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► **To cite this version:**

Pascal Boyer. Local Ihara's lemma and applications. International Mathematics Research Notices, 2021, <https://doi.org/10.1093/imrn/rnab298>. hal-01895168

HAL Id: hal-01895168

<https://hal.science/hal-01895168>

Submitted on 14 Oct 2018

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PERSISTENCE OF NON DEGENERACY: A LOCAL ANALOG OF IHARA'S LEMMA

by

Boyer Pascal

Abstract. — Persistence of non degeneracy is a phenomenon which appears in the theory of $\overline{\mathbb{Q}_l}$ -representations of the linear group: every irreducible submodule of the restriction to the mirabolic subgroup of a non degenerate irreducible representation is non degenerate. This is no more true in general, if we look at the modulo l reduction of some stable lattice. As in the Clozel-Harris-Taylor generalization of global Ihara's lemma, we show that this property, called non degeneracy persistence, remains true for lattices given by the cohomology of Lubin-Tate spaces.

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2010 Mathematics Subject Classification. — 11F70, 11F80, 11F85, 11G18, 20C08.

Key words and phrases. — Rapoport-Zink spaces, mod l representations, mirabolic group, non degenerate representations.

The authors thanks the ANR for his support through the project PerCoLaTor 14-CE25.

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Introduction

Before the “Ihara avoidance” argument of Taylor, the proof of Sato-Tate conjecture by Clozel Harris and Taylor, rested on a conjectural generalization in higher dimension of the classical Ihara’s lemma for GL_2 . Their formulation can be understood as some persistence of the non degeneracy property by reduction modulo l of automorphic representations, cf. conjecture 2.2.1.

Locally fix prime numbers $l \neq p$ and a finite extension K of \mathbb{Q}_p . Recall then [17] corollary 6.8, that any irreducible $\overline{\mathbb{Q}_l}$ -representation π of $GL_d(K)$ is homogeneous which means, cf. [17] definition 5.1, that its restriction to the mirabolic subgroup $M_d(K)$ of matrices such that the last row is $(0, \dots, 0, 1)$, is homogeneous in the sense that every irreducible sub- $M_d(K)$ -representation has the same level of degeneracy, cf. [17] 4.3 or [4] 3.5. In particular if π is non degenerate i.e. its level of degeneracy equals d , then any irreducible sub-representation of $\pi|_{M_d(K)}$ is also non degenerate. Modulo l , for π a irreducible non degenerate representation of $GL_d(K)$, they might exist stable lattices such that $\pi|_{M_d(K)} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}$ owns irreducible degenerate subspaces, cf. corollary 1.4.3.

We then propose to prove some persistence of non degeneracy phenomena in the cohomology groups of Lubin-Tate spaces. Let then consider a finite extension K/\mathbb{Q}_p with ring of integers \mathcal{O}_K . For $d \geq 1$, denote $\widehat{\mathcal{M}}_{LT, d, n}$ the pro-formal scheme representing the functor of isomorphism classes of deformations by quasi-isogenies of the formal \mathcal{O}_K -module over $\overline{\mathbb{F}_p}$ with height d and with level structure n . We denote $\mathcal{M}_{LT, d, n}$ its

generic fiber over \widehat{K}^{un} . For $\Lambda = \overline{\mathbb{Q}}_l, \overline{\mathbb{Z}}_l$ or $\overline{\mathbb{F}}_l$, consider both

$$\mathcal{U}_{LT,d,\Lambda}^{d-1} := \varinjlim_n H^{d-1}(\mathcal{M}_{LT,d,n} \widehat{\otimes}_{\widehat{K}^{un}} \widehat{K}, \Lambda)$$

and

$$\mathcal{V}_{LT,d,\Lambda}^{d-1} := \varinjlim_n H_c^{d-1}(\mathcal{M}_{LT,d,n} \widehat{\otimes}_{\widehat{K}^{un}} \widehat{K}, \Lambda).$$

There is a natural action of $GL_d(K) \times D_{K,d}^\times \times W_k$ on $\mathcal{U}_{LT,d,\overline{\mathbb{Q}}_l}^{d-1}$ and $\mathcal{V}_{LT,d,\Lambda}^{d-1}$, where $D_{K,d}$ (resp. W_k) is the central division algebra over K with invariant $1/d$ (resp. the Weil group of K). In this paper we focus on the action of $GL_d(K)$ and it appears, cf. [5], that every irreducible $GL_d(K)$ -subquotient of $\mathcal{U}_{LT,d,\overline{\mathbb{Q}}_l}^{d-1}$ and $\mathcal{V}_{LT,d,\overline{\mathbb{Q}}_l}^{d-1}$ is either a cuspidal or a generalized Steinberg representation, so it's always non degenerate. One can then ask if any irreducible $GL_d(K)$ -equivariant subspace of $\mathcal{U}_{LT,d,\overline{\mathbb{F}}_l}^{d-1}$ (resp. $\mathcal{V}_{LT,d,\overline{\mathbb{F}}_l}^{d-1}$) is still non degenerate or even more if any irreducible $M_d(K)$ -equivariant subspace is non degenerate.

Theorem. — (cf. theorem 4.1.5 and 4.2.4)

The persistence of non degeneracy property relatively to M_d holds for $\mathcal{V}_{LT,d,\overline{\mathbb{F}}_l}^{d-1}$ and $\mathcal{U}_{LT,d,\overline{\mathbb{F}}_l}^{d-1}$, i.e. any irreducible $M_d(K)$ -equivariant subspace is non degenerate.

Remark: in [8] we prove that $\mathcal{U}_{LT,d,\overline{\mathbb{Z}}_l}^i$ are free. As we don't want to use results of [8] in this paper, except in the last section, we rather use the modulo l reduction of its free quotient $\mathcal{U}_{LT,d,\overline{\mathbb{Z}}_l,free}^{d-1}$.

It should be not too difficult to prove that this theorem can be deduced from the higher dimensional global Ihara's lemma for unitary groups. In the other direction, from this local result, it's quite easy to prove the square integrable case of global Ihara's lemma, that is when the local component of the automorphic representation is isomorphic to a generalized Steingerg representations. This case is supposed to be the easiest one but we think the global results proved here could be useful to the general case.

In the last section, using results of [8], we also look at $\mathcal{U}_{LT,d,\overline{\mathbb{F}}_l}^{d-1-\delta}$ (resp. $\mathcal{V}_{LT,d,\overline{\mathbb{F}}_l}^{d-1+\delta}$ for $\delta > 0$). The situation is less pleasant to state but we can find cases where, cf. proposition 4.3.1 and the remarks before and after it, that irreducible subspaces must have minimal derivative order, but among the

irreducible quotients of such derivative order, the lattices of Lubin-Tate cohomology groups select the one with non degenerate highest derivative.

1. Review on the representation theory for $GL_n(\mathbb{Q}_p)$

We fix a finite extension K/\mathbb{Q}_p with residue field \mathbb{F}_q . We denote $|\cdot|$ its absolute value.

1.1. Induced representations. — For a representation π of $GL_d(K)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n \text{ val} \circ \det}.$$

1.1.1. Notations. — For π_1 and π_2 representations of respectively $GL_{n_1}(K)$ and $GL_{n_2}(K)$, we will denote by

$$\pi_1 \times \pi_2 := \text{ind}_{P_{n_1, n_1+n_2}(K)}^{GL_{n_1+n_2}(K)} \pi_1\left\{\frac{n_2}{2}\right\} \otimes \pi_2\left\{-\frac{n_1}{2}\right\},$$

the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \dots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d with Levi

$$GL_{r_1} \times GL_{r_2-r_1} \times \dots \times GL_{r_k-r_{k-1}}.$$

Recall that a representation ϱ of $GL_d(K)$ is called *cuspidal* (resp. *supercuspidal*) if it's not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true for $\overline{\mathbb{F}}_l$.

1.1.2. Definition. — (see [17] §9 and [6] §1.4) Let g be a divisor of $d = sg$ and π an irreducible cuspidal $\overline{\mathbb{Q}}_l$ -representation of $GL_g(K)$.

– The induced representation

$$\pi\left\{\frac{1-s}{2}\right\} \times \pi\left\{\frac{3-s}{2}\right\} \times \dots \times \pi\left\{\frac{s-1}{2}\right\}$$

holds a unique irreducible quotient (resp. subspace) denoted $\text{St}_s(\pi)$ (resp. $\text{Speh}_s(\pi)$); it's a generalized Steinberg (resp. *Speh*) representation.

– The induced representation $\text{St}_t(\pi\{\frac{-r}{2}\}) \times \text{Speh}_r(\pi\{\frac{t}{2}\})$ (resp. of $\text{St}_{t-1}(\pi\{\frac{-r-1}{2}\}) \times \text{Speh}_{r+1}(\pi\{\frac{t-1}{2}\})$) owns a unique irreducible subspace (resp. quotient), denoted $LT_\pi(t-1, r)$.

1.2. Reduction modulo l of a Steinberg representation. — Denote $e_l(q)$ the order of $q \in \mathbb{F}_l^\times$.

1.2.1. Notation. — For $\Lambda = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, let denote $\text{Scusp}_\Lambda(g)$ the set of equivalence classes of irreducible supercuspidal Λ -representations of $GL_g(K)$.

1.2.2. Proposition. — (cf. [15] III.5.10) Let π be a irreducible cuspidal representation of $GL_g(K)$ with a stable $\overline{\mathbb{Z}}$ -lattice⁽¹⁾, then its modulo l reduction is irreducible and cuspidal but not necessary supercuspidal.

1.2.3. Proposition. — [12] §2.2.3

Let π be a irreducible entire cuspidal representation, and $s \geq 1$. Then the modulo l reduction of $\text{Speh}_s(\pi)$ is irreducible.

1.2.4. Notation. — The Zelevinski line associated to some irreducible supercuspidal $\overline{\mathbb{F}}_l$ -representation ϱ , is the set $\{\varrho\{i\} / i \in \mathbb{Z}\}$. It's clearly a finite and we denote $\epsilon(\varrho)$ its cardinal which is a divisor of $e_l(q)$. We also introduce, cf. [16] p.51

$$m(\varrho) = \begin{cases} \epsilon(\varrho), & \text{si } \epsilon(\varrho) > 1; \\ l, & \text{sinon.} \end{cases}$$

1.2.5. Definition. — Consider a multiset⁽²⁾ $\underline{s} = \{\rho_1^{n_1}, \dots, \rho_r^{n_r}\}$ of irreducible supervespidual $\overline{\mathbb{F}}_l$ -representations. We then denote, following [16] V.7, $\text{St}(\underline{s})$ the unique non degenerate sub-quotient of the induced representation

$$\rho(\underline{s}) := \overbrace{\rho_1 \times \dots \times \rho_1}^{n_1} \times \dots \times \overbrace{\rho_r \times \dots \times \rho_r}^{n_r}.$$

Remark: Thanks to [16] V.7, every irreducible non degenerate $\overline{\mathbb{F}}_l$ -representation can be written like before.

1.2.6. Notation. — For $s \geq 1$ and ρ an irreducible cuspidal $\overline{\mathbb{F}}_l$ -representation, we denote $\underline{s}(\rho)$ for the multi-segment $\{\rho, \rho\{1\}, \dots, \rho\{s-1\}\}$ and, cf. [16] V.4, $\text{St}_s(\rho) := \text{St}(\underline{s}(\rho))$.

1.2.7. Proposition. — (cf. [16] V.4)

With the previous notation, the $\overline{\mathbb{F}}_l$ -representation $\text{St}_s(\varrho)$ is cuspidal if and only if $s = 1$ or $m(\varrho)l^k$ for some $k \geq 0$.

⁽¹⁾We say that π is entire.

⁽²⁾meaning we take into account the multiplicities

Remark: from [15] III-3.15 and 5.14, every irreducible cuspidal $\overline{\mathbb{F}}_l$ -representation can be written $\text{St}_s(\varrho)$ for some irreducible supercuspidal representation ϱ , and $s = 1$ or $s = m(\varrho)l^k$ with $k \geq 0$.

1.2.8. Notations. — Let ϱ be an irreducible cuspidal $\overline{\mathbb{F}}_l$ -representation of $GL_g(K)$. We then denote

- $g_{-1}(\varrho) := g$ and for $i \geq 0$, $g_i(\varrho) := m(\varrho)l^i g$;
- $\varrho_{-1} = \varrho$ and for all $i \geq 0$, $\varrho_i = \text{St}_{m(\varrho)l^i}(\varrho)$.
- $\text{Cusp}(\varrho, i)$ the set of equivalence classes of irreducible entire $\overline{\mathbb{Q}}_l$ -representations such that modulo l it is isomorphic to ϱ_i ,
- and $\text{Cusp}(\varrho) = \bigcup_{i \geq -1} \text{Cusp}(\varrho, i)$.

1.2.9. Notation. — Let $s \geq 1$ and ϱ an irreducible cuspidal $\overline{\mathbb{F}}_l$ -representation of $GL_g(K)$. We denote $\mathcal{I}_\varrho(s)$ the set of sequences (m_{-1}, m_0, \dots) of integers such that

$$s = m_{-1} + m(\varrho) \sum_{k=0}^{+\infty} m_k l^k.$$

We denote $\text{lg}_\varrho(s)$ the cardinal of $\mathcal{I}_\varrho(s)$. We then define a relation of order on $\mathcal{I}_\varrho(s)$ by

$$(m_{-1}, m_0, \dots) > (m'_{-1}, m'_0, \dots) \Leftrightarrow \exists k \geq -1 \text{ s.t. } \forall i > k : m_i = m'_i \text{ and } m_k > m'_k.$$

1.2.10. Definition. — For $\underline{i} = (i_{-1}, i_0, \dots) \in \mathcal{I}_\varrho(s)$, we define

$$\text{St}_{\underline{i}}(\varrho) := \text{St}_{i_{-1}}(\varrho_{-1}) \times \text{St}_{i_0}(\varrho_0) \times \dots \times \text{St}_{i_u}(\varrho_u)$$

where $i_k = 0$ for all $k > u$.

Remark: we will denote s_{\max} the maximal element of $\mathcal{I}_\varrho(s)$ so that $\text{St}_{s_{\max}}(\varrho)$ is non degenerate.

1.2.11. Theorem. — Consider π an entire irreducible cuspidal $\overline{\mathbb{Q}}_l$ -representation of $GL_g(K)$ and let ϱ be its modulo l reduction. In the Grothendieck group of $\overline{\mathbb{F}}_l$ -representations of $GL_{sg}(K)$, we have the following equality:

$$r_l(\text{St}_s(\pi)) = \sum_{\underline{i} \in \mathcal{I}_\varrho(s)} \text{St}_{\underline{i}}(\varrho).$$

Remark: for $s < m(\varrho)$, it's irreducible so, up to isomorphism, it possess an unique stable lattice, cf. [2] proposition 3.3.2 and the following remark.

