. Indeed, light vectors orientations remain unchanged by the Lorentz transformation – even if their norms can be modified – which is the property of eigenvectors. Here is the basis:

$$P = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

## VI. VELOCITY COMPOSITION LAW

Velocity composition law can be easily understood, if one gets inspired by figure 5, which represents length contractions. Indeed, instead of having  $X_1$  and  $X_1'$  separate in figure 5, let us merge this two points in figure 10 : it will be point  $X$ . As a result, one can observe two different systems of reference, at two different velocities:

-  $\theta_{01}$  is the image of the velocity between system (0) and system (1), from (0)'s point of view. The present  $P_0$  of  $O_0$  can observe the present  $P_1$  of  $O_1$  ;

-  $\theta_{12}$  is the image of the velocity between system (1) and system (2), from (1)'s point of view. The present  $P_1$  of  $O_1$  can observe the present  $P_2$  of  $O_2$ .

The question is : how to obtain  $\theta_{02}$ , which is the velocity between (0) and (2)? The aim is to observe  $P_2$  from system (0). The method is similar to figure 5 : a  $P_2'$  point has to be defined (which is exactly the same point as  $P_2$ ), horizontal, with  $P_1 P_2 = P_1 P_2'$ .  $P_2'$  will be observed by  $P_0'$ , with the following condition :  $(P_0' P_2') \parallel (P_0 P_1)$ . Indeed, it was as if  $O_2$  was briefly belonging to system (1) at point  $P_2'$ , which is simultaneous to  $P_1$ . So  $P_0'$  can observe  $P_2'$  with a  $(\frac{\pi}{2} - \theta_{01})$  angle.

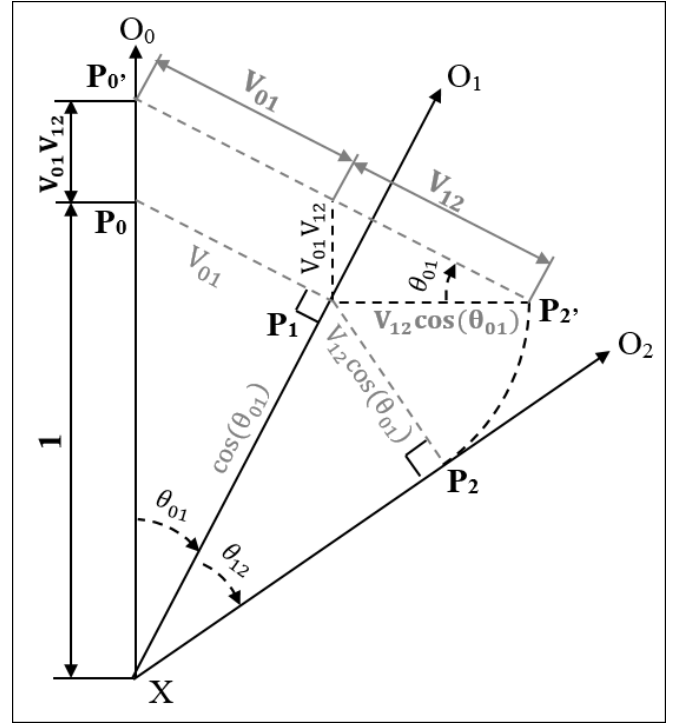


FIG. 10. Velocity composition law geometrically explained.

In figure 10, for practical reasons, distances have been normalized :  $X P_0 = 1$ , consequently:

$$\begin{aligned} \frac{P_0 P_1}{X P_0} &= \sin \theta_{01} = \frac{v_{01}}{c} = V_{01} \\ P_0 P_1 &= V_{01} \\ P_1 P_2 &= V_{12} \cos \theta_{01} \end{aligned}$$

Finally:

$$V_{02} = \frac{P_0' P_2'}{X P_0'} = \frac{V_{01} + V_{12}}{1 + V_{01} \cdot V_{12}} \quad (18)$$

As a partial conclusion, one can explain relativistic velocity composition with simple geometrics. The  $\theta_{02}$  angle can be drawn by using a simple pair of compasses. To go further, let us draw another way to explain it. Let us first define a new velocity function:

$$V_{01}^* = \gamma_{01} V_{01} = \frac{\sin \theta_{01}}{\cos \theta_{01}} = \tan \theta_{01} \quad (19)$$

Let us prove that equation (18) is equivalent to :

$$(18) \Leftrightarrow V_{02}^* = \gamma_{01} \cdot \gamma_{12} \cdot (V_{01} + V_{12}) \quad (20)$$

$$\Leftrightarrow V_{02}^* = \gamma_{12} \cdot V_{01}^* + \gamma_{01} \cdot V_{12}^* \quad (21)$$

