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Abstract. This paper is devoted to the study of shape optimization problems for the first eigenvalue of the elliptic operator with drift $L = -\Delta + V(x)\cdot\nabla$ with Dirichlet boundary conditions, where V is a bounded vector field. In the first instance, we prove the existence of a principal eigenvalue $\lambda_1(\Omega, V)$ for a bounded quasi-open set Ω which enjoys similar properties to the case of open sets. Then, given $m > 0$ and $\tau \geq 0$, we show that the minimum of the following non-variational problem

$$\min\Big\{\lambda_1(\Omega, V) \ : \ \Omega \subset D \text{ quasi-open}, \ |\Omega| \leq m, \ \|V\|_{L^\infty} \leq \tau\Big\}.$$

is achieved, where the box $D \subset \mathbb{R}^d$ is a bounded open set. The existence when V is fixed, as well as when V varies among all the vector fields which are the gradient of a Lipschitz function, are also proved.

The second interest and main result of this paper is the regularity of the optimal shape Ω^* solving the minimization problem

$$\min\Big\{\lambda_1(\Omega, \nabla\Phi) \ : \ \Omega \subset D \text{ quasi-open}, \ |\Omega| \leq m\Big\},$$

where Φ is a given Lipschitz function on D. We prove that the topological boundary $\partial\Omega^*$ is composed of a *regular part* which is locally the graph of a $C^{1,\alpha}$ function and a *singular part* which is empty if $d < d^*$, discrete if $d = d^*$ and of locally finite $\mathcal{H}^{d-d^*}$ Hausdorff measure if $d > d^*$, where $d^* \in \{5, 6, 7\}$ is the smallest dimension at which there exists a global solution to the one-phase free boundary problem with singularities. Moreover, if D is smooth, we prove that, for each $x \in \partial\Omega^* \cap \partial D$, $\partial\Omega^*$ is $C^{1,\alpha}$ in a neighborhood of x, for some $\alpha \leq 1/2$. This last result is optimal in the sense that $C^{1,1/2}$ is the best regularity that one can expect.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (here and after, "domain" means open connected set) and $V \in L^\infty(\Omega, \mathbb{R}^d)$ be a given vector field. We consider the elliptic operator with drift $L = -\Delta + V(x) \cdot \nabla$ with Dirichlet boundary conditions on $\partial\Omega$. In [4] Berestycki, Nirenberg and Varadhan showed that there is a positive real eigenvalue, not greater than the real part of any other eigenvalue of L. This eigenvalue is called *principal* or *first eigenvalue* of L and is denoted by $\lambda_1(\Omega, V)$.

Assume now that Ω is a C^2 bounded open set of Lebesgue measure $|\Omega| = m$ and let $V : \Omega \to \mathbb{R}^d$ be a vector field such that $\|V\|_{L^\infty} = \tau$ (where $\|V\|_{L^\infty}$ stands for the L^∞-norm of the Euclidean norm of V). In [28] Hamel, Nadirashvili and Russ introduced a new symmetrization technique to prove the lower bound

$$\lambda_1\left(B, \tau\frac{x}{|x|}\right) \leq \lambda_1(\Omega, V), \tag{1.1}$$

where B is the ball of Lebesgue measure m centered in zero, with an equality that holds if and only if, up to translation, $\Omega = B$ and $V(x) = \tau\frac{x}{|x|}$. In other words, $(B, \tau\frac{x}{|x|})$ is, up to translation, the unique solution of the shape optimization problem

$$\min\Big\{\lambda_1(\Omega, V) \ : \ \Omega \subset \mathbb{R}^d \text{ open}, \ |\Omega| = m, \ \|V\|_{L^\infty} \leq \tau\Big\}. \tag{1.2}$$

In the present paper, we fix a bounded open set $D \subset \mathbb{R}^d$. If $V \in L^\infty(D, \mathbb{R}^d)$, we first prove that, if $\Omega \subset D$ is a *quasi-open* set, there exists a principal eigenvalue $\lambda_1(\Omega, V)$, satisfying some properties similar to the case of open sets. Our motivation to do so is the study of shape optimization problems.

More precisely, the aim of the present paper is twofold. From one side, we develop a purely variational existence theory for shape optimization problems of the form (1.2). We consider the model problem

$$\min\Big\{\lambda_1(\Omega, V) \ : \ \Omega \subset D, \ |\Omega| \leq m, \ \|V\|_{L^\infty} \leq \tau\Big\}, \tag{1.3}$$

where $D \subset \mathbb{R}^d$ is a given bounded open set. We notice that in this case a symmetrization technique in the spirit of [28] cannot be applied since the geometry of D strongly affects the geometry of the admissible domains and it is impossible to determine explicitly the shape of the optimal domains or the precise analytic expression of the optimal vector fields. Moreover, in the case of a generic vector field V the principal eigenvalue $\lambda_1(\Omega, V)$ does *not* have a variational formulation but is only determined trough the solution of a certain PDE on Ω so, the minimization problem (1.3) cannot be expressed as a variational problem involving integral cost functionals. Our main result concerning the optimization problem (1.3) is the existence Theorem 4.3 in the class of quasi-open sets.

On the other hand, we study the case in which only the shape Ω is variable, while the vector field V is fixed. Precisely, we consider the shape optimization problem

$$\min\Big\{\lambda_1(\Omega, V) \ : \ \Omega \subset D, \ |\Omega| \leq m\Big\}, \tag{1.4}$$

where both the upper bound m of the Lebesgue measure of the domain Ω and the vector field V are fixed. In this case the geometry of the optimal sets is affected not only by the geometric

constraint $\Omega \subset D$ but also by the form of the vector field V. We notice that in this case it is the inclusion constraint that provides the compactness necessary for the existence of an optimal set. We show that the shape functional $\Omega \mapsto \lambda_1(\Omega, V)$ is lower semi-continuous with respect to the so-called γ-convergence of sets and then we obtain the existence of optimal sets by a general result of Buttazzo and Dal Maso [13]. Furthermore, when the vector field is the gradient of a Lipschitz function, we prove a regularity result for the optimal sets. Our main result is the following.

Theorem 1.1. *Let D be an open, bounded and connected set in $\mathbb{R}^d$. Let $m \in (0, |D|)$ and let the vector field $V : D \to \mathbb{R}^d$ be such that $\|V\|_{L^\infty} = \tau < +\infty$. Then the shape optimization problem*

$$\min\Big\{\lambda_1(\Omega, V) \ : \ \Omega \subset D \text{ quasi-open}, \ |\Omega| \le m\Big\} \tag{1.5}$$

admits a solution $\Omega^* \subset D$. Moreover, if the vector field is of the form $V = \nabla\Phi$, where $\Phi : D \to \mathbb{R}$ is a given Lipschitz function, then the solution is an open set of locally finite perimeter such that $|\Omega^*| = m$. Moreover, the boundary $\partial\Omega^*$ can be decomposed in the disjoint union of a regular set $Reg(\partial\Omega^*)$, which is locally the graph of a $C^{1,\alpha}$ function, and a singular set $Sing(\partial\Omega^*)$, which is empty if $d < d^*$, discrete (possibly empty) if $d = d^*$ and of locally finite $\mathcal{H}^{d-d^*}$ Hausdorff measure if $d > d^*$, where $d^* \in \{5, 6, 7\}$ is defined in Definition 5.34. Furthermore, if ∂D is $C^{1,1}$, then the regular part $Reg(\partial\Omega^*)$ contains $\partial\Omega^* \cap \partial D$, which means that, for all $x \in \partial\Omega^* \cap \partial D$, $\partial\Omega^*$ is $C^{1,\alpha}$ in a neighborhood of x.

Remark 1.2. We notice that the hypothesis on the connectedness of the box D could be dropped since the intersection of an optimal shape with a connected component of D is itself a solution in this component.

Remark 1.3 (On the optimality of the smoothness at contact points)**.** The regularity of the boundary of an optimal set Ω^* to the problem (1.5) at contact points of the free boundary with the box cannot exceed $C^{1,1/2}$. Indeed, Chang-Lara and Savin proved in [18] that the boundary of Ω_u, where u is a solution of the free boundary problem (5.36) in $\Omega_u = \Omega^*$, is at most $C^{1,1/2}$ regular.

As a consequence of Theorem 1.1 we get the regularity of every shape solution to the analogous problem where V varies among all the vector fields which are the gradient of a Lipschitz function.

Corollary 1.4. *Let D be an open, bounded and connected set in $\mathbb{R}^d$, $m \in (0, |D|)$ and $\tau > 0$. Then the shape optimization problem*

$$\min\Big\{\lambda_1(\Omega, \nabla\Phi) \ : \ \Omega \subset D \text{ quasi-open}, \ |\Omega| \le m, \ \Phi \in W^{1,\infty}(D), \ \|\nabla\Phi\|_{L^\infty} \le \tau\Big\} \tag{1.6}$$

admits a solution $(\Omega^*, \nabla\Phi^*)$. Moreover, Ω^* is an open set of locally finite perimeter such that $|\Omega^*| = m$. Furthermore, the boundary $\partial\Omega^*$ can be decomposed in the disjoint union of a regular set $Reg(\partial\Omega^*)$ and a singular set $Sing(\partial\Omega^*)$ with the properties stated in Theorem 1.1.

Outline of the proof and plan of the paper. Let us now briefly describe our strategy. We first extend the definition of $\lambda_1(\Omega, V)$ to the case of quasi-open sets, setting

$$\lambda_1(\Omega, V) := \sup\big\{\lambda_1(\tilde\Omega, V) \ : \ \tilde\Omega \text{ open}, \ \Omega \subset \tilde\Omega \subset D\big\}.$$

Considering a maximizing sequence and using γ-convergence arguments and resolvent estimates, we prove that $\lambda_1(\Omega, V) < +\infty$ is indeed an eigenvalue of L on Ω under Dirichlet boundary condition, and that the real part of any other eigenvalue of L is not smaller than $\lambda_1(\Omega, V)$.
We then consider three minimization problems for $\lambda_1(\Omega, V)$. First, the quasi-open set Ω is fixed and the vector field V varies under the constraint $\|V\|_{L^\infty(\Omega)} \le \tau$. In this case, we establish that this problem has a solution V^*, which satisfies

$$V^*(x) = -\tau\frac{\nabla u(x)}{|\nabla u(x)|} \ \text{ if } \ |\nabla u(x)| \neq 0 \ ; \qquad V^*(x) = 0 \ \text{ if } \ |\nabla u(x)| = 0 \ , \tag{1.7}$$

where u is the corresponding eigenfunction. The proof relies on γ-convergence arguments (since Ω can be approximated by smooth open sets in the sense of the γ-convergence) and the corresponding result for open sets. In the second minimization problem, both Ω and V vary, satisfying $\Omega \subset D$ and $|\Omega| \leq m$ and $\|V\|_{L^\infty(\Omega)} \leq \tau$, and the result states that this minimization problem has a solution (Ω^*, V^*), where V^* still satisfies (1.7). Finally, we restrict our attention to the minimization problem for $\lambda_1(\Omega, V)$ when Ω and V vary, satisfying the same constraints as before, assuming furthermore that V is the gradient of a Lipschitz function in D. Using a variational formulation for $\lambda_1(\Omega, V)$, which is available due to the form of V (namely $\lambda_1(\Omega, \nabla\Phi) = \min_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_D e^{-\Phi}|\nabla u|^2\,dx}{\int_D e^{-\Phi}u^2\,dx}$), we show that this third minimization problem has a solution (Ω^*, V^*).

We then study the regularity of the optimal set Ω^* in this last optimization issue, relying in an essential way on the variational formulation of $\lambda_1(\Omega, V)$. More precisely, we observe that, if $V = \nabla\Phi$ is given and $\Omega \subset D$ is a solution of (1.5), then there exists a corresponding eigenfunction u solving

$$\min\Big\{ \int_D e^{-\Phi}|\nabla u|^2\,dx \ :\ u \in H^1_0(D),\ u \geq 0,\ \big|\{u \neq 0\}\big| \leq m,\ \int_D e^{-\Phi}u^2\,dx = 1\Big\}. \tag{1.8}$$

Conversely, if u is a solution of (1.8), then the quasi-open set $\{u > 0\}$ is a solution of (1.5). This observation is crucial for our analysis, since we interpret the optimal domain Ω as the set where the eigenfunction u is positive and we are therefore led to regularity issues for a free boundary problem. Inspired by [6], we reformulate the problem (1.8) using the functional

$$J(v) := \int_D |\nabla v|^2 e^{-\Phi}dx - \lambda_m \int_D v^2 e^{-\Phi}dx$$

for $v \in H^1_0(D)$, where $\lambda_m := \int_D |\nabla u|^2 e^{-\Phi}\,dx$ for u solving (1.8). One easily checks that, if u is a solution of (1.8), then $J(u) \leq J(v)$ whenever $v \in H^1_0(D)$ and $|\Omega_v| \leq m$, where we set $\Omega_v := \{x \in \Omega;\ v(x) > 0\}$. We first focus on regularity properties of the solution u of (1.8). The function u is proved to be bounded, and we show that $|\Omega_u| = m$. Then, using regularity properties of the Lagrange multipliers Λ_u associated with the functional J, we prove that Ω_u is actually an open set of (locally) finite perimeter, and that u is (locally) Lipschitz and behaves like the distance to $\partial\Omega_u$ near $\partial\Omega_u \cap D$.

Finally, using blow-up type arguments inspired by [33], we prove that $|\nabla u| = \sqrt{\Lambda_u e^{\Phi}}$ on $\partial\Omega_u \cap D$ and $|\nabla u| \geq \sqrt{\Lambda_u e^{\Phi}}$ on $\partial\Omega_u \cap \partial D$ (if we assume some smoothness on D) in the viscosity sense, from which we derive the decomposition of $\partial\Omega_u$ as the union of a regular $C^{1,\alpha}$ part and a "singular" one with zero $d - d^*$ Hausdorff measure.

In the general case where the drift V is not assumed to be the gradient of a Lipschitz function, obtaining regularity results for the boundary of the optimal domain Ω is an open problem.

The paper is organized as follows. After giving some general useful results about γ-convergence in Section 2, we define and prove the main properties of $\lambda_1(\Omega, V)$ when Ω is merely a quasi open set in Section 3. Section 4 is devoted to the proof of the existence of optimal domains and vector fields for the various optimization problems we consider, while, in Section 5, we establish the regularity results in Theorem 1.1.

2. Preliminaries

In this section we recall the main definitions and the properties of the quasi-open sets and the γ-convergence.

2.1. **Capacity, quasi-open sets and quasi-continuous functions.**

The *capacity* of a set $E \subset \mathbb{R}^d$ is defined as

$$\mathrm{cap}(E) := \inf\big\{ \|u\|^2_{H^1} \ :\ u \in H^1(\mathbb{R}^d),\ u \geq 1 \text{ in a neighborhood of } E\big\},$$

where $H^1(\mathbb{R}^d)$ is the Sobolev space equipped with the norm $\|u\|^2_{H^1} = \displaystyle\int_{\mathbb{R}^d} \left(|\nabla u|^2 + u^2\right) dx$.

We say that a property holds *quasi-everywhere* (q.e.) if it holds on the complementary of a set of zero capacity.

A set $\Omega \subset \mathbb{R}^d$ is said to be *quasi-open* if there exists a decreasing sequence $(\omega_n)_{n\geq 1}$ of open sets such that, for every $n \geq 1$, $\Omega \cup \omega_n$ is an open set and $\lim_{n\to\infty} \mathrm{cap}(\omega_n) = 0$.

A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be *quasi-continuous* if there exists a decreasing sequence $(\omega_n)_{n\geq 1}$ of open sets such that $\lim_{n\to\infty} \mathrm{cap}(\omega_n) = 0$ and the restriction of u to $\mathbb{R}^d \setminus \omega_n$ is continuous.

It is well-known (see for instance [24, Theorem 1, Section 4.8]) that, for every $u \in H^1(\mathbb{R}^d)$, there exists a quasi-continuous representative $\tilde{u}$ of u, which is unique up to a set of zero capacity. From now on we will identify a function $u \in H^1(\mathbb{R}^d)$ with its quasi-continuous representative. We note that, by definition of a quasi-open set and a quasi-continuous function, for every $u \in H^1(\mathbb{R}^d)$, the set $\Omega_u := \{u > 0\} = \{x \in \mathbb{R}^d \mid u(x) > 0\}$ is a quasi-open set ([29, Proposition 3.3.41]). On the other hand, for every quasi-open set Ω, there exists a function $u \in H^1(\mathbb{R}^d)$ such that $\Omega = \Omega_u$ up to a set of zero capacity that is, the quasi-open sets are superlevel sets of Sobolev functions.

For any set $E \subset \mathbb{R}^d$, the Sobolev space $H^1_0(E) \subset H^1(\mathbb{R}^d)$ is defined as

$$H^1_0(E) := \left\{u \in H^1(\mathbb{R}^d) \ : \ u = 0 \ \text{ q.e. in } \ \mathbb{R}^d \setminus E\right\}.$$

Note that, whenever E is open, this definition coincides with the usual definition of $H^1_0(E)$ as the closure of $C^\infty_c(E)$ with respect to the norm $\|\cdot\|_{H^1}$, $C^\infty_c(E)$ being the set of smooth functions compactly supported in E (see for instance [29, Theorem 3.3.42]). For any set $E \subset \mathbb{R}^d$ there is a quasi-open set $\tilde{E} \subset \mathbb{R}^d$ such that $\mathrm{cap}(\tilde{E} \setminus E) = 0$ and $H^1_0(\tilde{E}) = H^1_0(E)$. Roughly speaking, the quasi-open sets are the natural domains for the Sobolev space H^1_0. We notice that, for every quasi-open set E, $H^1_0(E)$ is a closed subspace of $H^1(\mathbb{R}^d)$; if $E_1 \subset E_2$ are two quasi-open sets, then $H^1_0(E_1) \subset H^1_0(E_2)$ and the two sets E_1 and E_2 coincide q.e. if and only if $H^1_0(E_1) = H^1_0(E_2)$.

2.2. **PDEs on quasi-open sets.** Let $D \subset \mathbb{R}^d$ be a given open set and $\Omega \subset D$ be a quasi-open set of finite Lebesgue measure. For every quasi-open set $\Omega \subset D$ and every function $f \in L^2(\Omega)$, the Lax-Milgram theorem and the Poincaré inequality ensure that there is a unique solution $u \in H^1_0(\Omega)$ of the problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u \in H^1_0(\Omega),$$

where the PDE is intended in the weak sense

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f\varphi \, dx \ , \quad \text{for every} \quad \varphi \in H^1_0(\Omega).$$

In particular, taking $u = \varphi$, we notice that $\|\nabla u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$. Now since Ω has a finite Lebesgue measure, there is a constant C_Ω such that $\|u\|_{H^1} \leq C_\Omega \|\nabla u\|_{L^2}$ for every $u \in H^1_0(\Omega)$. Thus, we get that $\|u\|_{H^1} \leq C_\Omega \|f\|_{L^2}$.

The resolvent operator $R^{-\Delta}_\Omega : L^2(D) \to L^2(D)$ is defined as $R^{-\Delta}_\Omega(f) := u$ and is a linear, continuous, self-adjoint, positive operator such that $R^{-\Delta}_\Omega(L^2(D)) \subset H^1_0(\Omega)$. Moreover, thanks to the compact embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, the resolvent $R^{-\Delta}_\Omega$ is also compact.

The usual comparison and weak maximum principles hold in this setting. Precisely, we have:

- if $f \in L^2(D)$ is a positive function and $\Omega_1 \subset \Omega_2 \subset D$ are two quasi-open sets , then $w_{\Omega_1} \leq w_{\Omega_2}$.
- if Ω is a quasi-open set and $f, g \in L^2(\Omega)$ are such that $f \leq g$ in Ω, then $R^{-\Delta}_\Omega(f) \leq R^{-\Delta}_\Omega(g)$.

In the sequel we denote by w_Ω (and sometimes also by $R^{-\Delta}_\Omega(1)$) the solution of

$$-\Delta w_\Omega = 1 \quad \text{in} \quad \Omega, \qquad w_\Omega \in H^1_0(\Omega).$$

This function is sometimes called *torsion* or *energy* function and is useful, in particular, to define the topology of the γ-convergence on the family of quasi-open sets, which is the purpose of the

next section. In the following proposition we summarize the main properties of the function w_Ω (see for instance [35, Proposition 3.50, Remark 3.53, Lemma 3.125, Proposition 3.72]).

Proposition 2.1 (Properties of the torsion function w_Ω)**.**

(1) There is a dimensional constant $C_d > 0$ such that

$$\|\nabla w_\Omega\|_{L^2} \le C_d|\Omega|^{\frac{d+2}{2d}} \qquad \text{and} \qquad \|w_\Omega\|_{L^\infty} \le C_d|\Omega|^{2/d}. \tag{2.1}$$

(2) Let $\Omega_1, \Omega_2 \subset D$ be two quasi-open sets. Then we have the estimate

$$\int_D (w_{\Omega_1} - w_{\Omega_1\setminus\Omega_2})\,dx \le \mathrm{cap}(\Omega_2)\,\|w_{\Omega_1}\|^2_{L^\infty(\Omega_1)}. \tag{2.2}$$

(3) $H^1_0(\Omega) = H^1_0(\{w_\Omega > 0\})$. In particular, $\Omega = \{w_\Omega > 0\}$ up to a set of zero capacity.

In the sequel we make the convention to extend to D any vector field $V \in L^\infty(\Omega, \mathbb{R}^d)$ and any function $u \in H^1_0(\Omega)$ by letting it equal to 0 on $D \setminus \Omega$ so that $V \in L^\infty(D, \mathbb{R}^d)$ and $u \in H^1_0(D)$.

We notice that, given a drift $V \in L^\infty(\Omega, \mathbb{R}^d)$, the bilinear form associated to the operator $L = -\Delta + V \cdot \nabla$ may not be coercive on $H^1_0(\Omega)$. Thus, in order to define the resolvent of $L = -\Delta + V \cdot \nabla$, we consider a large enough constant $c > 0$ (depending only on $\|V\|_{L^\infty(\Omega)}$), for which there exists a positive constant $\delta > 0$ such that

$$\delta \int_D \big(|\nabla u|^2 + u^2\big)\,dx \le \int_D \big(|\nabla u|^2 + (V \cdot \nabla u)\,u + c\,u^2\big)\,dx\ , \quad \text{for every} \quad u \in H^1_0(\Omega). \tag{2.3}$$

The bilinear form associated to the operator $L' = L + c$ is hence coercive on $H^1_0(\Omega)$. Note that

$$\text{if} \quad \|V\|_{L^\infty} \le \tau\ , \quad \text{then we can take any} \quad 0 < \delta < 1 \quad \text{and} \quad c \ge \delta + \frac{\tau^2}{4(1-\delta)}.$$

Therefore, thanks to Lax Milgram theorem, we define the resolvent $R^{L'}_\Omega : L^2(D) \to L^2(D)$ as the compact (non self-adjoint) operator, which maps $f \in L^2(\Omega)$ to the unique solution of the problem

$$L'u = f \quad \text{in} \quad \Omega\ , \qquad u \in H^1_0(\Omega),$$

which is intended in the weak sense

$$\int_\Omega \big(\nabla u \cdot \nabla\varphi + (V \cdot \nabla u)\varphi + c\,u\,\varphi\big)\,dx = \int_\Omega f\varphi\,dx\ , \quad \text{for every} \quad \varphi \in H^1_0(\Omega).$$

2.3. **The γ-convergence and the weak-γ-convergence.** In this subsection we briefly recall the definition and the main properties of the γ-convergence of (quasi-open) sets.

Definition 2.2. Let $D \subset \mathbb{R}^d$ be a given open set of finite Lebesgue measure, $(\Omega_n)_{n\ge1}$ be a sequence of quasi-open sets and let Ω be a quasi-open set, all included in D. We say that

- Ω_n γ-converges to Ω, if w_{Ω_n} converges to w_Ω strongly in $L^2(D)$;
- Ω_n weak-γ-converges to Ω, if there exists $w \in H^1_0(D)$ such that $\Omega = \{w > 0\}$ and w_{Ω_n} converges to w in $L^2(D)$.

Though the γ-convergence is not compact on the family of quasi-open sets (see for instance [19] and [29, § 3.2.6] for an example), it is easy to see that the weak-γ-convergence is: by (2.1), up to a subsequence, w_{Ω_n} weakly converges in $H^1_0(D)$ to some $w \in H^1_0(D)$ and hence Ω_n weak-γ-converges to the quasi-open set $\Omega := \{w > 0\}$. To deal with the non-compactness of the γ-convergence we will use the following Lemma (see for example [12] and [29, Lemma 4.7.11]).

Lemma 2.3. *Let $(\Omega_n)_{n\ge1} \subset D$ be a sequence of quasi-open sets that weak-γ-converges to the quasi-open set $\Omega \subset D$. Then there exists a subsequence of $(\Omega_n)_{n\ge1}$, still denoted by $(\Omega_n)_{n\ge1}$, and a sequence $(\tilde\Omega_n)_{n\ge1} \subset D$ of quasi-open sets satisfying $\Omega_n \subset \tilde\Omega_n$, such that $\tilde\Omega_n$ γ-converges to Ω.*

The following lemma is a direct consequence of the definition of the weak-γ-convergence and the fact that for every quasi-open set $\Omega = \{w_\Omega > 0\}$ (the detailed proof can be found for example in [12] and [35, Lemma 2.2.21]).

