Convergence rate of an asymptotic preserving scheme for the diffusive limit of the p-system with damping

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Abstract. This paper aims to establish the convergence rate of approximate solutions of the p-system with damping towards its diffusive limit. We consider an approximation obtained with a full discrete Asymptotic Preserving Finite Volume scheme. We study the discrete diffusive limit and establish an exact formulation of the convergence rate. To access such an issue, we estimate the error between approximate solutions of the hyperbolic system and the approximate diffusive limit using a discrete version of the relative entropy method.

Keywords. Asymptotic Preserving scheme; numerical convergence rate; relative entropy

AMS subject classifications. 65M08; 65M12

1 Introduction

This paper is devoted to the numerical behaviour of asymptotic regimes satisfied by approximated solutions of the p-system. The system of interest reads:

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = -\sigma u, \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

where $\tau > 0$ is the specific volume of gas and $u \in \mathbb{R}$ stands for the velocity. Here $\sigma > 0$ denotes the friction parameter. Regarding the pressure law, $p(\tau) > 0$ is assumed to be smooth enough; namely $C^2(\mathbb{R}_+^*)$. Moreover, in order to enforce the system (1.1) to be hyperbolic, we also impose $p'(\tau) < 0$ for all $\tau > 0$.

To simplify the notations, we set $w = {}^t(\tau, u)$ assumed to belong to the following phase space:

$$\Omega = \left\{ {}^t(\tau, u);\ \tau > 0,\ u \in \mathbb{R} \right\}. \tag{1.2}$$

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The system (1.1) is endowed with an entropy inequality given as follows:

$$\partial_t \eta(w) + \partial_x \psi(w) \leq -\sigma u^2. \tag{1.3}$$

The entropy-entropy flux pair (η, ψ) is defined by:

$$\eta(w) = \frac{u^2}{2} - P(\tau), \qquad \psi(w) = up(\tau), \tag{1.4}$$

where the function P defines an internal energy given by:

$$P(\tau) = \int_{\tau^*}^{\tau} p(s)ds, \tag{1.5}$$

with $\tau^* > 0$ a fixed reference specific volume.

In the present work, we study the long time behaviour of solutions to system (1.1) with a dominant friction. Such behaviours are of particular interest since long time solutions coincide with solutions of diffusive regimes. During the two last decades, several works studied this kind of mathematical difficulties.

In the pioneer work of Hsiao and Liu [20], the authors established that the time asymptotic regime of the system (1.1) coincides with the nonlinear porous media equation given by:

$$\begin{cases} \partial_t \overline{\tau} = \partial_x \overline{u}, \\ \overline{u} = -\dfrac{1}{\sigma} \partial_x p(\overline{\tau}), \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.6}$$

or equivalently

$$\partial_t \overline{\tau} + \frac{1}{\sigma} \partial_{xx} p(\overline{\tau}) = 0 \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

More precisely, they proved that the solution w of (1.1) converges to $\overline{w} = {}^t(\overline{\tau}, \overline{u})$ solution of (1.6) when time goes to infinity. In addition, they exhibited an estimation of the L^∞-convergence rate in $O(t^{-1/2})$. Next, by adopting energy estimate techniques, Nishihara and co-authors improved this convergence rate in [32] (see also the companion papers [30, 31]). A generalization has been proposed by Bianchini *et al* in [3] to deal with general entropy dissipative hyperbolic systems of balance laws. Under the Shizuta-Kawashima condition, they established that the solutions under interest converge to a constant equilibrium state. In addition they exhibited the asymptotic convergence rate. Moreover, we refer the reader to the recent paper of Mei [26] where a wide litterature devoted to the long time asymptotic behaviour of the $p-$system with damping (1.1) is given.

Another method to analyze the convergence from w solution of (1.1) to $\overline{w}$ solution of (1.6) consists in introducing a relevant time rescaling (see [24, 26]). To address such an issue, let us note $\varepsilon > 0$ a small parameter to govern the long time and the dominant friction. From now on, because of the high friction, let us emphasize that the velocity is naturally controlled by ε. As a consequence, the rescaled system writes:

$$\begin{cases} \partial_t \tau^\varepsilon - \partial_x u^\varepsilon = 0, \\ \varepsilon^2 \partial_t u^\varepsilon + \partial_x p(\tau^\varepsilon) = -\sigma u^\varepsilon, \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{1.7}$$

This rescaled system is now endowed with the following rescaled entropy inequality:

$$\partial_t \eta^\varepsilon(\tau^\varepsilon, u^\varepsilon) + \partial_x \psi(\tau^\varepsilon, u^\varepsilon) \leq -\sigma (u^\varepsilon)^2, \tag{1.8}$$

where the rescaled entropy reads:

$$\eta^\varepsilon(\tau, u) = \varepsilon^2 \frac{u^2}{2} - P(\tau). \tag{1.9}$$

From a formal Chapman Enskog expansion, let us note that the solutions of (1.7) clearly coincide with the solutions of the asymptotic equation (1.6) in the limit of ε to zero.

This work is of course dedicated to the nonlinear formulation (1.1), but let us emphasize that, as soon as the pressure law is assumed to be linear, namely $p(\tau) = -a\tau$, we recognize the well-known Goldstein Taylor model, arising from the two velocities kinetic model [22, 27]. Indeed let us set $f^\pm$ the distribution functions associated with constant particle velocities ± 1 governed by the following system:

$$\begin{cases} \partial_t f^+ + \dfrac{1}{\varepsilon} \partial_x f^+ = \dfrac{1}{\varepsilon^2} \left(f^- - f^+ \right), \\ \partial_t f^- - \dfrac{1}{\varepsilon} \partial_x f^- = \dfrac{1}{\varepsilon^2} \left(f^+ - f^- \right). \end{cases}$$

The macroscopic variables for this model are the mass density $\rho = f^+ + f^-$ and the current $j = \dfrac{1}{\varepsilon}(f^+ - f^-)$. Since f^+ and f^- can be expressed in terms of ρ and j, we obtain the following macroscopic model:

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \varepsilon^2 \partial_t j - \partial_x \rho = -2j. \end{cases}$$

We easily recognize the initial model (1.7) with a linear pressure.

Recently, Lattanzio and Tzavaras [24] rigorously proved this convergence and exhibited the rate by adopting the well known relative entropy approach. Relative entropy is a usefull tool to compare the difference (in a sense to be prescribed) between two solutions. The notion of relative entropy for hyperbolic systems of conservation laws was introduced in the pioneer works of DiPerna [10] and Dafermos [8]. It was used to study rigorously the convergence from kinetic models to their hydrodynamic limit [14, 33]. Later, in [9], Dafermos adopted this method to establish a stability result for classical solutions in the class of entropy weak solutions. Next, in [34], Tzavaras applied a similar relative entropy method to study the convergence of hyperbolic systems with stiff relaxation towards the corresponding hyperbolic limit. Based on the same ideas, Lattanzio and Tzavaras considered in [24] the case of diffusive relaxation. They treated several hyperbolic systems with source term of type (1.7) which converge to a diffusive problem when ε goes to zero. In particular, they established the convergence of solutions to the p-system (1.7) towards solutions of the porous media equation (1.6).

To be more precise, as soon as the hyperbolic system (1.7) is supplemented with an entropy function η^ε, the associated relative entropy is defined as the quadratic term of the Taylor expansion between $w^\varepsilon = {}^t(\tau^\varepsilon, u^\varepsilon)$ solution of (1.7) and $\overline{w} = {}^t(\overline{\tau}, \overline{u})$ solution of (1.6) as follows:

$$\eta^\varepsilon(w|\overline{w}) = \eta^\varepsilon(w) - \eta^\varepsilon(\overline{w}) - \nabla \eta^\varepsilon(\overline{w}) \cdot (w - \overline{w}). \tag{1.10}$$

Since it will be useful in the sequel, let us introduce the space integral of the relative entropy:

$$\phi^\varepsilon(t) = \int_{\mathbb{R}} \eta^\varepsilon(w|\overline{w})dx. \tag{1.11}$$

We emphasize that this quantity has a similar behaviour than $\|w-\overline{w}\|_{L^2}^2$ as long as η^ε is a convex function.

Lattanzio and Tzavaras estimated this quantity and obtained the following convergence rate:

$$\phi^\varepsilon(t) \leq C\left(\phi^\varepsilon(0)+\varepsilon^4\right), \quad t\in[0,T), \tag{1.12}$$

where C is a positive constant depending only on T, the properties of the pressure function p and estimates on the smooth diffusive limit $\overline{w}$.

From a numerical point of view, the derivation of a suitable discretization of (1.7) in order to get a consistent approximation of (1.6) in the limit of ε to zero is essential to ensure an accurate approximation. In the two last decades, numerous works have been devoted to both elaboration and analysis of so-called Asymptotic Preserving (AP) schemes. This notion was introduced by Jin *et al.* in [21, 22] in the kinetic framework. Indeed they remarked that a particular attention has to be paid on the discretization of the hyperbolic system in order to recover the correct asymptotic diffusive regime. For instance, Naldi and Pareschi proposed several discretizations of a two velocities kinetic equation in [27, 28] in order to obtain a discretization of the heat equation at the limit. In the same mind, we also refer to the work of Gosse and Toscani [16]. Numerous applications arising from complex physics require the development of AP schemes: gas dynamics [4], radiative transfert M1 model [5, 6], chemotaxis [2, 15, 18, 29], hydrodynamics [11, 17].

In general, the asymptotic preserving property is obtained by performing a formal Chapmann Enskog expansion. Recently, Mathis and Therme adopted in [25] the relative entropy technique to analyse the diffusive limit of a finite volume scheme for the linear Goldstein Taylor system on a bounded domain. In [1], a similar approach was considered to study the long time asymptotic behaviour of a semi discrete scheme for the nonlinear system (1.7). In addition, the authors established a convergence rate, according to (1.12). Let us underline that the time discretization may introduce additionnal error of approximation which make wrong some semi discrete results. We also mention that the relative entropy can be applied in other numerical contexts (see for instance [7, 23]). Recently, in [12, 13], authors used a discrete relative entropy estimate to control the convergence error of the approximation of the barotropic Navier-Stokes equations.

The main objective of the present paper is the full derivation of the asymptotic convergence rate given by (1.12) in the context of the numerical scheme proposed by Jin *et al.* [22]. At the discrepancy with [1], the time discretization involves new error terms, which necessitate a particular attention to be controlled. To address such an issue, in the next section, the scheme under consideration is detailed and our main result is stated. The sequel of the paper is devoted to the establishment of the convergence rate. To give short keys of the method, we first derive the discrete evolution law satisfied by the relative entropy. Next, we exhibit the evolution of the discrete space average of the relative entropy which is governed by a dissipative equation. The result is then obtained

by relevant estimations of the dissipative terms. This proof is the aim of Section 3 while Section 4 is devoted to the proof of some technical lemmas.
In the last section, several numerical experiments illustrate the relevance of the main result.

2 Scheme and convergence rate result

For the sake of completeness, let us introduce the here adopted scheme given in [1, 22] to approximate the weak solutions of (1.7). Such numerical approach is based on a suitable reformulation of (1.7) as follows:

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = -\dfrac{\sigma}{\varepsilon^2} u - \dfrac{1-\varepsilon^2}{\varepsilon^2} \partial_x p(\tau). \end{cases}$$

Considering this reformulation, a 2-step splitting technique is derived. The first step consists in approximating the solutions of the following hyperbolic system:

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = 0, \end{cases} \tag{2.13}$$

while, in the second step we approximate the solutions of:

$$\begin{cases} \partial_t \tau = 0, \\ \partial_t u = -\dfrac{\sigma}{\varepsilon^2} u - \dfrac{1-\varepsilon^2}{\varepsilon^2} \partial_x p(\tau). \end{cases} \tag{2.14}$$

Now, we detail the discretization of each step. We consider a uniform mesh made of cells $(x_{i-1/2}, x_{i+1/2})_{i\in\mathbb{Z}}$ of constant size Δx. We denote by Δt the time increment, with $t^{n+1} = t^n + \Delta t$ for all $n \in \mathbb{N}$.

Concerning the first step, we use a HLL scheme [19] given by:

$$\begin{cases} \tau_i^{n+1/2} = \tau_i^n + \dfrac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \dfrac{\lambda \Delta t}{2\Delta x}\left(\tau_{i+1}^n - 2\tau_i^n + \tau_{i-1}^n\right), \\ u_i^{n+1/2} = u_i^n - \dfrac{\Delta t}{2\Delta x}\left(p(\tau_{i+1}^n) - p(\tau_{i-1}^n)\right) + \dfrac{\lambda \Delta t}{2\Delta x}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \end{cases} \tag{2.15}$$

where we have set:

$$\lambda \leq \sup_{0\leq n\leq N} \max_{i\in\mathbb{Z}} \left(\sqrt{-p'(\tau_i^n)}\right).$$

As usual the time step is restricted according to the following CFL like condition:

$$\frac{\Delta t}{\Delta x}\lambda \leq \frac{1}{2}.$$

Next, concerning the second step, the system (2.14) is approximated with an implicit method:

$$\begin{cases} \tau_i^{n+1} = \tau_i^{n+1/2}, \\ \dfrac{u_i^{n+1} - u_i^{n+1/2}}{\Delta t} = -\dfrac{\sigma}{\varepsilon^2} u_i^{n+1} - \dfrac{1-\varepsilon^2}{\varepsilon^2} \dfrac{p\left(\tau_{i+1}^{n+1}\right) - p\left(\tau_{i-1}^{n+1}\right)}{2\Delta x}. \end{cases} \tag{2.16}$$

From (2.15) and (2.16), we get the complete scheme which reads:

$$\begin{cases} \tau_i^{n+1} = \tau_i^n + \dfrac{\lambda\Delta t}{2\Delta x}\left(\tau_{i+1}^n - 2\tau_i^n + \tau_{i-1}^n\right) + \dfrac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right), \\ u_i^{n+1} = u_i^n - \dfrac{\sigma\Delta t}{\varepsilon^2}u_i^{n+1} - \dfrac{\Delta t}{2\Delta x}\left(\left(p(\tau_{i+1}^n) - p(\tau_{i-1}^n)\right) + \dfrac{(1-\varepsilon^2)}{\varepsilon^2}\left(p\left(\tau_{i+1}^{n+1}\right) - p\left(\tau_{i-1}^{n+1}\right)\right)\right) \\ \qquad + \dfrac{\lambda\Delta t}{2\Delta x}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right). \end{cases} \tag{2.17}$$

Equipped with (2.17) to approximate the solutions of (1.7), we now exhibit a discretization of the diffusive limit system (1.6). This asymptotic scheme is defined as the limit of (2.17) when ε tends to zero. Straightforward computations give the following approximation of $\overline{w}$ solution of (1.6):

$$\begin{cases} \overline{\tau}_i^{n+1} = \overline{\tau}_i^n + \dfrac{\Delta t}{2\Delta x}\left(\overline{u}_{i+1}^n - \overline{u}_{i-1}^n\right) + \dfrac{\lambda\Delta t}{2\Delta x}\left(\overline{\tau}_{i+1}^n - 2\overline{\tau}_i^n + \overline{\tau}_{i-1}^n\right), \\ \overline{u}_i^{n+1} = -\dfrac{1}{2\sigma\Delta x}\left(p(\overline{\tau}_{i+1}^{n+1}) - p(\overline{\tau}_{i-1}^{n+1})\right). \end{cases} \tag{2.18}$$

Because $\overline{\tau}_{i+1}^n - 2\overline{\tau}_i^n + \overline{\tau}_{i-1}^n = O(\Delta x^2)$ coincides with numerical viscosity, this scheme is naturally consistent with the limit regime equations (1.6).