Here is the demonstration:

$$\begin{aligned} V_{02}^* &= \tan \theta_{02} = \pm \left( \frac{1}{\sin^2 \theta_{02}} - 1 \right)^{-1/2} \\ &= \pm \left[ \left( \frac{1 + V_{01} \cdot V_{12}}{V_{01} + V_{12}} \right)^2 - 1 \right]^{-1/2} \end{aligned}$$



$$\begin{aligned}
&= \frac{V_{01} + V_{12}}{\sqrt{1 - (V_{01} + V_{12})^2 + (V_{01} \cdot V_{12})^2}} \\
&= \frac{1}{\sqrt{(1 - V_{01}^2)} \sqrt{(1 - V_{12}^2)}} (V_{01} + V_{12}) \\
V_{02}^* &= \gamma_{01} \cdot \gamma_{12} \cdot (V_{01} + V_{12})
\end{aligned}$$

Hence, from equation (21):

$$\tan \theta_{02} = \frac{\tan \theta_{01}}{\cos \theta_{12}} + \frac{\tan \theta_{12}}{\cos \theta_{01}} \quad (22)$$

From this trigonometrical result of velocity composition, one can draw the figure below.

For practical reasons, distances have been normalized :

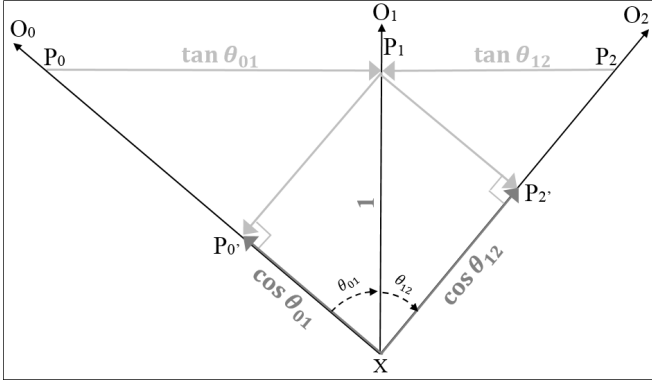


FIG. 11. The “domino effect” in velocity composition law, [1], [2].

$XP_1 = 1$ , consequently:

$$\begin{aligned}
P_0P_1 &= \tan \theta_{01} \\
P_1P_2 &= \tan \theta_{12} \\
XP'_0 &= \cos \theta_{01} \\
XP'_2 &= \cos \theta_{12}
\end{aligned}$$

Here is an analysis of figure 11, which is called “domino effect”:

- from its present  $P_0$ ,  $O_0$  is observing  $O_1$  in  $P_1$ , which is observing  $O_2$  in  $P_2$  ;
- from its present  $P_2$ ,  $O_2$  is observing  $O_1$  in  $P_1$ , which is observing  $O_0$  in  $P'_0$  ;

The final result is the addition of these last two effects:

$$\begin{aligned}
V_{02}^* &= \frac{P_0P_1}{XP_1} \cdot \frac{XP_1}{XP'_2} + \frac{P_1P_2}{XP_1} \cdot \frac{XP_1}{XP'_0} \\
V_{02}^* &= \frac{P_0P_1}{XP'_2} + \frac{P_1P_2}{XP'_0} \quad (23)
\end{aligned}$$

As a conclusion, equations (22) and (23) are equivalent. This result will be very useful to study stellar aberration.

## VII. STELLAR ABERRATION

Moreover, one can consider stellar aberration as a kind of “velocity composition” in 2 dimensions of space and 1

dimension of time. For the record, due to stellar aberration, the angle of the source of light is dependent from the velocity of the observer. The faster the observer is, the more horizontal the source seems to be.

