

On the modeling of elastic waveguides coupled to infinite media

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- 3 Forced response
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- 5 Numerical results
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Context

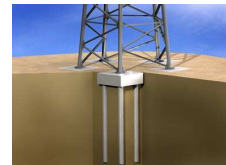
- **Non Destructive Evaluation** of elongated structures that are buried, completely or partially (including surface waves)
- Many examples in civil engineering and geophysics



Prestressing cable



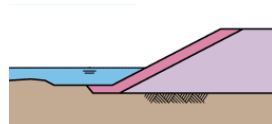
Rockbolt



Piles



Pavement

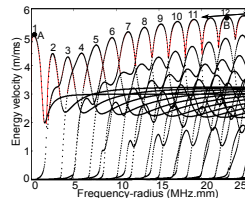


Seawall

Wave modes

Generality:

- Guided wave propagation: dispersive and multimodal
- Dispersion curves required
- Modeling tools needed
- Waves=modes \rightarrow eigenvalue problem

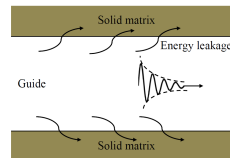


Energy velocity vs. frequency

Waveguides coupled to infinite media

- unbounded in the transverse direction
- terminology: **open waveguides**
(as opposed to *closed* waveguides, in vacuum)

Computation of eigenmodes in unbounded problems

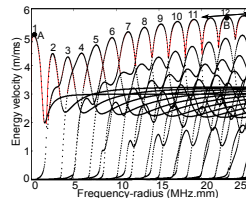


Open waveguide

Wave modes

Generality:

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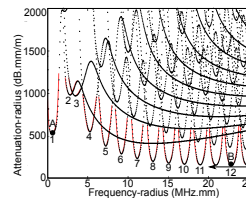


Energy velocity vs. frequency

Waveguides coupled to infinite media

- unbounded in the transverse direction
- terminology: **open waveguides** (as opposed to *closed* waveguides, in vacuum)
- leakage loss: **leaky modes**, attenuation to minimize for NDE

Computation of eigenmodes in unbounded problems



Attenuation vs. frequency

State-of-the-art (elasticity)

- plates, cylinders: analytical approaches (Thomson-Haskell, ©Disperse, ...)
- **arbitrary cross section**: finite element discretization of cross-section (**Semi-Analytical Finite Element** method), but...

... how to handle the infinite transverse direction?

No discretization of the surrounding medium:

- SAFE + Boundary Element Method^a (nonlinear eigenproblem)
- SAFE + dashpot boundary condition^b (approximation)

Discretization of the truncated surrounding medium:

- SAFE + Absorbing Layer^c (slowly varying visco. → large thickness)
- **SAFE + Perfectly Match Layer (PML)**^d

^aMazzotti et al., Ultrasonics, 2013

^bGravenkamp et al., JASA, 2014

^cCastaings and Lowe, JASA, 2008

^dTreysse et al., Wave Motion, 2014 ; Nguyen et al., JSV, 2015

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SAFE-PML approach

- Harmonic elastodynamic equilibrium equations:

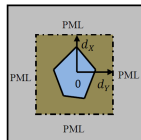
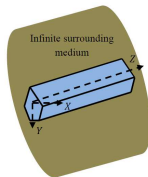
$$\begin{cases} \tilde{\nabla} \cdot \tilde{\sigma}(\tilde{\mathbf{u}}) + \tilde{\rho}\omega^2\tilde{\mathbf{u}} = \tilde{\mathbf{f}} \\ \tilde{\sigma}(\tilde{\mathbf{u}}) = \tilde{\mathbf{C}} : \tilde{\epsilon}(\tilde{\mathbf{u}}), \quad \tilde{\epsilon}(\tilde{\mathbf{u}}) = \frac{1}{2}(\tilde{\nabla}\tilde{\mathbf{u}} + \tilde{\nabla}^T\tilde{\mathbf{u}}) \end{cases} \quad (1)$$

with **PML** for transverse coordinates (x, y) :

$$\tilde{x} = \int_0^x \gamma_x(s) ds \quad \text{with} \quad \begin{cases} \gamma_x = 1 & \text{if } |x| \leq d_x \\ \text{Im } \gamma_x > 0 & \text{if } |x| > d_x \end{cases} \quad (\text{idem for } \tilde{y})$$