As usual, according to [1, 24], the discrete numerical solution is assumed to satisfy the following limits for all $n \in \mathbb{N}$:

$$\begin{aligned} \lim_{i\to\pm\infty} \tau_i^n &= \lim_{i\to\pm\infty} \overline{\tau}_i^n = \tau_\pm, \\ \lim_{i\to\pm\infty} u_i^n &= \lim_{i\to\pm\infty} \overline{u}_i^n = 0. \end{aligned} \tag{2.19}$$

For the sake of simplicity in the notations, we set $w_i^n = {}^t(\tau_i^n, u_i^n)$ and $\overline{w}_i^n = {}^t(\overline{\tau}_i^n, \overline{u}_i^n)$. In addition, let us introduce $\Delta X_i^n = X_i^n - \overline{X}_i^n$ and the following discrete derivative operators:

$$\begin{aligned} \delta_t X_i^{n+1/2} &= \frac{X_i^{n+1} - X_i^n}{\Delta t}, & \delta_x X_i^n &= \frac{X_{i+1}^n - X_{i-1}^n}{2\Delta x}, \\ \delta_{xx} X_i^n &= \frac{X_{i+1}^n - 2X_i^n + X_{i-1}^n}{\Delta x^2}, & \tilde{\delta}_x X_{i+1/2}^n &= \frac{X_{i+1}^n - X_i^n}{\Delta x}. \end{aligned} \tag{2.20}$$

Endowed with these notations, both schemes (2.17) and (2.18) now read:

$$\begin{cases} \delta_t \tau_i^{n+1/2} = \delta_x u_i^n + \dfrac{\lambda}{2}\Delta x \delta_{xx}\tau_i^n, \\ \varepsilon^2 \delta_t u_i^{n+1/2} = -\sigma u_i^{n+1} - \delta_x p\left(\tau^{n+1}\right)_i + \varepsilon^2 \Delta t \delta_{tx} p(\tau)_i^{n+1/2} + \dfrac{\lambda\varepsilon^2}{2}\Delta x \delta_{xx} u_i^n, \end{cases} \tag{2.21}$$

and

$$\begin{cases} \delta_t \overline{\tau}_i^{n+1/2} = \delta_x \overline{u}_i^n + \dfrac{\lambda}{2}\Delta x \delta_{xx}\overline{\tau}_i^n, \\ \overline{u}_i^{n+1} = -\dfrac{1}{\sigma}\delta_x p\left(\overline{\tau}^{n+1}\right)_i. \end{cases} \tag{2.22}$$

In order to present the required asymptotic convergence rate from (2.21) to (2.22), we define the measuring distance between w_i^n and $\overline{w}_i^n$ as follows:

$$\phi_\varepsilon^n = \sum_{i\in\mathbb{Z}} \eta_i^{\varepsilon,n}\Delta x, \tag{2.23}$$

where $\eta_i^{\varepsilon,n}$ is the discrete relative entropy given by:

$$\eta_i^{\varepsilon,n} = \eta^\varepsilon(w_i^n|\overline{w}_i^n) = \frac{\varepsilon^2}{2}(u_i^n - \overline{u}_i^n)^2 - P(\tau_i^n|\overline{\tau}_i^n), \tag{2.24}$$

and where $P(\tau|\overline{\tau})$ writes:

$$P(\tau|\overline{\tau}) = P(\tau) - P(\overline{\tau}) - p(\overline{\tau})(\tau - \overline{\tau}). \tag{2.25}$$

Moreover, to simplify the forthcoming estimations, we introduce the two following norms:

$$\begin{aligned} \|X^n\|_{L_x^2} &= \left(\sum_{i\in\mathbb{Z}} (X_i^n)^2 \Delta x\right)^{1/2}, \\ \|X^n\|_{L_x^\infty} &= \sup_{i\in\mathbb{Z}} |X_i^n|. \end{aligned} \tag{2.26}$$

Equipped with these notations, we now give the expected convergence result.

Theorem 2.1 *Let $T = (N+1)\Delta t$ the final time. Let w_i^n be given by the scheme (2.21) and $\overline{w}_i^n$ be given by the asymptotic scheme (2.22).*
We assume the existence of a positive constant K such that:

$$\|\delta_t \overline{u}^{n+1/2}\|_{L_x^2}^2, \quad \|\delta_{xx}\overline{u}^n\|_{L_x^2}^2, \quad \|\delta_t \overline{\tau}^{n+1/2}\|_{L_x^\infty}, \quad \|\tilde{\delta}_x \overline{\tau}^n\|_{L_x^\infty} \leq K. \tag{2.27}$$

We assume the existence of a positive constant L_τ such that specific volumes are bounded as follows:

$$\frac{1}{L_\tau} \leq \tau_i^n, \overline{\tau}_i^n \leq L_\tau \ \forall\ i \in \mathbb{Z},\ 0 \leq n \leq N. \tag{2.28}$$

We assume the existence of a positive constant L_p such that the pressure p and its three first derivatives are bounded as follows:

$$\begin{aligned} &\frac{1}{L_p} \leq p(\tau) \leq L_p, && -L_p \leq p'(\tau) \leq -\frac{1}{L_p}, \\ &\frac{1}{L_p} \leq p''(\tau) \leq L_p, && -L_p \leq p^{(3)}(\tau) \leq -\frac{1}{L_p}, \end{aligned} \quad \forall\ \tau \in [1/L_\tau, L_\tau]. \tag{2.29}$$

We assume that the time step Δt is restricted according to the following parabolic CFL condition:

$$\frac{\Delta t}{\Delta x^2} \leq C_p, \tag{2.30}$$

where

$$C_p = \frac{\sigma}{4L_p^3 + 45L_p^2 + 14/3L_p + 32/3L_pL_\tau^2 + 49/3L_pL_\tau + 4/K + 6}. \tag{2.31}$$

We moreover assume that ε and Δt satisfy:

$$\varepsilon^2 \leq \min\left(\frac{\sigma}{C_p\left(2 + 15L_p^2\right)}, \frac{\sigma}{8\lambda}\Delta x\right), \tag{2.32}$$

and

$$\Delta t \leq \frac{1}{K}. \tag{2.33}$$

Then the following convergence rate holds:

$$\phi_\varepsilon^{N+1} \leq M\left(\phi_\varepsilon^0 + \|u^0 - \overline{u}^0\|_{L_x^2}^2 + \varepsilon^4\right), \tag{2.34}$$

where M is a positive constant only depending on T and the parameters σ, λ, K, L_τ and L_p.

The estimation (2.34) contains the convergence rate ε^4 and the term $\phi_\varepsilon^0 + \|u^0 - \overline{u}^0\|_{L_x^2}^2$ which corresponds to the error at time $t=0$ for general initial data. In the sequel, well prepared initial data must satisfy

$$\phi_\varepsilon^0 + \|u^0 - \overline{u}^0\|_{L_x^2}^2 = 0. \tag{2.35}$$

As expected, according to the convergence rate (1.12) established by Lattanzio and Tzavaras in [24], the above result ensures an ε^4 convergence rate as long as the initial data are well prepared. To reach this result, hypotheses have been imposed. Similarly to [24] (see also [1]), assumptions must be put on $\overline{w}_i^n$. Essentially, these conditions allow a control of the numerical viscosity. Because of the *expected smoothness* satisfied by $\overline{w}_i^n$, these assumptions are natural. Moreover, in [1, 24], conditions are imposed on the two first derivatives of the pressure. Once again, to control the numerical viscosity due to the time discretization, we here need an estimation of the third derivative of the pressure. Let us underline that the classical pressure function $p(\tau) = \tau^{-\gamma}$ with $\gamma > 1$ satisfies these conditions. Actually, the most restrictive assumption stated in Theorem 2.1 concerns τ_i^n since we impose (2.28). Let us underline that hypotheses (2.32) and (2.33) are not restrictive at all. Finally, the parabolic CFL condition (2.30) is relevant according to the numerical experiments performed with the scheme (2.17) and natural since we consider an explicit discretization of a diffusive problem. This CFL restriction is discussed in Section 5 devoted to numerical experiments.

3 Proof of Theorem 2.1

In order to establish the convergence rate (2.34), several technical lemmas are needed. They will be proved in the next section.

We first exhibit an evolution law satisfied by the measuring distance ϕ_ε^n defined by (2.23). To address such an issue, in the following lemma, we state a discrete evolution law satisfied by the relative entropy.

Lemma 3.1 *Let w_i^n be given by (2.21) and $\overline{w}_i^n$ be given by (2.22). The discrete relative entropy $\eta_i^{\varepsilon,n}$, defined by (2.24), verifies the following expression:*

$$\begin{aligned}\delta_t \eta_i^{\varepsilon,n+1/2} + \frac{1}{\Delta x}\left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1}\right) = &-\sigma\left(\Delta u_i^{n+1}\right)^2 - \frac{\varepsilon^2}{2}\Delta t\left(\delta_t \Delta u_i^{n+1/2}\right)^2 \\ &+ R_i^{n,1} + R_i^{n,2} + R_i^{n,3} + R_i^{n,4},\end{aligned} \tag{3.36}$$

with the residues given by:

$$\begin{aligned}
R_i^{n,1} =& -\varepsilon^2 \Delta u_i^{n+1} \delta_t \overline{u}_i^{n+1/2} + \varepsilon^2 \Delta t \Delta u_i^{n+1} \delta_{tx} p(\overline{\tau})_i^{n+1/2} + \frac{\lambda}{2} \varepsilon^2 \Delta x \Delta u_i^{n+1} \delta_{xx} \overline{u}_i^n, \\
R_i^{n,2} =& \varepsilon^2 \Delta t \Delta u_i^{n+1} \delta_{tx} \Delta p(\tau)_i^{n+1/2} + \frac{\lambda}{2} \varepsilon^2 \Delta x \Delta u_i^{n+1} \delta_{xx} \Delta u_i^n, \\
R_i^{n,3} =& \Delta t \Delta p(\tau_i^{n+1}) \delta_{tx} \Delta u_i^{n+1/2} - \frac{\lambda}{2} \Delta x \Delta p(\tau_i^{n+1}) \delta_{xx} \Delta \tau_i^n, \\
R_i^{n,4} =& \left(-Q_i^{n+1/2} + \overline{Q}_i^{n+1/2} + \overline{q}_i^{n+1/2} \Delta \tau_i^n \right) \delta_t \overline{\tau}_i^{n+1/2} + \left(p(\tau_i^{n+1}) - Q_i^{n+1/2} \right) \delta_t \Delta \tau_i^{n+1/2},
\end{aligned} \tag{3.37}$$

where we have set:

$$Q_i^{n+1/2} = \frac{P(\tau_i^{n+1}) - P(\tau_i^n)}{\tau_i^{n+1} - \tau_i^n}, \quad \overline{Q}_i^{n+1/2} = \frac{P(\overline{\tau}_i^{n+1}) - P(\overline{\tau}_i^n)}{\overline{\tau}_i^{n+1} - \overline{\tau}_i^n}, \quad \overline{q}_i^{n+1/2} = \frac{p(\overline{\tau}_i^{n+1}) - p(\overline{\tau}_i^n)}{\overline{\tau}_i^{n+1} - \overline{\tau}_i^n}, \tag{3.38}$$

and where the discrete entropy flux writes:

$$\psi_{i+1/2}^{n+1} = \frac{1}{2} \left(\Delta u_{i+1}^{n+1} \Delta p(\tau_i^{n+1}) + \Delta u_i^{n+1} \Delta p(\tau_{i+1}^{n+1}) \right). \tag{3.39}$$

According to [1, 24], the relation (3.36) is now numerically integrated in time and space. By summation of (3.36), we obtain:

$$\begin{aligned}
\delta_t \phi_\varepsilon^{n+1/2} + \sum_{i \in \mathbb{Z}} \left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1} \right) =& -\sigma \sum_{i \in \mathbb{Z}} \left(\Delta u_i^{n+1} \right)^2 \Delta x - \frac{\varepsilon^2}{2} \Delta t \sum_{i \in \mathbb{Z}} \left(\delta_t \Delta u_i^{n+1/2} \right)^2 \Delta x \\
&+ \sum_{i \in \mathbb{Z}} R_i^{n,1} \Delta x + \sum_{i \in \mathbb{Z}} R_i^{n,2} \Delta x + \sum_{i \in \mathbb{Z}} R_i^{n,3} \Delta x + \sum_{i \in \mathbb{Z}} R_i^{n,4} \Delta x.
\end{aligned} \tag{3.40}$$