Lemma 2.4 (Lower semi-continuity of the Lebesgue measure)**.** *Let $(\Omega_n)_{n\geq 1}$ be a sequence of quasi-open sets in D weak-γ-converging to $\Omega \subset D$, then $|\Omega| \leq \liminf_{n\to+\infty} |\Omega_n|$.*

As was shown in [9] and [12], the following theorem, first proved in [13], is an immediate consequence of Lemma 2.3 and Lemma 2.4.

Theorem 2.5 (Buttazzo-Dal Maso [13])**.** *Let $\mathcal{F}$ be a functional on the quasi-open sets, which is:*
- *decreasing with respect to the inclusion of sets;*
- *lower semi-continuous with respect to the γ-convergence.*

Then, for every bounded open set $D \subset \mathbb{R}^d$ and every $0 < m \leq |D|$, the shape optimization problem

$$\min\Big\{\mathcal{F}(\Omega) \ : \ \Omega \text{ quasi-open}, \ \Omega \subset D, \ |\Omega| \leq m\Big\}$$

has a solution.

We will not be able to apply directly Theorem 2.5 to establish the existence of optimal sets for both the problems (1.4) and (1.3) in the class of quasi-open sets. Instead, in Section 4, we will use an argument based only on Lemma 2.3 and Lemma 2.4, but before that we will need to extend the definition of $\lambda_1(\Omega, V)$ to the class of quasi-open sets. We do this in Section 3, where we will use several times the following approximation result.

Lemma 2.6 (Approximation with open and smooth sets)**.** *Let $\Omega \subset D$ be a quasi-open set. Then:*
(1) there is a sequence of open sets $(\Omega_n)_{n\geq 1}$ that γ-converges to Ω and is such that $\Omega \subset \Omega_n \subset D$ and $\lim_{n\to+\infty} |\Omega_n| = |\Omega|$;
(2) there is a sequence $(\Omega_n)_{n\geq 1}$ of smooth (C^∞) open sets contained in D, that γ-converges to Ω.

Proof. The result is well-known; here we give the proof for the readers' convenience.

(1) Let $(\omega_n)_{n\geq 1}$ be a sequence of open sets such that $\lim_{n\to\infty} \mathrm{cap}(\omega_n) = 0$ and $\Omega_n = (\Omega \cup \omega_n) \cap D$ is an open set. Then, (2.2) applied to the sets Ω_n and $\omega_n \setminus \Omega$ together with the second estimate in (2.1) show that w_{Ω_n} converges to w_Ω in $L^1(D)$. Moreover, up to a subsequence, w_{Ω_n} weakly converges in $H^1(D)$ thanks to the first estimate in (2.1). Since the embedding $H^1_0(D) \hookrightarrow L^2(D)$ is compact, there is a subsequence which converges strongly in $L^2(D)$. By uniqueness of the limit in $L^1(D)$, it has to be w_Ω. Thus, w_{Ω_n} converges in $L^2(D)$ to w_Ω and so, Ω_n γ-converges to Ω. Observe also that one has $\lim_{n\to+\infty} |\Omega_n| = |\Omega|$ since $\lim_{n\to+\infty} |\omega_n| = 0$.

(2) Firstly, assume that Ω is an open set. Let $(\Omega_n)_{n\geq 1}$ be an increasing sequence of smooth open sets included in Ω which Hausdorff converges to Ω. Then, up to a subsequence, $w_n := w_{\Omega_n}$ weakly converges in $H^1_0(D)$ to some $w \in H^1_0(D)$. But Ω_n, Ω are open sets such that $\Omega_n \subset \Omega$, and since the convergence of Ω_n to Ω is Hausdorff, we can pass to the limit in the equation

$$-\Delta w_n = 1 \quad \text{in} \quad \Omega_n$$

to see that w satisfies

$$-\Delta w = 1 \quad \text{in} \quad \Omega.$$

This also shows that the sequence of norms $\|w_n\|_{H^1(D)}$ converges to $\|w\|_{H^1(D)}$, so that the convergence of w_n to w is strong in $H^1(D)$. Finally, since $\Omega_n \subset \Omega$, we get that $w \in H^1_0(\Omega)$ and hence that $w = w_\Omega$. Therefore, the sequence of smooth open sets Ω_n γ-converges to Ω.

If now Ω is merely a quasi-open set, we can approximate Ω by a sequence of open sets which γ-converges to Ω thanks to *(1)*. Hence, by approximating these open sets by open smooth sets as above, we get a sequence of smooth open sets which γ-converges to Ω. Recall that the topology of the γ-convergence is metrizable (see for example [9]). □

Remark 2.7. In general, we cannot approximate a quasi-open set (or even an open set) $\Omega \subset D$ by a sequence of smooth (say of class C^1) open sets $(\Omega_n)_{n\geq 1}$ which γ-converges to Ω and such that $\Omega_n \supset \Omega$. Indeed, let $(\xi_n)_{n\geq 1}$ be a dense sequence in $D = (0,1)^2 \subset \mathbb{R}^2$ and pick a sequence $(r_n)_{n\geq 1}$ of positive numbers such that $\sum_{n\geq 1} \pi r_n^2 < 1$. Set $\Omega := \cup_{n\geq 1} B_{r_n}(\xi_n) \subset D$. We now claim that if $\Omega_n \supset \Omega$ is a smooth open set, then necessarily $\Omega_n \supset D$. To see this, let $x^0 \in D \subset \overline{\Omega} \subset \overline{\Omega}_n$.

Then if $x^0 \in \partial\Omega_n$, there exist $r > 0$ and a smooth, say of class C^1, function $f : \mathbb{R}^d \to \mathbb{R}$ such that, up to reorienting the axis, we have $\Omega_n \cap B_r(x^0) = \{x \in B_r(x^0) \ : \ x_d > f(x_1, \cdots, x_{d-1})\}$. It follows that $\overline{\Omega}_n^c \cap B_r(x^0) \subset D$ is a nonempty open set which does not intersect Ω_n. This is in contradiction with $\Omega_n \supset \Omega$ since Ω is a dense open set in D. Hence $x \in \Omega_n$ and this shows that $D \subset \Omega_n$. Now, suppose that $(\Omega_n)_{n\geq 1}$ is a sequence of smooth sets such that $D \supset \Omega_n \supset \Omega$. Then $\Omega_n = D$ for every $n \geq 1$. Furthermore, the weak maximum principle implies $w_\Omega < w_D = w_{\Omega_n}$ in D, where the first inequality is strict since $|\Omega| < |D| = 1$. Therefore, w_{Ω_n} cannot strongly converge to w_Ω in $L^2(D)$.

We now give a characterization of the γ-convergence in terms of convergence of resolvent operators. The following theorem is a generalization of [29, Lemma 4.7.3] for the operator L'.

Theorem 2.8. *Let $D \subset \mathbb{R}^d$ be a bounded open set, $(\Omega_n)_{n\geq 1} \subset D$ be a sequence of quasi-open sets and $\Omega \subset D$ be a quasi-open set. The following assertions are equivalent :*
(1) the sequence $(\Omega_n)_{n\geq 1}$ γ-converges to Ω;
(2) for every sequence $(f_n)_{n\geq 1} \in L^2(D)$ weakly converging in $L^2(D)$ to $f \in L^2(D)$, the sequence $\left(R^{L'}_{\Omega_n}(f_n)\right)_{n\geq 1}$ converges to $R^{L'}_{\Omega}(f)$ strongly in $L^2(D)$;
(3) the sequence of operators $\left(R^{L'}_{\Omega_n}\right)_{n\geq 1} \in \mathcal{L}(L^2(D))$ converges to $R^{L'}_{\Omega}$ in the operator norm $\|\cdot\|_{\mathcal{L}(L^2(D))}$.

Proof. It is plain to see that the equivalence between *(2)* and *(3)* holds for all sequence of compact operators defined on Hilbert spaces. It then remains to prove that *(1)* and *(2)* are equivalent.
(1)⇒(2). Let $f_n \in L^2(D)$ be a sequence $L^2(D)$-weakly converging to $f \in L^2(D)$. Then $\|f_n\|_{L^2}$ is uniformly bounded. Moreover, writing $u_n = R^{L'}_{\Omega_n}(f_n)$ we have

$$\int_D f_n u_n \, dx = \int_D \left(|\nabla u_n|^2 + V \cdot \nabla u_n \ u_n + c u_n^2\right) dx.$$

Thanks to (2.3) this gives

$$\frac{1}{2}\int_D (f_n^2 + u_n^2) \, dx \geq \delta \int_D (|\nabla u_n|^2 + u_n^2) \, dx,$$

and therefore

$$\int_D f_n^2 \, dx \geq (2\delta - 1) \int_D (|\nabla u_n|^2 + u_n^2) \, dx.$$

Taking $\delta \in (1/2, 1)$, this shows that the sequence $\|u_n\|_{H^1(D)}$ is bounded.

Assume now that the conclusion of *(2)* does not hold. Then there exists $\varepsilon > 0$ such that, up to a subsequence, $\left\|R^{L'}_{\Omega_n}(f_n) - R^{L'}_{\Omega}(f)\right\|_{L^2(D)} \geq \varepsilon$. Moreover, up to a subsequence, u_n weakly converges in $H^1(D)$ to some $u \in H^1_0(D)$, and therefore $g_n = f_n - V \cdot \nabla u_n - c u_n$ weakly converges in $L^2(D)$ to $g = f - V \cdot \nabla u - cu$. Theorem 2.8 being true for the Laplacian (see [14, Proposition 3.4]), we conclude that $R^{-\Delta}_{\Omega_n}(g_n)$ strongly converges in $L^2(D)$ to $R^{-\Delta}_{\Omega}(g)$. Thus $R^{L'}_{\Omega_n}(f_n) = R_{\Omega_n}(g_n)$ and $R^{L'}_{\Omega}(f) = R_\Omega(g)$ imply that $R^{L'}_{\Omega_n}(f_n)$ strongly converges in $L^2(D)$ to $R^{L'}_{\Omega}(f)$, which yields a contradiction and therefore proves *(2)* .
(2)⇒(1). Let $(f_n)_{n\geq 1} \in L^2(D)$ be a sequence weakly converging in $L^2(D)$ to $f \in L^2(D)$. Set $w_n := R^{-\Delta}_{\Omega_n}(f_n)$ and $w := R^{-\Delta}_{\Omega}(f)$. We claim that $w_n \to w$ strongly in $L^2(D)$, which, according to [14] and [29, Lemma 4.7.3], implies that Ω_n γ-converges to Ω. Assume by contradiction that it is not the case, and pick up $\varepsilon > 0$ and an increasing function $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ such that

$$\left\|w_{\varphi(n)} - w\right\|_{L^2(D)} \geq \varepsilon \qquad \text{for every} \qquad n \geq 1. \tag{2.4}$$

Since the sequence $(w_n)_{n\geq 1}$ is bounded in $H^1_0(D)$, up to a subsequence, there exists a function $z \in H^1_0(D)$ such that $w_{\varphi(n)}$ converges to z weakly in $H^1_0(D)$ and strongly in $L^2(D)$. Now, since

$$L' w_n = f_n + V \cdot \nabla w_n + c w_n := g_n \quad \text{ in } \quad \Omega_n,$$

and $w_n \in H^1_0(\Omega_n)$, $w_n = R^{L'}_{\Omega_n}(g_n)$. But $g_{\varphi(n)} \rightharpoonup g := f + V\cdot\nabla z + cz$ weakly in $L^2(D)$, so that, by assumption (2), $w_{\varphi(n)} \to R^{L'}_{\Omega}(g)$ strongly in $L^2(D)$. Then the convergence of $w_{\varphi(n)}$ to z yields that $z = R^{L'}_{\Omega}(g)$, is a solution of

$$L'z = f + V\cdot\nabla z + cz \quad \text{in} \quad \Omega, \qquad z \in H^1_0(\Omega),$$

or, in other words, $z = R^{-\Delta}_{\Omega}(f) = w$. Thus, (2.4) provides a contradiction, therefore showing that $w_n \to w$ strongly in $L^2(D)$, which means that *(1)* holds. □

3. The principal eigenvalue on quasi-open sets

For a bounded domain $\Omega \subset \mathbb{R}^d$ and $V \in L^\infty(\Omega, \mathbb{R}^d)$, the principal eigenvalue $\lambda_1(\Omega, V)$, of the (non self-adjoint) elliptic operator $L = -\Delta + V\cdot\nabla$ on Ω with Dirichlet boundary condition on $\partial\Omega$, was defined in [4] by

$$\lambda_1(\Omega, V) = \sup\big\{\lambda \in \mathbb{R} \ : \ \exists \phi \in W^{2,d}(\Omega) \quad \text{such that} \quad \phi > 0 \quad \text{and} \quad -L\phi + \lambda\phi \le 0 \quad \text{in} \quad \Omega\big\},$$

where it was proved that $\lambda_1(\Omega, V) \in \mathbb{R}$ has the following properties:

(i) There is a positive eigenfunction $u : \Omega \to \mathbb{R}$ such that $u \in W^{2,p}_{loc}(\Omega)$, for all $p \in [1, +\infty)$, and

$$Lu = \lambda_1(\Omega, V)u \quad \text{in} \quad \Omega, \qquad u \in H^1_0(\Omega), \qquad \int_\Omega u^2\,dx = 1,$$

(see [4, Theorem 2.1]).

(ii) $\lambda_1(\Omega, V) < \mathrm{Re}\,(\lambda)$ for every eigenvalue $\lambda \neq \lambda_1(\Omega, V)$ of L in Ω (see [4, Theorem 2.3]).

(iii) The functional $\Omega \mapsto \lambda_1(\Omega, V)$ is decreasing with respect to the domain inclusion.

In the sequel we extend the definition of $\lambda_1(\Omega, V)$ to quasi-open sets. We first recall that the definition can be extended to an arbitrary open set $\Omega \subset D$ by

$$\lambda_1(\Omega, V) = \inf \lambda_1(\Omega_n, V),$$

where the infimum is taken over all the connected component Ω_n of Ω. Now, in view of property *(iii)* above, for any quasi-open set $\Omega \subset D$, we define

$$\lambda_1(\Omega, V) := \sup\big\{\lambda_1(\tilde\Omega, V) \ : \ \tilde\Omega \text{ open}, \ \Omega \subset \tilde\Omega \subset D\big\}. \tag{3.1}$$

Remark 3.1. Notice that, these two definitions coincide for open sets.

Remark 3.2. The functional $\Omega \mapsto \lambda_1(\Omega, V)$, defined on the family of quasi-open sets, is still non-increasing with respect to the set inclusion, that is $\lambda_1(\Omega_2, V) \le \lambda_1(\Omega_1, V)$, whenever $\Omega_1 \subset \Omega_2$.

We will show that $\lambda_1(\Omega, V)$ is finite and is an eigenvalue of L in Ω satisfying the minimality property *(ii)*. Recall that, for a quasi-open set of finite Lebesgue measure $\Omega \subset D$, we say that $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $L = -\Delta + V\cdot\nabla$ in Ω if there is an eigenfunction $u : \mathbb{R}^d \to \mathbb{C}$, (weak) solution to the problem

$$-\Delta u + V\cdot\nabla u = \lambda u \quad \text{in} \quad \Omega, \qquad u \in H^1_0(\Omega; \mathbb{C}), \qquad \int_\Omega |u|^2\,dx = 1. \tag{3.2}$$

Let now $c > 0$ be the constant from Subsection 2.2 and $L' = L + c$. Note that $\lambda \in \mathbb{C}$ is an eigenvalue of L in Ω, if and only if, $\lambda + c$ is an eigenvalue of L' in Ω. By the argument from Subsection 2.2, we have that the bilinear form associated to the operator L' is coercive and so, $R^{L'}_{\Omega}$ is a compact operator on $L^2(D)$. In particular, the spectrum is a discrete set of eigenvalues with no accumulation points except zero and $\lambda \in \mathbb{C}$ is an eigenvalue of L in the sense of (3.2) if and only if $(\lambda + c)^{-1}$ is an eigenvalue of $R^{L'}_{\Omega}$.

The following theorem shows that most of the properties of the principal eigenvalue on an open set still hold for $\lambda_1(\Omega, V)$ if $\Omega \subset D$ is merely a quasi-open set.

Theorem 3.3. *Let D be a bounded open set, $V \in L^\infty(D, \mathbb{R}^d)$ and $\Omega \subset D$ be a non-empty quasi-open set. Then*

(1) $\lambda_1(\Omega, V)$ is well-defined that is, $\lambda_1(\Omega, V) < +\infty$.

(2) $\lambda_1(\Omega, V)$ is an eigenvalue of L in Ω; there is a (non-trivial) real-valued eigenfunction u such that

$$Lu = \lambda_1(\Omega, V)u \quad in \quad \Omega\ , \qquad u \in H_0^1(\Omega), \qquad \int_\Omega u^2\, dx = 1.$$

(3) If $\lambda \in \mathbb{C}$ is an eigenvalue of L in Ω, then $\lambda_1(\Omega, V) \le Re\,(\lambda)$.

In order to prove Theorem 3.3 we will need the following two lemmas. The key estimate for the proof of Theorem 3.3 *(1)* is contained in the following lemma inspired by [4, Proposition 5.1].

Lemma 3.4. *Let $V \in L^\infty(D, \mathbb{R}^d)$ and $\Omega \subset D$ be an open set. Suppose that there is $\tau > 0$ such that $\|V\|_{L^\infty(\Omega)} \le \tau < 2\sqrt{\lambda_1(\Omega, V)}$. Then*

$$\lambda_1(\Omega, 0) \ge \lambda_1(\Omega, V) - \tau\sqrt{\lambda_1(\Omega, V)}. \tag{3.3}$$

Proof. Let us first suppose that Ω is connected. For convenience, set $\lambda := \lambda_1(\Omega, V)$. By the definition of the first eigenvalue of $-\Delta$ on domains, it is enough to find some $\phi > 0$ in Ω such that $-\Delta\phi \ge (\lambda - \tau\sqrt{\lambda})\phi$ in Ω. Since Ω is an open set, from [4, Theorem 2.1], there exists a positive eigenfunction ϕ_V for the first eigenvalue of L in Ω, that is, $\phi_V > 0$ in Ω and $L\phi_V = \lambda\phi_V$. Set $\phi := \phi_V^\alpha$ for some $\alpha \in (0, 1)$ to be chosen later. Then, in Ω, we have

$$\begin{aligned}
-\Delta\phi - \lambda\phi &= -\alpha(\Delta\phi_V)\phi_V^{\alpha-1} - \alpha(\alpha-1)|\nabla\phi_V|^2\phi_V^{\alpha-2} - \lambda\phi_V^\alpha \\
&= \left[\lambda(\alpha-1) - \alpha V \cdot \frac{\nabla\phi_V}{\phi_V} + \alpha(1-\alpha)\frac{|\nabla\phi_V|^2}{\phi_V^2}\right]\phi_V^\alpha \\
&\ge \left[\lambda(\alpha-1) - \alpha\tau\frac{|\nabla\phi_V|}{\phi_V} + \alpha(1-\alpha)\frac{|\nabla\phi_V|^2}{\phi_V^2}\right]\phi_V^\alpha.
\end{aligned}$$

The function $x \mapsto -\alpha\tau x + \alpha(1-\alpha)x^2$ reaches its minimum at $x = \tau/(2(1-\alpha))$. Therefore, we get

$$-\Delta\phi - \lambda\phi \ge \left[\lambda(\alpha-1) - \alpha\frac{\tau^2}{4(1-\alpha)}\right]\phi_V^\alpha = \left[\lambda(\alpha-1) - \alpha\frac{\tau^2}{4(1-\alpha)}\right]\phi.$$

Since $\alpha \in (0, 1)$ is arbitrary, we can choose it so that it maximizes the term in the brackets of the above estimate, that is, such that $1 - \alpha = \tau/(2\sqrt{\lambda})$. Note that, by hypothesis on τ, we have $\alpha \in (0, 1)$. It follows

$$-\Delta\phi - \lambda\phi \ge \left[-\tau\sqrt{\lambda} + \frac{\tau^2}{4}\right]\phi \ge -\tau\sqrt{\lambda}\,\phi,$$

which proves the claim in the case when Ω is connected.

In the general case, let $(\Omega_n)_{n\ge1}$ be the connected components of Ω. Then, for every V, we have

$$\lambda_1(\Omega, V) = \inf_n \lambda_1(\Omega_n, V).$$

Then, we have, for all n,

$$\lambda_1(\Omega_n, 0) \ge \lambda_1(\Omega_n, V) - \tau\sqrt{\lambda_1(\Omega_n, V)} \ge \lambda_1(\Omega, V) - \tau\sqrt{\lambda_1(\Omega, V)},$$

where the last inequality is due to the fact that $x \mapsto x - \tau\sqrt{x}$ is a non-increasing function on the interval $[\lambda_1(\Omega, V), +\infty)$. □

The next lemma is a direct consequence of the classical result [31, Theorem 3.16] on the convergence of a spectrum of closed operators with suitable properties. We will use it in the proof of Theorem 3.3 *(3)*.

Lemma 3.5. *Let H be a separable Hilbert space and $(T_n)_{n\geq 1} \in \mathcal{L}(H)$ a sequence of compact operators converging to the compact operator $T \in \mathcal{L}(H)$ in the operator norm $\|\cdot\|_{\mathcal{L}(H)}$. Suppose that $\lambda \in \mathbb{C}\setminus\{0\}$ is an (isolated) eigenvalue of T and let $r>0$ be such that $B_r(\lambda)\cap\sigma(T)=\{\lambda\}$. Then, there is $n_0\geq 1$ such that for every $n\geq n_0$ there is an eigenvalue $\lambda_n \in \sigma(T_n)\cap B_{r/2}(\lambda)$.*

We are now in position to prove Theorem 3.3.

Proof of Theorem 3.3. Consider a maximizing sequence $(\Omega_n)_{n\geq 1}$ for (3.1), that is, a sequence of open sets $(\Omega_n)_{n\geq 1}$ such that

$$\lambda_1(\Omega,V)=\lim_{n\to\infty}\lambda_1(\Omega_n,V) \qquad \text{and} \qquad \Omega\subset\Omega_n\subset D \quad \text{for every} \quad n\geq 1.$$

We first show that we can assume that Ω_n γ-converges to Ω. Let ω_n be a sequence of open sets such that $\Omega\cup\omega_n$ is open and $\operatorname{cap}(\omega_n)\to 0$. We set $\tilde{\Omega}_n := \Omega_n\cap(\Omega\cup\omega_n)=\Omega\cup(\omega_n\cap\Omega_n)$. By (3.1) and the inclusion $\Omega\subset\tilde{\Omega}_n\subset\Omega_n$ we have $\lambda_1(\Omega_n,V)\leq\lambda_1(\tilde{\Omega}_n,V)\leq\lambda_1(\Omega,V)$, so we get

$$\lambda_1(\Omega,V)=\lim_{n\to\infty}\lambda_1(\tilde{\Omega}_n,V) \qquad \text{and} \qquad \Omega\subset\tilde{\Omega}_n\subset D \quad \text{for every} \quad n\geq 1.$$

Thus, we may consider $\tilde{\Omega}_n$ in place of Ω_n as a maximizing sequence for (3.1). Finally, as in Lemma 2.6, $\tilde{\Omega}_n$ γ-converges to Ω thanks to the estimate (2.2) applied to the sets $\tilde{\Omega}_n$ and $\Omega_n\cap\omega_n$.

We now prove claim (1). Indeed, suppose by contradiction that

$$\lambda_1(\Omega,V)=\lim_{n\to\infty}\lambda_1(\Omega_n,V)=+\infty.$$

Then, by Lemma 3.4 we have that

$$\lim_{n\to\infty}\lambda_1(\Omega_n,0)=+\infty.$$

Now, since $\Omega\mapsto\lambda_1(\Omega,0)$ is decreasing and $\Omega\subset\Omega_n$, we get that $\lambda_1(\Omega,0)=+\infty$. By the variational characterization

$$\lambda_1(\Omega,0)=\min_{u\in H^1_0(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2\,dx}{\int_\Omega u^2\,dx},$$

we get that $H^1_0(\Omega)=\{0\}$, which implies that $\Omega=\emptyset$ (or, equivalently, $\operatorname{cap}\Omega=0$), which is absurd.

We now prove (2). Let $u_n\in H^1_0(\Omega_n)\subset H^1_0(D)$ be the (normalized) eigenfunction associated to $\lambda_1(\Omega_n,V)$. Then we have

$$L'u_n=\big(\lambda_1(\Omega_n,V)+c\big)u_n \quad \text{in} \quad \Omega_n, \qquad u_n\in H^1_0(\Omega_n), \qquad \int_{\Omega_n}u_n^2\,dx=1.$$

Multiplying the above equation by u_n, integrating over Ω_n and using the estimate (2.3) we get

$$\delta\|u_n\|^2_{H^1}\leq\lambda_1(\Omega_n,V)+c \quad \text{for every} \quad n\geq 1.$$

In particular, since $\lambda_1(\Omega,V)<\infty$, we get that $(u_n)_{n\geq 1}$ is uniformly bounded in $H^1_0(D)$ and so, up to a subsequence, we may assume that u_n converges, weakly in $H^1_0(D)$ and strongly in $L^2(D)$, to a function $u\in H^1_0(D)$. Moreover, Ω_n γ-converges to Ω and so, $R^{L'}_{\Omega_n}$ converges in norm to $R^{L'}_{\Omega}$. Thus,

$$u=\lim_{n\to\infty}u_n=\lim_{n\to\infty}(\lambda_1(\Omega_n,V)+c)R^{L'}_{\Omega_n}(u_n)=(\lambda_1(\Omega,V)+c)R^{L'}_{\Omega}(u),$$

which concludes the proof of *(2)*.