1.3. Restriction to the mirabolic group. — In this paragraph, we want to state some of the main results of [4] §4. Recall first some notations of [4] §3. The mirabolic subgroup $M_d(K)$ of $GL_d(K)$ is the set of matrices with last row $(0, \dots, 0, 1)$: we denote

$$V_d(K) = \{(m_{i,j} \in P_d(K) : m_{i,j} = \delta_{i,j} \text{ for } j < n)\}.$$

its unipotent radical. We fix a non trivial character ψ of K and let θ the character of $V_d(K)$ defined by $\theta((m_{i,j})) = \psi(m_{d-1,d})$. For $G = GL_r(K)$ or $M_r(K)$, we denote $\text{Alg}(G)$ the abelian category of algebraic representations of G and, following [4], we introduce

$$\Psi^- : \text{Alg}(M_d(K)) \longrightarrow \text{Alg}(GL_{d-1}(K)), \quad \Phi^- : \text{Alg}(M_d) \longrightarrow \text{Alg}(M_{d-1}(K))$$

defined by $\Psi^- = r_{V_d,1}$ (resp. $\Phi^- = r_{V_d,\theta}$) the functor of V_{d-1} coinvariants (resp. (V_{d-1}, θ) -coinvariants), cf. [4] 1.8. We also introduce the normalize compact induced functor

$$\Psi^+ := i_{V,1} : \text{Alg}(GL_{d-1}(K)) \longrightarrow \text{Alg}(M_d(K)),$$

$$\Phi^+ := i_{V,\theta} : \text{Alg}(M_{d-1}(K)) \longrightarrow \text{Alg}(M_d(K)).$$

1.3.1. Proposition. — ([4] p451)

- The functors Ψ^- , Ψ^+ , Φ^- and Φ^+ are exact.
- $\Phi^- \circ \Psi^+ = \Psi^- \circ \Phi^+ = 0$.
- Ψ^- (resp. Φ^+) is left adjoint to Ψ^+ (resp. Φ^-) and the following adjunction maps

$$\text{Id} \longrightarrow \Phi^- \Phi^+, \quad \Psi^+ \Psi^- \longrightarrow \text{Id},$$

are isomorphisms meanwhile

$$0 \rightarrow \Phi^+ \Phi^- \longrightarrow \text{Id} \longrightarrow \Psi^+ \Psi^- \rightarrow 0.$$

1.3.2. Definition. — For $\tau \in \text{Alg}(M_d(K))$, the representation

$$\tau^{(k)} := \Psi^- \circ (\Phi^-)^{k-1}(\tau)$$

is called the k -th derivative of τ . If $\tau^{(k)} \neq 0$ and $\tau^{(m)} = 0$ for all $m > k$, then $\tau^{(k)}$ is called the highest derivative of τ .

1.3.3. Notation. — (cf. [17] 4.3) Let $\pi \in \text{Alg}(GL_d(K))$ (or $\pi \in \text{Alg}(M_d(K))$). The maximal number k such that $(\pi|_{M_d(K)})^{(k)} \neq (0)$ is called the level of non-degeneracy of π and denoted $\lambda(\pi)$.

Remark: cf [4] 3.5, there exists a natural filtration $0 \subset \tau_d \subset \dots \subset \tau_1 = \tau$ with

$$\tau_k = (\Phi^+)^{k-1} \circ (\Phi^-)^{k-1}(\tau) \text{ and } \tau_k/\tau_{k+1} = (\Phi^+)^{k-1} \circ_P \text{si}^+(\tau^{(k)}).$$

In particular for τ irreducible there is exactly one k such that $\tau^{(k)} \neq (0)$ and then $\tau \simeq (\Phi^+)^{k-1} \circ_P \text{si}^+(\tau^{(k)})$.

1.3.4. Notation. — *In the particular case where $k = d$, there is so a unique irreducible representation τ_{nd} with derivative of order d .*

Remark: Note then by [4] 4.4, for every irreducible supercuspidal representation π of $GL_d(K)$, we have

$$\pi|_{M_d(K)} \simeq \tau_{nd}.$$

Note also that for τ an irreducible entire $M_d(K)$ representation, its reduction modulo l is still irreducible. We can then understand theorem 1.2.11 as giving a partition of $\text{St}_t(\pi)|_{M_d(K)}$ that associates to each part an irreducible constituent of $r_l(\text{St}_t(\pi))$.

Consider first the following embedding $GL_r(K) \times M_s(K) \hookrightarrow M_{r+s}(K)$ sending

$$A \times M \mapsto \begin{pmatrix} A & U \\ 0 & M \end{pmatrix}.$$

By inducing we then define a functor

$$\rho \times \tau \in \text{Alg}(GL_r(K)) \times \text{Alg}(M_s(K)) \mapsto \rho \times \tau \in \text{Alg}(M_{r+s}(K)).$$

Secondly we consider $M_r(K) \times GL_s(K) \hookrightarrow M_{r+s}(K)$ sending

$$\begin{pmatrix} A & V \\ 0 & 1 \end{pmatrix} \times B \mapsto \begin{pmatrix} A & U & V \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we define

$$\tau \times \rho \in \text{Alg}(M_r(K)) \times \text{Alg}(GL_s(K)) \mapsto \tau \times \rho\nu^{-1/2} \in \text{Alg}(M_{r+s}(K)),$$

to be the compact induced representation.

1.3.5. Proposition. — (cf. [4] 4.13) *Let $\rho \in \text{Alg}(GL_r(K))$, $\sigma \in \text{Alg}(GL_t(K))$ and $\tau \in \text{Alg}(M_s(K))$.*

(a) *In $\text{Alg}(M_{r+t}(K))$, we have*

$$0 \rightarrow (\rho|_{M_r(K)}) \times \sigma \rightarrow (\rho \times \sigma)|_{M_{r+t}(K)} \rightarrow \rho \times (\sigma|_{M_t(K)}) \rightarrow 0.$$

(b) *If Ω is one of the functors Ψ^\pm, Φ^\pm , then $\rho \times \Omega(\tau) \simeq \Omega(\rho \times \tau)$.*

(c) $\Psi^-(\tau \times \rho) \simeq \Psi^-(\tau) \times \rho$ and

$$0 \rightarrow \Phi^-(\tau) \times \rho \longrightarrow \Phi^-(\tau \times \rho) \longrightarrow \Phi^-(\tau) \times (\rho|_{M_r(K)}) \rightarrow 0.$$

(d) Suppose $r > 0$. Then for any non-zero $M_{s+t}(K)$ -submodule $\omega \subset \tau \times \rho$, we have $\Phi^-(\omega) \neq (0)$.

1.3.6. Definition. — ([17] 5.1) A representation $\tau \in \text{Alg}(M_d(K))$ is called homogeneous if for all non-zero submodule $\sigma \subset \tau$, we have $\lambda(\sigma) = \lambda(\tau)$.

1.3.7. Proposition. — (cf. [17] 6.8) Let π be an irreducible representation of $GL_d(K)$. Then $\pi|_{M_d(K)}$ is homogeneous.

In the sequel we will use, in some sense dually, the group $P_d(F_v)$ with first column equals to $(1, 0, \dots, 0)$. The map $g \mapsto \sigma({}^t g^{-1})\sigma^{-1}$ where σ is the matrix permutation associated to the cycle $(1\ 2\ \dots\ n)$, induces an isomorphism between $P_d(F_v)$ and $M_d(F_v)$. After twisting with this isomorphism, we obtain analogs of the previous results with for example the following short exact sequence

$$0 \rightarrow \rho \times (\sigma|_{P_t(K)}) \longrightarrow (\rho \times \sigma)|_{P_{r+t}(K)} \longrightarrow (\rho|_{P_r(K)}) \times \sigma \rightarrow 0, \quad (1.3.8)$$

where the first representation is the compact induction relatively to

$$\begin{pmatrix} 1 & 0 & V_{t-1} \\ 0 & GL_r & U \\ 0 & 0 & GL_{t-1} \end{pmatrix},$$

and the second one is the induction from

$$\begin{pmatrix} P_r & U \\ 0 & GL_t \end{pmatrix}.$$

We will particularly use the following case.

1.3.9. Lemma. — Let π be an irreducible cuspidal representation of $GL_g(K)$. Then as a representation of $P_{(t+s)g}(K)$, we have isomorphisms

$$\text{St}_t(\pi\{-\frac{s}{2}\})|_{P_{tg}(K)} \times \text{Speh}_s(\pi\{\frac{t}{2}\}) \simeq LT_\pi(t-1, s)|_{P_{(t+s)g}(K)},$$

and

$$\text{St}_t(\pi\{-\frac{s}{2}\}) \times \text{Speh}_s(\pi\{\frac{t}{2}\})|_{P_{sg}(K)} \simeq LT_\pi(t, s-1)|_{P_{(t+s)g}(K)}.$$

Proof. — Recall the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{St}_t(\pi\{-\frac{s}{2}\}) \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\})|_{P_{sg}(K)} \\ \rightarrow \left(\mathrm{St}_t(\pi\{-\frac{s}{2}\}) \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}) \right)|_{P_{(t+s)g}(K)} \\ \rightarrow \mathrm{St}_t(\pi\{-\frac{s}{2}\})|_{P_{tg}(K)} \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}) \rightarrow 0, \end{aligned}$$

For every $k \geq 0$, about the k -th derivative we have

$$\left(\mathrm{St}_t(\pi\{-\frac{s}{2}\})|_{P_{tg}(K)} \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}) \right)^{(k)} \simeq \left(\mathrm{St}_t(\pi\{-\frac{s}{2}\})|_{P_{tg}(K)} \right)^{(k)} \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}).$$

Remind that the k -th derivative of $\mathrm{St}_t(\pi)$ is zero except if k is of the shape δg with $0 \leq \delta \leq t$ in which case it is isomorphic to $\mathrm{St}_{t-\delta}(\pi\{\frac{\delta}{2}\})$. Then arguing by induction on t , we deduce that $\mathrm{St}_t(\pi\{-\frac{s}{2}\})|_{P_{tg}(K)} \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\})$ and $LT_{\pi_v}(t-1, s)|_{P_{(t+s)g}(K)}$ have the same derivative and they are all of degree $\leq tg$. Consider then

$$\begin{aligned} LT_{\pi_v}(t-1, s)|_{P_{(t+s)g}(K)} \hookrightarrow \left(\mathrm{St}_t(\pi\{-\frac{s}{2}\}) \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}) \right)|_{P_{(t+s)g}(K)} \\ \rightarrow \mathrm{St}_t(\pi\{-\frac{s}{2}\})|_{P_{tg}(K)} \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}) \end{aligned}$$

and denote $K \hookrightarrow \mathrm{St}_t(\pi\{-\frac{s}{2}\}) \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\})|_{P_{sg}(K)}$ its kernel. By [17] proposition 5.3 and corolory 6.8, $\mathrm{St}_t(\pi\{-\frac{s}{2}\}) \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\})|_{P_{sg}(K)}$ is homogenous, i.e. every $P_{(t+s)g}(K)$ -equivariant irreducible subspace has a derivative of order $(t+1)g$, but we just saw that the derivative of $LT_{\pi}(t-1, s)|_{P_{(t+s)g}(K)}$ are of order $\leq tg$, so that

$$LT_{\pi}(t-1, s) \hookrightarrow \mathrm{St}_t(\pi\{-\frac{s}{2}\})|_{P_{tg}(K)} \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}).$$

As they have the same derivative, this injection is an isomorphism. \square

1.4. Some lattices of Steinberg representations. — Let π be an irreducible cuspidal $\overline{\mathbb{Q}}_l$ -representation of $GL_g(K)$, supposed to be entire. As its reduction modulo l , denoted ϱ , is still irreducible, up to isomorphism, it has an unique stable lattice, cf. [2] proposition 3.3.2 and its following remark.

1.4.1. Definition. — (cf. [7]) Given a stable lattice of $\text{St}_t(\pi)$, the surjection (resp. the embedding)

$$\text{St}_t(\pi) \times \pi\{t\} \twoheadrightarrow \text{St}_{t+1}(\pi), \quad \text{resp. } \text{St}_{t+1}(\pi) \hookrightarrow \text{St}_t(\pi\{1\}) \times \pi$$

gives a stable lattice of $\text{St}_{t+1}(\pi)$ so that inductively starting from $t = 1$, we construct a lattice denoted $RI_{\bar{\mathbb{Z}}_l, -}(\pi, t)$ (resp. $RI_{\bar{\mathbb{Z}}_l, +}(\pi, t)$). We then denote

$$RI_{\bar{\mathbb{F}}_l, -}(\pi, t) := RI_{\bar{\mathbb{Z}}_l, -}(\pi, t) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l, \quad \text{resp. } RI_{\bar{\mathbb{F}}_l, +}(\pi, t) := RI_{\bar{\mathbb{Z}}_l, +}(\pi, t) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l.$$

1.4.2. Proposition. — (cf. [7] propositions 3.2.2 and 3.2.7) For every $0 \leq k \leq \text{lg}_\varrho(s)$, there exists a unique length k sub-representation $V_{\varrho, \pm}(s; k)$ of $RI_{\bar{\mathbb{F}}_l, \pm}(\overleftarrow{[s-1]_\pi})$

$$(0) = V_{\varrho, \pm}(s; 0) \subsetneq V_{\varrho, \pm}(s; 1) \subsetneq \cdots \subsetneq V_{\varrho, \pm}(s; \text{lg}_\varrho(s)) = RI_{\bar{\mathbb{F}}_l, \pm}(\pi, s),$$

such that the image of $V_{\varrho, -}(s; k)$ (resp. $V_{\varrho, +}(s; k)$) in the Grothendieck group verifies the following property: all its irreducible constituents are strictly greater (resp. smaller) than any irreducible constituent of

$$W_{\varrho, -}(s; k) := V_{\varrho, -}(s; \text{lg}_\varrho(s)) / V_{\varrho, -}(s; k)$$

(resp. $W_{\varrho, +}(s; k) := V_{\varrho, +}(s; \text{lg}_\varrho(s)) / V_{\varrho, +}(s; k)$), relatively to the relation of order of 1.2.9.

1.4.3. Corollary. — If $\text{lg}_\varrho(s) \geq 2$, then we have two irreducible subspaces of $RI_{\bar{\mathbb{Z}}_l, +}(\pi, s)|_{P_{sg}(K)} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$ which are

- first some irreducible $P_{sg}(K)$ -subspaces of $\text{St}_{\underline{s}}(\varrho)$ which is necessary degenerate,
- and the non degenerate irreducible $P_{sg}(K)$ -representation, τ_{nd} which is a subspace of $\text{St}_{\underline{s_{\max}}}(\varrho)|_{P_{sg}(K)}$.

1.4.4. Proposition. — The only irreducible subspace of the modulo l reduction of $RI_{\bar{\mathbb{Z}}_l, -}(\pi, s)|_{P_{sg}(K)}$ is the non degenerate one τ_{nd} .

Proof. — From the previous section, we have

$$RI_{\bar{\mathbb{Z}}_l, -}(\pi, s)|_{P_{sg}(K)} \simeq RI_{\bar{\mathbb{Z}}_l, -}(\pi\{\frac{-1}{2}\}, s-1) \times (\pi\{\frac{s-2}{2}\})|_{P_{sg}(K)},$$

so that the result follows by induction using proposition 1.3.5. \square

2. Review on the geometric objects

2.1. Lubin-Tate spaces. — Let \mathcal{O}_K the ring of integers of K , \mathcal{P}_K its maximal ideal, ϖ_K an uniformizer and $\kappa = \mathcal{O}_K/\mathcal{P}_K$ the residue field of cardinal $q = p^f$. Let K^{nr} be the maximal unramified extension of K and \widehat{K}^{nr} its completion with ring of integers $\mathcal{O}_{\widehat{K}^{nr}}$. Let $\Sigma_{K,d}$ be the \mathcal{O}_K -formal module of Barsotti-Tate over $\overline{\mathbb{F}}_p$ with height d , cf. [13] §II. We consider the category \mathcal{C} of \mathcal{O}_K -artinian local algebra with residue field $\overline{\kappa}$.

