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Abstract. We determine the asymptotic behavior of the Green function for zero-drift random walks confined to multidimensional convex cones. As a consequence, we prove that (up to a multiplicative constant) there is a unique positive discrete harmonic function for these processes, in other words the Martin boundary is reduced to a singleton.

1. Introduction and main results

The main purpose of the present paper is to determine the asymptotic behavior of the Green functions for zero drift random walks confined to multidimensional convex cones. We deduce that there is a unique (up to a multiplicative constant) positive discrete harmonic function for these processes; in other words the Martin boundary is reduced to a singleton.

Context. Random walks conditioned to stay in multidimensional cones are a very popular topic in probability. Indeed, they appear naturally in various situations: nonintersecting paths [44, 22, 15], which can be seen as random walks in Weyl chambers, random walks in the quarter plane [24, 40], queueing theory [12], branching processes and random walks in random environment [1], finance [13], modeling of some populations in biology [7], etc. As these random walk models are in bijection with many other discrete models (maps, permutations, trees, Young tableaux, partitions), they are also intensively studied in combinatorics [11, 9, 18].

Let us now briefly review the literature regarding asymptotics of Green functions and Martin boundary for killed random walks in cones (see [43] for a general introduction to Martin boundary theory). In the one-dimensional case, Doney [17] describes the harmonic functions and the Martin boundary of a random walk $\{S(n)\}$ on $\mathbb{Z}$ killed on the negative half-line (obviously there is essentially a unique cone in dimension 1, namely $\mathbb{N} = \{0, 1, 2, \ldots\}$). Alili and Doney [2] extend this result to the corresponding space-time random walk $\{(S(n), n)\}$.

In the higher dimensional case, let us start by quoting the famous Ney and Spitzer result [38] on the Green function asymptotics and the Martin boundary of drifted, unconstrained random walks in $\mathbb{Z}^d$. The Martin boundary is homeomorphic to the unit sphere $\mathbb{S}^{d-1}$. By large deviation techniques and Harnack inequalities, Ignatiouk-Robert

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[28, 29], then Ignatiouk-Robert and Loree [31], find the Martin boundary of random walks in half-spaces $\mathbb{N} \times \mathbb{Z}^{d-1}$ and orthants $\mathbb{N}^d$, with non-zero drift and killed at the boundary; they also derive the asymptotics of ratios of Green functions. For small step walks in the quarter plane, Lecouvey and Raschel [34] show that generating functions of harmonic functions are strongly related to certain conformal mappings.

The results are rarer for zero-drift random walks, and typically require a strong underlying structure: the random walks are Cartesian products in [39]; they are associated with Lie algebras in [4, 5]; certain reflection groups are supposed to be finite in [6]. Varopoulos [46, 47] derives upper and lower bounds for the tail probability under an additional assumption that the increments of the random walk are bounded. He also proves various statements on the growth or harmonic functions. Raschel [40, 41] obtains the asymptotics of the Green function and the Martin boundary in the case of small step quadrant random walks related to finite reflection groups. Bouaziz, Mustapha and Sifi [8] prove the existence and uniqueness of the positive harmonic function for random walks satisfying finite range, centering and ellipticity conditions, killed at the boundary of the orthant $\mathbb{N}^d$. Interestingly the results of [8] can be applied to (weakly) inhomogeneous random walks. Ignatiouk-Robert [30] shows the uniqueness of the harmonic function in a convex cone, under the assumption that the first exit time has infinite expectation.

Notations and assumptions on cones and random walks. We now present our hypotheses, which are of three types: some of them only concern the random walk (namely $(H1)$, $(H2)$ and $(H3)$), the assumption $(H4)$ is a restriction on the cone, while the last ones ($(H5)$ and $(H6)$) concern the behavior of the random walk in the cone.

Consider a random walk $\{S(n)\}_{n\geqslant 1}$ on $\mathbb{R}^d$, $d \geqslant 1$, where

$$S(n) = X(1) + \cdots + X(n)$$

and $\{X(n)\}_{n\geqslant 1}$ is a family of independent and identically distributed (i.i.d) copies of a random variable $X = (X_1, \ldots, X_d)$. The increments are supposed to be lattice and

$(H1)$ $\mathbb{E}(X_i) = 0$,
$(H2)$ $\mathrm{cov}(X_i, X_j) = \delta_{i,j}$,
$(H3)$ the random walk is aperiodic.

Notice that $(H2)$ is not a restriction: we may always perform a linear transform so as to decorrelate the random walk (obviously this linear transform changes the cone in which the walk is defined).

Denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$ and by Σ an open, connected subset of $\mathbb{S}^{d-1}$. Let K be the cone generated by the rays emanating from the origin and passing through Σ, i.e., $\Sigma = K \cap \mathbb{S}^{d-1}$; see Figure 1 for two examples. By standard analytic results (see, e.g., [27, Sec. 6.3]), there exists up to a scalar multiplication a unique function u harmonic on K, i.e., $\Delta u = 0$, such that $u_{\partial K} = 0$, ∂K denoting the boundary of K. The function u is called the réduite of K, it is homogeneous (or radial) in the sense that there exists $p > 0$ such that $u(tx) = t^p u(x)$ for all $t > 0$ and $x \in K$. The homogeneity exponent p is called the exponent of the cone K.

In this paper, we suppose the following:

$(H4)$ the cone K is convex.

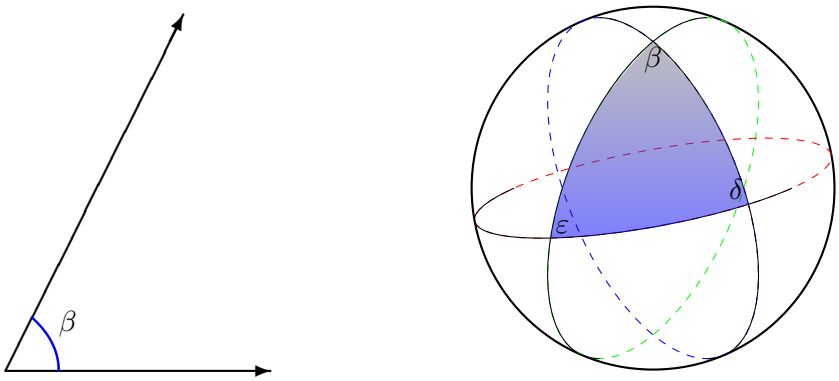

FIGURE 1. In dimension 2, Σ is an arc of circle and the cone K is a wedge of opening β. In dimension 3, any section $\Sigma \subset \mathbb{S}^2$ can be taken. The picture on the right gives the example of a spherical triangle on the sphere $\mathbb{S}^2$, corresponding to the orthant $K = \mathbb{N}^3$ (after possible decorrelation of the coordinates, see $(H2)$)

We shall also assume a moment condition on the increments:

$(H5)$ $\mathbb{E}(|X|^{r(p)}) < \infty$ for some $r(p) > 2p + d - 2 + 2(1 - p/2)^+$ and $\mathbb{E}(|X|^{2+\delta}) < \infty$, for some $\delta > 0$;

Our final hypothesis deals with the behavior of the random walk in the cone K. First, we require a form of irreducibility for the random walk, which is an adaptation to unbounded random walks of the concept of reachability condition from infinity introduced in [10]. From now on, we fix an origin $x_0 \in \mathbb{R}^d$ and denote by Λ the lattice generated by the random walk starting at x_0.

$(H6)$ There exists a constant $R > 0$ such that for any $z \in K \cap \Lambda$, $|z| \geqslant R$, there exists a path with positive probability in $K \cap B(z, R)$ which starts in $z + K$ and ends at z.

There are several simple situations where the latter condition is satisfied. In particular, this is the case when $\mathbb{P}(X \in -K) > 0$, or when K is additionally $\mathcal{C}^2$ (see [49]).

We do not require in this paper the existence of a bigger cone K' with $\partial K \setminus \{0\} \subset \mathrm{int}(K')$ and such that the réduite u can be extended to a harmonic function on K'. This condition, which was necessary in [16], is removed in Appendix B under the moment assumption $(H5)$.

Exit time, Green functions, harmonic functions and reverse random walk. Let τ_x be the exit time from the cone K of the random walk with starting point $x \in K$, i.e.,

$$\tau_x = \inf\{n \geqslant 1 : x + S(n) \notin K\}. \tag{1}$$

By definition, the Green function is

$$G_K(x, y) = G(x, y) = \sum_{n=0}^{\infty} \mathbb{P}(x + S(n) = y, \tau_x > n). \tag{2}$$

A function $h : K \longrightarrow \mathbb{R}$ is said to be (discrete) harmonic with respect to K and $\{S(n)\}$ if for every $x \in K$ and $n \geqslant 1$,

$$h(x) = \mathbb{E}(h(x + S(n)), \tau_x > n).$$

Remark that the above identity for $n = 1$ implies all the other relations for $n \geqslant 2$. In the sequel a harmonic function with respect to K and $\{S(n)\}$ will be simply called a harmonic function.

Denisov and Wachtel proved [16, Thm 1] the existence of a positive harmonic function $V : K \longrightarrow \mathbb{R}_+$ defined by

$$V(x) = \lim_{n\to\infty} \mathbb{E}(u(x + S(n)), \tau_x > n). \tag{3}$$

This harmonic function is of central importance in the present paper, since it will ultimately be identified with the Martin boundary of the random walk in K.

We denote by $\{S'(n)\}_{n\geqslant 1}$ the reverse random walk, which is the sum of the increments $\{X'(n)\}_{n\geqslant 1}$, i.i.d, independent from $\{X(n)\}_{n\geqslant 1}$ and such that $X'(n)$ is distributed as $-X$. In the sequel, every quantity involving S' will be denoted similarly as the same quantity involving S, with a prime added at the right.

Structure and sketch of the results. Our paper is organized as follows:

- Section 2: Gaussian estimates for the heat kernel in a cone and coupling approach of [16]. Some proofs are postponed to Appendix A
- Section 3: Theorem 1 and Corollary 2 giving uniform extensions of the local limit theorem [16, Thm 6]
- Section 4: Theorem 3 on the Green function asymptotics along any direction of the cone; Theorem 4 on the Martin boundary reduced to one point
- Appendix B: proof that the hypothesis (H^*) done in [16] may be removed

In the remaining part of the introduction we present all these results in more details.

Local limit theorem at the fluctuation scale. The first result of the paper is a uniform extension of the local limit theorem [16, Thm 6]. This extension can be made more or less explicit depending on the proximity with the boundary of the cone with respect to the fluctuation scale. Let $\varepsilon > 0$ be a small parameter. Following [16] we set

$$K_{n,\varepsilon} = \{x \in K : d(x, \partial K) \geqslant n^{1/2-\varepsilon}\}. \tag{4}$$

Since the typical fluctuations of the random walk at time n are expected to be of order $\sqrt{n}$, $K_{n,\varepsilon}$ represents the set of points of the cone whose distance to the boundary at time n is more than $n^{-\varepsilon}$ times the typical fluctuations.

For $x \in K$ and $n \geqslant 1$, set $t_{x,\varepsilon}(n) = \inf\{m \geqslant 0 : x + S(m) \in K_{n,\varepsilon}\}$. Then $t_{x,\varepsilon}(n)$ is a stopping time with respect to $\{S(m)\}$, which gives the first time at which the random walk started at x is far from the boundary compared to the order of fluctuation $\sqrt{n}$; we denote then by $x_\varepsilon(n)$ the position $x + S(t_{x,\varepsilon}(n))$. Define similarly $t'_{x,\varepsilon}(n)$ and $x'_\varepsilon(n)$ for the reverse random walk.

Theorem 1. *Assume (H1)–(H6). For $A > 0$ let $K_{n,\varepsilon}^A$ denote the set $K_{n,\varepsilon} \cap B(0, A\sqrt{n})$. Then there exists a constant κ which depends only on K such that for any fixed $x \in K$,*

$$\mathbb{P}(x + S(n) = y, \tau_x > n) \underset{n\to\infty}{\sim} \kappa V(x) n^{-p/2-d/2} \mathbb{E}(u(y'_\varepsilon/\sqrt{n}), \tau'_{y,\varepsilon}(n) \leqslant \tau'_y) \exp(-|y|^2/(2n)),$$

uniformly on $y \in K$ such that $|y| \leqslant A\sqrt{n}$.

Remark that this result implies the local limit theorem of [16], since for a fixed value of y, [16, Thm 1] yields

$$n^{p/2} \mathbb{E}(u(y'_\varepsilon(n)/\sqrt{n}), \tau'_{y,\varepsilon}(n) \leqslant \tau'_y) \exp(-|y|^2/(2n)) \xrightarrow[n\to\infty]{} V'(y).$$

Theorem 1 is sharp in the sense that $\mathbb{E}(u(y'_\varepsilon(n)/\sqrt{n}), \tau'_{y,\varepsilon}(n) \leqslant \tau'_y)$ changes a lot with n. However, there is a nice simplification when endpoints are located in the domain $K_{n,\varepsilon}$.

Corollary 2. *Assume (H1)–(H6). Then for fixed $A > 0$ and $x \in K$,*

$$\mathbb{P}(x + S(n) = y, \tau_x > n) \underset{n\to\infty}{\sim} \kappa_0 V(x) n^{-p/2-d/2} u(y/\sqrt{n}) \exp(-|y|^2/(2n)),$$

uniformly on $y \in K_{n,\varepsilon}$ with $|y| \leqslant A\sqrt{n}$.

The proof of Theorem 1 follows the same pattern as the proof of [16, Thm 6]: all intermediate steps of our proof are improved versions of the corresponding intermediate steps in the proof of [16, Thm 6]. In [16], these steps mainly consist in a coupling of the random walk with a standard Brownian motion, and then of estimates of the survival probability of the random walk started at a given $x \in K$, or of the transition probability between two fixed points $x, y \in K$, when n goes to infinity. The main novelty in the proof of Theorem 1 is to allow x and y to change with n. To that purpose we use Gaussian estimates on the heat kernel on K. We will mainly use the results of [27, Sec. 5 and Sec. 6.3]. See also [46] for a complementary approach on the subject.

Green function asymptotics along the boundary and Martin boundary. Set

$$K_\varepsilon = \{y \in K : d(y, \partial K) \geqslant 2|y|^{1-2\varepsilon}\},$$

and for $y \in K$, introduce the stopping time $\theta_y = \inf\{n \geqslant 1 : y + S'(n) \in K_\varepsilon\}$. We denote also by y_ε the random element $y + S'(\theta_y)$.

We prove that $G(x, y)$ factorizes asymptotically as a product of two quantities, one depending only on x and the other one only on y.

Theorem 3. *Assume (H1)–(H6). There exists a bounded function $F : K_\varepsilon \longrightarrow \mathbb{R}^+$ such that as $|y|$ goes to infinity,*

$$G(x, y) \underset{|y|\to\infty}{\sim} V(x) \mathbb{E}(F(y_\varepsilon), \tau'_y > \theta_y).$$

The latter result easily implies the identification of the Martin boundary of S killed when exiting K.

Theorem 4. *Assume (H1)–(H6). The Martin kernel of S killed on the boundary of K is reduced to one point, which corresponds to the function V in* (3)*. In particular, there is up to a scaling constant a unique positive harmonic function killed at the boundary of K.*

The proof of Theorem 3 consists in the splitting of the sum $G(x,y)$ in (2) in two terms $\Sigma_1 + \Sigma_2$. The first (resp. second) term is given by the contribution in the large deviation (resp. asymptotic) regime. The difficulty is to prove that the first term is dominated by the second one: in order to achieve this, we use a coupling of the random walk with a Brownian motion, with stronger bounds than the ones initially used in [16]. The drawback is the need of stronger moment assumptions on the increments, which is the main reason why the assumption $(H5)$ is used instead of the classical moment condition of [16], namely $\mathbb{E}(|X|^{r(p)}) < \infty$ with $r(p) = p$ if $p > 2$ and $r(p) = 2 + \delta$ for some $\delta > 0$ if $p \leqslant 2$.

Applications of our results and construction of harmonic functions. As we recalled in the introduction, walks in cones are of central importance in enumerative combinatorics. Given a cone (typically, the orthant $\mathbb{N}^d$) and a set of steps, the question is to compute (in a exact or asymptotic way) the number of excursions, i.e., the number of paths of length n going from x to y and confined to the cone. See [11, 9, 18] for modern contributions

All our results concerning the local probability $\mathbb{P}(x + S(n) = y, \tau_x > n)$ (especially Theorem 1 and Corollary 2) can very easily be turned into combinatorial results, giving the asymptotic behavior of the number of excursions. This discussion (in particular, the precise combinatorial traduction of the probabilistic results) is detailed in [16, Sec. 1.5], so we refer to [16] for any further information.

Our second possible application is a Ney and Spitzer theorem for walks in cones. More precisely, Ney and Spitzer consider in [38] random walks with non-zero drift in $\mathbb{Z}^d$ and prove that the Martin boundary is homeomorphic to the unit sphere $\mathbb{S}^{d-1}$. In [31, 29], Ignatiouk-Robert and Loree prove that for random walks in $\mathbb{N}^d$ with a drift whose all entries are non-zero, the Martin boundary is homeomorphic to $\mathbb{S}^{d-1} \cap \mathbb{R}^d_+$. However, the question of a general non-zero drift (i.e., with zero entries allowed) is left opened in [31, 29]. Our results should allow to complete the picture; this will be the topic of future research.

Theorem 4 shows that there is a unique positive harmonic function, namely the function V defined in (3). The formula (3) uses the réduite of the cone and the exit time of the cone, and is thus in some sense explicit. However, the question of finding an algebraic expression of V in terms of the variable x remains.

Let us mention a few cases where the construction (or better, an explicit expression) of V is known. For walks with small steps in the quarter plane, generating functions of harmonic functions are expressed in terms of conformal mappings in [41, 34]. For multidimensional Weyl chambers, harmonic functions usually admit an expression as modified Vandermonde determinants [22, 15]. In the half-space and orthant there is in [31, 29, 19] a construction of harmonic functions à la Ney and Spitzer (with a correction if the random walk is allowed to jump over the boundary). To conclude, let us mention product form harmonic functions, which we can compute using less

dimensional harmonic functions. This is typically the case for Cartesian product of Markov chains [39]. There is also in [9] an interesting concept of Hadamard walks, which generalizes the Cartesian product, but for which the harmonic function should still admit a product form.

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2. Preliminary results

We give in this section a brief review on the Gaussian estimates for the heat kernel in a cone and on the coupling approach of [16].

2.1. **Heat kernel in a cone and Gaussian estimates.** For $t > 0$, let us denote by $K_t : K \times K \longrightarrow \mathbb{R}^+$ the heat kernel on K with Dirichlet boundary conditions at time t. Namely, K_t is the solution of the equation

$$\begin{cases} \partial_t K_t(x,\cdot) + \Delta K_t(x,\cdot) = 0, & x \in K, \\ \lim\limits_{t\to 0} K_t(x,y) = \delta_x, & x \in K, \\ K_t(x,y) = 0, & x \in K,\ y \in \partial K, \end{cases}$$

where the limit on the second line is understood in the distributional sense. The kernel K_t is symmetric in x and y.

We denote by $k_t : K \longrightarrow \mathbb{R}^+$ the survival probability of a standard Brownian motion at time t. If τ_x^{bm} denotes the survival time of a standard Brownian motion starting at x, i.e., $\tau_x^{\mathrm{bm}} =\in \{t \geqslant 0 : x + B_t \notin \mathbb{C}\}$, then

$$k_t(x) = \mathbb{P}(\tau_x^{\mathrm{bm}} \geqslant t).$$

It satisfies the formula (see [27, Eq. (5.7)])

$$k_t(x) = \int_K K_t(x,y)dy.$$

The functions K_t and k_t are homogeneous in time, in the sense that

$$k_t(x) = k(x/\sqrt{t}) \quad \text{and} \quad K_t(x,y) = t^{-d/2}K(x/\sqrt{t}, y/\sqrt{t})$$

for $x, y \in K, t > 0$. We simply write K for K_1 and k for k_1.

We review here some inequalities concerning the heat kernel in a cone and deduce some useful estimates. Let u denote the réduite of the convex cone K, whose formula is given by

$$u(x) = |x|^p m\left(\frac{x}{|x|}\right),$$

with $p \geqslant 1$ and $m : \Sigma \longrightarrow \mathbb{R}$ is a function which is $\mathcal{C}^2$ in the interior of Σ and such that $m_{\partial\Sigma} = 0$. Note that the gradient of u is locally bounded (see Lemma 26).

The first important Gaussian estimates give upper and lower bounds of k_t in terms of the réduite u of K. For $x \in K$ and $t > 0$, let x_t denote an arbitrary point of K such that $|x_t - x| \leqslant t$ and $d(x_t, \partial K) \geqslant c_0 t$ for some constant c_0 independent of x and t (we take the same notation as in [27, Eq. (4.29)]). In all the results involving elements of

type x_t for some $x \in K$ and $t > 0$, the constants given do not depend on the particular choice of x_t. Moreover, since $d(x_t, \partial K) \geqslant c_0 t$ by definition, [16, Lem. 19] yields that

$$u(x_t) \geqslant ct^p \tag{5}$$

from some constant $c > 0$.

Theorem 5 (Thm 5.14 of [27]). *There exist positive constants c_1 and C_1 such that for all $(x,t) \in K \times \mathbb{R}_{>0}$,*

$$c_1 \frac{u(x)}{u(x_{\sqrt{t}})} \leqslant k_t(x) \leqslant C_1 \frac{u(x)}{u(x_{\sqrt{t}})}.$$

We will also use an equivalent version of the latter theorem, which relies on the properties of the harmonic function u. Indeed, by [27, 4.20] there exist constants c, C such that

$$cu(x_t)^2 \leqslant \frac{\int_{B(x,t)\cap K} u(s)^2 ds}{V(x,t)} \leqslant Cu(x_t)^2,$$

where $V(x,t)$ denotes the volume of $B(x,t) \cap K$. This yields the alternative estimate

$$c_1' \sqrt{\frac{V(x,\sqrt{t})}{\int_{B(x,\sqrt{t})\cap K} u(s)^2 ds}} u(x) \leqslant k_t(x) \leqslant C_1' \sqrt{\frac{V(x,\sqrt{t})}{\int_{B(x,\sqrt{t})\cap K} u(s)^2 ds}} u(x). \tag{6}$$

The second estimates concern the heat kernel itself.

Theorem 6 (Thm 5.11 and Thm 5.15 in [27]). *There exist c_2, c_3, C_2, C_3, C_4 positive constants and $0 < \alpha < 1$ such that*

$$c_2 \frac{k_t(x)k_t(y)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp(-|x-y|^2/(C_3 t)) \leqslant K_t(x,y) \leqslant C_2 \frac{k_t(x)k_t(y)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp(-|x-y|^2/(c_3 t))$$

for all $y, y' \in K$, and

$$\left| \frac{K_t(x,y)}{u(y)} - \frac{K_t(x,y')}{u(y')} \right| \leqslant C_4 \left(\frac{|y-y'|}{\sqrt{t}} \right)^{\alpha} \frac{K_{2t}(x,y)}{u(y)},$$

for all $y, y' \in K$ such that $|y - y'| \leqslant \sqrt{t}$.

Moreover, there exists $\beta, C_5 > 0$ such that

$$|\partial_t K_t(x,y)| \leqslant C_5 \frac{k_t(x)k_t(y)}{t\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} (1 + |x-y|^2/t)^{\beta+1} \exp(-|x-y|^2/(4t)),$$

for all $t > 0$ and $x, y \in K$.

The above result is actually much more general, since it holds for every inner uniform domain (see [27]). The above estimates can actually be simplified by the following inequality

$$\inf_{z\in K} V(z,\sqrt{t}) \geqslant ct^{d/2}, \tag{7}$$

for some constant c independent of $t > 0$ (see Lemma 28 for a proof of this inequality).

We can deduce from the Gaussian estimates several other estimates on the heat kernel which are needed to prove Theorem 1 and Theorem 3. All results concerning these heat kernel estimates are proven in Appendix A.

2.2. **The coupling approach of Denisov and Wachtel.** All the results of this subsection are borrowed from [16], and the interested reader should refer to [16] for the proofs. The local limit theorems obtained by Denisov and Wachtel in [16, Thm 5 and Thm 6] rely on a coupling of the random walk with a Brownian motion [16, Lem. 17], based on an important work of Götze and Zaitsev [26, Thm 4]. Namely, suppose that $\mathbb{E}(|X|^{2+\delta}) < \infty$ for some $0 < \delta < 1$. Then, one can define a random walk with the same probability distribution as $S(n)$ and a Brownian motion $B(t)$ on the same probability space such that, for any γ satisfying $0 < \gamma < \frac{\delta}{2(2+\delta)}$,

$$\mathbb{P}(\sup_{u \leqslant n} |B(u) - S(\lfloor u \rfloor)| \geqslant n^{1/2-\gamma}) \leqslant C n^{2\gamma+\gamma\delta-\delta/2} \tag{8}$$

for some constant $C > 0$. This coupling is particularly useful in the case of random walks in cones because the distribution of the Brownian motion has an explicit expression in this situation [14, 3].

For example, the value of the kernel $K_t(x, \cdot)$ satisfies the uniform asymptotic formula [16, Lem. 18]

$$K_t(x,y) \underset{\substack{|x| \leqslant \theta_t \sqrt{t}, \\ |y| \leqslant \sqrt{t/\theta_t}}}{\sim} \chi_0 t^{-d/2-p} u(x) u(y) e^{-|y|^2/(2t)}, \tag{9}$$

where θ_t is any function of t converging to zero as t goes to infinity, and χ_0 is a positive constant. Likewise, the survival probability of the Brownian motion in K satisfies the uniform asymptotic formula

$$k_t(x) \underset{|x| \leqslant \theta_t \sqrt{t}}{\sim} \chi t^{-p/2} u(x), \tag{10}$$

with the same θ_t as before and a positive constant χ.