Proof of (3). Suppose that $\lambda\in\mathbb{C}$ is an eigenvalue of L on Ω such that $\operatorname{Re}(\lambda)<\lambda_1(\Omega,V)$. Then, $(\lambda+c)^{-1}\in\mathbb{C}$ is a (non-zero) eigenvalue of the compact operator $R^{L'}_{\Omega}$. Applying Lemma 3.5, we can assume that for n large enough, there is an eigenvalue λ_n of L on Ω_n such that $\operatorname{Re}(\lambda_n)<\lambda_1(\Omega_n,V)$, which is a contradiction with [4, Theorem 2.3] . □

Remark 3.6 (On the sign of the first eigenfunction)**.** In particular, as a consequence of the proof of Theorem 3.3 *(2)*, there is an eigenfunction u of L on the quasi-open set Ω, which is non-negative, being the limit of non-negative functions. We notice that u does not need to be strictly positive as Ω might be disconnected.

We conclude this section with a proposition on the continuity of $\lambda_1(\cdot, V)$ with respect to the γ-convergence.

Proposition 3.7. *Let $D \subset \mathbb{R}^d$ be a bounded open set, $V \in L^\infty(D;\mathbb{R}^d)$ and $(\Omega_n)_{n\geq 1} \subset D$ be a sequence of quasi-open sets that γ-converges to the quasi-open set $\Omega \subset D$. Then*

$$\lambda_1(\Omega, V) = \begin{cases} \lim\limits_{n\to\infty} \lambda_1(\Omega_n, V), & \text{if} \quad \Omega \neq \emptyset, \\ +\infty, & \text{if} \quad \Omega = \emptyset. \end{cases}$$

Proof. Let $\tau = \|V\|_{L^\infty(D)}$ and δ and c be as in (2.3). Set $L' = L + c$.
Suppose first that the sequence $(\lambda_1(\Omega_n, V))_{n\geq 1}$ is bounded. Reasoning as in the proof of Theorem 3.3 *(2)* we get that, up to a subsequence, $\lambda_1(\Omega_n, V)$ converges to an eigenvalue $\lambda \in \mathbb{R}$ of L on Ω. Now, by the argument of Theorem 3.3 *(3)* and Lemma 3.5, we have that λ satisfies the property *(3)* of Theorem 3.3, so $\lambda = \lambda_1(\Omega, V)$, which concludes the proof since the sequence $(\lambda_1(\Omega_n, V))_{n\geq 1}$ is bounded .
Next, suppose that the sequence $(\lambda_1(\Omega_n, V))_{n\geq 1}$ is unbounded. Applying Lemma 3.4, we get that, up to a subsequence, $\lim\limits_{n\to\infty} \lambda_1(\Omega_n, 0) = +\infty$. Since $R_{\Omega_n}^{-\Delta}$ are self-adjoint compact operators, we get that

$$\lim_{n\to\infty} \|R_{\Omega_n}^{-\Delta}\|_{\mathcal{L}(L^2(D))} = \lim_{n\to\infty} \frac{1}{\lambda_1(\Omega_n, 0)} = 0.$$

Finally, the γ-convergence gives that $R_\Omega^{-\Delta}(\Omega) \equiv 0$ and so, $H_0^1(\Omega) = \{0\}$ and $\mathrm{cap}(\Omega) = 0$. □

Remark 3.8. In view of Proposition 3.7 we set $\lambda_1(\emptyset, V) = +\infty$.

Putting together Theorem 3.3 and Proposition 3.7 we obtain the following result.

Corollary 3.9. *Let Ω be a bounded quasi-open set and $V \in L^\infty(\Omega;\mathbb{R}^d)$. Then, there is an eigenvalue $\lambda_1(\Omega, V) \in \mathbb{R}$ of $L = -\Delta + V \cdot \nabla$ in Ω such that:*

$$\begin{aligned} \lambda_1(\Omega, V) &= \min\left\{ Re\,\lambda \ : \ \lambda \in \mathbb{C} \text{ is an eigenvalue of } L \text{ on } \Omega \right\} \\ &= \sup\left\{ \lambda_1(\tilde{\Omega}) \ : \ \tilde{\Omega} \text{ is an open set containing } \Omega \right\} \\ &= \lim_{n\to\infty} \lambda_1(\Omega_n, V), \text{where } (\Omega_n)_{n\geq 1} \text{ is any sequence of (smooth) open sets } \gamma\text{-converging to } \Omega. \end{aligned}$$

Proof. The first two inequalities are due to Theorem 3.3. For the third one it is sufficient to note that for every quasi-open set Ω there is a sequence of smooth open sets γ-converging to Ω and to apply Proposition 3.7. □

Remark 3.10 (Faber-Krahn with drift for quasi-open sets)**.** As further consequence of Corollary 3.9 we can extend the Hamel-Nadirashvili-Russ inequality to the class of (bounded) quasi-open sets. Precisely, for every bounded quasi-open set $\Omega \subset \mathbb{R}^d$ with $|\Omega| > 0$ and every $\tau > 0$, we have

$$\lambda_1\left(B, \tau\frac{x}{|x|}\right) \leq \lambda_1(\Omega, V) \quad \text{for every} \quad V \in L^\infty(\Omega;\mathbb{R}^d) \quad \text{with} \quad \|V\|_{L^\infty} \leq \tau, \tag{3.4}$$

where B is the ball centered in zero of the same Lebesgue measure as Ω. Indeed, let $\Omega \subset \mathbb{R}^d$ be a bounded quasi-open set and $V \in L^\infty(\Omega, \mathbb{R}^d)$ be such that $\|V\|_{L^\infty} \leq \tau$ (in what follows we assume that V is extended by zero outside Ω). Let $(\Omega_n)_{n\geq 1}$ be a sequence of bounded open sets which γ-converges to Ω and such that $|\Omega_n|$ converges to $|\Omega|$ (see Lemmalem approx qo). Denote by B_{r_n} (resp. B) the ball centred at 0 whose Lebesgue measure is $|B_{r_n}| = |\Omega_n|$ (resp. $|B| = |\Omega|$). Then, since Ω_n is an open set, we have $\lambda_1(B_{r_n}, \tau e_r) \leq \lambda_1(\Omega_n, v)$ thanks to [27, Remark 6.10]. Moreover, B_{r_n} γ-converges to B (since $|B_{r_n}| \to |B|$ and hence B_{r_n} converges to B in the sense of Hausdorff; see [29, Proposition 3.4.2]). Therefore, Corollary 3.9 implies that $\lambda_1(B_{r_n}, \tau e_r)$ converges to $\lambda_1(B, \tau e_r)$ and similarly, $\lambda_1(\Omega_n, V) \to \lambda_1(\Omega, V)$. Passing to the limit we get (3.4).

4. Existence of optimal domains

In this section we prove the existence of optimal domains for the cost functional $\lambda_1(\Omega, V)$. We first consider the case when the drift V is fixed, for which the existence follows by the result of the previous section and a classical theorem in shape optimization. The case when both the domain Ω and the drift V may vary requires more careful analysis and the rest of the section is dedicated to the proof of Theorem 4.3. In the end of the section (Theorem 4.5) we also prove that a solution (Ω, V) exists also in the class of vector fields V obtained as gradients of Lipschitz continuous functions.

Theorem 4.1 (Existence of optimal sets for a fixed vector field)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set and $V \in L^\infty(D;\mathbb{R}^d)$. Then, for every $0 < m \le |D|$, there is an optimal domain, solution of the problem* (1.5)*.*

Proof. By Remark 3.2 and Proposition 3.7 we get that $\Omega \mapsto \lambda_1(\Omega, V)$ is γ-continuous and decreasing with respect to the set inclusion. The claim follows by Theorem 2.5. □

4.1. **Optimal drifts on a fixed domain.** Let $\Omega \subset \mathbb{R}^d$ be a fixed bounded quasi-open set and $\tau > 0$ be given. We consider the following variational minimization problem

$$\min\Big\{\lambda_1(\Omega, V) \ : \ V \in L^\infty(\Omega, \mathbb{R}^d),\ \|V\|_{L^\infty} \le \tau\Big\}. \tag{4.1}$$

Theorem 4.2. *The problem* (4.1) *has a solution, which satisfies*

$$V_*(x) = -\tau\frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{if } \ |\nabla u(x)| \neq 0\ ; \qquad V_*(x) = 0 \quad \text{if } \ |\nabla u(x)| = 0\ , \tag{4.2}$$

where u is the eigenfunction of $L = -\Delta + V_* \cdot \nabla$ in Ω, associated to the eigenvalue $\lambda_1(\Omega, V_*)$.

Proof. Let $(\Omega_n)_{n\ge1}$ be a sequence of smooth, say of class $C^{2,\alpha}$ for some $0 < \alpha < 1$, open sets which γ-converges to Ω (see Remark 2.6). Since Ω_n is smooth, we already know (see [28, theorem 1.5]) that the problem (4.1) for the fixed domain Ω_n has a solution V_n. Moreover, if u_n is the associated eigenfunction of $-\Delta + V_n \cdot \nabla$ in Ω_n, that is, u_n is defined by

$$-\Delta u_n + V_n \cdot \nabla u_n = \lambda_1(\Omega_n, V_n)u_n \quad \text{in} \quad \Omega_n, \qquad u_n \in H^1_0(\Omega_n), \qquad \int_{\Omega_n} u_n^2\,dx = 1,$$

then the optimal vector field V_n is unique and is given by

$$V_n(x) = \begin{cases} -\tau\dfrac{\nabla u_n(x)}{|\nabla u_n(x)|} & \text{if } |\nabla u_n(x)| \neq 0, \\ 0 & \text{if } |\nabla u_n(x)| = 0. \end{cases}$$

In particular, u_n is a solution of

$$-\Delta u_n - \tau\,|\nabla u_n| = \lambda_1(\Omega_n, V_n)u_n \quad \text{in} \quad \Omega_n, \qquad u_n \in H^1_0(\Omega_n), \qquad \int_{\Omega_n} u_n^2\,dx = 1.$$

We first claim that the sequence $(\lambda_1(\Omega_n, V_n))_{n\ge1}$ is bounded. Indeed, by optimality of V_n, one has $\lambda_1(\Omega_n, V_n) \le \lambda_1(\Omega_n, 0)$, which is nothing but the principal eigenvalue of $-\Delta$ on Ω_n with Dirichlet boundary condition. But since Ω_n γ-converges to Ω, Proposition 3.7 yields that $\lambda_1(\Omega_n, 0) \to \lambda_1(\Omega, 0)$ so that the sequence $(\lambda_1(\Omega_n, 0))_{n\ge1}$ is bounded, proving our claim.
Therefore, up to a subsequence, $\lambda_1(\Omega_n, V_n)$ converges to some $\lambda \in \mathbb{R}$ and u_n has a uniformly bounded norm in $H^1_0(D)$, which yields a function $u \in H^1_0(D)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ weakly in } H^1_0(D) \text{ and } u_n \to u \text{ strongly in } L^2(D). \tag{4.3}$$

Since the sequence $|\nabla u_n|$ is bounded in $L^2(D)$, up to a subsequence, $-\tau\,|\nabla u_n| \rightharpoonup z$ weakly in $L^2(D)$ for some function $z \in L^2(D)$. Therefore, $f_n := \lambda_1(\Omega_n, v_n)u_n + \tau|\nabla u_n|$ weakly converges in

$L^2(D)$ to $f := \lambda u - z$. Thanks to theorem 2.8 (applied to $-\Delta$), $u_n = R_{\Omega_n}^{-\Delta}(f_n)$ strongly converges in $L^2(D)$ to $R_\Omega^{-\Delta}(f)$. By (4.3), we have $u = R_\Omega^{-\Delta}(f)$ and hence $u \in H_0^1(\Omega)$. Furthermore

$$\begin{aligned}\int_D |\nabla u|^2 \, dx &= \int_D (-zu + \lambda u^2) \, dx \\ &= \lim_{n\to+\infty} \int_D (\tau|\nabla u_n| u_n + \lambda_1(\Omega_n, V_n) u_n^2) \, dx = \lim_{n\to+\infty} \int_D |\nabla u_n|^2 \, dx,\end{aligned}$$

where the first line is due to the fact that $u \in H_0^1(\Omega)$ and $-\Delta u = \lambda u - z$ in Ω. This proves that u_n converges strongly in $H^1(D)$ to u, that $|\nabla u_n|$ strongly converges in $L^2(D)$ to $|\nabla u|$, and hence that $z = -\tau|\nabla u|$. Therefore u satisfies

$$-\Delta u + V_* \cdot \nabla u = -\Delta u - \tau \, |\nabla u| = \lambda u \quad \text{in} \quad \Omega, \qquad u \in H_0^1(\Omega), \qquad \int_\Omega u^2 \, dx = 1,$$

where $V_* \in L^\infty(D, \mathbb{R}^d)$ is given by (4.2). This shows that λ is an eigenvalue of the operator $L = -\Delta + V_* \cdot \nabla$ in Ω. In particular, we have $\|V_*\|_\infty \le \tau$ and $\lambda_1(\Omega, V_*) \le \lambda$. On the other hand, by the minimality of V_n, we have $\lambda_1(\Omega_n, V_n) \le \lambda_1(\Omega_n, V_*)$. Hence, letting $n \to \infty$, we get that $\lambda \le \lambda_1(\Omega, V_*)$, which yields $\lambda = \lambda_1(\Omega, V_*)$ and concludes the proof of the theorem. □

4.2. **Shape optimization problem over domains and vector fields.** Let $D \subset \mathbb{R}^d$ be a bounded open set, $0 < m \le |D|$ and $\tau > 0$. We consider the shape optimization problem

$$\min\big\{\lambda_1(\Omega, V) \ : \ \Omega \subset D \text{ quasi-open}, \ |\Omega| \le m, \ \|V\|_{L^\infty} \le \tau\big\}. \tag{4.4}$$

Theorem 4.3 (Existence of optimal sets and optimal vector fields)**.** *Let $\tau \ge 0$ and $m \in (0, |D|)$. Then the problem* (4.4) *has a solution (Ω^*, V^*), where V^* is given by* (4.2)*.*

Proof. Let (Ω_n, V_n) be a minimizing sequence for (4.4) and let

$$\underline{\lambda} := \inf\big\{\lambda_1(\Omega, V) \ : \ \Omega \subset D \text{ quasi-open}, \ |\Omega| \le m, \ \|V\|_{L^\infty} \le \tau\big\} = \lim_{n\to\infty} \lambda_1(\Omega_n, V_n),$$

Since the topology of the weak γ-convergence is compact, we can assume that, up to a subsequence, Ω_n weakly γ-converges to a quasi-open set $\Omega \subset D$. Then, let $\tilde\Omega_n$ be a sequence of quasi-open sets as in Lemma 2.3. Denote by $\tilde V_n$ the optimal vector field given by Theorem 4.2 on $\tilde\Omega_n$, and let $u_n \in H_0^1(\tilde\Omega_n)$ be a solution of

$$-\Delta u_n + \tilde V_n \cdot \nabla u_n = \lambda_1(\tilde\Omega_n, \tilde V_n) u_n \quad \text{in} \quad \tilde\Omega_n, \qquad u_n \in H_0^1(\tilde\Omega_n), \qquad \int_D u_n^2 \, dx = 1.$$

By the minimality of $\tilde V_n$ and the inclusion $\Omega_n \subset \tilde\Omega_n$, we have

$$0 < \lambda_1(\tilde\Omega_n, \tilde V_n) \le \lambda_1(\tilde\Omega_n, V_n) \le \lambda_1(\Omega_n, V_n) \qquad \text{for every} \quad n \ge 1.$$

Therefore, up to a subsequence, $\lambda_1(\tilde\Omega_n, \tilde V_n)$ converges to some $\tilde\lambda$ such that $\tilde\lambda \le \underline{\lambda}$. In particular, $(u_n)_{n\ge1}$ is uniformly bounded in $H_0^1(D)$ and so, up to a subsequence, u_n weakly converges in $H_0^1(D)$ to some $u \in H_0^1(D)$. Now, since $\tilde\Omega_n$ γ-converges to Ω, we can argue as in the end of the proof of Theorem 4.2 to conclude that the convergence of u_n to u is strong in $H^1(D)$. This yields that u is not identically zero and satisfies

$$-\Delta u + V \cdot \nabla u = \Delta u - \tau \, |\nabla u| = \tilde\lambda \, u \quad \text{in} \quad \Omega, \qquad u \in H_0^1(\Omega), \qquad \int_D u^2 \, dx = 1,$$

where $V \in L^\infty(D, \mathbb{R}^d)$ is given by (4.2). Furthermore, thanks to Lemma 2.4, we have that $|\Omega| \le m$. Hence, $\underline{\lambda} \le \tilde\lambda$. Thus, we get that $\tilde\lambda = \underline{\lambda}$ and hence that $\underline{\lambda} = \lambda_1(\Omega, V)$, which proves that the couple (Ω, V) is a solution of (4.4). □

Remark 4.4. If the box D contains a ball $B \subset D$ such that $|B| = m$, then by Remark 3.10 a solution of (4.4) is given by $\lambda_1(B, \tau x/|x|)$.

We now consider a shape optimization problem in the more restrictive class of couples (Ω, V), in which the vector field V is a gradient of a Lipschitz function. Precisely, given a bounded open set $D \subset \mathbb{R}^d$, $\tau \geq 0$ and $m \in (0, |D|)$, we consider the shape optimization problem

$$\min\Big\{\lambda_1(\Omega, \nabla\Phi) \ : \ \Omega \subset D \text{ quasi-open}, \ \Phi \in W^{1,\infty}(D), \ |\Omega| \leq m, \ \|\nabla\Phi\|_{L^\infty} \leq \tau\Big\}. \tag{4.5}$$

In this case the argument from Theorem 4.3 does not apply since the optimal vector field from Theorem 4.2 may not be the gradient of a Lipschitz function. On the other hand, the functional $\lambda_1(\Omega, \nabla\Phi)$ is variational so we can use a more direct approach. Indeed, for every $\lambda \in \mathbb{R}$ and $u \in H^1_0(\Omega)$ we have

$$-\Delta u + \nabla\Phi \cdot \nabla u = \lambda u \quad \text{in} \quad \Omega \quad \Leftrightarrow \quad -\mathrm{div}\,(e^{-\Phi}\nabla u) = \lambda e^{-\Phi} u \quad \text{in} \quad \Omega\,,$$

and since the operator $A = -\mathrm{div}\,(e^{-\Phi}\nabla\cdot)$ is self-adjoint, we get that

$$\lambda_1(\Omega, \nabla\Phi) = \min_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_D e^{-\Phi}|\nabla u|^2\,dx}{\int_D e^{-\Phi} u^2\,dx}. \tag{4.6}$$

Theorem 4.5 (Existence of optimal sets and optimal potentials)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set, $\tau \geq 0$ and $m \in (0, |D|)$. Then the problem* (4.5) *has a solution.*

Proof. Suppose that (Ω_n, Φ_n) is a minimizing sequence for (4.5) and let $\lambda_n = \lambda_1(\Omega_n, \nabla\Phi_n)$. Given $x_0 \in D$, we may suppose that $\Phi_n(x_0) = 0$ for every $n \geq 1$. Thus, up to a subsequence, Φ_n converges uniformly in $\overline{D}$ to a function $\Phi \in W^{1,\infty}(D)$ such that $\Phi(x_0) = 0$ and $\|\nabla\Phi\|_{L^\infty} \leq \tau$. Let u_n be the solution of

$$-\Delta u_n + \nabla\Phi_n \cdot \nabla u_n = \lambda_n u_n \quad \text{in} \quad \Omega_n, \qquad u_n \in H^1_0(\Omega_n), \qquad \int_D u_n^2\,dx = 1.$$

Then, u_n is uniformly bounded in $H^1_0(D)$ an so, up to a subsequence, u_n converges weakly in $H^1_0(D)$ and strongly in $L^2(D)$ to a function $u \in H^1_0(D)$. Thus, we have

$$\int_D e^{-\Phi} u^2\,dx = \lim_{n\to\infty} \int_D e^{-\Phi_n} u_n^2\,dx \qquad \text{and} \qquad \int_D e^{-\Phi}|\nabla u|^2\,dx \leq \liminf_{n\to\infty} \int_D e^{-\Phi_n}|\nabla u_n|^2\,dx.$$

Now, choosing $\Omega := \{u > 0\}$ and applying (4.6), we get

$$\lambda_1(\Omega, \Phi) \leq \frac{\int_D e^{-\Phi}|\nabla u|^2\,dx}{\int_D e^{-\Phi} u^2\,dx} \leq \liminf_{n\to\infty} \frac{\int_D e^{-\Phi_n}|\nabla u_n|^2\,dx}{\int_D e^{-\Phi_n} u_n^2\,dx} = \liminf_{n\to\infty} \lambda_1(\Omega_n, \Phi_n).$$

Now, in order to conclude, it is sufficient to notice that by choosing a subsequence, we may assume that u_n converges to u pointwise a.e., so we get

$$|\Omega| = |\{u > 0\}| \leq \liminf_{n\to\infty} |\{u_n > 0\}| \leq \liminf_{n\to\infty} |\Omega_n| \leq m,$$

which proves that (Ω, Φ) is a solution of (4.5). □

5. Regularity of the optimal sets

In this section we study the regularity of the boundary $\partial\Omega$ of the optimal sets Ω. We only consider the case $V = \nabla\Phi$, with $\Phi \in W^{1,\infty}(D)$, since in this case the optimization problem (1.5) is equivalent to a free boundary problem for the first eigenfunction u on the optimal set Ω. The regularity for a generic vector field $V \in L^\infty(D)$ remains an open problem essentially due to the lack of variational characterization of the eigenvalue $\lambda_1(\Omega, V)$. We start with the following lemma.

Lemma 5.1 (Reduction to a free boundary problem)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set, $0 < m < |D|$, $\tau > 0$, $\Phi \in W^{1,\infty}(D)$, with $\|\nabla\Phi\|_{L^\infty} \leq \tau$, and $V = \nabla\Phi$. Suppose that the quasi-open set $\Omega \subset D$ is a solution of* (1.5)*. Then every corresponding first eigenfunction u_Ω of the operator $-\Delta + V\cdot\nabla$ on Ω is a solution to the variational problem*

$$\min\Big\{\int_D |\nabla u|^2 e^{-\Phi}\,dx \ : \ u \in H^1_0(D), \ \big|\{u \neq 0\}\big| \leq m, \ \int_D e^{-\Phi} u^2\,dx = 1\Big\}. \tag{5.1}$$

Conversely, if u is a solution of (5.1), then the quasi-open set $\{u \neq 0\}$ is a solution of (1.5).

Proof. The proof is a straightforward consequence of the variational formula (4.6). □

Remark 5.2. It turns out that if u is a solution of (5.1), then $u \geq 0$ in D (see Lemma 5.8 below).

The rest of this section is dedicated to the regularity of the free boundary $\partial\Omega_u \cap D$ and of the whole boundary $\partial\Omega_u$ if D is smooth, of a solution u of (5.1), where we recall that, for any function $v \in H^1_0(D)$ we denote by Ω_v the (quasi-open) set $\{v > 0\}$.

The first regularity result for a free boundary problem formulated as the minimum of a variational functional is due to Alt and Caffarelli [2]. Nowaday, there is a well established regularity theory for the solutions to free boundary problems. Let us briefly reassume the main steps of the proof.

1. Regularity of the solution. Establish the continuity of the solution u and prove that u behaves as the distance function $\mathrm{dist}(\cdot, \partial\Omega_u)$ in a neighborhood to the boundary $\partial\Omega_u$.

2. Optimality condition. Prove that the solution u satisfies an elliptic equation in Ω_u and an optimality condition (in some weak sense) $|\nabla u| = Ce^{\Phi/2}$ on the free boundary $\partial\Omega_u \cap D$ and $|\nabla u| \geq Ce^{\Phi/2}$ on $\partial\Omega_u \cap \partial D$, where C is a constant.

3. Regularity of the free boundary. Prove that the boundary $\partial\Omega_u$ can be decomposed as a disjoint union of a regular part, $\mathrm{Reg}\,(\partial\Omega_u)$, and a (small) singular part $\mathrm{Sing}\,(\partial\Omega_u)$.