2.1.1. Definition. — *The functor which associates to an object R of \mathcal{C} , the set of isomorphism classes of deformation by quasi-isogenies over R of $\Sigma_{K,d}$, equipped with a n -level structure, is pro-representable by formal scheme $\widehat{\mathcal{M}}_{LT,d,n} = \prod_{h \in \mathbb{Z}} \widehat{\mathcal{M}}_{LT,d,n}^{(h)}$ where $\widehat{\mathcal{M}}_{LT,d,n}^{(h)}$ represents the subfunctor of deformations by quasi-isogenies with height h .*

Remark: each of the $\widehat{\mathcal{M}}_{LT,d,n}^{(h)}$ is non canonically isomorphic to the formal scheme $\widehat{\mathcal{M}}_{LT,d,n}^{(0)}$ denoted $\mathrm{Spf} \mathrm{Def}_{d,n}$ in [5]. We will use the notations without hat for the Berkovich generic fibers which are \widehat{K}^{nr} -analytic spaces in the sense of [3] and we note $\mathcal{M}_{LT,n}^{d/K} := \mathcal{M}_{LT,d,n} \widehat{\otimes}_{\widehat{K}^{nr}} \widehat{K}$.

The group of quasi-isogenies of $\Sigma_{K,d}$ is isomorphic to the unity group $D_{K,d}^\times$ of the central division algebra over K with invariant $1/d$, which then acts on $\mathcal{M}_{LT,n}^{d/K}$. For all $n \geq 1$, we have a natural action of $GL_d(\mathcal{O}_K/\mathcal{P}_K^n)$ on the level structures and then on $\mathcal{M}_{LT,n}^{d/K}$; this action can be extended to $GL_d(K)$ on the projective limit $\varprojlim_n \mathcal{M}_{LT,n}^{d/K}$ which is then equipped with the action of $GL_d(K) \times D_{K,d}^\times$ which factorises by $(GL_d(K) \times D_{K,d}^\times)/K^\times$ where K^\times is embedded diagonally.

2.1.2. Definition. — *Let $\Psi_{K,\Lambda,d,n}^i \simeq H^i(\mathcal{M}_{LT,d,n}^{(0)} \widehat{\otimes}_{\widehat{K}^{nr}} \widehat{K}, \Lambda)$, the Λ -module of finite type associated, by the vanishing cycle theory of Berkovich, to the structural morphism $\widehat{\mathcal{M}}_{LT,d,n}^{(0)} \rightarrow \mathrm{Spf} \widehat{\mathcal{O}}_K^{nr}$.*

We also introduce $\mathcal{U}_{K,\Lambda,d,n}^i := H^i(\mathcal{M}_{LT,n}^{d/K}, \Lambda)$ and $\mathcal{U}_{K,\Lambda,d}^i = \varinjlim_n \mathcal{U}_{K,\Lambda,d,n}^i$ as well as the cohomology groups with compact supports

$$\mathcal{V}_{K,\Lambda,d,n}^i := H_c^i(\mathcal{M}_{LT,n}^{d/K}, \Lambda), \text{ and } \mathcal{V}_{K,\Lambda,d}^i = \varinjlim_n \mathcal{V}_{K,\Lambda,d,n}^i.$$

As $\mathfrak{K}_n := \text{Ker}(GL_d(\mathcal{O}_K) \rightarrow GL_d(\mathcal{O}_K/\mathcal{P}_K^n))$ is pro- p for all $n \geq 1$, then we have $\mathcal{U}_{K,\Lambda,d,n}^i = (\mathcal{U}_{K,\Lambda,d}^i)^{\mathfrak{K}_n}$ and $\mathcal{V}_{K,\Lambda,d,n}^i = (\mathcal{V}_{K,\Lambda,d}^i)^{\mathfrak{K}_n}$.

The description of the $\mathcal{U}_{K,\overline{\mathbb{Q}}_l,d}^i$ is given in [5] theorem 2.3.5. We will denote $\mathcal{U}_{K,\overline{\mathbb{Z}}_l,d,\text{free}}^i$ (resp. $\mathcal{V}_{K,\overline{\mathbb{Z}}_l,d,\text{free}}^i$) the free quotient which is the full of $\mathcal{U}_{K,\overline{\mathbb{Z}}_l,d}^i$ (resp. $\mathcal{V}_{K,\overline{\mathbb{Z}}_l,d}^i$) by the main result of [8].

2.2. Global Ihara's lemma. — Let $F = F^+E$ be a CM field with E/\mathbb{Q} quadratic imaginary. For \overline{B}/F a central division algebra with dimension d^2 equipped with a involution of second specie $*$ and $\beta \in \overline{B}^{*-1}$, consider the similitude group \overline{G}/\mathbb{Q} defined for any \mathbb{Q} -algebra R by

$$\overline{G}(R) := \{(\lambda, g) \in R^\times \times (\overline{B}^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp\beta} = \lambda\}$$

with $\overline{B}^{op} = \overline{B} \otimes_{F,c} F$ where $c = *_|_F$ is the complex conjugation and \sharp_β the involution $x \mapsto x^{\sharp\beta} = \beta x^* \beta^{-1}$. For $p = uu^c$ decomposed in E , we have

$$\overline{G}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (\overline{B}_v^{op})^\times$$

where w describes the places of F above u . We suppose:

- the associated unitary group $\overline{G}_0(\mathbb{R})$ being compact,
- for any place x of \mathbb{Q} inert or ramified in E , then $G(\mathbb{Q}_x)$ is quasi-split.
- There exists a place v_0 of F above u such that $\overline{B}_{v_0} \simeq D_{v_0,d}$ is the central division algebra over the completion F_{v_0} of F at v_0 , with invariant $\frac{1}{d}$.

Consider a finite set S of \mathbb{Q} -places containing the ramification places of B . Let denote $\mathbb{T}_S/\overline{\mathbb{Z}}_l$ the unramified Hecke algebra of G . For a cohomological minimal prime ideal $\tilde{\mathfrak{m}}$ of \mathbb{T}_S , we can associate both near equivalence class of $\overline{\mathbb{Q}}_l$ -automorphic representation $\Pi_{\tilde{\mathfrak{m}}}$ and a Galois representation

$$\rho_{\tilde{\mathfrak{m}}} : G_F := \text{Gal}(\overline{F}/F) \rightarrow GL_d(\overline{\mathbb{Q}}_l)$$

such that the eigenvalues of the frobenius morphism at an unramified place w are given by the Satake's parameter of the local component $\Pi_{\tilde{\mathfrak{m}},w}$ of $\Pi_{\tilde{\mathfrak{m}}}$. The semi-simple class of the reduction modulo l of $\rho_{\tilde{\mathfrak{m}}}$ depends only of the maximal ideal \mathfrak{m} of \mathbb{T} containing $\tilde{\mathfrak{m}}$.

2.2.1. Conjecture. — (*Generalized Ihara's lemma*) Consider

- an open compact subgroup \overline{U} of $\overline{G}(\mathbb{A})$ such that outside S its local component is the maximal compact subgroup;
- a place $w_0 \notin S$ decomposed in E ;

– a maximal \mathfrak{m} of \mathbb{T}_S such that $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

Let $\bar{\pi}$ be an irreducible sub-representation of $C^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}) / \overline{U}^{w_0}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, where $\overline{U} = \overline{U}_{w_0} \overline{U}^{w_0}$, then its local component $\bar{\pi}_{w_0}$ at w_0 is generic.

Attached to \overline{G} is a zero dimensional tower of Shimura variety $\text{Sh}_{\overline{G}, \overline{U}}$ over F , indexed by the open compact subgroup \overline{U} of $\overline{G}(\mathbb{A})$, such that $H^0(\text{Sh}_{\overline{G}, \overline{U}} \times_F \overline{F}, \overline{\mathbb{F}}_l) \simeq C^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}) / \overline{U}^{w_0}, \overline{\mathbb{F}}_l)$.

2.3. KHT-Shimura varieties. — Consider now the similitude group G/\mathbb{Q} such that

- $G(\mathbb{A}^\infty) = \overline{G}(\mathbb{A}^{\infty, p}) \times \left(\mathbb{Q}_{p_{v_0}}^\times GL_d(F_{v_0}) \times \prod_{\substack{w|u \\ w \neq v_0}} (\overline{B}_w^{op})^\times \right)$,
- the signature of $G(\mathbb{R})$ are $(1, d-1) \times (0, d) \times \cdots \times (0, d)$.

For all open compact subgroup U^p of $G(\mathbb{A}^{\infty, p})$ and $m = (m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$, we consider

$$U^p(m) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^r \text{Ker}(\mathcal{O}_{B_{v_i}}^\times \longrightarrow (\mathcal{O}_{B_{v_i}} / \mathcal{P}_{v_i}^{m_i})^\times).$$

We then denote \mathcal{I} for the set of these $U^p(m)$ such that it exists a place x for which the projection from U^p to $G(\mathbb{Q}_x)$ doesn't contain any element with finite order except the identity, cf. [13] below of page 90.

Attached to each $I \in \mathcal{I}$ is a Shimura variety $X_I \rightarrow \text{Spec } \mathcal{O}_v$ of type Kottwitz-Harris-Taylor. The projective system $X_{\mathcal{I}} = (X_I)_{I \in \mathcal{I}}$ is then equipped with a Hecke action of $G(\mathbb{A}^\infty)$, the transition morphisms $r_{J, I} : X_J \rightarrow X_I$ being finite flat and even etale when $m_1(J) = m_1(I)$.

2.3.1. Notations. — (cf. [5] §1.3) Let $I \in \mathcal{I}$,

- the special fiber of X_I will be denoted $X_{I, s}$ and its geometric special fiber $X_{I, \bar{s}} := X_{I, s} \times \text{Spec } \overline{\mathbb{F}}_p$.
- For $1 \leq h \leq d$, let $X_{I, \bar{s}}^{\geq h}$ (resp. $X_{I, \bar{s}}^{=h}$) be the closed (resp. open) Newton stratum of height h , defined as the subscheme where the connected component of the universal Barsotti-Tate group is of rank greater or equal to h (resp. equal to h).

Remark: $X_{I, \bar{s}}^{\geq h}$ is of pure dimension $d - h$. For $1 \leq h < d$, the Newton stratum $X_{I, \bar{s}}^{=h}$ is geometrically induced under the action of the parabolic

subgroup $P_{h,d-h}(\mathcal{O}_v)$ in the sense where there exists a closed subscheme $X_{I,\bar{s},\bar{1}_h}^{=h}$ stabilized by the Hecke action of $P_{h,d-h}(\mathcal{O}_v)$ and such that

$$X_{\mathcal{I},\bar{s}}^{=h} \simeq X_{\mathcal{I},\bar{s},\bar{1}_h}^{=h} \times_{P_{h,d-h}(\mathcal{O}_v)} GL_d(\mathcal{O}_v).$$

Let denote $\mathcal{G}(h)$ the universal Barsotti-Tate group over $X_{I,\bar{s},\bar{1}_h}^{=h}$:

$$0 \rightarrow \mathcal{G}(h)^c \rightarrow \mathcal{G}(h) \rightarrow \mathcal{G}(h)^{et} \rightarrow 0$$

where $\mathcal{G}(h)^c$ (resp. $\mathcal{G}(h)^{et}$) is connected (resp. étale) of dimension h (resp. $d-h$). Denote $\iota_{m_1} : (\mathcal{P}_v^{-m_1}/\mathcal{O}_v)^d \rightarrow \mathcal{G}(h)[p^{m_1}]$ the universal level structure. If we denote $(e_i)_{1 \leq i \leq d}$ the canonical basis of $(\mathcal{P}_v^{-m_1}/\mathcal{O}_v)^d$, then the Newton stratum $X_{I,\bar{s},\bar{1}_h}^{=h}$ is defined by asking $\{\iota_{m_1}(e_i) : 1 \leq i \leq h\}$ to be a Drinfeld basis of $\mathcal{G}(h)^c[p^{m_1}]$.

2.3.2. Notation. — *In the following, we won't make any distinction between an element $a \in GL_d(F_v)/P_{h,d-h}(F_v)$ and the subspace $\langle a(e_1), \dots, a(e_h) \rangle$ generated by the image through a of the first h vectors e_1, \dots, e_h of the canonical basis of F_v^d . Let denote $P_a(F_v) := aP_{h,d-h}(F_v)a^{-1}$ the parabolic subgroup of elements of $GL_d(F_v)$ stabilizing $a \subset F_v^d$.*

For an ideal $I \in \mathcal{I}$, the element $a \in GL_d(F_v)/P_{h,d-h}(F_v)$ gives a direct factor a_{m_1} of $(\mathcal{P}_v^{-m_1}/\mathcal{O}_v)^d$ and so a stratum $X_{I,\bar{s},a}^{=h}$ which is defined by asking for a basis (f_1, \dots, f_h) of a_{m_1} , that $\{\iota_{m_1}(f_i) : 1 \leq i \leq h\}$ is a Drinfeld basis of $\mathcal{G}(h)^c[p^{m_1}]$. We also denote $X_{I,\bar{s},a}^{\geq h}$ its closure in $X_{I,\bar{s}}^{\geq h}$. Such a stratum is said pure compared to the following situation. For a pure stratum $X_{\mathcal{I},\bar{s},c}^{=h}$ and $h' \geq h$, let denote

$$X_{\mathcal{I},\bar{s},c}^{=h'} := \coprod_{\substack{a: \dim a=h' \\ c \subset a}} X_{\mathcal{I},\bar{s},a}^{=h'}$$

and $X_{\mathcal{I},\bar{s},c}^{\geq h'}$ its closure.

2.4. Harris-Taylor perverse sheaves. — We recall now some notations about Harris-Taylor local systems of [13]. Let π_v be a irreducible cuspidal $\overline{\mathbb{Q}}_l$ -representation of $GL_g(F_v)$. Fix $t \geq 1$ such that $tg \leq d$. The Jacquet-Langlands correspondence associates to $\text{St}_t(\pi_v)$, an irreducible representation $\pi_v[t]_D$ of $D_{v,tg}^\times$ which thanks to Igusa varieties, gives a

local system on $X_{\mathcal{L}, \bar{s}, \bar{1}_h}^{-=tg}$

$$\mathcal{L}(\pi_v[t]_D)_{\bar{1}_h} = \bigoplus_{i=1}^{e_{\pi_v}} \mathcal{L}_{\bar{\mathbb{Q}}_l}(\rho_{v,i})_{\bar{1}_h}$$

where $(\pi_v[t]_D)|_{\mathcal{D}_{v,h}^\times} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$ with $\rho_{v,i}$ irreducible. The Hecke action of $P_{tg,d-tg}(F_v)$ is then given through its quotient $GL_{d-tg} \times \mathbb{Z}$.

2.4.1. Notations. — For Π_t any representation of GL_{tg} and $\Xi : \frac{1}{2}\mathbb{Z} \rightarrow \bar{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2}) = q^{1/2}$, we introduce

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\bar{1}_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}(\pi_v, \Pi_t) := \left(\mathcal{L}(\pi_v[t]_D)_{\bar{1}_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}} \right) \times_{P_{tg,d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

is given

- by the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$, and
- the action of $(g^{\infty,v}, g_v^{et}, \text{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ on $\mathcal{L}_{\bar{\mathbb{Q}}_l}(\pi_v[t]_D)_{\bar{1}_h} \otimes \Xi^{\frac{tg-d}{2}}$.

