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Analysis of the incurred but not reported/infinite server queue process with semi-Markovian multivariate discounted inputs

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Abstract: We consider a general k dimensional discounted infinite server queues process (alternatively, an Incurred But Not Reported (IBNR) claim process) where the multivariate inputs (claims) are given by a k dimensional finite state Markov chain and the arrivals follow a renewal process. After deriving a multidimensional integral equation for the moment generating function jointly to the state of the input at time t given the initial state of the input at time 0, asymptotic results for the first and second (matrix) moments of the process are provided. In particular, when the interarrival or service times are exponentially distributed, transient expressions for the first two moments are obtained. Also, the moment generating function for the process with deterministic interarrival times is considered to provide more explicit expressions. Finally, we demonstrate the potential of the present model by showing how it allows us to study a semi-Markov modulated infinite queues process where the customers (claims) arrival and service (reporting delay) times depend on the state of the process immediately before and on the switching times.

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1. Introduction

It is an important task to study probability modeling of aggregate risk processes in various areas such as actuarial science, queueing theory, and reliability theory. For instance, modeling aggregate losses is a fundamental and essential task in the insurance business since risk premiums and reserves are calculated based on these quantities. In particular, the research on aggregation of correlated risks (multivariate risks) is striving to develop techniques to estimate the combined effect of different types of risks on the firm or system. Furthermore, it is essential to focus on those risks observed at different times with each arrival adjusted by adding a random delay which lead to study the multivariate discounted Incurred But Not Reported (IBNR) claims in insurance and also the total number of customers in an infinite server queue with correlated batch arrivals. In the renewal arrival process, [13] provides explicit expressions for the joint moments of multivariate aggregate discounted IBNR claims which are recursively obtainable, and then [11] develop asymptotic approximation methods to study these joint

moments and also provided some queueing theoretic applications, including the workload of the queue and infinite server queues in tandem.

The present paper considers the model which is an extension of the one given in [11]. In each batch arrival the model consists of multivariate queues (claims) which are modeled by some finite Markov chain and a renewal process is assumed for the batch arrivals. The Markovian assumption for a vector of queues (claims) enable us to study the infinite server queue/IBNR process in more realistic situations such that arrival times and service times are dependent on the states of the external semi-Markovian process immediately before and on the switching times, as will be illustrated later in Section 6. It is natural to model that the arrival process and service time are modulated by some external process, in particular, when this process impacts on the intensity of claim arrival processes and in turn, the types of service time. For example, the number of multiple types of claims in catastrophe insurance varies depending on the environmental condition and also it could lead to different types of reporting/settlement time delays.

To the best of our knowledge, there is no study of the current setting of the model (especially in the presence of a discounting factor) in the literature of queueing theory. Instead, similar settings of the model such as Markov modulated infinite server systems are found. For example, in [4] the particle arrives according to a Poisson process and the Poisson arrival rate and the distribution of service times are dependent on the state of an external Markov process (background process). When the interarrival times in our model is exponential, the one-dimensional case in Section 6 is similar to the one studied in [4]. In a system with multiple infinite-server queues, [8] consider that both the arrival rates and the parameter of the exponentially distributed service times are modulated by a common background process. In [3], a similar model but a single queue with a Poisson arrival is revisited to study the asymptotic behavior of the number of customers in the system in the large-deviations regime. In some papers, arrival and service rates in an infinite server queue are governed by an external semi-Markov process. See [5] and [7] for instance. [10] study $M/M/\infty$ queue model modulated by an external continuous-time Markov Chain.

The remainder of the paper is structured as follows: In Section 2, we provide the description of the mathematical model. After deriving some general results on the (joint) moment generating function (mgf)/Laplace Transform (LT) and first two moments in Section 3, we show in Section 4 that the limiting second order joint moments are explicitly available when the service times are (potentially degenerated) exponentially distributed and the transient moments can be obtained when the interarrivals are exponentially distributed. Some numerical illustrations for the limiting behaviour of the first and second joint moments are provided at the end of Section 4.1. Section 5 is concerned with the particular case of deterministic interarrivals, where we show that the mgf has a simple expression as a matrix product (see Theorem 17). Application to a model related to the queue/IBNR process is provided in Section 6. It is assumed that the queue/IBNR process is modulated by an external semi-markovian process, such that arrivals and service times depend on the state of the modulating process immediately before and after it switches states.