Thanks to the coupling (8), the above asymptotic results can be transferred [16, Lem. 20] to the random walk $\{S(n)\}$ when the random walk starts far enough from the boundary, compared to the typical scale that we are considering. For example, for ε small enough and with $K_{n,\varepsilon}$ defined in (4),

$$\mathbb{P}(\tau_x > n) \underset{\substack{x \in K_{n,\varepsilon} \\ |x| \leqslant \theta_n \sqrt{n}}}{\sim} \chi u(x) n^{-p/2}. \tag{11}$$

Let us review how this technique yields exact asymptotics for $\mathbb{P}(\tau_x > n)$, independently of the initial position of x. The matter is then to find the asymptotic formula when x is to close to the boundary. In order to deal with these issues, Denisov and Wachtel introduce the stopping time $t_{x,\varepsilon}(n) = \inf\{m \geqslant 1 : x + S(m) \in K_{n,\varepsilon}\}$. Then, they prove [16, Lem. 14] that $t_x^\varepsilon(n)$ is small on the event that $\tau_x \geqslant n^{1-\varepsilon}$. Namely, there exists a constant $C > 0$ such that

$$\mathbb{P}(t_{x,\varepsilon} \geqslant n^{1-\varepsilon}, \tau_x \geqslant n^{1-\varepsilon}) \leqslant \exp(-C n^\varepsilon). \tag{12}$$

Therefore, on an exit time of order n, the random walk spends most of its time in a regime which can be controlled by the coupling (8), and one can thus show that

$$\mathbb{P}(\tau_x > n) \underset{t \to \infty}{\sim} \chi n^{-p/2} \mathbb{E}(u(x_{n,\varepsilon}), t_{x,\varepsilon}(n) \leqslant \tau_x, t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}),$$

where $x_{n,\varepsilon} = x + S(t_{x,\varepsilon}(n))$. The last step of their method is to adapt the definition of $V(x)$ in (3) to the stopping time $t_{x,\varepsilon}(n)$, which yields [16, Lem. 21]

$$\lim_{n\to\infty} \mathbb{E}(u(x_\varepsilon), t_{x,\varepsilon}(n) \leqslant \tau_x, t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}) = V(x), \tag{13}$$

and the exact asymptotic formula given in [16, Thm 1]:

$$\mathbb{P}(\tau_x > n) \underset{t\to\infty}{\sim} \chi n^{-p/2} V(x). \tag{14}$$

The local limit theorems are then obtained from the previous results; we do not review their proof here, since the pattern is roughly the same as the one of the proof of Theorem 1 in the following section. Let us stress however that these proofs rely on some important estimates on the local probability of a non-constrained random walk. Namely [16, Lem. 29], there exist positive constants a and C such that for all $u \geqslant 0$,

$$\limsup_{n\to\infty} n^{d/2} \sup_{|z-x|\geqslant u\sqrt{n}} \mathbb{P}(x + S(n) = z) \leqslant C \exp(-au^2). \tag{15}$$

In particular [16, Lem. 27], using (15) at $u = 0$ together with (14) yields the existence of a positive constant $C(x)$ for each $x \in K$ such that

$$\sup_{y\in K} \mathbb{P}(x + S(n) = y, \tau_x \geqslant n) \leqslant C(x) n^{-p/2-d/2}. \tag{16}$$

3. Local limit theorems at the fluctuations scale

This section is dedicated to the proof of Theorem 1, which gives the uniform asymptotics of the local probability $\mathbb{P}(x + S(n) = y, \tau_x > n)$ as n goes to infinity and $y \leqslant A\sqrt{n}$ (throughout the section, $A > 0$ is a fixed parameter which is expected to be large).

Uniform convergence of the exit time and the conditioned distribution far from the boundary. In this subsection we give asymptotics of the distribution of $(y + S(n), \tau_y > n)$ uniformly for all

$$y \in K_{n,\varepsilon}^A = \{z \in K : |z| \leqslant A\sqrt{n},\, d(z, K) \geqslant n^{1/2-\varepsilon}\}.$$

Proposition 7. *Let $B > 0$, $0 < s < 1$ and $D \subset \mathbb{R}^d$ a bounded convex domain containing 0 in its interior. Then there exists $\varepsilon > 0$ such that*

$$\mathbb{P}(\tau_y > n) \underset{n\to\infty}{\sim} k_n(y)$$

and

$$\mathbb{P}(y + S(n) \in \sqrt{n}(x + tD), \tau_y > n) \underset{n\to\infty}{\sim} \int_{x+tD} K(y/\sqrt{n}, z)dz,$$

uniformly for all $y \in K_{n,\varepsilon}^A$, $x \in K \cap B(0, B)$ and $s \leqslant t \leqslant 1$.

Let us start by giving an upper bound for $u(y_1)$, which will be useful for the proof of Proposition 7.

Lemma 8. *There exists a constant $C > 0$ such that for all $y \in K$, we can choose y_1 with $u(y_1) \leqslant u(y) \vee C(|y| \vee 1)^{p-1}$.*

Proof. Recall that y_1 is an arbitrary point such that $d(y, y_1) \leqslant 1$ and $d(y_1, \partial K) \geqslant c_0$. Hence, when $d(y, \partial K) \geqslant c_0$, we can choose $y_1 = y$, and thus $u(y_1) = u(y)$.

Suppose now that $d(y, \partial K) \leqslant c_0$. If $d(y, \partial K) \leqslant c_0$, then any choice of y_1 must satisfy $d(y_1, K) \geqslant c_0$, which yields in particular $|y_1| \geqslant c_0$. Since $y_1 \in B(y, 1)$, we have also $d(y_1, \partial K) \leqslant c_0 + 1$, and thus (52) gives

$$u(y_1) \leqslant C d(y_1, \partial K) \big|y_1\big|^{p-1} \leqslant C(|y| \vee c_0)^{p-1}$$

for some constant C. Thus, in any case we can choose y_1 such that

$$u(y_1) \leqslant u(y) \vee C(|y| \vee 1)^{p-1}. \qquad \square$$

Proof of Proposition 7. The proof follows closely the one of [16, Lem. 20]; since we have to take care of the extra condition of uniformity, we choose to rewrite it completely.

Fix $0 < \gamma \leqslant \frac{\delta}{2(2+\delta)}$ and choose $x_0 \in \mathbb{R}^d$ and $R_0 > 0$ such that $|x_0| = 1$, $x_0 + K \subset K$ and $d(R_0 x_0 + K, \partial K) > 1$. Let $y \in K^A_{n,\varepsilon}$, and set $y^\pm := y \pm R_0 x_0 n^{1/2-\gamma}$. Set

$$A_n = \left\{ \sup_{u \leqslant n} |B(u) - S(\lfloor u \rfloor)| \leqslant n^{1/2-\gamma} \right\},$$

where B is the Brownian motion coupled to S in Section 2.2. By (8), $\mathbb{P}(A_n^c) \leqslant n^{-r}$, where $r = \delta/2 - 2\gamma - \gamma\delta$. Moreover (cf the proof of [16, Lem. 20]), for n large enough

$$\{\tau_y > n\} \cap A_n \subset \{\tau^{\mathrm{bm}}_{y^+} > n\} \quad \text{and} \quad \{\tau^{\mathrm{bm}}_{y^-} > n\} \cap A_n \subset \{\tau_y > n\},$$

which yields

$$k_n(y^-) + O(n^{-r}) \leqslant \mathbb{P}(\tau_y > n) \leqslant k_n(y^+) + O(n^{-r}). \tag{17}$$

In order to conclude the first part of the proposition, we have to show that, uniformly on $y \in K^A_{n,\varepsilon}$, $k_n(y^-) \sim k_n(y^+)$ and $n^{-r} = o(k_n(y))$ as n goes to infinity. On the one hand, by homogeneity of k and by Proposition 30,

$$|k_n(y) - k_n(y^\pm)| = |k(y/\sqrt{n}) - k(y^\pm/\sqrt{n})| \leqslant C_\alpha (1 + |y/\sqrt{n}|^{p-1}) |y - y^\pm|^\alpha / n^{\alpha/2} \leqslant C A^{p-1} n^{-\gamma\alpha}$$

for some constant $C > 0$. On the other hand, by the estimates of Theorem 5 and Lemma 8, there exists a constant c such that

$$k(y) \geqslant c \frac{u(y)}{u_1(y)} \geqslant c \left(1 \wedge \frac{u(y)}{(|y| \vee 1)^{p-1}} \right)$$

for $y \in K$. If the cone K is $\mathcal{C}^2$, by [16, Lem. 19] there exists a constant C such that $u(y) \geqslant C|y|^{p-1} d(y, \partial K)$. Thus, for $y \in K^A_{n,\varepsilon}$ and n large enough,

$$k(y/\sqrt{n}) \geqslant c \frac{n^{-\varepsilon} |y/\sqrt{n}|^{p-1}}{(|y/\sqrt{n}| \vee 1)^{p-1}} \geqslant c' n^{-\varepsilon} \big(n^{-(p-1)\varepsilon} \wedge 1 \big) \geqslant c'' n^{-p\varepsilon} \tag{18}$$

for some $c'' > 0$. If K is convex, by [16, Lem. 19] there exists a constant C such that $u(y) \geqslant C d(y, \partial K)^p$. Thus, for $y \in K^A_{n,\varepsilon}$ and n large enough,

$$k(y/\sqrt{n}) \geqslant cC \frac{n^{-p\varepsilon}}{C(|y/\sqrt{n}| \vee 1)^{p-1}} \geqslant c' n^{-p\varepsilon} \left(1 \wedge \left(\frac{1}{A} \right)^{p-1} \right) \geqslant c'' n^{-p\varepsilon} \tag{19}$$

for some $c'' > 0$. Suppose that ε is such that $(\gamma\alpha \wedge r) > p\varepsilon$. Then $n^{-\gamma\alpha} = o(k_n(y))$ and $n^{-r} = o(k_n(y))$, which proves that $k_n(y) \sim k_n(y^{\pm})$ and $n^{-r} = o(k_n(y))$. Therefore, (17) yields that uniformly in $y \in K^A_{n,\varepsilon}$,

$$\mathbb{P}(\tau_y > n) \underset{n\to\infty}{\sim} k_n(y).$$

We now turn to the second asymptotics in Proposition 7. Likewise, for $y \in K^A_{n,\varepsilon}$, $x \in K \cap B(0,B)$, $s \leqslant t \leqslant 1$ and $D \subset K$, we have by [16, Eq. (46)]

$$\int_{\sqrt{n}D^-_{x,t}} K_n(y^-,z)dz + O(n^{-r}) \leqslant \mathbb{P}(y+S(n) \in \sqrt{n}D_{x,t}, \tau_y > n) \leqslant \int_{\sqrt{n}D^+_{x,t}} K_n(y^+,z)dz + O(n^{-r}), \tag{20}$$

where

$$\left\{ \begin{array}{ll} D_{x,t} & = x + tD, \\ D^+_{x,t} & = \{z \in K : d(z, D_{x,t}) \leqslant 2n^{-\gamma}\}, \\ D^-_{x,t} & = \{z \in D_{x,t} : d(z, \partial D_{x,t}) \geqslant 2n^{-\gamma}\}. \end{array} \right.$$

By homogeneity,

$$\int_{\sqrt{n}D^{\pm}_{x,t}} K_n(y^{\pm},z)dz = \int_{D^{\pm}_{x,t}} K(y^{\pm}/\sqrt{n},z)dz,$$

and by Proposition 30,

$$\left|K(y^{\pm}/\sqrt{n},z) - K(y/\sqrt{n},z)\right| \leqslant Cn^{-\gamma\alpha}(1+|y/\sqrt{n}|)^{p-1}\exp(-|y/\sqrt{n}-z|^2/(2c_3)),$$

which implies

$$\begin{aligned} \left| \int_{D^{\pm}_{x,t}} K(y^{\pm}/\sqrt{n},z)dz - \int_{D_{x,t}} K(y/\sqrt{n},z)dz \right| & \\ \leqslant C(1+A)^{p-1}n^{-\gamma\alpha} \int_{D^+_{x,t}} \exp(-|y/\sqrt{n}-z|^2/(2c_3))dz + 2\,\mathrm{Vol}(D^+_{x,t} \setminus D^-_{x,t}). & \end{aligned}$$

The definition of $D^-_{x,t}$ and $D^+_{x,t}$ yields

$$\mathrm{Vol}(D^+_x \setminus D^-_x) = \mathrm{Vol}(\{z \in K : d(z,\partial D_{x,t}) < 2n^{-\gamma}\}) \leqslant \mathrm{Vol}(\{z \in \mathbb{R}^d : d(z, t\partial D) < 2n^{-\gamma}\}).$$

Hence, applying Steiner formula (Theorem 46 in [36, Chap. 16]) to the convex tD gives

$$\mathrm{Vol}(D^+_{x,t} \setminus D^-_{x,t}) \underset{n\to\infty}{\sim} 2n^{-\gamma}t^{d-1}\lambda(D)$$

uniformly on all $x \in K \cap B(0,B)$ and $s \leqslant t \leqslant 1$, where $\lambda(D)$ denotes the surface area of D. Thus, since $\alpha \leqslant 1$, there exists a constant C'' independent of t such that

$$\left| \int_{D^{\pm}_{x,t}} K(y^{\pm}/\sqrt{n},z)dz - \int_{D_{x,t}} K(y/\sqrt{n},z)dz \right| \leqslant C''n^{-\gamma\alpha}.$$

Therefore, (20) yields

$$\begin{aligned} \int_{D_{x,t}} K(y/\sqrt{n},z)dz + O(n^{-r}) + O(n^{-\alpha\gamma}) \leqslant \mathbb{P}(y+S(n) \in \sqrt{n}D_{x,t}, \tau_y > n) & \\ \leqslant \int_{D_{x,t}} K(y/\sqrt{n},z)dz + O(n^{-r}) + O(n^{-\alpha\gamma}). & \end{aligned}$$

It remains to shows that as n goes to infinity, $n^{-(r\wedge\alpha\gamma)} = o(\int_{D_{x,t}} K(y/\sqrt{n}, z)dz)$. By Theorem 6 and Lemma 7, we have

$$\begin{aligned} K(y/\sqrt{n}, z) \geqslant & c_2 \frac{k(y/\sqrt{n})k(z)}{\sqrt{V(y/\sqrt{n}, 1)V(z, 1)}} \exp(-|y/\sqrt{n} - z|^2/C_3) \\ \geqslant & ck(y/\sqrt{n})k(z) \exp(-|y/\sqrt{n} - z|^2/C_3) \end{aligned}$$

for $y, z \in K$, $s \leqslant t \leqslant 1$ and some constant $c > 0$. Hence, if we set $M := \sup\{|z| : z \in D\}$, then $D_{x,t} \subset B(0, B + tM)$ and thus

$$\int_{D_{x,t}} K(y/\sqrt{n}, z)dz \geqslant Ck(y/\sqrt{n}) \exp(-(B + tM)^2/C_3) \int_{D_{x,t}} k(z)dz.$$

Recalling from (18) and (19) that $k(z/\sqrt{n}) \geqslant c''n^{-p\varepsilon}$ for $z \in K_{n,\varepsilon}$, we get

$$\int_{D_{x,t}} K(y/\sqrt{n}, z)dz \geqslant c'n^{-2p\varepsilon} \operatorname{Vol}\Big(D_{x,t} \cap \frac{1}{\sqrt{n}} K_{n,\varepsilon}\Big) \tag{21}$$

for some constant $c' > 0$. Since 0 is in the interior of D, there exists $u > 0$ such that $B(0, u) \subset D$. Hence, by Lemma 7,

$$\operatorname{Vol}(D_{x,t} \cap K) \geqslant V(x, tu) \geqslant c(tu)^d \geqslant c(su)^d$$

for some constant c independent of $x \in B(0, B) \cap K$. Since

$$\operatorname{Vol}\Big((K \cap B(0, B + M)) \setminus \frac{1}{\sqrt{n}} K_{n,\varepsilon}\Big)$$

goes to 0 as n goes to infinity, there exists $\delta > 0$ such that

$$\operatorname{Vol}\Big(D_{x,t} \cap \frac{1}{\sqrt{n}} K_{n,\varepsilon}\Big) \geqslant \operatorname{Vol}(D_{x,t} \cap K) - \operatorname{Vol}\Big((K \cap B(0, B + M)) \setminus \frac{1}{\sqrt{n}} K_{n,\varepsilon}\Big) \geqslant \delta$$

for all $x \in B(0, B) \cap K$, $s \leqslant t \leqslant 1$ and n large enough. The latter inequality together with (21) gives

$$\int_{D_{x,t}} K(y/\sqrt{n}, z)dz \geqslant c'\delta n^{-2p\varepsilon}$$

for n large enough, $y \in K^A_{n,\varepsilon}$, $x \in B(0, B) \cap K$ and $s \leqslant t \leqslant 1$. Choosing $\varepsilon < (r \wedge \alpha\gamma)/(2p)$ thus yields that $n^{-(r\wedge\alpha\gamma)} = o(\int_{D_{x,t}} K(y/\sqrt{n}, z)dz)$. For these values of ε, we have

$$\mathbb{P}(y + S(n) \in \sqrt{n}D_{x,t}, \tau_y > n) \sim \int_{D_{x,t}} K(y/\sqrt{n}, z)dz$$

uniformly on y, x and t satisfying the conditions of the statement. □

The latter proposition can be turned into a uniform version of the convergence in distribution.

Corollary 9. *Let $B, \eta > 0$. Then*

$$\sup_{\substack{x\in K_{n,\varepsilon}^A \\ y\in B(0,B)\cap K}} \sup_{\substack{f\in \mathrm{Lip}(B(y,r)) \\ \eta\leqslant r\leqslant d(y,\partial K)}} \frac{1}{\|f\|_\infty}\Bigg| \mathbb{E}\left(f\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\right) - \frac{1}{k(x/\sqrt{n})}\int_{B(y,r)} K(x/\sqrt{n},z)f(z)dz\Bigg| \xrightarrow[n\to\infty]{} 0,$$

where $\mathrm{Lip}(B(y,r))$ *denotes the set of all Lipschitz functions on* $B(y,r)$ *with Lipschitz constant equal to* 1.

Proof. For $p \geqslant 1$ and $D \subset K \cap B(0,2B)$, denote by $\mathcal{F}_p(D)$ the set of functions with support in D which are constant on each set

$$R_{\vec{k}}^p := \left\{ x \in \mathbb{R}^d : \frac{k_i}{p} \leqslant x_i < \frac{k_i+1}{p} \right\}$$

for $\vec{k} \in \mathbb{Z}^d$. Note that if $f \in \mathcal{F}_p(D)$, $f_{|R_{\vec{k}}}$ can be non zero only if $R_{\vec{k}}^p \subset D$.

For $D \subset K$ and $f \in \mathrm{Lip}(D)$, define for each $p \geqslant 1$ the function

$$f_p := \sum_{\vec{k}\in\mathbb{Z}^d, R_{\vec{k}}^p \subset D} f((1/2,\dots,1/2)+\vec{k})\mathbf{1}_{R_{\vec{k}}^p}$$

and the set $D^{(p)} = \bigcup_{\substack{\vec{k}\in\mathbb{Z}^d \\ R_{\vec{k}}^p\subset D}} R_{\vec{k}}^p$. Then, $f_p \in \mathcal{F}_p(D)$ and by the Lipschitz continuity of f,

$$\|f_p - f_{|D^{(p)}}\|_\infty \leqslant \frac{\sqrt{d}}{2p}.$$

Let $0 < \delta < \eta$ and let p_0 be such that $\frac{\sqrt{d}}{2p_0} \leqslant \delta$ and $\mathrm{Vol}(B(0,r) \setminus B(0, r - \frac{\sqrt{d}}{2p_0})) \leqslant \delta$ for all $\eta \leqslant r \leqslant B$. Since each set $R_{\vec{k}}^{p_0}$ is a translation of the convex set $R_{(0,\dots,0)}^{p_0}$ and there are finitely many $R_{\vec{k}}^{p_0}$ included in K, Proposition 7 yields that

$$\frac{1}{\|f\|_\infty}\mathbb{E}\left(f\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\right) \underset{n\to\infty}{\sim} \frac{1}{\|f\|_\infty k(x/\sqrt{n})}\int_K K(x/\sqrt{n},z)f(z)dz$$

uniformly on all $x \in K_{n,\varepsilon}^A$ and $f \in \mathcal{F}_{p_0}(D)$, with $D \subset K \cap B(0,2B)$. Hence, there exists $n_0 \geqslant 1$ such that for $n \geqslant n_0$,

$$\frac{1}{\|f\|_\infty}\left|\mathbb{E}\left(f\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\right) - \frac{1}{k(x/\sqrt{n})}\int_K K(x/\sqrt{n},z)f(z)dz\right| \leqslant \delta \qquad (22)$$

for all $x \in K_{n,\varepsilon}^A$ and $f \in \mathcal{F}_{p_0}(D)$, with $D \subset K \cap B(0,2B)$. Set $s = \eta/B$, and by Proposition 7 let $n_1 \geqslant 1$ be such that for $n \geqslant n_1$,

$$\begin{aligned}\frac{1-\delta}{k(x/\sqrt{n})}\int_{y+tB(0,B)} K(x/\sqrt{n},z)dz &\leqslant \mathbb{P}(x+S(n) \in \sqrt{n}(y+tB(0,B))|\tau_x > n) \\ &\leqslant \frac{1+\delta}{k(x/\sqrt{n})}\int_{y+tB(0,B)} K(x/\sqrt{n},z)dz, \quad (23)\end{aligned}$$

for all $x \in K_{n,\varepsilon}$, $y \in K \cap B(0,B)$ and $s \leqslant t \leqslant 1$.

Suppose from now that $n \geqslant (n_0 \vee n_1)$, $x \in K^A_{n,\varepsilon}$, $y \in K \cap B(0,B)$ and $f \in \mathrm{Lip}(B(y,r))$, with $\eta \leqslant r \leqslant d(y, \partial K)$. Suppose without loss of generality that $\|f\|_\infty = 1$. Then,

$$\begin{aligned}\Big|\mathbb{E}\Big(f\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\Big) - \mathbb{E}\left(f_{p_0}\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\right)\Big| \\ \leqslant &\|f_{|B(y,r)^{(p_0)}} - f_{p_0}\|_\infty \mathbb{P}(x+S(n) \in B(y,r)^{(p_0)}|\tau_x > n) \\ &+ \|f\|_\infty \mathbb{P}(x+S(n) \in B(y,r) \setminus B(y,r)^{(p_0)}|\tau_x > n) \\ \leqslant &\delta + \mathbb{P}(x+S(n) \in B(y,r) \setminus B(y,r)^{(p_0)}|\tau_x > n),\end{aligned}$$

where we have used on the last inequality that $\|f_{|B(y,r)^{(p_0)}} - f_{p_0}\|_\infty \leqslant \delta$ by the choice of p_0, $\mathbb{P}(x+S(n) \in B(y,r)^{(p)}|\tau_x > n) \leqslant 1$ and $\|f\|_\infty = 1$. By the choice of $n \geqslant n_1$ and the fact that $B(y,r) \setminus B(y,r)^{(p_0)} \subset B(y,r) \setminus B(y, r-\delta)$, (23) yields

$$\begin{aligned}&\mathbb{P}(x+S(n) \in B(y,r) \setminus B(y,r)^{(p_0)}|\tau_x > n) \\ &\quad\leqslant \frac{1+\delta}{k(x/\sqrt{n})}\int_{B(y,r)} K(x/\sqrt{n},z)dz - \frac{1-\delta}{k(x/\sqrt{n})}\int_{B(y,r-\delta)} K(x/\sqrt{n},z)dz \\ &\quad\leqslant \frac{2\delta}{k(x/\sqrt{n})}\int_{B(y,r-\delta)} K(x/\sqrt{n},z)dz + \frac{1+\delta}{k(x/\sqrt{n})}\int_{B(y,r)\setminus B(y,r-\delta)} K(x/\sqrt{n},z)dz \\ &\quad\leqslant 2\delta + (1+\delta)C\,\mathrm{Vol}(B(y,r) \setminus B(y,r-\delta)),\end{aligned}$$

for some constant C independent of x coming from Theorem 6. Since there exists $K > 0$ such that $\mathrm{Vol}(B(y,r) \setminus B(y,r-\delta)) \leqslant K\delta$ for $r \leqslant B$, there exists $C' > 0$ independent of x, y, r, f and n such that

$$\Big|\mathbb{E}\Big(f\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\Big) - \mathbb{E}\left(f_{p_0}\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\right)\Big| \leqslant C'\delta. \tag{24}$$

Since $n \geqslant n_0$ and $f_{p_0} \in \mathcal{F}_{p_0}(B(y,r))$, (22) together with (24) imply that

$$\left|\mathbb{E}\Big(f\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\Big) - \frac{1}{k(x/\sqrt{n})}\int_K K(x/\sqrt{n},z) f_{p_0}(z)dz\right| \leqslant (C'+1)\delta. \tag{25}$$

Finally, for the same reasons as in the proof of (24),

$$\begin{aligned}\left|\frac{1}{k(x/\sqrt{n})}\int_K K(x/\sqrt{n},z) f_{p_0}(z)dz - \frac{1}{k(x/\sqrt{n})}\int_K K(x/\sqrt{n},z) f(z)dz\right| \\ \leqslant &\|f_{|B(y,r)^{(p)}} - f_p\|_\infty + C\,\mathrm{Vol}(B(y,r) \setminus B(y,r-\delta)) \\ \leqslant &C'\delta. \end{aligned}\tag{26}$$

Combining (25) and (26) yields that for $n \geqslant n_0 \vee n_1$,

$$\begin{aligned}\sup_{\substack{x \in K^A_{n,\varepsilon} \\ y \in B(0,B)\cap K}} \sup_{\substack{f \in \mathrm{Lip}(B(y,r)) \\ \eta \leqslant r \leqslant d(y,\partial K)}} \frac{1}{\|f\|_\infty}\Bigg|\mathbb{E}\left(f\big((x+S(n))/\sqrt{n}\big)|\tau_x > n\right) \\ - \frac{1}{k(x/\sqrt{n})}\int_{B(y,r)} K(x/\sqrt{n},z) f(z)dz\Bigg| \leqslant (1+2C')\delta. \quad \square\end{aligned}$$

As another consequence of Proposition 7, we can give an estimate on the scaling of exit times.