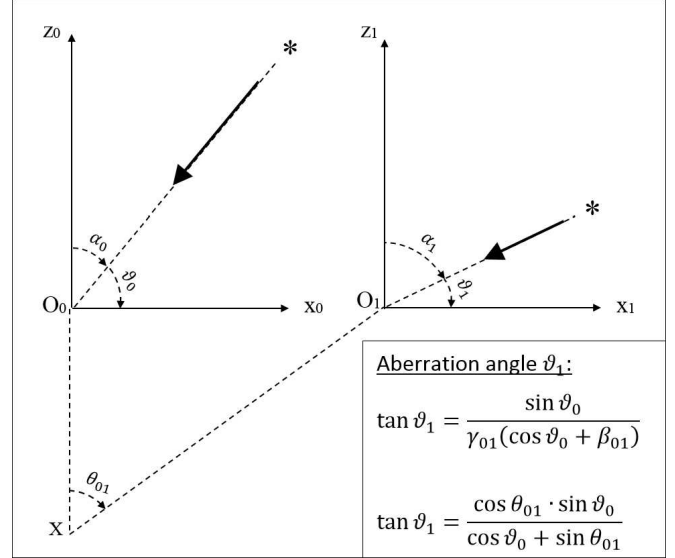


FIG. 12. Stellar aberration angle  $\vartheta$  considered in classical books on relativity . In theory, if  $O_0$  (resp.  $O_1$ ) is the observer,  $(O_1x_1)$  and  $(O_1z_1)$  [resp.  $(O_0x_0)$  and  $(O_0z_0)$ ] are not supposed to be perpendicular to each other. In order to simplify the figure, as we are focusing on the same star (symbol “\*”), two observers ( $O_0$  and  $O_1$ ) are simultaneously defined here.

In figure 12, where time and space are in the same plane for practical reasons, one can see the relativistic equation of stellar aberration, which is obtained by applying the Lorentz transformation to the momentum-energy four-vector of a photon coming from the star (symbol “\*” on this figure). The first interesting thing is : if the star is just above  $O_0$ , with an angle  $\vartheta_0 = \frac{\pi}{2}$ , the angle  $\vartheta_1$  will be equal to  $(\frac{\pi}{2} - \theta_{01})$ . Consequently, let us simply define new angles in this article :  $\alpha_0 = \frac{\pi}{2} - \vartheta_0$  and  $\alpha_1 = \frac{\pi}{2} - \vartheta_1$ , which are angles between the vertical axis and the star, see figure 12. With this new definition, if  $\alpha_0 = \frac{\pi}{2}$ ,  $\alpha_1 = \theta_{01}$ . So, in the case of a star initially above us, without any sophisticated device, just by observing this star, one can almost directly measure the relative velocity of our system of reference, obtained by the sine of  $\alpha_1$  (a fraction of speedlight). More generally and more interesting, the equation of figure 12 becomes much simpler:

$$\begin{aligned}
\tan \alpha_1 &= \frac{1}{\tan \vartheta_1} \\
&= \frac{\cos \vartheta_0 + \sin \theta_{01}}{\sin \vartheta_0 \cdot \cos \theta_{01}} \\
&= \frac{\sin \alpha_0 + \sin \theta_{01}}{\cos \alpha_0 \cdot \cos \theta_{01}}
\end{aligned}$$

Finally, equations (22) and (24) are similar:

$$\tan \alpha_1 = \frac{\tan \alpha_0}{\cos \theta_{01}} + \frac{\tan \theta_{01}}{\cos \alpha_0} \quad (24)$$

As a conclusion, stellar aberration is a kind of velocity composition, and one could define an artificial “velocity” ( $\beta_{\alpha i} = V_{\alpha i} = \sin \alpha_i$ ) and an artificial Lorentz factor ( $\gamma_{\alpha i} = \cos^{-1} \alpha_i$ ), with the following equation similar to (18):

$$V_{\alpha 1} = \frac{V_{01} + V_{\alpha 0}}{1 + V_{01} \cdot V_{\alpha 0}} \quad (25)$$

And with the following equation similar to (20):

$$V_{\alpha 1}^* = \gamma_{01} \cdot \gamma_{\alpha 0} \cdot (V_{01} + V_{\alpha 0}) \quad (26)$$

### VIII. GENERAL DOPPLER-FIZEAU EFFECT

After considering the longitudinal Doppler-Fizeau effect in subsection V.2, let us consider it in 2 dimensions of space. The use of previous sections, especially VI and VII, will be essential. In figure 13, space and time are in the same plane.

The electromagnetic wave, with a wavelength equal to  $\lambda_0$ , is emitted from the system of reference 1, in  $\alpha_0$  direction from 0’s point of view, and in  $\alpha_1$  direction from its own point of view, which is not represented:

- the first top is at X in the *corrisphere* or at  $P_{L\alpha 0}$  in the *platosphere* ;

- the second top is at  $P_1$  in the *corrisphere* or at  $P_{1'}$  in the in the *platosphere*.