- Change from complex to real coordinates:

$$\tilde{x} \mapsto x : \quad \frac{\partial}{\partial \tilde{x}} = \frac{1}{\gamma_x} \frac{\partial}{\partial x}, \quad d\tilde{x} = \gamma_x dx \quad (\text{idem for } \tilde{y}) \quad (2)$$



User-defined parameters of a PML

attenuation function $\gamma_x(x)$, distance to the core d_x , thickness h_x (idem for y)

- Spatial Fourier transform along z :

$$\hat{\mathbf{u}}(k) = \int_{-\infty}^{+\infty} \mathbf{u}(z) e^{-ikz} \quad (3)$$

- FE discretization of the cross-section (x, y)

⇒ SAFE-PML formulation

$$\left(\mathbf{K}_1 - \omega^2 \mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2 \mathbf{K}_3 \right) \hat{\mathbf{U}} = \hat{\mathbf{F}} \quad (4)$$

Eigenvalue problem (free response)

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Characteristics of the eigenvalue problem:

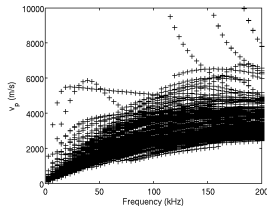
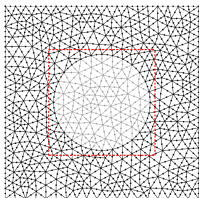
- \mathbf{K}_1 , \mathbf{K}_2 , \mathbf{K}_3 , \mathbf{M} are **complex** due to γ_x , γ_y
- not self-adjoint
- quadratic for k

Linearization of the EVP:

$$(\mathbf{A} - k\mathbf{B})\mathbf{x} = \hat{\mathbf{0}} \quad \text{with } \mathbf{x} = \begin{bmatrix} \hat{\mathbf{U}} \\ k\hat{\mathbf{U}} \end{bmatrix} \quad (5)$$

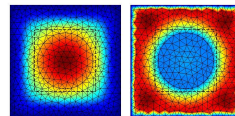
Example of dispersion curves

Validation test case: steel cylinder into concrete (Castaings and Lowe JASA 2008)



Cross-section mesh (left), phase velocity curves before (middle)

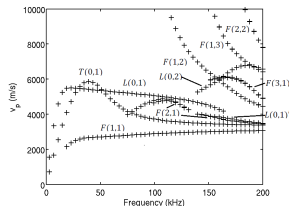
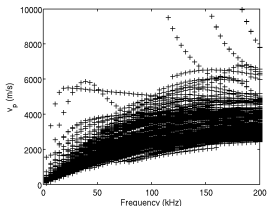
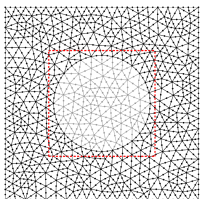
- Many 'PML modes'...



Leaky mode (left) and PML mode (right)

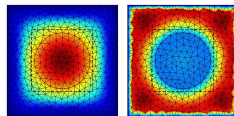
Example of dispersion curves

Validation test case: steel cylinder into concrete (Castaings and Low JASA 2008)



Cross-section mesh (left), phase velocity curves before (middle) and after modal filtering (right)

- Many 'PML modes'...
- Efficient modal filtering based on kinetic energy ratio:
 $E_{PML}/E_{TOT} > \text{threshold}$



Leaky mode (left) and PML mode (right)

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 - Biorthogonality and solution
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Biorthogonality and solution

Knowing \mathbf{U}_m , we want to solve $(\mathbf{K}_1 - \omega^2 \mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2 \mathbf{K}_3) \hat{\mathbf{U}} = \hat{\mathbf{F}}$ using the modal expansion: $\hat{\mathbf{U}} = \sum_m \alpha_m \mathbf{U}_m$

Biorthogonality relationship with PML

- fundamental property of the EVP: \mathbf{K}_1 , \mathbf{K}_3 , \mathbf{M} are symmetric
→ eigensolutions (k_m, \mathbf{U}_m) and $(-k_m, \mathbf{U}_{-m})$ of opposite travelling direction
- a general **biorthogonality relationship**, applicable to lossy waveguides^a:

$$\frac{i\omega}{4} (\mathbf{U}_m^T \mathbf{F}_{-n} - \mathbf{U}_{-n}^T \mathbf{F}_m) = Q_{m,-m} \delta_{mn}$$