Corollary 10. *Let $0 < s < 1$. There exists a constant $C > 0$ such that*

$$\limsup_{n\to\infty} \sup_{\substack{x\in K^A_{n,\varepsilon} \\ m\geqslant sn}} \frac{\mathbb{P}(\tau_x > m)}{\mathbb{P}(\tau_x > n)} \leqslant C.$$

Proof. Let $0 < \eta > 1$. By Proposition 7 there exists $n_0 \geqslant 1$ such that for $n \geqslant n_0$ and $x \in K^A_{n,\varepsilon}$,

$$(1-\eta)k(x/\sqrt{n}) \leqslant \mathbb{P}(\tau_x > n) \leqslant (1+\eta)k(x/\sqrt{n}).$$

Then, for $n > n_0/s$ and $m \geqslant sn$, we have

$$\frac{\mathbb{P}(\tau_x > m)}{\mathbb{P}(\tau_x > n)} \leqslant \frac{1+\eta}{1-\eta}\frac{k(x/\sqrt{m})}{k(x/\sqrt{n})}.$$

Since $\frac{\sqrt{m}}{\sqrt{n}} \geqslant \sqrt{s}$, Lemma 29 gives a constant $C > 0$ independent of $m \geqslant sn$ such that $\frac{k(x/\sqrt{m})}{k(x/\sqrt{n})} \leqslant C$ for all $x \in K$. Therefore,

$$\limsup_{n\to\infty} \sup_{\substack{x\in K^A_{n,\varepsilon} \\ m\geqslant sn}} \frac{\mathbb{P}(\tau_x > m)}{\mathbb{P}(\tau_x > n)} \leqslant \frac{1+\eta}{1-\eta}C. \qquad \square$$

Local limit theorems far from the boundary. The goal of this subsection is to improve the local limit theorem [16, Thm 6] in order to get a more uniform result, which will correspond to Theorem 1 restricted to $x \in K^A_{n,\varepsilon}$. As in the previous subsection, the pattern follows closely the one of [16, Thm 6] and is divided into two steps:

- The first step, given by the following proposition, is a more general local limit theorem analogue to [16, Thm 5]:

Proposition 11. *Uniformly on $x \in K^A_{n,\varepsilon}$,*

$$\sup_{y\in K}\left| n^{d/2}\mathbb{P}(x + S(n) = y|\tau_x > n) - \frac{K(x/\sqrt{n}, y/\sqrt{n})}{k(x/\sqrt{n})}\right| \to 0.$$

- The second step is a derivation of the more specific local limit theorem analogue to [16, Thm 6]:

Proposition 12. *Let $x \in K$. Then there exists $\kappa > 0$ such that uniformly on $y \in K^A_{n,\varepsilon}$,*

$$\mathbb{P}(x + S(n) = y, \tau_x > n) \sim \kappa V(x) n^{-d/2-p} u(y) \exp\left(-\frac{|y|^2}{2n}\right).$$

Let us thus first begin with the general local limit theorem given in Proposition 11. The proof follows the one of [16, Thm 5], adding simultaneously the heat kernel estimates; since the proof of [16, Thm 5] is already quite technical, we choose to divide the proof of Proposition 11 into three lemmas. As in [16], let us divide K in three regions which depend on the choice of $x \in K^A_{n,\varepsilon}$ and two positive parameters $M, \eta > 0$ (the dependence does not appear in the notations):

- $K^{(1)} := \{y \in K, |y - x| > M\sqrt{n}\}$,

- $K^{(2)} := \{y \in K, |y-x| \leqslant M\sqrt{n}, d(y,\partial K) \leqslant 2\eta\sqrt{n}\}$, and
- $K^{(3)} := \{y \in K, |y-x| \leqslant M\sqrt{n}, d(y,\partial K) > 2\eta\sqrt{n}\}$.

We estimate then the probability $\mathbb{P}(x+S(n)=y|\tau_x > n)$ for y belonging to each one of these three regions.

Lemma 13. *One has*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(1)}} n^{d/2}\mathbb{P}(x+S(n)=y|\tau_x>n) = 0.$$

Lemma 14. *For each $M>0$,*

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(2)}} n^{d/2}\mathbb{P}(x+S(n)=y|\tau_x>n) = 0.$$

Lemma 15. *For each $M>0$,*

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(3)}} \left| n^{d/2}\mathbb{P}(x+S(n)=y|\tau_x>n) - \frac{K(x/\sqrt{n}, y/\sqrt{n})}{k(x/\sqrt{n})}\right| = 0.$$

The rest of the subsection is devoted to the proofs of Propositions 11 and 12, as well as of Lemmas 13, 14 and 15.

Proof of Lemma 13. Set $m = \lfloor n/2\rfloor$. Then, for $x \in K$,

$$\begin{aligned}\mathbb{P}(x+S(n)=y|\tau_x>n) &= \frac{\mathbb{P}\big(x+S(n)=y, \tau_x>n, |S(m)|>(M/2)\sqrt{n}\big)}{\mathbb{P}(\tau_x>n)} \\ &\quad + \frac{\mathbb{P}\big(x+S(n)=y, \tau_x>n, |S(m)|\leqslant (M/2)\sqrt{n}\big)}{\mathbb{P}(\tau_x>n)}. \quad (27)\end{aligned}$$

Considering the position of the random walk at time m yields that

$$\begin{aligned}&\mathbb{P}\big(x+S(n)=y, \tau_x>n, |S(m)|>(M/2)\sqrt{n}\big) \\ &\qquad = \sum_{|z-x|\geqslant (M/2)\sqrt{n}} \mathbb{P}(x+S(m)=z, \tau_x>m)\mathbb{P}(z+S(n-m)=y, \tau_z>n-m).\end{aligned}$$

Hence, the first term in the right hand-side of (27) is bounded by

$$\begin{aligned}&\frac{\mathbb{P}(x+S(m)\in B(x,(M/2)\sqrt{n})^c, \tau_x>m)}{\mathbb{P}(\tau_x>n)} \sup_{|z-x|\geqslant (M/2)\sqrt{n}} \mathbb{P}(z+S(n-m)=y) \\ &\qquad \leqslant Cn^{-d/2}\frac{\mathbb{P}(x+S(m)\in B(x,(M/2)\sqrt{n})^c, \tau_x>m)}{\mathbb{P}(\tau_x>n)} \quad (28)\end{aligned}$$

for some constant $C>0$, where we have used (15) with the parameter $u=0$ in the second line. By Proposition 7,

$$\frac{\mathbb{P}(x+S(m)\in B(x,(M/2)\sqrt{n})^c, \tau_x>m)}{\mathbb{P}(\tau_x>n)} \sim \frac{\int_{|z-x/\sqrt{m}|>M/\sqrt{2}} K(x/\sqrt{m}, z)dz}{k(x/\sqrt{n})}$$

as n goes to infinity, uniformly on $x \in K^A_{n,\varepsilon}$. By Theorem 6 and Lemma 7, there exists a constant $C' > 0$ such that

$$\frac{\int_{|z-x/\sqrt{n}|>M/\sqrt{2}} K(x/\sqrt{m},z)dz}{k(x/\sqrt{n})} \leqslant \frac{C'k(x/\sqrt{m})}{k(x/\sqrt{n})} \int_{|z-x/\sqrt{m}|>M/\sqrt{2}} \exp(-|z-x/\sqrt{m}|^2/c_3)dz.$$

By Lemma 29, there exists a constant $C_{\sqrt{2}^{-1}}$ such that $\frac{k(\sqrt{2}x/\sqrt{n})}{k(x/\sqrt{n})} \leqslant C^{-1}_{\sqrt{2}^{-1}}$ for $x \in K$, which yields

$$\sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(1)}} \frac{\mathbb{P}(x+S(m)\in B(x,(M/2)\sqrt{n})^c, \tau_x > m)}{\mathbb{P}(\tau_x > n)} \leqslant C''\mathbb{P}(|B_{c_3/2}| > M/\sqrt{2})$$

for some constant $C'' > 0$, where B is a d-dimensional Brownian motion starting at 0. Hence, combining the latter inequality with (28) gives

$$\limsup_{n\to+\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(1)}} n^{d/2} \frac{\mathbb{P}(x+S(n)=y, \tau_x > n, |S(m)| \geqslant (M/2)\sqrt{n})}{\mathbb{P}(\tau_x > n)} \leqslant C\mathbb{P}(|B_{c_3/2}| > M/\sqrt{2}),$$

which yields

$$\lim_{M\to\infty} \limsup_{n\to+\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(1)}} n^{d/2} \frac{\mathbb{P}(x+S(n)=y, \tau_x > n, |S(m)| \geqslant (M/2)\sqrt{n})}{\mathbb{P}(\tau_x > n)} = 0. \tag{29}$$

Likewise, we have

$$\frac{\mathbb{P}(x+S(n)=y, \tau_x > n, |S(m)| \leqslant (M/2)\sqrt{n})}{\mathbb{P}(\tau_x > n)} \leqslant \frac{\mathbb{P}(\tau_x \geqslant m)}{\mathbb{P}(\tau_x \geqslant n)} \sup_{|z-y|\geqslant (M/2)\sqrt{n}} \mathbb{P}(z+S(n-m)=y).$$

By Corollary 10, $\limsup_{n\to\infty} \sup_{x\in K^A_{n,\varepsilon}} \frac{\mathbb{P}(\tau_x>m)}{\mathbb{P}(\tau_x>n)} \leqslant C$ for some constant $C > 0$, and by (15), there exist positive constants C' and a such that

$$\sup_{|z-y|\geqslant (M/2)\sqrt{n}} m^{d/2}\mathbb{P}(z+S(m)=y) \leqslant C'\exp(-aM^2/2).$$

Therefore, since $n/m \sim 2$ as n goes to infinity,

$$\limsup_{n\to\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(1)}} n^{d/2} \frac{\mathbb{P}(x+S_n=y, \tau_x > n, |S(m)| \leqslant (M/2)\sqrt{n})}{\mathbb{P}(\tau_x > n)} \leqslant C2^{d/2}C'\exp(-aM^2/2),$$

which yields

$$\lim_{M\to\infty} \limsup_{n\to\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(1)}} n^{d/2} \frac{\mathbb{P}(x+S_n=y, \tau_x > n, |S(m)| \leqslant (M/2)\sqrt{n})}{\mathbb{P}(\tau_x > n)} = 0. \tag{30}$$

Combining (29) and (30) yields the statement of the lemma. □

Proof of Lemma 14. Let $y \in K^{(2)}$ and set $m = \lfloor n/4 \rfloor$. We recall that S' denote the reverse random walk $-S$, and τ' the corresponding survival time. Then for $x, y \in K$,

$$\begin{aligned}
&\mathbb{P}(x+S(n) = y | \tau_x > n) \\
&\quad = \frac{1}{\mathbb{P}(\tau_x > n)} \sum_{z \in K} \mathbb{P}(x + S(2m) = z, \tau_x > 2m) \mathbb{P}(z + S(n-2m) = y, \tau_z > n - 2m) \\
&\quad = \sum_{z_1, z_2 \in K} \frac{1}{\mathbb{P}(\tau_x > n)} \mathbb{P}(x + S(m) = z_1, \tau_x > m) \times \\
&\qquad\qquad \mathbb{P}(z_1 + S(m) = z_2, \tau_{z_1} > m) \mathbb{P}(y + S'(n-2m) = z_2, \tau'_y > n - 2m).
\end{aligned}$$

By (15) with $u = 0$, there exists a constant C independent of z_2 such that

$$\mathbb{P}(z_1 + S(m) = z_2, \tau_{z_1} > m) \leqslant \mathbb{P}(z_1 + S(m) = z_2) \leqslant C m^{-d/2} \leqslant C(n/4)^{-d/2}.$$

Hence, setting $C' = C/(2^{-d})$ yields

$$\begin{aligned}
&\mathbb{P}(x + S(n) = y | \tau_x > n) \\
&\leqslant C' n^{-d/2} \sum_{z_1 \in K} \frac{1}{\mathbb{P}(\tau_x > n)} \mathbb{P}(x + S(m) = z_1, \tau_x > m) \sum_{z_2 \in K} \mathbb{P}(y + S'(n-2m) = z_2, \tau'_y > n - 2m) \\
&\leqslant C' n^{-d/2} \frac{\mathbb{P}(\tau_x > m)}{\mathbb{P}(\tau_x > n)} \mathbb{P}(\tau'_y > n - 2m).
\end{aligned}$$

By Corollary 10, $\limsup_{n \geqslant 0} \sup_{x \in K^A_{n,\varepsilon}} \frac{\mathbb{P}(\tau_x > m)}{\mathbb{P}(\tau_x > n)} \leqslant C''$ for some constant C'', and by [16, Eq. (78)], there exists a function $g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ converging to zero at zero such that

$$\limsup_{n \to \infty} \sup_{y \in K^{(2)}} \mathbb{P}(\tau'_y > n - m) \leqslant g(\eta).$$

Hence,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \sup_{\substack{x \in K^A_{n,\varepsilon} \\ y \in K^{(2)}}} \frac{\mathbb{P}(\tau_x > m)}{\mathbb{P}(\tau_x > n)} \mathbb{P}(\tau'_y > n - m) = 0,$$

which implies that

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \sup_{\substack{x \in K^A_{n,\varepsilon} \\ y \in K^{(2)}}} n^{d/2} \mathbb{P}(x + S(n) = y | \tau_x > n) = 0. \qquad \square$$

Proof of Lemma 15. Set $m = \lfloor \eta^3 n \rfloor$ and for $y \in K^{(3)}$, set

$$K_1(y) = \{ z \in K : |z - y| < \eta \sqrt{n} \}.$$

Then, we can write

$$\begin{aligned}
&n^{d/2}\mathbb{P}(x+S(n)=y,\tau_x>n)\\
&\quad=\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}\Bigg(\sum_{z\in K\setminus K_1(y)}\mathbb{P}(x+S(n-m)=z,\tau_x>n-m)\mathbb{P}(z+S(m)=y,\tau_z>m)\\
&\qquad+\sum_{z\in K_1(y)}\mathbb{P}(x+S(n-m)=z,\tau_x>n-m)\mathbb{P}(z+S(m)=y)\\
&\qquad-\sum_{z\in K_1(y)}\mathbb{P}(x+S(n-m)=z,\tau_x>n-m)\mathbb{P}(z+S(m)=y,\tau_z\leqslant m)\Bigg)\\
&\quad:=\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}(\Sigma_1+\Sigma_2-\Sigma_3).
\end{aligned}$$

By (15) with $u=\eta^{-1/2}$, there exist constants C,a independent of y such that for $z\notin K_1(y)$ (i.e., for $z\in\mathbb{R}^d$ such that $|z-y|\geqslant\eta\sqrt{n}\geqslant\eta^{-1/2}\sqrt{m}$),

$$\mathbb{P}(z+S(m)=y,\tau_z>m)\leqslant\mathbb{P}(z+S(m)=y)\leqslant Cm^{-d/2}\exp(-a/\eta).$$

Thus

$$\frac{n^{d/2}\Sigma_1}{\mathbb{P}(\tau_x>n)}\leqslant C\frac{\mathbb{P}(\tau_x>n-m)}{\mathbb{P}(\tau_x>n)}\eta^{-3d/2}\exp(-a/\eta).$$

By Corollary 10, $\limsup_{n\to\infty}\sup_{x\in K^A_{n,\varepsilon}}\frac{\mathbb{P}(\tau_x>n-m)}{\mathbb{P}(\tau_x>n)}\leqslant C'$ for some constant $C'>0$, which implies

$$\lim_{\eta\to0}\limsup_{n\to\infty}\sup_{x\in K^A_{n,\varepsilon}}C\frac{\mathbb{P}(\tau_x>n-m)}{\mathbb{P}(\tau_x>n)}\eta^{-3d/2}\exp(-a/\eta)\leqslant C'C\lim_{\eta\to0}\eta^{-3d/2}\exp(-a/\eta)=0.$$

Hence,

$$\lim_{\eta\to0}\limsup_{n\to\infty}\sup_{y\in K^{(3)},\,x\in K^A_{n,\varepsilon}}\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}\Sigma_1=0. \tag{31}$$

Likewise, by [16, Eq. (76)], there exist constants a,C such that for $y\in K^{(3)}$ and $z\in K_1(y)$

$$\mathbb{P}(x+S(m)=y,\tau_z\leqslant m)\leqslant Cm^{-d/2}\exp(-a/\eta).$$

Therefore,

$$\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}\Sigma_2\leqslant C\frac{\mathbb{P}(\tau_x>n-m)}{\mathbb{P}(\tau_x>n)}\eta^{-3d/2}\exp(-a/\eta).$$

Applying the same method as for Σ_1 yields

$$\lim_{\eta\to0}\limsup_{n\to\infty}\sup_{y\in K^{(3)},\,x\in K^A_{n,\varepsilon}}\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}\Sigma_3=0. \tag{32}$$

We now estimate the term Σ_2, which will eventually give the main contribution to the probability $n^{d/2}\mathbb{P}(x+S(n)=y,\tau_x>n)$. By the regular local limit theorem [42, Prop. 7.9],

$$\sup_{z\in K}\left|(2\pi m)^{d/2}\mathbb{P}(z+S(m)=y)-\exp(-|y-z|^2/(2m))\right|\xrightarrow[m\to\infty]{}0.$$

Hence,

$$\frac{1}{\mathbb{P}(\tau_x > n)} \sum_{z\in K_1(y)} \mathbb{P}(x + S(n-m) = z, \tau_x > n-m)\mathbb{P}(z + S(m) = y)$$
$$= \frac{1}{\mathbb{P}(\tau_x > n)} \sum_{z\in K_1(y)} \mathbb{P}(x + S(n-m) = z, \tau_x > n-m)(2\pi m)^{-d/2} \exp\big(-|y-z|^2/(2m)\big)$$
$$+ \frac{\mathbb{P}(\tau_x > n-m)}{\mathbb{P}(\tau_x > n)} o(m^{-d/2}), \tag{33}$$

where $o(m^{-d/2})$ is uniform on all $x \in K^A_{n,\varepsilon}$ and $y \in K^{(3)}$. On the one hand, using Corollary 10 yields

$$\limsup_{n\to\infty} \sup_{x\in K^A_{n,\varepsilon},\, y\in K^{(3)}} n^{d/2} o(m^{-d/2}) \frac{\mathbb{P}(\tau_x > n-m)}{\mathbb{P}(\tau_x > n)} = 0. \tag{34}$$

For n large enough,

$$\eta \leqslant \eta \frac{\sqrt{n}}{\sqrt{n-m}} \leqslant \frac{\eta}{\sqrt{1-\eta^3/2}},$$

and the function

$$f_n := u \mapsto \Big(\frac{n}{2\pi m}\Big)^{d/2} \exp\left(-\frac{(n-m)|u|^2}{2m}\right)$$

is uniformly bounded by $(2\pi\eta^3)^{-d/2}$ and Lipschitz with uniform Lipschitz constant $(2\pi\eta^3)^{-d/2}\eta^{-3}$. Applying Corollary 9 to the set of function $(f_n)_{n\geqslant 1}$ and the set of disks of radius $\left(\eta\frac{\sqrt{n}}{\sqrt{n-m}}\right)_{n\geqslant 1}$ yields

$$\sup_{\substack{x\in K^A_{n,\varepsilon}\\ y\in K^{(3)}}} \left| \mathbb{E}\left(\mathbf{1}_{B\left(y/\sqrt{n-m},\eta\frac{n}{n-m}\right)} f_n\big((x+S(n-m))/\sqrt{n-m}\big) | \tau_x > n-m\right)\right.$$
$$\left. - \frac{1}{k(x/\sqrt{n-m})} \int_{B\left(y/\sqrt{n-m},\eta\frac{\sqrt{n}}{\sqrt{n-m}}\right)} K(x/\sqrt{n-m}, z) f_n(z) dz \right| \xrightarrow[n\to\infty]{} 0.$$

Hence, expanding the expectation in the latter equation, doing the change of variable $u \leftrightarrow \frac{\sqrt{n-m}}{\sqrt{n}} u$ and using the scaling property of K_t give

$$\sup_{\substack{x\in K^A_{n,\varepsilon}\\ y\in K^{(3)}}} \left| \frac{n^{d/2}}{\mathbb{P}(\tau_x > n)} \sum_{z\in K_1(y)} \mathbb{P}(x + S(n-m) = z, \tau_x > n-m)(2\pi m)^{-d/2} \exp\big(-|y-z|^2/(2m)\big)\right.$$
$$\left. - \frac{n^d}{k(x/\sqrt{n-m})(2\pi(n-m)m)^{d/2}} \int_{|u|\leqslant\eta} K_{\frac{n-m}{n}}\left(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}} + u\right) \exp\left(-\frac{n|u|^2}{2m}\right) du \right| \xrightarrow[n\to\infty]{} 0. \tag{35}$$

As n goes to infinity, $\frac{n^2}{m(n-m)}$ converges to $\frac{1}{\eta^3(1-\eta^3)}$ and $\frac{n-m}{n}$ to $1-\eta^3$. Lemma 32 yields that

$$\frac{1}{k(x/\sqrt{n-m})}K_{\frac{n-m}{n}}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{n|u|^2}{2m}\right)$$
$$\underset{n\to\infty}{\sim}\frac{1}{k_{1-\eta^3}(x/\sqrt{n})}K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)$$

uniformly on $x\in K_{n,\varepsilon}^A$, $y\in K^{(3)}$ and $u\in B(0,\eta)$. Hence, using (34) with (35) in (33) yields

$$\limsup_{n\to\infty}\sup_{\substack{x\in K_{n,\varepsilon}^A\\ y\in K^{(3)}}}\left|\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}\Sigma_2\right.$$
$$\left.-\frac{(2\pi\eta^3)^{-d/2}}{(1-\eta^3)^{d/2}k_{1-\eta^3}(x/\sqrt{n})}\int_{|u|\leqslant\eta}K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du\right|=0.$$

Let us show an asymptotic formula for the second term of the latter equation as η goes to zero. Let $\theta>0$. By Lemma 32, there exists $\eta_0>0$ such that for $\eta\leqslant\eta_0$,

$$\frac{(1-\theta)}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\leqslant\frac{1}{k_{1-\eta^3}(x/\sqrt{n})}K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\leqslant\frac{(1+\theta)}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)$$

for all $x\in K_{n,\varepsilon}^A$, $y\in K^{(3)}$, $u\in B(0,\eta)$ and $n\geqslant 1$. Moreover, it comes from Remark 31 that $\frac{1}{k(x)}K(x,\cdot)$ is Hölder continuous on $K\cap B(0,M+\eta)$ with exponent α and some constant C independent of $x\in K\cap B(0,A)$, and thus uniformly continuous with uniform continuous bound independent of $x\in K\cap B(0,A)$. Hence, since the measure with density

$$\mathbf{1}_{|u|\leqslant\eta}(2\pi\eta^3)^{-d/2}\exp\left(\frac{-|u|^2}{2\eta^3}\right)du$$

converges weakly to a Dirac at 0 as η goes to 0, we have

$$\frac{(1-\eta^3)(2\pi\eta^3)^{-d/2}}{k(x/\sqrt{n})}\int_{|u|\leqslant\eta}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du\underset{\eta\to0}{\sim}\frac{1}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}\right)$$

uniformly on all $n\geqslant 1$, $x\in K_{n,\varepsilon}^A$ and $y\in K^{(3)}$. Hence, there exists $0<\eta_1<\eta_0$ such that for $\eta<\eta_1$,

$$\frac{(2\pi\eta^3)^{-d/2}}{(1-\eta^3)k_{1-\eta^3}(x/\sqrt{n})}\int_{|u|\leqslant\eta}K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du$$
$$\leqslant(1+\theta)\frac{(2\pi\eta^3)^{-d/2}}{k(x/\sqrt{n})}\int_{|u|\leqslant\eta}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du$$
$$\leqslant(1+\theta)^2\frac{1}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}\right),$$

and similarly,

$$\frac{(2\pi\eta^3)^{-d/2}}{(1-\eta^3)k_{1-\eta^3}(x/\sqrt{n})}\int_{|u|\leqslant\eta} K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du \geqslant \frac{(1-\theta)^2}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}\right)$$

for all $n\geqslant 1$, $x\in K^A_{n,\varepsilon}$ and $y\in K^{(3)}$. Thus,

$$\lim_{\eta\to 0}\sup_{\substack{n\geqslant 1,\, x\in K^A_{n,\varepsilon}\\ y\in K^{(3)}}}\left|\frac{(2\pi\eta^3)^{-d/2}}{(1-\eta^3)k_{1-\eta^3}(x/\sqrt{n})}\int_{|u|\leqslant\eta} K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du - \frac{1}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}\right)\right| = 0.$$

This yields finally

$$\begin{aligned}
&\lim_{\eta\to 0}\limsup_{n\to\infty}\sup_{\substack{x\in K^A_{n,\varepsilon}\\ y\in K^{(3)}}}\left|\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}\Sigma_2 - \frac{1}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}\right)\right|\\
&\leqslant \lim_{\eta\to 0}\limsup_{n\to\infty}\sup_{\substack{x\in K^A_{n,\varepsilon}\\ y\in K^{(3)}}}\left|\frac{n^{d/2}}{\mathbb{P}(\tau_x>n)}\Sigma_2\right.\\
&\qquad\left. - \frac{(2\pi\eta^3)^{-d/2}}{(1-\eta^3)^{d/2}k_{1-\eta^3}(x/\sqrt{n})}\int_{|u|\leqslant\eta} K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du\right|\\
&+\lim_{\eta\to 0}\limsup_{n\to\infty}\sup_{\substack{x\in K^A_{n,\varepsilon}\\ y\in K^{(3)}}}\left|\frac{(2\pi\eta^3)^{-d/2}}{(1-\eta^3)^{d/2}k_{1-\eta^3}(x/\sqrt{n})}\int_{|u|\leqslant\eta} K_{1-\eta^3}\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}+u\right)\exp\left(-\frac{|u|^2}{2\eta^3}\right)du\right.\\
&\qquad\left. - \frac{1}{k(x/\sqrt{n})}K\left(\frac{x}{\sqrt{n}},\frac{y}{\sqrt{n}}\right)\right| = 0.
\end{aligned}$$

Combining the latter equation with (31) and (32), we get

$$\lim_{\eta\to 0}\limsup_{n\to\infty}\sup_{y\in K^{(3)},\, x\in K^A_{n,\varepsilon}}\left|n^{d/2}\mathbb{P}(x+S(n)=y|\tau_x>n) - \frac{K\left(x/\sqrt{n},y/\sqrt{n}\right)}{k(x/\sqrt{n})}\right| = 0. \quad \square$$

With the help of Lemmas 13, 14 and 15, we can achieve the proof of Proposition 11.