We also introduce

$$HT(\pi_v, \Pi_t)_{\bar{1}_h} := \widetilde{HT}(\pi_v, \Pi_t)_{\bar{1}_h}[d - tg],$$

and the perverse sheaf

$$P(t, \pi_v)_{\bar{1}_h} := j_{\bar{1}_h, !*}^{-=tg} HT(\pi_v, \text{St}_t(\pi_v))_{\bar{1}_h} \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT(\pi_v, \Pi_t)$ and $P(t, \pi_v)$, where

$$j = h = i^h \circ j^{\geq h} : X_{\mathcal{L}, \bar{s}}^{-=h} \hookrightarrow X_{\mathcal{L}, \bar{s}}^{\geq h} \hookrightarrow X_{\mathcal{L}, \bar{s}}$$

and \mathbb{L}^\vee is the local Langlands correspondence.

Remark: recall that π'_v is said inertially equivalent to π_v if there exists a character $\zeta : \mathbb{Z} \rightarrow \bar{\mathbb{Q}}_l^\times$ such that $\pi'_v \simeq \pi_v \otimes (\zeta \circ \text{val} \circ \det)$. Note then that $P(t, \pi_v)$ depends only on the inertial class of π_v and

$$P(t, \pi_v) = e_{\pi_v} \mathcal{P}(t, \pi_v)$$

where $\mathcal{P}(t, \pi_v)$ is an irreducible perverse sheaf.

the weight filtration of $j_{c,!}^{-tg} HT_c(\pi_v, \text{St}_t(\pi_v))$, so that we obtain

$$\begin{aligned} & {}^p\mathcal{H}^{-1} i_{c'}^{tg+1,*} \left({}^p j_{c,!}^{-tg} HT_c(\pi_v, \text{St}_t(\pi_v)) \right) \rightarrow \\ & {}^p j_{\langle c,c' \rangle,!}^{=(t+1)g} HT_{\langle c,c' \rangle} \left(\pi_v, \left(\text{St}_t(\pi_v \left\{ \frac{-1}{2} \right\}) \right)_{|P_{c,c'}(F_v)} \otimes \pi_v \left\{ \frac{t}{2} \right\} \right) \otimes \Xi^{\frac{1}{2}} \end{aligned}$$

where by lemma 1.3.9,

$$\left(\text{St}_t(\pi_v \left\{ \frac{-1}{2} \right\}) \right)_{|P_{c,c'}(F_v)} \times \pi_v \left\{ \frac{t}{2} \right\} \simeq \text{St}_{t+1}(\pi_v)_{|P_{c,c'}(F_v)}.$$

We then obtain a surjection from ${}^p\mathcal{H}^{-1} i_{c'}^{1,*} ({}^p j_{c,!}^{-tg} HT_c(\pi_v, \text{St}_t(\pi_v)))$ to the expected perverse sheaf so that now it suffices to prove that their cohomology sheaves have the same germs at every geometric points.

Let then z be a geometric point of $X_{\mathcal{I},\bar{s},c'}^=h$. The germ at z of the i th sheaf of cohomology of ${}^p\mathcal{H}^{-1} i_{c'}^{1,*} ({}^p j_{c,!}^{-tg} HT_c(\pi_v, \text{St}_t(\pi_v)))$ is isomorphic to those of $(i-1)$ -th of ${}^p j_{c,!}^{-tg} HT_c(\pi_v, \text{St}_t(\pi_v))$. Then by [5], this germ is

- zero if (h, i) is not of the shape $((t+\delta)g, (t+\delta)g - d - \delta)$ with $(t+\delta)g \leq d$,
- otherwise it's isomorphic to the germ at z of $HT(\pi_v, \Pi)$ where Π is the normalized induced representation

$$\Pi := \left(\text{St}_t(\pi_v \left\{ -\frac{\delta}{2} \right\}) \right)_{|P_{c,c'}(F_v)} \times \text{Speh}_{\delta}(\pi_v \left\{ \frac{t}{2} \right\}) \simeq LT(t, \delta - 1, \pi_v)_{|P_{c,c'}(F_v)},$$

the last isomorphis being given by lemma 1.3.9.

About the germs at z of the i -th sheaf of cohomology of the first term of the short exact sequence of the statement, it has the same condition of cancellation and otherwise we obtain the germ at z of $HT(\pi_v, \Pi')$ with

$$\Pi' := \text{St}_{t+1}(\pi_v)_{|P_{c,c'}(F_v)} \left\{ \frac{1-\delta}{2} \right\} \times \text{Speh}_{\delta-1}(\pi_v \left\{ \frac{t+1}{2} \right\})$$

which, by lemma 1.3.9, is isomorphic to $\Pi \simeq LT_{\pi_v}(t, \delta - 1)_{|P_{c,c'}(F_v)}$. \square

3. Some coarse filtrations of $\Psi_{\mathcal{I}}$

3.1. Filtrations of free perverse sheaves. — Let $S = \text{Spec } \mathbb{F}_q$ and X/S of finite type, then the usual t -structure on $\mathcal{D}(X, \overline{\mathbb{Z}}_l) := D_c^b(X, \overline{\mathbb{Z}}_l)$ is

$$\begin{aligned} A \in {}^p\mathcal{D}^{\leq 0}(X, \overline{\mathbb{Z}}_l) &\Leftrightarrow \forall x \in X, \mathcal{H}^k i_x^* A = 0, \forall k > -\dim \overline{\{x\}} \\ A \in {}^p\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l) &\Leftrightarrow \forall x \in X, \mathcal{H}^k i_x^! A = 0, \forall k < -\dim \{x\} \end{aligned}$$

where $i_x : \text{Spec } \kappa(x) \hookrightarrow X$ and $\mathcal{H}^k(K)$ is the k -th sheaf of cohomology of K .

3.1.1. Notation. — Let denote ${}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l)$ the heart of this t -structure with associated cohomology functors ${}^p\mathcal{H}^i$. For a functor T we denote ${}^pT := {}^p\mathcal{H}^0 \circ T$.

The category ${}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l)$ is abelian equipped with a torsion theory $(\mathcal{T}, \mathcal{F})$ where \mathcal{T} (resp. \mathcal{F}) is the full subcategory of objects T (resp. F) such that $l^N 1_T$ is trivial for some large enough N (resp. $l \cdot 1_F$ is a monomorphism). Applying Grothendieck-Verdier duality, we obtain

$$\begin{aligned} {}^{p+}\mathcal{D}^{\leq 0}(X, \overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\leq 1}(X, \overline{\mathbb{Z}}_l) : {}^p\mathcal{H}^1(A) \in \mathcal{T}\} \\ {}^{p+}\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l) : {}^p\mathcal{H}^0(A) \in \mathcal{F}\} \end{aligned}$$

with heart ${}^{p+}\mathcal{C}(X, \overline{\mathbb{Z}}_l)$ equipped with its torsion theory $(\mathcal{F}, \mathcal{T}[-1])$.

3.1.2. Definition. — (cf. [9] §1.3) Let

$$\mathcal{F}(X, \overline{\mathbb{Z}}_l) := {}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{C}(X, \overline{\mathbb{Z}}_l) = {}^p\mathcal{D}^{\leq 0}(X, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l)$$

the quasi-abelian category of free perverse sheaves over X .

Remark: for an object L of $\mathcal{F}(X, \overline{\mathbb{Z}}_l)$, we will consider filtrations

$$L_1 \subset L_2 \subset \cdots \subset L_e = L$$

such that for every $1 \leq i \leq e - 1$, $L_i \hookrightarrow L_{i+1}$ is a strict monomorphism, i.e. L_{i+1}/L_i is an object of $\mathcal{F}(X, \overline{\mathbb{Z}}_l)$.

Consider an open subscheme $j : U \hookrightarrow X$ and $i : F := X \setminus U \hookrightarrow X$. Then

$${}^{p+}j_! \mathcal{F}(U, \Lambda) \subset \mathcal{F}(X, \Lambda) \quad \text{and} \quad {}^p j_* \mathcal{F}(U, \Lambda) \subset \mathcal{F}(X, \Lambda).$$

Moreover if j is affine then $j_!$ is t -exact and $j_! = {}^p j_! = {}^{p+} j_!$.

3.1.3. Lemma. — Consider $L \in \mathcal{F}(X, \Lambda)$ such that $j_! j^* L \in \mathcal{F}(X, \Lambda)$. Then $i_* {}^p\mathcal{H}^{-\delta} i^* L$ is trivial for every $\delta \neq 0, 1$; for $\delta = 1$ it belongs to $\mathcal{F}(X, \Lambda)$.

Proof. — Start from the following distinguished triangle $j_! j^* L \rightarrow L \rightarrow i_* i^* L \rightsquigarrow$. From the perversity of L and $j_! j^* L$, the long exact sequence of perverse cohomology is

$$0 \rightarrow i_* {}^p\mathcal{H}^{-1} i^* L \rightarrow {}^p j_! j^* L \rightarrow L \rightarrow i_* {}^p\mathcal{H}^0 i^* L \rightarrow 0.$$

The freeness of $i_* {}^p\mathcal{H}^{-1} i^* L$ then follows from those of i_* and i^* of $i_* {}^p\mathcal{H}^0 i^* L$. ${}^p j_! j^* L = j_! j^* L$. \square

3.1.4. Definition. — A bimorphism of $\mathcal{F}(X, \Lambda)$, written $L \hookrightarrow L'$, is both a monomorphism and an epimorphism. If moreover the cokernel in ${}^{p+}\mathcal{C}(X, \Lambda)$ is of dimension strictly less than those of the support of L , we will write $L \hookrightarrow_+ L'$.

For a free $L \in \mathcal{F}(X, \Lambda)$, we consider the following diagram

$$\begin{array}{ccccc}
 & & L & & \\
 & \nearrow^{\text{can}_{!,L}} & & \searrow^{\text{can}_{*,L}} & \\
 {}^{p+}j_!j^*L & \twoheadrightarrow & {}^{p+}j_!j^*L & \xrightarrow{+} & {}^{p+}j_!j^*L & \twoheadrightarrow & {}^{p+}j_!j^*L
 \end{array}$$

where below is, cf. the remark following 1.3.12 de [9], the canonical factorisation of ${}^{p+}j_!j^*L \rightarrow {}^{p+}j_!j^*L$ and where the maps $\text{can}_{!,L}$ and $\text{can}_{*,L}$ are given by the adjunction property.

3.1.5. Notation. — (cf. lemma 2.1.2 of [9]) We introduce the filtration $\text{Fil}_{U,!}^{-1}(L) \subset \text{Fil}_{U,!}^0(L) \subset L$ with

$$\text{Fil}_{U,!}^0(L) = \text{Im}_{\mathcal{F}}(\text{can}_{!,L}) \quad \text{and} \quad \text{Fil}_{U,!}^{-1}(L) = \text{Im}_{\mathcal{F}}\left((\text{can}_{!,L})|_{\mathcal{P}_L}\right),$$

where $\mathcal{P}_L := i_*{}^p\mathcal{H}_{\text{libre}}^{-1}i^*j_*j^*L$ is the kernel of $\text{Ker}_{\mathcal{F}}\left({}^{p+}j_!j^*L \rightarrow {}^{p+}j_!j^*L\right)$.

Remark: we have $L/\text{Fil}_{U,!}^0(L) \simeq i_*{}^{p+}i^*L$ and ${}^{p+}j_!j^*L \hookrightarrow_+ \text{Fil}_{U,!}^0(L)/\text{Fil}_{U,!}^{-1}(L)$, which gives, cf. lemma 1.3.13 of [9], a commutative triangle

$$\begin{array}{ccc}
 {}^{p+}j_!j^*L & \xrightarrow{+} & \text{Fil}_{U,!}^0(L)/\text{Fil}_{U,!}^{-1}(L) \\
 & \searrow^{+} & \downarrow^{+} \\
 & & {}^{p+}j_!j^*L.
 \end{array}$$

3.1.6. Notation. — (cf. [9] 2.1.4) Dually there is a cofiltration $L \rightarrow \text{CoFil}_{U,*,0}(L) \rightarrow \text{CoFil}_{U,*,1}(L)$ where

$$\text{CoFil}_{U,*,0}(L) = \text{Coim}_{\mathcal{F}}(\text{can}_{*,L}) \quad \text{and} \quad \text{CoFil}_{U,*,1}(L) = \text{Coim}_{\mathcal{F}}(p_L \circ \text{can}_{*,L}),$$

with $p_L : {}^{p+}j_!j^*L \rightarrow Q_L := i_*{}^p\mathcal{H}_{\text{libre}}^0i^*j_*j^*L$.

Remark: the kernel $\text{cogr}_{U,*,1}(L)$ of $\text{CoFil}_{U,*,0}(L) \rightarrow \text{CoFil}_{U,*,1}(L)$ verifies

$${}^{p+}j_!j^*L \hookrightarrow_+ \text{cogr}_{U,*,1}(L) \hookrightarrow_+ {}^{p+}j_!j^*L.$$

The kernel $\text{cogr}_{U,*,0}(L)$ of $L \rightarrow \text{CoFil}_{U,*,0}(L)$ is isomorphic to $i_*{}^pi^!L$.

Consider now X equipped with a stratification

$$X = X^{\geq 1} \supset X^{\geq 2} \supset \dots \supset X^{\geq d},$$

and let $L \in \mathcal{F}(X, \overline{\mathbb{Z}}_l)$. For $1 \leq h < d$, let denote $X^{1 \leq h} := X^{\geq 1} - X^{\geq h+1}$ and $j^{1 \leq h} : X^{1 \leq h} \hookrightarrow X^{\geq 1}$. We then define

$$\mathrm{Fil}_!^r(L) := \mathrm{Im}_{\mathcal{F}} \left({}^{p+}j_!^{1 \leq r} j^{1 \leq r,*} L \longrightarrow L \right),$$

which gives a filtration

$$0 = \mathrm{Fil}_!^0(L) \subset \mathrm{Fil}_!^1(L) \subset \mathrm{Fil}_!^2(L) \cdots \subset \mathrm{Fil}_!^{d-1}(L) \subset \mathrm{Fil}_!^d(L) = L.$$

Dually, the following

$$\mathrm{CoFil}_{*,r}(L) = \mathrm{Coim}_{\mathcal{F}} \left(L \longrightarrow {}^p j_*^{1 \leq r} j^{1 \leq r,*} L \right),$$

define a cofiltration

$$\begin{aligned} L = \mathrm{CoFil}_{\mathfrak{S},*,d}(L) &\twoheadrightarrow \mathrm{CoFil}_{\mathfrak{S},*,d-1}(L) \twoheadrightarrow \cdots \\ &\cdots \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{S},*,1}(L) \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{S},*,0}(L) = 0, \end{aligned}$$

and a filtration

$$0 = \mathrm{Fil}_*^{-d}(L) \subset \mathrm{Fil}_*^{-d+1}(L) \subset \cdots \subset \mathrm{Fil}_*^0(L) = L$$

where

$$\mathrm{Fil}_*^{-r}(L) := \mathrm{Ker}_{\mathcal{F}}(L \twoheadrightarrow \mathrm{CoFil}_{*,r}(L)).$$

Note these two constructions are exchanged by Grothendieck-Verdier duality,

$$D\left(\mathrm{CoFil}_{\mathfrak{S},!,-r}(L)\right) \simeq \mathrm{Fil}_{\mathfrak{S},*}^{-r}(D(L)) \text{ and } D\left(\mathrm{CoFil}_{\mathfrak{S},*,r}(L)\right) \simeq \mathrm{Fil}_{\mathfrak{S},!}^r(D(L)).$$

We can also refine the previous filtrations with the help of $\mathrm{Fil}_{U,!}^{-1}(L)$, cf. [9] proposition 2.3.3, to obtain exhaustive filtrations

$$\begin{aligned} 0 = \mathrm{Fill}_!^{-2^{d-1}}(L) &\subset \mathrm{Fill}_!^{-2^{d-1}+1}(L) \subset \cdots \\ &\cdots \subset \mathrm{Fill}_!^0(L) \subset \cdots \subset \mathrm{Fill}_!^{2^{d-1}-1}(L) = L, \end{aligned} \quad (3.1.7)$$

such that the graduate $\mathrm{grr}^k(L)$ are simple over $\overline{\mathbb{Q}}_l$, simples, i.e. verify ${}^p j_{!*}^{=h} j^{=h,*} \mathrm{grr}^k(L) \hookrightarrow_+ \mathrm{grr}^k(L)$ for some h . Dually using $\mathrm{coFil}_{U,*,1}(L)$, we construct a cofiltration

$$L = \mathrm{CoFill}_{*,2^{d-1}}(L) \twoheadrightarrow \mathrm{CoFill}_{*,2^{d-1}-1}(L) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{CoFill}_{*,-2^{d-1}}(L) = 0$$

and a filtration $\text{Fill}_*^{-r}(L) := \text{Ker}_{\mathcal{F}}(L \rightarrow \text{CoFill}_{*,r}(L))$. These two constructions are exchanged by duality

$$D(\text{CoFill}_{*,r}(L)) \simeq \text{Fill}_*^r(D(L)) \text{ and } D(\text{CoFill}_{l,r}(L)) \simeq \text{Fill}_*^r(D(L))$$

and can be mixed if we want to.