Since (2.19) holds, we have:

$$\sum_{i \in \mathbb{Z}} \left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1} \right) = 0.$$

Using the definition of the discrete L^2 norm (2.26), we numerically integrate (3.40) with respect to time to get:

$$\begin{aligned}
\sum_{n=0}^{N} \delta_t \phi_\varepsilon^{n+1/2} \Delta t =& -\sigma \sum_{n=0}^{N} \| \Delta u^{n+1} \|_{L_x^2}^2 \Delta t - \frac{\varepsilon^2}{2} \Delta t \sum_{n=0}^{N} \| \delta_t \Delta u^{n+1/2} \|_{L_x^2}^2 \Delta t \\
&+ \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} R_i^{n,1} \Delta x \Delta t + \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} R_i^{n,2} \Delta x \Delta t + \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} R_i^{n,3} \Delta x \Delta t + \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} R_i^{n,4} \Delta x \Delta t.
\end{aligned}$$

Since we have:

$$\sum_{n=0}^{N} \delta_t \phi_\varepsilon^{n+1/2} \Delta t = \phi_\varepsilon^{N+1} - \phi_\varepsilon^0,$$

we easily obtain:

$$\phi_\varepsilon^{N+1} = \phi_\varepsilon^0 - \sigma \sum_{n=0}^{N} \|\Delta u^{n+1}\|_{L_x^2}^2 \Delta t - \frac{\varepsilon^2}{2}\Delta t \sum_{n=0}^{N} \|\delta_t \Delta u^{n+1/2}\|_{L_x^2}^2 \Delta t + \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,1}\Delta x \Delta t$$
$$+ \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,2}\Delta x \Delta t + \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,3}\Delta x \Delta t + \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,4}\Delta x \Delta t. \tag{3.41}$$

Now our objective is to state an estimation of ϕ_ε^n of the following form:

$$\phi_\varepsilon^{N+1} \le \alpha\left(\phi_\varepsilon^0 + \|\Delta u^0\|_{L_x^2}^2 + \varepsilon^4\right) + \beta \sum_{n=0}^{N} \phi_\varepsilon^n \Delta t, \tag{3.42}$$

where α and β are two positive constants only depending on the final time $T=(N+1)\Delta t$ and the parameters σ, λ, K, L_τ and L_p. Indeed, as soon as such an inequality is obtained, a discrete Gronwall lemma immediatly yields:

$$\phi_\varepsilon^{N+1} \le \alpha e^{\beta T}\left(\phi_\varepsilon^0 + \|\Delta u^0\|_{L_x^2}^2 + \varepsilon^4\right),$$

and the proof is thus achieved.

We now need to obtain a control of each residue, essentially by either ε^4, initial data, or the nonpositive terms in (3.41). In the following, we give the sequence of required estimations.

Lemma 3.2 *Assume that $\|\delta_t \overline{u}^{n+1/2}\|_{L_x^2}^2 \le K$, $\|\delta_{tx} p(\overline{\tau})^{n+1/2}\|_{L_x^2}^2 \le K$ and $\|\delta_{xx}\overline{u}^n\|_{L_x^2}^2 \le K$ for all $0\le n\le N$, with K a positive constant. Then for all $\theta>0$, we have:*

$$\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,1}\Delta x\Delta t \le \theta \sum_{n=0}^{N} \|\Delta u^{n+1}\|_{L_x^2}^2 \Delta t + \frac{KT}{4\theta}\left(1+\sigma\Delta t + \frac{\lambda}{2}\Delta x\right)^2 \varepsilon^4. \tag{3.43}$$

Lemma 3.3 *Assume that $\|\delta_t \overline{\tau}^{n+1/2}\|_{L_x^\infty} \le K$ for all $0\le n\le N$, with K a positive constant and the conditions on the pressure (2.29). Then we have:*

$$\begin{aligned}\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,2}\Delta x\Delta t \le& \frac{\varepsilon^2}{\Delta x}\left(\frac{\Delta t}{2\Delta x}+\lambda\right)\sum_{n=0}^{N}\|\Delta u^{n+1}\|_{L_x^2}^2\Delta t \\ &+ \frac{\varepsilon^2}{\Delta x}\left(\frac{3L_p^2\Delta t}{\Delta x}\left(1+\frac{K^2\Delta t^2}{4}\right)+\lambda\right)\sum_{n=0}^{N}\|\Delta u^n\|_{L_x^2}^2\Delta t \\ &+ \frac{3L_p^3\varepsilon^2\Delta t}{\Delta x^2}\left(8\lambda^2+2\lambda^2K^2\Delta t^2+K^2\Delta x^2\right)\sum_{n=0}^{N}\phi_\varepsilon^n\Delta t.\end{aligned} \tag{3.44}$$

Lemma 3.4 *Assume that $\|\tilde{\delta}_x \overline{\tau}^n\|_{L_x^\infty} \le K$ for all $0\le n\le N$, with K a positive constant* *and the conditions on the pressure* (2.29). *Then we have:*

$$\begin{aligned}\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,3}\Delta x\Delta t \leq{}& \frac{1}{2}\phi_\varepsilon^{N+1} + \frac{L_p^3\Delta t}{\Delta x^2}\sum_{n=0}^{N}\|\Delta u^{n+1}\|_{L_x^2}^2\Delta t \\ &+ \frac{\Delta t}{\Delta x^2}\left(9L_p^2\left(1+\frac{K^2\Delta t^2}{4}\right)+\frac{1}{2}\right)\sum_{n=0}^{N}\|\Delta u^n\|_{L_x^2}^2\Delta t + \frac{L_p^3\Delta t}{2\Delta x^2}\|\Delta u^0\|_{L_x^2}^2 \\ &+ \left(\frac{9L_p^3\Delta t}{\Delta x^2}\left(8\lambda^2+2\lambda^2K^2\Delta t^2+K^2\Delta x^2\right)+1+2L_p\left(\frac{\lambda^2\Delta t}{\Delta x^2}+2\lambda K L_p\right)\right)\sum_{n=0}^{N}\phi_\varepsilon^n\Delta t.\end{aligned} \tag{3.45}$$

Lemma 3.5 *Assume that* $\|\delta_t\overline{\tau}^{n+1/2}\|_{L_x^\infty}\leq K$ *for all* $0\leq n\leq N$, *with* K *a positive constant, the conditions on specific volumes* (2.28) *and the conditions on the pressure* (2.29). *Then we have:*

$$\begin{aligned}\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}} R_i^{n,4}\Delta x\Delta t \leq{}& \Bigg(L_p\left(\frac{7}{3}L_\tau+K\Delta t+\frac{1}{6}K^2\Delta t^2+\frac{7}{4}KL_\tau\Delta t+\frac{8}{3}L_\tau^2\right) \\ &+\Delta t\Bigg)\frac{\Delta t}{\Delta x^2}\sum_{n=0}^{N}\|\Delta u^n\|_{L_x^2}^2\Delta t + L_p\Bigg(\frac{8\lambda^2\Delta t}{\Delta x^2}\Bigg(L_p\left(\frac{7}{3}L_\tau+K\Delta t+\frac{1}{6}K^2\Delta t^2+\frac{7}{4}KL_\tau\Delta t\right. \\ &\left.+\frac{8}{3}L_\tau^2\right)+\Delta t\Bigg)+KL_p\left(KL_p\left(\frac{1}{2}+\frac{1}{3}K\Delta t\right)^2+K\Delta t+2L_\tau+1\right)\Bigg)\sum_{n=0}^{N}\phi_\varepsilon^n\Delta t.\end{aligned} \tag{3.46}$$

After tedious computations, the estimations of the four above lemmas (3.43)–(3.46) allow us to rewrite (3.41) as follows:

$$\begin{aligned}\phi_\varepsilon^{N+1}\leq{}&\phi_\varepsilon^0+\frac{1}{2}\phi_\varepsilon^{N+1}+a_1\sum_{n=0}^{N}\|\Delta u^{n+1}\|_{L_x^2}^2\Delta t+a_2\sum_{n=0}^{N}\|\Delta u^n\|_{L_x^2}^2\Delta t \\ &+a_3\varepsilon^4+a_4\sum_{n=0}^{N}\phi_\varepsilon^n\Delta t+a_5\|\Delta u^0\|_{L_x^2}^2,\end{aligned} \tag{3.47}$$

where a_1, a_2, a_3, a_4 and a_5 write:

$$\begin{aligned}a_1 ={}& -\sigma+\theta+\frac{\varepsilon^2}{\Delta x}\left(\frac{C_p}{2}\Delta x+\lambda\right)+L_p^3C_p, \\ a_2 ={}& \frac{\varepsilon^2}{\Delta x}\left(3L_p^2C_p\Delta x\left(1+\frac{K^2\Delta t^2}{4}\right)+\lambda\right)+C_p\left(9L_p^2\left(1+\frac{K^2\Delta t^2}{4}\right)+\frac{1}{2}\right) \\ &+C_p\left(L_p\left(\frac{7}{3}L_\tau+K\Delta t+\frac{1}{6}K^2\Delta t^2+\frac{7}{4}KL_\tau\Delta t+\frac{8}{3}L_\tau^2\right)+\Delta t\right), \\ a_3 ={}& \frac{KT}{4\theta}\left(1+\sigma\Delta t+\frac{\lambda}{2}\Delta x\right)^2, \\ a_4 ={}& \left(3+\varepsilon^2\right)3L_p^3C_p\left(8\lambda^2+2\lambda^2K^2\Delta t^2+K^2\Delta x^2\right)+1+2L_p\left(\lambda^2C_p+2\lambda KL_p\right) \\ &+L_p\Bigg[8\lambda^2C_p\left(L_p\left(\frac{7}{3}L_\tau+K\Delta t+\frac{1}{6}K^2\Delta t^2+\frac{7}{4}KL_\tau\Delta t+\frac{8}{3}L_\tau^2\right)+\Delta t\right)\end{aligned}$$

$$+KL_p\left(KL_p\left(\frac{1}{2}+\frac{1}{3}K\Delta t\right)^2+K\Delta t+2L_\tau+1\right)\Bigg],$$
$$a_5=\frac{L_p^3C_p}{2},$$

with C_p given by (2.31).

A particular attention must be paid on $a_2\sum_{n=0}^{N}\|\Delta u^n\|^2_{L^2_x}\Delta t$. Performing a discrete integration by parts, this sum rewrites:

$$\begin{aligned}a_2\sum_{n=0}^{N}\|\Delta u^n\|^2_{L^2_x}\Delta t=&a_2\sum_{n=0}^{N}\left(\|\Delta u^n\|^2_{L^2_x}-\|\Delta u^{n+1}\|^2_{L^2_x}\right)\Delta t+a_2\sum_{n=0}^{N}\|\Delta u^{n+1}\|^2_{L^2_x}\Delta t\\ =&-a_2\|\Delta u^{N+1}\|^2_{L^2_x}\Delta t+a_2\|\Delta u^0\|^2_{L^2_x}\Delta t+a_2\sum_{n=0}^{N}\|\Delta u^{n+1}\|^2_{L^2_x}\Delta t\\ \leq&\ a_2\|\Delta u^0\|^2_{L^2_x}\Delta t+a_2\sum_{n=0}^{N}\|\Delta u^{n+1}\|^2_{L^2_x}\Delta t.\end{aligned}$$

Hence we have:

$$\begin{aligned}\frac{1}{2}\phi_\varepsilon^{N+1}\leq&\phi_\varepsilon^0+(a_1+a_2)\sum_{n=0}^{N}\|\Delta u^{n+1}\|^2_{L^2_x}\Delta t+a_3\varepsilon^4+a_4\sum_{n=0}^{N}\phi_\varepsilon^n\Delta t\\ &+(a_2\Delta t+a_5)\|\Delta u^0\|^2_{L^2_x}.\end{aligned}\tag{3.48}$$

Choosing $\theta=C_p$ and under the conditions (2.30) and (2.32), we have

$$a_1+a_2\leq 0,$$

and the inequality (3.42) is obtained. The proof of Theorem 2.1 is achieved.

4 Proof of technical lemmas

This section is devoted to the proof of technical lemmas 3.1 to 3.5. For sake of clarity in the presentation, we first establish intermediate results and then we give successively the required proofs.

Lemma 4.1 *Let us assume the hypotheses of lemmas 3.2 to 3.5.*

(i) Let $(X_i^n)_{i\in\mathbb{Z},\ 0\leq n\leq N}$ be given. Thus we have:

$$\begin{gathered}\|\delta_x X^n\|_{L^2_x}\leq\frac{1}{\Delta x}\|X^n\|_{L^2_x},\quad\|\tilde\delta_x X^n\|_{L^2_x}\leq\frac{2}{\Delta x}\|X^n\|_{L^2_x},\\ \|\delta_{xx}X^n\|_{L^2_x}\leq\frac{4}{\Delta x^2}\|X^n\|_{L^2_x}.\end{gathered}\tag{4.49}$$

(ii) We have the following estimations:

$$\|\Delta\tau^n\|^2_{L^2_x}\leq 2L_p\phi_\varepsilon^n,\tag{4.50}$$
$$\Delta t\|\delta_t\Delta\tau^{n+1/2}\|^2_{L^2_x}\leq\frac{2\Delta t}{\Delta x^2}\left(\|\Delta u^n\|^2_{L^2_x}+8\lambda^2L_p\phi_\varepsilon^n\right).\tag{4.51}$$

(iii) The quantity $\tilde{\delta}_x \Delta p(\tau^n)_{i+1/2}$ satisfies the following relation:

$$\begin{aligned}\tilde{\delta}_x \Delta p(\tau^n)_{i+1/2} = &\tilde{\delta}_x \Delta \tau^n_{i+1/2} \int_0^1 p'(\tau_i^n + s\Delta x \tilde{\delta}_x \tau^n_{i+1/2}) ds \\ &+ \tilde{\delta}_x \overline{\tau}^n_{i+1/2} \Delta \tau_i^n \int_0^1 I^n_{i+1/2,s} ds + \Delta x \tilde{\delta}_x \overline{\tau}^n_{i+1/2} \tilde{\delta}_x \Delta \tau^n_{i+1/2} \int_0^1 s I^n_{i+1/2,s} ds,\end{aligned} \tag{4.52}$$

where

$$I^n_{i+1/2,s} = \int_0^1 p'' \left(\overline{\tau}_i^n + s\Delta x \tilde{\delta}_x \overline{\tau}^n_{i+1/2} + t\left(\Delta \tau_i^n + s\Delta x \tilde{\delta}_x \Delta \tau^n_{i+1/2}\right)\right) dt. \tag{4.53}$$

(iv) The quantity $\Delta t \|\delta_t \Delta p(\tau)^{n+1/2}\|^2_{L^2_x}$ is estimated as follows:

$$\begin{aligned}\Delta t \|\delta_t \Delta p(\tau)^{n+1/2}\|^2_{L^2_x} \le &\frac{6L_p^2 \Delta t}{\Delta x^2}\left(1 + \frac{K^2 \Delta t^2}{4}\right) \|\Delta u^n\|^2_{L^2_x} \\ &+ \frac{6L_p^3 \Delta t}{\Delta x^2}\left(8\lambda^2 + 2\lambda^2 K^2 \Delta t^2 + K^2 \Delta x^2\right) \phi_\varepsilon^n.\end{aligned} \tag{4.54}$$

Proof. **(Proof of Lemma 4.1.)** The proof of *(i)* follows from direct computations using the definition of the discrete derivative operators.