Proof of Proposition 11. Note first that by Theorem 6, we have

$$\sup_{\substack{y\in K\\ |y-x|\geqslant M}}\frac{K(x,y)}{k(x)} \xrightarrow[M\to\infty]{} 0. \tag{36}$$

By Theorem 5, $k(y)\leqslant C_1\frac{u(y)}{u(y_1)}$ for $y\in K$, and by (5) there exists a constant $c>0$ such $u(y_1)>c$ for $y\in K$. Thus, there exists a constant C such that $k(y)\leqslant Cu(y)$ for all

$y \in K$. Hence, by (52), there exists a constant C' such that

$$k(y) \leqslant C' d(y, \partial K)|y|^{p-1},$$

which yields that for each $M > 0$

$$\sup_{\substack{y \in K,\, |y| \leqslant A+M \\ d(y,\partial K) < \eta}} k(y) \xrightarrow[\eta \to 0]{} 0.$$

Therefore, Theorem 6 implies that for each $M > 0$,

$$\sup_{\substack{y \in K,\, |x-y| \leqslant M \\ d(y,\partial K) < \eta}} \frac{K(x,y)}{k(x)} \xrightarrow[\eta \to 0]{} 0, \tag{37}$$

uniformly in $x \in K$, $|x| \leqslant A$. Let $\delta > 0$. By Lemma 13 and (36), there exists $M > 0$ such that

$$\limsup_{n \to \infty} \sup_{\substack{x \in K_{n,\varepsilon}^A \\ y \in K^{(1)}}} n^{d/2} \mathbb{P}(x + S(n) = y | \tau_x > n) \leqslant \delta$$

and

$$\sup_{\substack{x \in K_{n,\varepsilon}^A \\ y \in K^{(1)}}} \frac{K(x/\sqrt{n}, y/\sqrt{n})}{k(x/\sqrt{n})} \leqslant \delta,$$

where we recall that $K^{(1)} = \{y \in K : |y - x| > M\sqrt{n}\}$. Hence, for this value of M,

$$\limsup_{n \to \infty} \sup_{\substack{x \in K_{n,\varepsilon}^A \\ y \in K^{(1)}}} \left| n^{d/2} \mathbb{P}(x + S(n) = y | \tau_x > n) - \frac{K(x/\sqrt{n}, y/\sqrt{n})}{k(x/\sqrt{n})} \right| \leqslant 2\delta.$$

Then, by Lemma 14, (37) and Lemma 15, there exists $\eta > 0$ such that defining $K^{(2)}$ and $K^{(3)}$ with this value of η and the value of M chosen above gives

$$\limsup_{n \to \infty} \sup_{\substack{x \in K_{n,\varepsilon}^A \\ y \in K^{(2)}}} n^{d/2} \mathbb{P}(x + S(n) = y | \tau_x > n) \leqslant \delta, \qquad \sup_{\substack{x \in K_{n,\varepsilon}^A \\ y \in K^{(2)}}} \frac{K(x/\sqrt{n}, y/\sqrt{n})}{k(x/\sqrt{n})} \leqslant \delta,$$

and

$$\limsup_{n \to \infty} \sup_{\substack{x \in K_{n,\varepsilon}^A \\ y \in K^{(3)}}} \left| n^{d/2} \mathbb{P}(x + S(n) = y | \tau_x > n) - \frac{K(x/\sqrt{n}, y/\sqrt{n})}{k(x/\sqrt{n})} \right| \leqslant \delta.$$

Hence, for these values of M and η, there exists n_0 such that for $n \geqslant n_0$

$$\left| n^{d/2} \mathbb{P}(x + S(n) = y | \tau_x > n) - \frac{K(x/\sqrt{n}, y/\sqrt{n})}{k(x/\sqrt{n})} \right| \leqslant 3\delta$$

for all $x \in K_{n,\varepsilon}^A$ and $y \in K^{(1)} \cup K^{(2)} \cup K^{(3)} = K$. □

We now turn to the proof of Proposition 12, which gives the exact asymptotics of the probability $\mathbb{P}(x + S(n) = y, \tau_x > n)$, for y varying with n. Proposition 12 is an extension of [16, Thm 6].

We notice that the hypothesis of Proposition 12 stating that y has to remain in $K_{n,\varepsilon}^A$ will be removed in the next subsection, where the boundary case will be considered.

Proof of Proposition 12. Set $m = \lfloor n/2 \rfloor$. Classically,

$$\mathbb{P}(x+S(n) = y, \tau_x > n) = \sum_{z\in K} \mathbb{P}(x+S(n-m) = z, \tau_x > n-m)\mathbb{P}(y+S'(m) = z, \tau'_y > m).$$

Let $B > 0$. On the one hand, by (16) and Proposition 7, there exists a constant $C(x)$ such that

$$\begin{aligned}\Sigma_1(B,n) :=& \sum_{z\in K,\, |z|>(A+B)\sqrt{n}} \mathbb{P}(x + S(n-m) = z, \tau_x > n-m)\mathbb{P}(y + S'(m) = z, \tau'_y > m)\\ \leqslant& C(x)2^{p/2+d/2}n^{-p/2-d/2}\mathbb{P}(y + S'(m) > (B+A)\sqrt{n}, \tau'_y > m)\\ \leqslant& C\int_{|w|>A+B/\sqrt{2}} K(\sqrt{2}y/\sqrt{n}, w)dw\end{aligned}$$

for some constant $C > 0$. Hence, by Theorem 6,

$$\lim_{B\to\infty}\lim_{n\to\infty}\sup_{y\in K^A_{n,\varepsilon}} \frac{n^{p/2+d/2}}{k(y/\sqrt{n})}\Sigma_1(B,n) = 0. \tag{38}$$

On the other hand, by applying Proposition 11 to $S'(m)$ and then [16, Thm 5] to $S(n-m)$, we obtain that

$$\begin{aligned}&\sum_{z\in K,\, |z|\leqslant(A+B)\sqrt{n}} \mathbb{P}(x + S(n-m) = z, \tau_x > n-m)\mathbb{P}(y + S'(m) = z, \tau'_y > m)\\ =&\kappa 2^{p+d/2}V(x)n^{-p/2-d}\sum_{z\in K,\, |z|\leqslant(A+B)\sqrt{n}} u(\sqrt{2}z/\sqrt{n})K(\sqrt{2}y/\sqrt{n}, \sqrt{2}z/\sqrt{n})\exp\left(-\frac{|z|^2}{n}\right) + o(R_n)\\ :=&\Sigma_2(B,n) + o(R_n),\end{aligned}$$

with κ being a constant coming from [16, Thm 5] and

$$R_n = \mathbb{P}(\tau_x > n-m)n^{-d/2}k(y/\sqrt{m}) + (n-m)^{-p/2-d/2}k(y/\sqrt{m}).$$

Using (14), we get

$$o(R_n) = o(n^{-p/2-d/2}k(y/\sqrt{m})). \tag{39}$$

Moreover, as n goes to infinity, by the Hölder continuity of $K(z,\cdot)$ on $K\cap B(0, A+B)$, we have the uniform convergence of the Riemann integral

$$\sup_{y\in K^A_{n,\varepsilon}}\left| n^{-d/2}\sum_{z\in K,\, |z|\leqslant(A+B)\sqrt{n}} u(\sqrt{2}z/\sqrt{n})K(\sqrt{2}y/\sqrt{n}, \sqrt{2}z/\sqrt{n})\exp\left(-\frac{|z|^2}{n}\right) - 2^{p/2}\int_{|w|\leqslant A+B} u(w)K(\sqrt{2}y/\sqrt{n}, \sqrt{2}w)e^{-|w|^2}dw\right| \xrightarrow[n\to\infty]{} 0,$$

which yields

$$\Sigma_2(B,n) \sim_{n\to\infty} \kappa 2^{p+d}V(x)n^{-p/2-d/2}\int_{|w|\leqslant A+B} u(w)K(\sqrt{2}y/\sqrt{n}, \sqrt{2}w)e^{-|w|^2}dw \tag{40}$$

uniformly on $y \in K^A_{n,\varepsilon}$. In particular, by Theorem 6, there exist $C > 0$ and $n_0 \geqslant 1$ such that

$$\Sigma_2(B,n) \geqslant Cn^{-p/2-d/2}k(\sqrt{2}y/\sqrt{n})$$

for $n \geqslant n_0$ and $y \in K^A_{n,\varepsilon}$. Thus, combining (38), (39) and (40) yields

$$\mathbb{P}(x+S(n) = y, \tau_x > n) \sim_{n\to\infty} \kappa 2^{p+d/2} V(x) n^{-p/2-d/2} \int_{w\in K} u(w) K(\sqrt{2}y/\sqrt{n}, \sqrt{2}w) e^{-|w|^2} dw.$$

It remains to compute the above integral. Note first that by homogeneity of K,

$$K(\sqrt{2}y/\sqrt{n}, \sqrt{2}w) = K_{1/2}(y/\sqrt{n}, w).$$

Moreover, by [16, Lem. 18],

$$K_t(x,w) \underset{\substack{x\to 0 \\ |w|\leqslant |x|^{-1/2}}}{\sim} \kappa' u(x) u(w) \exp\left(-\frac{|w|^2}{2t}\right) t^{-p-d/2} \tag{41}$$

for some constant κ'. Hence,

$$\int_{w\in K} u(w) K_{1/2}(y/\sqrt{n}, w) e^{-|w|^2} dw = \lim_{x\to 0} \frac{1}{\kappa' u(x) 2^{p+d/2}} \int_{w\in K} K_{1/2}(x,w) K_{1/2}(y/\sqrt{n}, w) dw.$$

Since K is symmetric and is a Markov kernel,

$$\int_{w\in K} u(w) K_{1/2}(y/\sqrt{n}, w) e^{-|w|^2} dw = \lim_{x\to 0} \frac{1}{\kappa' u(x) 2^{p+d/2}} K(x, y/\sqrt{n}).$$

Finally, using (41) again in the latter equality yields

$$\int_{w\in K} u(w) K_{1/2}(y/\sqrt{n}, w) e^{-|w|^2} dw = u(y/\sqrt{n}) 2^{-p-d/2} \exp\left(-\frac{|y|^2}{2n}\right)$$

and

$$\mathbb{P}(x + S(n) = y, \tau_x > n) \underset{n\to\infty}{\sim} \kappa V(x) n^{-p/2-d/2} u(y/\sqrt{n}) \exp\left(-\frac{|y|^2}{2n}\right). \qquad \square$$

Local limit theorem close to the boundary. We generalize the previous results to the case of an arbitrary element of K. We first give a useful generalization of [16, Lem. 24] under stronger moment conditions on the increments.

Lemma 16. *Let $0 \leqslant r \leqslant p$ and $A > 0$, and suppose that the increments X admit moments of order $q > r + 2$. Set*

$$S(x,n)^+ = \sup_{1\leqslant \ell \leqslant n^{1-\varepsilon}} \{|S(\ell)| : \tau_x > \ell\}.$$

Then, for each $s < (q-r)/2$, $s' < (q-2)/2$ and $\beta \in ((p/2-1) \wedge 0, p/2)$, there exists $C > 0$ such that

$$\mathbb{E}\left((S(x,n)^+)^r, S(x,n)^+ \geqslant n^{1/2-\varepsilon/8}\right) \leqslant C n^{-s} n^{(1-(p/2-\beta))} (1+|x|)^{p-2\beta}$$

for all $x \in K$. In particular, uniformly on $x \in K$, $|x| \leqslant A\sqrt{n}$,

$$\mathbb{E}\left((S(x,n)^+)^r, S(x,n)^+ \geqslant n^{1/2-\varepsilon/8}\right) \leqslant C n^{-s+1}.$$

The proof of this lemma follows closely the one of [16, Lem. 24], and we invite the reader to refer to this reference for the proof of several facts used in the following argument.

Proof. Let $\ell_n \leqslant n^{1-\varepsilon}$ be the time where $S(x,n)^+$ is reached by the random walk. As in the proof of [16, Lem. 24], we set $\mu_n = \min\{j \geqslant 1 : |X_j| \geqslant n^{1/2-\varepsilon/4}\}$ and write

$$\begin{aligned}\mathbb{E}\left((S(x,n)^+)^r, S(x,n)^+ \geqslant n^{1/2-\varepsilon/8}\right) =& \mathbb{E}\left((S(x,n)^+)^r, S(x,n)^+ \geqslant n^{1/2-\varepsilon/8}, \mu_n > \ell_n\right)\\ &+ \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{E}\left((S(x,n)^+)^r, S(x,n)^+ \geqslant n^{1/2-\varepsilon/8}, j \leqslant \ell_n, \mu_n = j\right)\\ =& E_1 + E_2.\end{aligned}$$

Using Fuk-Nagaev inequalities [25] as in [16, Lem. 24], the term E_1 is bounded by $\exp(-Cn^{\varepsilon'})$ for some positive constants C and ε' independent of $x \in K$. Then, we bound the second term as

$$\begin{aligned}E_2 \leqslant C \sum_{j=1}^{n^{1-\varepsilon}} &\mathbb{E}(|S(\ell_n) - X(j)|^r, \tau_x \geqslant j, \ell_n \geqslant j, \mu_n = j)\\ &+ \mathbb{E}(|X(j)|^r, \tau_x \geqslant j-1, \mu_n = j) + C\mathbb{E}(|S(j-1)|^r, \tau_x \geqslant j, \mu_n = j).\end{aligned}$$

Using Doob and Rosenthal inequalities as in [16, Lem. 24], we get

$$\begin{aligned}\sum_{j=1}^{n^{1-\varepsilon}} &\mathbb{E}(|S(\ell_n) - X(j)|^r, \tau_x \geqslant j, \ell_n \geqslant j), \mu_n = j)\\ &\leqslant Cn^{r(1-\varepsilon)/2} \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{P}(\tau_x \geqslant j)\mathbb{P}(|X(j)| \geqslant n^{1/2-\varepsilon/4})\\ &\leqslant Cn^{r(1-\varepsilon)/2} n^{-q(1/2-\varepsilon/4)} \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{P}(\tau_x \geqslant j).\end{aligned}$$

Then, by Markov's inequality,

$$\begin{aligned}\sum_{j=1}^{n^{1-\varepsilon}} \mathbb{E}(|X(j)|^r, \tau_x \geqslant j-1, \mu_n = j) &\leqslant \mathbb{E}(|X|^r, |X| \geqslant n^{1/2-\varepsilon/4}) \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{P}(\tau_x \geqslant j-1)\\ &\leqslant Cn^{-(q-r)(1/2-\varepsilon/4)} \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{P}(\tau_x \geqslant j-1).\end{aligned}$$

Using again Fuk-Nagaev inequalities as in [16, Lemma 24] gives

$$\mathbb{E}(|x + S(j-1)|^r, \tau_x \geqslant j, \mu_n = j) \leqslant Cn^{-(q-r)(1/2-\varepsilon/8)} \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{P}(\tau_x \geqslant j-1).$$

Hence, we have

$$E_2 \leqslant Cn^{-(q-r)/2+g(\varepsilon)} \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{P}(\tau_x \geqslant j-1),$$

with $g : \mathbb{R} \to \mathbb{R}$ linear. Then, we have

$$\begin{aligned} E_2 \leqslant & Cn^{-(q-r)/2+g(\varepsilon)+(p/2-\beta)(1-\varepsilon)} \sum_{j=1}^{n^{1-\varepsilon}} \mathbb{P}(\tau_x \geqslant j-1) \\ \leqslant & Cn^{-(q-r)/2+g(\varepsilon)+(1-(p/2-\beta))} \sum_{j=1}^{n^{1-\varepsilon}} j^{-1+(p/2-\beta)} \mathbb{P}(\tau_x \geqslant j-1) \\ \leqslant & Cn^{-(q-r)/2+g(\varepsilon)+(1-(p/2-\beta))} \mathbb{E}(\tau_x^{p/2-\beta}) \leqslant Cn^{-(q-r)/2+g(\varepsilon)+(1-(p/2-\beta))}(1+|x|)^{p-2\beta}, \end{aligned}$$

where we have used [35, Thm 3.1] in the last inequality. This yields the first part of the lemma for ε small enough. Moreover, for any $s < (q-r-2)/2$ we have by choosing ε

$$E_2 \leqslant Cn^{-s+1},$$

for all $x \in K$, $|x| \leqslant A\sqrt{n}$. □

We now prove an estimate on the survival probability for the reverse random walk S'; recall that τ'_z denotes the exit time of S'.

Lemma 17. *There exists $c > 0$ such that for all $z \in K$ large enough all $n \geqslant 1$,*

$$\mathbb{P}(\tau'_z > n) \geqslant cn^{-p/2}.$$

Proof. Let H be a hyperplane such that $H \cap K \subset \partial K$; such an H exists thanks to the convexity assumption on K. Let v be the unit vector orthogonal to H and pointing toward K. For $z \in K$, let us denote by $d_v(z)$ the vertical distance from z to ∂K parallel to v: namely,

$$d_v(z) = \sup\{\lambda > 0 : z - \lambda v \in K\}.$$

Since $z \in K$ one has $d_v(z) \geqslant 0$ and for $t > 0$, $d_v(tv) = t$. Moreover, if $y \in z + K$, then $y - d_v(z)v \in z - d_v(z)v + K \subset K$, which yields

$$d_v(y) \geqslant d_{v(z)} + \sup\{\lambda > 0 : y - \lambda v \in z + K\}. \tag{42}$$

Since the random walk satisfies $(H6)$, there exists $R > 0$ such that for $z \in \Lambda \cap K$ with $|z| \geqslant R$, there exists a path in $B(z, R) \cap K$ with positive probability with respect to S' which starts at z and ends in $z + K$. Let $t > R$ be such that $tv \in z_0 + K$ for some $z_0 \in K \cap \Lambda$, $|z_0| \geqslant R$. Let Γ' be a path in $B(z_0, R) \cap K$ from z_0 to an endpoint $z_0 + x \in z_0 + K$ with $\mathbb{P}(\Gamma') > 0$ with respect to S'. By convexity of K and the fact that $tv \in z_0 + K$, the translation of Γ' by $tv - z_0$ yields a path Γ in $B(tv, R) \cap K$ from tv to an endpoint $tv + x \in tv + K$ with $\mathbb{P}(\Gamma) > 0$ with respect to S'. Concatenating n times Γ gives a path in K from tv to $tv + nx$ with positive probability with respect to S'. Let us set

$$\gamma := \frac{d(x, \partial K)}{|x| + |tv|}.$$

Then for every $M > 0$ there exists $n \geqslant 1$ such that $\mathbb{P}(|tv+S(n)| \geqslant M, d(tv+S(n), \partial K) \geqslant \gamma|tv + S(n)|$. Hence by [16, Thm 1], $V'(tv) > 0$ and there exists $c > 0$ such that

$$\mathbb{P}\left(\tau'_{tv} > n\right) \geqslant cn^{-p/2} \tag{43}$$

for all $n \geqslant 1$.

When C is a cone distinct from K containing x, denote by $\tau'_{x,C}$ the exit time for the random walk $x+S'$ in C. Suppose that $z \in K$ is such that $d_v(z) > t$. Since K is convex, $z - tv \in K$ and thus $z - tv + K \subset K$. Then (43) yields

$$\mathbb{P}(\tau'_z > n) \geqslant \mathbb{P}\left(\tau'_{z,z-tv+K} > n\right) \geqslant cn^{-p/2} \tag{44}$$

for all $n \geqslant 1$, since by translation $\mathbb{P}(\tau'_{z,z-tv+K} > n) = \mathbb{P}(\tau'_{tv,K} > n)$.

Let us show that there exists $\rho > 0$ such that for all $z \in K \cap \Lambda$ with $|z| \geqslant R$ there exists $n \geqslant 1$ with $\mathbb{P}(d_v(z + S(n)) \geqslant t, \tau_z > n) \geqslant \rho$. Let $z \in K \cap \Lambda$ with $|z| \geqslant R$, and let Γ_1 be a path in $B(z,R) \cap K$ from z to $z + K$ with positive probability for S', whose existence is given by $(H6)$. Let n_0 be the maximum number of points of the lattice inside $B(z, 2R)$ for $z \in \mathbb{R}^d$ and set

$$\eta := \min_{\substack{\gamma \subset B(0,2R),\, l(\gamma) \leqslant n_0 \\ \mathbb{P}(\gamma) > 0}} \mathbb{P}(\gamma),$$

$l(\gamma)$ denoting the length of γ, where the probability is understood with respect to S'. Moreover, set

$$\delta := \min\{d_v(z) : z \in \Lambda \cap K, |z| \leqslant 2R\}.$$

Since there are at most n_0 points of the lattice in $B(z,R)$ we may assume $l(\Gamma_1) \leqslant n_0$, which yields $\mathbb{P}(\Gamma_1) > \eta$. Let z_1 be the endpoint of Γ_1. Since $z_1 \in z + K \cap \Lambda$ and $z \in \Lambda$, then

$$\sup\{\lambda > 0 : z_1 - \lambda v \in z + K\} \geqslant \delta.$$

Hence, (42) yields

$$d_v(z_1) > d_v(z) + \delta.$$

Repeating the operation for $z_1, z_2, \dots$ at most $\lfloor \frac{t}{\delta} \rfloor$ times yields ultimately a path Γ in K from z to $z' \in K$ of length m such that $d_v(z') \geqslant t$ and $\mathbb{P}(\Gamma) \geqslant \eta^{\lfloor \frac{t}{\delta} \rfloor}$. Then, the Markov property and (44) yield

$$\mathbb{P}(\tau_z > n) \geqslant \eta^{\lfloor \frac{t}{\delta} \rfloor} \mathbb{P}(\tau_{z'} > n - m) \geqslant \eta^{\lfloor \frac{t}{\delta} \rfloor} c(n-m)^{-p/2} \geqslant c' n^{-p/2},$$

for some constant c' independent of z. □

We are now able to prove Theorem 1. Recall that $t'_{y,\varepsilon}(n) := \inf\{m \geqslant 0 : y + S(m) \in K_{n,\varepsilon}\}$ denotes the first time that $y + S(n)$ reaches $K_{n,\varepsilon}$, and $y'_\varepsilon(n) := y + S(t'_{y,\varepsilon}(n))$.

Proof of Theorem 1. Let $y \in K$. Then, since $\mathbb{P}(x + S(n) = y, \tau_x > n) = \mathbb{P}(y + S'(n) = x, \tau'_y > n)$,

$$\begin{aligned}
\mathbb{P}(x + S(n) = y, \tau_x > n) &= \mathbb{P}(y + S'(n) = x, t'_{y,\varepsilon}(n) < n^{1-\varepsilon}, \tau'_y > n) \\
&\quad + \mathbb{P}(y + S'(n) = x, t'_{y,\varepsilon}(n) < n^{1-\varepsilon}, |S'(t'_{y,\varepsilon}(n))| > n^{1/2-\varepsilon/8}, \tau'_y > n) \\
&\quad + \mathbb{P}(y + S'(n) = x, t'_{y,\varepsilon}(n) \geqslant n^{1-\varepsilon}, |S'(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}, \tau'_y > n).
\end{aligned}$$

By (12),

$$\mathbb{P}(y + S'(n) = x, t'_{y,\varepsilon}(n) \geqslant n^{1-\varepsilon}, \tau'_y > n) \leqslant \mathbb{P}(t'_{y,\varepsilon}(n) \leqslant n^{1-\varepsilon}, \tau'_y > n^{1-\varepsilon}) \leqslant \exp(-Cn^\varepsilon)$$

for some constant $C > 0$. Moreover, by Lemma 16 and the moment condition on the increments, we get

$$\begin{aligned}\mathbb{P}(t'_{y,\varepsilon}(n) \leqslant n^{1-\varepsilon}, |S'(t'_{y,\varepsilon}(n))| > n^{1/2-\varepsilon/8}, \tau'_y > n) \\ \leqslant \mathbb{P}\Big(\sup_{1\leqslant \ell \leqslant n^{1-\varepsilon}} (|S'(\ell)|, \tau_x > \ell) > n^{1/2-\varepsilon/8}\Big) \leqslant Cn^{-(r-2)/2} \quad (45)\end{aligned}$$

for some $r > 2p + d - 2 + 2(1 - p/2)^+ \geqslant p + 2$. Thus, (16) yields

$$\begin{aligned}&\mathbb{P}(y + S'(n) = x, t'_{y,\varepsilon}(n) \geqslant n^{1-\varepsilon}, |S(t'_{y,\varepsilon}(n))| > n^{1/2-\varepsilon/8}, \tau'_y > n) \\ &\leqslant \mathbb{P}(t'_{y,\varepsilon}(n) \leqslant n^{1-\varepsilon}, |S'(t'_{y,\varepsilon}(n))| > n^{1/2-\varepsilon/8}, \tau'_y > n) \sup_{\substack{z\in K \\ n-n^{1-\varepsilon}\leqslant m\leqslant n}} \mathbb{P}(x + S(m) = z, \tau_x > m) \\ &\leqslant C' n^{-(r-2)-p/2-d/2}\end{aligned}$$

for some constant $C' > 0$. By Proposition 12 and the strong Markov property of S',

$$\begin{aligned}\mathbb{P}(y + S'(n) = x, t'_{y,\varepsilon}(n) < n^{1-\varepsilon}, |S(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}, \tau'_y > n) \\ = \mathbb{E}(\mathbb{P}(x + S(n - t'_{y,\varepsilon}(n)) = y'_\varepsilon(n), \tau_x > n - t'_{y,\varepsilon}(n)), \\ t'_{y,\varepsilon}(n) < n^{1-\varepsilon}, |S(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}, \tau'_y > t'_{y,\varepsilon}(n)) \\ \sim \kappa V(x) n^{-p-d/2} \mathbb{E}(u(y'_\varepsilon(n)) \exp(-|y|^2/(2n)), t'_{y,\varepsilon}(n) \leqslant n^{1-\varepsilon}, \\ |S(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}, \tau'_y > t'_{y,\varepsilon}(n))\end{aligned}$$

uniformly on $y \in K$, $y \leqslant A\sqrt{n}$. Using the definition of $t^\varepsilon_y(n)$ and the lower bound $u(z) \geqslant cd(z, \partial K)^p$ for some constant $c > 0$, we get

$$u(y'_\varepsilon(n)) \exp(-|y|^2/(2n)) \geqslant cn^{p(1/2-\varepsilon)}$$

on $\{t'_{y,\varepsilon}(n) \leqslant n^{1-\varepsilon}, \tau'_y > t'_{y,\varepsilon}(n), |S(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}\}$. Hence, there exists c' such that for n large enough,

$$\begin{aligned}\mathbb{P}(y + S'(n) = x, t^\varepsilon_y(n) < n^{1-\varepsilon}, \tau'_y > n) \\ \geqslant c' n^{-p/2-p\varepsilon-d/2} \mathbb{P}(t^\varepsilon_y(n) \leqslant n^{1-\varepsilon}, \tau'_y > t^\varepsilon_y(n), |S(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}).\end{aligned}$$