- \equiv discretized version of Auld's real orthogonality

^aTreysède and Laguerre, JASA, 2013, $\mathbf{F}_m = (\mathbf{K}_2^T + ik_m \mathbf{K}_3) \mathbf{U}_m$

Biorthogonality and solution

Knowing \mathbf{U}_m , we want to solve $(\mathbf{K}_1 - \omega^2 \mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2 \mathbf{K}_3) \hat{\mathbf{U}} = \hat{\mathbf{F}}$ using the modal expansion: $\hat{\mathbf{U}} = \sum_m \alpha_m \mathbf{U}_m$

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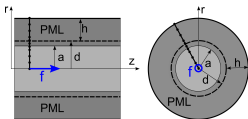
^aTreysède and Laguerre, JASA, 2013, $\mathbf{F}_m = (\mathbf{K}_2^T + ik_m \mathbf{K}_3) \mathbf{U}_m$

- Biorthogonality relationship → unique modal coefficients $\alpha_m = -\frac{\omega \mathbf{U}_{-m}^T \hat{\mathbf{F}}}{4Q_{m,-m}(k_m - k)}$
- Inverse spatial Fourier transform thanks to Cauchy's residues theorem:

$$\mathbf{U}(z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbf{U}}(k, \omega) e^{ikz} dk = \sum_{m=1}^M \underbrace{\frac{i\omega \mathbf{U}_m \mathbf{U}_{-m}^T}{4Q_{m,-m}}}_{\mathbf{E}_m: \text{excitability}} \hat{\mathbf{F}}(k_m) e^{ik_m z}$$

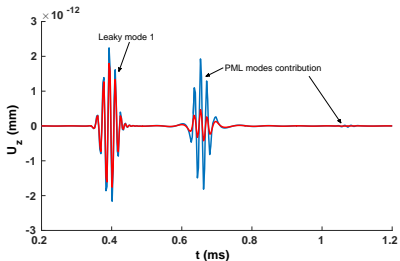
- Solution in the time domain (IFFT): $\mathbf{U}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{U}(z, \omega) e^{-i\omega t} d\omega$

Example of response



Axisymmetric SAFE-PML model of open waveguide
 PML parameters: $\hat{\gamma} = 4 + 4i$, $h = 4a$, $d = a$, $M = 51$

Steel into grout, low frequency: no trapped modes, 1 leaky mode, PML modes



$u_z(r=0)$ as a function of time at $z = 175a$
 Elastic (blue) and viscoelastic (red) materials

Problem: PML modes are not intrinsic to the physics...

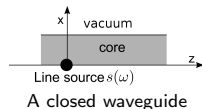
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 - Step 1: spectrum of open waveguides
 - Step 2: including a PML
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Step 0: back to closed waveguides

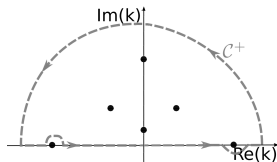
Assumptions:

- two-dimensional
- out-of-plane line source
- **scalar** (*i.e.* satisfying Helmholtz)
- **closed**



The solution is of the form:

$$\hat{u}(x, k) = \frac{a(x, k)}{b(k)} \Rightarrow u(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a(x, k)}{b(k)} e^{ikz} dk = \sum \text{modes}^1 \quad (\text{residue theorem})$$



Integration path, ●: poles (roots of $b(k)$, eigenvalues)

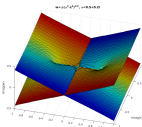
¹In closed waveguides, modes are propagative ($k_m \in \mathbb{R}$), evanescent ($k_m \in i\mathbb{R}$) or inhomogeneous

Step 1: spectrum of open waveguides (Malichewsky, 1987 ; Collin, 1991; ...)

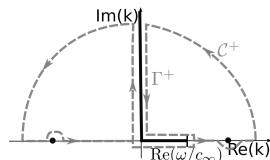
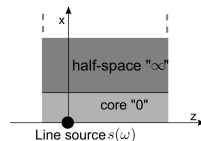
The solution in the half-space is of the form:

$$u_{\infty}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a(k)}{b(k)} e^{i\alpha_{\infty} x} e^{ikz} dk$$

with $\alpha_{\infty} = \pm \sqrt{\omega^2/c_{\infty}^2 - k^2}$: multivalued



Riemann surface for α_{∞}



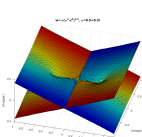
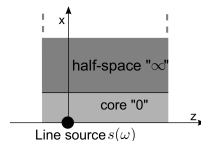
Integration path, Γ^+ : branch cut
●: poles of the proper sheet

Step 1: spectrum of open waveguides (Malichewsky, 1987 ; Collin, 1991; ...)