The first step is usually obtained by an appropriate construction of the competitor against which the optimality of u is tested. Now, in our case, this might appear as a difficult task since the constraints $\displaystyle\int_D e^{-\Phi}u^2\,dx = 1$ and especially $|\{u > 0\}| \leq m$ are quite restrictive. Similar technical obstruction was overcome by Aguilera, Alt and Caffarelli in [1], Briançon [7], and Briançon and Lamboley [6]. In order to overcome this difficulty, we essentially adopt the approach from [7] and [6] which allows to replace the measure contraint with a quasi-minimality condition at small scales (Subsection 5.5). We then obtain the Lipschitz continuity (Subsection 5.6) and the non-degeneracy (Subsection 5.7) of the solution. In Subsection 5.8 we prove that the blow-up limits of u are global solutions of the one-phase Alt-Caffarelli free boundary problem and in Subsection 5.10 we deduce the optimality condition (in viscosity sense) on the free boundary and we prove our main regularity result (Proposition 5.32).

5.1. **Boundedness of the eigenfunctions.** In this subsection we give a bound on the L^∞ norm of the eigenfunctions on generic bounded quasi-open sets. We first prove that if u is a solution of a PDE with sufficiently integrable right-hand side, then u is bounded. Then we use and iterate an interpolation argument to improve the integrability of the eigenfunctions.

Lemma 5.3. *Let $D \subset \mathbb{R}^d$ be a bounded open set, $\Omega \subset D$ be a quasi-open set and $\Phi \in W^{1,\infty}(D)$. Let $f \in L^p(D)$ for some $p \in (d/2, +\infty]$ and let $u \in H^1_0(\Omega)$ be the solution of*

$$-\operatorname{div}(e^{-\Phi}\nabla u) = f \quad \text{in} \quad \Omega, \qquad u \in H^1_0(\Omega). \tag{5.2}$$

Then, there is a dimensional constant C_d such that

$$\|u\|_{L^\infty} \leq \frac{C_d\, e^{\max\Phi}}{2/d - 1/p}|\Omega|^{2/d-1/p}\|f\|_{L^p},$$

where $\max\Phi = \|\Phi\|_{L^\infty(D)}$.

Proof. We first assume that f is a non-negative function. We notice that $u \geq 0$ on Ω and that u is a minimum in $H^1_0(\Omega)$ of the functional

$$J(u) := \frac{1}{2}\int_\Omega e^{-\Phi}|\nabla u|^2\,dx - \int_\Omega f u\,dx.$$

The rest of the proof follows precisely as in [35, Lemma 3.51]. For every $0 < t < \|u\|_{L^\infty}$ and $\varepsilon > 0$, we consider the test function $u_{t,\varepsilon} = u \wedge t + (u - t - \varepsilon)_+$. The inequality $J(u) \leq J(u_{t,\varepsilon})$

gives that

$$\frac{1}{2}\int_{\{t<u\le t+\varepsilon\}} e^{-\Phi}|\nabla u|^2\,dx \le \int_{\mathbb{R}^d} f\,(u-u_{t,\varepsilon})\,dx \le \varepsilon\int_{\{u>t\}} f\,dx \le \varepsilon\|f\|_{L^p}|\{u>t\}|^{\frac{p-1}{p}},$$

and, using the co-area formula and passing to the limit as $\varepsilon\to 0$, we get

$$\int_{\{u=t\}} |\nabla u|\,d\mathcal{H}^{d-1} \le 2e^{\max\Phi}\|f\|_{L^p}|\{u>t\}|^{\frac{p-1}{p}}. \tag{5.3}$$

Now, setting $\varphi(t) := |\{u>t\}|$ and using the co-area formula again as well as the Cauchy-Schwarz inequality, we obtain

$$\varphi'(t) = -\int_{\{u=t\}} \frac{1}{|\nabla u|}\,d\mathcal{H}^{d-1} \le -\left(\int_{\{u=t\}} |\nabla u|\,d\mathcal{H}^{d-1}\right)^{-1} Per(\{u>t\})^2,$$

which, together with the isoperimetric inequality $|\{u>t\}|^{\frac{d-1}{d}} \le C_d Per(\{u>t\})$ and (5.3), gives

$$\varphi'(t) \le -\frac{C_d}{e^{\max\Phi}\|f\|_{L^p}}\varphi(t)^{\frac{d-2}{d}+\frac{1}{p}}.$$

Setting $\alpha = \frac{d-2}{d}+\frac{1}{p} < 1$ and $C = C_d\|f\|_{L^p}^{-1}e^{-\max\Phi}$, we have $\varphi' \le -C\varphi^\alpha$. If $t_{\max} := \sup\{t>0;\ \varphi(s)>0 \text{ for all } s\in[0,t)\} \le +\infty$, then $\varphi'(t)\varphi(t)^{-\alpha} \le -C$ for all $t\in[0,t_{\max})$, so that $0\le\varphi(t)\le\big(|\Omega|^{1-\alpha}-(1-\alpha)Ct\big)^{\frac{1}{1-\alpha}}$ for all $t\in[0,t_{\max})$. This shows that $t_{\max}<+\infty$ and that

$$\|u\|_{L^\infty} \le t_{\max} \le \frac{1}{C}\frac{|\Omega|^{2/d-1/p}}{2/d-1/p},$$

which concludes the proof when f is non-negative. For a general function f, the proof now follows by applying the estimate in Lemma 5.3 to both the positive and the negative parts of f. □

Lemma 5.4 (Boundedness of the eigenfunctions)**.** *Let $D\subset\mathbb{R}^d$ be a bounded open set, $\Omega\subset D$ be a quasi-open set, $\Phi\in W^{1,\infty}(D)$ and $V=\nabla\Phi$. Let $R: L^2(\Omega)\to L^2(\Omega)$ be the resolvent operator of $-\Delta+V\cdot\nabla$ on Ω. Then, there are constants $n\in\mathbb{N}$, depending only on d, and $C\in\mathbb{R}$, depending on d, $|\Omega|$ and $\|\Phi\|_{L^\infty}$, such that*

$$R^n(L^2(\Omega))\subset L^\infty(\Omega) \qquad \text{and} \qquad \|R^n\|_{\mathcal{L}(L^2(\Omega);L^\infty(\Omega))}\le C.$$

In particular, if u is a first eigenfunction of $-\Delta+V\cdot\nabla$ on Ω normalized by $\|u\|_{L^2}=1$, then $u\in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty}\le C\lambda_1^n(\Omega,V).$$

Proof. Let us first notice that if $d\le 3$, then $d/2<2$ and so, taking $n=1$, the claim follows directly by Lemma 5.3. If $d>3$, then setting $2^*=\frac{2d}{d-2}$, we have

$$R: L^2(\Omega)\to L^{2^*}(\Omega) \qquad \text{and} \qquad R: L^d(\Omega)\to L^\infty(\Omega).$$

Thus, interpolating between 2 and d, we get

$$\|R\|_{\mathcal{L}(L^p;L^q)}\le C, \quad \text{where} \quad p\in[2,d] \quad \text{and} \quad q=\frac{pd}{d-p}\ge\frac{pd}{d-2}, \tag{5.4}$$

where C depends only on d, $|\Omega|$ and $\|\Phi\|_{L^\infty}$. Now, it is sufficient to notice that $R^k\in\mathcal{L}(L^2;L^{q_k})$, where $q_k=2\Big(\frac{d}{d-2}\Big)^k$. For k big enough we have that $q_k>d/2$ and so, $R^{k+1}\in\mathcal{L}(L^2;L^\infty)$, which proves the first part of the claim. Finally, in order to get the estimate on u, it is sufficient to notice that $R(u)=\lambda_1^{-1}(\Omega,V)u$ and $R^n(u)=\lambda_1^{-n}(\Omega,V)u$. □

5.2. **Pointwise definition of the solutions.** When we deal with Sobolev functions we usually reason up to a choice of certain representative of the function. Even if this representative is defined quasi-everywhere, there still might be a set of zero capacity where the function is not defined. Of course, this interferes with the notion of a free boundary in the sense that we cannot just consider the topological boundary of Ω_u without specifying the representative of u that we work with. Fortunately, the eigenfunctions of the quasi-open sets are defined pointwise everywhere, that is *every* point is a Lebesgue point.

Lemma 5.5. *Let $D \subset \mathbb{R}^d$ be a bounded open set, $\Omega \subset D$ a quasi-open set and $\Phi \in W^{1,\infty}(D)$ a given Lipschitz function. Let $f \in L^\infty(D)$ and $u \geq 0$ be a solution to the problem (5.2) in Ω_u.*

(1) Then, $\operatorname{div}(e^{-\Phi}\nabla u) + f \geq 0$ in D, in sense of distributions. In particular, $\operatorname{div}(e^{-\Phi}\nabla u)$ is a (signed) Radon measure on D.

(2) If $x_0 \in D$ is an arbitrary point in such that $B_r(x_0)$ is a ball included in D, then we can define the value of u at x_0 by

$$u(x_0) = \lim_{r\to 0} \fint_{\partial B_r(x_0)} u(x)\, d\mathcal{H}^{d-1}(x) = \lim_{r\to 0} \fint_{B_r(x_0)} u(x)\, dx.$$

Moreover, we have the identity

$$\fint_{\partial B_r(x_0)} u e^{-\Phi} d\mathcal{H}^{d-1} - u(x_0)e^{-\Phi(x_0)} = \frac{1}{d\omega_d}\int_0^r s^{1-d} \operatorname{div}(e^{-\Phi}\nabla u)(B_s(x_0))\, ds - \frac{1}{d\omega_d}\int_0^r s^{2-d}\, ds \int_{\partial B_s} u\nabla\Phi\cdot\nu e^{-\Phi}\, d\mathcal{H}^{d-1}. \quad (5.5)$$

Proof. (1) For $n \in \mathbb{N}$ define $p_n : \mathbb{R} \to \mathbb{R}$ by

$$p_n(s) = 0, \quad \text{for} \quad s \leq 0; \qquad p_n(s) = ns, \quad \text{for} \quad s \in [0, 1/n]; \qquad p_n(s) = 1, \quad \text{for} \quad s \geq 1/n.$$

Since p_n is Lipschitz continuous, we have $p_n(u) \in H^1_0(\Omega)$ and $\nabla p_n(u) = p_n'(u)\nabla u$. Let $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ in D. Using $\varphi p_n(u)$ as a test function in (5.2), we get

$$\int_D p_n(u)\nabla u\cdot\nabla\varphi\, e^{-\Phi} dx \leq \int_D \big(p_n(u)\nabla u\cdot\nabla\varphi + \varphi p_n'(u)|\nabla u|^2\big)e^{-\Phi} dx = \int_D f\varphi p_n(u)\, dx.$$

which, letting $n \to \infty$, gives the first claim.

In order to prove *(2)*, we suppose that $x_0 = 0$ and we calculate

$$\begin{aligned}
\frac{d}{ds}\fint_{\partial B_s} u e^{-\Phi} d\mathcal{H}^{d-1} &= \frac{d}{ds}\fint_{\partial B_1} u(s\xi)e^{-\Phi(s\xi)} d\mathcal{H}^{d-1} \\
&= \fint_{\partial B_1} \big[\xi\cdot\nabla u(s\xi) - u(s\xi)\xi\cdot\nabla\Phi(s\xi)\big] e^{-\Phi(s\xi)} d\mathcal{H}^{d-1} \\
&= \frac{s^{1-d}}{d\omega_d}\operatorname{div}(e^{-\Phi}\nabla u)(B_s) - \frac{s^{2-d}}{d\omega_d}\int_{\partial B_s} u\nabla\Phi\cdot\nu e^{-\Phi}\, d\mathcal{H}^{d-1}.
\end{aligned}$$

Then, integrating from ρ to r ($\rho < r$), using the inequality from *(1)* and the fact that $u \in L^\infty(D)$ by Lemma 5.4, we get

$$\begin{aligned}
\fint_{\partial B_r} u e^{-\Phi} d\mathcal{H}^{d-1} - \fint_{\partial B_\rho} u e^{-\Phi} d\mathcal{H}^{d-1} &= \frac{1}{d\omega_d}\int_\rho^r s^{1-d}\operatorname{div}(e^{-\Phi}\nabla u)(B_s(x_0))\, ds \quad (5.6)\\
&\quad - \frac{1}{d\omega_d}\int_\rho^r s^{2-d}\, ds \int_{\partial B_s} u\nabla\Phi\cdot\nu e^{-\Phi}\, d\mathcal{H}^{d-1} \\
&\geq -\Big(\frac{1}{2d}\|f\|_{L^\infty} + \frac{1}{2}\|u\|_{L^\infty}\,\|\nabla\Phi\|_{L^\infty} e^{-\min\Phi}\Big)(\rho^2 - r^2) \\
&:= -C(\rho^2 - r^2)
\end{aligned}$$

where $C > 0$. This shows that the function $r \mapsto \fint_{\partial B_r} ue^{-\Phi} d\mathcal{H}^{d-1} - Cr^2$ is non-decreasing. In particular, the limit $\ell(x_0) = \lim_{r\to 0} \fint_{\partial B_r(x_0)} ue^{-\Phi} d\mathcal{H}^{d-1}$ exists and we set $u(x_0) := e^{\Phi(x_0)}\ell(x_0)$. Finally, (5.5) follows by letting $\rho \to 0$ in (5.6). □

As a direct consequence of Lemma 5.5 and (5.5) we get the following strong maximum principle.

Lemma 5.6 (Strong maximum principle)**.** *Let $D \subset \mathbb{R}^d$ be an open connected set and $u \in H^1_0(D)$ satisfy $u \geq 0$. Assume that $\operatorname{div}(e^{-\Phi}\nabla u) \in L^\infty(D)$ satisfies $\operatorname{div}(e^{-\Phi}\nabla u) \leq 0$. Then, if u is not identically vanishing in D, then u is strictly positive in D.*

Proof. Set $A := \{x_0 \in D;\ u(x_0) = 0\}$. If $x_0 \in A$, then (5.5) implies that $u(x) = 0$ for almost every $x \in B_r(x_0)$ whenever $B_r(x_0) \subset D$. But, if $x \in B_r(x_0)$, since x is a Lebesgue point for u, $u(x) = 0$. Thus, A is open.
Consider now a sequence $(x_n)_{n\geq 1} \in A$ converging to $x_0 \in D$. For some n large enough, there exists a ball $B_r(x_n) \subset D$ containing x_0. Since u vanishes everywhere in $B_r(x_n)$, $u(x_0) = 0$, which proves that A is closed in D. We conclude by the connectedness of D. □

A consequence of Lemma 5.5 is the fact that the set $\Omega_u = \{u > 0\}$ and the (topological) free boundary $\partial\Omega_u \cap D$ are well defined. Below we prove that the topological boundary coincides with the measure theoretic one.

Lemma 5.7. *Let $u \in H^1_0(D)$, $u \geq 0$ in D, be a solution of (5.1), $x_0 \in \partial\Omega_u$ and let $r > 0$ be such that $D_r(x_0) := B_r(x_0) \cap D$ is connected. Then we have $0 < |\Omega_u \cap B_r(x_0)|$. Moreover, if $x_0 \in \partial\Omega_u \cap D$, we have $|\Omega_u \cap B_r(x_0)| < |D_r(x_0)|$.*

Proof. The first inequality comes from the fact that every point is a Lebesgue point for u. To show the second one, we argue by contradiction and assume that $|\Omega_u \cap B_r(x_0)| = |D_r(x_0)|$ for some $r > 0$. We claim that u is a solution of

$$-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda_m u e^{-\Phi} \quad \text{in} \quad D_r(x_0), \qquad \text{where} \quad \lambda_m := \int_D |\nabla u|^2 e^{-\Phi}\, dx.$$

Indeed, let v be the solution of

$$-\operatorname{div}(e^{-\Phi}\nabla v) = \lambda_m u e^{-\Phi} \quad \text{in} \quad D_r(x_0), \qquad v = u \quad \text{in} \quad D\backslash B_r(x_0).$$

Then Lemma 5.6 implies that $v > 0$ in $D_r(x_0)$. Since $|\Omega_v| = |\Omega_u|$, the optimality of u gives

$$\frac{\int_D |\nabla v|^2 e^{-\Phi}\, dx}{\int_D v^2 e^{-\Phi}\, dx} \geq \int_D |\nabla u|^2 e^{-\Phi}\, dx = \frac{\int_D |\nabla u|^2 e^{-\Phi}\, dx}{\int_D v^2 e^{-\Phi}\, dx} + \lambda_m \left(1 - \frac{\int_D u^2 e^{-\Phi}\, dx}{\int_D v^2 e^{-\Phi}\, dx}\right),$$

which implies

$$0 \geq \int_D \left(|\nabla u|^2 - |\nabla v|^2\right) e^{-\Phi} dx + \lambda_m \int_D \left(v^2 - u^2\right) e^{-\Phi} dx = \int_D \left(|\nabla(u-v)|^2 + \lambda_m (u-v)^2\right) e^{-\Phi} dx,$$

where the last equality follows by the definition of v and the fact that $v - u \in H^1_0(D_r(x_0))$. This implies that $u = v$ almost everywhere and hence, by Lemma 5.5, that $u = v$ everywhere. Therefore, we have $u > 0$ in $B_r(x_0)$, which is in contradiction with $x_0 \in \partial\Omega_u \cap D$. □

Lemma 5.8 (Saturation of the constraint)**.** *Let $D \subset \mathbb{R}^d$ be an open connected set, $\Phi \in W^{1,\infty}(D)$, m and τ be as in Lemma 5.1. Then every solution u of (5.1) is such that $u \geq 0$ on D and $|\Omega_u| = m$ (up to a change of sign). In particular, every solution Ω of (1.5) is such that $|\Omega| = m$.*

Proof. Let u be a solution of (5.1) and set

$$u_1 = \frac{u_+}{\left(\int_D u_+^2 e^{-\Phi}\right)^{1/2}} \text{ and } u_2 = \frac{u_-}{\left(\int_D u_-^2 e^{-\Phi}\right)^{1/2}}.$$

We first prove that either u_1 or u_2 is a solution of (5.1). It is obvious if $u = u_+$ or $u = u_-$. Otherwise, we have $u_+ \neq 0$ and $u_- \neq 0$, and the claim follows from the estimate

$$\inf\left(\frac{\int_D |\nabla u_+|^2 e^{-\Phi}dx}{\int_D u_+^2 e^{-\Phi}dx}, \frac{\int_D |\nabla u_-|^2 e^{-\Phi}dx}{\int_D u_-^2 e^{-\Phi}dx}\right) \leq \frac{\int_D \left(|\nabla u_+|^2 + |\nabla u_-|^2\right) e^{-\Phi}dx}{\int_D \left(u_+^2 + u_-^2\right) e^{-\Phi}dx} = \frac{\int_D |\nabla u|^2 e^{-\Phi}dx}{\int_D u^2 e^{-\Phi}dx}.$$

Up to changing u into $-u$, we assume that u_1 is a solution of (5.1). Now, suppose by contradiction that $|\Omega_u| < m$. Then, for every ball $B_r(x_0) \subset D$ such that $|\Omega_u| + |B_r| \leq m$, u_1 is a solution of

$$-\operatorname{div}(e^{-\Phi}\nabla u_1) = \lambda_m e^{-\Phi} u_1 \quad \text{in} \quad B_r(x_0).$$

By the strong maximum principle, we get $u > 0$ in $B_r(x_0)$, which is a contradiction. This proves both the saturation of the constraint and the positivity of u. □

5.3. **A free-boundary problem with measure constraint.** We now follow the strategy adopted in [6, 7]. In particular, the proof of Theorem 5.16 below is very close to the one of Theorem 1.5 in [6]. Note that the approach is local and that a result analogous to Theorem 5.16 with perturbations in D is vain (see Remark 1.6 in [6]).

Let $u \in H_0^1(D)$ be a solution of (5.1) and let $\lambda_m = \displaystyle\int_D |\nabla u|^2 e^{-\Phi}\,dx$. For any $v \in H_0^1(D)$ we set

$$J(v) := \int_D |\nabla v|^2 e^{-\Phi}dx - \lambda_m \int_D v^2 e^{-\Phi}dx. \tag{5.7}$$

Remark 5.9. It is plain to see that, when $u \in H_0^1(D)$ is a solution of (5.1),

$$J(u) = \min\left\{J(v) \ : \ v \in H_0^1(D),\ |\Omega_v| \leq m\right\}. \tag{5.8}$$

For a ball $B_r(x_0) \subset \mathbb{R}^d$ we define the admissible set

$$\mathcal{A}(u, x_0, r) := \left\{v \in H_0^1(D) \ : \ u - v \in H_0^1(B_r(x_0))\right\}.$$

Remark 5.10 (Coercivity of J)**.** We notice that the set $\{v \in \mathcal{A}(u, x_0, r) \ : \ J(v) < C\}$ is weakly compact in $H_0^1(D)$. Precisely, if $u \in H_0^1(D)$, $\Phi \in W^{1,\infty}(D)$ and J be given by (5.7), then there is a constant $r_0 > 0$, depending on d, Φ, λ_m and D such that for all $r \leq r_0$,

$$\int_{B_r(x_0)} |\nabla v|^2\,dx \leq 2e^{\max\Phi} J(v) + (1 + 4\lambda_m e^{\max\Phi - \min\Phi})\,\|u\|^2_{H^1(D)}, \quad \forall v \in \mathcal{A}(u, x_0, r). \tag{5.9}$$

Indeed, let $v \in \mathcal{A}(u, x_0, r)$ with $r \leq r_0$. We have

$$\begin{aligned}\int_D v^2\,dx &\leq 2\int_D (v-u)^2\,dx + 2\int_D u^2\,dx \leq \frac{2}{\lambda_1(B_r(x_0))}\int |\nabla(v-u)|^2 dx + 2\int_D u^2 dx \\ &\leq \frac{4r_0^2}{\lambda_1(B_1)}\int_D (|\nabla v|^2 + |\nabla u|^2)\,dx + 2\int_D u^2\,dx,\end{aligned}$$

where the last inequality is due to the (-2)-homogeneity of $\lambda_1(B_r)$ and the fact that $r \leq r_0$. Choosing r_0 small enough (depending only on d, λ_m, $\|\nabla\Phi\|_{L^\infty}$ and the diameter of D) we get

$$\begin{aligned}\int_{B_r(x_0)} |\nabla v|^2\,dx &\leq e^{\max\Phi} J(v) + \lambda_m e^{\max\Phi - \min\Phi}\int_D v^2\,dx \\ &\leq e^{\max\Phi} J(v) + \frac{1}{2}\int_{B_r(x_0)} (|\nabla v|^2 + |\nabla u|^2)\,dx + 2\lambda_m e^{\max\Phi - \min\Phi}\int_D u^2\,dx.\end{aligned}$$

which concludes the proof of (5.9).

As a consequence, we obtain the following result, which gives us the existence of a solution to a local version of the minimization problem (5.8) with some different measure constraint.

Lemma 5.11. *Let $u \in H_0^1(D)$ be a solution of the problem (5.8). Let $B_r(x_0) \subset \mathbb{R}^d$ be a ball and let $\widetilde{m} > |\Omega_u \setminus D_r(x_0)|$. Then:*

(1) the problem

$$\min\Big\{J(v)\ :\ v\in\mathcal{A}(u,x_0,r),\ |\Omega_v|\le\widetilde{m}\Big\} \tag{5.10}$$

has a solution,
(2) If $D_r(x_0):=B_r(x_0)\cap\Omega_u$ is connected and $|\Omega_u\cup D_r(x_0)|>\tilde{m}$, then $|\Omega_v|=\widetilde{m}$,
(3) there exists $r_0>0$ such that, for every $r<r_0$, every solution v of (5.10) is non-negative.

Proof. For 1, it is enough to notice that, by Remark 5.10, J is bounded from below in $\mathcal{A}(u,x_0,r)$. Then, if $(v_n)_{n\ge1}$ is a minimizing sequence for (5.10), by (5.9) v_n is bounded in H^1 and so a minimizer exists by the semicontinuity of J (notice that, up to a subsequence, there exists $v\in L^2$ such that $v_n\to v$ strongly in L^2 and almost everywhere, so that $\mathbb{1}_{\Omega_v}\le\underline{\lim}\,\mathbb{1}_{\Omega_{v_n}}$).
For 2, if $D_r(x_0)$ is connected and $|\Omega_u\cup D_r(x_0)|>\tilde{m}$, we argue as in the proof of Lemma 5.8 to conclude that $|\Omega_v|=\widetilde{m}$. For 3, let v be a solution of (5.10). Then, by the optimality of v and the fact that $v^+\in\mathcal{A}(u,x_0,r)$ and $\Omega_{v^+}\subset\Omega_v$, one has

$$J(v^+)+J(v^-)=J(v)\le J(v^+),$$

which means that $J(v^-)\le0$. Therefore,

$$\begin{aligned}\int_{B_r(x_0)}|\nabla v^-|^2e^{-\Phi}\,dx&\le\lambda_m\int_{B_r(x_0)}|v^-|^2e^{-\Phi}\,dx\\&\le\lambda_me^{2r\tau}C_dr^2\int_{B_r(x_0)}|\nabla v^-|^2e^{-\Phi}\,dx,\end{aligned}$$

where the second inequality is due to the fact that $\max_{B_r(x_0)}\Phi-\min_{B_r(x_0)}\Phi\le2r\tau$ and the variational characterization and the scaling of $\lambda_1(B_r,0)=C_dr^{-2}$. Thus, for r small enough ($r\le r_0$ with r_0 depending only on τ, λ_m and d), $v^-=0$.