By (12) and (45),

$$\begin{aligned}\mathbb{P}(t^\varepsilon_y \leqslant n^{1-\varepsilon}, \tau'_y > t^\varepsilon_y, |S(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}) \\ \geqslant \mathbb{P}(t^\varepsilon_y \leqslant n^{1-\varepsilon}, \tau'_y > n^{1-\varepsilon}) - Cn^{-(r-1)/2} \\ \geqslant \mathbb{P}(\tau'_y > n^{1-\varepsilon}) - (\exp(-Cn^\varepsilon) - n^{-(r-1)/2}).\end{aligned}$$

Since $r > p + 2$ by $(H5)$, Lemma 17 yields that

$$\mathbb{P}(y + S'(n) = x, t^\varepsilon_y < n^{1-\varepsilon}, \tau'_y > n, |S(t'_{y,\varepsilon}(n))| \leqslant n^{1/2-\varepsilon/8}) \geqslant cn^{-p(1/2-\varepsilon)-d/2-p/2} \quad (46)$$

for some constant c'' independent of $y \in K$, $|y| \leqslant A\sqrt{n}$. Therefore, since $r > p + 2$, we have

$$p(1/2 - \varepsilon) + d/2 + p/2 < p/2 + d/2 + (r - 2)/2,$$

and thus

$$\mathbb{P}(y+S'(n)=x, t'_{y,\varepsilon}(n)\leqslant n^{1-\varepsilon}, |S(t'_{y,\varepsilon}(n))|>n^{1/2-\varepsilon/8}, \tau'_y>n) \\ = o\left(\mathbb{P}(y+S'(n)=x, t^{\varepsilon}_y(n)<n^{1-\varepsilon}, \tau'_y>n)\right).$$

Hence, finally

$$\mathbb{P}(x+S(n)=y,\tau_x>n)\sim$$
$$\kappa V(x) n^{-p-d/2}\mathbb{E}(u(y'_\varepsilon(n))\exp(-|y|^2/(2n)), t'_{y,\varepsilon}(n)\leqslant n^{1-\varepsilon}, \tau'_y>t'_{y,\varepsilon}(n), |S(t'_{y,\varepsilon}(n))|\leqslant n^{1/2-\varepsilon/8}).$$

We have again

$$\begin{aligned}\mathbb{E}(u(y'_\varepsilon(n))\exp(-|y'_\varepsilon(n)|^2/(2n)), t'_{y,\varepsilon}(n)\geqslant n^{1-\varepsilon}, \tau'_y>t'_{y,\varepsilon}(n)) \\ \leqslant n^{p/2}\sup_{y\in K}\left|u(y)\exp(-|y|^2/2)\right|\mathbb{P}(\tau^{\varepsilon}_y\geqslant n^{1-\varepsilon}, \tau'_y>t^{\varepsilon}_y) \\ \leqslant Cn^{p/2}\exp(-Cn^{\varepsilon})\end{aligned}$$

for some constant $C>0$, and

$$\begin{aligned}\mathbb{E}(u(y'_\varepsilon(n))\exp(-|y'_\varepsilon(n)|^2/(2n)), t'_{y,\varepsilon}(n)\leqslant n^{1-\varepsilon}, \tau'_y>t'_{y,\varepsilon}(n), |S(t'_{y,\varepsilon}(n))|\geqslant n^{1/2-\varepsilon/8}) \\ \leqslant n^{p/2}\sup_{y\in K}\left|u(y)\exp(-|y|^2/2)\right|\mathbb{P}(\tau^{\varepsilon}_y\leqslant n^{1-\varepsilon}, \tau'_y>t^{\varepsilon}_y, |S(t'_{y,\varepsilon}(n))|\geqslant n^{1/2-\varepsilon/8}) \\ \leqslant Cn^{(p+2-r)/2}\end{aligned}$$

for some constant $C>0$, with $r>p+2$. Hence, by (46),

$$\mathbb{P}(x+S(n)=y,\tau_x>n)\sim\kappa V(x)n^{-p-d/2}\mathbb{E}(u(y'_\varepsilon(n))\exp(-|y|^2/(2n)), \tau'_y>t'_{y,\varepsilon}(n)).\ \square$$

4. Identification of the Martin boundary

In this section, we identify the Martin boundary for K convex and $d\geqslant 2$, with the Denisov and Wachtel harmonic function (3).

Estimate of the local probability in middle and large range. In this subsection we give a bound on the probability $\mathbb{P}(x+S(n)=y,\tau_x>n)$ when $|x-y|$ is of order $(n\log(n))^{1/2}$.

We will use the coupling of Zaitsev and Götze (see [26, Thm 4] and [16, Lem. 17]) for random walks having increments satisfying to $(H5)$. Suppose that X has moments of order $r(p)$ with $r(p)>2p+d-2+2(1-p/2)^+$ and $r(p)>2+\delta$. By [23, Thm 4], there exists a constant K such that for $\gamma\leqslant 1/2-1/r(p)$,

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant n}|S(\lfloor k\rfloor)-B(k))\geqslant n^{1/2-\gamma}\right)\leqslant Kn^{-r}, \tag{47}$$

with $r=r(p)(1/2-\gamma)-1$. Since we are only considering the case $d\geqslant 2$, we have $r>p/2$ for γ small enough (which we assume from now on).

Lemma 18. *There exist $\kappa,\varepsilon,C>0$ such that for all n large enough and $A\sqrt{n}\leqslant t\leqslant n^{1/2+\kappa}$,*

$$\mathbb{P}(|S(n)|>t,\tau_y>n)\leqslant C\left(u(y/\sqrt{n})\exp(-t^2/(2c_3n))\vee n^{-r}\right)$$

and for $y \in K_{n,\varepsilon}$ such that $|y| \leqslant n^{1/2+\kappa}$,

$$\mathbb{P}(\tau_y > n) \leqslant Cu(y/\sqrt{n}).$$

Proof. Using the same construction as in the proof of Proposition 7 and using the same notations (see (20)), we get

$$\mathbb{P}(|S(n)| > t, \tau_y > n) \leqslant \int_{|z-y/\sqrt{n}|>t/\sqrt{n}-2n^{-\gamma}} K(y^+/\sqrt{n}, z)dz + O(n^{-r}).$$

Using the upper bound of Theorem 6 yields

$$\int_{|z-y/\sqrt{n}|>t/\sqrt{n}-2n^{-\gamma}} K(y^+/\sqrt{n}, z)dz \leqslant Ck(y^+/\sqrt{n}) \int_{|z-y/\sqrt{n}|>t/\sqrt{n}-2n^{-\gamma}} C_2 \exp(-|z-y^+/\sqrt{n}|^2/(2c_3))dz. \quad (48)$$

By the local Hölder continuity of k given by Lemma 30,

$$k(y^+/\sqrt{n}) \leqslant k(y/\sqrt{n}) + C_\alpha \left(|y|/\sqrt{n}\right)^{p-1} n^{-\alpha\gamma} \leqslant u(y/\sqrt{n}) + n^{(p-1)\kappa} n^{-\alpha\gamma}.$$

Since $u(y/\sqrt{n}) \geqslant d(y/\sqrt{n}, K)^p \geqslant n^{-p\varepsilon}$, choosing κ such that $\alpha\gamma - (p-1)\kappa > 0$ and ε such that $\varepsilon \leqslant (\alpha\gamma - (p-1)\kappa)/p$ yields that for some $C' > 0$ and $y \in K_{n,\varepsilon}^{n^\kappa}$,

$$k(y^+/\sqrt{n}) \leqslant C'u(y/\sqrt{n}).$$

Hence, after integrating in (48) over the angular coordinate, we get that for some C'',

$$\int_{|z-y/\sqrt{n}|>t/\sqrt{n}-2n^{-\gamma}} K(y^+/\sqrt{n}, z)dz \leqslant C''u(y/\sqrt{n}) \int_{z>t/\sqrt{n}-(2+R)n^{-\gamma}} \exp(-|z|^2/(2c_3))dz.$$

The latter inequality for $t = 0$ gives the second inequality of the statement. For the first statement, notice that there exists $C > 0$ such that $\int_x^\infty \exp(-z^2)dz \leqslant C\exp(-x^2)$ and so that choosing $\kappa < \gamma$ yields $\exp((t/\sqrt{n} - (2+R)n^{-\gamma})^2/(2c_3)) \sim \exp((t/\sqrt{n})^2/(2c_3))$ for $t \leqslant n^{1/2+\kappa}$. Finally we get that for some constant $C''' > 0$,

$$\int_{|z-y/\sqrt{n}|>t/\sqrt{n}-2n^{-\gamma}} K(y^+/\sqrt{n}, z)dz \leqslant C'''u(y/\sqrt{n}) \exp((t/\sqrt{n})^2/(2c_3)). \qquad \square$$

We can extend the latter result by relaxing the condition $y \in K_{n,\varepsilon}$.

Lemma 19. *Let $x \in K$. There exists $C > 0$ such that*

$$\mathbb{P}(|S(n)| \geqslant t, \tau_x \geqslant n) \leqslant C\left(V(x)n^{-p/2}\exp(-t^2/(2nc_3)) \vee n^{-r}\right)$$

for all $t \leqslant n^{1/2+\kappa}$, where κ is defined in Lemma 18.

Proof. Let $t_{x,\varepsilon}(n)$ be the stopping time $\inf\{n \geqslant 1 : x + S(n) \in K_{n,\varepsilon}\}$ and $x_\varepsilon(n) = x + S(t_{x,\varepsilon}(n))$. Then, applying [16, Section 4] to Lemma 18, we get

$$\begin{aligned}\mathbb{P}(|S(n)| \geqslant t, \tau_x \geqslant n) \leqslant C\Big(& n^{-p/2} \exp(-t^2/(2c_3 n)) \\ & \mathbb{E}\left(u(x_\varepsilon(n)), \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right) \vee n^{-r}\Big) \\ & + n^{-p/2} O\left(\mathbb{E}\left(|x_\varepsilon(n)|^p, |x_\varepsilon(n)| > \theta_n\sqrt{n}, \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right)\right) \\ & + O(\exp(-Cn^{\varepsilon'}),\end{aligned}$$

where $\theta_n = n^{-\varepsilon/8}$ and ε' is small enough. Using Lemma 16 with $r = p$ and $q = r(p)$ gives

$$n^{-p/2}\mathbb{E}\left(|x_\varepsilon(n)|^p, |x_\varepsilon(n)| > \theta_n\sqrt{n}, \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right) = O(n^{-(2p+d-4)/2-\delta}),$$

since $2p + d - 2 + 2\delta < r(p)$. Since $(2p + d - 4)/2 \geqslant r$, Lemma 18 yields

$$n^{-p/2}\mathbb{E}\left(|x_\varepsilon(n)|^p, |x_\varepsilon(n)| > \theta_n\sqrt{n}, \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right) = o(n^{-r})$$

for $t \leqslant n^{1/2+\kappa}$. Since, by [16, Lem. 21],

$$\lim_{n\to\infty} \mathbb{E}\left(u(x_\varepsilon(n)), \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right) = V(x),$$

the result is deduced. □

Lemma 20. *There exist C and n_0 such that for $n \geqslant n_0$, all $y \in K_{n,\varepsilon}$ such that $|y| \leqslant n^{1/2+\kappa}$ and all $z \in K$,*

$$\mathbb{P}(y + S(n) = z, \tau_y > n) \leqslant Cn^{-d/2}u(y/\sqrt{n}).$$

Proof. Let $m := \lfloor n/2 \rfloor$. Then

$$\begin{aligned}\mathbb{P}(y + S(n) = z, \tau_y > n) &= \sum_{z' \in K} \mathbb{P}(y + S(m) = z', \tau_y > m)\mathbb{P}(z' + S(n-m) = z, \tau_{z'} > n - m) \\ &\leqslant C\mathbb{P}(\tau_y > m)m^{-d/2},\end{aligned}$$

where we have used (15) with $u = 0$ to bound $\mathbb{P}(z' + S(m) = z, \tau_{z'} > n - m)$. Thus, by Lemma 18, there exists n_0 such that for $n \geqslant n_0$ and $y \in K_{n,\varepsilon}$ with $|y| \leqslant n^{1/2+\kappa}$,

$$\mathbb{P}(y + S(n) = z, \tau_y > n) \leqslant Cn^{-d/2}u(y/\sqrt{n}).$$ □

Putting the previous results together yields the following estimate on the local probability at middle range.

Proposition 21. *Let $x \in K$. There exists C such that*

$$\mathbb{P}(x + S(n) = y, \tau_x > n) \leqslant CV(x)u(y)n^{-p/2-d/2}\left(n^{-p/2}\exp(-|x-y|^2/(c_3 n)) \vee n^{-r}\right)$$

for all $y \in K_{n,\varepsilon}$ such that $|y - x| \leqslant n^{1/2+\kappa}$.

Proof. Let $m := \lfloor n/2 \rfloor$. Then we have

$$\begin{aligned}
&\mathbb{P}(x + S(n) = y, \tau_x > n) \\
&= \sum_{z \in K, \, |z-x| \geqslant |y-x|/2} \mathbb{P}(x + S(m) = z, \tau_x > m) \mathbb{P}(y + S'(n-m) = y, \tau'_y > n - m) \\
&+ \sum_{z \in K, \, |z-x| < |y-x|/2} \mathbb{P}(x + S(m) = z, \tau_x > m) \mathbb{P}(y + S'(n-m) = y, \tau'_y > n - m) \\
&= M_1 + M_2.
\end{aligned}$$

By Lemma 20 and Lemma 19, the first sum is bounded by

$$\begin{aligned}
M_1 &\leqslant C u(y/\sqrt{n}) n^{-d/2} \mathbb{P}(|S(n)| > |x-y|/2, \tau_x > n) \\
&\leqslant C V(x) u(y/\sqrt{n}) n^{-d/2} \left(n^{-p/2} \exp(-|y-x|^2/(c_3 n)) \vee n^{-r} \right),
\end{aligned}$$

where we have used in the last inequality the hypothesis $|y-x|/2 \leqslant n^{1/2+\kappa}$ in order to apply Lemma 19. Similarly, by (16) and Lemma 18, the second sum is bounded by

$$\begin{aligned}
M_2 &\leqslant C V(x) n^{-d/2-p/2} \mathbb{P}(|S'(m)| > |x-y|/2, \tau'_y > n) \\
&\leqslant C V(x) n^{-d/2-p/2} \left(u(y/\sqrt{n}) \exp(-|y-x|^2/(c_3 n)) \vee n^{-r} \right) \\
&\leqslant C V(x) u(y) n^{-d/2-p/2} \left(n^{-p/2} \exp(-|y-x|^2/(c_3 n)) \vee n^{-r} \right),
\end{aligned}$$

where we have used in the last inequality the hypothesis that $|y-x|/2 \leqslant n^{1/2+\kappa}$ in order to apply Lemma 18, as well as the fact that $u(y) \geqslant 1$ for $y \in K_{n,\varepsilon}$ and n large enough. The result is then deduced by summing the bounds on M_1 and M_2. □

Uniqueness of the harmonic function. The Green function (see (2)) is defined for $x, y \in K$ by

$$G(x,y) = \sum_{n=0}^{\infty} \mathbb{P}(x + S(n) = y, \tau_x > n).$$

We begin this subsection by giving an estimate of the truncated Green function

$$\sum_{n=0}^{|y-x|^{2-\nu}} \mathbb{P}(x + S(n) = y, \tau_x > n),$$

for $\nu > 0$.

Proposition 22. *For $\nu > 0$, there exist positive constants C, and a such that*

$$\sum_{n=0}^{|y-x|^{2-\nu}} \mathbb{P}(x + S(n) = y, \tau_x > n) \leqslant C |y-x|^{-a}$$

with $a > 2p + d - 2$ if $d \geqslant 2$ and $a > 0$ if $d = 1$.

Proof. We follow the proof of [16, Lem. 24] and introduce the stopping time

$$\mu = \inf\{i \geqslant 1 : |X_i| \geqslant |y-x|^{1-\nu/\alpha}\},$$

with α large to be chosen later. Let $n \leqslant |y-x|^{2-\nu}$. Then

$$\mathbb{P}(x+S(n) = y, \tau_x > n) = \mathbb{P}(x+S(n) = y, \tau_x > n, \mu > n) + \mathbb{P}(x+S(n) = y, \tau_x > n, \mu \leqslant n).$$

On the one hand, using Fuk-Nagaev inequalities [25] as in [16, Corollary 23] yields

$$\begin{aligned}\mathbb{P}(x+S(n)=y,\tau_x>n,\mu>n) &\leqslant \mathbb{P}(|S_n|\geqslant |x-y|/2, \sup_{k\leqslant n}|X_k|\leqslant |y-x|^{1-\nu/\alpha})\\ &\leqslant \left(\frac{n\sqrt{d}e}{|x-y|^{2-\nu/\alpha}/2}\right)^{|x-y|^{\nu/\alpha}/(2\sqrt{d})}\\ &\leqslant \left(\frac{|x-y|^{2-\nu}\sqrt{d}e}{|x-y|^{2-\nu/\alpha}/2}\right)^{2|x-y|^{\nu/\alpha}/(2\sqrt{d})} \leqslant \exp(-C|x-y|^{\nu/\alpha})\end{aligned}$$

for y large enough. On the other hand, recall that since X admits moments of order $r(p) > 2p + d - 2 + 2(1 - p/2)^+$,

$$\begin{aligned}&\mathbb{P}(x+S(n)=y,\tau_x>n,\mu\leqslant n)\\ &\qquad\leqslant \sum_{k=1}^{n}\mathbb{P}(\tau_x>k-1,|X_k|\geqslant |y-x|^{1-\nu/\alpha}, y+S'(n-k)=x+S(k))\\ &\qquad\leqslant CV(x)\frac{\mathbb{E}(|X|^{r(p)})}{|y-x|^{(1-\nu/\alpha)r(p)}}\sum_{k=1}^{n}k^{-p/2}(n+1-k)^{-d/2},\end{aligned}$$

where we have used the Markov property of the random walk, applied (15) with $u = 0$ to $S'(n-k)$ and then (16) in the last inequality. Hence, we get

$$\begin{aligned}\sum_{n=0}^{|y-x|^{2-\nu}}\mathbb{P}(x+S(n)=y,\tau_x>n) &\leqslant |y-x|^{2-\nu}\exp(-C|x-y|^{\nu/\alpha})\\ &\quad + CV(x)\frac{\mathbb{E}(|X|^{r(p)})}{|y-x|^{(1-\nu/\alpha)r(p)}}\sum_{n=1}^{|y-x|^{2-\nu}}\sum_{k=1}^{n}k^{-p/2}(n+1-k)^{-d/2}\\ &\leqslant C'|y-x|^{-(1-\nu/\alpha)r(p)}\sum_{k=1}^{|y-x|^{2-\nu}}k^{-p/2}\sum_{k=1}^{|y-x|^{2-\nu}}k^{-d/2}.\end{aligned}$$

We now divide the proof according to the value of d. If $d = 1$, then $p = 1$ and $q = 1$ and

$$\sum_{k=1}^{|y-x|^{2-\nu}}k^{-p/2}\sum_{k=1}^{|y-x|^{2-\nu}}k^{-d/2}\sim |y-x|^{2-\nu},$$

which yields, since $r(p) \geqslant 2 + \delta$ for some $\delta > 0$,

$$\sum_{n=0}^{|y-x|^{2-\nu}}\mathbb{P}(x+S(n)=y,\tau_x>n)\leqslant C|y-x|^{-(1-\nu/\alpha)(2+\delta)}|y-x|^{2-\nu}\leqslant C|y-x|^{2-\nu-(2+\delta)(1-\nu/\alpha)}.$$

Choosing $\alpha > 2$ yields

$$\sum_{n=0}^{|y-x|^{2-\nu}}\mathbb{P}(x+S(n)=y,\tau_x>n)\leqslant C|y-x|^{-\delta-(1-2/\alpha)\nu}.$$

If now $d \geqslant 2$, then

$$\sum_{k=1}^{|y-x|^{2-\nu}} k^{-p/2} \sum_{k=1}^{|y-x|^{2-\nu}} k^{-d/2} \sim C \log|y-x|^{\mathbf{1}_{d=2}+\mathbf{1}_{p=2}} \left(|y-x|^{2-\nu}\right)^{(1-p/2)\wedge 0}$$

for some constant $C > 0$. Hence,

$$\sum_{n=0}^{|y-x|^{2-\nu}} \mathbb{P}(x+S(n) = y, \tau_x > n) \leqslant C|y-x|^{-(1-\nu/\alpha)r(p)+(2-\nu)((1-p/2)\wedge 0)} \log|y-x|^{\mathbf{1}_{d=2}+\mathbf{1}_{p=2}}.$$

Since $r(p) > 2p + d - 2 + 2(1-p/2)^+$, for α large enough we have

$$\sum_{n=0}^{|y-x|^{2-\nu}} \mathbb{P}(x + S(n) = y, \tau_x > n) = o(|y-x|^{-2p-d+2}). \qquad \square$$

For $N \geqslant 1$, set

$$G_N(x,y) = \sum_{n=N}^{\infty} \mathbb{P}(x + S(n) = y, \tau_x > n).$$

Proposition 23. *Set $\rho = 1 - 2\varepsilon$ and let $x \in K$. Then there exist $R > 0$ and $\nu > 0$ such that uniformly on y such that $|y| \to \infty$ and $d(y, \partial K) \geqslant R|y|^\rho$, we have for $\delta < \nu$*

$$G_{|y-x|^{2-\delta}}(x,y) \sim V(x)F(y),$$

where $F(y)$ is a function independent of x. Moreover, there exists a constant C such that

$$F(y) \geqslant Cu(y)y^{-2p-d+2},$$

and F is bounded on its domain of definition.

Proof. Let $A > 1$. Let $\eta > 0$ and following Proposition 12 let n_η be such that

$$\left| \frac{\mathbb{P}(x + S(n) = y, \tau_x > n)}{\kappa V(x) n^{-p/2-d/2} u(y/\sqrt{n}) \exp(-|y|^2/(2n))} - 1 \right| \leqslant \eta \tag{49}$$

for $n \geqslant n_\eta$ and $y \in K^A_{n,\varepsilon}$.

Set $R = A^\rho$ and let $y \in K$ be such that $d(y, \partial K) \geqslant R|y|^\rho$. Then for $n \leqslant A^2|y|^2$ we have

$$d(y, \partial K) \geqslant R|y|^\rho \geqslant RA^{-\rho} n^{\rho/2} \geqslant n^{1/2-\varepsilon},$$

and thus $y \in K_{n,\varepsilon}$. Let $\nu > 0$ be such that $(2-\nu)(1/2+\kappa) > 1$ (with κ as in Lemma 18). Suppose that $\delta < \nu$, and set $N_1 = |y-x|^{2-\delta}$. Let N_2 be the first integer n such that $\exp(-|y-x|^2/(2c_3 n)) \geqslant n^{-r+p/2}$. Recall that $r = r(p)(1/2-\gamma) - 1$ and that $r > p/2$ for $d \geqslant 2$, so that N_2 exists. For $d \geqslant 2$ and y large enough, $N_2 \geqslant N_1$, since

$$\exp\left(-|y-x|^2 / \left(2c_3 \frac{|y-x|^2}{K \log|y-x|}\right)\right) = |y-x|^{-K/(2c_3)} \leqslant \left(\frac{|y-x|^2}{K \log|x-y|}\right)^{-r+p/2}$$

for K large enough.

Set $P(n) := \mathbb{P}(x + S(n) = y, \tau_x > n)$. Then, for $\delta < \nu$

$$\begin{aligned} &G_{|y-x|^{2-\delta}}(x,y) \\ &= \sum_{n=N_1}^{N_2} P(n) + \sum_{n=N_2+1}^{A^{-2}\lfloor |y-x|^2 \rfloor} P(n) + \sum_{n=A^{-2}\lfloor |y-x|^2 \rfloor+1}^{A^2\lfloor |y-x|^2 \rfloor} P(n) + \sum_{n=A^2\lfloor |y-x|^2 \rfloor+1}^{\infty} P(n) \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \end{aligned}$$

In the above decomposition of $G(x,y)$, Σ_1 and Σ_2 are expected to be negligible terms, whereas Σ_3 and Σ_4 should give the main contributions.

Study of Σ_2. First, by the choice of N_2 and Proposition 21,

$$\Sigma_2 \leqslant CV(x)u(y) \sum_{n=N_2+1}^{A^{-2}\lfloor |y-x|^2 \rfloor} n^{-p-d/2} \exp(-|y-x|^2/(2c_3 n)).$$

Set $g_{k,B}(t) = t^{-k}\exp(-B/t)$ with $B, k > 0$. Then

$$g'_{k,B}(t) = (Bt^{-k-2} - kt^{-k-1})\exp(-B/t),$$

and thus $g_{k,B}$ is increasing on $[0, B/k]$. Applying the latter property to $k = p + d/2$ and $B = |y-x|^2/(2c_3)$ yields that if $A^2 > 2c_3(p + d/2)$ (which we assume from now on), then

$$\Sigma_2 \leqslant C(\lfloor |y-x|^2/A^2 \rfloor)^{-p-d/2+1} u(y) \exp(-A^2/(2c_3)).$$

Study of Σ_1. By the choice ν and $\delta < \nu$, $|y-x| \leqslant n^{1/2+\kappa}$ for $n \geqslant |y-x|^{2-\delta}$ and y large enough. Applying Proposition 21 to Σ_1 yields then

$$\begin{aligned} \Sigma_1 \leqslant& CV(x)u(y)\frac{|x-y|^2}{A^2}\left(|y-x|^{2-\nu}\right)^{-r-p/2-d/2} \\ \leqslant& C\frac{V(x)u(y)}{A^2}|y-x|^{-(2r+p+d-2)+f(\nu)}, \end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ is linear. Since $r > p/2$ for $d \geqslant 2$ (see the definition of r in (47)), choosing ν small enough yields

$$\Sigma_1 \leqslant C\frac{V(x)u(y)}{A^2}|y-x|^{-(2p+d-2+u)},$$

with $u = 2r - p > 0$, for y large enough.