Concerning *(ii)*, from the definition of ϕ_ε^n and $\eta_i^{\varepsilon,n}$ we have:

$$\phi_\varepsilon^n = \sum_{i \in \mathbb{Z}} \left(\frac{\varepsilon^2}{2} (\Delta u_i^n)^2 - P(\tau_i^n | \overline{\tau}_i^n)\right) \Delta x \ge - \sum_{i\in\mathbb{Z}} P(\tau_i^n|\overline{\tau}_i^n) \Delta x.$$

Moreover the definition of $P(\tau_i^n|\overline{\tau}_i^n)$ and a Taylor expansion directly gives:

$$P(\tau_i^n|\overline{\tau}_i^n) = (\Delta \tau_i^n)^2 \int_0^1 (1-s) p'(\overline{\tau}_i^n + s\Delta\tau_i^n) ds.$$

Since the pressure function satisfies (2.29), we get:

$$P(\tau_i^n|\overline{\tau}_i^n) \le -\frac{1}{2L_p}(\Delta\tau_i^n)^2,$$

to write

$$\phi_\varepsilon^n \ge \frac{1}{2L_p} \sum_{i\in\mathbb{Z}} (\Delta \tau_i^n)^2 \Delta x,$$

and the inequality (4.50) is proved.

Concerning the establishment of (4.51), from (2.21) and (2.22), we have the evolution law satisfied by $\Delta\tau_i^n$ as follows:

$$\delta_t \Delta \tau_i^n = \delta_x \Delta u_i^n + \frac{\lambda}{2} \Delta x \delta_{xx} \Delta \tau_i^n.$$

Then the following equality holds:

$$\Delta t \|\delta_t \Delta \tau^{n+1/2}\|^2_{L^2_x} = \Delta t \sum_{i\in\mathbb{Z}} \left(\delta_x \Delta u_i^n + \frac{\lambda}{2}\Delta x \delta_{xx} \Delta \tau_i^n\right)^2 \Delta x.$$

Since for all $a,b\in\mathbb{R}$ we have $(a+b)^2\le 2(a^2+b^2)$, the following estimation is satisfied:

$$\Delta t\|\delta_t\Delta\tau^{n+1/2}\|^2_{L^2_x}\le 2\Delta t\sum_{i\in\mathbb{Z}}\left((\delta_x\Delta u_i^n)^2+\frac{\lambda^2}{4}\Delta x^2\,(\delta_{xx}\Delta\tau_i^n)^2\right)\Delta x.$$

By applying estimation (4.49) on $\|\delta_x X^n\|_{L^2_x}$ and on $\|\delta_{xx}X^n\|_{L^2_x}$, we have:

$$\Delta t\|\delta_t\Delta\tau^{n+1/2}\|^2_{L^2_x}\le 2\frac{\Delta t}{\Delta x^2}\left(\|\Delta u_i^n\|^2_{L^2_x}+4\lambda^2\|\Delta\tau_i^n\|^2_{L^2_x}\right).$$

The expected result (4.51) easily comes from inequality (4.50).

Concerning *(iii)*, by definition of the discrete operator $\tilde\delta_x$, we have:

$$\tilde\delta_x\Delta p(\tau^n)_{i+1/2}=\frac{1}{\Delta x}\left(\left(p(\tau_{i+1}^n)-p(\tau_i^n)\right)-\left(p(\overline\tau_{i+1}^n)-p(\overline\tau_i^n)\right)\right).$$

Moreover, from a Taylor expansion, the two following equalities are obtained:

$$p(\tau_{i+1}^n)=p(\tau_i^n)+\Delta x\tilde\delta_x\tau_{i+1/2}^n\int_0^1 p'(\tau_i^n+s\partial_x\tilde\delta_x\tau_{i+1/2}^n)ds,$$
$$p(\overline\tau_{i+1}^n)=p(\overline\tau_i^n)+\Delta x\tilde\delta_x\overline\tau_{i+1/2}^n\int_0^1 p'(\overline\tau_i^n+s\partial_x\tilde\delta_x\overline\tau_{i+1/2}^n)ds,$$

to write

$$\begin{aligned}\tilde\delta_x\Delta p(\tau^n)_{i+1/2}=&\tilde\delta_x\tau_{i+1/2}^n\int_0^1 p'(\tau_i^n+s\partial_x\tilde\delta_x\tau_{i+1/2}^n)ds\\&-\tilde\delta_x\overline\tau_{i+1/2}^n\int_0^1 p'(\overline\tau_i^n+s\partial_x\tilde\delta_x\overline\tau_{i+1/2}^n)ds.\end{aligned}$$

Since $\tilde\delta_x\tau_{i+1/2}^n=\tilde\delta_x\Delta\tau_{i+1/2}^n+\tilde\delta_x\overline\tau_{i+1/2}^n$, we obtain:

$$\begin{aligned}\tilde\delta_x\Delta p(\tau^n)_{i+1/2}=&\tilde\delta_x\Delta\tau_{i+1/2}^n\int_0^1 p'(\tau_i^n+s\partial_x\tilde\delta_x\tau_{i+1/2}^n)ds\\&+\tilde\delta_x\overline\tau_{i+1/2}^n\int_0^1\left(p'(\tau_i^n+s\partial_x\tilde\delta_x\tau_{i+1/2}^n)-p'(\overline\tau_i^n+s\Delta x\tilde\delta_x\overline\tau_{i+1/2}^n)\right)ds.\end{aligned}\tag{4.55}$$

Similarly, applying a Taylor expansion, we have:

$$p'(\tau_i^n+s\partial_x\tilde\delta_x\tau_{i+1/2}^n)=p'(\overline\tau_i^n+s\partial_x\tilde\delta_x\overline\tau_{i+1/2}^n)+\left(\Delta\tau_i^n+s\Delta x\tilde\delta_x\Delta\tau_{i+1/2}^n\right)I_{i+1/2,s}^n,$$

with $I_{i+1/2,s}^n$ defined by (4.53). Plugging the above quantity in (4.55) yields to the expected result (4.52).

We achieve the proof of Lemma 4.1 by establishing *(iv)*. Using the same type of arguments based on Taylor expansions, we get:

$$\delta_t\Delta p(\tau_i)^{n+1/2}=\delta_t\Delta\tau_i^{n+1/2}\int_0^1 p'(\tau_i^n+s\Delta t\delta_t\tau_i^{n+1/2})ds$$

$$+\delta_t\overline{\tau}_i^{n+1/2}\Delta\tau_i^n\int_0^1 I_{i,s}^{n+1/2}ds+\Delta t\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1 sI_{i,s}^{n+1/2}ds,$$

where

$$I_{i,s}^{n+1/2}=\int_0^1 p''\left(\overline{\tau}_i^n+s\Delta t\delta_t\overline{\tau}_i^{n+1/2}+t\left(\Delta\tau_i^n+s\Delta t\delta_t\Delta\tau_i^{n+1/2}\right)\right)dt.$$

Thus, by definition (2.26) of $\|.\|_{L^2_x}$, we obtain:

$$\Delta t\|\delta_t\Delta p(\tau)^{n+1/2}\|^2_{L^2_x}=\Delta t\sum_{i\in\mathbb{Z}}\left(\delta_t\Delta\tau_i^{n+1/2}\int_0^1 p'(\tau_i^n+s\Delta t\delta_t\tau_i^{n+1/2})ds\right.$$
$$\left.+\delta_t\overline{\tau}_i^{n+1/2}\Delta\tau_i^n\int_0^1 I_{i,s}^{n+1/2}ds+\Delta t\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1 sI_{i,s}^{n+1/2}ds\right)^2\Delta x.$$

For all $a,b,c\in\mathbb{R}$, we have $(a+b+c)^2\leq 3\left(a^2+b^2+c^2\right)$, so that the above inequality becomes:

$$\Delta t\|\delta_t\Delta p(\tau)^{n+1/2}\|^2_{L^2_x}\leq 3\Delta t\sum_{i\in\mathbb{Z}}\left(\left(\delta_t\Delta\tau_i^{n+1/2}\int_0^1 p'(\tau_i^n+s\Delta t\delta_t\tau_i^{n+1/2})ds\right)^2\right.$$
$$\left.+\left(\delta_t\overline{\tau}_i^{n+1/2}\Delta\tau_i^n\int_0^1 I_{i,s}^{n+1/2}ds\right)^2+\left(\Delta t\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1 sI_{i,s}^{n+1/2}ds\right)^2\right)\Delta x.$$

Moreover, from assumptions (2.29) on the pressure law, we have:

$$\int_0^1 p'(\tau_i^n+s\Delta t\delta_t\tau_i^{n+1/2})ds\leq L_p,\quad \int_0^1 I_{i,s}^{n+1/2}ds\leq L_p,\quad \int_0^1 sI_{i,s}^{n+1/2}ds\leq L_p/2,$$

thus

$$\Delta t\|\delta_t\Delta p(\tau)^{n+1/2}\|^2_{L^2_x}\leq 3L_p^2\Delta t\left(\left(1+\frac{\Delta t^2}{4}\|\delta_t\overline{\tau}^{n+1/2}\|^2_{L^\infty_x}\right)\|\delta_t\Delta\tau^{n+1/2}\|^2_{L^2_x}\right.$$
$$\left.+\|\delta_t\overline{\tau}^{n+1/2}\|^2_{L^\infty_x}\|\Delta\tau^n\|^2_{L^2_x}\right).$$

Finally, since $\|\delta_t\overline{\tau}^{n+1/2}\|_{L^\infty_x}\leq K$ by (2.27), combined with (4.50) and (4.51), the inequality (4.54) is obtained. The proof of Lemma 4.1 is thus achieved. □

Proof. **(Proof of Lemma 3.1.)** By definition of the discrete relative entropy (2.24) and the discrete operator δ_t given by (2.20), we have:

$$\delta_t\eta_i^{\varepsilon,n+1/2}=\frac{\varepsilon^2}{2\Delta t}\left(\left(\Delta u_i^{n+1}\right)^2-\left(\Delta u_i^n\right)^2\right)-\frac{1}{\Delta t}\left(P(\tau_i^{n+1}|\overline{\tau}_i^{n+1})-P(\tau_i^n|\overline{\tau}_i^n)\right).$$

Involving the definition of the relative pressure $P(\tau|\overline{\tau})$ given in (2.25), and since for all $a,b\in\mathbb{R}$ we have $a^2-b^2=2a(a-b)-(a-b)^2$, we obtain:

$$\delta_t\eta_i^{\varepsilon,n+1/2}=\frac{\varepsilon^2}{2\Delta t}\left(2\Delta u_i^{n+1}\left(\Delta u_i^{n+1}-\Delta u_i^n\right)-\left(\Delta u_i^{n+1}-\Delta u_i^n\right)^2\right)$$
$$-\frac{1}{\Delta t}\left(P(\tau_i^{n+1})-P(\overline{\tau}_i^{n+1})-p(\overline{\tau}_i^{n+1})\Delta\tau_i^{n+1}-P(\tau_i^n)+P(\overline{\tau}_i^n)+p(\overline{\tau}_i^n)\Delta\tau_i^n\right).$$

We reformulate the above relation as follows:

$$\begin{aligned}\delta_t\eta_i^{\varepsilon,n+1/2} =& \varepsilon^2\Delta u_i^{n+1}\delta_t\Delta u_i^{n+1/2} - \frac{\varepsilon^2}{2}\Delta t\left(\delta_t\Delta u_i^{n+1/2}\right)^2 - \frac{P(\tau_i^{n+1})-P(\tau_i^n)}{\tau_i^{n+1}-\tau_i^n}\delta_t\tau_i^{n+1/2}\\ &+\frac{P(\overline{\tau}_i^{n+1})-P(\overline{\tau}_i^n)}{\overline{\tau}_i^{n+1}-\overline{\tau}_i^n}\delta_t\overline{\tau}_i^{n+1/2} + p(\overline{\tau}_i^{n+1})\delta_t\Delta\tau_i^{n+1/2} + \frac{p(\overline{\tau}_i^{n+1})-p(\overline{\tau}_i^n)}{\overline{\tau}_i^{n+1}-\overline{\tau}_i^n}\delta_t\overline{\tau}_i^{n+1/2}\Delta\tau_i^n.\end{aligned} \tag{4.56}$$

Moreover, from both schemes (2.21) and (2.22), we derive the following evolution law for Δu_i^n:

$$\begin{aligned}\varepsilon^2\delta_t\Delta u_i^{n+1/2} =& -\sigma\Delta u_i^{n+1} - \delta_x\Delta p\left(\tau^{n+1}\right)_i + \varepsilon^2\Delta t\delta_{tx}p(\tau)_i^{n+1/2}\\ &+\frac{\lambda}{2}\varepsilon^2\Delta x\delta_{xx}u_i^n - \varepsilon^2\delta_t\overline{u}_i^{n+1/2}.\end{aligned}$$