2. Model description

Let $\{N_t, t \geq 0\}$ be a renewal process associated with a non decreasing sequence $(T_i)_{i \in \mathbb{N}}$ with $T_0 = 0$, such that $(T_i - T_{i-1})_{i \geq 1}$ is independent and identically distributed (iid). Also let $\tau = T_1$ with cumulative distribution function (cdf) F and LT $\mathcal{L}^\tau(u) = \mathbb{E}(e^{-u\tau})$ for $u \geq 0$. We

introduce a stationary ergodic finite Markov chain $(X_i)_{i \in \mathbb{N}}$ with a state space $\mathcal{S} = \{0, \dots, K\}^k$ for some $K \in \mathbb{N}$ and $k \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, so that X_i is for all i of the form $X_i = (X_{i1}, \dots, X_{ik})$ with $X_{ij} \in \{0, \dots, K\}$ for $j = 1, \dots, k$. Then for $\delta \geq 0$, the discounted process $\{Z(t) = Z(t; \delta) \in \mathbb{R}^k, t \geq 0\}$ is a vector of k processes $Z(t) = (Z_1(t), \dots, Z_k(t))$ with each process defined as

$$Z_j(t) = \sum_{i=1}^{N_t} X_{ij} e^{-\delta(L_{ij}+T_i)} \mathbb{1}_{[t < L_{ij}+T_i]} = \sum_{i=1}^{\infty} X_{ij} e^{-\delta(L_{ij}+T_i)} \mathbb{1}_{[T_i \leq t < L_{ij}+T_i]}, \quad (1)$$

where $(L_{ij})_{i \in \mathbb{N}, j=1, \dots, k}$ is a sequence of independent random variables (rvs) such that $(L_{i1}, \dots, L_{ik})_{i \in \mathbb{N}}$ is iid (although L_{i1}, \dots, L_{ik} may have different distributions). We set (L_1, \dots, L_k) to be a generic random vector distributed as the (L_{i1}, \dots, L_{ik}) 's, with each L_i having a Laplace transform denoted by $\mathcal{L}^{L_i}(u) = \mathbb{E}(e^{-uL_i})$ for $u \geq 0$. As in [11], we let $\tilde{Z}(t) = \tilde{Z}(t; \delta) := e^{\delta t} Z(t; \delta)$. The processes described in (1) are viewed as different quantities of interest in the following two areas. In actuarial science, when severities of the claims of different types occurring due to a common accident or catastrophe event and there are some time delays for insurers to hear (or settle) these claims, X_{ij} represents amounts of j -type of claim arriving at the i th event and X_i is a vector of multivariate claims arriving in the same event. In this case, $Z_j(t)$ in (1) is regarded as discounted IBNR amounts of j -type of claim by time t and $Z(t)$ is a vector of multivariate discounted IBNR claim processes of k types of claims. In queueing theory, and especially when $\delta = 0$, X_{ij} represents the number of customers arriving in queue $j \in \{1, \dots, k\}$ at time T_i , each of those customers with same service time L_{ij} . Hence, X_{ij} and L_{ij} will in what follows be invariably referred to the claim/batch sizes and delay/service times respectively. In particular, when $\delta = 0$, this model is a generalisation of the Model II in [8] which considers the case where τ is exponentially distributed, i.e. when the set of queues is modulated by a common continuous-time Markov chain, as will be discussed in Section 4.