Study of Σ_3. By (49) we have

$$\begin{aligned}\Sigma_3 \geqslant&(1-\eta)\sum_{n=\lfloor |y-x|^2/A^2\rfloor+1}^{\lfloor A^2|y|^2\rfloor}\kappa V(x)n^{-p-d/2}u(y)\exp(-|y|^2/(2n))\\ \geqslant&(1-\eta)\sum_{n=\lfloor |y-x|^2\rfloor+1}^{\lfloor A^2|y|^2\rfloor}\kappa V(x)n^{-p-d/2}u(y)\exp(-|y|^2/(2n))\\ \geqslant&cu(y)\exp(-|y|^2/(2|y-x|^2))\sum_{n=\lfloor |y-x|^2\rfloor+1}^{\lfloor A^2|y|^2\rfloor}n^{-p-d/2}\\ \geqslant&cu(y)\exp(-|y|^2/(2|y-x|^2))(\lfloor |y-x|^2\rfloor^{-p-d/2+1}-\lfloor A^2|y|^2+1\rfloor^{-p-d/2+1}),\end{aligned}$$

for some $c>0$ and y large enough. Hence,

$$\begin{aligned}\frac{\Sigma_1+\Sigma_2}{\Sigma_3}&\leqslant C\frac{u(y)|y-x|^{-2p-d+2-u}+(\lfloor |y-x|^2/A^2\rfloor)^{-p-d/2+1}u(y)\exp(-A^2/(2c_3))}{u(y)\exp(-|y|^2/(2|y-x|^2))(\lfloor |y-x|^2\rfloor^{-p-d/2+1}-\lfloor A^2|y|^2+1\rfloor^{-p-d/2+1})}\\ &\xrightarrow[|y|\to\infty]{} C\exp(-A^2/(2c_3))\frac{A^{2p+d-2}}{(1-\eta)(1-A^{-2p-d+2})},\end{aligned}$$

where C is independent of A. Since the right-hand side above goes to 0 as A goes to infinity, we can choose A such that $\frac{\Sigma_1+\Sigma_2}{\Sigma_3}\leqslant\eta$ for $|y|$ large enough.

Conclusion. Hence

$$G(x,y)\sim\Sigma_3+\Sigma_4$$

uniformly on y going to infinity. Set

$$F(y)=\kappa\sum_{n\geqslant\lfloor |y-x|^2/A^2\rfloor}n^{-p-d/2}\mathbb{E}\left(u\big(y'_\varepsilon(n)\big)\exp(-|y|^2/(2n))\right),t'_{y,\varepsilon}<\tau'_y).$$

Then by Theorem 1, for $|y|$ large enough satisfying the hypotheses of the statement,

$$\left|\frac{G(x,y)}{V(x)F(y)}-1\right|\leqslant\eta.$$

Moreover, the bound on Σ_3 above proves that for y large enough satisfying the conditions of the statement,

$$F(y)\geqslant Cu(y)|y|^{-2p-d+2}.$$

For the upper bound, note that by definition of $F(y)$, we have for $d\geqslant 2$

$$F(y)\leqslant\kappa\sum_{n=1}^{\infty}n^{-p/2-d/2}\sup_{z\in K}u(z)\exp(-|z|^2/2),$$

which gives a general upper bound for F. □

We can improve the above asymptotic behavior by removing the condition on the distance to the boundary of K. Set

$$K_\varepsilon:=\{y\in K:d(y,\partial K)\geqslant R|y|^\rho\},$$

where ρ and R are given in Proposition 23. For $y \in K$, let θ_y be the stopping time $\inf\{n \geqslant 0 : y + S(n) \in K_\varepsilon\}$. Remark that $\theta_y = t_{y,\varepsilon}(|y|^{2-2\varepsilon})$. Hence, noting that $t_{y,\varepsilon}(n)$ is increasing in n, we get with [16, Lem. 14]

$$\mathbb{P}(\theta_y > n^{1-\varepsilon}, \tau_y > n) \leqslant \mathbb{P}(t_{y,\varepsilon}(n) > n^{1-\varepsilon}, \tau_y > n) \leqslant C \exp(n^{-\varepsilon}) \tag{50}$$

for $n \geqslant |y|^{2-2\varepsilon}$. Applying Lemma 16 to the stopping time θ_y and the moment condition $\mathbb{E}(|X|^{r(p)}) < \infty$ yields that there exists $C > 0$ and $\alpha > 0$ such that

$$\mathbb{P}\big(|S'_{\theta_y}| \geqslant |y-x|^{1-\varepsilon/\alpha}, \theta_y \leqslant |y-x|^{2-\varepsilon}, \tau_y > |y-x|^{2-\varepsilon}\big) \leqslant C|y-x|^{-s} \tag{51}$$

with $s > (2-2\varepsilon)(2p+d-4+2(1-p/2)^+)/2$.

Theorem 24. *As $|y|$ goes to infinity,*

$$G(x,y) \sim V(x)\mathbb{E}(F(y+S_{\theta_y}), \tau_y > \theta_y).$$

Proof. Split the Green function as

$$\begin{aligned} G(x,y) = &\sum_{n=1}^{|y-x|^{2-2\varepsilon}-1} \mathbb{P}(y+S'(n) = x, \tau'_y > n) \\ &+ \sum_{n=|y-x|^{2-2\varepsilon}}^{\infty} \mathbb{P}(y+S'(n) = x, \tau'_y > n, \theta_y \geqslant |y-x|^{2-2\varepsilon}) \\ &+ \sum_{n=|y-x|^{2-2\varepsilon}}^{\infty} \mathbb{P}(y+S'(n) = x, \tau'_y > n, \theta_y \leqslant |y-x|^{2-3\varepsilon}, |S'_{\theta_y}| \geqslant |y-x|^{1-\varepsilon/\alpha}) \\ &+ \sum_{n=|y-x|^{2-2\varepsilon}}^{\infty} \mathbb{P}(y+S'(n) = x, \tau'_y > n, \theta_y \leqslant |y-x|^{2-3\varepsilon} \wedge \tau'_y, |S'_{\theta_y}| \leqslant |y-x|^{1-\varepsilon/\alpha}) \\ :=&T_1 + T_2 + T_3 + T_4, \end{aligned}$$

and we shall study successively the terms T_1, T_2, T_3 and T_4.

Study of T_1 and T_2. By Proposition 21, $T_1 \leqslant C|y-x|^{-a}$ for some $a > 2p+d-2$. Next,

$$\begin{aligned} T_2 = &\sum_{n \geqslant |y-x|^{2-2\varepsilon}} \mathbb{P}(x+S(n) = y, \tau_x > n, \theta_y \geqslant n^{1-\varepsilon}) \\ &+ \sum_{n \geqslant |y-x|^{2-2\varepsilon}} \mathbb{P}(x+S(n) = y, \tau_x > n, |y-x|^{2-3\varepsilon} \leqslant \theta_y \leqslant n^{1-\varepsilon}). \end{aligned}$$

By (50), the first term is bounded by $\sum_{n\geqslant|y-x|^{2-2\varepsilon}} C\exp(-n^{\varepsilon}) \leqslant C\exp(-|y-x|^{\varepsilon'})$ for some $C>0$ and some $0<\varepsilon'<\varepsilon$. Moreover, by (16) and (50),

$$\begin{aligned}
&\sum_{n\geqslant|y-x|^{2-2\varepsilon}} \mathbb{P}(x+S(n)=y,\tau_x>n,|y-x|^{2-3\varepsilon}\leqslant\theta_y\leqslant n^{1-\varepsilon})\\
&=\sum_{n\geqslant|y-x|^{2-2\varepsilon}} \mathbb{E}(x+S(n-\theta_y)=y+S'(\theta_y),\tau_x>n-\theta_y,\tau'_y>\theta_y,|y-x|^{2-3\varepsilon}\leqslant\theta_y\leqslant n^{1-\varepsilon})\\
&\leqslant\sum_{n\geqslant|y-x|^{2-2\varepsilon}} C(n-n^{1-\varepsilon})^{-d/2-p/2}\mathbb{P}(\theta_y>|y-x|^{2-3\varepsilon},\tau_y\geqslant|y-x|^{2-3\varepsilon})\\
&\leqslant C\exp(-|y-x|^{\varepsilon(2-3\varepsilon)}),
\end{aligned}$$

so that, finally,

$$T_2\leqslant C\exp(-|y-x|^{\varepsilon'}),$$

for some constant $C>0$ and $0<\varepsilon'<\varepsilon$.

Study of T_3. By (16), we have for $n\geqslant|x-y|^{2-2\varepsilon}$ and y large enough

$$\begin{aligned}
\mathbb{P}(y+S'(n)=x,\tau'_y>n,\theta_y\leqslant|y-x|^{2-3\varepsilon},|S'_{\theta_y}|\geqslant|y-x|^{1-\varepsilon/\alpha})&\\
\leqslant\mathbb{E}((n-\theta_y)^{-p/2-d/2},\tau'_y>\theta_y,\theta_y\leqslant|y-x|^{2-3\varepsilon},|S'_{\theta_y}|\geqslant|y-x|^{1-\varepsilon/\alpha})&\\
\leqslant Cn^{-p/2-d/2}\mathbb{P}(\tau'_y>\theta_y,\theta_y\leqslant|y-x|^{2-3\varepsilon},|S'_{\theta_y}|\geqslant|y-x|^{1-\varepsilon/\alpha})&\\
\leqslant Cn^{-p/2-d/2}|y-x|^s.&
\end{aligned}$$

Hence,

$$T_3\leqslant C\sum_{n=|x-y|^{2-2\varepsilon}}^{\infty} n^{-p/2-d/2}|y-x|^s\leqslant|y-x|^{-(2-2\varepsilon)(p/2+d/2-1)-s}.$$

By the definition of s given in (51),

$$\begin{aligned}
(2-2\varepsilon)(p/2+d/2-1)+s>&(p+d-2)+(2p+d-4+2(1-p/2)^+)+f(\varepsilon)\\
=&2p+d-2+(p+d-4+2(1-p/2)^+)+g(\varepsilon),
\end{aligned}$$

with g linear. Since $p+d-4+2(1-p/2)^+\geqslant 0$ for all $p\geqslant 1$ and $d\geqslant 2$,

$$T_3=o(|x-y|^{-b})$$

with $b>2p+d-2$ for ε small enough.

Study of T_4. By Proposition 23, we have

$$\begin{aligned}
T_4=&\mathbb{E}(G_{|x-y|^{2-2\varepsilon}}(x,y+S'(\theta_y)),\tau'_y>\theta_y,\theta_y\leqslant|y-x|^{2-3\varepsilon},|S(\theta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha})\\
\geqslant&C\mathbb{E}(u(y+S'(\theta_y))|y+S'(\theta_y)|^{-2p-d+2},\tau'_y>\theta_y,\theta_y\leqslant|y-x|^{2-3\varepsilon},|S(\theta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha})\\
\geqslant&C|y|^{-2p-d+2}|y|^{p(1-2\varepsilon)}\mathbb{P}(\tau'_y>\theta_y,\theta_y\leqslant|y-x|^{2-3\varepsilon},|S(\theta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}),
\end{aligned}$$

where we have used the fact that $y+S'(\theta_y)\in K_\varepsilon$ and $|S(\theta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}$ to give a lower bound on $u(y+S'(\theta_y))$. Hence

$$T_4\geqslant|y|^{-2p-d+2+p(1-2\varepsilon)}(\mathbb{P}(\tau'_y>|y-x|^{2-3\varepsilon})-C\exp(-|y|^{\varepsilon})-K|y-x|^s).$$

By Lemma 17, $\mathbb{P}(\tau'_y > |y-x|^{2-3\varepsilon}) \geqslant C|y-x|^{-p/2(2-3\varepsilon)}$ and $s \geqslant p/2(2-3\varepsilon)$ for ε small enough, which yields

$$T_4 \geqslant c|y|^{-2p-d+2+p(1-2\varepsilon)-p/2(2-3\varepsilon)} \geqslant c|y|^{-2p-d+2+g(\varepsilon)},$$

with g linear. Hence, for ε small enough,

$$T_1 + T_2 + T_3 = o(T_4).$$

Moreover, by Proposition 23,

$$\begin{aligned}\mathbb{E}(G_{|y-x|^{2-2\varepsilon}}(x, y+S'(\theta_y)), \tau'_y > \theta_y, \theta_y \leqslant |y-x|^{2-3\varepsilon}, |S(\theta_y)| \leqslant |y-x|^{1-\varepsilon/\alpha}) \\ \sim V(x)\mathbb{E}(F(x, y+S'(\theta_y)), \tau'_y > \theta_y, \theta_y \leqslant |y-x|^{2-3\varepsilon}, |S(\theta_y)| \leqslant |y-x|^{1-\varepsilon/\alpha}),\end{aligned}$$

as y goes to infinity, which yields

$$G(x,y) \sim V(x)\mathbb{E}(F(x, y+S'(\theta_y)), \tau'_y > \theta_y, \theta_y \leqslant |y-x|^{2-3\varepsilon}, |S(\theta_y)| \leqslant |y-x|^{1-\varepsilon/\alpha}).$$

Since we also have

$$\mathbb{E}(F(x, y+S'(\theta_y)), \tau'_y > \theta_y, (\theta_y \geqslant |y-x|^{2-3\varepsilon}) \cup (|S(\theta_y)| \geqslant |y-x|^{1-\varepsilon/\alpha})) = o(T_4)$$

for the same reasons as before, the result is deduced. □

The uniqueness of the harmonic function is then a straightforward deduction of the latter theorem.

Corollary 25. *The Martin boundary of S killed on the boundary of K is reduced to a singleton, and there exists a unique harmonic function (up to multiplication by a constant).*

Proof. Let $x_0, x \in K$ and let (y_n) be a sequence in K going to infinity. Then, by Theorem 24,

$$\frac{G(x,y_n)}{G(x_0,y_n)} \sim \frac{V(x)\mathbb{E}(F(y_n + S_{\theta_{y_n}}), \tau'_{y_n} > \theta_{y_n})}{V(x_0)\mathbb{E}(F(y_n + S_{\theta_{y_n}}), \tau'_{y_n} > \theta_{y_n})} = \frac{V(x)}{V(x_0)}.$$

The Martin boundary is thus reduced to a singleton. □

Appendix A. Regularity and estimates for the heat kernel in a cone

We prove in this appendix several inequalities concerning the heat kernel in a cone. We start with the boundedness of the gradient of u.

Lemma 26. *There exists $C > 0$ such that*

$$|\nabla u(z)| \leqslant C|z|^{p-1}.$$

Proof. If S is a sphere and $s \in S$, denotes by $n(s)$ the outward normal vector at s. Since K is either $\mathcal{C}^2$ or convex, by [46, Eq. (0.2.3)] there exists a constant C' such that for all $z \in K$,

$$u(z) \leqslant C' d(z,K)|z|^{p-1}. \tag{52}$$

Set $r = d(z, \partial K)$. If v is a unit vector in $\mathbb{R}^d$, $\partial_v u$ is harmonic and we have

$$\begin{aligned}|\partial_v u(z)| =& \frac{1}{\mathrm{Vol}(B(z,r))}\left|\int_{B(z,r)} \partial_v u(s) ds\right| \\ =& \frac{1}{\mathrm{Vol}(B(z,r))}\left|\int_{\partial B(z,r)} u(s)\langle v, n(s)\rangle ds\right| \\ \leqslant& \frac{\mathrm{Vol}(\partial B(z,r))}{\mathrm{Vol}(B(z,r))} C''|z|^{p-1} r ds\end{aligned}$$

for some positive constant C'', where we have used (52) at the last line. Hence, since $\frac{\mathrm{Vol}(\partial B(z,r))}{\mathrm{Vol}(B(z,r))} = C'''\frac{1}{r}$ for some C''', we have

$$|\partial_v u(z)| \leqslant C|z|^{p-1}$$

for some constant $C > 0$. □

The latter result yields in the next lemma the Hölder continuity of u. In the following statement, u is said locally Lipschitz with local Lipschitz constant $K(z)$ at $z \in K$ if $u_{|B(z,1)}$ is Lipschitz with Lipschitz constant $K(z)$.

Lemma 27. *If $p \geqslant 1$, then u is locally Lipschitz at $z \in K$ with local Lipschitz constant $|z|^{p-1}$. If $p \leqslant 1$, then u is Hölder continuous with exponent p.*

Proof. Suppose that $p \geqslant 1$ and let $y, y' \in K$ with $|y - y'| \leqslant 1$. Then, by Lemma 26,

$$\big|u(y) - u(y')\big| = \left|\int_0^1 \langle \nabla u(y' + t(y - y')), y - y'\rangle dt\right| \leqslant |y' - y| C \int_0^1 |y' + t(y - y')|^{p-1} dt.$$

Since $p \geqslant 1$, there exists a constant C' such that $\int_0^1 |y' + t(y - y')|^{p-1} dt \leqslant C'|y'|^{p-1}$, so that

$$|u(y) - u(y')| \leqslant C''|y'|^{p-1}|y - y'|$$

for some positive constant C''.

Suppose now that $p \leqslant 1$ and let $y, y' \in K$ with $|y'| \geqslant |y|$. Set $y'' = \frac{|y|}{|y'|}y'$, so that $|y''| = |y|$. Since $|y| \leqslant |y'|$ and y'' is proportional to y', one has

$$\langle y' - y'', y'' - y\rangle \geqslant 0,$$

so that $|y' - y''| \leqslant |y - y'|$ and $|y'' - y'| \leqslant |y - y'|$.

On the first hand, joining y to y'' by a geodesic on the sphere of radius $|y|$ yields that $|u(y'') - u(y)| \leqslant C|y|^{p-1}|y - y''|$ for some constant C. Since y'' and y are on the same sphere, there exists a constant K (independent of the radius of the sphere) such that $|y - y''| \leqslant K|y|$, and thus $|y|^{p-1} \leqslant K^{1-p}|y - y''|^{p-1}$. Hence

$$|u(y'') - u(y)| \leqslant CK^{1-p}|y - y''|^p \leqslant C'|y - y'|^p$$

for some constant $C' > 0$.

On the other hand, if we set $M = \|u_{|K\cap\mathbb{S}^{d-1}}\|_\infty$, the scaling property of u yields that

$$|u(y') - u(y'')| \leqslant M(|y'|^p - |y''|^p).$$

Since $p \leqslant 1$, $(|y'|^p - |y''|^p) \leqslant p2^{1-p}|y' - y''|^p$, and thus

$$|u(y') - u(y'')| \leqslant C'|y' - y''|^p \leqslant C'|y' - y|^p$$

for some constant $C'' > 0$. Finally,

$$|u(y) - u(y')| \leqslant |u(y) - u(y'')| + |u(y'') - u(y')| \leqslant (C' + C'')|y - y'|^p. \qquad \square$$

The following lemma gives the proof of (7).

Lemma 28. *There exists $c > 0$ such that for all $t > 0$,*

$$\inf_{z \in K} V(z, \sqrt{t}) > ct^{d/2}.$$

Proof. Write K as the epigraph of a Lipschitz function $\phi : H \longrightarrow \mathbb{R}$ with Lipschitz constant L, where H is a hyperplane of $\mathbb{R}^d$. Let v be the unit vector normal to H pointing toward the epigraph of ϕ; then one has $x + z \in K$ for all $x \in K$ and $z \in \mathbb{R}^d$ such that $\langle z, v\rangle \geqslant \sqrt{1 - L^{-2}}|z|$. Therefore, for some positive constant c, one has

$$V(x, \sqrt{t}) \geqslant \mathrm{Vol}\{|z| \leqslant \sqrt{t}, \langle z, v\rangle \geqslant \sqrt{1 - L^{-2}}|z|\} > ct^{d/2}.$$

Note that if K is convex, then $z + \big(B(0, \sqrt{t}) \cap K\big) \subset K$ for all $z \in K$, so that we can simply choose $c = V(0, 1)$. $\square$

From the Gaussian estimates we can deduce the Hölder regularity of k and K. In order to obtain this regularity, we first need to give a uniform bound for the ratio of k_t and $k_{t'}$ for $t' \leqslant t$. This is done in the following lemma, using the fundamental volume doubling property of the réduite u (see [27, Thm 4.19]): there exists a constant $D > 0$ such that

$$\int_{B(x,2r)\cap K} u(z)^2 dz \leqslant D \int_{B(x,r)K} u(z)^2 dz \tag{53}$$

for all $x \in K, r > 0$.

Lemma 29. *For all $0 < t < 1/2$, there exists $c_6^t > 0$ such that*

$$c_6^t \leqslant \frac{k(sx)}{k(x)} \leqslant 1$$

for all $x \in K$ and $s \in [t, 1 - t]$.

Proof. Since $k(sx) = k_{s^{-2}}(x) = \mathbb{P}(\tau_x^{\mathrm{bm}} > s^{-2})$, $k(sx)$ is increasing with s and we only have to prove the bound for $s = t$. Let $n \geqslant 1$ be such that $0 < 2^{-n} \leqslant t$. Let us prove the result for $r := 2^{-n}$, which implies the result for t. By (6), we have

$$c' \sqrt{\frac{V(rx, 1)}{V(x, 1)}} \sqrt{\frac{\int_{B(x,1)\cap K} u(z)^2 dz}{\int_{B(rx,1)\cap K} u(z)^2 dz}} \frac{u(rx)}{u(x)} \leqslant \frac{k(rx)}{k(x)} \leqslant 1$$

for $c' = \frac{c_1'}{C_1'}$. By Lemma 7 and the fact that $V(z, 1) \leqslant \mathrm{Vol}(B(0, 1))$ for all $z \in K$,

$$\sqrt{\frac{V(rx, 1)}{V(x, 1)}} \geqslant \sqrt{\frac{c}{\mathrm{Vol}(B(0, 1))}}.$$

Moreover, by the scaling property of u,

$$\int_{B(rx,1)\cap K} u(z)^2 dz = r^d \int_{B(x,r^{-1})\cap K} u(rz)^2 dz = r^{d+2p} \int_{B(x,r^{-1})\cap K} u(z)^2 dz,$$

where we have used that $r^{-1}\left(B(rx,1)\cap K\right) = B(x,r^{-1})\cap K$ by the cone property. Using n times the doubling volume property (53) yields

$$\int_{B(x,r^{-1})\cap K} u(z)^2 dz = \int_{B(x,2^n)\cap K} u(z)^2 dz \leqslant D^n \int_{B(x,1)\cap K} u(z)^2 dz,$$

independently of $x \in K$. Therefore,

$$\int_{B(rx,1)\cap K} u(z)^2 dz \leqslant r^{d+2p} D^n \int_{B(x,1)\cap K} u(z)^2 dz,$$

and thus

$$\sqrt{\frac{\int_{B(x,1)\cap K} u(z)^2 dz}{\int_{B(rx,1)\cap K} u(z)^2 dz}} \geqslant 2^{-nd/2-np} D^{-n/2}.$$

Finally, we have

$$c 2^{-nd/2-np} D^{-n/2} \sqrt{\frac{c'}{\mathrm{Vol}(B(0,1))}} \leqslant \frac{k(rx)}{k(x)} \leqslant 1. \qquad \square$$

We say that a function f is locally Hölder at $z \in K$ with exponent α and constant $C(z)$ if $|f(z') - f(z)| \leqslant C(z)|z'-z|^\alpha$ for z' such that $|z - z'| \leqslant 1$. The function is said globally Hölder on $D \subset K$ with exponent α and constant C if $|f(z') - f(z)| \leqslant C|z'-z|^\alpha$ for $z, z' \in K$.

Proposition 30. *There exist exponents $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \chi$, as well as a constant $C_\alpha > 0$, such that k and $\frac{1}{k(x)}K(x,\cdot)$ for $x \in K$ are locally Hölder at each z in K with exponent α and with respective constants (where α is given by Theorem 6)*

$$C_\alpha(1+|x'|^\chi) \quad \text{and} \quad C_\alpha(1+|z|^\chi)\exp(-|z-x|^2/(2c_3)).$$

By symmetry, $K(\cdot, y)$ is also locally α-Hölder at $z \in K$ with Hölder constant

$$C_\alpha(1+|z|^\chi)\exp(-|z-y|^2/(2c_3)).$$

Proof. We suppose without loss of generality that $\alpha \leqslant p$. Let us start by the Hölder continuity of $\frac{1}{k(x)}K(x,\cdot)$. Let $x, y, y' \in K$ be such that $|y - y'| \leqslant 1$ and $|y'| \geqslant |y|$. By the second part of Theorem 6, we have

$$\left|\frac{K(x,y)}{u(y)} - \frac{K(x,y')}{u(y')}\right| \leqslant C_4\left(|y-y'|\right)^\alpha \frac{K_2(x,y)}{u(y)}.$$

Hence

$$\frac{1}{k(x)}\left|K(x,y) - K(x,y')\right| \leqslant C_4 \frac{1}{k(x)}|y-y'|^\alpha K_2(x,y) + \frac{1}{k(x)}K(x,y')\left|\frac{u(y)}{u(y')} - 1\right|.$$

Using the first part of Theorem 6 and then Theorem 5 gives

$$\frac{1}{k(x)}K(x,y') \leqslant C_2 \frac{k(y')}{\sqrt{V(x,1)V(y',1)}}\exp(-|x-y'|^2/c_3) \leqslant C\frac{u(y')}{u(y_1')}\exp(-|x-y'|^2/c_3),$$

for some constant C, where we have also used in the last inequality that $V(z,1) \geqslant c$ for some positive constant c independent of z (see the previous proof for a proof of this fact). Likewise,

$$\begin{aligned}\frac{1}{k(x)}K_2(x,y) \leqslant & C_2 \frac{k(x/\sqrt{2})}{k(x)} \frac{k(y/\sqrt{2})}{\sqrt{V(x,2)V(y,2)}} \exp(-|x-y|^2/(2c_3)) \\ \leqslant & C \frac{k(x/\sqrt{2})}{k(x)} \exp(-|x-y|^2/(2c_3)) \leqslant C \exp(-|x-y|^2/(2c_3)),\end{aligned}$$

where we have used Lemma 29 in the last inequality. Thus

$$\begin{aligned}&\frac{1}{k(x)} \left|K(x,y) - K(x,y')\right| \\ &\quad \leqslant C|y-y'|^\alpha \exp(-|x-y|^2/(2c_3)) + C\frac{1}{u(y_1')} \exp(-|x-y'|^2/c_3) \left|u(y) - u(y')\right|.\end{aligned}$$

Let $\beta = p \wedge 1$. By Lemma 27, for $y, y' \in K$ such that $|y-y'| \leqslant 1$, we have for some constant $C' > 0$

$$|u(y) - u(y')| \leqslant C|y'|^{p-\beta}|y-y'|^\beta.$$

Since $d(y_1', \partial K) \geqslant c_0$ by definition, [16, Lem. 19] yields that from some constant $c > 0$

$$u(y_1') \geqslant c c_0^p. \tag{54}$$

Hence, with $\chi = p - \beta$, there exists $C^{(3)}$ such that

$$\frac{1}{u(y_1')} \left|u(y) - u(y')\right| \leqslant C^{(3)}|y'|^\chi |y-y'|^\beta.$$

Therefore, there exists $C^{(4)} > 0$ such that

$$\frac{1}{k(x)} \left|K(x,y) - K(x,y')\right| \leqslant C^{(4)}|y-y'|^\alpha (1 + |y'|^\chi) \exp(-|x-y|^2/(2c_3)),$$

which proves the first part of the lemma. Since k is bounded by 1, $K(x,\cdot)$ is also locally α-Hölder with the same constants, and by symmetry the same holds for $K(\cdot, y)$. Therefore, for $x, x' \in K$ such that $|x - x'| \leqslant 1$,

$$\begin{aligned}\left|k(x) - k(x')\right| \leqslant & \int_K |K(x,y) - K(x',y)| dy \\ \leqslant & C^{(4)}(1 + |x'|^\chi)|x-x'|^\alpha \int_K \exp(-|x'-y|^2/(2c_3)) dy \leqslant C_\alpha (1 + |x'|^\chi)|x-x'|^\alpha\end{aligned}$$

for some constant $C_\alpha \geqslant 0$. □

Remark 31. Since $\exp(-|z-y|^2/(2c_3)) \leqslant 1$ for all $x, z \in K$, Proposition 30 implies $\frac{1}{k(x)}K(x,\cdot)$ and $K(\cdot, x)$ are globally Hölder on any bounded subset D of K with a Hölder constant $C(D)$ independent of $x \in K$.