Arguing the notations $Q_i^{n+1/2}$, $\overline{Q}_i^{n+1/2}$, $\overline{q}_i^{n+1/2}$ defined in (3.38), and plugging the above relation equality within (4.56), $\delta_t\eta_i^{\varepsilon,n+1/2}$ now reads:

$$\begin{aligned}\delta_t\eta_i^{\varepsilon,n+1/2} =& -\frac{\varepsilon^2}{2}\Delta t\left(\delta_t\Delta u_i^{n+1/2}\right)^2\\ &+\Delta u_i^{n+1}\left(-\sigma\Delta u_i^{n+1} - \delta_x\Delta p\left(\tau^{n+1}\right)_i + \varepsilon^2\Delta t\delta_{tx}p(\tau)_i^{n+1/2} + \frac{\lambda}{2}\varepsilon^2\Delta x\delta_{xx}u_i^n - \varepsilon^2\delta_t\overline{u}_i^{n+1/2}\right)\\ &-Q_i^{n+1/2}\delta_t\tau_i^{n+1/2} + \overline{Q}_i^{n+1/2}\delta_t\overline{\tau}_i^{n+1/2} + p(\overline{\tau}_i^{n+1})\delta_t\Delta\tau_i^{n+1/2} + \overline{q}_i^{n+1/2}\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}.\end{aligned}$$

We develop the second term to get:

$$\begin{aligned}\delta_t\eta_i^{\varepsilon,n+1/2} =& -\sigma\left(\Delta u_i^{n+1}\right)^2 - \Delta u_i^{n+1}\delta_x\Delta p\left(\tau^{n+1}\right)_i + \varepsilon^2\Delta t\Delta u_i^{n+1}\delta_{tx}p(\tau)_i^{n+1/2}\\ &+\frac{\lambda}{2}\varepsilon^2\Delta x\Delta u_i^{n+1}\delta_{xx}u_i^n - \varepsilon^2\Delta u_i^{n+1}\delta_t\overline{u}_i^{n+1/2} - \frac{\varepsilon^2}{2}\Delta t\left(\delta_t\Delta u_i^{n+1/2}\right)^2\\ &-Q_i^{n+1/2}\delta_t\tau_i^{n+1/2} + \overline{Q}_i^{n+1/2}\delta_t\overline{\tau}_i^{n+1/2} + p(\overline{\tau}_i^{n+1})\delta_t\Delta\tau_i^{n+1/2} + \overline{q}_i^{n+1/2}\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}.\end{aligned} \tag{4.57}$$

The objective now is to correctly control the above quantity according to (3.36). To address such an issue, we reformulate the discrete entropy flux, defined by (3.39), as follows:

$$\begin{aligned}\frac{1}{\Delta x}\left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1}\right) =& \frac{1}{2\Delta x}\left(\Delta u_{i+1}^{n+1}\Delta p(\tau_i^{n+1}) + \Delta u_i^{n+1}\Delta p(\tau_{i+1}^{n+1})\right.\\ &\left.-\Delta u_i^{n+1}\Delta p(\tau_{i-1}^{n+1}) - \Delta u_{i-1}^{n+1}\Delta p(\tau_i^{n+1})\right)\\ =& \Delta p(\tau_i^{n+1})\frac{\Delta u_{i+1}^{n+1} - \Delta u_{i-1}^{n+1}}{2\Delta x} + \Delta u_i^{n+1}\frac{\Delta p(\tau_{i+1}^{n+1}) - \Delta p(\tau_{i-1}^{n+1})}{2\Delta x}\\ =& \Delta p(\tau_i^{n+1})\delta_x\Delta u_i^{n+1} + \Delta u_i^{n+1}\delta_x\Delta p\left(\tau^{n+1}\right)_i,\end{aligned} \tag{4.58}$$

to write

$$\begin{aligned}-\Delta u_i^{n+1}\delta_x\Delta p\left(\tau^{n+1}\right)_i =& -\frac{1}{\Delta x}\left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1}\right)\\ &+\Delta t\Delta p(\tau_i^{n+1})\delta_{tx}\Delta u_i^{n+1/2} + \Delta p(\tau_i^{n+1})\delta_x\Delta u_i^n.\end{aligned}$$

Moreover, using (2.21) and (2.22) we have:

$$\delta_x\Delta u_i^n = \delta_t\Delta\tau_i^{n+1/2} - \frac{\lambda}{2}\Delta x\delta_{xx}\Delta\tau_i^n,$$

and (4.58) becomes:

$$-\Delta u_i^{n+1}\delta_x \Delta p\left(\tau^{n+1}\right)_i = -\frac{1}{\Delta x}\left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1}\right) + \Delta t \Delta p(\tau_i^{n+1})\delta_{tx}\Delta u_i^{n+1/2}$$
$$+\Delta p(\tau_i^{n+1})\delta_t \Delta \tau_i^{n+1/2} - \frac{\lambda}{2}\Delta x \Delta p(\tau_i^{n+1})\delta_{xx}\Delta \tau_i^n.$$

We plug the above expression within (4.57) to obtain:

$$\begin{aligned}
\delta_t \eta_i^{\varepsilon,n+1/2} + \frac{1}{\Delta x}\left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1}\right) = & -\sigma\left(\Delta u_i^{n+1}\right)^2 - \frac{\varepsilon^2}{2}\Delta t\left(\delta_t \Delta u_i^{n+1/2}\right)^2 \\
& + \Delta t \Delta p(\tau_i^{n+1})\delta_{tx}\Delta u_i^{n+1/2} + \Delta p(\tau_i^{n+1})\delta_t \Delta \tau_i^{n+1/2} - \frac{\lambda}{2}\Delta x \Delta p(\tau_i^{n+1})\delta_{xx}\Delta \tau_i^n \\
& + \varepsilon^2 \Delta t \Delta u_i^{n+1}\delta_{tx} p(\tau)_i^{n+1/2}) + \frac{\lambda}{2}\varepsilon^2 \Delta x \Delta u_i^{n+1}\delta_{xx} u_i^n \\
& - \varepsilon^2 \Delta u_i^{n+1}\delta_t \overline{u}_i^{n+1/2} - Q_i^{n+1/2}\delta_t \tau_i^{n+1/2} + \overline{Q}_i^{n+1/2}\delta_t \overline{\tau}_i^{n+1/2} \\
& + p(\overline{\tau}_i^{n+1})\delta_t \Delta \tau_i^{n+1/2} + \overline{q}_i^{n+1/2}\Delta \tau_i^n \delta_t \overline{\tau}_i^{n+1/2}.
\end{aligned} \tag{4.59}$$

We remark that, using (3.37), we have:

$$\Delta t \Delta p(\tau_i^{n+1})\delta_{tx}\Delta u_i^{n+1/2} - \frac{\lambda}{2}\Delta x \Delta p(\tau_i^{n+1})\delta_{xx}\Delta \tau_i^n = R_i^{n,3},$$

and

$$\Delta p(\tau_i^{n+1})\delta_t \Delta \tau_i^{n+1/2} - Q_i^{n+1/2}\delta_t \tau_i^{n+1/2} + \overline{Q}_i^{n+1/2}\delta_t \overline{\tau}_i^{n+1/2} + p(\overline{\tau}_i^{n+1})\delta_t \Delta \tau_i^{n+1/2}$$
$$+\overline{q}_i^{n+1/2}\Delta \tau_i^n \delta_t \overline{\tau}_i^{n+1/2} = R_i^{n,4},$$

and (4.59) rewrites as follows:

$$\begin{aligned}
\delta_t \eta_i^{\varepsilon,n+1/2} + \frac{1}{\Delta x}\left(\psi_{i+1/2}^{n+1} - \psi_{i-1/2}^{n+1}\right) = & -\sigma\left(\Delta u_i^{n+1}\right)^2 - \frac{\varepsilon^2}{2}\Delta t\left(\delta_t \Delta u_i^{n+1/2}\right)^2 \\
& - \varepsilon^2 \Delta u_i^{n+1}\delta_t \overline{u}_i^{n+1/2} + \varepsilon^2 \Delta t \Delta u_i^{n+1}\delta_{tx} p(\overline{\tau})_i^{n+1/2} + \frac{\lambda}{2}\varepsilon^2 \Delta x \Delta u_i^{n+1}\delta_{xx}\overline{u}_i^n \\
& + \varepsilon^2 \Delta t \Delta u_i^{n+1}\delta_{tx}\Delta p(\tau)_i^{n+1/2} + \frac{\lambda}{2}\varepsilon^2 \Delta x \Delta u_i^{n+1}\delta_{xx}\Delta u_i^n \\
& + R_i^{n,3} + R_i^{n,4}.
\end{aligned}$$

We recognize $R_i^{n,1}$ and $R_i^{n,2}$ in the second and third lines and the expected evolution law (3.36) is obtained. □

Proof. **(Proof of Lemma 3.2.)** Arguing the definition (3.37) of $R_i^{n,1}$, we have:

$$\sum_{i\in\mathbb{Z}} R_i^{n,1}\Delta x = -\varepsilon^2 \sum_{i\in\mathbb{Z}} \Delta u_i^{n+1}\delta_t \overline{u}_i^{n+1/2}\Delta x + \varepsilon^2 \Delta t \sum_{i\in\mathbb{Z}} \Delta u_i^{n+1}\delta_{tx} p(\overline{\tau})_i^{n+1/2}\Delta x$$
$$+\frac{\lambda}{2}\varepsilon^2 \Delta x \sum_{i\in\mathbb{Z}} \Delta u_i^{n+1}\delta_{xx}\overline{u}_i^n \Delta x.$$

A Cauchy Schwarz inequality immediatly gives:

$$\sum_{i\in\mathbb{Z}} R_i^{n,1}\Delta x \leq \varepsilon^2 \|\Delta u^{n+1}\|_{L_x^2}\|\delta_t \overline{u}^{n+1/2}\|_{L_x^2} + \varepsilon^2 \Delta t \|\Delta u^{n+1}\|_{L_x^2}\|\delta_x \delta_t p(\overline{\tau})^{n+1/2}\|_{L_x^2}$$

$$+\frac{\lambda}{2}\varepsilon^2\Delta x\|\Delta u^{n+1}\|_{L^2_x}\|\delta_{xx}\overline{u}^n\|_{L^2_x}.$$

Considering assumptions (2.27) together with the limit scheme (2.22) we obtain:

$$\sum_{i\in\mathbb{Z}}R_i^{n,1}\Delta x\leq\sqrt{K}\varepsilon^2\left(1+\sigma\Delta t+\frac{\lambda}{2}\Delta x\right)\|\Delta u^{n+1}\|_{L^2_x}.$$

Then for all $\theta>0$, a Young inequality yields:

$$\sum_{i\in\mathbb{Z}}R_i^{n,1}\Delta x\leq\theta\|\Delta u^{n+1}\|^2_{L^2_x}+\frac{K}{4\theta}\left(1+\sigma\Delta t+\frac{\lambda}{2}\Delta x\right)^2\varepsilon^4.$$

The expected estimation (3.43) comes from a discrete integration with respect to time. ∎

Proof. **(Proof of Lemma 3.3.)** Arguing the definition (3.37) we have:

$$\sum_{i\in\mathbb{Z}}R_i^{n,2}\Delta x=\varepsilon^2\Delta t\sum_{i\in\mathbb{Z}}\Delta u_i^{n+1}\delta_{tx}\Delta p(\tau)_i^{n+1/2}\Delta x+\frac{\lambda}{2}\varepsilon^2\Delta x\sum_{i\in\mathbb{Z}}\Delta u_i^{n+1}\delta_{xx}\Delta u_i^n\Delta x.$$

Once again, a Cauchy Schwarz inequality gives:

$$\sum_{i\in\mathbb{Z}}R_i^{n,2}\Delta x\leq\varepsilon^2\Delta t\|\Delta u^{n+1}\|_{L^2_x}\|\delta_{tx}\Delta p(\tau)^{n+1/2}\|_{L^2_x}+\frac{\lambda}{2}\varepsilon^2\Delta x\|\Delta u^{n+1}\|_{L^2_x}\|\delta_{xx}\Delta u^n\|_{L^2_x}.$$

Moreover, applying item *(i)* of Lemma 4.1 for $\|\delta_x\left(\delta_t\Delta p(\tau)^{n+1/2}\right)\|_{L^2_x}$ and $\|\delta_{xx}\Delta u^n\|_{L^2_x}$, we have:

$$\sum_{i\in\mathbb{Z}}R_i^{n,2}\Delta x\leq\frac{\varepsilon^2\Delta t}{\Delta x}\|\Delta u^{n+1}\|_{L^2_x}\|\delta_t\Delta p(\tau)^{n+1/2}\|_{L^2_x}+\frac{2\lambda\varepsilon^2}{\Delta x}\|\Delta u^{n+1}\|_{L^2_x}\|\Delta u^n\|_{L^2_x}.$$

A Young inequality yields:

$$\begin{aligned}\sum_{i\in\mathbb{Z}}R_i^{n,2}\Delta x\leq&\frac{\varepsilon^2}{2}\left(\frac{\Delta t}{\Delta x^2}\|\Delta u^{n+1}\|^2_{L^2_x}+\Delta t\|\delta_t\Delta p(\tau)^{n+1/2}\|^2_{L^2_x}\right)\\&+\frac{\lambda\varepsilon^2}{\Delta x}\left(\|\Delta u^{n+1}\|^2_{L^2_x}+\|\Delta u^n\|^2_{L^2_x}\right).\end{aligned}$$

Using (4.54), we get:

$$\begin{aligned}\sum_{i\in\mathbb{Z}}R_i^{n,2}\Delta x\leq&\frac{\varepsilon^2}{\Delta x}\left(\frac{\Delta t}{2\Delta x}+\lambda\right)\|\Delta u^{n+1}\|^2_{L^2_x}+\frac{\varepsilon^2}{\Delta x}\left(\frac{3L_p^2\Delta t}{\Delta x}\left(1+\frac{K^2\Delta t^2}{4}\right)+\lambda\right)\|\Delta u^n\|^2_{L^2_x}\\&+\frac{3L_p^3\varepsilon^2\Delta t}{\Delta x^2}\left(8\lambda^2+2\lambda^2K^2\Delta t^2+K^2\Delta x^2\right)\phi_\varepsilon^n.\end{aligned}$$