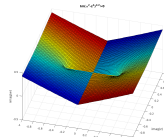
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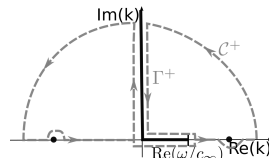
with $\alpha_{\infty} = \pm \sqrt{\omega^2/c_{\infty}^2 - k^2}$: multivalued



Riemann surface for α_{∞}



The sheet $\text{Im}(\alpha_{\infty}) > 0$



Integration path, Γ^+ : branch cut
●: poles of the proper sheet

The 'proper' sheet for \int_{C^+} to vanish: $\text{Im}(\alpha_{\infty}) > 0$
 $\text{Im}(\alpha_{\infty}) = 0$: discontinuity (branch cut)

Solution for $z > 0$:

$$u(x, z) = \underbrace{\sum \text{trapped modes}}_{\text{poles of the proper sheet}} + \underbrace{\int_{\Gamma^+} \text{radiation modes}}_{\text{branch cut contribution}}$$

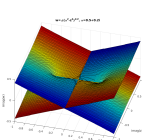
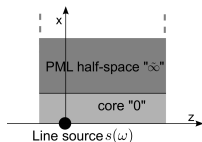
^aTrapped mode do not exist if $c_0 > c_{\infty}$, i.e. for usual configurations...

Step 2: including a PML (Gallezot and al., JASA-EL, 2017)

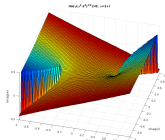
Assumptions: infinite PML, $\gamma(x) = \gamma$ in the PML

The solution in the half-space is now:

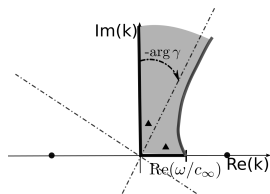
$$u_\infty(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a(k)}{b(k)} e^{i\gamma\alpha_\infty x} e^{ikz} dk$$



Riemann surface for α_∞



The sheet $\text{Im}(\gamma\alpha_\infty) > 0$



PML branch cut rotation, \blacktriangle : poles of the improper sheet without PML

The new sheet for $\int_{\mathcal{C}_+}$ to vanish: $\text{Im}(\gamma\alpha_\infty) > 0$

New branch cut: $\text{Im}(\gamma\alpha_\infty) = 0$

$$u(x, z) = \sum \text{trap. modes} + \underbrace{\sum \text{revealed leaky modes}}_{\text{poles such that } \text{Im}(\alpha_\infty) < 0!} + \underbrace{\int_{\tilde{\Gamma}_+} \text{radiation modes}^a}_{\text{new branch cut contribution}}$$

^aLeaky modes are a good approximation of the initial continuum, only near the core

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 - Remaining issues
 - The homogeneous test case
 - Back to the waveguide configuration
- 6 Conclusion

Remaining issues:

- What about elastodynamics? (non-scalar waveguide)
- What about PML truncation to a finite thickness?

With a finite thickness PML:

$$u(x, z) = \sum \text{trapped} + \sum \text{revealed leaky} + \underbrace{\sum \text{'PML modes'}}_?$$

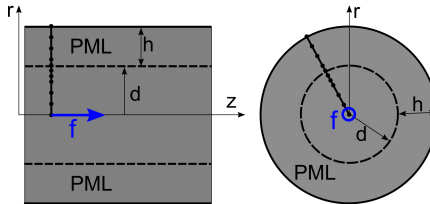
PML modes:

- discrete set
- dependent on user-defined parameters (PML thickness,...)

→ **non-intrinsic to the physics**

- does the discrete set of PML modes still have any meaning?
- contribution of PML modes vs. leaky modes?