We end this section by a uniform (in time) estimate of the convergence of the heat kernel.

Lemma 32. *Let $t_0 > 0$ and $M > 0$. Uniformly in $t > t_0$, and $x, y \in K$ with $|y-x| \leqslant M$,*

$$K_{t+h}(x,y) \sim_{h\to 0} K_t(x,y)$$

and uniformly in $t_0 < t < 2t_0$ and $x \in K$,

$$k_{t+h}(x) \sim_{h\to 0} k(x).$$

Proof. For the first convergence, let $t > t_0$ and $x, y \in K$ with $|y-x| \leqslant M$. Then for $h \geqslant 0$,

$$|K_{t+h}(x,y) - K_t(x,y)| \leqslant \int_0^h |\partial_t K_{t+r}(x,y)| dr.$$

Hence, by the last part of Theorem 6, for $|h| \leqslant t_0/2$,

$$\begin{aligned}&|K_{t+h}(x,y) - K_t(x,y)|\\ &\leqslant C_5 \int_0^h \frac{k_{t+r}(x)k_{t+r}(y)}{(t+r)\sqrt{V(x,\sqrt{t+r})V(y,\sqrt{t+r})}}(1+|x-y|^2/(t+r))^{\beta+1}\exp(-|x-y|^2/(4(t+r)))dr.\end{aligned}$$

Applying now the lower bound in the first part of Theorem 6, we get

$$\begin{aligned}|K_{t+h}(x,y) - K_t(x,y)| \leqslant &(C_5/c_2)\int_0^h K_{t+r}(x,y)\frac{1}{t+r}(1+|x-y|^2/(t+r))^{\beta+1}\\ &\exp(-|x-y|^2/(4(t+r)) + |x-y|^2/(C_3(t+r)))dr\\ \leqslant &C\int_0^h \frac{1}{t+r}K_{t+r}(x,y)(1+M^2/(t_0/2))^{\beta+1}\exp(M^2/(c(t_0/2)))dr,\end{aligned}$$

where C and c are positive constants. Since $K_t(x,y)$ is decreasing in t,

$$|K_{t+h}(x,y) - K_t(x,y)| \leqslant CK_{t+h}(x,y)\left(\frac{1}{t^2} - \frac{1}{(t+h)^2}\right),$$

and $K_{t+h}(x,y) \sim K_t(x,y)$ uniformly in $t > t_0$, and $x, y \in K$ with $|y-x| \leqslant M$ as h goes to zero. The case $h < 0$ is similar.

For the second convergence let $\delta > 0$ and $M > 0$ be such that

$$\int_{|z|>M} \exp(-|z|^2/(2c_3(2t_0+h_0)))dz \leqslant \delta$$

and let $0 < h_0 < t_0$ be such that for all $t > t_0$, $x, y \in K$ with $|y-x| > M$, and $h \in \mathbb{R}$ such that $|h| \leqslant h_0$,

$$(1-\delta)K_t(x,y) \leqslant K_{t+h} \leqslant (1+\delta)K_t(x,y).$$

Then for such $x \in K$, $t, h > 0$, we have by the first part of the lemma

$$\begin{aligned}
k_{t+h}(x) &= \int_{|z|\leqslant M} K_{t+h}(x,y)dy + \int_{|z|>M} K_{t+h}(x,y)dy \\
&\leqslant (1+\delta)\int_{|z|\leqslant M} K_t(x,y)dy + C_2\frac{k_{t+h}(x)}{c(t+h)^d}\int_{|z|>M}\exp(-|z|^2/(2c_3(2t_0+h_0))dz \\
&\leqslant (1+\delta)k_t(x) + (1+\delta)C_2\frac{k_t(x)}{ct^d}\int_{|z|>M}\exp(-|z|^2/(2c_3(2t_0+h_0))dz \\
&\qquad + C_2\frac{k_{t+h}(x)}{c(t+h)^d}\int_{|z|>M}\exp(-|z|^2/(2c_3(2t_0+h_0))dz \\
&\leqslant \left((1+\delta) + \frac{(1+\delta)C_2\delta}{c(t_0-h_0)^d}\right)k_t(x) + \frac{C_2\delta}{c(t_0-h_0)^d}k_{t+h}(x),
\end{aligned}$$

where we have used Theorem 6 and Lemma 7 in the second and third lines. Hence,

$$k_{t+h}(x) \leqslant \frac{(1+\delta) + \frac{(1+\delta)C_2\delta}{c(t_0-h_0)^d}}{1 - \frac{C_2\delta}{c(t_0-h_0)^d}}k_t(x) \leqslant (1+\delta')k_t(x)$$

for all $x \in K, t_0 \leqslant t \leqslant 2t_0$, where δ' goes to zero as δ goes to zero. The lower bound is obtained similarly. □

Appendix B. Extension of the results of [16] to the case of arbitrary convex cones

In [16], Denisov and Wachtel obtain fundamental results on the long-term behavior of a random walk in a convex or $\mathcal{C}^2$ cone. As we mentioned in Section 2.2, these results are the main ingredients in the proofs of the present paper. Unfortunately, the results of [16] are only valid when the cone K satisfies the following condition:

(H^*) There exists a bigger cone K' with $\partial K \setminus \{0\} \subset \mathrm{int}(K')$, such that the réduite u can be extended to a harmonic function on K'.

This condition is a strong limitation on the geometry of the cone, since it implies in particular that ∂K is piecewise analytic. In this appendix, we prove that the condition (H^*) is not necessary to get the results of [16] for a convex cone under the moment conditions ($H5$).

Define $f : K \longrightarrow \mathbb{R}$ by

$$f(x) = \mathbb{E}(u(x+X)) - u(x),$$

where we have set $u(x) = 0$ for $x \notin K$. By [16, Remark 4], the condition (H^*) is only needed in [16] to prove that

$$\mathbb{E}\left(\sum_{\ell=0}^{\tau_x} |f(x+S(\ell))|\right) < C(1+|x|^{p-r}) \tag{55}$$

for some positive constants C and r and for all $x \in K$. All other results are deductions of the bound (55) and other arguments which do not involve (H^*). The main result of Appendix B is thus the extension of (55) to an arbitrary convex cone.

Proposition 33. *Suppose that K is convex. There exist C and $r > 0$ such that for all $x \in K$,*

$$\mathbb{E}\left(\sum_{\ell=0}^{\tau_x - 1} |f(x + S(\ell))|\right) \leqslant C(1 + |x|)^{p-r}.$$

Let us first give some new bounds on the higher derivatives of u.

Lemma 34. *There exists $C > 0$ such that for all $x \in K$,*

$$|u_{x_i x_j}(x)| \leqslant \frac{C}{d(x, \partial K)} |x|^{p-1},$$
$$|u_{x_i x_j x_j}(x)| \leqslant \frac{C}{d(x, \partial K)^2} |x|^{p-1}.$$

Proof. The proof follows closely the ones of [16, Lem. 7] and Lemma 26. By Lemma 26 we have

$$|u_{x_i}(x)| \leqslant C|x|^{p-1}$$

for some constant $C > 0$ independent of i and $x \in K$. Hence, by harmonicity of u,

$$\begin{aligned}
|u_{x_i x_j}(x)| =& \frac{1}{\mathrm{Vol}(B(x, d(x, \partial K)))} \left| \int_{B(x, d(x, \partial K))} u_{x_i x_j} \right| \\
=& \frac{1}{\mathrm{Vol}(B(x, d(x, \partial K)))} \left| \int_{\partial B(x, d(x, \partial K))} u_{x_j} \nu_i ds \right| \\
\leqslant& \frac{\mathrm{Vol}(\partial B(x, d(x, \partial K)))}{\mathrm{Vol}(B(x, d(x, \partial K)))} \max_{\partial B(x, d(x, \partial K))} |u_{x_j}| \\
\leqslant& \frac{C}{d(x, \partial K)} |x|^{p-1},
\end{aligned}$$

for $x \in K$, which proves the first inequality. The second inequality is deduced similarly by using the harmonicity of $u_{x_i x_j x_k}$ on the ball $B(x, d(x, \partial K)/2)$. □

Let $\widetilde{K}$ be the set $\{x \in K : d(x, \partial K) \geqslant |x|^{1/2+\delta}\}$, with $\delta > 0$.

Lemma 35. *There exists $C > 0$ such that*

$$|f(x)| \leqslant C(1 + |x|)^{p-2-\delta}$$

for $x \in \widetilde{K}$, and

$$|f(x)| \leqslant C \min\left(1, \frac{1}{d(x, \partial K)}\right)(1 + |x|^{p-1})$$

for $x \in K \setminus \widetilde{K}$.

Proof. Let $x \in \widetilde{K}$. Applying Lemma 34 to $z \in K$ such that $d(z, \partial K) \geqslant |z|^{1/2+\delta}/2$ yields

$$\sup_{1 \leqslant i,j \leqslant d} |u_{x_i x_j}(x)| \leqslant C|x|^{p-3/2-\delta} \quad \text{and} \quad \sup_{\substack{1 \leqslant i,j,k \leqslant d \\ y \in B(x, |x|^{1/2+\delta}/2)}} |u_{x_i x_j x_k}(y)| \leqslant C|x|^{p-2-2\delta}.$$

Then, applying Taylor's theorem to the third order to u at x and using the latter inequality yield

$$\left|u(x+y)-u(y)-\langle\nabla u(x),y\rangle-\frac{1}{2}\sum_{i,j=1}^{d}u_{x_ix_j}(x)y_iy_j\right|\leqslant\sup_{\substack{1\leqslant i,j,k\leqslant d\\ z\in B(x,|x|^{1/2+\delta}/2)}}\frac{1}{6}|u_{x_ix_jx_k}(z)||y|^3$$
$$\leqslant C|x|^{p-2-\delta}|y|^3,$$

for $y\in B(x,|x|^{1/2+\delta}/2)$. Since $f(x)=\mathbb{E}(u(x+X)-u(x)$, we have therefore as in the proof of [16, Lem. 8]

$$\begin{aligned}|f(x)|\leqslant&|\mathbb{E}(u(x+X)-u(x),|X|\leqslant|x|^{1/2+\delta}/2)|+|\mathbb{E}(u(x+X)-u(x),|X|\geqslant|x|^{1/2+\delta}/2)|\\ \leqslant&|\mathbb{E}(\langle\nabla u(x),X\rangle+\sum_{i,j=1}^{d}u_{x_ix_j}(x)X_iX_j,|X|\leqslant|x|^{1/2+\delta}/2)|\\ &+C|x|^{p-2-\delta}\mathbb{E}(|X|^3,|X|\leqslant|x|^{1/2+\delta}/2)+C\mathbb{E}(|x|^p+|x+X|^p,|X|\geqslant|x|^{1/2+\delta}/2)\\ \leqslant&|\mathbb{E}(\langle\nabla u(x),X\rangle+\sum_{i,j=1}^{d}u_{x_ix_j}(x)X_iX_j,|X|\geqslant|x|^{1/2+\delta}/2)|\\ &+C|x|^{p-2-\delta}\mathbb{E}(|X|^3,|X|\leqslant|x|^{1/2+\delta}/2)+C\mathbb{E}(|x|^p+|x+X|^p,|X|\geqslant|x|^{1/2+\delta}/2),\end{aligned}$$

where we have used the bound on u on the last term of the second inequality, and the harmonicity of u together with the zero-mean and standard variance hypotheses on X to get the first term of the last inequality. By Markov's inequality and the moment assumption $\mathbb{E}(|X|^4)<\infty$ (note that $r(p)>2p+d-2+2(1-p/2)^+\geqslant 4$ for $d\geqslant 3$ and $p\geqslant 1$), we get

$$\begin{aligned}&|\mathbb{E}(\nabla\langle u(x),X\rangle+\sum_{i,j=1}^{d}u_{x_ix_j}(x)X_iX_j,|X|\geqslant|x|^{1/2+\delta}/2)|\\ &\qquad\leqslant C\mathbb{E}(|x|^{p-1}|X|+|x|^{p-3/2-\delta}|X|^2,|X|\geqslant|x|^{1/2+\delta}/2)\\ &\qquad\leqslant Cx^{p-2-2\delta},\end{aligned}$$

and

$$\mathbb{E}(|x|^p+|x+X|^p,|X|\geqslant|x|^{1/2+\delta}/2)\leqslant C\max(|x|^{p-2-2\delta},1).$$

Since $\mathbb{E}(|X|^3,|X|\leqslant|x|^{1/2+\delta}/2)\leqslant\mathbb{E}(|X|^3)$, we get for $x\in\widetilde{K}$

$$f(x)\leqslant C(1+|x|^{p-2-2\delta}).$$

For the second part of the proof, we first decompose $f(x)$ as

$$\begin{aligned}f(x)=&\,\mathbb{E}(u(x+X)-u(x),x+X\notin K)\\ &+\mathbb{E}(u(x+X)-u(x),|X|\geqslant d(x,\partial K)/2,x+X\in K)\\ &+\mathbb{E}(u(x+X)-u(x),|X|\leqslant d(x,\partial K)/2).\end{aligned}$$

In the first term, $u(x+X)=0$ and necessarily $|X| \geqslant d(x,\partial K)$, thus

$$\begin{aligned}|\mathbb{E}(u(x+X)-u(x), x+X \notin K)| \leqslant & |u(x)|\mathbb{P}(|X| \geqslant d(x,\partial K)) \\ \leqslant & Cd(x,\partial K)|x|^{p-1}\mathbb{P}(|X| \geqslant d(x,\partial K)) \\ \leqslant & \frac{C}{d(x,\partial K)}|x|^{p-1}\mathbb{E}(|X|^2).\end{aligned}$$

In the second term, we can use Taylor's expansion at zero order to get

$$\begin{aligned}\mathbb{E}(u(x+X)-u(x), |X| \geqslant & d(x,\partial K)/2, x+X \in K) \\ & \leqslant \mathbb{E}\left(\sup_{z\in[x,x+X]} \|\nabla u(z)\||X|, |X| \geqslant d(x,\partial K)/2\right) \\ & \leqslant C\mathbb{E}(\max(|x|^{p-1}, |x+X|^{p-1}), |X| \geqslant d(x,\partial K)/2) \\ & \leqslant C|x|^{p-1}\mathbb{P}(|X| \geqslant d(x,\partial K)/2) + \mathbb{E}(|X|^{p-1} \geqslant d(x,\partial K)/2).\end{aligned}$$

Using again Markov's inequality yields the bound

$$\mathbb{E}(u(x+X)-u(x), |X| \geqslant d(x,\partial K)/2, x+X \in K) \leqslant \frac{C}{d(x,\partial K)}\mathbb{E}(|X|^p)\max(1,|x|^{p-1}).$$

The last term is handled as in the first part of the proof, with a Taylor's expansion up to the second order in the ball $B(x, d(x,\partial K/2))$.

In order to conclude, we show that $f(x) \leqslant C(1+d(x,\partial K))(1+|x|^{p-1})$. Using the bound $u(z) \leqslant Cd(z,\partial K)|z|^{p-1}$, we get

$$|f(x)| \leqslant Cd(x,\partial K)|x|^{p-1} + \mathbb{E}(|x+X|^{p-1}d(x+X,\partial K)).$$

The first term on the right-hand side above is bounded by $C(1+d(x,\partial K))(1+|x|^{p-1})$. The second term is bounded as

$$\begin{aligned}\mathbb{E}(|x+X|^{p-1}d(x+X,\partial K)) \leqslant & C\mathbb{E}((|x|^{p-1}+|X|^{p-1})(d(x,\partial K))+|X|)) \\ \leqslant & d(x,\partial K)|x|^{p-1} + \mathbb{E}(|X|^p) + d(x,\partial K)\mathbb{E}(|X|^{p-1}) \\ & + |x|^{p-1}\mathbb{E}(|X|) \\ \leqslant & (1+d(x,\partial K))(1+|x|^{p-1}).\end{aligned}$$

□

In order to prove Proposition 33, we write

$$\mathbb{E}\left(\sum_{\ell=0}^{\tau_x-1}|f(x+S(\ell))|\right) = \mathbb{E}\left(\sum_{\ell=0}^{\infty}\mathbf{1}_{\tau_x\geqslant\ell}|f(x+S(\ell))|\right) = \sum_{\ell=0}^{\infty}\mathbb{E}(|f(x+S(\ell))|, \tau_x > \ell).$$

We will show that $\mathbb{E}(|f(x+S(\ell))|, \tau_x > \ell) \leqslant C(1+|x|^{p-r})\ell^{-(1+\alpha)}$ for some $C, \alpha > 0$ independent of x, by using a coupling of the random walk S with a Brownian motion. First, let us split the latter expectation as

$$\begin{aligned}\mathbb{E}(|f(x+S(\ell))|, \tau_x > \ell) = & \mathbb{E}(|f(x+S(\ell))|, \tau_x > \ell, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma}) \\ & + \mathbb{E}(|f(x+S(\ell))|, \tau_x > \ell, d(x+S(\ell),\partial K) \leqslant \ell^{1/2-\gamma}) \\ = & A_\ell + C_\ell\end{aligned}$$

with γ small enough to be chosen later. Then, we split A_ℓ for ε small enough as

$$\begin{aligned}A_\ell =&\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) > \ell^{1-\varepsilon}, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma})\\ &+ \mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, |S(t_{x,\varepsilon}(\ell)) > \theta_\ell\sqrt{\ell}, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma})\\ &+ \mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, S(t_{x,\varepsilon}(\ell)) \leqslant \theta_\ell\sqrt{\ell}, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma})\\ :=&B_\ell^{(1)} + B_\ell^{(2)} + B_\ell^{(3)},\end{aligned}$$

where $t_{x,\varepsilon}(\ell)$ is the usual stopping time $\inf\{n \geqslant 1 : x + S(n) \in K_{\ell,\varepsilon}\}$.

The proof of Proposition 33 is then a straightforward combination of Lemma 36, Lemma 37 and Equation (56).

Lemma 36. *For any $\gamma > 0$, there exist $C > 0$ and $\eta > 0$ such that*

$$\sum_{\ell \geqslant 0} C_\ell \leqslant C(1+|x|)^{p-\eta}.$$

Proof. Let us decompose C_ℓ as

$$\begin{aligned}C_\ell =&\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, d(x+S(\ell),\partial K) \leqslant \ell^{1/2-\gamma}, |S(\ell)| \leqslant \ell^{1/2+\delta'})\\ &+\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, d(x+S(\ell),\partial K) \leqslant \ell^{1/2-\gamma}, |S(\ell)| > \ell^{1/2+\delta'}),\end{aligned}$$

with δ' small enough to be adjusted later. We have by Lemma 35

$$\begin{aligned}&\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, d(x+S(\ell),\partial K) \leqslant \ell^{1/2-\gamma}, |S(\ell)| \leqslant \ell^{1/2+\delta'})\\ &\qquad\leqslant C \sum_{\substack{y\in K\cap\Lambda\\ d(y,\partial K)\leqslant \ell^{1/2-\gamma}\\ |y-x|\leqslant \ell^{1/2+\delta'}}} |y|^{p-1}\left(1 \vee \frac{1}{d(y,\partial K)}\right)\mathbb{P}(x+S(\ell)=y, \tau_x > \ell)\\ &\qquad\leqslant C \sum_{\substack{y\in K\cap\Lambda\\ d(y,\partial K)\leqslant \ell^{1/2-\gamma}\\ |y-x|\leqslant \ell^{1/2+\delta'}}} |y|^{p-1}\left(1 \vee \frac{1}{d(y,\partial K)}\right)\ell^{-d/2}\mathbb{P}(\tau_x > \ell/3)\mathbb{P}(\tau'_y > \ell/3),\end{aligned}$$

where we have used the decomposition

$$\begin{aligned}&\mathbb{P}(x+S(\ell)=y, \tau_x > \ell)\\ &\quad= \sum_{z,z'\in K} \mathbb{P}(x+S(\lfloor\tfrac{\ell}{3}\rfloor)=z, \tau_x > \tfrac{\ell}{3})\mathbb{P}(z+S(\lfloor\tfrac{\ell}{3}\rfloor)=z', \tau_z > \tfrac{\ell}{3})\mathbb{P}(y+S'(\ell-2\lfloor\tfrac{\ell}{3}\rfloor)=z', \tau'_y > \tfrac{\ell}{3})\\ &\quad\leqslant C\ell^{-d/2}\mathbb{P}(\tau_x > \ell/3)\mathbb{P}(\tau'_y > \ell/3),\end{aligned}$$

and (15) with $u = 0$ to bound the middle term in the latter sum. For each $y \in K$, let $y^\perp \in \partial K$ be a boundary point such that $|y - y^\perp| = d(y, \partial K)$, let H_y be a hyperplane tangent to K at $y^\perp$ (which exists because K is convex), and denote by H^+ the half-space delimited by H and containing K. Then,

$$\tau'_y \leqslant \tau_y^H,$$

where τ_y^H is the exit time for the random walk $y + S'$ in the domain H^+. It follows from [49] that

$$\mathbb{P}(\tau_z^H > \ell) \leqslant Cd(z,H)\ell^{-1/2}$$

uniformly for $d(z,H)=o(\sqrt{\ell})$, which implies

$$\mathbb{P}(\tau'_y>\ell)\leqslant Cd(y,\partial K)\ell^{-1/2}$$

for $y\in K$ such that $d(y,\partial K)\leqslant n^{1/2-\gamma}$. Hence, we finally get

$$\begin{aligned}
&\mathbb{E}(|f(x+S(\ell))|,\tau_x\geqslant\ell,d(x+S(\ell),\partial K)\leqslant\ell^{1/2-\gamma},|S(\ell)|\leqslant\ell^{1/2+\delta'})\\
&\leqslant C\sum_{\substack{y\in K\cap\Lambda\\ d(y,\partial K)\leqslant\ell^{1/2-\gamma}\\ |y-x|\leqslant\ell^{1/2+\delta'}}}|y|^{p-1}\left(1\vee\frac{1}{d(y,\partial K)}\right)C\ell^{-d/2-1/2}\mathbb{P}(\tau_x>\ell/3)d(y,\partial K)\\
&\leqslant C\sum_{\substack{y\in K\cap\Lambda\\ d(y,\partial K)\leqslant\ell^{1/2-\gamma}\\ |y-x|\leqslant\ell^{1/2+\delta'}}}|y|^{p-1}C\ell^{-d/2-1/2}\mathbb{P}(\tau_x>\ell/3)\\
&\leqslant C\ell^{-d/2-1/2}(|x|+\ell^{1/2+\delta'})^{p-1}\mathbb{P}(\tau_x>\ell/3)\,\mathrm{Vol}\{y\in B(x,\ell^{d/2+\delta'}),d(y,\partial K)\leqslant\ell^{1/2-\gamma}\}.
\end{aligned}$$

The volume of the region $\{y\in B(x,\ell^{d/2+\delta'}),d(y,\partial K)\leqslant\ell^{1/2-\gamma}\}$ is asymptotically bounded by $C\ell^{d(1/2+\delta')-\gamma}$, which yields

$$\begin{aligned}
\mathbb{E}(|f(x+S(\ell))|,\tau_x\geqslant\ell,d(x+S(\ell),\partial K)\leqslant\ell^{1/2-\gamma},|x+S(\ell)|\leqslant\ell^{1/2+\delta'})&\\
\leqslant C\ell^{-1/2-\gamma+o(\delta')}(|x|+\ell^{1/2+\delta'})^{p-1}\mathbb{P}(\tau_x>\ell/3)&\\
\leqslant C\ell^{p/2-1-\gamma+o(\delta')}\mathbb{P}(\tau_x>\ell/3)+C\ell^{-1/2-\gamma+o(\delta')}\mathbb{P}(\tau_x>\ell/3).&
\end{aligned}$$

Hence, for δ' small enough, we have

$$\begin{aligned}
\mathbb{E}(|f(x+S(\ell))|,\tau_x\geqslant\ell,d(x+S(\ell),\partial K)\leqslant\ell^{1/2-\gamma},|x+S(\ell)|\leqslant\ell^{1/2+\delta'})&\\
\leqslant C(\ell^{p/2-1-\kappa}+|x|^{p-1}\ell^{-1/2-\kappa})\mathbb{P}(\tau_x>\ell/3),&
\end{aligned}$$

for some $\kappa>0$. Using [35, Thm 3.1] with the fact that $p\geqslant 1$ yields