A discrete integration with respect to time gives the required estimation (3.44). ∎

Proof. **(Proof of Lemma 3.4.)** Arguing the definition (3.37) we have:

$$\sum_{i\in\mathbb{Z}}R_i^{n,3}\Delta x=A^n+B^n,$$

where

$$\begin{aligned} A^n =& \Delta t \sum_{i\in\mathbb{Z}} \Delta p(\tau_i^{n+1})\delta_{tx}\Delta u_i^{n+1/2}\Delta x, \\ B^n =& -\frac{\lambda}{2}\Delta x \sum_{i\in\mathbb{Z}} \Delta p(\tau_i^{n+1})\delta_{xx}\Delta\tau_i^n\Delta x. \end{aligned}$$

We first provide the estimation for A^n. A discrete integration by parts with respect to time gives:

$$\begin{aligned} \sum_{n=0}^N A^n\Delta t =& -\Delta t\sum_{n=0}^N\sum_{i\in\mathbb{Z}} \delta_t\Delta p(\tau_i)^{n+1/2}\delta_x\Delta u_i^n\Delta x\Delta t \\ &+\Delta t\sum_{i\in\mathbb{Z}}\Delta p(\tau_i^{N+1})\delta_x\Delta u_i^{N+1}\Delta x - \Delta t\sum_{i\in\mathbb{Z}}\Delta p(\tau_i^0)\delta_x\Delta u_i^0\Delta x. \end{aligned}$$

Once again, we apply Cauchy Schwarz inequality:

$$\begin{aligned} \sum_{n=0}^N A^n\Delta t \le& \Delta t\sum_{n=0}^N \|\delta_t\Delta p(\tau)^{n+1/2}\|_{L^2_x}\|\delta_x\Delta u^n\|_{L^2_x}\Delta t \\ &+\Delta t\|\Delta p(\tau^{N+1})\|_{L^2_x}\|\delta_x\Delta u^{N+1}\|_{L^2_x} + \Delta t\|\Delta p(\tau^0)\|_{L^2_x}\|\delta_x\Delta u^0\|_{L^2_x}, \end{aligned}$$

and Young inequality, which leads to:

$$\begin{aligned} \sum_{n=0}^N A^n\Delta t \le& \sum_{n=0}^N \frac{1}{2}\left(\Delta t\|\delta_t\Delta p(\tau)^{n+1/2}\|^2_{L^2_x} + \Delta t\|\delta_x\Delta u^n\|^2_{L^2_x}\right)\Delta t + \frac{\theta_1}{2}\|\Delta p(\tau^{N+1})\|^2_{L^2_x} \\ &+\frac{1}{2\theta_1}\Delta t^2\|\delta_x\Delta u^{N+1}\|^2_{L^2_x} + \frac{\theta_2}{2}\Delta t\|\Delta p(\tau^0)\|^2_{L^2_x} + \frac{1}{2\theta_2}\Delta t\|\delta_x\Delta u^0\|^2_{L^2_x}, \end{aligned}$$

where θ_1, $\theta_2 > 0$ will be fixed further.
From a Taylor expansion of p, we have for all $0\le n\le N$:

$$\|\Delta p(\tau^n)\|^2_{L^2_x} = \sum_{i\in\mathbb{Z}}\left(\Delta\tau_i^n\int_0^1 p''(\overline{\tau}_i^n + s\Delta\tau_i^n)ds\right)^2\Delta x.$$

Moreover, properties (2.29) on p give:

$$\|\Delta p(\tau^n)\|^2_{L^2_x} \le L_p^2\|\Delta\tau^n\|^2_{L^2_x},$$

and from (4.50) we obtain:

$$\|\Delta p(\tau^n)\|^2_{L^2_x} \le 2L_p^3\phi_\varepsilon^n.$$

By plugging the previous estimation for $n = N+1$ and $n = 0$ in the expression of A^n and by applying estimation (4.49), we get:

$$\begin{aligned} \sum_{n=0}^N A^n\Delta t \le& \frac{\Delta t}{2}\sum_{n=0}^N \|\delta_t\Delta p(\tau)^{n+1/2}\|^2_{L^2_x}\Delta t + \frac{\Delta t}{2\Delta x^2}\sum_{n=0}^N\|\Delta u^n\|^2_{L^2_x}\Delta t \\ &+\theta_1 L_p^3\phi_\varepsilon^{N+1} + \frac{\Delta t^2}{2\Delta x^2\theta_1}\|\Delta u^{N+1}\|^2_{L^2_x} + \theta_2\Delta t L_p^3\phi_\varepsilon^0 + \frac{1}{2\Delta x^2\theta_2}\Delta t\|\Delta u^0\|^2_{L^2_x}. \end{aligned}$$

We fix $\theta_1 = \dfrac{1}{2L_p^3}$ and $\theta_2 = \dfrac{1}{L_p^3}$ to get $1/2$ as coefficient of ϕ_ε^{N+1} and Δt as coefficient of ϕ_ε^0. We thus get:

$$\begin{aligned}\sum_{n=0}^{N} A^n \Delta t \leq & \frac{\Delta t}{2} \sum_{n=0}^{N} \|\delta_t \Delta p(\tau)^{n+1/2}\|_{L_x^2}^2 \Delta t + \frac{\Delta t}{2\Delta x^2} \sum_{n=0}^{N} \|\Delta u^n\|_{L_x^2}^2 \Delta t \\ & + \frac{1}{2}\phi_\varepsilon^{N+1} + \frac{L_p^3 \Delta t^2}{\Delta x^2} \|\Delta u^{N+1}\|_{L_x^2}^2 + \Delta t \phi_\varepsilon^0 + \frac{L_p^3}{2\Delta x^2} \Delta t \|\Delta u^0\|_{L_x^2}^2.\end{aligned}$$

Since $\|\Delta u^{n+1}\|_{L_x^2}^2$ and ϕ_ε^n are nonnegative for all $n = 0, \ldots, N$, we clearly have:

$$\|\Delta u^{N+1}\|_{L_x^2}^2 \leq \sum_{n=0}^{N} \|\Delta u^{n+1}\|_{L_x^2}^2 \quad \text{and} \quad \phi_\varepsilon^0 \leq \sum_{n=0}^{N} \phi_\varepsilon^n,$$

and we obtain:

$$\begin{aligned}\sum_{n=0}^{N} A^n \Delta t \leq & \frac{\Delta t}{2} \sum_{n=0}^{N} \|\delta_t \Delta p(\tau)^{n+1/2}\|_{L_x^2}^2 \Delta t + \frac{\Delta t}{2\Delta x^2} \sum_{n=0}^{N} \|\Delta u^n\|_{L_x^2}^2 \Delta t \\ & + \frac{1}{2}\phi_\varepsilon^{N+1} + \frac{L_p^3 \Delta t}{\Delta x^2} \sum_{n=0}^{N} \|\Delta u^{n+1}\|_{L_x^2}^2 \Delta t + \sum_{n=0}^{N} \phi_\varepsilon^n \Delta t + \frac{L_p^3}{2\Delta x^2} \Delta t \|\Delta u^0\|_{L_x^2}^2.\end{aligned}$$

We conclude by using the estimation (4.54) to write:

$$\begin{aligned}\sum_{n=0}^{N} A^n \Delta t \leq & \frac{1}{2}\phi_\varepsilon^{N+1} + \frac{L_p^3 \Delta t}{\Delta x^2} \sum_{n=0}^{N} \|\Delta u^{n+1}\|_{L_x^2}^2 \Delta t \\ & + \frac{\Delta t}{2\Delta x^2} \left(6L_p^2 \left(1 + \frac{K^2 \Delta t^2}{4} \right) + 1 \right) \sum_{n=0}^{N} \|\Delta u^n\|_{L_x^2}^2 \Delta t \\ & + \left(\frac{3L_p^3 \Delta t}{\Delta x^2} \left(8\lambda^2 + 2\lambda^2 K^2 \Delta t^2 + K^2 \Delta x^2 \right) + 1 \right) \sum_{n=0}^{N} \phi_\varepsilon^n \Delta t + \frac{L_p^3 \Delta t}{2\Delta x^2} \|\Delta u^0\|_{L_x^2}^2.\end{aligned}$$

Now we provide an estimation for B^n. We add and substract $\Delta p(\tau_i^n)$ and we proceed to a discrete integration by parts in space to get:

$$B^n = -\frac{\lambda}{2} \Delta x \Delta t \sum_{i\in\mathbb{Z}} \delta_t \Delta p(\tau_i)^{n+1/2} \delta_{xx} \Delta \tau_i^n \Delta x + \frac{\lambda}{2} \Delta x \sum_{i\in\mathbb{Z}} \tilde{\delta}_x \Delta p(\tau^n)_{i+1/2} \tilde{\delta}_x \Delta \tau_{i+1/2}^n \Delta x.$$

Applying the formula (4.52) we obtain:

$$\begin{aligned}B^n = & -\frac{\lambda}{2} \Delta x \Delta t \sum_{i\in\mathbb{Z}} \delta_t \Delta p(\tau_i)^{n+1/2} \delta_{xx} \Delta \tau_i^n \Delta x \\ & + \frac{\lambda}{2} \Delta x \sum_{i\in\mathbb{Z}} \left(\tilde{\delta}_x \Delta \tau_{i+1/2}^n \right)^2 \int_0^1 p'(\tau_i^n + s\Delta x \tilde{\delta}_x \tau_{i+1/2}^n) ds \Delta x \\ & + \frac{\lambda}{2} \Delta x \sum_{i\in\mathbb{Z}} \Delta \tau_i^n \tilde{\delta}_x \overline{\tau}_{i+1/2}^n \tilde{\delta}_x \Delta \tau_{i+1/2}^n \int_0^1 I_{i+1/2,s}^n ds \Delta x\end{aligned}$$

$$+\frac{\lambda}{2}\Delta x^2\sum_{i\in\mathbb{Z}}\tilde{\delta}_x\overline{\tau}^n_{i+1/2}\left(\tilde{\delta}_x\Delta\tau^n_{i+1/2}\right)^2\int_0^1 sI^n_{i+1/2,s}ds\Delta x,$$

where $I^n_{i+1/2,s}$ is given by (4.53). We now estimate each term of the previous expression to obtain an estimation of B^n. Using assumptions (2.29) on p, we have:

$$\frac{\lambda}{2}\Delta x\sum_{i\in\mathbb{Z}}\left(\tilde{\delta}_x\Delta\tau^n_{i+1/2}\right)^2\int_0^1 p'(\tau^n_i+s\Delta x\tilde{\delta}_x\tau^n_{i+1/2})ds\Delta x\leq 0,$$

$$\int_0^1 I^n_{i+1/2,s}ds\leq L_p \quad\text{and}\quad \int_0^1 sI^n_{i+1/2,s}ds\leq L_p/2.$$

Moreover, the Cauchy Schwarz inequality allows us to write the following estimation of B^n:

$$\begin{aligned}B^n\leq&\frac{\lambda}{2}\Delta x\Delta t\|\delta_t\Delta p(\tau)^{n+1/2}\|_{L^2_x}\|\delta_{xx}\Delta\tau^n\|_{L^2_x}\\&+\frac{\lambda L_p}{2}\Delta x\|\tilde{\delta}_x\overline{\tau}^n\|_{L^\infty_x}\|\Delta\tau^n\|_{L^2_x}\|\tilde{\delta}_x\Delta\tau^n\|_{L^2_x}+\frac{\lambda L_p}{4}\Delta x^2\|\tilde{\delta}_x\overline{\tau}^n\|_{L^\infty_x}\|\tilde{\delta}_x\Delta\tau^n\|^2_{L^2_x}.\end{aligned}$$

Using regularity assumptions (2.27) on $\overline{\tau}$ and estimations (4.49), we get:

$$B^n\leq\frac{2\lambda\Delta t}{\Delta x}\|\delta_t\Delta p(\tau)^{n+1/2}\|_{L^2_x}\|\Delta\tau^n\|_{L^2_x}+2\lambda KL_p\|\Delta\tau^n\|^2_{L^2_x}.$$

Now, we consider the Young inequality and estimations (4.54) and (4.50) to write:

$$\begin{aligned}B^n\leq&\frac{6L_p^2\Delta t}{\Delta x^2}\left(1+\frac{K^2\Delta t^2}{4}\right)\|\Delta u^n\|^2_{L^2_x}\\&+2L_p\left(\frac{3L_p^2\Delta t}{\Delta x^2}(8\lambda^2+2\lambda^2K^2\Delta t^2+K^2\Delta x^2)+\frac{\lambda^2\Delta t}{\Delta x^2}+2\lambda KL_p\right)\phi^n_\varepsilon.\end{aligned}$$

The expected result (3.45) is recovered after summing in time the estimation of B^n and combining it with the estimation of A^n. □

Proof. **(Proof of Lemma 3.5.)** Arguing the definition (3.37), we have:

$$R^{n,4}_i=\left(-Q^{n+1/2}_i+\overline{Q}^{n+1/2}_i+\overline{q}^{n+1/2}_i\Delta\tau^n_i\right)\delta_t\overline{\tau}^{n+1/2}_i+\left(p(\tau^{n+1}_i)-Q^{n+1/2}_i\right)\delta_t\Delta\tau^{n+1/2}_i.$$