□

5.4. **An internal variation optimality condition.** Let $D\subset\mathbb{R}^d$ be a bounded open set, $u\in H^1_0(D)$ and $\xi\in C^\infty_c(D;\mathbb{R}^d)$. The first variation $\delta J(u)[\xi]$, of J at u in the direction ξ, is given by

$$\delta J(u)[\xi]:=\lim_{t\to0}\frac{J(u_t)-J(u)}{t},\quad\text{where}\quad u_t(x):=u(x+t\xi(x)).$$

A straightforward computation gives that

$$\delta J(u)[\xi]:=\int_D\Big[2D\xi(\nabla u)\cdot\nabla u+(|\nabla u|^2-\lambda_mu^2)(\nabla\Phi\cdot\xi-\operatorname{div}\xi)\Big]e^{-\Phi}\,dx. \tag{5.11}$$

We prove in Proposition 5.12 the existence of an Euler-Lagrange multiplier for every solution u of (5.8). This, using a local internal variation of the boundary of the optimal set Ω_u, we derive an optimal boundary condition for u (see Lemma 5.29).

Proposition 5.12 (Euler-Lagrange equation)**.** *Let u be a solution of (5.8). Then, there exists $\Lambda_u>0$ such that*

$$\delta J(u)[\xi]=\Lambda_u\int_{\Omega_u}\operatorname{div}\xi\,dx\qquad\text{for every}\qquad\xi\in C^\infty_c(D;\mathbb{R}^d). \tag{5.12}$$

Moreover, for every $x_0\in\partial\Omega_u\cap\partial D$ and every $r>0$, we have

$$\delta J(u)[\xi]\ge\Lambda_u\int_{\Omega_u}\operatorname{div}\xi\,dx,$$

for every $\xi\in C^\infty_c(B_r(x_0),\mathbb{R}^d)$ such that $(Id+\xi)^{-1}(D_r(x_0))\subset D_r(x_0)$.

Proof. Let $\xi\in C^\infty_c(D;\mathbb{R}^d)$ and $u_t(x)=u(x+t\xi(x))$. Then we have

$$|\Omega_{u_t}|=|\Omega_u|-t\int_{\Omega_u}\operatorname{div}\xi\,dx+o(t). \tag{5.13}$$

Step 1. We first notice that if $B_r(x_0) \subset \mathbb{R}^d$ is a ball such that

$$D_r(x_0) := B_r(x_0) \cap D \quad \text{is connected} \qquad \text{and} \qquad 0 < |D_r(x_0) \cap \Omega_u| < |D_r(x_0)|,$$

then there is a vector field $\xi_0 \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ such that $\displaystyle\int_{\Omega_u} \operatorname{div} \xi_0 \, dx = 1$. Indeed, if this is not the case, then we have

$$\int_{\Omega_u} \operatorname{div} \xi \, dx = 0 \qquad \text{for every} \qquad \xi \in C_c^\infty(D_r(x_0); \mathbb{R}^d).$$

For every ball $B_\rho(x_1) \subset D_r(x_0)$, take a vector field of the form $\xi(x) = (x - x_1)\phi_\varepsilon(x)$ with $0 \le \phi_\varepsilon \le 1$ on $B_\rho(x_1)$, ϕ radially decreasing in $B_\rho(x_1)$ with $|\nabla \phi_\varepsilon| \le C(\rho\varepsilon)^{-1}$, $\phi_\varepsilon = 1$ on $B_{\rho(1-\varepsilon)}(x_1)$ and $\phi_\varepsilon = 0$ on $\partial B_\rho(x_1)$. Then we have $\displaystyle\int_{\Omega_u} \big(d\phi_\varepsilon(x) + (x - x_1) \cdot \nabla \phi_\varepsilon(x)\big)\, dx = 0$ and, passing to the limit as $\varepsilon \to 0$, we get

$$d|\Omega_u \cap B_\rho(x_1)| - \rho\, \mathcal{H}^{d-1}(\Omega_u \cap \partial B_\rho(x_1)) = 0.$$

In particular, we get that the map $\rho \mapsto \rho^{-d}|\Omega_u \cap B_\rho(x_1)|$ is constant. Since the above identity holds for all balls $B_\rho(x_1) \subset D_r(x_0)$, we get that $|\Omega_u \cap D_r(x_0)| = 0$ or $|\Omega_u \cap D_r(x_0)| = |D_r(x_0)|$, which concludes the proof of the claim.

Step 2. We now prove the first statement of the proposition. Let $\xi_0 \in C_c^\infty(D; \mathbb{R}^d)$ be as in Step 1 and $\xi \in C_c^\infty(D; \mathbb{R}^d)$. There are two cases:

If $\displaystyle\int_{\Omega_u} \operatorname{div} \xi \, dx = 0$, define $\xi_1 = \xi + \eta\xi_0$ with $\eta > 0$ so that $\displaystyle\int_{\Omega_u} \operatorname{div} \xi_1 \, dx = \eta$. Set $u_t(x) = u(x + t\xi_1(x))$. Then, for t small enough, $u_t \in H_0^1(D)$ and $|\Omega_{u_t}| \le |\Omega_u| = m$ and

$$J(u_t) = J(u) + t\, \delta J(u)[\xi_1] + o(t).$$

By the minimality of u we have $J(u) \le J(u_t)$ and so, $\delta J(u)[\xi_1] \ge 0$. Therefore,

$$\delta J(u)[\xi] \ge -\eta\, \delta J(u)[\xi_0] \quad \text{for every} \quad \eta > 0,$$

and hence, we get $\delta J(u)[\xi] \ge 0$. Taking $-\xi$ instead of ξ we have that $\delta J(u)[\xi] = 0$, and hence (5.12) holds for any $\Lambda_u \ge 0$.

If $\displaystyle\int_{\Omega_u} \operatorname{div} \xi \, dx \neq 0$, define $\xi_2 := \xi - \xi_0 \int_{\Omega_u} \operatorname{div} \xi \, dx$. Then $\displaystyle\int_{\Omega_u} \operatorname{div} \xi_2 \, dx = 0$ and, by the preceding case, we have $\delta J(u)[\xi_2] = 0$. On the other hand, $\delta J(u)[\xi_2] = \delta J(u)[\xi] - \delta J(u)[\xi_0] \displaystyle\int_{\Omega_u} \operatorname{div} \xi \, dx$, which proves (5.12) with $\Lambda_u := \delta J(u)[\xi_0]$. Moreover, for t small enough, $u_t(x) = u(x + t\xi(x)) \in H_0^1(D)$ and, by the minimality of u, we have

$$J(u) \le J(u_t) = J(u) + t\Lambda_u + o(t),$$

which proves that $\Lambda_u \ge 0$. The strict inequality follows by a general result (Proposition A.1) for minimizers of J with respect to internal perturbations.

Step 3. Let $x_0 \in \partial\Omega_u \cap \partial D$, $r > 0$ and $\xi_0 \in C_c^\infty(D; \mathbb{R}^d)$ be as in Step 1 so that we have $\delta J(u)[\xi_0] = \Lambda_u$. For any $\xi \in C_c^\infty(B_r(x_0), \mathbb{R}^d)$ such that $(Id + \xi)^{-1}(D_r(x_0)) \subset D_r(x_0)$, we set $\xi_1 = \xi - (1 - \eta)\xi_0 \int_{\Omega_u} \operatorname{div} \xi \, dx$ where η is some positive constant. Note that the vector field ξ_1 is such that $u_t(x) = u(x + t\xi_1(x)) \in H_0^1(D)$ for small $t > 0$ and $\int_{\Omega_u} \operatorname{div} \xi_1 \, dx = \eta > 0$. Therefore, using the minimality of u, we have for every $t > 0$ small enough

$$J(u) \le J(u_t) = J(u) + t\delta J(u)[\xi_1] + o(t),$$

so that we get $\delta J(u)[\xi_1] \ge 0$. It follows that $\delta J(u)[\xi] \ge (1 - \eta)\Lambda_u$ for every $\eta > 0$, which concludes the proof. □

In the following lemma we show that the Lagrange multipliers, associated to the solutions of variational problems with measure constraint in a fixed ball $B_r(x_0)$, are continuous with respect to variations of the measure constraint around m. This lemma will be used several times in the proof of the optimality of the blow-up limits.

Lemma 5.13 (Convergence of the Lagrange multipliers)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set, $u \in H^1_0(D)$ be a solution of (5.8) and Λ_u be the constant from (5.12). Let $B_r(x_0) \subset \mathbb{R}^d$ be a ball such that*

$$D_r(x_0) := B_r(x_0) \cap D \quad \text{is connected} \qquad \text{and} \qquad 0 < |D_r(x_0) \cap \Omega_u| < |D_r(x_0)|.$$

Let the sequence $(m_n)_{n\geq 1}$ be such that $\lim_{n\to\infty} m_n = m$. Then, for n big enough, there is a solution $u_n \in \mathcal{A}(u, x_0, r)$ of the problem

$$\min\Big\{J(v) \,:\, v \in \mathcal{A}(u, x_0, r),\ |\Omega_v| \leq m_n\Big\}. \tag{5.14}$$

Moreover, up to a subsequence, we have:

(a) for every n there is a Lagrange multiplier $\Lambda_{u_n} > 0$ for which (5.12) holds for u_n in $D_r(x_0)$;
(b) for every n there is a vector field $\xi_n \in C^\infty_c(D_r(x_0); \mathbb{R}^d)$ such that

$$\frac{d}{dt}\Big|_{t=0} J(u_n^t) = \Lambda_{u_n} \qquad \text{and} \qquad \frac{d}{dt}\Big|_{t=0} |\Omega_{u_n^t}| = -1 \qquad \text{where} \qquad u_n^t(x) := u_n(x + t\xi_n(x)); \tag{5.15}$$

(c) u_n converges strongly in $H^1_0(D)$ and pointwise almost everywhere to a function $u_\infty \in \mathcal{A}(u, x_0, r)$ which is a solution of (5.10);
(d) the sequence of characteristic functions $\mathbb{1}_{\Omega_{u_n}}$ converges to $\mathbb{1}_{\Omega_{u_\infty}}$ pointwise almost everywhere and strongly in $L^2(D)$;
(e) if we have $0 < |\Omega_u \setminus B_r(x_0)| < |D \setminus B_r(x_0)|$, then $\lim_{n\to\infty} \Lambda_{u_n} = \Lambda_u$.

Furthermore, if D is of class $C^{1,1}$ and $m_n < m$ for every n large enough, then all these properties still hold even if the assumption $|\Omega_u \cap D_r(x_0)| < |D_r(x_0)|$ is not satisfied.

Proof. First of all, we notice that since $|\Omega_u \setminus D_r(x_0)| < m < |\Omega_u \cup D_r(x_0)|$, we may assume that the same holds for every m_n, for n large enough. Thus, by Lemma 5.11, the problem (5.14) has a solution u_n such that $|\Omega_{u_n}| = m_n$. Then, it follows that u_n satisfies

$$0 < |\Omega_{u_n} \cap D_r(x_0)| < |D_r(x_0)|. \tag{5.16}$$

Therefore, by step 1 in the proof of Proposition 5.12, there exists a vector field $\xi_n \in C^\infty_c(D_r(x_0); \mathbb{R}^d)$ such that $\displaystyle\int_{\Omega_{u_n}} \operatorname{div} \xi_n \, dx = 1$, and, reasoning as in Proposition 5.12, there exists $\Lambda_{u_n} > 0$ such that

$$\delta J(u_n)[\xi] = \Lambda_{u_n} \int_{\Omega_{u_n}} \operatorname{div} \xi \, dx \qquad \text{for every} \qquad \xi \in C^\infty_0(D_r(x_0), \mathbb{R}^d). \tag{5.17}$$

Moreover, taking $u_n^t(x) = u_n(x + t\xi_n(x))$, we obtain (5.15). This proves (a) and (b). We notice that the only difference with Proposition 5.12 is that in the present case, u_n is only a solution of a variational problem in $B_r(x_0)$.

Let now n be fixed and $\xi_0 \in C^\infty_c(B_r(x_0); \mathbb{R}^d)$ be the vector field, from the proof of Proposition 5.12, associated to u. Then, taking $u_t(x) := u(x + t\xi_0(x))$, we have that

$$\frac{d}{dt}\Big|_{t=0} |\Omega_{u_t}| = -\int_{\Omega_u} \operatorname{div} \xi_0 \, dx = -1,$$

and so, for n large enough, there is a unique $t_n \in \mathbb{R}$ such that $|\Omega_{u_n}| = m_n = |\Omega_{u_{t_n}}|$. In particular, there are constants C and n_0, depending on u and ξ_0, but not on n, such that

$$J(u_n) \leq J(u_{t_n}) \leq C \quad \text{for every} \quad n \geq n_0.$$

Then, by Remark 5.10, $(u_n)_{n\geq 1}$ is uniformly bounded in $H^1_0(D)$, so up to a subsequence, u_n converges weakly in H^1, strongly in L^2 and pointwise a.e. to a function $u_\infty \in \mathcal{A}(u, x_0, r)$. Now, since the pointwise convergence implies $\mathbb{1}_{\Omega_{u_\infty}} \leq \liminf \mathbb{1}_{\Omega_{u_n}}$, we get that $|\Omega_{u_\infty}| \leq \liminf m_n = m$. In particular, $J(u) \leq J(u_\infty)$. On the other hand, the weak H^1 convergence of u_n gives that

$$J(u_\infty) \leq \liminf_{n\to\infty} J(u_n) \leq \liminf_{n\to\infty} J(u_{t_n}) = J(u),$$

so, we get $J(u_\infty) = J(u)$, u_∞ is a solution of (5.10), $|\Omega_{u_\infty}| = m$ (by the saturation of the constraint). Moreover, $J(u_n) \to J(u_\infty)$ since we have

$$\limsup_{n\to\infty} J(u_n) \le \limsup_{n\to\infty} J(u_{t_n}) = J(u) \le J(u_\infty) \le \liminf_{n\to\infty} J(u_n).$$

But u_n strongly converges in $L^2(D)$ to u_∞ so that it gives $\int_D e^{-\Phi}|\nabla u_n|^2\,dx \to \int_D e^{-\Phi}|\nabla u_\infty|^2\,dx$, which means that the convergence of u_n to u is strong in $H^1_0(D)$.
We now check that the convergence of $\mathbb{1}_{\Omega_{u_n}}$ to $\mathbb{1}_{\Omega_{u_\infty}}$ is strong in L^2. Indeed, for all non-negative function $\varphi \in L^2(D)$, the Fatou lemma shows that

$$\int_D \mathbb{1}_{\Omega_{u_\infty}}\varphi \le \int_D \underline{\lim}\, \mathbb{1}_{\Omega_{u_n}}\varphi \le \underline{\lim} \int_D \mathbb{1}_{\Omega_{u_n}}\varphi. \tag{5.18}$$

Â Up to a subsequence, there exists $h \in L^2(D)$ such that $\mathbb{1}_{\Omega_{u_n}} \rightharpoonup h$ weakly in $L^2(D)$. Thus, (5.18) yields $\mathbb{1}_{\Omega_{u_\infty}} \le h$. Moreover, $\|h\|_2 \le \underline{\lim}\, \big\|\mathbb{1}_{\Omega_{u_\infty}}\big\|_2$. As a consequence, $\|h\|_2 = m^{1/2}$, which entails that $\mathbb{1}_{\Omega_{u_n}} \to h$ strongly in $L^2(D)$. Since $\mathbb{1}_{\Omega_{u_\infty}} \le h$, we conclude that $\mathbb{1}_{\Omega_{u_n}} \to \mathbb{1}_{\Omega_{u_\infty}}$ strongly in $L^2(D)$, and so, up to a subsequence $\mathbb{1}_{\Omega_{u_n}}$ converges to $\mathbb{1}_{\Omega_{u_\infty}}$ pointwise almost everywhere. This proves (c) and (d).

In order to prove (e), we first notice that u and u_∞ are both solutions of (5.8) since $J(u_\infty) = J(u)$. Therefore, there is a Lagrange multiplier Λ_∞ such that

$$\delta J(u_\infty)[\xi] = \Lambda_\infty \int_{\Omega_{u_\infty}} \operatorname{div}\xi\,dx \qquad \text{for every} \qquad \xi \in C^\infty_c(D;\mathbb{R}^d), \tag{5.19}$$

Moreover, by (c) and (d), we get that

$$\delta J(u_\infty)[\xi] = \lim_{n\to\infty} \delta J(u_n)[\xi] \qquad \text{and} \qquad \int_{\Omega_{u_\infty}} \operatorname{div}\xi\,dx = \lim_{n\to\infty}\int_{\Omega_{u_n}} \operatorname{div}\xi\,dx,$$

for every $\xi \in C^\infty_c(D_r(x_0);\mathbb{R}^d)$. Now, choosing $\xi \in C^\infty_c(D_r(x_0);\mathbb{R}^d)$ such that $\displaystyle\int_{\Omega_{u_\infty}} \operatorname{div}\xi\,dx \neq 0$ and using (5.19) and (5.17) we get that Λ_{u_n} converges to Λ_∞. Finally, if we have $0 < |\Omega_u \setminus B_r(x_0)| < |D \setminus B_r(x_0)|$, there exists $\xi \in C^\infty_c(D \setminus B_r(x_0);\mathbb{R}^d)$ such that $\displaystyle\int_{\Omega_{u_\infty}} \operatorname{div}\xi\,dx \neq 0$, so that $\Lambda_\infty = \Lambda_u$ since $u = u_\infty$ outside the ball $B_r(x_0)$.

The proof of the last statement of the Proposition is very similar. We have $|\Omega_u \setminus D_r(x_0)| < m = |\Omega_u \cup D_r(x_0)|$ so that, since $m_n < m$, we have $|\Omega_u \setminus D_r(x_0)| < m_n < |\Omega_u \cup D_r(x_0)|$ for every n large enough. It follows from Lemma 5.11 that the problem (5.14) has a solution u_n with $|\Omega_{u_n}| = m_n$ and such that (5.16) holds. Note also that there exists a vector field $\xi_0 \in C^\infty_0(B_r(x_0),\mathbb{R}^d)$ such that $(Id + t\xi_0)^{-1}(D_r(x_0)) \subset D_r(x_0)$ for every small $t > 0$ and $\int_{\Omega_u} \operatorname{div}\xi_0\,dx = 1$ (consider a smooth extension of the normal to the boundary of D on $\partial D \cap B_{r/2}(x_0)$). Moreover, we have $t_n > 0$ (since $m_n < m$) and hence $u_{t_n} \in H^1_0(D)$. The remainder of the proof is unchanged. □

5.5. **Almost optimality of u at small scales.** Let u be a solution of (5.1) in $D \subset \mathbb{R}^d$. For $x_0 \in \mathbb{R}^d$ and $h > 0$, we define the upper and the lower Lagrange multipliers, $\mu_-(h,x_0,r) \ge 0$ and $\mu_+(h,x_0,r) \ge 0$, by

$$\mu_+(h,x_0,r) = \inf\{\mu \ge 0 \ :\ J(u) + \mu|\Omega_u| \le J(v) + \mu|\Omega_v|,\ \forall v \in \mathcal{A}(u,x_0,r),\ m \le |\Omega_v| \le m+h\},$$
$$\mu_-(h,x_0,r) = \sup\{\mu \ge 0 \ :\ J(u) + \mu|\Omega_u| \le J(v) + \mu|\Omega_v|,\ \forall v \in \mathcal{A}(u,x_0,r),\ m-h \le |\Omega_v| \le m\}.$$

Remark 5.14. We notice that if $B_r(x_0) \subset \mathbb{R}^d$ is a ball such that $D_r(x_0) := D \cap B_r(x_0)$ is connected and $0 < |D_r(x_0) \cap \Omega_u| < |D_r(x_0)|$, then

$$\mu_-(h,x_0,r) \le \Lambda_u \le \mu_+(h,x_0,r) \qquad \text{for every} \qquad h > 0.$$

Indeed, by Step 1 of the proof of Proposition 5.12, there is a vector field $\xi \in C^\infty_c(D_r(x_0);\mathbb{R}^d)$ such that $\displaystyle\int_{\Omega_u} \operatorname{div}\xi\,dx = 1$. Let $u_t(x) = u(x + t\xi(x))$. Then for $|t|$ small enough $u_t \in \mathcal{A}(u,x_0,r)$

and $m-h<|\Omega_{u_t}|<m+h$. Moreover, for every $\mu\geq 0$ we have

$$J(u_t)+\mu|\Omega_{u_t}|=J(u)+t\Lambda_u+\mu(|\Omega_u|-t)+o(t). \tag{5.20}$$

Now, if $t>0$ is small enough and $\Lambda_u<\mu$, then $m>|\Omega_{u_t}|$ and, by (5.20), $J(u_t)+\mu|\Omega_{u_t}|<J(u)+\mu|\Omega_u|$, which proves that $\Lambda_u\geq\mu_-(h,x_0,r)$. Analogously, if $t<0$ and $\Lambda_u>\mu$, then $m<|\Omega_{u_t}|$ and again $J(u_t)+\mu|\Omega_{u_t}|<J(u)+\mu|\Omega_u|$, which gives that $\Lambda_u\leq\mu_+(h,x_0,r)$.

Remark 5.15 (Monotonicity of μ_+ and μ_-)**.** We notice that the following inclusion holds:

$$\mathcal{A}(u,x,r)\subseteq\mathcal{A}(u,x_0,r_0)\quad\text{for every}\quad B_r(x)\subset B_{r_0}(x_0).$$

In particular, for every $0<h\leq h_0$ and every $B_r(x)\subset B_{r_0}(x_0)$, we have

$$\mu_-(h_0,x_0,r_0)\leq\mu_-(h,x,r)\qquad\text{and}\qquad\mu_+(h,x,r)\leq\mu_+(h_0,x_0,r_0).$$

Theorem 5.16 (Convergence of the upper and the lower Lagrange multipliers)**.** *Let u be a solution of* (5.1) *in the bounded open set $D\subset\mathbb{R}^d$ and let Λ_u be given by Proposition* 5.12*. Then there exists a constant $r_0>0$, which depends only on τ,λ_m and d, with the following property: for every ball $B_r(x_0)\subset\mathbb{R}^d$ centred at $x_0\in\partial\Omega_u$ with $r\leq r_0$ and such that*

$$D_r(x_0):=B_r(x_0)\cap D\ \text{ is connected}\qquad\text{and}\qquad 0<|\Omega_u\cap D_r(x_0)|<|D_r(x_0)|, \tag{5.21}$$

we have

$$\lim_{h\to0}\mu_+(h,x_0,r_0)=\lim_{h\to0}\mu_-(h,x_0,r_0)=\Lambda_u.$$

If, moreover, D is of class $C^{1,1}$, then there exists a constant $r_1>0$, which depends only on τ,λ_m,d and D, such that, for every ball $B_r(x_0)$ centred at $x_0\in\partial\Omega_u\cap\partial D$ with $r\leq r_1$, we have

$$\lim_{h\to0}\mu_-(h,x_0,r_0)=\Lambda_u.$$

Proof of Theorem 5.16*:* Let $x_0\in\partial\Omega_u$ be such that (5.21) holds and let $h>0$ be small. We set for simplicity $r=r_0$, $B_r(x_0)=B_r$, $\mu_+(h):=\mu_+(h,x_0,r)$ and $\mu_-(h):=\mu_-(h,x_0,r)$. We proceed in three steps.

Step 1. We first prove that $\mu_+(h)$ is finite. Let, for any $n\in\mathbb{N}$, $v_n\in\mathcal{A}(u,x_0,r)$ be a solution of the variational problem

$$\min\big\{J(v)+n(|\Omega_v|-m)_+\ :\ v\in\mathcal{A}(u,x_0,r),\ |\Omega_v|\leq m+h\big\}. \tag{5.22}$$

If there exists n such that $|\Omega_{v_n}|\leq m$, then $\mu_+(h)\leq n$ and hence $\mu_+(h)$ is finite. Indeed, by the minimality of u and the definition of v_n, we have for every $v\in\mathcal{A}(u,x_0,r)$ such that $m\leq|\Omega_v|\leq m+h$

$$J(u)+n|\Omega_u|\leq J(v_n)+n|\Omega_u|\leq J(v)+n|\Omega_v|,$$

so that $\mu_+(h)\leq n$ and the inequality $\mu_+(h)<\infty$ holds.

Suppose, by contradiction, that $|\Omega_{v_n}|>m$ for every n. First notice that since $J(v_n)$ is bounded from below (see Remark 5.10) and $J(v_n)+n(|\Omega_{v_n}|-m)\leq J(u)$, we have that $|\Omega_{v_n}|\to m$ as $n\to\infty$. Since v_n is a solution of (5.14) with $m_n:=|\Omega_{v_n}|$, there is a Lagrange multiplier Λ_{v_n} such that (5.12) holds for v_n and a vector field ξ_n such that (5.15) holds for $v_n^t(x)=v_n(x+t\xi_n(x))$. For $t>0$ small enough, $v_n^t\in\mathcal{A}(u,x_0,r)$ and $m<|\Omega_{v_n^t}|<m+h$. Then, by the minimality of v_n we have

$$J(v_n)+n(|\Omega_{v_n}|-m)\leq J(v_n^t)+n(|\Omega_{v_n^t}|-m)=J(v_n)+t\Lambda_{v_n}+n(|\Omega_{v_n}|-t-m)+o(t),$$

which implies $n\leq\Lambda_{v_n}$, in contradiction with $\lim_{n\to\infty}\Lambda_{u_n}=\Lambda_u$ from Lemma 5.13.