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}\mathbb{E}(|f(x+S(\ell))|,\tau_x\geqslant\ell,d(x+S(\ell),\partial K)\leqslant\ell^{1/2-\gamma},|x+S(\ell)|\leqslant\ell^{1/2+\delta'})\\
&\qquad\leqslant C\sum_{\ell=0}^{\infty}(\ell^{p/2-1-\kappa}+|x|^{p-1}\ell^{-1/2-\kappa})\mathbb{P}(\tau_x>\ell/3)\\
&\qquad\leqslant C\left(\mathbb{E}(\tau_x^{p/2-\kappa})+|x|^{p-1}\mathbb{E}(\tau_x^{1/2-\kappa})\right)\leqslant C(1+|x|)^{p-2\kappa}
\end{aligned}$$

for some $C>0$ and $\kappa>0$. For the second term of C_ℓ, we have

$$\begin{aligned}
\mathbb{E}(|f(x+S(\ell))|,\tau_x\geqslant\ell,d(x+S(\ell),\partial K)\leqslant\ell^{1/2-\gamma},|S(\ell)|\geqslant\ell^{1/2+\delta'})&\\
\leqslant C\mathbb{E}(|x+S(\ell)|^{p-1},\tau_x\geqslant\ell,|S(\ell)|\geqslant\ell^{1/2+\delta'})&\\
\leqslant C|x|^{p-1}\mathbb{P}(\tau_x\geqslant\ell,|S(\ell)|\geqslant\ell^{1/2+\delta'})+C\mathbb{E}(|S(\ell)|^{p-1},\tau_x\geqslant\ell,|S(\ell)|\geqslant\ell^{1/2+\delta'}).&
\end{aligned}$$

By Lemma 16 with $\beta=p/2-1/2+\kappa$, the first term is bounded as

$$\begin{aligned}
C|x|^{p-1}\mathbb{P}(\tau_x\geqslant\ell,|S(\ell)|\geqslant\ell^{1/2+\delta'})&\leqslant C|x|^{p-1}(1+|x|)^{p-2(p-1+\kappa)}\ell^{-r(p)/2+1-(p/2-p/2+1/2-\kappa)+o(\delta')}\\
&\leqslant C(1+|x|)^{p-2\kappa}\ell^{-r(p)/2+1/2+\kappa+o(\delta')}.
\end{aligned}$$

Since $r(p) > 2p + d - 2 + 2(1-p/2)^+$, taking κ and δ' small enough yields

$$C|x|^{p-1}\mathbb{P}(\tau_x \geqslant \ell, |S(\ell)| \geqslant \ell^{1/2+\delta'}) \leqslant C(1+|x|)^{p-2\kappa}\ell^{1/2-p-d/2} \leqslant C(1+|x|)^{p-2\kappa}\ell^{-1-\eta}$$

for some $\eta > 0$ and $C > 0$. Next, by Lemma 16 with β small enough, the second term is bounded as

$$\mathbb{E}(|S(\ell)|^{p-1}, \tau_x \geqslant \ell, |S(\ell)| \geqslant \ell^{1/2+\delta'}) \leqslant C(1+|x|)^{(p-2\beta)\wedge 2}\ell^{-(r(p)-(p-1))/2+(1-p/2+\beta)^+ + o(\delta')}.$$

Likewise, since $r(p) > 2p + d - 2 + 2(1-p/2)^+$, choosing β and δ' small enough and using the fact that $d \geqslant 3$ yields

$$\ell^{-(s-(p-1))/2+(1-p/2+\beta)^+ + o(\delta')} \leqslant \ell^{-1-\eta}$$

for some $\eta > 0$. Hence,

$$\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, d(x+S(\ell), \partial K) \leqslant \ell^{1/2-\gamma}, |S(\ell)| \geqslant \ell^{1/2+\delta'}) \leqslant C(1+|x|)^{p-r}\ell^{-1-\eta}$$

for some $r, \eta > 0$. Therefore,

$$\sum_{\ell=0}^{\infty} C_\ell \leqslant C(1+|x|)^{p-r}$$

for some constant $C > 0$. □

Lemma 37. *There exists a constant $C > 0$ such that for some η small enough,*

$$\sum_{\ell=0}^{\infty} B_\ell^{(1)} + B_\ell^{(2)} \leqslant C(1+|x|^{p-\eta}).$$

Proof. Since $f(x) \leqslant C(1+|x|^{p-1})$ on K, the first term is bounded by

$$B_1^{(\ell)} \leqslant C\mathbb{E}(|x+S(\ell)|^{p-1}, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) > \ell^{1-\varepsilon}, d(x+S(\ell), \partial K) \geqslant \ell^{1/2-\gamma}).$$

Using Hölder's inequality together with [16, Lem. 14] gives

$$\begin{aligned} B_\ell^{(1)} &\leqslant \left(\mathbb{E}(|x+S(\ell)|^{p-1/2}, \tau_x > \ell, d(x+S(\ell), \partial K) \geqslant \ell^{1/2-\gamma})\right)^{\frac{p-1}{p-1/2}} \mathbb{P}(\tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \geqslant \ell^{1-\varepsilon})^{\frac{1}{2p-1}} \\ &\leqslant \mathbb{E}\left(\sup_{1\leqslant \ell \leqslant \tau_x} |x+S(\ell)|^{p-1/2}\right)^{\frac{p-1}{p-1/2}} \mathbb{P}(\tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \geqslant \ell^{1-\varepsilon})^{\frac{1}{2p-1}} \\ &\leqslant C(1+|x|)^{p-1}\exp(-C'\ell^{\varepsilon'}) \end{aligned}$$

for some $C, C', \varepsilon' > 0$, where we have used [35, Thm 3.1] in the last inequality. Hence,

$$\sum_{\ell=0}^{\infty} B_\ell^{(1)} \leqslant C(1+|x|^{p-1})$$

for some constant $C > 0$.

Likewise, we have for η small enough

$$\begin{aligned} B_\ell^{(2)} &\leqslant \mathbb{E}((x+S(\ell))^{p-1}, \tau_x \geqslant \ell, t_{x,\varepsilon} \leqslant \ell^{1-\varepsilon}, |S(t_{x,\varepsilon}(\ell))| \geqslant \theta_\ell\sqrt{\ell}, d(x+S(\ell), \partial K) \geqslant \ell^{1/2-\gamma}) \\ &\leqslant \mathbb{E}\left(\sup_{1\leqslant \ell \leqslant \tau_x} |x+S(\ell)|^{p-\eta}\right)^{\frac{p-1}{p-\eta}} \mathbb{P}(\tau_x \geqslant \ell, t_{x,\varepsilon} \leqslant \ell^{1-\varepsilon}, |S(t_{x,\varepsilon}(\ell))| \geqslant \theta_\ell\sqrt{\ell})^{\frac{1-\eta}{p-\eta}} \\ &\leqslant C(1+|x|)^{p-1}\mathbb{P}(\tau_x \geqslant \ell, t_{x,\varepsilon} \leqslant \ell^{1-\varepsilon}, |S(t_{x,\varepsilon}(\ell))| \geqslant \theta_\ell\sqrt{\ell})^{\frac{1-\eta}{p-\eta}}. \end{aligned}$$

We bound the right-hand side of the last inequality using Lemma 16 with β small enough to get

$$\mathbb{P}(\tau_x \geqslant \ell, t_{x,\varepsilon} \leqslant \ell^{1-\varepsilon}, |S(t_{x,\varepsilon}(\ell))| \geqslant \theta_\ell \sqrt{\ell}) \leqslant C\ell^{-r(p)/2}(1+|x|)^{(p-2\beta)\wedge 2}\ell^{(1-(p/2-\beta))\wedge 0}$$

for $r(p) > 2p + d - 2 + 2(1-p/2)^+$. Therefore, choosing η small enough and β small enough yields

$$B_\ell^{(2)} \leqslant C(1+|x|)^{p-1}(1+|x|)^{(1-2\beta/p)\wedge 2/p}\ell^{-r(p)/2p+(1/p-(1/2-\beta/p))^+} \leqslant C(1+|x|)^{p-r}\ell^{-1-r'}$$

for some $r, r' > 0$. Hence,

$$\sum_{\ell=0}^{\infty} B_\ell^{(2)} \leqslant C(1+|x|)^{p-r}. \qquad \square$$

It remains to handle the term $B_\ell^{(3)}$. Since X admits a moment $r > 0$, we use [23, Thm 4] to deduce that for $0 < \gamma' < 1/2$ and $r > 3$ there exists C and a coupling of S with a Brownian motion B such that

$$\mathbb{P}(\sup_{1\leqslant k\leqslant n} |B(k) - S(k)| \geqslant n^{1/2-\gamma'}) \leqslant C_r n^{1-r(1/2-\gamma')}$$

for all $n \in \mathbb{N}$. Hence, with our moment assumptions, we can choose such a coupling with $r = r(p) > 2p + d - 2 + 2(1-p/2)^+$. Set

$$\chi_\ell = \left\{ \sup_{1\leqslant k\leqslant \ell} |B(k) - S(k)| \leqslant \ell^{1/2-\gamma'} \right\}.$$

Lemma 38. *There exist positive constants C and s such that*

$$\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, x_\varepsilon(\ell) \leqslant \theta_\ell\sqrt{\ell}, \chi_\ell^c, d(x+S(\ell), \partial K) \geqslant \ell^{1/2-\gamma}) \leqslant C(1+|x|^{p-1})\ell^{-1-s}.$$

Proof. By Lemma 35, $|f(z)| \leqslant |z|^{p-1}\ell^{-1/2+\gamma}$ on $\{z \in K : d(z, \partial K) \geqslant \ell^{1/2-\gamma}\}$. Hence we have

$$\begin{aligned}\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, &x_\varepsilon(\ell) \leqslant \theta_\ell\sqrt{\ell}, \chi_\ell^c, d(x+S(\ell), \partial K) \geqslant \ell^{1/2-\gamma}) \\ &\leqslant \mathbb{E}(|x+S(\ell)|^{p-1}\chi^c)\ell^{-1/2+\gamma} \\ &\leqslant C|x|^{p-1}\mathbb{P}(\chi_\ell^c)\ell^{-1/2+\gamma} + \mathbb{E}(|S(\ell)|^{p-1}\chi_\ell^c)\ell^{-1/2+\gamma}.\end{aligned}$$

Since $(r(p)-2)/2 > 1$ (note that $p \geqslant 1$ when K is convex), for γ small enough we have $|x|^{p-1}\mathbb{P}(\chi_\ell^c)\ell^{-1/2+\gamma} \leqslant C\ell^{-1-\eta}$ for some constants $C, \eta > 0$. Moreover, since X admits moments of order $r(p)$, the Hölder inequality for $q = r(p)/(p-1)$ yields

$$\ell^{-1/2+\gamma}\mathbb{E}(|S(\ell)|^{p-1}\chi_\ell^c) \leqslant \ell^{-1/2+\gamma}\mathbb{E}(|S(\ell)|^{r(p)})^{1/q}\mathbb{P}(\chi_\ell^c)^{1-1/q}.$$

Einmahl's theorem yields

$$\mathbb{P}(\chi_\ell^c)^{1-1/q} \leqslant \ell^{(2-r(p)(1-2\gamma'))/2(1-1/q)},$$

and Rosenthal inequality gives

$$\mathbb{E}(|S(\ell)|^{q(p-1)})^{q^{-1}} \sim C\ell^{(p-1)/2}$$

for some constant $C > 0$. Hence,

$$\ell^{-1/2+\gamma}\mathbb{E}(|S(\ell)|^{p-1}\chi_\ell^c) \leqslant C\ell^{-r}$$

with

$$\begin{aligned} r &= (1-1/q)(r(p)(1-2\gamma')-2)/2-(p-1)/2+1/2-\gamma \\ &= \left(1-\frac{p-1}{r(p)}\right)(r(p)-2)/2-(p-1)/2+1/2+o(\gamma,\gamma') \\ &= r(p)/2-p+1/2+\frac{p-1}{r(p)}+o(\gamma). \end{aligned}$$

By the condition $r(p) > 2p+d-2+2(1-p/2)^+$, we obtain

$$r > d/2-1/2+(1-p/2)^+ + (p-1)/r(p)+o(\gamma,\gamma').$$

Hence, since $d \geqslant 3$, we get

$$r > 1+\eta$$

for γ small enough and some $\eta > 0$. Hence,

$$\mathbb{E}(|S(\ell)|^p \chi_\ell^c, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma}) \leqslant C\ell^{-1-\eta},$$

so that

$$\begin{aligned} \mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, x_\varepsilon(\ell) \leqslant \theta_\ell\sqrt{\ell}, \chi_\ell^c, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma}) \\ \leqslant C(1+|x|^{p-1})\ell^{-1-\eta}, \end{aligned}$$

and a summation concludes the proof. □

We conclude with the main part $B_\ell^{(3)}$, which consists in the part of the random walk which is close to the coupled Brownian motion:

$$\begin{aligned} &\mathbb{E}(|f(x+S(\ell))|, \tau_x \geqslant \ell, t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, x_\varepsilon(\ell) \leqslant \theta_\ell\sqrt{\ell}, \chi_\ell, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma}) \\ &= \mathbb{E}\Big(\mathbb{E}\big(|f(x_\varepsilon(\ell)+S(\ell-t_\varepsilon(\ell)))|, \chi_\ell, \tau_x > \ell-t_\varepsilon(\ell),, d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma}\big), \tau_x \geqslant t_{x,\varepsilon}(\ell), \\ &\qquad t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, x_\varepsilon(\ell) \leqslant \theta_\ell\sqrt{\ell}\Big). \end{aligned}$$

Lemma 39. *There exist positive constants C and r such that*

$$\mathbb{E}\big(|f(x_\varepsilon(\ell)+S(\ell-t_\varepsilon(\ell)))|, \chi_\ell, \tau_x > \ell-t_\varepsilon(\ell), d(x+S(\ell),\partial K) \geqslant \ell^{1/2-\gamma}\big) \leqslant C\ell^{-1-r}|x_\varepsilon(\ell)|^p$$

uniformly on the event $\{\tau_x \geqslant t_{x,\varepsilon}(\ell), t_{x,\varepsilon}(\ell) \leqslant \ell^{1-\varepsilon}, x_\varepsilon(\ell) \leqslant \theta_\ell\sqrt{\ell}\}$.

Proof. Let us temporarily set $m = \ell - t_\varepsilon(\ell)$ and $z = x_\varepsilon(\ell)$. We divide the proof according to the value of $z+S(m)$. For the remaining part of the proof we fix $0 < \gamma < \gamma'$ and we set

- $A_1 := \{d(z+S(m),\partial K) \geqslant 2|z+S(m)|^{1/2+\delta}, |z+S(m)| \geqslant m^{1/2-\gamma}\}$,
- $A_2 := \{m^{1/2-\gamma} < d(z+S(m),\partial K) < 2|z+S(m)|^{1/2+\delta}\}$.

Case A_1: For m large enough, on the event χ_ℓ, the inequalities

$$d(z+S(m),\partial K) \geqslant 2|z+S(m)|^{1/2+\delta} \quad \text{and} \quad |z+S(m)| \geqslant m^{1/2-\gamma}$$

imply that $d(z+B(m),\partial K) \geqslant |z+B(m)|^{1/2+\delta}$. Hence, on the set $A_1 \cap \chi_\ell$, we have by Lemma 35

$$f(z+S(m)) \leqslant C|z+S(m)|^{p-2-\delta} \leqslant C'|z+B(m)|^{p-2-\delta},$$

where we have used the fact that $|z+S(m)| \geqslant m^{1/2-\gamma'}$ and $|S(m)-B(m)| \leqslant m^{1/2-\gamma}$ on the last inequality. Recall from Section 3 that for $z \in K_{\ell,\varepsilon}$, z^+ denotes the element $z + R_0 x_0 \ell^{1/2-\gamma'}$, so that we have

$$\{\tau_z > \ell, \chi_\ell\} \subset \{\tau^{\rm bm}_{z^+} > \ell, \chi_\ell\}.$$

Since $|z+B(m)| \leqslant |z^+ + B(m)|$, we have

$$\begin{aligned}\mathbb{E}\big(|f(z+S(m))|, \chi_\ell, A_1, \tau_z > m\big) &\leqslant \mathbb{E}\big(|z+B(m)|^{p-2-\delta}, \chi_\ell, A_1, \tau_z > m\big)\\ &\leqslant \mathbb{E}\big(|z^+ + B(m)|^{p-2-2\delta}, \chi_\ell, A_1, \tau^{\rm bm}_{z^+} > m\big)\\ &\leqslant \mathbb{E}\big(|z^+ + B(m)|^{p-2-2\delta}, \tau^{\rm bm}_{z^+} > m\big).\end{aligned}$$

By [16, Lem. 18], we thus have uniformly on $z \in K$, $|z| \leqslant \theta_m \sqrt{m}$,

$$\begin{aligned}\mathbb{E}\big(|f(z+S(m))|, \chi_\ell, A_1, \tau_z > m\big) &\leqslant C m^{-p-d/2} \int_K u(z^+) u(y) |y|^{p-2-2\delta} \exp(-y^2/(2m)) dy\\ &\leqslant C u(z^+) m^{-1-\delta} \int_K u(y) |y|^{p-2-2\delta} \exp(-y^2/2) dy,\end{aligned}$$

where we have made the change of variables $y \leftrightarrow y/\sqrt{m}$ in the last inequality. Therefore, there exists $C>0$ such that

$$\mathbb{E}\big(|f(z+S(m))|, \chi_\ell, A_1, \tau_z > m\big) \leqslant C u(z^+) m^{-1-\delta}.$$

Case A_2: On the set $A_2 \cap \chi_\ell$, because $\gamma < \gamma'$ we have for m large enough

$$\frac{1}{d(z+S(m), \partial K)} \leqslant C \frac{1}{d(z^+ + B(m), \partial K)},$$

so that, by Lemma 35 and the fact that $|z^+ + B(m)| \geqslant m^{1/2-\gamma}$ on A_2, we have

$$\begin{aligned} f(z+S(m)) &\leqslant C \frac{1}{d(z+S(m), \partial K)} |z+S(m)|^{p-1}\\ &\leqslant C \frac{1}{d(z^+ + B(m), \partial K)} \left(|z^+ + B(m)|^{p-1} + m^{(p-1)(1/2-\gamma)} \right)\\ &\leqslant C \frac{|z^+ + B(m)|^{p-1}}{d(z^+ + B(m), \partial K)}.\end{aligned}$$

Then, following the same proof as in the first case gives

$$\begin{aligned}&\mathbb{E}\big(|f(z+S(m))|, \chi_\ell, A_2, \tau_z > m\big)\\ &\quad\leqslant C \mathbb{E}\big(|z^+ + B(m)|^{p-1}/d(z^+ + B(m), \partial K), \chi_\ell, A_2, \tau_z > m\big)\\ &\quad\leqslant C \mathbb{E}\big(|z^+ + B(m)|^{p-1}/d(z^+ + B(m)), \partial K), A_2, \tau^{\rm bm}_{z^+} > m\big)\\ &\quad\leqslant C m^{-p-d/2} \int_{\substack{m^{1/2-\gamma'} < d(y,\partial K)\\ d(y,\partial K) \leqslant 2|y|^{1/2+\delta}}} u(z^+) \frac{u(y)}{d(y,\partial K)} |y|^{p-1} \exp(-y^2/(2m)) dy.\end{aligned}$$

Let us introduce the set

$$\partial K_m := \left\{ y \in K : d(y, \partial K) \leqslant 2 m^{-1/4+\delta} |y|^{1/2+\delta} \right\}.$$

Using the fact that $u(y) \leqslant Cd(y,\partial K)|y|^{p-1}$ on K, the change of variable $y \leftrightarrow y/\sqrt{m}$ yields

$$\mathbb{E}\big(|f(z+S(m))|,\chi_\ell,A_2,\tau_z>m\big) \leqslant Cu(z^+)m^{-1}\int_{\partial K_m}|y|^{2(p-1)}\exp(-y^2/2)dy.$$

Since $|y|^{1/2+\delta}\leqslant|y|$ for $y\in K$ such that $|y|\geqslant m^{1/2-\gamma'}$, we have

$$\begin{aligned}\int_{\partial K_m}|y|^{2(p-1)}\exp(-y^2/2)dy &\leqslant \int_{d(y,\partial K)\leqslant m^{-1/4+\delta}|y|}|y|^{2(p-1)}\exp(-y^2/2)dy\\ &\leqslant \mathrm{Vol}(w\in\Sigma, d(w,\partial\Sigma)\leqslant m^{-1/4+\delta})\int_{\mathbb{R}^+}|y|^{2(p-1)}\exp(-y^2/2)dy\\ &\leqslant Cm^{-1/4+\delta},\end{aligned}$$

where we recall that Σ denotes the intersection $\Sigma\cap\mathbb{S}^{d-1}$ and we have used Steiner's formula (see Theorem 46 in [36, Ch. 16]) to compute the infinitesimal expansion for the volume of the tubular region around the boundary of a Riemannian manifold with convex or $\mathcal{C}^2$ boundary. This gives

$$\mathbb{E}\big(|f(z+S(m))|,\chi_\ell,A_2,\tau_z>m\big)\leqslant u(z^+)Cm^{-1-1/4+\delta}$$

for some $C>0$. Summing the terms coming from A_1 and A_2 gives the bound

$$\mathbb{E}\big(|f(z+S(m))|,\chi_\ell,\tau_z>m,d(z+S(m),\partial K)\geqslant\ell^{1/2-\gamma}\big)\leqslant Cu(z^+)m^{-1-r}\leqslant Cm^{-1-r}|z^+|^p$$

for some $r>0$ and uniformly on $z\in K_{\ell,\varepsilon}$ with $|z|\leqslant\theta_m\sqrt{m}$. Since $z\in K_{\ell,\varepsilon}$, $|z|\geqslant\ell^{1/2-\varepsilon}$, so that the condition $\gamma>\varepsilon$ yields

$$|z^+|\leqslant|z|+m^{1/2-\gamma}\leqslant 2|z|$$

for ℓ large enough. Since $m\sim\ell$ as ℓ goes to infinity on the event $\{t_{x,\varepsilon}(\ell)\leqslant\ell^{1-\varepsilon}\}$, the result is deduced. □

We can now give a bound on $B_\ell^{(3)}$.

$$\begin{aligned}\mathbb{E}(|f(x+S(\ell))|,&\tau_x\geqslant\ell,t_{x,\varepsilon}(\ell)\leqslant\ell^{1-\varepsilon},|S(t_{x,\varepsilon}(\ell))|\leqslant\theta_\ell\sqrt{\ell},\chi_\ell,d(z+S(m),\partial K)\geqslant\ell^{1/2-\gamma})\\ &\leqslant C\ell^{-1-r}\mathbb{E}(|x_\varepsilon(\ell)|^p,\tau_x\geqslant t_{x,\varepsilon}(\ell),t_{x,\varepsilon}(\ell)\leqslant\ell^{1-\varepsilon},|S(t_{x,\varepsilon}(\ell))|\leqslant\theta_\ell\sqrt{\ell})\\ &\leqslant C\ell^{-1-r}\mathbb{E}(|x+S(t_{x,\varepsilon}(\ell))|^p,\tau_x\geqslant t_{x,\varepsilon}(\ell),t_{x,\varepsilon}(\ell)\leqslant\ell^{1-\varepsilon},|S(t_{x,\varepsilon}(\ell))|\leqslant\theta_\ell\sqrt{\ell})\end{aligned}$$

Then, by the condition $|S(t_{x,\varepsilon}(\ell))|\leqslant\theta_\ell\sqrt{\ell}$,

$$\begin{aligned}&\mathbb{E}(|x+S(t_{x,\varepsilon}(\ell))|^p,\tau_x\geqslant t_{x,\varepsilon}(\ell),t_{x,\varepsilon}(\ell)\leqslant\ell^{1-\varepsilon},|S(t_{x,\varepsilon}(\ell))|\leqslant\theta_\ell\sqrt{\ell})\\ &=\mathbb{E}(|x+S(t_{x,\varepsilon}(\ell))|^{p-r}|x+S(t_{x,\varepsilon}(\ell))|^r,\tau_x\geqslant t_{x,\varepsilon}(\ell),t_{x,\varepsilon}(\ell)\leqslant\ell^{1-\varepsilon},|S(t_{x,\varepsilon}(\ell))|\leqslant\theta_\ell\sqrt{\ell})\\ &\leqslant C\left||x|+\theta_\ell\sqrt{\ell}\right|^r\mathbb{E}\left(\sup_{1\leqslant n\leqslant\tau_x}|x+S(n)|^{p-r}\right)\leqslant C(|x|^r+\ell^{r/2})(1+|x|)^{p-r},\end{aligned}$$

where we have used the general bound $\mathbb{E}(\sup_{1\leqslant n\leqslant\tau_x}|x+S(n)|^\beta)\leqslant C(1+|x|)^\beta$ for $\beta<p$ given by [35, Thm 3.1]. Finally, we have

$$\mathbb{E}(|f(x+S(\ell))|,\tau_x\geqslant\ell,t_{x,\varepsilon}(\ell)\leqslant\ell^{1-\varepsilon},|S(t_{x,\varepsilon}(\ell))|\leqslant\theta_\ell\sqrt{\ell},\chi_\ell)\leqslant C(1+|x|)^p\ell^{-1-r/2}.$$

In particular,

$$\sum_{\ell=|x|^{1/2}}^{\infty} B_\ell^{(3)} \leqslant C(1+|x|)^p \sum_{\ell=|x|^{1/2}}^{\infty} \ell^{-1-r/2} \leqslant C(1+|x|)^{p-r/4}.$$

for some constant $C>0$. By [35, Thm 3.1],

$$\begin{aligned}\sum_{\ell=0}^{|x|^{1/2}} B_\ell^{(3)} \leqslant \sum_{\ell=0}^{|x|^{1/2}} \mathbb{E}(|f(x+S(\ell))|, \tau_x > \ell) \leqslant & C \sum_{\ell=0}^{|x|^{1/2}} \mathbb{E}(|x+S(\ell)|^{p-1}, \tau_{x>\ell}) \\ \leqslant & C|x|^{1/2}\mathbb{E}\left(\sup_{1\leqslant \ell \leqslant \tau_x} |x+S(\ell)|^{p-1}\right), \\ \leqslant & C(1+|x|)^{p-1/2}.\end{aligned}$$

Therefore, using the latter bounds with Lemma 38 finally yields

$$\sum_{\ell=0}^{\infty} B_\ell^{(3)} \leqslant C(1+|x|)^{p-r} \tag{56}$$

for some positive constants C and r.

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