According to a sequence of Taylor expansions, we write:

$$\begin{aligned}\overline{Q}^{n+1/2}_i=&\int_0^1\left[p(\overline{\tau}^n_i)+\Delta t(1-s)\delta_t\overline{\tau}^{n+1/2}_i\overline{p'_s}\right]ds,\\\overline{q}^{n+1/2}_i=&\int_0^1\left[p'(\overline{\tau}^n_i)+\Delta t(1-s)\delta_t\overline{\tau}^{n+1/2}_i\overline{p''_s}\right]ds,\\p(\tau^{n+1}_i)=&\int_0^1\Big[p(\overline{\tau}^n_i)+\Delta\tau^n_ip'(\overline{\tau}^n_i)+\Delta t\delta_t\overline{\tau}^{n+1/2}_i\overline{p'_s}+\Delta t\delta_t\Delta\tau^{n+1/2}_i\overline{p'_s}\\&+\Delta t^2s\delta_t\overline{\tau}^{n+1/2}_i\delta_t\Delta\tau^{n+1/2}_i\overline{p''_s}+\Delta t\Delta\tau^n_i\delta_t\overline{\tau}^{n+1/2}_i\overline{p''_s}+T^n_{is1}+T^n_{is2}\Big]ds,\\Q^{n+1/2}_i=&\int_0^1\Big[p(\overline{\tau}^n_i)+\Delta\tau^n_ip'(\overline{\tau}^n_i)+\Delta t(1-s)\delta_t\overline{\tau}^{n+1/2}_i\overline{p'_s}+\Delta t(1-s)\delta_t\Delta\tau^{n+1/2}_i\overline{p'_s}\end{aligned}$$

$$+\Delta t^2 s(1-s)\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\overline{p''_s}+\Delta t(1-s)\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\overline{p''_s}+T_{is1}^n$$
$$+(1-s)T_{is2}^n]\,ds,$$

where we have for all $s\in[0,1]$:

$$\overline{p'_s}=p'(\overline{\tau}_i^n+s\Delta t\delta_t\overline{\tau}_i^{n+1/2}),$$
$$\overline{p''_s}=p''(\overline{\tau}_i^n+s\Delta t\delta_t\overline{\tau}_i^{n+1/2}),$$
$$T_{is1}^n=(\Delta\tau_i^n)^2(1-s)p''(\overline{\tau}_i^n+s\Delta\tau_i^n),$$
$$\begin{aligned}T_{is2}^n=&\Delta t\Delta\tau_i^n\delta_t\Delta\tau_i^{n+1/2}\overline{p''_s}+\Delta t^2 s\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\overline{p''_s}\\&+\Delta t\left(\delta_t\overline{\tau}_i^{n+1/2}+\delta_t\Delta\tau_i^{n+1/2}\right)\left(s\Delta\tau_i^{n+1}+(1-s)\Delta\tau_i^n\right)^2\int_0^1(1-t)p^{(3)}(m_{st})dt,\end{aligned}$$

with $m_{st}=\overline{\tau}_i^n+t\Delta\tau_i^n+s\Delta t\left(\delta_t\overline{\tau}_i^{n+1/2}+t\delta_t\Delta\tau_i^{n+1/2}\right)$ for all $s,t\in[0,1]$.
Equipped with these formulas, we can rewrite $R_i^{n,4}$ as follows:

$$\begin{aligned}R_i^{n,4}=\int_0^1\Big[&\Big(-p(\overline{\tau}_i^n)-\Delta\tau_i^n p'(\overline{\tau}_i^n)-\Delta t(1-s)\delta_t\overline{\tau}_i^{n+1/2}\overline{p'_s}-\Delta t(1-s)\delta_t\Delta\tau_i^{n+1/2}\overline{p'_s}\\&-\Delta t^2 s(1-s)\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\overline{p''_s}-\Delta t(1-s)\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\overline{p''_s}-T_{is1}^n-(1-s)T_{is2}^n\\&+p(\overline{\tau}_i^n)+\Delta t(1-s)\delta_t\overline{\tau}_i^{n+1/2}\overline{p'_s}+\Delta\tau_i^n p'(\overline{\tau}_i^n)+\Delta t(1-s)\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\overline{p''_s}\Big)\delta_t\overline{\tau}_i^{n+1/2}\\&+s\Big(\Delta t\delta_t\overline{\tau}_i^{n+1/2}\overline{p'_s}+\Delta t\delta_t\Delta\tau_i^{n+1/2}\overline{p'_s}+\Delta t^2 s\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\overline{p''_s}\\&+\Delta t\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\overline{p''_s}+T_{is2}^n\Big)\delta_t\Delta\tau_i^{n+1/2}\Big]ds.\end{aligned}$$

After simplifications, we get:

$$\begin{aligned}R_i^{n,4}=&\Delta t\delta_t\Delta\tau_i^{n+1/2}\Bigg[\delta_t\overline{\tau}_i^{n+1/2}\int_0^1(2s-1)\overline{p'_s}ds+\delta_t\Delta\tau_i^{n+1/2}\int_0^1 s\overline{p'_s}ds\\&-\Delta t\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\int_0^1 s(1-s)\overline{p''_s}ds+\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1\Delta t s^2\overline{p''_s}ds\\&+\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\int_0^1 s\overline{p''_s}ds\Bigg]+T_i^n,\end{aligned}$$

where

$$T_i^n=\int_0^1\left[-(T_{is1}^n+(1-s)T_{is2}^n)\delta_t\overline{\tau}_i^{n+1/2}+sT_{is2}^n\delta_t\Delta\tau_i^{n+1/2}\right]ds. \tag{4.60}$$

Moreover, performing a Taylor expansion on $\overline{p'_s}$, we get:

$$\begin{aligned}\int_0^1(2s-1)\overline{p'_s}ds=&\int_0^1(2s-1)\left(p'(\overline{\tau}_i^n)+s\Delta t\delta_t\overline{\tau}_i^{n+1/2}\int_0^1 p''(\overline{\tau}_i^n+st\Delta t\delta_t\overline{\tau}_i^{n+1/2})dt\right)ds\\=&\Delta t\delta_t\overline{\tau}_i^{n+1/2}\int_0^1 s(1-s)\overline{p''_s}ds,\end{aligned}$$

and

$$\int_0^1 s\overline{p'_s}ds=\int_0^1 s\left(p'(\overline{\tau}_i^n)+s\Delta t\delta_t\overline{\tau}_i^{n+1/2}\int_0^1 p''(\overline{\tau}_i^n+st\Delta t\delta_t\overline{\tau}_i^{n+1/2})dt\right)ds$$

$$=\frac{1}{2}p'(\overline{\tau}_i^n)+\frac{\Delta t}{2}\delta_t\overline{\tau}_i^{n+1/2}\int_0^1(1-s^2)\overline{p''_s}ds.$$

We can rewrite $R_i^{n,4}$ as follows:

$$\begin{aligned}R_i^{n,4}=&\Delta t\delta_t\Delta\tau_i^{n+1/2}\left[\Delta t\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\int_0^1 s(1-s)\overline{p''_s}ds+\frac{1}{2}\delta_t\Delta\tau_i^{n+1/2}p'(\overline{\tau}_i^n)\right.\\&+\frac{\Delta t}{2}\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1(1-s^2)\overline{p''_s}ds-\Delta t\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\int_0^1 s(1-s)\overline{p''_s}ds\\&\left.+\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1\Delta t s^2\overline{p''_s}ds+\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\int_0^1 s\overline{p''_s}ds\right]+T_i^n.\end{aligned}$$

After simplifications, we obtain:

$$\begin{aligned}R_i^{n,4}=&\frac{\Delta t}{2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2p'(\overline{\tau}_i^n)+\frac{\Delta t^2}{2}\delta_t\overline{\tau}_i^{n+1/2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1(1+s^2)\overline{p''_s}ds\\&+\Delta t\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1 s\overline{p''_s}ds+T_i^n.\end{aligned}$$

Now we focus on T_i^n defined by (4.60). Setting $I_s=\displaystyle\int_0^1(1-t)p^{(3)}(m_{st})dt$ and developing T_i^n, we get:

$$\begin{aligned}T_i^n=&\Delta t\Delta\tau_i^n\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 s\overline{p''_s}ds-\Delta t\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1(1-s)\overline{p''_s}ds\\&+\Delta t^2\left(\delta_t\Delta\tau_i^{n+1/2}\right)^3\int_0^1 s^2\overline{p''_s}ds-\Delta t^2\delta_t\overline{\tau}_i^{n+1/2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 s(1-s))\overline{p''_s}ds\\&-\Delta t(\Delta\tau_i^n)^2\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\int_0^1(1-s)I_sds\\&-2\Delta t^2\Delta\tau_i^n\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\delta_t\Delta\tau_i^{n+1/2}\int_0^1 s(1-s)I_sds\\&-\Delta t^3\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 s^2(1-s)I_sds\\&+\Delta t(\Delta\tau_i^n)^2\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1(2s-1)I_sds\\&+2\Delta t^2\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 s(2s-1)I_sds\\&+\Delta t^3\delta_t\overline{\tau}_i^{n+1/2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^3\int_0^1 s^2(2s-1)I_sds\\&+\Delta t(\Delta\tau_i^n)^2\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 sI_sds+2\Delta t^2\Delta\tau_i^n\left(\delta_t\Delta\tau_i^{n+1/2}\right)^3\int_0^1 s^2I_sds\\&+\Delta t^3\left(\delta_t\Delta\tau_i^{n+1/2}\right)^4\int_0^1 s^3I_sds-(\Delta\tau_i^n)^2\delta_t\overline{\tau}_i^{n+1/2}\int_0^1(1-s)p''(\overline{\tau}_i^n+s\Delta\tau_i^n)ds.\end{aligned}$$

We plug this development in the expression of $R_i^{n,4}$ to write:

$$R_i^{n,4}=\frac{\Delta t}{2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2p'(\overline{\tau}_i^n)$$

$$
\begin{aligned}
&+\Delta t\Delta\tau_i^n\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 s\overline{p_s''}ds+\Delta t\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1(2s-1)\overline{p_s''}ds\\
&+\Delta t^2\left(\delta_t\Delta\tau_i^{n+1/2}\right)^3\int_0^1 s^2\overline{p_s''}ds+\frac{\Delta t^2}{2}\delta_t\overline{\tau}_i^{n+1/2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1\left(3s^2-2s+1\right)\overline{p_s''}ds\\
&-\Delta t\left(\Delta\tau_i^n\right)^2\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\int_0^1(1-s)I_sds\\
&-2\Delta t^2\Delta\tau_i^n\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\delta_t\Delta\tau_i^{n+1/2}\int_0^1 s(1-s)I_sds\\
&-\Delta t^3\left(\delta_t\overline{\tau}_i^{n+1/2}\right)^2\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 s^2(1-s)I_sds\\
&+\Delta t\left(\Delta\tau_i^n\right)^2\delta_t\overline{\tau}_i^{n+1/2}\delta_t\Delta\tau_i^{n+1/2}\int_0^1(2s-1)I_sds\\
&+2\Delta t^2\Delta\tau_i^n\delta_t\overline{\tau}_i^{n+1/2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 s(2s-1)I_sds\\
&+\Delta t^3\delta_t\overline{\tau}_i^{n+1/2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^3\int_0^1 s^2(2s-1)I_sds\\
&+\Delta t\left(\Delta\tau_i^n\right)^2\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 sI_sds+2\Delta t^2\Delta\tau_i^n\left(\delta_t\Delta\tau_i^{n+1/2}\right)^3\int_0^1 s^2I_sds\\
&+\Delta t^3\left(\delta_t\Delta\tau_i^{n+1/2}\right)^4\int_0^1 s^3I_sds-\left(\Delta\tau_i^n\right)^2\delta_t\overline{\tau}_i^{n+1/2}\int_0^1(1-s)p''(\overline{\tau}_i^n+s\Delta\tau_i^n)ds.
\end{aligned}
$$

Assumptions (2.29) on p, which give in particular that I_s is negative, imply that:

$$
\begin{aligned}
&\frac{\Delta t}{2}\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2p'(\overline{\tau}_i^n)\le 0,\\
&\Delta t\left(\Delta\tau_i^n\right)^2\left(\delta_t\Delta\tau_i^{n+1/2}\right)^2\int_0^1 sI_sds\le 0,\\
&\Delta t^3\left(\delta_t\Delta\tau_i^{n+1/2}\right)^4\int_0^1 s^3I_sds\le 0.
\end{aligned}
$$

In addition, assumptions (2.28) on τ and (2.29) on p, p'' and $p^{(3)}$ together with a Cauchy Schwarz inequality allow us to write, after summing in space:

$$
\begin{aligned}
\sum_{i\in\mathbb{Z}}R_i^{n,4}\Delta x\le&\frac{1}{2}L_pL_\tau\Delta t\|\delta_t\Delta\tau^{n+1/2}\|_{L_x^2}^2\\
&+\frac{1}{2}L_p\Delta t\|\delta_t\overline{\tau}^{n+1/2}\|_{L_x^\infty}\|\Delta\tau^n\|_{L_x^2}\|\delta_t\Delta\tau^{n+1/2}\|_{L_x^2}\\
&+\frac{2}{3}L_pL_\tau\Delta t\|\delta_t\Delta\tau^{n+1/2}\|_{L_x^2}^2+\frac{1}{2}L_p\Delta t^2\|\delta_t\overline{\tau}^{n+1/2}\|_{L_x^\infty}\|\delta_t\Delta\tau^{n+1/2}\|_{L_x^2}^2\\
&+\frac{1}{2}L_p\Delta t\|\delta_t\overline{\tau}^{n+1/2}\|_{L_x^\infty}^2\|\Delta\tau^n\|_{L_x^2}^2\\
&+\frac{1}{3}L_p\Delta t^2\|\delta_t\overline{\tau}^{n+1/2}\|_{L_x^\infty}^2\|\Delta\tau^n\|_{L_x^2}\|\delta_t\Delta\tau^{n+1/2}\|_{L_x^2}\\
&+\frac{1}{12}L_p\Delta t^3\|\delta_t\overline{\tau}_i^{n+1/2}\|_{L_x^\infty}^2\|\delta_t\Delta\tau^{n+1/2}\|_{L_x^2}^2+L_pL_\tau\|\delta_t\overline{\tau}^{n+1/2}\|_{L_x^\infty}\|\Delta\tau^n\|_{L_x^2}^2\\
&+\frac{1}{2}L_pL_\tau\Delta t^2\|\delta_t\overline{\tau}^{n+1/2}\|_{L_x^\infty}\|\delta_t\Delta\tau^{n+1/2}\|_{L_x^2}^2
\end{aligned}
$$

$$+\frac{3}{8}L_pL_\tau\Delta t^2\|\delta_t\overline{\tau}^{n+1/2}\|_{L^\infty_x}\|\delta_t\Delta\tau^{n+1/2}\|^2_{L^2_x}$$
$$+\frac{4}{3}L_pL_\tau^2\Delta t\|\delta_t\Delta\tau^{n+1/2}\|^2_{L^2_x}+\frac{1}{2}L_p\|\delta_t\overline{\tau}^{n+1/2}\|_{L^\infty_x}\|\Delta\tau^n\|^2_{L^2_x}.$$