3.2. Supercuspidal decomposition of $\Psi_{\mathcal{I}}$. —

3.2.1. Notation. — For $I \in \mathcal{I}$, let

$$\Psi_{I,\Lambda} := R\Psi_{\eta_v,I}(\Lambda[d-1])\left(\frac{d-1}{2}\right)$$

be the vanishing cycle autodual perverse sheaf on $X_{I,\bar{s}}$. When $\Lambda = \bar{\mathbb{Z}}_l$, we will simply write $\Psi_{\mathcal{I}}$.

Recall the following result of [13] relating $\Psi_{\mathcal{I}}$ with Harris-Taylor local systems.

3.2.2. Proposition. — (cf. [13] proposition IV.2.2 et le §2.4 de [5])
There is an isomorphism $G(\mathbb{A}^{\infty,v}) \times P_{h,d-h}(F_v) \times W_v$ -equivariant

$$\text{ind}_{(D_{v,h}^{\times})^0 \varpi_v^{\mathbb{Z}}}^{D_{v,h}^{\times}} (\mathcal{H}^{h-d-i} \Psi_{\mathcal{I},\bar{\mathbb{Z}}_l})|_{X_{\mathcal{I},\bar{s},\bar{1}_h}^{=h}} \simeq \bigoplus_{\bar{\tau} \in \mathcal{R}_{\bar{\mathbb{F}}_l}(h)} \mathcal{L}_{\bar{\mathbb{Z}}_l, \bar{1}_h}(\mathcal{U}_{\bar{\tau},\mathbb{N}}^{h-1-i}),$$

where

- $\mathcal{R}_{\bar{\mathbb{F}}_l}(h)$ is the set of equivalence classes of irreducible $\bar{\mathbb{F}}_l$ -representations of $D_{v,h}^{\times}$;
- for $\bar{\tau} \in \mathcal{R}_{\bar{\mathbb{F}}_l}(h)$ and V a $\bar{\mathbb{Z}}_l$ -representation of $D_{v,h}^{\times}$, then $V_{\bar{\tau}}$ denotes, cf. [11] §B.2, the direct factor of V whose irreducible subquotients are isomorphic to a subquotient of $\bar{\tau}|_{\mathcal{D}_{v,h}^{\times}}$ where $\mathcal{D}_{v,h}$ is the maximal order of $D_{v,h}$.
- With the previous notation, $\mathcal{U}_{\bar{\tau},\mathbb{N}}^i := (\mathcal{U}_{F_v, \bar{\mathbb{Z}}_l, d}^i)_{\bar{\tau}}$.
- The matching between the system indexed by \mathcal{I} and those by \mathbb{N} is given by the application $m_1 : \mathcal{I} \rightarrow \mathbb{N}$.

Remark: for $\bar{\tau} \in \mathcal{R}_{\bar{\mathbb{F}}_l}(h)$, and a lifting τ which by Jacquet-Langlands correspondence can be written $\tau \simeq \pi[t]_D$ for π irreducible cuspidal, let $\varrho \in \text{Scusp}_{\bar{\mathbb{F}}_l}(g)$ be in the supercuspidal support. Then the inertial class of ϱ depends only on $\bar{\tau}$ and we will use the following notation.

3.2.3. Notation. — With the previous notation, we denote V_{ϱ} for $V_{\bar{\tau}}$.

The description of the various filtration of previous section applied to $\Psi_{\mathcal{I}, \overline{\mathbb{Q}}_l}$ is given in [9] §3.4. Over $\overline{\mathbb{Z}}_l$, first note that $\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l}$ is an object of $\mathcal{F}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l)$. Indeed, by [1] proposition 4.4.2, $\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l}$ is an object of ${}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l)$. By [14] variant 4.4 of theorem 4.2, we have $D\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l} \simeq \Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l}$, so that

$$\Psi_{\mathcal{I}, \overline{\mathbb{Z}}_l} \in {}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l) = \mathcal{F}(X_{\mathcal{I}, \overline{s}}, \overline{\mathbb{Z}}_l).$$

We can then deduce from the description of the filtrations of $\Psi_{\mathcal{I}, \overline{\mathbb{Q}}_l}$ the same sort of description except that first we have no control on the bi-morphism ${}^p j_{!*} j^{=h,*} \text{gr}^k(L) \hookrightarrow_+ \text{gr}^k(L)$ and secondly all the contribution relatively to irreducible cuspidal $\overline{\mathbb{Q}}_l$ -representations should be considered altogether. About this last point, we have the following result.

3.2.4. Proposition. — *We have a decomposition*

$$\Psi_{\mathcal{I}} \simeq \bigoplus_{g=1}^d \bigoplus_{\varrho \in \text{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_{\varrho}$$

with $\Psi_{\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \text{Cusp}(\varrho)} \Psi_{\pi_v}$ where the irreducible constituent of Ψ_{π_v} are exactly the perverse Harris-Taylor sheaves attached to π_v .

Remark: the graduate $\text{gr}_!^h(\Psi_{\varrho})$ of the previous filtration of stratification of Ψ_{ϱ} verify

$$j^{=h,*} \text{gr}_!^h(\Psi_{\varrho}) \simeq \begin{cases} 0 & \text{si } g \nmid h \\ \mathcal{L}_{\overline{\mathbb{Z}}_l}(\varrho[t]_D) & \text{pour } h = tg. \end{cases}$$

Proof. — We argue by induction on r such that. there exists a decomposition

$$\text{Fill}_!^r(\Psi_{\mathcal{I}}) = \bigoplus_{g=1}^d \bigoplus_{\varrho \in \text{Scusp}_{\overline{\mathbb{F}}_l}(g)} \text{Fil}_{!, \varrho}^r(\Psi_{\mathcal{I}}).$$

The case $r = 0$ being trivial, we suppose it's true for $r - 1$. From $j^{=r,*} \text{gr}_!^r(\Psi_{\mathcal{I}}) \simeq \bigoplus_{g|r} \bigoplus_{\varrho \in \text{Scusp}_{\overline{\mathbb{F}}_l}(g)} \mathcal{L}_{\overline{\mathbb{Z}}_l}(\varrho[t]_D)$, we obtain

$$\text{gr}_!^r(\Psi_{\mathcal{I}}) \simeq \bigoplus_{g|r} \bigoplus_{\varrho \in \text{Scusp}_{\overline{\mathbb{F}}_l}(g)} \text{gr}_{!, \varrho}^r(\Psi_{\mathcal{I}})$$

with $j_{!*}^{-r} \mathcal{L}_{\overline{\mathbb{Z}}_l}(\varrho[t]_D)[d - r] \twoheadrightarrow \text{gr}_{!, \varrho}^r(\Psi_{\mathcal{I}})$ where the irreducible constituents of $\text{gr}_{!, \varrho}^r(\Psi_{\mathcal{I}}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are of type ϱ .

Consider two free perverse sheaves A_1 and A_2 and let A be an extension

$$0 \rightarrow A_1 \rightarrow A \rightarrow A_2 \rightarrow 0,$$

supposed to be split over $\overline{\mathbb{Q}}_l$. Denote then the pull back A'_2

$$\begin{array}{ccc} A'_2 \hookrightarrow & \dashrightarrow & A \\ \downarrow & & \downarrow \\ A_2 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \hookrightarrow & \longrightarrow & A \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \end{array}$$

so that

$$\begin{array}{ccccc} & & A_1 & \xlongequal{\quad} & A_1 & & (3.2.5) \\ & & \downarrow & & \downarrow & & \\ & & A & \longrightarrow & A'_1 & & \\ & & \downarrow & & \downarrow & & \\ A'_2 \hookrightarrow & A & \longrightarrow & A'_1 & & & \\ \parallel & & & & & & \\ A'_2 \hookrightarrow & A_2 & \longrightarrow & T & & & \end{array}$$

where $T = 0$ if and only the extension A is split. Now if A_1 (resp. A_2) is supposed to be a Harris-Taylor perverse sheaf of type ϱ_1 (resp. ϱ_2) with ϱ_1 and ϱ_2 not belonging to the same Zelevinsky line. Then the action of the Weil group on $T[l]$ seen as a quotient of A'_1 (resp. of A_2) is isotypic relatively to the galois representation associated to ϱ_1 (resp. ϱ_2) by the Langlands-Vigneras correspondence, which imposes that $T = 0$.

By applying this general remark to $\mathrm{gr}_{i,\varrho_2}^r(\Psi_{\mathcal{I}})$, we conclude it's in a direct sum with $\mathrm{Fil}_{i,\varrho_1}^{r-1}(\Psi_{\mathcal{I}})$, which, by varying ϱ_1 and ϱ_2 , proves the result. \square

3.3. Filtrations with the use of $j_{\neq c}$. — Let denote

$$\bar{j} : X_{\mathcal{I},\bar{\eta}} \hookrightarrow X_{\mathcal{I}} \hookleftarrow X_{\mathcal{I},\bar{s}} : \bar{i},$$

and consider the following t -structure on $\overline{X}_{\mathcal{I}} := X_{\mathcal{I}} \times_{\mathrm{Spec} \mathcal{O}_v} \mathrm{Spec} \overline{\mathcal{O}}_v$ obtained by glueing

$$\left({}^p D^{\leq -1}(X_{\mathcal{I},\bar{\eta}}, \overline{\mathbb{Z}}_l), {}^p D^{\geq -1}(X_{\mathcal{I},\bar{\eta}}, \overline{\mathbb{Z}}_l) \right) \quad \text{and} \quad \left({}^p D^{\leq 0}(X_{\mathcal{I},\bar{s}}, \overline{\mathbb{Z}}_l), {}^p D^{\geq 0}(X_{\mathcal{I},\bar{s}}, \overline{\mathbb{Z}}_l) \right).$$

The functors $\bar{j}^!$ and $\bar{j}_* = {}^p \bar{j}_*$ are then t -exact with

$$0 \rightarrow \Psi_{\mathcal{I}} \rightarrow \bar{j}^! \overline{\mathbb{Z}}_l[d-1] \left(\frac{d-1}{2} \right) \rightarrow \bar{j}_* \overline{\mathbb{Z}}_l[d-1] \left(\frac{d-1}{2} \right) \rightarrow 0.$$

Consider now the affine morphism $\bar{j}_{\neq c} : \overline{X}_{\mathcal{I}} \setminus X_{\mathcal{I},\bar{s},c}^{\geq 1} \hookrightarrow \overline{X}_{\mathcal{I}}$.

3.3.1. Lemma. — *The perverse sheaf $\Psi_c := i_{c,*}^1 p\mathcal{H}^0 i_c^{1,*}(\Psi_{\mathcal{I}})$ is free.*

Proof. — Let $\bar{F} := \bar{j}_* \bar{\mathbb{Z}}_l[d-1](\frac{d-1}{2}) := \bar{j}_* \bar{\mathbb{Z}}_l[d-1](\frac{d-1}{2})$ over $X_{\mathcal{I},\bar{\eta}}$. Denote $i_c^1 : X_{\bar{\mathcal{I},\bar{s},c}}^{\geq 1} \hookrightarrow X_{\bar{\mathcal{I},\bar{s}}}^{\geq 1}$, and $\bar{i}_c := \bar{i} \circ i_c^{\geq 1}$. As $\Psi_{\mathcal{I}} = p\mathcal{H}^{-1} \bar{i}^* \bar{j}_* \bar{\mathbb{Z}}_l[d-1](\frac{d-1}{2})$, we have to prove that $i_c^{1,*} p\mathcal{H}^{-1} \bar{i}^* \bar{F}$ is perverse for the t -structures p and $p+$. Consider the spectral sequence

$$E_2^{r,s} = p\mathcal{H}^r i_c^{1,*} \left(p\mathcal{H}^s \bar{i}^* \bar{F} \right) \Rightarrow p\mathcal{H}^{r+s} \bar{i}_c^* \bar{F}.$$

As \bar{j} is affine, by lemma3.1.3, we know that $p\mathcal{H}^s \bar{i}^* \bar{F}$ is trivial for $s < -1$. The epimorphism $\bar{j}_! \bar{j}^* \bar{F} \rightarrow \bar{F}$, gives also that $p\mathcal{H}^0 \bar{i}^* \bar{F} = 0$ so that the previous spectral sequence degenerates at E_2 with

$$p\mathcal{H}^r \bar{i}_c^* \bar{F} \simeq p\mathcal{H}^{r+1} i_c^{1,*} (p\mathcal{H}^{-1} \bar{i}^* \bar{F}).$$

In the same way as $\bar{j}_{\neq c} : X_{\mathcal{I}} \setminus X_{\bar{\mathcal{I},\bar{s},c}}^{\geq 1} \hookrightarrow X_{\mathcal{I}}$ is affine, then, by lemma3.1.3, $p\mathcal{H}^r \bar{i}_c^* \bar{F}$ is trivial for $r < -1$ and free for $r = -1$ which finishes the proof. \square

The decomposition of 3.2.4 gives $\Psi_c \simeq \bigoplus_{g=1}^d \bigoplus_{\varrho \in \text{Scusp}_{\bar{\mathbb{F}}_l}(g)} \Psi_{\varrho,c}$. For any $\varrho \in \text{Scusp}_{\bar{\mathbb{F}}_l}(g)$ we then have the following short exact sequence of free perverse sheaves

$$0 \rightarrow j_{\neq c,!} j_{\neq c}^* \Psi_{\varrho} \rightarrow \Psi_{\varrho} \rightarrow \Psi_{\varrho,!c} \rightarrow 0, \quad (3.3.2)$$

where $j_{\neq c} : X_{\bar{\mathcal{I},\bar{s}}}^{\geq 1} \setminus X_{\bar{\mathcal{I},\bar{s},c}}^{\geq 1} \hookrightarrow X_{\bar{\mathcal{I},\bar{s}}}^{\geq 1}$. Consider the filtration of stratification

$$0 = \text{Fil}_*^{-d}(\Psi_{\varrho,!c}) \subset \text{Fil}_*^{1-d}(\Psi_{\varrho,!c}) \subset \cdots \subset \text{Fil}_*^0(\Psi_{\varrho,!c}) = \Psi_{\varrho,!c}.$$