As a monochromatic plane wave is considered here, one

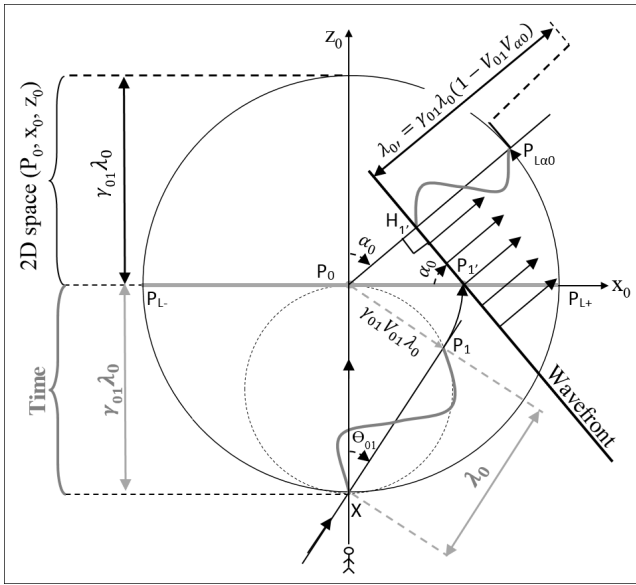


FIG. 13. Relativistic general Doppler-Fizeau effect geometrically explained.

can draw a wavefront, which allows us to represent  $H_{1'}$ , the orthogonal projection of  $P_{1'}$  on  $(P_0 P_{L\alpha 0})$ :

$$P_0 H_{1'} = P_0 P_1 \sin \alpha_0 = P_0 P_1 V_{\alpha 0} = \gamma_{01} \lambda_0 V_{01} V_{\alpha 0} \quad (27)$$

From equation (27), one can deduce the Doppler wavelength  $\lambda_{0'}$ , function of  $\alpha_0$  and  $\theta_{01}$ , from 0’s point of view:

$$\begin{aligned} \lambda_{0'} &= H_{1'} P_{L\alpha 0} = P_0 P_{L\alpha 0} - P_0 H_{1'} \\ \lambda_{0'} &= \lambda_0 \gamma_{01} (1 - V_{01} V_{\alpha 0}) \\ \lambda_{0'} &= \frac{\lambda_0}{\cos \theta_{01}} (1 - \sin \theta_{01} \sin \alpha_0) \end{aligned} \quad (28)$$

Thanks to equation (25), one can also calculate this Doppler wavelength in function of  $\alpha_1$  and  $\theta_{01}$ . But, as light is not coming from the star, but from the system of reference 1, spatial angles ( $\alpha_0$  and  $\alpha_1$ ) will be shifted by  $\pi$  in this equation, to correctly apply “velocity” composition:

$$\begin{aligned} \sin \alpha_0 &= -\sin (\alpha_0 + \pi) = -\frac{\sin (\alpha_1 + \pi) - \sin \theta_{01}}{1 - \sin (\alpha_1 + \pi) \cdot \sin \theta_{01}} \\ \sin \alpha_0 &= \frac{\sin \alpha_1 + \sin \theta_{01}}{1 + \sin \alpha_1 \cdot \sin \theta_{01}} \end{aligned} \quad (29)$$

From equation (29), here is the result:

$$\begin{aligned} \lambda_{0'} &= \frac{\lambda_0}{\cos \theta_{01}} \left( 1 - \sin \theta_{01} \frac{\sin \alpha_1 + \sin \theta_{01}}{1 + \sin \alpha_1 \cdot \sin \theta_{01}} \right) \\ &= \frac{\lambda_0}{\cos \theta_{01}} \frac{1 - \sin^2 \theta_{01}}{1 + \sin \alpha_1 \cdot \sin \theta_{01}} \end{aligned} \quad (30)$$

$$\lambda_{0'} = \frac{\lambda_0 \cdot \cos \theta_{01}}{1 + \sin \alpha_1 \cdot \sin \theta_{01}} \quad (30)$$

$$\lambda_{0'} = \frac{\lambda_0}{\gamma_{01} \cdot (1 + V_{\alpha 1} \cdot V_{01})} \quad (31)$$

Let us now test equation (31), with  $\alpha_1 = 0^\circ$ , which is the transversal Doppler effect:

$$\begin{aligned} \lambda_{0'} &= \frac{\lambda_0}{\gamma_{01}} \\ \sin \alpha_0 &= \sin \theta_{01} \Rightarrow \alpha_0 = \theta_{01} \end{aligned}$$

Longitudinal Doppler effect of subsection V.2 can be calculated with  $\alpha_1 = \alpha_0 = \pm \frac{\pi}{2}$ . In fact,  $\alpha_1 = \alpha_0$  because it was as if there was an artificial Speedlight in equation (25):  $V_{\alpha} = \sin \frac{\pi}{2} = 1$ .

### IX. DISCUSSION AND CONCLUSIONS

This original geometrical approach could not only be helpful for students, but also for researchers. This new way of thinking might indirectly play a role in future discoveries.

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