The homogeneous test case



Axisymmetric SAFE-PML model of a homogeneous elastic medium

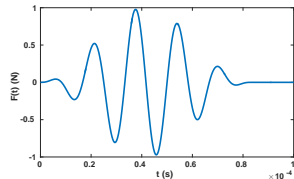
- Fully 3D homogeneous: no waveguide!
- No trapped, no leaky \rightarrow only PML modes...
- Analytical solution available (see e.g. Aki and Richards, 1980)

Excitation: point force, $\mathbf{f}(r, z, t) = \frac{F(t)}{2\pi r} \delta(r) \delta(z) \mathbf{e}_z$

Material: grout

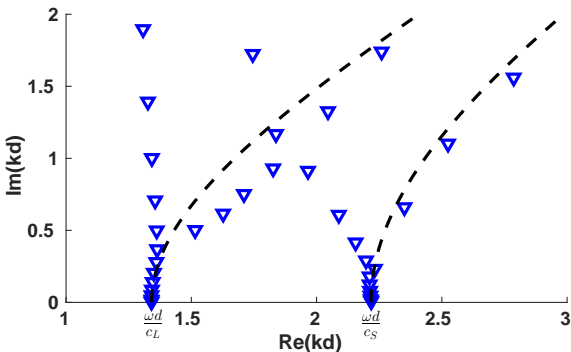
PML parameters:

- complex thickness $d + \hat{\gamma}h = d + (4 + 4i) \times 4d$
- number of modes in modal expansion $M = 50$



$F(t)$: sinus 60kHz, 5 cycles, Hanning window

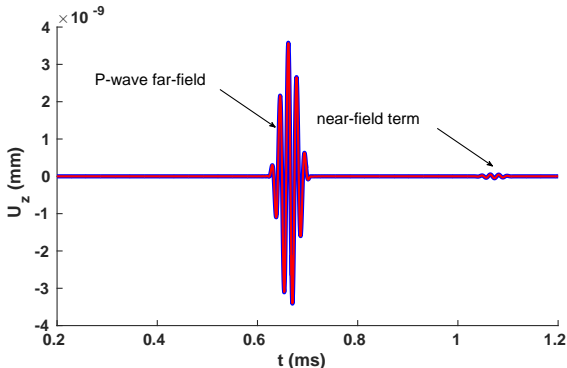
SAFE-PML Eigenspectrum



Discrete set of PML modes at a given frequency ($f=60\text{kHz}$)
(dashed lines: theoretical branchcuts with infinite PML)

- Elastodynamics \rightarrow 2 continua (1 for longitudinal waves + 1 for shear waves)
- Discrete PML modes = discretization of the continua

Response to point force

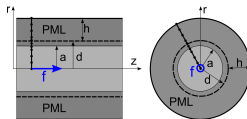


$u_z(r=0)$ as a function of time at a given distance ($z = 175d$)
SAFE-PML solutions (red) and analytical solutions (blue)

- Discrete PML modes enables to reassemble the exact solution^a

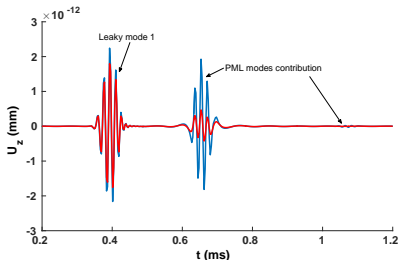
^aA proof in scalar waveguides: Olyslager, SIAM J. Appl. Math., 2004

Back to the waveguide configuration



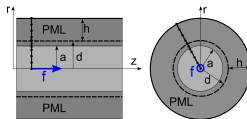
Axisymmetric SAFE-PML model of open waveguide
 PML parameters: $\hat{\gamma} = 4 + 4i$, $h = 4a$, $d = a$, $M = 51$

Steel into grout, low frequency: no trapped modes, only 1 leaky + 50 PML modes



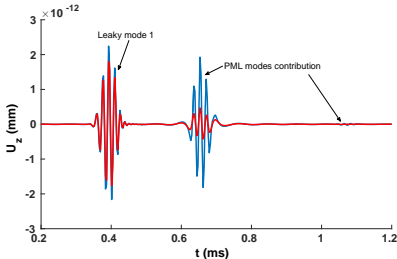
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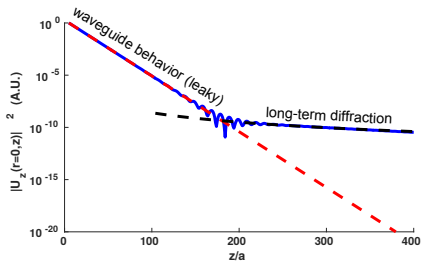


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Energy decay of the solution vs. distance.
 Solution for 51 modes (blue), 1 leaky mode (red)
 and geometrical decay (black).