Step 2. $\lim_{h\to0}\mu_+(h)=\Lambda_u$. Let $(h_n)_{n\geq1}$ be a decreasing sequence such that $h_n\to0$. Since $\Lambda_u\leq\mu_+(h)$ and $h\mapsto\mu_+(h)$ is non-decreasing, it is sufficient to prove that $\lim_{n\to\infty}\mu_+(h_n)=\Lambda_u$. Fix $\varepsilon\in(0,\Lambda_u)$ and let $0<\alpha_n:=\mu_+(h_n)-\varepsilon<\mu_+(h_n)$. Let u_n be the solution of the problem

$$\min\big\{J(v)+\alpha_n(|\Omega_v|-m)^+\ :\ v\in\mathcal{A}(u,x_0,r),\ |\Omega_v|\leq m+h_n\big\}.$$

Notice that $|\Omega_{u_n}|>m$, since otherwise we would have $J(u)\leq J(u_n)+\alpha_n(|\Omega_{u_n}|-m)^+$, which contradicts the definition of $\mu_+(h_n)$. For n large enough, (5.21) holds with u_n, and since u_n is

solution of (5.14) with $m_n = |\Omega_{u_n}|$, by Proposition 5.12, there is a Lagrange multiplier $\Lambda_{u_n} \geq 0$ and a vector field ξ_n such that (5.15) holds for $u_n^t(x) := u_n(x + t\xi_n(x))$. By the minimality of u_n, for $t > 0$ small enough, we have

$$J(u_n) + \alpha_n(|\Omega_{u_n}| - m) \leq J(u_n^t) + \alpha_n(|\Omega_{u_n^t}| - m) = J(u_n) + t\Lambda_{u_n} + \alpha_n(|\Omega_{u_n}| - t - m) + o(t),$$

which shows that $\Lambda_{u_n} \geq \alpha_n$. By Lemma 5.13 we have

$$\lim_{n\to\infty} \mu_+(h_n) - \varepsilon = \lim_{n\to\infty} \alpha_n \leq \lim_{n\to\infty} \Lambda_{u_n} = \Lambda_u,$$

which proves the claim since $\varepsilon > 0$ is arbitrary.

Step 3. $\lim_{h\to 0} \mu_-(h) = \Lambda_u$. We prove this result for any $x_0 \in \partial\Omega_u$, which will conclude the proof of the Theorem. Note that the smoothness of D implies that there exists a constant $c_D > 0$ such that $D_r(x_0)$ is connected for every $r \leq r_D$ and every $x_0 \in \partial\Omega_u \cap \partial D$.

Let $\varepsilon > 0$ and $(h_n)_{n\in\mathbb{N}}$ be a decreasing infinitesimal sequence. We will show that $\Lambda_u - \varepsilon \leq \lim_{n\to\infty} \mu_-(h_n)$. Let u_n be a solution of the problem

$$\min\big\{J(v) + (\mu_-(h_n) + \varepsilon)(|\Omega_v| - (m - h_n))^+ \ : v \in \mathcal{A}(u, x_0, r),\ |\Omega_v| \leq m\big\}. \tag{5.23}$$

Up to replacing u_n by u_n^+, we can assume that $u_n \geq 0$ in B_r (the argument is similar to the proof of Lemma 5.11). We claim that

$$m - h_n \leq |\Omega_{u_n}| < m. \tag{5.24}$$

Suppose that $|\Omega_{u_n}| = m$. By the minimality of u and u_n we get

$$J(u) + (\mu_-(h_n) + \varepsilon)|\Omega_u| \leq J(u_n) + (\mu_-(h_n) + \varepsilon)|\Omega_{u_n}| \leq J(v) + (\mu_-(h_n) + \varepsilon)|\Omega_v|,$$

for every $v \in \mathcal{A}(u, x_0, r)$ such that $m - h_n \leq |\Omega_v| \leq m$, which contradicts the definition of $\mu_-(h_n)$. Now, if $|\Omega_{u_n}| < m - h_n$, we have $J(u_n) \leq J(u_n + t\varphi)$ for every $\varphi \in C_c^\infty(D_r(x_0))$ with sufficiently small compact support. Thus u_n solves the PDE $-\operatorname{div}(e^{-\Phi}\nabla u_n) = \lambda_m e^{-\Phi} u_n$ in $D_r(x_0)$. Since $u_n \geq 0$ in $D_r(x_0)$, by the strong maximum principle, we have that either $u_n \equiv 0$ or $u_n > 0$ in $D_r(x_0)$, in contradiction with (5.16). Thus, we proved (5.24).

We have that u_n is solution of (5.14) with $m_n := |\Omega_{u_n}|$ which converges to m as $n \to \infty$. By Lemma 5.13, we have an Euler-Lagrange equation for u_n in B_r for some Λ_{u_n}. Let $\xi_n \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ be the vector field from Lemma 5.13 (b) and let $u_n^t(x) = u_n(x + t\xi_n(x))$. For negative $t < 0$ and $|t|$ small enough, $u_n^t \in \mathcal{A}(u, x_0, r)$ and $|\Omega_{u_n}| \leq |\Omega_{u_n^t}| < m$. Thus, by the minimality of u_n, we get

$$J(u_n) + (\mu_-(h_n) + \varepsilon)(|\Omega_{u_n}| - (m - h_n)) \leq J(u_n) + \Lambda_{u_n} t + (\mu_-(h_n) + \varepsilon)(|\Omega_{u_n}| - t - (m - h_n)) + o(t),$$

which implies that $\Lambda_{u_n} \leq \mu_-(h_n) + \varepsilon$. Now, by Lemma 5.13, we get

$$\Lambda_u = \lim_{n\to\infty} \Lambda_{u_n} \leq \lim_{n\to\infty} \mu_-(h_n) + \varepsilon,$$

which conclude the proof. □

Remark 5.17 (Quasi-minimality at small scales)**.** Suppose that $D \subset \mathbb{R}^d$ is just a bounded open set. By the monotonicity of μ_+ and μ_- with respect to the inclusion (Remark 5.15) and a covering argument we get that for every compact set $\mathcal{K} \subset D$ there is $r(\mathcal{K}) > 0$ such that: for every $\varepsilon > 0$ there is $h > 0$ such that

$$\mu_+(h, x, r) - \varepsilon \leq \Lambda_u \leq \mu_-(h, x, r) + \varepsilon \quad \text{for every} \quad x \in \mathcal{K} \cap \partial\Omega_u \quad \text{and every} \quad 0 < r \leq r(\mathcal{K}).$$

If, moreover, D is of class $C^{1,1}$, then then exists $r_D > 0$ such that, for every $\varepsilon > 0$ there exists $h > 0$ such that: for every $0 < r \leq r_D$ and every $x_0 \in \partial\Omega_u$ we have

$$\begin{aligned} \mu_+(h, x, r) - \varepsilon \leq \Lambda_u \leq \mu_-(h, x, r) + \varepsilon \quad &\text{if } |\Omega_u \cap D_r(x_0)| < |D_r(x_0)|, \\ \Lambda_u \leq \mu_-(h, x, r) + \varepsilon \quad &\text{otherwise.} \end{aligned}$$

5.6. **Lipschitz continuity of the eigenfunctions on the optimal sets.** In this subsection we prove that the solutions of (5.1) are (locally) Lipschitz continuous in D. For $\delta > 0$ we set $D_\delta = \{x \in D \,:\, d(x, \partial D) > \delta\}$ and let $\mu > 0$ be given. By Theorem 5.16 and Remark 5.17 we get that if u is a solution of (5.1) and $\mu > \Lambda_u$, then there is $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u \cap D_\delta$, we have

$$J(u) + \mu|\Omega_u| \le J(v) + \mu|\Omega_v| \quad \text{for every} \quad v \in \mathcal{A}(u, x_0, r_0) \quad \text{such that} \quad |\Omega_v| \ge |\Omega_u|. \tag{5.25}$$

Note that the condition $|\Omega_v| \le |\Omega_u| + h$ can be dropped by choosing r_0 such that $|B_{r_0}| \le h$. We will prove that if $u \in H^1(B_{r_0})$ is bounded, nonnegative and satisfies (5.5) and (5.25), then u is Lipschitz in D_δ. In particular, we will obtain the following proposition.

Proposition 5.18 (Lipschitz continuity of the eigenfunction)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set. Let $\tau \ge 0$, $m \in (0, |D|)$ and $\Phi \in W^{1,\infty}(D)$. Then, every solution of* (5.1) *is locally Lipschitz continuous in D. More precisely, it is Lipschitz in D_δ for all $\delta > 0$. If, moreover, the box D is of class $C^{1,1}$, then u (extended by 0 outside D) is Lipschitz continuous in $\mathbb{R}^d$.*

The proof is based on the following lemma, whose (more general) two-phase counterpart can be found in [3], [5] and [10].

Lemma 5.19. *Let u be a solution of* (5.8) *and let $r_0 > 0$ be such that u satisfies* (5.25) *for some $\mu > \Lambda_u$. Then, there is a constant $C > 0$ such that for every $x_0 \in \partial\Omega_u \cap D_\delta$ we have*

$$|\operatorname{div}(e^{-\Phi}\nabla u)|(B_r(x)) \le Cr^{d-1} \qquad \text{for every ball} \qquad B_{2r}(x) \subset B_{r_0}(x_0). \tag{5.26}$$

Proof. Let $x = 0$ and $\eta \in C_c^\infty(B_{2r})$ be such that

$$0 \le \eta \le 1, \qquad \eta = 1 \ \text{ in } \ B_r, \qquad \|\nabla\eta\|_{L^\infty} \le \frac{C}{r}.$$

Using $u + t\eta$ as a test function for J, and setting $\langle f, g\rangle := \displaystyle\int_D fg\,dx$, we get

$$2\langle \operatorname{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}, \eta\rangle \le tJ(\eta) + \frac{\mu}{t}|B_{2r}| \le C\left(t\|\nabla\eta\|_{L^2}^2 + \frac{r^d}{t}\right)$$

where the constant $C > 0$ depends on d, Φ and μ. Now, minimizing over $t > 0$ and using the estimate $\|\nabla\eta\|_{L^2} \le C_d r^{\frac{d}{2}-1}$, we get

$$\langle \operatorname{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}, \eta\rangle \le Cr^{d-1}.$$

By Lemma 5.5, we have that $\operatorname{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}$ is a positive Radon measure. Thus, the inequality $\eta \ge \mathbb{1}_{B_r}$ and the boundedness of u imply

$$|\operatorname{div}(e^{-\Phi}\nabla u)|(B_r) \le \lambda_m \int_{B_r} u e^{-\Phi}\,dx + \langle \operatorname{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}, \mathbb{1}_{B_r}\rangle \le Cr^{d-1}. \qquad \square$$

Proof of Proposition 5.18. Let u be a solution of (5.1). We proceed in four steps.
Step 1. Ω_u is open. Let $\bar{x} \in \partial\Omega_u \cap D$. We will prove that $u(\bar{x}) = 0$. Let $r_1 > 0$ be such that $B_{r_1}(\bar{x}) \subset D$ and let $x_n \in B_{r_1/2}(\bar{x})$ be a sequence converging to $\bar{x}$ such that $u(x_n) = 0$ (see Lemma 5.7). By Lemma 5.19 and Lemma 5.5, for every n and every $r \le r_1/2$ we have

$$\fint_{\partial B_r(x_n)} u e^{-\Phi}\,d\mathcal{H}^{d-1} \le u(x_n)e^{-\Phi(x_n)} + Cr = Cr.$$

passing to the limit as $n \to \infty$, we get that

$$\fint_{\partial B_r(\bar{x})} u e^{-\Phi}\,d\mathcal{H}^{d-1} \le Cr \quad \text{for every} \quad r \le r_1/2,$$

which, passing again to the limit as $r \to 0$, proves that $u(\bar{x}) = 0$.

Step 2. Gradient estimate in Ω_u. For every ball $B_r(\bar{x}) \subset \Omega_u$ there is a constant C, depending only on Φ, d and λ_m, such that

$$\|\nabla u\|_{L^\infty(B_{r/2}(\bar{x}))} \leq \frac{C}{r}\|u\|_{L^\infty(B_r(\bar{x}))}. \tag{5.27}$$

Indeed, suppose that $\bar{x} = 0$ and set $\Phi_r(x) := \Phi(rx)$ and $u_r(x) = u(rx)$. Then u_r is a solution of

$$\operatorname{div}(e^{-\Phi_r}\nabla u_r) + \lambda_m e^{-\Phi_r} u_r = 0 \quad \text{in} \quad B_1,$$

and so, the interior Schauder estimate (see for example [26, Theorem 6.2 and Theorem 9.19]) gives

$$\|\nabla u_r\|_{L^\infty(B_{1/2})} \leq \|u_r\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u_r\|_{L^\infty(B_1)},$$

which, after rescaling, is precisely (5.27).

Step 3. Proof of the local Lipschitz continuity. Let $\bar{x} \in \Omega_u \cap D_\delta$ and set $r := \operatorname{dist}(\bar{x}, \partial\Omega_u)$. Let $r_0 \in (0, \delta/2)$ be such that u satisfies (5.25) for every $\bar{y} \in \partial\Omega_u \cap D_{\delta/2}$. If $r \geq r_0/6$, the estimate (5.27) gives $|\nabla u(\bar{x})| \leq C_{r_0}$. Now, if $r \leq r_0/6$, let $\bar{y} \in \partial\Omega_u$ be such that $r = |\bar{x} - \bar{y}|$ and let $\bar{z} \in B_r(\bar{x})$. Then, by (5.5) and since $\bar{y} \in \partial\Omega_u \cap D_{\delta/2}$, we have have for every $s \leq r$

$$u(\bar{z}) \leq C\left(\fint_{\partial B_s(\bar{z})} u\, d\mathcal{H}^{d-1} + s\right).$$

Now, multiplying by s^{d-1} and then integrating from 0 to r the above inequality, we get

$$\begin{aligned} u(\bar{z}) &\leq C\left(\fint_{B_r(\bar{z})} u\, d\mathcal{H}^{d-1} + r\right) \leq C\left(\fint_{B_{3r}(\bar{y})} u\, d\mathcal{H}^{d-1} + r\right) \\ &\leq C\left(r^{-d}\int_0^{3r} s^{d-1}\, ds \fint_{\partial B_s(\bar{y})} u\, d\mathcal{H}^{d-1} + r\right) \leq Cr \end{aligned}$$

for every $\bar{z} \in B_r(\bar{x})$, where in the last inequality we use Lemma (5.5) and Lemma 5.19. Finally, using the estimate (5.27) this gives

$$|\nabla u(\bar{x})| \leq \|\nabla \bar{u}\|_{L^\infty(B_{r/2}(\bar{x}))} \leq \frac{C}{r}\|u\|_{L^\infty(B_r(\bar{x}))} \leq C. \tag{5.28}$$

This proves that $|\nabla u|$ is bounded in D_δ.

Step 4. Global Lipschitz estimate. We first notice that since D is $C^{1,1}$ regular, the radius r_0 for which (5.25) holds does not depend on the point $x_0 \in \partial\Omega_u$. Now, let $\bar{x} \in \Omega_u \setminus D_{r_0}$ and set $r := \operatorname{dist}(\bar{x}, \partial\Omega_u)$. We consider the projection $\bar{y}$ of $\bar{x}$ on $\partial\Omega_u$ and we distinguish two cases. If $r \leq 6\operatorname{dist}(\bar{x}, \partial D)$, then we apply the estimate from Step 3 and we get that $|\nabla u(\bar{x})| \leq C$. If $r \geq 6\operatorname{dist}(\bar{x}, \partial D)$, we consider the solution w to the problem

$$-\operatorname{div}(e^{-\Phi}\nabla w) = 1 \quad \text{in} \quad D, \qquad w \in H^1_0(D),$$

which is Lipschitz continuous in $\mathbb{R}^d$ since D is of class $C^{1,1}$ (see for example [26, Theorem 9.13]). Moreover, by the strong maximum principle, we have that $u \leq Cw$ for some constant C depending on λ_m, d and Φ. Therefore, setting $r_1 = \operatorname{dist}(\bar{x}, \partial D)$, we have for every $\bar{z} \in B_{r_1}(\bar{x})$,

$$u(\bar{z}) \leq Cw(\bar{z}) \leq C|\bar{z} - \bar{y}| \leq Cr_1,$$

and we conclude by the gradient estimate (5.27). □

5.7. **Non-degeneracy of the eigenfunctions and finiteness of the perimeter of Ω_u.** Let u be a solution of (5.1) in the bounded open set $D \subset \mathbb{R}^d$. Let $x_0 \in \partial\Omega_u$ and $r_0(x_0)$ be such that for every $0 < r \leq r(x_0)$ the set $D_r(x_0) := B_r(x_0) \cap D$ is connected. Notice that such an $r(x_0)$ trivially exists if $x_0 \in \partial\Omega_u \cap D$, while in the general case it is sufficient to assume some a priori regularity of the box D. Then, by Remark 5.17, for every $\mu < \Lambda_u$ there is some $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u$, we have

$$J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v| \quad \text{for every} \quad v \in \mathcal{A}(u, x_0, r_0) \quad \text{such that} \quad |\Omega_v| \leq |\Omega_u|. \tag{5.29}$$

This property was first exploited by Alt and Caffarelli to prove the non-degeneracy of the solutions. More recently, it was exploited by Bucur who introduced the notion of a *shape subsolution* which found application to several shape optimization problems (see for example [8] and [11]).

The next lemma is a fundamental step in the proof of the regularity of the free boundary since it allows to prove that the blow-up limits (see Subsection 5.8) are non trivial. It is the analogue of the non-degeneracy estimate from [2] and the proof is based on the same idea.

Lemma 5.20 (Non-degeneracy of the eigenfunctions)**.** *Let u be a solution of* (5.1) *in the bounded open set $D \subset \mathbb{R}^d$. Suppose that $x_0 \in \partial\Omega_u$, $0 < \mu < \Lambda_u$ and $r_0 > 0$ are such that* (5.29) *holds. Then there are constants $c > 0$ and $r_1 > 0$ which depend only on τ, λ_m, μ and d, such that for every ball $B_{2r}(x) \subset B_{r_0}(x_0)$ with $r \le r_1$, we have that if $\|u\|_{L^\infty(B_{2r}(x))} \le cr$, then $u = 0$ on $B_r(x)$.*

Proof. Let r, x be such that $B_{2r}(x) \subset B_{r_0}(x_0)$ with $r \le r_1$ and $\|u\|_{L^\infty(B_{2r}(x))} < cr$. Assume $x = 0$ and $r_1 = r$. Let $\eta \in H^1(B_{2r})$ be the solution of the problem

$$-\operatorname{div}(e^{-\Phi}\nabla\eta) = \beta e^{-\Phi} \quad \text{in} \quad B_{2r}\backslash B_r, \qquad \eta = 0 \quad \text{in} \quad B_r, \qquad \eta = cr \quad \text{in} \quad D\backslash B_{2r},$$

for some $\beta > 0$ which will be chosen later. Note that (see for instance [26, Theorem 9.11 and 9.15]) we have the following estimate

$$\|\nabla\eta\|_{L^\infty(B_r)} \le C_d\left(\beta e^{-\min\Phi} r + \frac{1}{r}\|\eta\|_{L^\infty(B_{2r})}\right) \le C_d(\beta e^{-\min\Phi} r + c). \tag{5.30}$$

Consider the test function $\tilde{u} \in H^1_0(D)$ defined as

$$\tilde{u} = u \wedge \eta \quad \text{in} \quad B_{2r}, \qquad \tilde{u} = u \quad \text{in} \quad D\backslash B_{2r}.$$

Moreover, by (5.29), we get

$$\int_D |\nabla u|^2 e^{-\Phi} dx - \lambda_m \int_D u^2 e^{-\Phi} dx + \mu|\Omega_u| \le \int_D |\nabla\tilde{u}|^2 e^{-\Phi} dx - \lambda_m \int_D \tilde{u}^2 e^{-\Phi} dx + \mu|\Omega_{\tilde{u}}|.$$

Now, set $E(u,r) := \displaystyle\int_{B_r} |\nabla u|^2 e^{-\Phi} dx + \mu|\Omega_u \cap B_r|$ and use that $|\Omega_u| - |\Omega_{\tilde{u}}| = |\Omega_u \cap B_r|$ to obtain

$$E(u,r) \le \int_{B_{2r}\backslash B_r} (|\nabla\tilde{u}|^2 - |\nabla u|^2) e^{-\Phi} dx + 4c\lambda_m \int_{B_{2r}\backslash B_r} (u - \tilde{u}) e^{-\Phi} dx + \lambda_m \int_{B_r} u^2 e^{-\Phi} dx. \tag{5.31}$$

We first estimate the first term of the right hand side of the inequality above. We have

$$|\nabla\tilde{u}|^2 - |\nabla u|^2 = -|\nabla(\tilde{u} - u)|^2 + 2\nabla\tilde{u}\cdot\nabla(\tilde{u} - u) \le 2\nabla\tilde{u}\cdot\nabla(\tilde{u} - u). \tag{5.32}$$

Moreover, integrating by parts and using that $(u - \eta)_+ = 0$ on ∂B_{2r}, we get

$$\begin{aligned}\int_{B_{2r}\backslash B_r} \nabla\tilde{u}\cdot\nabla(\tilde{u} - u) e^{-\Phi} dx &= -\int_{B_{2r}\backslash B_r} \nabla\eta\cdot\nabla[(u - \eta)_+] e^{-\Phi} dx \\ &\le -\beta\int_{B_{2r}\backslash B_r} (u - \eta)_+ e^{-\Phi} dx + \|\nabla\eta\|_{L^\infty(\partial B_r)} \int_{\partial B_r} u e^{-\Phi} d\mathcal{H}^{d-1}.\end{aligned} \tag{5.33}$$

We now set $\beta = 2c\lambda_m$ so that, combining (5.31), (5.32) and (5.33) we have

$$E(u,r) \le 2C_d(\beta e^{-\min\Phi} r + c) \int_{\partial B_r} u e^{-\Phi} d\mathcal{H}^{d-1} + \lambda_m \int_{B_r} u^2 e^{-\Phi} dx.$$

Now, for every $s \in (0, r]$, we have by the $W^{1,1}$ trace inequality in B_s

$$\begin{aligned}\int_{\partial B_s} u e^{-\Phi} d\mathcal{H}^{d-1} &\le e^{-\min\Phi} C_d\left(\int_{B_s} |\nabla u|\, dx + \frac{1}{s}\int_{B_s} u\, dx\right) \\ &\le e^{-\min\Phi} C_d\left(\frac{1}{2}\int_{B_s} |\nabla u|^2 dx + \frac{1}{2}|\Omega_u \cap B_s| + c|\Omega_u \cap B_s|\right) \\ &\le C\left(\int_{B_s} |\nabla u|^2 e^{-\Phi} dx + \mu|\Omega_u \cap B_s|\right) \le CE(u,s) \le CE(u,r),\end{aligned}$$

where we have set $C = e^{-\min\Phi} C_d \max\left\{e^{\max\Phi}, \frac{1}{\mu}(1+2c)\right\}$. Moreover, since the above inequality holds for every $s \in (0,r]$, we have

$$\int_{B_r} u e^{-\Phi} dx = \int_0^r ds \int_{\partial B_s} u e^{-\Phi} d\mathcal{H}^{d-1} \leq rCE(u,r).$$

Finally, using the bound (5.30), we get

$$E(u,r) \leq \left(2C_d(\beta e^{-\min\Phi} r + c) + r^2 c\lambda_m\right) CE(u,r),$$

which, for c and r small enough, implies that $E(u,r) = 0$ and concludes the proof. □

Another consequence of property (5.29) is that the optimal sets have finite perimeter. This fact is of independent interest but it can alo be used to estimate the dimension of the singular set of the free boundary (see Subsection 5.10). The local finiteness of the perimeter was also obtained in [2] in the case of the Laplacian by a completely different approach. Here we use an argument which is the local version of an estimate that was used in [8] to prove that some optimal shapes have finite perimeter.