3.3.3. Proposition. — *The graduates $\text{gr}_*^h(\Psi_{\varrho,!c})$ verify the following properties*

- *it is trivial if h is not equal to some $-g_i(\varrho) + 1 > -d$ for $i \geq -1$;*
- *for such $i \geq -1$ with $g_i(\varrho) \leq d$, then*

$$\text{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho,!c}) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \text{Cusp}(\varrho,i)} \text{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!c})$$

where $\text{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!c})$ is the push forward

$$\begin{array}{ccc} \Psi_{\pi_v} & \longrightarrow & \Psi_{\pi_v,!c} \\ \downarrow & & \downarrow \\ \text{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v}) & \dashrightarrow & \text{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!c}). \end{array}$$

Remark: in particular the graduates $\mathrm{gr}_!^h(\mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c}))$ of the filtration of stratification are

- trivial if h is not of the shape $tg_i(\varrho) \leq d$,
- and for $h = tg_i(\varrho) \leq d$, we have, if we consider for simplicity $c = \overline{1}_1$

$$\mathrm{gr}_!^{tg_i(\varrho)}(\mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c})) \simeq \mathrm{ind}_{P_{1,h-1,d-h}(F_v)}^{P_{1,d-1}(F_v)} \mathcal{P}(t, \pi_v)_{\overline{1}_h} \left(\frac{1-t}{2} \right).$$

Proof. — This is trivially a statement on Ψ_{π_v} and the precise description of Ψ_{π_v} . For simplicity we suppose $c = \overline{1}_1$. From [9], the graduate $\mathrm{gr}_!^h(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v}))$ of the filtration $\mathrm{Fil}_!^\bullet(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v}))$ are trivial for all $h \neq tg_i(\varrho) \leq d$ and

$$\mathrm{gr}_!^{tg_i(\varrho)}(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) \simeq \mathcal{P}(t, \pi_v) \left(\frac{1-t}{2} \right).$$

3.3.4. Lemma. — For all $0 \leq r \leq d$, the graduate of

$$\mathrm{Fil}_!^h \left(i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \mathrm{Fil}_!^r(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) \right)$$

are trivial if $h \neq tg_i(\varrho) \leq r$, and otherwise isomorphic to

$$\mathrm{ind}_{P_{1,h-1,d-h}(F_v)}^{P_{1,d-1}(F_v)} \mathcal{P}(t, \pi_v)_{\overline{1}_h} \left(\frac{1-t}{2} \right).$$

Proof. — Note first that $i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \mathcal{P}(t, \pi_v) \simeq \mathcal{P}(t, \pi_v)_c$. In particular the following statement is true for $r \leq g_i(\varrho)$. We then argue by induction through the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Fil}_!^{r-1}(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) &\longrightarrow \mathrm{Fil}_!^r(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) \\ &\longrightarrow \mathrm{gr}_!^r(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) \rightarrow 0. \end{aligned}$$

If r is not of the shape $tg_i(\varrho)$ there is nothing to prove, otherwise as

- the irreducible constituents of $i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \mathrm{Fil}_!^{r-1}(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v}))$ are, by induction, intermediate extensions of Harris-Taylor local systems on $X_{\overline{\mathcal{I}}, \overline{s}}^i$ for $i \leq r$,
- and $i_{c,*}^1 {}^p\mathcal{H}^{-1} i_c^{1,*} \mathcal{P}(t, \pi_v)$ is supported on $X_{\overline{\mathcal{I}}, \overline{s}}^{\geq r+1}$,

then the cone map $i_{c,*}^1 {}^p\mathcal{H}^{-1} i_c^{1,*} \mathcal{P}(t, \pi_v) \longrightarrow i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \mathrm{Fil}_!^{r-1}(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v}))$ is trivial. The result follows then from the short exact sequence

$$\begin{aligned} 0 \rightarrow i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \mathrm{Fil}_!^{r-1}(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) &\longrightarrow i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \mathrm{Fil}_!^r(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) \\ &\longrightarrow i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \mathrm{gr}_!^r(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})) \rightarrow 0. \end{aligned}$$

□

It suffices now to prove that the epimorphism

$$i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} \Psi_{\pi_v} \rightarrow i_{c,*}^1 {}^p\mathcal{H}^0 i_c^{1,*} (\text{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v}))$$

is an isomorphism. For that it suffices to prove that, for every geometric point z , the germs at z of the sheaves cohomology groups of these two perverse sheaves, are the same.

Let then z be a geometric point of $X_{\mathcal{L},\bar{s},\bar{1}_h}^{=h}$. By [5], the germ at z of the i -th sheaf of cohomology $\mathcal{H}^i j_{\bar{1}_{kg},*}^{-kg} HT_{\bar{1}_{kg}}(\pi_v, \text{St}_k(\pi_v)) \otimes \Xi^{\frac{1-k}{2}}$ is zero if (h, i) is not of the shape $(tg - d, tg - d + k - t)$ with $k \leq t \leq \lfloor \frac{d}{g} \rfloor$ and otherwise isomorphic to those of

$$HT_{\bar{1}_{tg}}(\pi_v, \text{St}_k(\pi_v \{ \frac{k-t}{2} \})) \otimes \text{Speh}_{t-k}(\pi_v \{ \frac{k}{2} \}) \otimes \Xi^{\frac{1+t-2k}{2}}.$$

We then deduce that the fiber at z of $j_{\bar{1}_1,*}^{-kg} HT_{\bar{1}_1}(\pi_v, \text{St}_k(\pi_v)) \otimes \Xi^{\frac{1-k}{2}}$ is isomorphic to those of

$$HT_{\bar{1}_{tg}} \left(\pi_v, \left(\text{St}_k(\pi_v \{ \frac{k-t}{2} \}) \right)_{|P_{1,kg-1}(F_v)} \times \text{Speh}_{t-k}(\pi_v \{ \frac{k}{2} \}) \right) \otimes \Xi^{\frac{1+t-2k}{2}},$$

where we induce from $P_{1,kg-1}(F_v) \otimes GL_{(t-k)g}(F_v)$ to $P_{1,tg-1}(F_v)$. Moreover considering the weights, we see that the spectral sequence computing the fibers of sheaves of cohomology of $\Psi_{\pi_v,c}$ from those of $\text{gr}_!^k(\Psi_{\pi_v,c})$ degenerate at E_1 . From 1.3.9, we have

$$\left(\text{St}_k(\pi_v \{ \frac{k-t}{2} \}) \right)_{|P_{1,kg-1}(F_v)} \times \text{Speh}_{t-k}(\pi_v \{ \frac{k}{2} \}) \simeq (LT_{\pi}(k, t-1-k)_{\pi_v})_{|P_{1,tg-1}(F_v)}$$

so that, by the main result of [5], the fiber at z of $\mathcal{H}^i(\Psi_{\pi_v,c})$ is isomorphic to those of $\mathcal{H}^i({}^p\mathcal{H}^0 i_c^{1,*} \Psi_{\pi_v})$, so we are done. □

Dually we have

$$0 \rightarrow \Psi_{\varrho,*,c} \rightarrow \Psi_{\varrho} \rightarrow j_{\neq c,*} j_{\neq c}^* \Psi_{\varrho} \rightarrow 0, \quad (3.3.5)$$

such that the graduates $\text{gr}_!^h(\Psi_{\varrho,*,c})$ verify the following properties

- it is trivial if h is not equal to some $g_i(\varrho) \leq d$ for $i \geq -1$;
- for such $i \geq -1$ with $g_i(\varrho) \leq d$, then

$$\text{gr}_!^{g_i(\varrho)}(\Psi_{\varrho,*,c}) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \text{Cusp}(\varrho,i)} \text{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$$

where $\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$ is the pull back

$$\begin{array}{ccc} \mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c}) & \hookrightarrow & \Psi_{\pi_v,*,c} \\ \downarrow & & \downarrow \\ \mathrm{Fil}_!^{g_i(\varrho)}(\Psi_{\pi_v}) & \hookrightarrow & \Psi_{\pi_v}. \end{array}$$

By [5], the image of $\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$ in the Grothendieck group is $\sum_{t=1}^{\lfloor \frac{d}{g_i(\varrho)} \rfloor} \mathcal{P}(t, \pi_v)_c(\frac{t-1}{2})$, more precisely $\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$ has a filtration $\mathrm{Fil}^k(\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c}))$ for $0 \leq k \leq s_i(\varrho) := \lfloor \frac{d}{g_i(\varrho)} \rfloor$ with graduates $\mathrm{gr}^k(\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})) \simeq \mathcal{P}(s_i(\varrho) - k + 1, \pi_v)_c(\frac{s_i(\varrho) - k}{2})$.

4. Non degeneracy property for submodules

4.1. The case of $\mathcal{V}_{\varrho, \mathbb{N}}^{d-1}$. — Recall first that, for a fixed irreducible $\overline{\mathbb{F}}_l$ -representations of $D_{v,d}^\times$, the notation $\mathcal{V}_{\varrho, \mathbb{N}}^{d-1}$ designates the direct factor of $\mathcal{V}_{F_v, \overline{\mathbb{Z}}_l, d}^{d-1}$ associated to ϱ in the sense of [11] §B.2. Let $i_z : z \hookrightarrow X_{\overline{\mathbb{Z}}, \overline{s}}^d$, be any supersingular point then, from the main theorem of Berkovitch, we have an isomorphism

$$\mathrm{ind}_{(D_{v,d}^\times)^0 \varpi_{\overline{\mathbb{Z}}}}^{D_{v,d}^\times} {}^p h^0 i_z^! \Psi_\varrho \simeq \mathcal{V}_{\varrho, \mathbb{N}}^{d-1}, \quad (4.1.1)$$

which is equivariant for $D_{v,d}^\times \times GL_d(F_v) \times W_v$.

Let $i_z : z \hookrightarrow X_{\overline{\mathbb{Z}}, \overline{s}, c}^d$, we are then lead to compute $h^0 i_z^! \Psi_\varrho$. We start with the short exact sequence (3.3.5)

$$0 \rightarrow \Psi_{\varrho,*,c} \rightarrow \Psi_\varrho \rightarrow j_{\neq c,*} j_{\neq c}^* \Psi_\varrho \rightarrow 0,$$

so that $h^0 z^! \Psi_\varrho \simeq h^0 z^! \Psi_{\varrho,*,c}$. Consider then another pure stratum $X_{\overline{\mathbb{Z}}, \overline{s}, c'}^{\geq 1}$ with $c' \neq c$.

4.1.2. Lemma. — *The perverse sheaf ${}^p \mathcal{H}^i i_{c'}^* \Psi_{\varrho,*,c}$ is zero for $i \neq 0$ and it's free for $i = 0$.*

Proof. — Note first that the result is true for Ψ_ϱ . Moreover for any perverse free sheaf P , we have ${}^p \mathcal{H}^i i_{c'}^* P = 0$ if $i \notin \{0, -1\}$ and it's free for $i = -1$. The result then follows easily from the long exact sequence associated to the previous short exact sequence when we apply $i_{c'}^*$. \square

In particular in the following short exact sequence

$$0 \rightarrow j_{\neq c'}^* ! j_{\neq c'}^* \Psi_{\varrho, *, c} \longrightarrow \Psi_{\varrho} \longrightarrow \Psi_{\varrho, *, c, !, c'} \rightarrow 0$$

the perverse sheaf $\Psi_{\varrho, *, c, !, c'}$ is free and can be written

$$0 \rightarrow i_{c', *}'^1 {}^p \mathcal{H}^0 i_{c'}^{1, *} \Psi_{\varrho, *, c} \longrightarrow \Psi_{\varrho, *, c, !, c'} \longrightarrow j_{\neq c, *} j_{\neq c'}^* \Psi_{\varrho} \rightarrow 0.$$

As $j_{\neq c'}$ is affine, $j_{\neq c'} ! j_{\neq c'}^*$ is a exact functor, so that from the previous section, $j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c}$ has a filtration $\text{Fil}^\bullet(j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c})$ with graduates $j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)} \Psi_{\varrho, *, c}$ such that

$$j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)}(\Psi_{\varrho, *, c}) \otimes_{\overline{\mathbb{Q}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \text{Cusp}(\varrho, i)} j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v, *, c}),$$

where $j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v, *, c})$ has a filtration whose graduates, by lemma 2.4.3, are the $\mathcal{P}(t, \pi_v)_c(\frac{t-1}{2})$ for $2 \leq t \leq s_i(\varrho)$ and $\mathcal{P}(1, \pi_v)_{c, \neq c'}$.

From what we have already seen, $i_{c', *}'^1 {}^p \mathcal{H}^0 i_{c'}^{1, *} \Psi_{\varrho, *, c}$ is a free perverse sheaf which, over $\overline{\mathbb{Q}}_l$, have irreducible constituents $\mathcal{P}(1, \pi_v)_{(c, c')}$ for $\pi_v \in \text{Scusp}(\varrho)$ an irreducible representation of $GL_g(F_v)$ with $g < d$.

- If c' is such that $z \notin X_{\overline{\mathcal{I}}, \overline{s}, c'}$, then ${}^p \mathcal{H}^0 i_z^! \Psi_{\varrho} \simeq {}^p \mathcal{H}^0 i_z^!(j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c})$.
- If z is a supersingular so that we can not find such a c' , we then have the following short exact sequence

$${}^p \mathcal{H}^0 i_z^!(j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c}) \hookrightarrow {}^p \mathcal{H}^0 i_z^! \Psi_{\varrho} \twoheadrightarrow \bigoplus_{\pi_v \in \text{Scusp}_i(\varrho)} {}^p \mathcal{H}^0 i_z^! \mathcal{P}_{\overline{\mathbb{Q}}_l}(1, \pi_v),$$

where in the direct sum, the index i is such that $g_i(\varrho) = d$ and $\mathcal{P}_{\overline{\mathbb{Q}}_l}(1, \pi_v)$ is, as the modulo l reduction of π_v is irreducible, the only, up to isomorphism, stable lattice of $\mathcal{P}(1, \pi_v)$.

We focus then on ${}^p \mathcal{H}^0 i_z^!(j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c})$ by constructing a new filtration of $j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c}$. We start with the previous filtration $\text{Fil}^\bullet(j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c})$ with graduates $j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)} \Psi_{\varrho, *, c}$. We can introduce a naive filtration of $j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)} \Psi_{\varrho, *, c}$ such that the graduates are some lattices of $HT_{c, \neq c'}(\pi_v, \pi_v)$ for π_v describing $\text{Scusp}_i(\varrho)$. By taking the image by $j_{\neq c'}^{=g_i(\varrho)}$ of this filtration gives a filtration of $j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)} \Psi_{\varrho, *, c}$ whose graduates are entire version of $j_{\neq c'} ! j_{\neq c'}^* \text{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v, *, c})$ for π_v describing $\text{Scusp}_i(\varrho)$. Then we can filtrate each of these graduate to obtain a filtration denoted $\text{Fill}^\bullet(j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c})$ whose graduates $\text{gr}_!^k(j_{\neq c'} ! j_{\neq c'}^* \Psi_{\varrho, *, c})$ are some entire version of the $\mathcal{P}(t, \pi_v)_c$ if $2 \leq t \leq s_i(\varrho)$ (resp. $\mathcal{P}(1, \pi_v)_{c, \neq c'}$), for $\pi_v \in$

$\text{Scusp}_i(\varrho)$ with $i \geq -1$: these entire perverse sheaves may depend of all the choices.