Notation. Let $P = (p(x, x'))_{(x, x') \in \mathcal{S}^2}$ and $\pi = (\pi(x))_{x \in \mathcal{S}}$ (written as a row vector) be respectively the transition matrix and stationary distribution of the Markov chain. We next define for all $r \geq 0$ and $s = (s_1, \dots, s_k) \in \mathbb{R}^k$,

$$\tilde{\pi}(s, r) := \text{diag} \left[\mathbb{E} \left(\exp \left\{ \sum_{j=1}^k s_j x_j e^{-\delta(L_j - r)} \mathbb{1}_{[L_j > r]} \right\} \right) \right], \quad x = (x_1, \dots, x_k) \in \mathcal{S}, \quad (2)$$

$$\tilde{Q}(s, r) := \tilde{\pi}(s, r) P', \quad (3)$$

where P' denotes the transpose of matrix P . We also introduce some notation in the following. I is the identity matrix, $\mathbf{0}$ is a column vector with zeroes, and $\mathbf{1}$ is a column vector with 1's, of appropriate dimensions. When a random variable (rv) X is exponentially distributed with mean $1/\beta$, it is denoted as $X \sim \mathcal{E}(\beta)$. Also, we let the $\mathcal{S} \times \mathcal{S}$ diagonal matrices

$$\begin{aligned} \Delta_i &:= \text{diag} [x_i, x = (x_1, \dots, x_k) \in \mathcal{S}], \quad i = 1, \dots, k, \\ \Delta_\pi &:= \text{diag}(\pi(x), x \in \mathcal{S}). \end{aligned} \quad (4)$$

The mgf of the process $\tilde{Z}(t) = \tilde{Z}(t; \delta)$ jointly to the state of X_{N_t} given the initial state of X_0 is denoted by

$$\tilde{\psi}(s, t) = \tilde{\psi}(s, t; \delta) := \left[\mathbb{E} \left(e^{\langle s, \tilde{Z}(t) \rangle} \mathbb{1}_{[X_{N_t} = y]} \middle| X_0 = x \right) \right]_{(x, y) \in \mathcal{S}^2}, \quad t \geq 0, \quad (5)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on \mathbb{R}^k . Note that $s = (s_1, \dots, s_k)$ is assumed to be such that $s_j \in \mathbb{R}$ for all $j = 1, \dots, k$ and such that (5) is well defined, i.e. the expectation is finite. Definition (5) may in fact very well include the case where the s_j 's are complex and purely imaginary, in which case $\tilde{\psi}(s, t)$ is the characteristic function of $\tilde{Z}(t)$ jointly to X_{N_t} ; this will particularly be the case in the proof of Theorem 17. Note also that X_0 in (5) has no direct physical interpretation here, as the claims sizes/customer batches are given by X_i , $i \geq 1$, and is rather introduced for technical purpose. We define the first and second (matrix) moments of $\tilde{Z}(t)$ jointly to the state of the Markov chain X_{N_t} as

$$\begin{aligned} M_i(t) &:= \left[\mathbb{E} \left(\tilde{Z}_i(t) \mathbb{1}_{[X_{N_t}=y]} \middle| X_0 = x \right) \right]_{(x,y) \in \mathcal{S}^2}, \quad i = 1, \dots, k, \\ M_{ii'}(t) &:= \left[\mathbb{E} \left(\tilde{Z}_i(t) \tilde{Z}_{i'}(t) \mathbb{1}_{[X_{N_t}=y]} \middle| X_0 = x \right) \right]_{(x,y) \in \mathcal{S}^2}, \quad i, i' = 1, \dots, k, \end{aligned} \quad (6)$$

respectively. We remark that the mgf defined in (5) is different from the one studied in [11] which does not consider Markovian assumption for a vector X_i and joint structure with the state X_{N_t} conditioning on the initial state X_0 .