Lemma 5.21 (Local finiteness of the perimeter)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set and u a solution of (5.1). Then Ω_u is a set of locally finite perimeter in D. Moreover, if D is of class $C^{1,1}$, then Ω_u is a set of finite perimeter.*

Proof. Let $x_0 \in \partial\Omega_u$ and $0 < \mu < \Lambda_u$ be fixed. Let $r > 0$ be such that (5.29) holds in $D_r(x_0) := B_r(x_0) \cap D$. Assume $x_0 = 0$ and $r_0 = r$. In the sequel we denote by $C > 0$ any constant, which does not depend on t or x_0. Let $t \in (0,1)$ and $\eta \in C_c^\infty(B_r)$ be such that

$$0 \leq \eta \leq 1, \qquad \eta = 1 \quad \text{in} \quad B_{r/2}, \qquad \eta = 0 \quad \text{in} \quad \mathbb{R}^d \backslash B_r, \qquad |\nabla\eta| \leq \frac{C}{r}.$$

We set $u_t := \eta(u-t)_+ + (1-\eta)u \in \mathcal{A}(u, x_0, r)$. By the optimality of u, we have

$$\int_{B_r} \left(|\nabla u|^2 - \lambda_m u^2\right) e^{-\Phi} dx + \mu|\Omega_u \cap B_r| \leq \int_{B_r} \left(|\nabla u_t|^2 - \lambda_m u_t^2\right) e^{-\Phi} dx + \mu|\Omega_{u_t} \cap B_r|.$$

We now estimate

$$\int_{B_r} \left(u^2 - u_t^2\right) e^{-\Phi} dx \leq \int_{B_r} 2\eta u \left(u - (u-t)_+\right) e^{-\Phi} dx \leq Ct,$$

$$\int_{B_r} \left(|\nabla u|^2 - |\nabla u_t|^2\right) e^{-\Phi} dx \geq \int_{\{0<u<t\}\cap B_{r/2}} |\nabla u|^2 e^{-\Phi} dx - Ct.$$

Therefore, combining the previous estimates and using $2ab \leq a^2 + b^2$ for all a, b, we obtain

$$\int_{\{0<u<t\}\cap B_{r/2}} |\nabla u|\, dx \leq C\left(\int_{\{0<u<t\}\cap B_{r/2}} |\nabla u|^2 e^{-\Phi} dx + \mu|\{0<u<t\} \cap B_{r/2}|\right) \leq Ct.$$

We now use the co-area formula to rewrite the above inequality as

$$\frac{1}{t}\int_0^t Per\left(\{u > s\}; B_{r/2}\right) ds \leq C.$$

Hence, there is a sequence $t_n \to 0$ such that $Per\left(\{u > t_n\}; B_{r/2}\right) ds \leq C$, which implies that $Per\left(\Omega_u; B_{r/2}\right) ds \leq C$. The last claim of the lemma follows by a covering argument. □

5.8. **Blow-up sequences and blow-up limits.** Let u be a solution of (5.1) in the bounded open set $D \subset \mathbb{R}^d$. For $r > 0$ and $x_0 \in \partial\Omega_u$, we define the rescaled function

$$u_{x_0,r}(x) := \frac{1}{r} u(x_0 + rx).$$

Now since u is Lipschitz continuous in some ball $B_{r_0}(x_0)$ (assume some regularity of the box if $x_0 \in \partial D$) we get that every sequence $(u_{x_0,r_n})_{n\geq 1}$ such that $r_n \to 0$ admits a subsequence (still denoted by r_n) that converges to a function $u_0 : \mathbb{R}^d \to \mathbb{R}$ uniformly on every compact set $K \subset \mathbb{R}^d$. We say that u_0 is a blow-up limit of u at x_0 and we use the notation $\mathcal{BU}_u(x_0)$ for the family of all blow-up limits of u at x_0. We notice that, due to the non-degeneracy of u, the blow-up limits are non-trivial. Precisely, $u_0 \neq 0$ and there is a constant $c > 0$ such that $\|u_0\|_{L^\infty(B_r)} \geq cr$.

The following proposition is standard. For a detailed proof we refer for example to [33, Proposition 4.5].

Proposition 5.22 (Convergence of the blow-up sequences)**.** *Let u be a solution of* (5.1) *and let $x_0 \in \partial\Omega_u$. Assume moreover that D is of class $C^{1,1}$ if $x_0 \in \partial D$. Let $u_0 \in \mathcal{BU}_u(x_0)$ and $u_n := u_{x_0,r_n}$ be a blow-up sequence such that $u_n \to u_0$ locally uniformly in $\mathbb{R}^d$ as $n \to \infty$. Then*

(1) The sequence $(u_n)_{n\geq 1}$ converges to u_0 strongly in $H^1_{loc}(\mathbb{R}^d)$.
(2) The sequence of characteristic functions $\left(\mathbb{1}_{\Omega_{u_n}}\right)_{n\geq 1}$ converges to $\mathbb{1}_{\Omega_{u_0}}$ in $L^1_{loc}(\mathbb{R}^d)$.
(3) The sequences of closed sets $\left(\overline{\Omega}_n\right)_{n\geq 1}$ and $(\Omega_n^c)_{n\geq 1}$ Hausdorff converge locally in $\mathbb{R}^d$ to $\overline{\Omega}_0$ and Ω_0^c, respectively.
(4) If $x_0 \in \partial\Omega_u \cap D$, then u_0 is a non-trivial global minimizer of the one-phase Alt-Caffarelli functional with $\Lambda = \Lambda_u e^{\Phi(x_0)}$ (see Definition 5.23 *below).*
If $x_0 \in \partial\Omega_u \cap \partial D$, then, up to a rotation, u_0 is a non-trivial global minimizer of the one-phase constrained Alt-Caffarelli functional with $\Lambda = \Lambda_u e^{\Phi(x_0)}$.

Definition 5.23 (Global minimizers of the one-phase problem)**.** Let $u_0 \in H^1_{loc}(\mathbb{R}^d)$ be a non-negative function.

- We say that u_0 is a global minimizer of the one-phase Alt-Caffarelli functional with $\Lambda > 0$, if

$$\int_B |\nabla u|^2 dx + \Lambda|\{u > 0\} \cap B| \leq \int_B |\nabla v|^2 dx + \Lambda|\{v > 0\} \cap B|, \tag{5.34}$$

  for every ball $B \subset \mathbb{R}^d$ and every function $v \in H^1(B)$ such that $u - v \in H^1_0(B)$.
- We say that u_0 is a global minimizer of the one-phase constrained Alt-Caffarelli functional with $\Lambda > 0$, if $\Omega_{u_0} \subset \{x_d > 0\}$ and (5.34) holds for every ball $B \subset \mathbb{R}^d$ and every function $v \in H^1(B)$ such that $u - v \in H^1_0(B)$ and $\Omega_v \subset \{x_d > 0\}$.

Remark 5.24 (Optimality of the blow-up sequence)**.** The last claim of Proposition 5.22 follows by a standard argument and by Theorem 5.16. Indeed, Theorem 5.16 implies that if $x_0 \in \partial\Omega_u \cap D$ and $B \subset \mathbb{R}^d$ are fixed, then for every $\varepsilon > 0$ there is $r_0 > 0$ such that for every $0 < r \leq r_0$ we have (setting $\Phi_r(x) = \Phi(x_0 + rx)$)

$$\int_B \left(|\nabla u_r|^2 - r^2\lambda_m u_r^2\right) e^{-\Phi_r} dx + (\Lambda_u - \varepsilon)|\Omega_{u_r} \cap B| \leq \int_B \left(|\nabla v|^2 - r^2\lambda_m v^2\right) e^{-\Phi_r} dx + (\Lambda_u - \varepsilon)|\Omega_v \cap B|,$$

for every $v \in H^1(B)$ such that $u_r - v \in H^1_0(B)$ and $|\Omega_v \cap B| \leq |\Omega_{u_r} \cap B|$;

$$\int_B \left(|\nabla u_r|^2 - r^2\lambda_m u_r^2\right) e^{-\Phi_r} dx + (\Lambda_u + \varepsilon)|\Omega_{u_r} \cap B| \leq \int_B \left(|\nabla v|^2 - r^2\lambda_m v^2\right) e^{-\Phi_r} dx + (\Lambda_u + \varepsilon)|\Omega_v \cap B|,$$

for every $v \in H^1(B)$ such that $u_r - v \in H^1_0(B)$ and $|\Omega_v \cap B| \geq |\Omega_{u_r} \cap B|$.
If D is of class $C^{1,1}$ and $x_0 \in \partial\Omega_u \cap \partial D$, then the same result holds with test functions $v \in H^1(B)$ such that $\Omega_v \subset \{x_d > 0\}$.

Remark 5.25 (Lebesgue density on the free boundary)**.** For every $\gamma \in [0,1]$ we define

$$\Omega_u^{(\gamma)} := \Big\{ x \in \mathbb{R}^d \ : \ \lim_{r\to 0} \frac{|\Omega_u \cap B_r(x)|}{|B_r|} = \gamma \Big\}.$$

We notice that, as a consequence of Proposition 5.22, we get that

$$\partial\Omega_u \cap D \cap \Omega_u^{(0)} = \emptyset \qquad \text{and} \qquad \partial\Omega_u \cap D \cap \Omega_u^{(1)} = \emptyset.$$

Indeed, the first equality follows by the non-degeneracy of u, while the second one follows from the fact that all the blow-up limits vanish in zero and are global solutions of the Alt-Caffarelli problem.

5.9. **Monotonicity formula.** We prove in this section a monotonicity formula for the Weiss' functional of a solution u to (5.1) at every point of the boundary $\partial\Omega_u$. Let u be a solution (5.1) and Λ_u be the constant given by Theorem 5.12. We define the Weiss' functional as

$$W(u,\Phi,x_0,r) = \frac{1}{r^d}\int_{B_r} |\nabla u|^2 e^{-\Phi} dx - \frac{1}{r^{d+1}}\int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1} + \frac{\Lambda_u}{r^d}|\Omega_u \cap B_r(x_0)|.$$

Lemma 5.26 (Weiss monotonicity formula)**.** *Let u be a solution* (5.1) *in the bounded open set D. Then, for every $x_0 \in \partial\Omega_u \cap D$ and every $0 < r < dist(x_0, \partial D)$, the function W satisfies the differential inequality*

$$\frac{d}{dr}W(u,\Phi,x_0,r) \geq \frac{2e^{-\max\Phi}}{r^{d+2}}\int_{\partial B_r} |\nabla u \cdot x - u|^2 \, d\mathcal{H}^{d-1} - C, \tag{5.35}$$

where $C > 0$ is a constant depending only on $\lambda_m, \Phi, L = \|\nabla u\|_{L^\infty}$ and d.
If, moreover, D is of class $C^{1,1}$, then there exists a constant $r_0 > 0$ depending only on D, such that, for every $x_0 \in \partial\Omega_u \cap \partial D$ and every $0 < r < r_0$ the inequality (5.35) *holds for some constant $C > 0$ which also depends on D.*

Proof. We first prove the claim when $x_0 \in \partial\Omega_u \cap D$. Assume $x_0 = 0$. We set

$$H(r) := \int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1} \qquad \text{and} \qquad D(r) := \int_{B_r} |\nabla u|^2 e^{-\Phi} dx,$$

$$H_\Phi(r) := \int_{\partial B_r} u^2 (n\cdot\nabla\Phi) e^{-\Phi} d\mathcal{H}^{d-1} \qquad \text{and} \qquad D_\Phi(r) := \int_{B_r} \left(|\nabla u|^2 - \lambda_m u^2\right)(n\cdot\nabla\Phi) e^{-\Phi} dx,$$

where $n(x) = x/r$ is the normal to the sphere ∂B_r at x. As in Proposition A.1 (notice that in Proposition A.1 D_Φ is defined differently) we have

$$D'(r) = \int_{\partial B_r} |\nabla u|^2 e^{-\Phi} d\mathcal{H}^{d-1} \quad \text{and} \quad H'(r) = \frac{d-1}{r}H(r) + 2D(r) - 2\lambda_m \int_{B_r} u^2 e^{-\Phi} dx - H_\Phi(r).$$

As in Step 2 of the proof of Proposition A.1, the optimality condition $\delta J(u)[\xi] = \Lambda_u \int_{\Omega_u} \operatorname{div}\xi \, dx$, applied to the vector field $\xi(x) = x\phi_\varepsilon(x)$, gives that

$$\begin{aligned}\Lambda_u \Big(d|\Omega_u \cap B_r| - r\mathcal{H}^{d-1}(\Omega_u \cap \partial B_r)\Big) &= -(d-2)D(r) + rD'(r) - 2r\int_{\partial B_r} (\partial_n u)^2 e^{-\Phi} d\mathcal{H}^{d-1} \\ &\quad + \lambda_m \left(d\int_{B_r} u^2 e^{-\Phi} dx - r\int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1}\right) + rD_\Phi(r),\end{aligned}$$

where $\partial_n u := n \cdot \nabla u$. We now calculate

$$\begin{aligned}\frac{d}{dr}W(u,\Phi,x_0,r) &= \frac{1}{r^d}D'(r) - \frac{d}{r^{d+1}}D(r) - \frac{1}{r^{d+1}}H'(r) + \frac{d+1}{r^{d+2}}H(r)\\ &\quad + \frac{\Lambda_u}{r^{d+1}}\Big(r\mathcal{H}^{d-1}(\Omega_u \cap \partial B_r) - d|\Omega_u \cap B_r|\Big)\\ &= \frac{2}{r^{d+2}}\int_{\partial B_r} |\nabla u \cdot x - u|^2\, e^{-\Phi} d\mathcal{H}^{d-1} + \frac{1}{r^{d+1}}H_\Phi(r) - \frac{1}{r^d}D_\Phi(r)\\ &\quad - \frac{\lambda_m}{r^{d+1}}\left((d+2)\int_{B_r} u^2 e^{-\Phi}dx - r\int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1}\right)\\ &\geq \frac{2e^{-\max\Phi}}{r^{d+2}}\int_{\partial B_r} |\nabla u \cdot x - u|^2\, d\mathcal{H}^{d-1} - C,\end{aligned}$$

which gives the claim if $x_0 \in \partial\Omega_u \cap D$.

Since D is of class $C^{1,1}$, D satisfies a uniform exterior ball condition, that is, there exists a constant $r_D > 0$ such that, for every $x_0 \in \partial D$, there exists a ball of radius r_D lying outside D and touching D at x_0. Let $x_0 \in \partial\Omega_u \cap \partial D$ and assume that $x_0 = 0$. Then, there exists a constant $c > 0$ which depends only on r_D such that $(Id + \tilde{\xi})^{-1}(D) \subset D$ where we have set $\tilde{\xi}(x) = (x - (x_0 - cr^2 n_{x_0}))\phi_\varepsilon(x)$ and where ϕ_ε is defined as in the step 2 of the proof of Proposition A.1. It follows that $u_t(x) := u(x + t\tilde{\xi}(x)) \in H^1_0(D)$ for every small $t > 0$. From Proposition 5.12 we have $\delta J(u)[\tilde{\xi}] \geq \Lambda_u \int_{\Omega_u} \operatorname{div}\tilde{\xi}\, dx$ which can be rewritten as

$$\delta J(u)[\xi] \geq \Lambda_u \int_{\Omega_u} \operatorname{div}\xi\, dx - cr^2\Big(\delta J(u)[\phi_\varepsilon n_{x_0}] - \Lambda_u \int_{\Omega_u} \operatorname{div}(\phi_\varepsilon n_{x_0})\, dx\Big),$$

where $\xi(x) = (x - x_0)\phi_\varepsilon(x)$. Then, letting ε go to 0, it follows that there exists a constant C depending only on λ_m, Φ, L, r_D and d such that

$$\delta J(u)[\xi] \geq \Lambda_u \int_{\Omega_u} \operatorname{div}\xi\, dx - Cr^{d+1}.$$

We now conclude the proof with the same computations as above. □

Lemma 5.27 (Homogeneity of the blow-up limits)**.** *Let u be a solution* (5.1) *in the bounded open set D and let $x_0 \in \partial\Omega_u$. Assume moreover that D is of class $C^{1,1}$ if $x_0 \in \partial\Omega_u \cap \partial D$. Then every blow-up limit $u_0 \in \mathcal{BU}_u(x_0)$ is one-homogeneous.*

Proof. Let $x_0 = 0$ and $W(u,\Phi,r) := W(u,\Phi,x_0,r)$. Recall that $u_r(x) = \frac{1}{r}u(rx)$ and $\Phi_r(x) = \Phi(rx)$. We first notice that for every $r > 0$ and $s > 0$ such that $rs \leq \operatorname{dist}(x_0, \partial D)$ we have

$$W(u_r, \Phi_r, s) = W(u, \Phi, rs).$$

Moreover, since the function $r \mapsto W(u,\Phi,t) + Cr$ is monotone, the limit

$$W(u,\Phi,0) := \lim_{r\to 0^+} W(u,\Phi,r)$$

exists (and is finite due to the Lipschtz continuity of u). On the other hand, for every blow-up sequence u_{r_n} with blow-up limit u_0, we have

$$W(u_0,\Phi(0),s) = \lim_{n\to\infty} W(u_{r_n},\Phi_{r_n}(0),s) = \lim_{n\to\infty} W(u,\Phi(0),r_n s) = W(u,\Phi(0),0).$$

Thus, the function

$$s \mapsto \frac{1}{s^d}\int_{B_s} |\nabla u_0|^2\, dx - \frac{1}{s^{d+1}}\int_{\partial B_s} u_0^2\, d\mathcal{H}^{d-1} + \frac{\Lambda_u e^{\Phi(0)}}{s^d}|\Omega_{u_0} \cap B_s|,$$

is constant. Now, by [36] (or, simply by applying (5.35) to $u = u_0$, $\lambda_m = 0$ and $\Phi = 0$), we have that u_0 is one-homogeneous. □

5.10. **Regularity of the free boundary.** In order to obtain our main regularity result, we first show that the optimality condition $|\nabla u|^2 = \Lambda_u e^{\Phi}$ on the free boundary $\partial\Omega_u \cap D$ and $|\nabla u|^2 \geq \Lambda_u e^{\Phi}$ on $\partial\Omega_u \cap \partial D$ holds in the viscosity sense.

Definition 5.28 (Optimality condition in viscosity sense)**.** Let D be an open set and $u \in C(D)$.

• We say that $\varphi \in C(D)$ touches u by below (resp. by above) at $x_0 \in D$ if $\varphi(x_0) = u(x_0)$ and $\varphi \leq u$ (resp. $\varphi \geq u$) in a neighborhood of x_0.

• Let Λ be a non-negative function on D and assume that u is non-negative. We say that u satisfies the boundary condition

$$|\nabla u| = \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega_u \cap D$$

in the viscosity sense if, for every $\varphi \in C^2(D)$ such that φ^+ touches u by below (resp. by above) at some $x_0 \in \partial\Omega_u \cap D$, we have $|\nabla\varphi|(x_0) \leq \sqrt{\Lambda}$ (resp. $|\nabla\varphi|(x_0) \geq \sqrt{\Lambda}$)).
Analogously, we say that u satisfies the boundary condition

$$|\nabla u| \geq \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega_u \cap \partial D$$

in the viscosity sense if, for every $\varphi \in C^2(D)$ such that φ^+ touches u by above at some $x_0 \in \partial\Omega_u \cap \partial D$, we have $|\nabla\varphi|(x_0) \geq \sqrt{\Lambda}$.

Lemma 5.29 (Optimality condition on the free boundary)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set of class $C^{1,1}$ and let u be a solution of* (5.1)*. Then u is a solution of the problem*

$$\begin{cases} -\operatorname{div}(e^{-\Phi}u) = \lambda_m u e^{-\Phi} & \text{in } \ \Omega_u, \\ |\nabla u| = \sqrt{\Lambda_u e^{\Phi}} & \text{on } \ \partial\Omega_u \cap D, \\ |\nabla u| \geq \sqrt{\Lambda_u e^{\Phi}} & \text{on } \ \partial\Omega_u \cap \partial D, \end{cases} \tag{5.36}$$

where the boundary conditions hold in the viscosity sense.

Proof. From Proposition 5.18 it follows that u is continuous in D. We only have to prove that u satisfies the two boundary conditions in the viscosity sense. We first show that $|\nabla u| = \sqrt{\Lambda_u e^{\Phi}}$ holds on $\partial\Omega_u \cap D$. Let $\varphi \in C^2(D)$ a function such that φ^+ touches u by below at $x_0 \in \partial\Omega_u \cap D$. Let r_n be an infinitesimal sequence and

$$u_n(x) = \frac{1}{r_n} u(x_0 + r_n x) \quad \text{and} \quad \varphi_n(x) = \frac{1}{r_n}\varphi^+(x_0 + r_n x).$$

Up to a subsequence, u_n converges locally uniformly to some $u_0 \in \mathcal{BU}_u(x_0)$, while φ_n converges to $\varphi_0(x) := x \cdot \nabla\varphi(x_0)$. Since $u_0 \geq \varphi_0$ we have $u_0 > 0$ in $\{x_d > 0\}$. Moreover, by Proposition 5.22, u_0 is a local minimum of the Alt-Caffarelli functional for $\Lambda = \Lambda_u e^{\Phi(x_0)}$ and by Lemma 5.27 u_0 is a one-homogeneous function. Thus, $u_0 = 0$ on the hyperplane $\{x_d = 0\}$ and, (see [2, Theorem 2.5]) u_0 satisfies (in the classical sense) the optimality condition

$$|\nabla u_0| = \sqrt{\Lambda} \quad \text{on} \quad \{x_d \neq 0\}.$$

Now, since φ_0 touches u_0 from below at $0 \in \partial\Omega_{u_0}$, we get that $|\nabla\varphi|(x_0) = |\nabla\varphi_0|(0) \leq |\nabla u_0|(0) = \sqrt{\Lambda}$. The proof when φ^+ touches u from above at x_0 is analogous.
We argue in the same way to prove the boundary condition $|\nabla u| \geq \sqrt{\Lambda}$ on $\partial\Omega_u \cap \partial D$. Note that, in this case, any blow-up $u_0 \in \mathcal{BU}_u(x_0)$ at some point $x_0 \in \partial\Omega_u \cap \partial D$ is solution of the Alt-Caffarelli constrained functional so that it satisfies the optimality condition

$$|\nabla u_0| \geq \sqrt{\Lambda} \quad \text{on} \quad \{x_d \neq 0\}.$$

□

Remark 5.30. Using an argument based on an internal variation of the boundary as in [2, Theorem 2.5] we can get in a weak sense the optimality boundary condition given in Lemma

5.29, namely: for every $x_0 \in \partial\Omega_u \cap D$, $r > 0$ such that $B_r(x_0) \subset D$ and $\xi \in C_0^\infty(B_r(x_0), \mathbb{R}^d)$ we have

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u>\varepsilon\}} (|\nabla u|^2 - \Lambda_u e^\Phi)\, e^{-\Phi}\, \xi \cdot \nu \, d\mathcal{H}^{d-1} = 0,$$

while for every $x_0 \in \partial\Omega_u \cap \partial D$, $r > 0$ such that $D_r(x_0)$ is connected and every $\xi \in C_0^\infty(B_r(x_0), \mathbb{R}^d)$ such that $(Id + \xi)^{-1}(D_r(x_0)) \subset D_r(x_0)$ we have

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u>\varepsilon\}} (|\nabla u|^2 - \Lambda_u e^\Phi)\, e^{-\Phi}\, \xi \cdot \nu \, d\mathcal{H}^{d-1} \geq 0.$$

Definition 5.31 (Regular and singular parts of the free boundary)**.** Assume that D is a bounded open set of class $C^{1,1}$. We define the regular part of the boundary of Ω_u, $Reg(\partial\Omega_u)$, as the set of points $x_0 \in \partial\Omega_u$ such that there exists a blow-up $u_0 \in \mathcal{BU}_u(x_0)$ of the form

$$u_0(x) = \sqrt{\Lambda_u e^{\Phi(x_0)}}\, (x \cdot \nu)_+ \qquad \text{if} \quad x_0 \in \partial\Omega_u \cap D, \tag{5.37}$$

$$u_0(x) = q(x_0)\, (x \cdot \nu)_+ \qquad \text{if} \quad x_0 \in \partial\Omega_u \cap \partial D, \tag{5.38}$$

where $\nu \in \partial B_1$ is some unit vector and $q : \Omega_u \cap \partial D \to \mathbb{R}$ is a function bounded from below by $\sqrt{\Lambda_u e^{\Phi(x_0)}}$. The singular part of the boundary is $Sing(\partial\Omega_u) := \partial\Omega_u \setminus Reg(\partial\Omega_u)$.

Proposition 5.32 (Regularity of the free boundary)**.** *Suppose that u is a solution of (5.1) in the bounded open set $D \subset \mathbb{R}^d$ of class $C^{1,1}$. Then, $Reg(\partial\Omega_u)$ is locally the graph of a $C^{1,\alpha}$ function for some $\alpha > 0$, $\partial\Omega_u \cap \partial D \subset Reg(\partial\Omega_u)$ and $\mathcal{H}^{d-1}(Sing(\partial\Omega_u)) = 0$. Moreover, if $d \leq 4$ then $Sing(\partial\Omega_u) = \emptyset$.*

Proof. We first notice that, since D is smooth, every point $x_0 \in \partial\Omega_u \cap \partial D$ is flat and hence every blow-up at $x_0 \in \partial\Omega_u \cap \partial D$ is of the form (5.38); in particular, $\partial\Omega_u \cap \partial D \subset Reg(\partial\Omega_u)$. Now, the regularity of the regular part of the free boundary $Reg(\partial\Omega_u) \cap D$ follows by Lemma 5.29 and the improvement of flatness Theorem from [21], while the regularity of $Reg(\partial\Omega_u) \cap \partial D$ follows from [18]. Thus, we only need to prove the estimate on $Sing(\partial\Omega_u)$.