4.1.3. Notation. — We pay special attention to the previous graduates concentrated on the supersingular locus. For $\pi_v \in \text{Scusp}_i(\varrho)$ and t such that $tg_i(\varrho) = d$, let then denote $\mathcal{P}_{\text{Fill},!,c}(t, \pi_v)$ the lattice obtained by the previous filtration of $j_{\neq c',!} j_{\neq c'}^* \Psi_{\pi_v,*,c}$.

Remark: we will come back to these lattices latter but, beside the issue about the lattices, the main difficulty about these graduates is that we only know that

$$\begin{aligned} p_{j_{c,!}^*}^{-tg_i(\varrho)} HT_c(\pi_v, \text{St}_t(\pi_v)) \left(\frac{t-1}{2} \right) \\ \hookrightarrow_+ \text{grr}^k(j_{\neq c',!} j_{\neq c'}^* \Psi_{\pi_v,*,c}) \hookrightarrow_+ \\ p_{j_{c,!}^*}^{-tg_i(\varrho)} HT_c(\pi_v, \text{St}_t(\pi_v)) \left(\frac{t-1}{2} \right). \end{aligned}$$

One of the main results of [8], is to deal with this last issue.

The orders of these graduates verify the following property: consider two indexes k and k' corresponding to respectively $\mathcal{P}(t, \pi_v)_c$ and $\mathcal{P}(t', \pi'_v)_c$:

- if $\pi_v \in \text{Scusp}_i(\varrho)$ and $\pi'_v \in \text{Scusp}_{i'}(\varrho)$ with $i > i'$, then $k > k'$;
- if $\pi_v = \pi'_v$ and $t < t'$ then $k > k'$.

Recall we want to compute ${}^p\mathcal{H}^0 i_z^! \Psi_\varrho$ and that from

$$0 \rightarrow \text{gr}_*^{1-d}(\Psi_\varrho) \longrightarrow \Psi_\varrho \longrightarrow \text{coFil}_{*,d-1}(\Psi_\varrho) \rightarrow 0$$

with $\text{coFil}_{*,1-d} \Psi_\varrho \hookrightarrow j_*^{1 \leq d-1} j^{1 \leq d-1,*} \Psi_\varrho$ so that ${}^p\mathcal{H}^0 i_z^! \Psi_\varrho \simeq {}^p\mathcal{H}^0 i_z^! \text{Fil}_*^{1-d}(\Psi_\varrho)$. Moreover $\text{Fil}_*^{1-d}(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is, in the Grothendieck group, equals to the sum of the irreducible constituents of $i_{c',*}^1 {}^p\mathcal{H}^0 i_{c'}^{1,*} \Psi_{\varrho,*,c}$ and $j_{\neq c',!} j_{\neq c'}^* \Psi_{\varrho,*,c}$, which means that in the previous filtration, we can modify the order of the graduates so that those concentrated in the supersingular points appear in the first positions.

Start first with the filtrations of $j_{\neq c',!} j_{\neq c'}^* \Psi_{\varrho,*,c}$, an consider a subquotient X of this filtration which can be written

$$0 \rightarrow A_1 \longrightarrow X \longrightarrow A_2 \rightarrow 0$$

where

- $A_2 = \mathcal{P}(t, \pi_v)$ with $\pi_v \in \text{Scusp}_i(\varrho)$ and $tg_i(\varrho) = d$,

- and A_1 is some free perverse sheaf, irreducible over $\overline{\mathbb{Q}}_l$, say of the form ${}^p j_{c,!} {}^{h,*} j_c {}^{h,*} A_1 \hookrightarrow_+ A_1 \hookrightarrow_+ {}^{p+} j_{c,!} {}^{h,*} j_c {}^{h,*} A_1$, with $h < d$ and $A_1 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathcal{P}(t', \pi'_v)_c$ with $\pi'_v \in \text{Scusp}_i(\varrho)$, $h = t' g_{i'}(\varrho)$ and $i' < i$.

We then have a diagram like (3.2.5) where T is supported on $X_{\overline{\mathbb{Z}}, \overline{s}}^{\leq d}$ so that $A_1 \hookrightarrow A'_1 \rightarrow T$ is obtained through

$${}^p j_{c,!} {}^{h,*} j_c {}^{h,*} A_1 \hookrightarrow_+ A_1 \hookrightarrow_+ A'_1 \hookrightarrow_+ {}^{p+} j_{c,!} {}^{h,*} j_c {}^{h,*} A_1.$$

Suppose by absurdity, that $T \neq (0)$. Then as a quotient of A'_1 (resp. A_2), T is equipped with an action of $P_c(F_v)$ (resp. $GL_d(F_v)$) and that this two action agrees on $P_c(F_v)$. Note then that

- as a quotient of A'_1 , by restricting the action to $P_c(F_v) \subset GL_h(F_v)$, then T has a irreducible subquotient with derivative of ordre $g_{i'}(\varrho)$,
- but as a quotient of A_2 all of its derivative have order $\geq g_i(\varrho) > g_{i'}(\varrho)$.

This gives us a contradiction and so T is trivial, i.e. $X = A_1 \oplus A_2$.

Remark: we also have to consider the case where $A_1 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathcal{P}(t', \pi'_v)_{c, \neq c'}$ in which case we only have an action of $P_{c,c'}(F_v)$ but as before when we restrict the action to $GL_h(F_v)$ we still have an action of $P_c(F_v) \subset GL_h(F_v)$ and the argument is the same.

By arguing also with the supersingular free perverse sheaves of $i_{c',*}^1 {}^p \mathcal{H}^0 i_{c'}^{1,*} \Psi_{\varrho,*,c}$, we are lead to the following statement.

4.1.4. Proposition. — *There exists a filtration of*

$$(0) = \text{Fil}^0(\Psi_{\varrho,*,c}) \subset \text{Fil}^1(\Psi_{\varrho,*,c}) \subset \text{Fil}^2(\Psi_{\varrho,*,c}) = \Psi_{\varrho,*,c}$$

such that

- the irreducible constituents of $\text{gr}^i(\Psi_{\varrho,*,c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ for $i = 1$ (resp. $i = 2$) are all with support in $X_{\overline{\mathbb{Z}}, \overline{s}}^{\leq d}$ (resp. are of the form $\mathcal{P}(t, \pi_v)_c$ with $\pi_v \in \text{Scusp}_i(\varrho)$ and $t g_i(\varrho) < d$).
- Moreover there is a filtration of

$$(0) = \text{Fil}^{-2}(\text{gr}^1(\Psi_{\varrho,*,c})) \subset \text{Fil}^{-1}(\text{gr}^1(\Psi_{\varrho,*,c})) \subset \dots \\ \dots \subset \text{Fil}^s(\text{gr}^1(\Psi_{\varrho,*,c})) = \text{gr}^1(\Psi_{\varrho,*,c})$$

whose graduate $\text{gr}^i(\text{gr}^1(\Psi_{\varrho,*,c}))$ are zero except if there exists t such that $t g_i(\varrho) = d$ in which case with the notation of 4.1.3,

$$\text{gr}^i(\text{gr}^1(\Psi_{\varrho,*,c})) \simeq \bigoplus_{\pi_v \in \text{Scusp}_i(\varrho)} \mathcal{P}_{\text{Fill},!,c}(t, \pi_v).$$

As for a perverse sheaf $\mathcal{P}(t, \pi_v)$ not concentrated in the supersingular point, we have ${}^p\mathcal{H}^0 i_z^! \mathcal{P}(t, \pi_v) = (0)$, then we deduce from the previous proposition a filtration

$$(0) = \text{Fil}^{-2}({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) \subset \text{Fil}^{-1}({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) \subset \cdots \\ \cdots \subset \text{Fil}^s({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) = {}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)$$

whose non zero graduates coincide with $i \geq -1$ such that there exists t with $tg_i(\varrho) = d$ with

$$\text{gr}^i({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) \simeq \bigoplus_{\pi_v \in \text{Scusp}_i(\varrho)} \mathcal{P}_{\text{Fill},!,c}(t, \pi_v)_z.$$

4.1.5. Theorem. — As a $\overline{\mathbb{Z}}[P_d(F_v) \times D_{v,d}^\times \times W_v]$ -module, $\mathcal{V}_{\varrho, \mathbb{N}}^{d-1}$ has a filtration with successive graduates $\text{gr}^i(\mathcal{V}_{\varrho, \mathbb{N}}^{d-1})$ for $-1 \leq i \leq s$ where

- s is maximal such that $g_s(\varrho)$ divides d ;
- $\text{gr}^i(\mathcal{V}_{\varrho, \mathbb{N}}^{d-1}) \simeq \bigoplus_{\pi_v \in \text{Scusp}_i(\varrho)} \Gamma_{GDW}(\pi_v)$ with $\Gamma_{GDW}(\pi_v) \simeq \Gamma_G(\pi_v) \otimes \Gamma_D \otimes \Gamma_W(\pi_v)$ where
 - Γ_D (resp. Γ_W) is a stable lattice of $\pi_v[t]_D$ (resp. $\mathbb{L}_{g_i(\varrho)}(\pi_v)$);
 - Γ_G is isomorphic to the stable lattice $(RI_{\overline{\mathbb{Z}}_l, -}(\pi_v, t))|_{P_d(F_v)}$ of definition 1.4.1.

Proof. — We argue by induction on d . As the result is trivial for $g_{-1}(\varrho)$ because there is, up to isomorphism, only one stable lattice, we suppose the result true for all $h < d$. From the isomorphism (4.1.1), and arguing like before on $j^{1 \leq h, *} \Psi_\varrho$ we can conclude that the lattices of $HT_c(\pi_v, \text{St}_t(\pi_v))(\frac{t-1}{2})$ of the graduates $\text{gr}^k(j_{\neq c'}, !j_{\neq c'}^* \Psi_{\varrho, *, c})$ are of the shape $\mathcal{L}_D \otimes (RI_{\overline{\mathbb{Z}}_l, -}(\pi_v, t))|_{P_d(F_v)} \otimes \Gamma_W$ where \mathcal{L}_D is some stable $\overline{\mathbb{Z}}_l$ -lattice sheaf of $\mathcal{L}(\pi_v[T]_D)_c$.

To prove the theorem, by the isomorphism (4.1.1), we have now to show that the lattice $\mathcal{P}_{\text{Fill},!,c}(t, \pi_v)$ is a tensorial product of stable lattices of respectively $P_d(F_v)$, $D_{v,d}^\times$ and W_v , where the lattice relatively of the action of $P_d(F_v)$ is isomorphic to $(RI_{\overline{\mathbb{Z}}_l, -}(\pi_v, t))|_{P_d(F_v)}$.

By hypothesis we have $\pi_v \in \text{Scusp}_i(\varrho)$ and $tg_i(\varrho) = d$. Recall also, using the exactness of $j_{c, \neq c'}, !$, that $\mathcal{P}_{\text{Fill},!,c}(t, \pi_v)$ fits in the following short exact sequence of lemma 2.4.3

$$0 \rightarrow \mathcal{P}_{\text{Fill},!,c}(t, \pi_v) \longrightarrow j_{\neq c'}, !j_{\neq c'}^* \text{gr}^k(j_{\neq c'}, !j_{\neq c'}^* \Psi_{\varrho, *, c}) \\ \longrightarrow j_{\neq c'}, !*j_{\neq c'}^* \text{gr}^k(j_{\neq c'}, !j_{\neq c'}^* \Psi_{\varrho, *, c}) \rightarrow 0$$

where

$$\mathrm{gr}^k(j_{\neq c'}!j_{\neq c'}^*\Psi_{\varrho,*,c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathcal{P}(t-1, \pi_v)_c \left(\frac{t-2}{2}\right) \hookrightarrow \mathcal{P}(t-1, \pi_v) \left(\frac{t-2}{2}\right).$$

Over $\overline{\mathbb{Z}}_l$, we have seen that $j^{=(t-1)g_i(\varrho),*} \mathrm{gr}^k(j_{\neq c'}!j_{\neq c'}^*\Psi_{\varrho,*,c})$ is a tensorial product of lattices where those relatively to the action of $P_{(t-1)g_i(\varrho)}(F_v)$ is $(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v, t-1))|_{P_{(t-1)g_i(\varrho)}(F_v)}$. Moreover we can write

$$\mathrm{gr}^k(j_{\neq c'}!j_{\neq c'}^*\Psi_{\varrho,*,c}) \hookrightarrow {}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1, \pi_v) \left(\frac{t-2}{2}\right)$$

where:

- the left exponent $p(c)$ means that for all geometric singular point z , the perverse sheaf looks like a p -intermediate extension, i.e. $\mathcal{H}^0 i_z^* ({}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1, \pi_v)) = (0)$;
- $j^{=(t-1)g_i(\varrho),*} ({}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1, \pi_v))$ is a tensorial stable lattice where the lattice associated to the action of $P_{(t-1)g_i(\varrho)}(F_v)$ is isomorphic to $(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v, t-1))|_{P_{(t-1)g_i(\varrho)}}$.

We then have

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{Fill},!,c}(t, \pi_v) & \hookrightarrow & j_{\neq c'}!j_{\neq c'}^* \mathrm{gr}^k(j_{\neq c'}!j_{\neq c'}^*\Psi_{\varrho,*,c}) \\ \downarrow \simeq & & \downarrow \\ {}^p\mathcal{H}^{-1} i_{c'}^* \mathcal{P}_{RI,\otimes}(t-1, \pi_v) \left(\frac{t-2}{2}\right) & \hookrightarrow & j_{\neq c'}!j_{\neq c'}^* {}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1, \pi_v) \left(\frac{t-2}{2}\right) \end{array}$$

where the lattice relatively to the action of $P_d(F_v)$ on the first term ${}^p\mathcal{H}^{-1} i_{c'}^* \mathcal{P}_{RI,\otimes}(t-1, \pi_v) \left(\frac{t-2}{2}\right)$ is given by the induced representation

$$\begin{array}{ccc} RI_{\overline{\mathbb{Z}}_l,-}(\pi_v \left\{ \frac{-1}{2} \right\}, t-1) \times (\pi_v \left\{ \frac{t-1}{2} \right\})|_{P_{g_i(\varrho)}(F_v)} & \hookrightarrow & (RI_{\overline{\mathbb{Z}}_l,-}(\pi_v \left\{ \frac{-1}{2} \right\}, t-1) \times \pi_v \left\{ \frac{t-1}{2} \right\})|_{P_d(F_v)} \\ & \searrow \sim & \downarrow \\ & & RI_{\overline{\mathbb{Z}}_l,-}(\pi_v, t)|_{P_d(F_v)}, \end{array}$$

which finishes the proof. \square

From propositions 1.4.2 and 1.4.2, we obtain the expected non degeneracy property.