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Conclusion

Using PML for the numerical modeling of open waveguides:

- rather simple to implement
- 3 discrete sets: trapped, leaky and PML modes
- trapped and leaky modes: intrinsic to the physics
- PML modes: contribution to long-term diffraction
what about alternative methods? (not possible with AL; Hu and Menyuk, AOP, 2009)
- provides **orthogonality for all types of modes, leaky included**
in theory: orthogonality holds for trapped and radiation modes only
→ the contribution of leaky is revealed in a natural way (**excitability**)

For NDE, in practice:

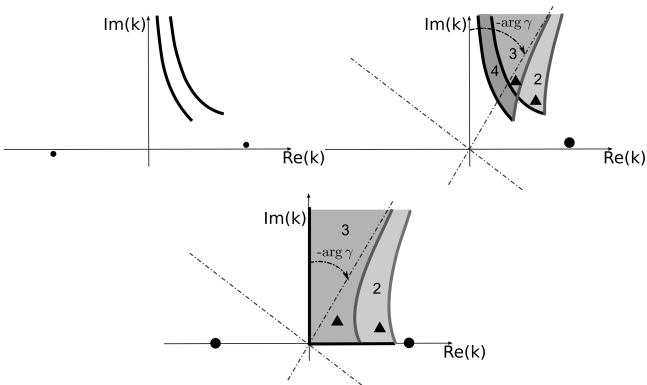
- attention is restricted to leaky modes (+trapped modes if existence)
 - the PML thickness can be greatly reduced
 - the PML can be set close to the core to limit the transverse growth of leaky
- **reduction of the computational domain**

Drawbacks and future works:

- PML parameters are user-defined → optimization?...
- many PML modes computed for a few leaky: computational time ↗
→ numerical strategies?...

The elastodynamic problem

Elastodynamics: compressional + shear waves \Rightarrow two transverse wavenumbers
4 Riemann sheets and 2 branch cuts \Rightarrow 2 continua of radiation modes²



Branch cuts and poles: viscoelastic medium without PML, with PML and pure elastic case³

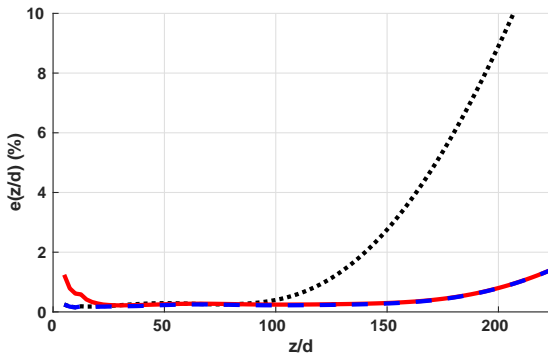
\rightarrow only 2 sheets revealed by PML in pure elastic medium

²Lapwood, Trans. Roy. Soc. A, 1949

³Gallezot and al., JASA-EL, 2017

Influence of PML parameters on error vs. distance

$$\text{Error vs. distance: } e(z) = \sqrt{\frac{\int |u_z^{\text{ref}}(z, \omega) - u_z^{\text{num}}(z, \omega)|^2 d\omega}{\int |u_z^{\text{ref}}(z, \omega)|^2 d\omega}}$$



Relative error as a function of the propagation distance. $\hat{\gamma} = 4 + 4j$. $h = 4d$, $M = 50$; $h = 4d$, $M = 30$; $h = 3d$, $M = 50$.

Influence of the argument

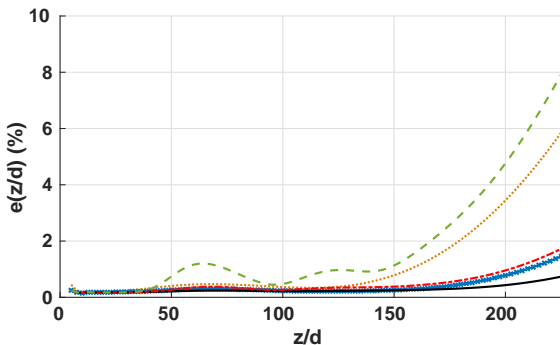


Figure : Relative error as a function of propagation distance for a complex thickness $25e^{j\theta}$. Dotted orange line: $\theta = 20$ deg ; crossed blue line: $\theta = 30$ deg ; black line: $\theta = 45$ deg ; dashed-dotted red line: $\theta = 60$ deg ; dashed green line: $\theta = 70$ deg.