Application of Young inequality, assumptions (2.27) on $\overline{\tau}$ and estimation (4.50) give:

$$\sum_{i\in\mathbb{Z}}R_i^{n,4}\Delta x\leq +KL_p^2\left(KL_p\left(\frac{1}{2}+\frac{1}{3}K\Delta t\right)^2+K\Delta t+2L_\tau+1\right)\phi_\varepsilon^n$$
$$+\Delta t\left(L_p\left(\frac{7}{6}L_\tau+\frac{1}{2}K\Delta t+\frac{1}{12}K^2\Delta t^2+\frac{7}{8}KL_\tau\Delta t+\frac{4}{3}L_\tau^2\right)+\frac{1}{2}\Delta t\right)\|\delta_t\Delta\tau^{n+1/2}\|^2_{L^2_x}.$$

Invoking (4.51) and a discrete integration in time lead to the required estimation (3.46).
□

5 Numerical experiments

In this section, numerical experiments are performed to illustrate Therorem 2.1. To address such an issue, three simulations, with distinct initial data and distinct pressure laws, are displayed on the interval $(-10,10)$. All the numerical results are considered at final time $T=0.5$. The friction coefficient is fixed to $\sigma=1$ and we have adopted standard Neumann boundary conditions. The pressure is under the form $p(\tau)=\tau^{-\gamma}$ with $\gamma>1$. For all the numerical tests, initial data are well prepared according to (2.35) while the initial velocity is fixed as follows :

$$u_i^0=-\frac{1}{\sigma}\frac{p(\tau_{i+1}^0)-p(\tau_{i-1}^0)}{2\Delta x}. \tag{5.61}$$

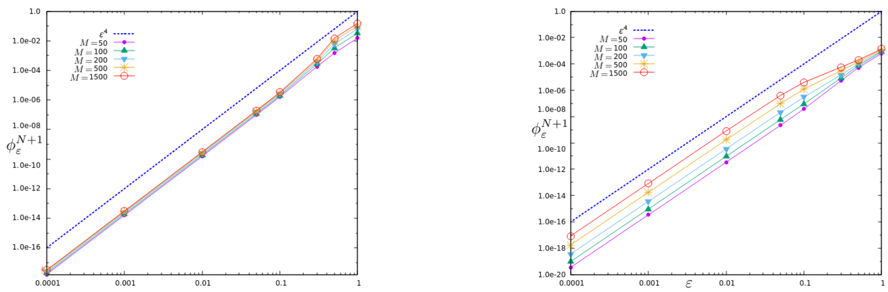
We compute the approximate solutions of system (1.7) with $\varepsilon\in\{1,5.10^{-1},3.10^{-1},1.10^{-1},5.10^{-2},1.10^{-2},1.10^{-3},1.10^{-4}\}$ and different numbers of cells $M\in\{50,100,200,500,1500\}$.
The time step Δt must be restricted according to a CFL condition. The restriction (2.30) can be adopted. However, constants involved in (2.27), (2.28) and (2.29) are clearly not optimal. Indeed, to improve the readability of Theorem 2.1, the number of constants have been minimized. In fact it is possible to get a better CFL condition with the introduction of numerous additional constants. The full evaluation of C_p defined by (2.31), according to the adopted numerical values, is of order 10^{-6}. Here we have considered less restrictive CFL condition as follows :

$$\Delta t=C_{CFL}\frac{\Delta x^2}{\max\limits_{i=0,\dots,M}(|p'(\tau_i^n)|,|p'(\overline{\tau}_i^n)|)}.$$

First, we present on Figure 1 simulations with $C_{CFL}=0.4$ and $\gamma=1.4$ for two different initial conditions as follows :

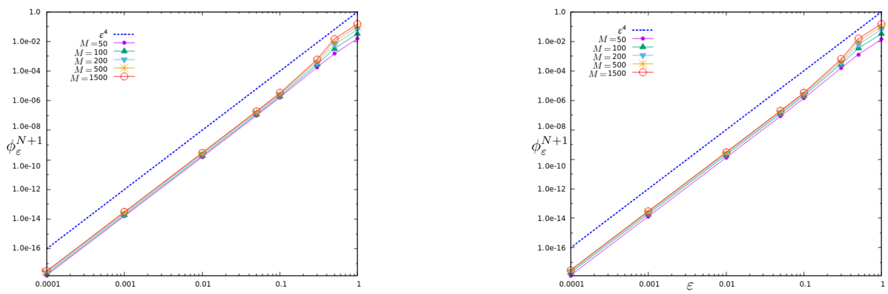
- Discontinuous initial condition :

$$\tau_0(x)=\begin{cases}1 & \text{if } x<0,\\ 2 & \text{else.}\end{cases} \tag{5.62}$$



(a) Discontinuous initial data (b) Smooth initial data

Figure 1: Space integral of the relative entropy ϕ_ε^{N+1} with respect to ε in log scale with $C_{CFL}=0.4$ and $\gamma=1.4$.



(a) $C_{CFL}=0.4$ (b) $C_{CFL}=0.25$

Figure 2: Space integral of the relative entropy ϕ_ε^{N+1} with respect to ε in log scale with the discontinuous initial data and $\gamma=1.4$.

- Smooth initial condition :

$$\tau_0(x)=2+\cos\left(\frac{\pi x}{10}\right). \tag{5.63}$$

We display the discrete space integral of the relative entropy at final time ϕ_ε^{N+1} with respect to ε in log scale. We can see that both for the discontinuous and smooth initial data, the slopes are of order ε^4, which is expected according to Theorem 2.1.

Next, we study the behaviour of the convergence rate for different values of the constant C_{CFL}. We adopt the discontinuous initial data given by (5.62). The pressure coefficient is equal to 1.4. Once again we present, on Figure 2, the discrete space integral of the relative entropy ϕ_ε^{N+1} with respect to ε in log scale for two distinct values of the CFL constant, $C_{CFL}=0.4$ and $C_{CFL}=0.25$. As expected, the slopes are of order ε^4.

Finally, on Figure 3, we display the convergence rate with the discontinuous initial data (5.62) and $C_{CFL}=0.4$ for two different pressure functions respectively defined by $\gamma=1.4$ and $\gamma=3.5$. Here again, we obtain a convergence rate of order ε^4.

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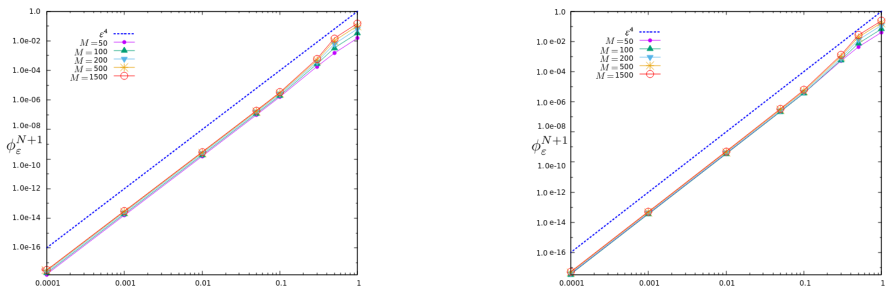

(a) $\gamma = 1.4$ (b) $\gamma = 3.5$

Figure 3: Space integral of the relative entropy ϕ_ε^{N+1} with respect to ε in log scale with the discontinuous initial data and $C_{CFL} = 0.4$.

References

[1] C. Berthon, M. Bessemoulin-Chatard, and H. Mathis. Numerical convergence rate for a diffusive limit of hyperbolic systems: p-system with damping. *SMAI J. Comput. Math.*, 2:99–119, 2016. 1, 2, 2, 2, 3

[2] C. Berthon, A. Crestetto, and F. Foucher. A well-balanced finite volume scheme for a mixed hyperbolic/parabolic system to model chemotaxis. *J. Sci. Comput.*, 67(2):618–643, 2016. 1

[3] S. Bianchini, B. Hanouzet, and R. Natalini. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Comm. Pure Appl. Math.*, 60(11):1559–1622, 2007. 1

[4] F. Bouchut, H. Ounaissa, and B. Perthame. Upwinding of the source term at interfaces for Euler equations with high friction. *Comput. Math. Appl.*, 53(3-4):361–375, 2007. 1

[5] C. Buet and S. Cordier. Asymptotic preserving scheme and numerical methods for radiative hydrodynamic models. *C. R. Math. Acad. Sci. Paris*, 338(12):951–956, 2004. 1

[6] C. Buet and B. Despres. Asymptotic preserving and positive schemes for radiation hydrodynamics. *J. Comput. Phys.*, 215(2):717–740, 2006. 1

[7] C. Cancès, H. Mathis, and N. Seguin. Relative entropy for the finite volume approximation of strong solutions to systems of conservation laws. *HAL*, 2016. 1

[8] C. M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, 70(2):167–179, 1979. 1

[9] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010. 1

[10] R. J. DiPerna. Uniqueness of solutions to hyperbolic conservation laws. *Indiana Univ. Math. J.*, 28(1):137–188, 1979. 1

[11] A. Duran, F. Marche, C. Berthon, and R. Turpault. Numerical discretizations for shallow water equations with source terms on unstructured meshes. In *Hyperbolic problems: theory, numerics, applications*, volume 8 of *AIMS Ser. Appl. Math.*, pages 541–549. Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2014. 1

[12] E. Feireisl, R. Hosek, D. Maltese, and A. Novotný. Unconditional convergence and error estimates for bounded numerical solutions of the barotropic Navier-Stokes system. *Numer. Methods Partial Differential Equations*, 33(4):1208–1223, 2017. 1

[13] T. Gallouët, D. Maltese, and A. Novotný. Discrete relative entropy for the compressible Stokes system. In *Finite volumes for complex applications VII. Methods and theoretical aspects*, volume 77 of *Springer Proc. Math. Stat.*, pages 383–392. Springer, Cham, 2014. 1

[14] F. Golse, C. D. Levermore, and L. Saint-Raymond. La méthode de l'entropie relative pour les limites hydrodynamiques de modèles cinétiques. In *Séminaire: Équations aux Dérivées Partielles, 1999–2000*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XIX, 23. École Polytech., Palaiseau, 2000. 1

[15] L. Gosse. Asymptotic-preserving and well-balanced schemes for the 1D Cattaneo model of chemotaxis movement in both hyperbolic and diffusive regimes. *J. Math. Anal. Appl.*, 388(2):964–983, 2012. 1

[16] L. Gosse and G. Toscani. An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations. *C. R. Math. Acad. Sci. Paris*, 334(4):337–342, 2002. 1

[17] L. Gosse and G. Toscani. Identification of asymptotic decay to self-similarity for one-dimensional filtration equations. *SIAM J. Numer. Anal.*, 43(6):2590–2606, 2006. 1

[18] L. Gosse and N. Vauchelet. Numerical high-field limits in two-stream kinetic models and 1D aggregation equations. *SIAM J. Sci. Comput.*, 38(1):A412–A434, 2016. 1

[19] A. Harten, P. D. Lax, and B. van Leer. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM Rev.*, 25(1):35–61, 1983. 2

[20] L. Hsiao and T.-P. Liu. Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping. *Comm. Math. Phys.*, 143(3):599–605, 1992. 1

[21] S. Jin. Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM J. Sci. Comput.*, 21(2):441–454, 1999. 1

[22] S. Jin, L. Pareschi, and G. Toscani. Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations. *SIAM J. Numer. Anal.*, 35(6):2405–2439, 1998. 1, 1, 2

[23] V. Jovanović and C. Rohde. Error estimates for finite volume approximations of classical solutions for nonlinear systems of hyperbolic balance laws. *SIAM J. Numer. Anal.*, 43(6):2423–2449, 2006. 1

[24] C. Lattanzio and A. E. Tzavaras. Relative entropy in diffusive relaxation. *SIAM J. Math. Anal.*, 45(3):1563–1584, 2013. 1, 1, 2, 2, 3

[25] H. Mathis and N. Therme. *Numerical Convergence for a Diffusive Limit of the Goldstein–Taylor System on Bounded Domain*, pages 225–233. Springer International Publishing, Cham, 2017. 1

[26] M. Mei. Best asymptotic profile for hyperbolic p-system with damping. *SIAM J. Math. Anal.*, 42(1):1–23, 2010. 1

[27] G. Naldi and L. Pareschi. Numerical schemes for kinetic equations in diffusive regimes. *Appl. Math. Lett.*, 11(2):29–35, 1998. 1, 1

[28] G. Naldi and L. Pareschi. Numerical schemes for hyperbolic systems of conservation laws with stiff diffusive relaxation. *SIAM J. Numer. Anal.*, 37(4):1246–1270, 2000. 1

[29] R. Natalini, M. Ribot, and M. Twarogowska. A numerical comparison between degenerate parabolic and quasilinear hyperbolic models of cell movements under chemotaxis. *J. Sci. Comput.*, 63(3):654–677, 2015. 1

[30] K. Nishihara. Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping. *J. Differential Equations*, 131(2):171–188, 1996. 1

[31] K. Nishihara. Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping. *J. Differential Equations*, 137(2):384–395, 1997. 1

[32] K. Nishihara, W. Wang, and T. Yang. L_p-convergence rate to nonlinear diffusion waves for p-system with damping. *J. Differential Equations*, 161(1):191–218, 2000. 1

[33] L. Saint-Raymond. Hydrodynamic limits: some improvements of the relative entropy method. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(3):705–744, 2009. 1

[34] A. E. Tzavaras. Relative entropy in hyperbolic relaxation. *Commun. Math. Sci.*, 3(2):119–132, 2005. 1