4.1.6. Corollary. — *Any irreducible $P_d(F_v)$ -equivariant subspace of $\mathcal{V}_{\varrho, \mathbb{N}}^{d-1}$, is non degenerate and so isomorphic to τ_{nd} .*

4.2. The case of $\mathcal{U}_{\varrho, \mathbb{N}}^{d-1}$. — In [8], we prove that for any supercuspidal $\overline{\mathbb{F}}_l$ -representation ϱ , then $\mathcal{U}_{\varrho, \mathbb{N}}^{d-1}$ is free. As at this stage we don't want to use [8], we introduce its free quotient $\mathcal{U}_{\varrho, \mathbb{N}, free}^{d-1}$. We then follow exactly the same steps than in the previous section, but dually. Precisely fix a supersingular $i_z : z \hookrightarrow X_{\overline{\mathbb{Z}}, \overline{s}}^d$, and we start with the $D_{v,d}^\times \times GL_d(F_v) \times W_v$ -equivariant isomorphism

$$\mathrm{ind}_{(D_{v,d}^\times)^0 \varpi_v^{\mathbb{Z}}}^{D_{v,d}^\times} {}^p h^0 i_z^* \Psi_\varrho \simeq \mathcal{U}_{\varrho, \mathbb{N}}^{d-1}, \quad (4.2.1)$$

so that we have to compute the free quotient of $h^0 i_z^* \Psi_\varrho$ through the short exact sequences

$$0 \rightarrow j_{\neq c, !} j_{\neq c}^* \Psi_\varrho \longrightarrow \Psi_\varrho \longrightarrow \Psi_{\varrho, !, c} \rightarrow 0,$$

and

$$0 \rightarrow i_{c', *}' {}^p \mathcal{H}^0 i_{c', !} \Psi_{\varrho, !, c} \longrightarrow \Psi_{\varrho, !, c} \longrightarrow j_{c, \neq c', *}' j_{c, \neq c'}^* \Psi_{\varrho, !, c} \rightarrow 0.$$

Using the exactness of $j_{\neq c', *}' j_{\neq c'}^*$, the filtration $\mathrm{Fil}_*(\Psi_{\varrho, !, c})$ of proposition 3.3.3, gives a filtration $\mathrm{Fil}^\bullet(j_{\neq c', *}' j_{\neq c'}^* \Psi_{\varrho, !, c})$ with graduates $j_{\neq c', *}' j_{\neq c'}^* \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho, !, c})$ such that

$$j_{\neq c', *}' j_{\neq c'}^* \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho, !, c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho, i)} j_{\neq c', *}' j_{\neq c'}^* \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v, !, c}),$$

where $j_{\neq c', *}' j_{\neq c'}^* \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v, !, c})$ has a filtration whose graduates, by lemma 2.4.3, are the $\mathcal{P}(t, \pi_v)_c(\frac{1-t}{2})$ for $2 \leq t \leq s_i(\varrho)$ and $\mathcal{P}(1, \pi_v)_{c, \neq c'}$. By arguing like before, we can manage to obtain then a filtration $\mathrm{Fill}^\bullet(j_{\neq c', *}' j_{\neq c'}^* \Psi_{\varrho, !, c})$ whose graduates are some entire version of the $\mathcal{P}(t, \pi_v)_c$ if $2 \leq t \leq s_i(\varrho)$ (resp. $\mathcal{P}(1, \pi_v)_{c, \neq c'}$), for $\pi_v \in \mathrm{Scusp}_i(\varrho)$ with $i \geq -1$.

Remark: Like before we don't pay attention about the position of these perverse sheafs between the p and $p+$ intermediate extensions, but we merely concentrate on the lattice of the associated local systems.

4.2.2. Notation. — For $\pi_v \in \mathrm{Scusp}_i(\varrho)$ and t such that $tg_i(\varrho) = d$, let then denote $\mathcal{P}_{\mathrm{Fill}, *, c}(t, \pi_v)$ the lattice obtained by the previous filtration of $j_{\neq c', *}' j_{\neq c'}^* \Psi_{\pi_v, !, c}$.

Using the same arguments of the previous section, we can gather these perverse sheaves $\mathcal{P}_{\mathrm{Fill}, *, c}(t, \pi_v)$ with modifying them to obtain the following result similar to 4.1.4.

4.2.3. Proposition. — *There exists a filtration of*

$$(0) = \text{Fil}^{-2}(\Psi_{\varrho,!},c) \subset \text{Fil}^{-1}(\Psi_{\varrho,!},c) \subset \text{Fil}^0(\Psi_{\varrho,!},c) = \Psi_{\varrho,!},c$$

such that

- the irreducible constituents of $\text{gr}^i(\Psi_{\varrho,!},c) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ for $i = -1$ (resp. $i = 0$) are all with support in $X_{\overline{\mathbb{Z}},\overline{s}}^d$ (resp. are of the form $\mathcal{P}(t, \pi_v)_c$ with $\pi_v \in \text{Scusp}_i(\varrho)$ and $tg_i(\varrho) < d$).
- Moreover there is a filtration of

$$(0) = \text{Fil}^{-s-1}(\text{gr}^0(\Psi_{\varrho,!},c)) \subset \text{Fil}^{-s}(\text{gr}^0(\Psi_{\varrho,!},c)) \subset \dots \\ \dots \subset \text{Fil}^{-1}(\text{gr}^0(\Psi_{\varrho,!},c)) = \text{gr}^0(\Psi_{\varrho,!},c)$$

whose graduate $\text{gr}^{-i}(\text{gr}^0(\Psi_{\varrho,!},c))$ are zero except if there exists t such that $tg_i(\varrho) = d$ in which case with the notation of 4.2.2,

$$\text{gr}^{-i}(\text{gr}^0(\Psi_{\varrho,!},c)) \simeq \bigoplus_{\pi_v \in \text{Scusp}_i(\varrho)} \mathcal{P}_{\text{Fill},*,c}(t, \pi_v).$$

Using (4.2.1) and arguing by induction we obtain the $\mathcal{U}_{\varrho,\mathbb{N}}^{d-1}$ -version of theorem 4.1.5.

4.2.4. Theorem. — *As a $\overline{\mathbb{Z}}[P_d(F_v) \times D_{v,d}^\times \times W_v]$ -module, $\mathcal{U}_{\varrho,\mathbb{N},\text{free}}^{d-1}$ has a filtration with successive graduates $\text{gr}^i(\mathcal{U}_{\varrho,\mathbb{N},\text{free}}^{d-1})$ for $-s \leq i \leq 1$ where*

- s is maximal such that $g_s(\varrho)$ divides d , and we denote for such index $s_i(\varrho) = \frac{d}{g_i(\varrho)}$;
- $\text{gr}^{-i}(\mathcal{U}_{\varrho,\mathbb{N},\text{free}}^{d-1}) \simeq \bigoplus_{\pi_v \in \text{Scusp}_i(\varrho)} \Gamma_{GDW}(\pi_v)$ with $\Gamma_{GDW}(\pi_v) \simeq \Gamma_G(\pi_v) \otimes \Gamma_D \otimes \Gamma_W(\pi_v)$ where
 - Γ_D (resp. Γ_W) is a stable lattice of $\pi_v[s_i(\varrho)]_D$ (resp. $\mathbb{L}_{g_i(\varrho)}(\pi_v)$);
 - Γ_G is isomorphic is a stable $P_d(F_v)$ -equivariant lattice of $\text{St}_{s_i(\varrho)}(\pi_v)$ such that every irreducible subspace of its modulo l reduction, is isomorphic to $r_l(\tau_{nd})$.

The only difference from previous section concerns the lattice Γ_G which is obtained through

$$\begin{array}{ccc} j_{\neq c'}, *j_{\neq c'}^*{}^{p+(ss)} \mathcal{P}_{RI,\otimes}(s_i(\varrho) - 1, \pi_v) \left(\frac{2-s_i(\varrho)}{2} \right) & \twoheadrightarrow & {}^{p+} \mathcal{H}^1 i_c^! \mathcal{P}_{RI,\otimes}(s_i(\varrho) - 1, \pi_v) \left(\frac{s_i(\varrho)-2}{2} \right) \\ \uparrow & & \simeq \uparrow \\ j_{\neq c'}, *j_{\neq c'}^* \text{gr}^k(j_{\neq c'}, *j_{\neq c'}^* \Psi_{\varrho,!},c) & \twoheadrightarrow & \mathcal{P}_{\text{Fill},*,c}(s_i(\varrho), \pi_v) \end{array}$$

and where, by induction, ${}^{p+}\mathcal{H}^1 i_c^! \mathcal{P}_{RI, \otimes}(s_i(\varrho) - 1, \pi_v)(\frac{s_i(\varrho)-2}{2})$ is given by $\Gamma'_G \times (\pi_v \{\frac{1-s_i(\varrho)}{2}\})|_{P_{g_i(\varrho)}(F_v)}$ where by the induction hypothesis Γ'_G is a $P_{(s_i(\varrho)-1)g_i(\varrho)}(F_v)$ -equivariant lattice of $\text{St}_{s_i(\varrho)-1}(\pi_v \{\frac{1}{2}\})$ such that every subspace of its modulo l reduction is isomorphic to $r_l(\tau_{nd})$. The persistance of non degeneracy property then follows from the exactness of Φ^- and Ψ^- and from proposition 1.3.5.

Remark: It's not so easy than in the previous situation, to identify the lattice as now we only have the following commutative diagram

$$\begin{array}{ccc}
 & & \text{St}_{s_i(\varrho)}(\pi_v)|_{P_d(F_v)} \\
 & & \downarrow \\
 \Gamma'_G \times (\pi_v \{\frac{1-s_i(\varrho)}{2}\})|_{P_{g_i(\varrho)}(F_v)} & \hookrightarrow & (\Gamma'_G \times \pi_v \{\frac{1-s_i(\varrho)}{2}\})|_{P_d(F_v)} \\
 & \searrow & \downarrow \\
 & & LT_{\pi_v}(s_i(\varrho) - 2, 1)|_{P_d(F_v)}.
 \end{array}$$

4.3. Other order of cohomology groups. — As the situations of $\mathcal{U}_{\varrho, \mathbb{N}}^{d-1-\delta}$ and $\mathcal{V}_{\varrho, \mathbb{N}}^{d-1+\delta}$ are dual, consider for example the case of $\mathcal{U}_{\varrho, \mathbb{N}}^{d-1-\delta}$ for $\delta > 0$. Start again from

$$0 \rightarrow j_{\neq c, !} j_{\neq c}^* \Psi_{\varrho} \rightarrow \Psi_{\varrho} \rightarrow \Psi_{\varrho, !, c} \rightarrow 0,$$

and with the filtration of $\text{Fil}_*^{\bullet}(\Psi_{\varrho, !, c})$ with graduates $\text{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho, !, c})$ which can be refine as before, such that to obtain graduate $\text{gr}^k(\Psi_{\varrho, !, c})$ verifying

$$\begin{aligned}
 {}^p j_{c, !} j_c^{:=tg_i(\varrho)} j_c^{:=tg_i(\varrho), *} \text{gr}^k(\Psi_{\varrho, !, c}) \\
 \hookrightarrow_+ \text{gr}_k(\Psi_{\varrho, *, c}) \hookrightarrow_+ \\
 {}^{p+} j_{c, !} j_c^{:=tg_i(\varrho)} j_c^{:=tg_i(\varrho), *} \text{gr}^k(\Psi_{\varrho, !, c}),
 \end{aligned}$$

with $\text{gr}^k(\Psi_{\varrho, !, c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathcal{P}(t, \pi_v)(\frac{1-t}{2})$. In [8], we prove

- $\text{gr}^k(\Psi_{\varrho, *, c}) \simeq {}^p j_{c, !} j_c^{:=tg_i(\varrho)} j_c^{:=tg_i(\varrho), *} \text{gr}^k(\Psi_{\varrho, !, c})$,
- and the sheaf cohomology group of $\text{gr}_k(\Psi_{\varrho, *, c})$ are free.

In particular for a supersingular point z , the spectral sequence computing $\mathcal{H}^{-\delta} i_z^* \Psi_{\varrho} \simeq \mathcal{H}^{-\delta} i_z^* \Psi_{\varrho, !, c}$ through the $\mathcal{H}^{\bullet} i_z^* \text{gr}^k(\Psi_{\varrho, *, c})$, degenerates at E_1 .

Note then that the $P_d(F_v)$ -lattice is given by the induced representation

$$\Gamma_G \times \text{Speh}_\delta(\pi_v \left\{ \frac{s_i(\varrho) - \delta - 1}{2} \right\})$$

where

- Γ_G is a stable $P_{(s_i(\varrho)-\delta)g_i(\varrho)}(F_v)$ -lattice of $\text{St}_{(s_i(\varrho)-\delta)g_i(\varrho)}(F_v)(\pi_v)$ such that any irreducible subspace is isomorphic to τ_{nd} ;
- $\text{Speh}_\delta(\pi_v)$ has, up to isomorphism, only one stable $GL_{\delta g_i(\varrho)}(F_v)$ -stable lattice.

Like in previous sections, we then obtain the following description of $\mathcal{U}_{\varrho, \mathbb{N}}^{d-1-\delta}$, which is free by the main result of [8].

4.3.1. Proposition. — *As a $\overline{\mathbb{Z}}[P_d(F_v) \times D_{v,d}^\times \times W_v]$ -module, $\mathcal{U}_{\varrho, \mathbb{N}}^{d-1-\delta}$ has a filtration with successive graduates $\text{gr}^i(\mathcal{U}_{\varrho, \mathbb{N}, \text{free}}^{d-1-\delta})$ for $-s \leq i \leq 1$ where*

- *s is maximal such that $g_s(\varrho)$ divides d and for $-1 \leq i \leq s$, we denote again $s_i(\varrho) = \frac{d}{g_i(\varrho)}$;*
- *$\text{gr}^{-i}(\mathcal{U}_{\varrho, \mathbb{N}, \text{free}}^{d-1-\delta}) \simeq \bigoplus_{\pi_v \in \text{Scusp}_i(\varrho)} \Gamma_{GDW}(\pi_v)$ with $\Gamma_{GDW}(\pi_v) \simeq \Gamma_G(\pi_v) \otimes \Gamma_D \otimes \Gamma_W(\pi_v)$ where*
 - *Γ_D (resp. Γ_W) is a stable lattice of $\pi_v[s_i(\varrho)]_D$ (resp. $\mathbb{L}_{g_i(\varrho)}(\pi_v)$);*
 - *Γ_G is isomorphic is a stable $P_d(F_v)$ -equivariant lattice of $LT_{\pi_v}(s_i(\varrho) - \delta - 1, \delta)$ such that any irreducible $P_d(F_v)$ -equivariant subspace of $\Gamma_G \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ has order of derivative equals to $g_i(\varrho)$.*

Remark: Consider the case where $s = -1$, that is $g_0(\varrho)$ does not divide d . Then we see that the non degeneracy property which would advocate that irreducible subspaces of $\mathcal{U}_{\varrho, \mathbb{N}}^{d-1-\delta} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ should be the less possible degenerate among all the others, is no more true for $\delta > 0$, even more this is the exact opposite as $g_i(\varrho)$ is the smallest derivative order of all irreducible subquotients of $(\Gamma_G \times \text{Speh}_\delta(\pi_v \left\{ \frac{s_i(\varrho) - \delta - 1}{2} \right\})) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. One way to keep trace of the non degeneracy property might be the following statement which follows trivially from the isomorphism $(\tau \times \pi)^{(k)} \simeq \tau^{(k)} \times \pi$ for τ (resp. π) a representation of $P_d(F_v)$ (resp. $GL_s(F_v)$), and the short exact sequence (1.3.8).

4.3.2. Proposition. — *Let τ be an irreducible subspace of the modulo l reduction of $\Gamma_G \times \text{Speh}_\delta(\pi_v \left\{ \frac{s_i(\varrho) - \delta - 1}{2} \right\})$. Then $\tau^{(g_i(\varrho))}$ is non degenerate.*

To sum up, we have seen that a irreducible subspace of $(\Gamma_G \times \mathrm{Speh}_\delta(\pi_v\{\frac{s_i(\varrho)-\delta-1}{2}\})) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is necessary with derivative order $g_i(\varrho)$, but among all of them it is the less degenerated one.

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