First, we notice that the reduced boundary $\partial^*\Omega_u \cap D \subset Reg(\partial\Omega_u)$. Indeed, let $x_0 \in \partial\Omega_u \cap D$ and $u_n := u_{x_0,r_n}$ be a blow-up sequence at x_0 converging to some $u_0 \in \mathcal{BU}_u(x_0)$ such that $\mathbb{1}_{\Omega_{u_n}}$ converges in $L^1_{loc}(\mathbb{R}^d)$ to $\mathbb{1}_{\Omega_{u_0}}$. If $x_0 \in \partial^*\Omega_u \cap D$, then Ω_{u_0} is a half-plane of the form $H = \{x \in \mathbb{R}^d \ : \ x \cdot \nu > 0\}$ for some $\nu \in \partial B_1$. Now since u_0 is a solution of the one-phase Alt-Caffarelli problem and is one-homogeneous, this implies that $\{u_0 > 0\} = \{x_d > 0\}$ (see [33, Remark 4.8]). Therefore, by uniqueness of the Cauchy problem and the optimal boundary condition from Lemma 5.29, it follows that u_0 is of the form (5.37).

Finally, since Ω_u has locally finite perimeter, the Federer Theorem and Remark 5.25 give that

$$\mathcal{H}^{d-1}\left(\partial\Omega_u \cap D \setminus (\partial^*\Omega_u)\right) = \mathcal{H}^{d-1}\left(\partial\Omega_u \cap D \setminus \left(\partial^*\Omega_u \cup \Omega_u^{(0)} \cup \Omega_u^{(1)}\right)\right) = 0,$$

which proves that $\mathcal{H}^{d-1}(Sing(\partial\Omega_u)) = 0$.

In order to prove the last claim we recall that every blow-up $u_0 \in \mathcal{BU}_u(x_0)$ is a solution of the one-phase Alt-Caffarelli problem. Thus, by [2] (for $d = 2$), [16] (for $d = 3$) and [30] (for $d = 4$), the free boundary $\partial\Omega_{u_0}$ is locally a graph of a smooth function and so the blow-up u_{00} of u_0 in 0 is of the form (5.37). Now since $u_{00} \in \mathcal{BU}_u(x_0)$ we get that $x_0 \in Reg(\partial\Omega_u)$. □

Remark 5.33. The smoothness of the free boundary can be improved under an additional regularity assumption on Φ. Indeed, if $\nabla\Phi \in C^{k+1,\alpha}(D; \mathbb{R}^d)$ for some $k \geq 1$ and $\alpha \in (0, 1)$, then by [32, Theorem 1], $Reg(\partial\Omega^*) \cap D$ is locally a graph of a $C^{k+1,\alpha}$ function.

5.11. **Further estimates on the dimension of the singular set.** We prove in this section that optimal sets to the problem (1.5) are d^*-regular, where d^* is defined below.

Definition 5.34. We define d^* as the smallest dimension which admits one-homogeneous global minimizers of the one-phase Alt-Caffarelli problem with (isolated) singularity in zero.

By [30] and [22] we know that $d^* \in \{5, 6, 7\}$. Weiss was first to prove that the monotonicity formula implies the dimension estimate

$$\dim_{\mathcal{H}} Sing(\partial\Omega_u) = \inf\{\alpha \geq 0 \ : \ \mathcal{H}^\alpha(Sing(\partial\Omega_u)) = 0\} \leq d - d^*$$

for every $d > d^*$ (see also [33] for an argument using only the monotonicity of W). Recently, using the innovative approach of Naber and Valtorta [34], Edelen and Engelstein [23] showed the the monotonicity formula implies the stronger estimate $\mathcal{H}^{d-d^*}(Sing(\partial\Omega_u)) < \infty$. Thus, as a consequence of Lemma 5.26, Lemma 5.27 and the results from [36], [33] and [23], we get

Proposition 5.35 (On the dimension of the singular set)**.** *Let u be a solution of* (5.1) *in the bounded open set $D \subset \mathbb{R}^d$. Then*

- $Sing(\partial\Omega_u) = \emptyset$ *if* $d < d^*$,
- $Sing(\partial\Omega_u)$ *is a discrete (locally finite) set if* $d = d^*$,
- $\mathcal{H}^{d-d^*}(Sing(\partial\Omega_u) \cap K) < \infty$ *for every compact set* $K \subset D$ *and* $d > d^*$.

Appendix A. Extremality conditions and Lebesgue density

In this section we prove Proposition A.1, which we use in Proposition 5.12 to show that the Lagrange multiplier Λ_u is strictly positive, but the result is of independent interest. For instance, it applies to optimal partition problems (see, for example, [20] and [17]). We first show that a function which is critical for the functional

$$J(u) := \int_D |\nabla u|^2 e^{-\Phi}\, dx - \lambda \int_D u^2 e^{-\Phi}\, dx, \tag{A.1}$$

with respect to internal variations that is

$$\delta J(u)[\xi] := \lim_{t\to 0} J(u(x + t\xi(x))) = 0 \quad \text{for every vector field} \quad \xi \in C_c^\infty(D; \mathbb{R}^d),$$

satisfies a monotonicity formula for the associated Almgren frequency function $N(r)$. Now, by the argument of Garofalo and Lin (see [25]) the monotonicity of the frequency function implies that u cannot decay too fast around the free boundary points. If, in addition, u is a solution of $-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda u e^{-\Phi}$ on the positivity set $\Omega_u = \{u > 0\}$, we can use a Caccioppoli inequality to show that if the Lebesgue density of Ω_u is too small, then the decay of u on the balls of radius r should be very fast. This, in combination with the monotonicity of the Almgren's frequency function, shows that the Lebesgue density of Ω_u should be bounded from below *everywhere* (and not only on the boundary of Ω_u). In particular, there cannot be points of zero Lebesgue density for Ω_u in D.

Proposition A.1. *Let $D \subset \mathbb{R}^d$ be a bounded open set and $\Phi \in W^{1,\infty}(D)$. Suppose that $\lambda \geq 0$ and $u \in H^1(D)$ is a nonnegative function such that*

(a) u is a solution of the equation

$$-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda e^{-\Phi} u \quad \text{in} \quad \Omega_u = \{u > 0\}; \tag{A.2}$$

(b) u satisfies the extremality condition

$$\delta J(u)[\xi] = 0 \quad \text{for every} \quad \xi \in C_c^\infty(D; \mathbb{R}^d),$$

where J is given by (A.1) *and its first variation in the direction ξ is given by*

$$\delta J(u)[\xi] := \int_D \Big[2D\xi(\nabla u)\cdot\nabla u + \big(|\nabla u|^2 - \lambda u^2\big)(\nabla\Phi\cdot\xi - \operatorname{div}\xi)\Big] e^{-\Phi} dx. \tag{A.3}$$

Then, $|D \setminus \Omega_u| = 0$.

A.1. **Reduction to the case $\lambda = 0$.** In this section we will show that it is sufficient to prove Proposition A.1 for $\lambda = 0$. The general case will then follow by an elementary substitution argument. In the next lemma we deal with the first variation of the functional J.

Lemma A.2. *Suppose that $D \subset \mathbb{R}^d$ is a bounded open set, $a : D \to \mathbb{R}$ is a given Lipschitz function such that $0 < \varepsilon \le a \le \varepsilon^{-1}$ on D. Let $\lambda > 0$ and let $\varphi \in H^2(D)$ be such that*

$$-\operatorname{div}(a\nabla\varphi) = \lambda a\varphi \quad \text{in} \quad D, \qquad \varphi \ge \varepsilon > 0 \quad \text{on} \quad D.$$

For any $u \in H^1(D)$, we set $\tilde a(x) := \varphi^2(x)a(x)$, $\tilde u := {}^{u}\!/_{\varphi}$,

$$J(u) := \int_D \big(|\nabla u|^2 - \lambda u^2\big)\, a(x)\, dx \qquad \text{and} \qquad \tilde J(u) := \int_D |\nabla u|^2 \tilde a(x)\, dx,$$

$$\delta J(u)[\xi] := \int_D \Big[2aD\xi(\nabla u)\cdot\nabla u - \big(|\nabla u|^2 - \lambda u^2\big)\operatorname{div}(a\xi)\Big]dx,$$

$$\delta\tilde J(u)[\xi] := \int_D \Big[2\tilde a D\xi(\nabla u)\cdot\nabla u - |\nabla u|^2\operatorname{div}(\tilde a\xi)\Big]dx \quad \text{for any} \quad \xi \in C_c^\infty(D;\mathbb{R}^d).$$

Then, for every $u \in H^1(D)$ and every $\xi \in C_c^\infty(D;\mathbb{R}^d)$, we have

$$\delta\tilde J(\tilde u)[\xi] = \delta J(u)[\xi] - 2\int_D \nabla\left(u\xi\cdot\nabla(\ln\varphi)\right)\cdot\nabla u\, a\, dx + 2\int_D \left(u\xi\cdot\nabla(\ln\varphi)\right)\lambda a u\, dx. \tag{A.4}$$

Proof. Notice that we may assume $u \in C^\infty(D)$. First we notice that an integration by parts gives

$$\begin{aligned}
\delta\tilde J(\tilde u)[\xi] &= \int_D 2\,\partial_i\xi_j\,\partial_i\tilde u\,\partial_j\tilde u\,\tilde a\, dx - \int_D |\nabla\tilde u|^2\operatorname{div}(\tilde a\xi)\, dx\\
&= -\int_D 2\,\xi_j\,\partial_i(\tilde a\,\partial_i\tilde u)\,\partial_j\tilde u\, dx - \int_D 2\,\xi_j\,\partial_i\tilde u\,\partial_{ij}\tilde u\,\tilde a\, dx - \int_D |\nabla\tilde u|^2\operatorname{div}(\tilde a\xi)\, dx\\
&= -\int_D 2\,\xi_j\,\partial_i(\tilde a\,\partial_i\tilde u)\,\partial_j\tilde u\, dx - \int_D \operatorname{div}(\tilde a|\nabla\tilde u|^2\xi)\, dx = -\int_D 2\,\xi_j\,\partial_i(\tilde a\,\partial_i\tilde u)\,\partial_j\tilde u\, dx\\
&= -\int_D 2(\xi\cdot\nabla\tilde u)\operatorname{div}(\tilde a\nabla\tilde u)\, dx.
\end{aligned}$$

and, analogously,

$$\delta J(u)[\xi] = -\int_D 2(\xi\cdot\nabla u)\operatorname{div}(a\nabla u)\, dx + \lambda\int_D u^2\operatorname{div}(a\xi)\, dx.$$

Now, since

$$\operatorname{div}(\tilde a\nabla\tilde u) = \operatorname{div}(a(\varphi\nabla u - u\nabla\varphi)) = \varphi\operatorname{div}(a\nabla u) - u\operatorname{div}(a\nabla\varphi) = \varphi(\operatorname{div}(a\nabla u) + \lambda a u),$$

we get

$$\begin{aligned}
\delta\tilde J(\tilde u)[\xi] &= -2\int_D \xi\cdot(\nabla u - \frac{u}{\varphi}\nabla\varphi)\big(\operatorname{div}(a\nabla u) + \lambda a u\big)\, dx\\
&= 2\int_D \xi\cdot\nabla\varphi\frac{u}{\varphi}\big(\operatorname{div}(a\nabla u) + \lambda a u\big)\, dx - 2\int_D (\xi\cdot\nabla u)\big(\operatorname{div}(a\nabla u) + \lambda a u\big)\, dx\\
&= -2\int_D \nabla\left(\frac{\xi\cdot\nabla\varphi}{\varphi}u\right)\cdot\nabla u\, a\, dx + 2\int_D \left(\frac{\xi\cdot\nabla\varphi}{\varphi}u\right)\lambda a u\, dx + \delta J(u)[\xi],
\end{aligned}$$

which is precisely (A.4). □

Let now $D \subset \mathbb{R}^d$ and $u \in H^1(D)$ be as in Proposition A.1 for some $\lambda > 0$. In order to prove that $|D \setminus \Omega_u| = 0$, it is sufficient to prove that $|(D \cap B) \setminus \Omega_u| = 0$ for any (small) ball $B \subset D$. Let now $x_0 \in D$ and let $R > 0$ be such that $\lambda_1(B_R(x_0), \nabla\Phi) = \lambda$. Such a radius exists, since the map $f(r) := \lambda_1(B_r(x_0), \nabla\Phi)$ is continuous, $f(0) = +\infty$ and $f(+\infty) = 0$. Notice also that we may assume Φ to be defined on the entire space $\mathbb{R}^d$. Let φ be the first eigenfunction on $B_R(x_0)$

and let $r = R/2$. Then, we can apply Lemma A.2 in the set $D \cap B_r(x_0)$ with $a = e^{-\Phi}$. Moreover, since u satisfies (A.2), we get that

$$\delta\tilde{J}(\tilde{u})[\xi] = \delta J(u)[\xi] = 0, \quad \text{for every} \quad \xi \in C_c^\infty(D \cap B_r(x_0); \mathbb{R}^d),$$

which proves that $\tilde{u} = u/\varphi$ satisfies hypothesis (b) for $\lambda = 0$. Finally, in order to prove that $\tilde{u}$ satisfies hypothesis (a), we notice that on $\Omega_u = \Omega_{\tilde{u}}$ we have (in a weak sense)

$$\operatorname{div}(\tilde{a}\nabla\tilde{u}) = \varphi \operatorname{div}(a\nabla u) - u \operatorname{div}(a\nabla\varphi) = \varphi\,(\operatorname{div}(a\nabla u) + \lambda a u) = 0.$$

A.2. **Proof of Proposition A.1 in the case $\lambda = 0$.** Let $\lambda = 0$. Then we have

$$J(u) := \int_D |\nabla u|^2 e^{-\Phi}\,dx, \tag{A.5}$$

$$\delta J(u)[\xi] := \int_D \Big[2D\xi(\nabla u)\cdot\nabla u + |\nabla u|^2(\nabla\Phi\cdot\xi - \operatorname{div}\xi)\Big]e^{-\Phi}dx. \tag{A.6}$$

Let $x_0 = 0 \in D$ and $\tau = \|\nabla\Phi\|_{L^\infty(D)}$. We set

$$H(r) := \int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1}, \qquad D(r) := \int_{B_r} |\nabla u|^2 e^{-\Phi} dx \qquad \text{and} \qquad N(r) := \frac{rD(r)}{H(r)}.$$

Step 1. Derivative of H. We calculate

$$\begin{aligned} H'(r) &= \frac{d-1}{r}H(r) + r^{d-1}\frac{d}{dr}\int_{\partial B_1} u^2(rx)e^{-\Phi(rx)}d\mathcal{H}^{d-1}(x) \\ &= \frac{d-1}{r}H(r) + 2\int_{\partial B_r} u\frac{\partial u}{\partial n}e^{-\Phi}d\mathcal{H}^{d-1} - \int_{\partial B_r} u^2(n\cdot\nabla\Phi)e^{-\Phi}d\mathcal{H}^{d-1} \\ &= \frac{d-1}{r}H(r) + 2\int_{B_r} |\nabla u|^2 e^{-\Phi}dx - \int_{\partial B_r} u^2(n\cdot\nabla\Phi)e^{-\Phi}d\mathcal{H}^{d-1}, \end{aligned}$$

which we rewrite as

$$H'(r) = \frac{d-1}{r}H(r) + 2D(r) - H_\Phi(r). \tag{A.7}$$

where we have set

$$H_\Phi(r) := \int_{\partial B_r} u^2(n\cdot\nabla\Phi)e^{-\Phi}d\mathcal{H}^{d-1} \qquad \text{and} \qquad |H_\Phi(r)| \le \tau H(r).$$

Step 2. Equidistribution of the energy. Let ϕ_ε be a radially decreasing function such that $0 \le \phi_\varepsilon \le 1$ on B_r, $\phi_\varepsilon = 1$ on $B_{r(1-\varepsilon)}$, $\phi_\varepsilon = 0$ on ∂B_r and $|\nabla\varphi_\varepsilon| \le C(r\varepsilon)^{-1}$. The vector field $\xi(x) := x\phi_\varepsilon(x)$ satisfies $\operatorname{div}\xi(x) = d\phi_\varepsilon(x) + x\cdot\nabla\phi_\varepsilon$ and $\partial_i\xi_j = \delta_{ij}\phi_\varepsilon(x) + x_j\partial_i\phi_\varepsilon(x)$. Since $\lambda = 0$ we have

$$\begin{aligned} \delta J(u)[\xi] &= \int_D \Big[2D\xi(\nabla u)\cdot\nabla u + |\nabla u|^2(\nabla\Phi\cdot\xi - \operatorname{div}\xi)\Big]e^{-\Phi}dx \\ &= \int_D \Big[2|\nabla u|^2\phi_\varepsilon + 2(x\cdot\nabla u)(\nabla\phi_\varepsilon\cdot\nabla u) - |\nabla u|^2(d\phi_\varepsilon(x) + x\cdot\nabla\phi_\varepsilon)\Big]e^{-\Phi}dx \\ &\quad + \int_D |\nabla u|^2(\nabla\Phi\cdot x)\phi_\varepsilon e^{-\Phi}dx, \end{aligned}$$

and passing to the limit as $\varepsilon \to 0$, rearraging the terms and using the property (b), we get

$$\begin{aligned} 0 = -(d-2)\int_{B_r} |\nabla u|^2 e^{-\Phi}dx + r\int_{\partial B_r} |\nabla u|^2 e^{-\Phi}d\mathcal{H}^{d-1} \\ - 2r\int_{\partial B_r}\left(\frac{\partial u}{\partial n}\right)^2 e^{-\Phi}d\mathcal{H}^{d-1} + \int_{B_r} |\nabla u|^2(\nabla\Phi\cdot x)e^{-\Phi}dx, \end{aligned}$$

which we rewrite as

$$-(d-2)D(r) + rD'(r) = 2r\int_{\partial B_r}\left(\frac{\partial u}{\partial n}\right)^2 e^{-\Phi}d\mathcal{H}^{d-1} - rD_\Phi(r),$$

where

$$D_\Phi(r) := \frac{1}{r}\int_{B_r} |\nabla u|^2 (\nabla\Phi\cdot x) e^{-\Phi} dx \qquad \text{and} \qquad |D_\Phi(r)| \le \tau D(r).$$

Step 3. The derivative of N. We notice that $N(r)$ is only defined for r such that $H(r) > 0$. In what follows we fix $r_0 > 0$ such that $B_{r_0}(x_0) \subset D$ and $H(r_0) > 0$. Since $u \in H^1(D)$, there is an interval $(a, b) \ni r_0$, on which $H > 0$.

$$\begin{aligned} N'(r) &= \frac{D(r)H(r) + rD'(r)H(r) - rD(r)H'(r)}{H^2(r)} \\ &= \frac{D(r)H(r) + rD'(r)H(r) - rD(r)\left(\frac{d-1}{r}H(r) + 2D(r) - H_\Phi(r)\right)}{H^2(r)} \\ &= \frac{-(d-2)D(r)H(r) + rD'(r)H(r) - 2rD^2(r) + rD(r)H_\Phi(r)}{H^2(r)} \\ &= \frac{2r}{H^2(r)}\left(H(r)\int_{\partial B_r}\left(\frac{\partial u}{\partial n}\right)^2 e^{-\Phi} d\mathcal{H}^{d-1} - D^2(r)\right) + \frac{r\left(D(r)H_\Phi(r) - D_\Phi(r)H(r)\right)}{H^2(r)} \end{aligned} \tag{A.8}$$

Now we notice that, since u solves (A.2) on Ω_u, we have

$$D(r) = \int_{B_r} |\nabla u|^2 e^{-\Phi} dx = \int_{\partial B_r} u\frac{\partial u}{\partial n} e^{-\Phi} d\mathcal{H}^{d-1},$$

and so, by the Cauchy-Schwarz inequality and (A.8) we obtain

$$N'(r) \ge \frac{r\left(D(r)H_\Phi(r) - D_\Phi(r)H(r)\right)}{H^2(r)} \ge -2\tau N(r). \tag{A.9}$$

Step 4. A bound on $N(r)$. Using the estimate (A.9) from the previous step we get that the function $r \mapsto e^{2\tau r}N(r)$ is non-decreasing in r and so

$$N(r) \le e^{2\tau(r_0 - r)}N(r_0) \le e^{2\tau r_0}N(r_0) \quad \text{for every} \quad a < r \le r_0.$$

Step 5. Strict positivity and doubling inequality for $H(r)$. By the step 4 we have

$$\frac{d}{dr}\left[\log\left(\frac{H(r)}{r^{d-1}}\right)\right] = 2\frac{N(r)}{r} - \frac{H_\Phi(r)}{H(r)} \le \frac{2e^{2\tau r_0}N(r_0)}{r} + \tau, \tag{A.10}$$

and integrating we get

$$\log\left(\frac{H(r_0)}{r_0^{d-1}}\right) - \log\left(\frac{H(r)}{r^{d-1}}\right) \le \log\left(\frac{r_0}{r}\right) 2e^{2\tau r_0}N(r_0) + \tau r_0, \quad \text{for every} \quad a < r \le r_0.$$

In particular, $H > 0$ on every interval $[\varepsilon r_0, r_0]$ and so, $H > 0$ on $(0, r_0]$ and we might take $a = 0$. Moreover, integrating once again the inequality (A.10) from $r < r_0/2$ to $2r$, we get

$$\log\left(\frac{H(2r)}{H(r)}\right) \le ((d-1)\log 2 + \tau r_0) + 2\log 2\, e^{2\tau r_0}N(r_0) \qquad \text{for every} \qquad 0 < r \le \frac{r_0}{2}.$$

Taking $r_0 \le 1$, there is a constant C, depending only on d and τ, such that

$$H(2r) \le C\exp(CN(r_0))H(r) \qquad \text{for every} \qquad 0 < r \le \frac{r_0}{2}. \tag{A.11}$$

Integrating once more in r we get

$$\int_{B_{2r}} u^2 e^{-\Phi}\, dx \le C\exp(CN(r_0))\int_{B_r} u^2 e^{-\Phi}\, dx \qquad \text{for every} \qquad 0 < r \le \frac{r_0}{2}. \tag{A.12}$$

Step 6. Caccioppoli inequality and conclusion. Let $r \in (0, r_0/2]$ and let $\phi \in C_0^\infty(B_{2r})$ be such that $\phi = 1$ in B_r, $\phi = 0$ on ∂B_{2r}, $0 \le \phi \le 1$ and $|\nabla\phi| \le 2/r$ on $B_{2r} \setminus B_r$. Using the fact that u is a

solution of $-\operatorname{div}(e^{-\Phi}\nabla u) = 0$ in Ω_u, we get the following Caccioppoli inequality:

$$\begin{aligned}\int_{B_r} |\nabla u|^2 e^{-\Phi}\,dx &\le \int_{B_{2r}} |\nabla(u\phi)|^2 e^{-\Phi}\,dx = \int_{B_{2r}} \left(u^2|\nabla\phi|^2 + \nabla u\cdot\nabla(u\phi^2)\right) e^{-\Phi}\,dx\\ &= \int_{B_{2r}} u^2|\nabla\phi|^2 e^{-\Phi}\,dx - \int_{B_{2r}} u\phi^2 \operatorname{div}\left(e^{-\Phi}\nabla u\right)\,dx = \int_{B_{2r}} u^2|\nabla\phi|^2 e^{-\Phi}\,dx.\\ &\le \frac{4}{r^2}\int_{B_{2r}} u^2 e^{-\Phi}\,dx. \end{aligned} \tag{A.13}$$

On the other hand, there are dimensional constants C_d and $\varepsilon_d > 0$ such that, if $|\Omega_u \cap B_r| \le \varepsilon_d |B_r|$, then the following inequality does hold (see [15, Lemma 4.4])

$$\int_{B_r} u^2\,dx \le C_d r^2 \left(\frac{|\Omega_u \cap B_r|}{|B_r|}\right)^{2/d} \int_{B_r} |\nabla u|^2\,dx,$$

which, taking $C := C_d \exp(\max\Phi - \min\Phi)$, implies

$$\int_{B_r} u^2 e^{-\Phi}\,dx \le C r^2 \left(\frac{|\Omega_u \cap B_r|}{|B_r|}\right)^{2/d} \int_{B_r} |\nabla u|^2 e^{-\Phi}\,dx.$$

This, together with (A.13) and the doubling inequality (A.12), gives that there are constants C_1 and C_2, depending only on d and τ such that

$$\min\left\{\varepsilon_d, C_1\exp(-C_2 N(r_0))\right\} \le \frac{|\Omega_u \cap B_r|}{|B_r|} \qquad \text{for every} \qquad 0 < r \le \frac{r_0}{2},$$

where to be precise we recall that we assumed $r_0 \le 1$. In particular, we have a lower density bound for Ω_u at *every* point of D, which implies that $|D \setminus \Omega_u| = 0$ and concludes the proof. □

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Emmanuel Russ:
Université Grenoble Alpes, CNRS UMR 5582, Institut Fourier
100 rue des Mathématiques, F-38610 Gières, France
E-mail address: emmanuel.russ@univ-grenoble-alpes.fr

Baptiste Trey:
Université Grenoble Alpes, CNRS UMR 5582, Institut Fourier
100 rue des Mathématiques, F-38610 Gières, France
E-mail address: baptiste.trey@etu.univ-grenoble-alpes.fr

Bozhidar Velichkov:
Université Grenoble Alpes, CNRS UMR 5224
700 avenue Centrale, F-38401 Domaine Universitaire de Saint-Martin-d'Hères, France
E-mail address: bozhidar.velichkov@univ-grenoble-alpes.fr