We finish this introductory section by giving some results of independent interest that will be used in the rest of the paper. The following lemma is important for some computations on Markov chains, and may be found in [6, Lemma 1]:

Lemma 1. *Let $(S_n)_{n \in \mathbb{N}}$ be a stationary Markov chain with a state space E , the transition matrix P and (stationary) distribution $\pi = (\pi(x))_{x \in E}$. For all functions f_1, \dots, f_{l+1} we have*

$$\mathbb{E}(f_1(S_1) \cdots f_l(S_l)) = \mathbf{1}' \prod_{i=0}^{l-2} Q_{f_{l-i}} \pi_{f_1}, \quad (7)$$

where $Q_{f_i} := \text{diag}(f_i(z), z \in E)P'$ for $i = 1, \dots, l$, and $\pi_{f_1} := \text{diag}(f_1(z), z \in E)\pi'$. Here $\mathbf{1}$ represents a column vector with 1's, of appropriate dimension.

The following is a direct consequence of (7). Let e_x (resp. e_y) be the column vector of which z th entry is $\delta_x(z)$ (resp. $\delta_y(z)$). One has then for all x, y in E that

$$\mathbb{E}(f_1(S_1) \cdots f_l(S_l) \mathbb{1}_{[S_l=y]} \middle| S_1 = x) = e'_y \prod_{i=0}^{l-2} Q_{f_{l-i}} \text{diag}(f_1(z), z \in E) e_x,$$

which, because it is a scalar, is equal to its transpose, i.e.

$$\mathbb{E}(f_1(S_1) \cdots f_l(S_l) \mathbb{1}_{[S_l=y]} \middle| S_1 = x) = e'_x \text{diag}(f_1(z), z \in E) \prod_{i=2}^l Q'_{f_i} e_y,$$

which immediately implies the following corollary.

Corollary 2. *Under the same notation as in Lemma 1, one has the matrix equality*

$$\left[\mathbb{E}(f_1(S_1) \cdots f_l(S_l) \mathbb{1}_{[S_l=y]} \middle| S_1 = x) \right]_{(x,y) \in E^2} = \text{diag}(f_1(z), z \in E) \prod_{i=2}^l Q'_{f_i}.$$

3. General results

3.1. The Laplace transform

The aim of this subsection is to establish some properties verified by the mgf $\tilde{\psi}(s, t)$ in (5).

Proposition 3. *The mgf of $\tilde{Z}(t)$ defined by (5) satisfies*

$$\tilde{\psi}(s, t) = \mathbb{E} \left(\prod_{i=1}^{N_t} \tilde{Q}(s, t - T_i)' \right) = \bar{F}(t)I + \mathbb{E} \left(\mathbb{1}_{[N_t > 0]} \prod_{i=1}^{N_t} \tilde{Q}(s, t - T_i)' \right) \quad (8)$$

for all $s \in \mathbb{R}^k$, $t \geq 0$, with the usual convention $\prod_{i=1}^{N_t} \tilde{Q}(s, t - T_i)' = I$ if $N_t = 0$. Besides, it satisfies the following multidimensional integral equation:

$$\tilde{\psi}(s, t) = \bar{F}(t)I + \int_0^t \tilde{Q}(s, t - y)' \tilde{\psi}(s, t - y) dF(y), \quad \forall s \in \mathbb{R}^k, t \geq 0. \quad (9)$$

Proof. Decomposing according to $N_t = 0$ and $N_t > 0$ yields that

$$\tilde{\psi}(s, t) = [\mathbb{P}(X_0 = y | X_0 = x) \mathbb{P}(N_t = 0)]_{(x, y) \in \mathcal{S}^2} + \left[\mathbb{E} \left(e^{\langle s, \tilde{Z}(t) \rangle} \mathbb{1}_{[X_{N_t} = y]} \mathbb{1}_{[N_t > 0]} \middle| X_0 = x \right) \right]_{(x, y) \in \mathcal{S}^2}. \quad (10)$$

Note that $\mathbb{P}(X_0 = y | X_0 = x) \mathbb{P}(N_t = 0) = \delta_{x, y} \bar{F}(t)$ where $\delta_{x, y}$ is the Kronecker symbol, equal to 1 iff $x = y$ and 0 else, so that the first term on the right-hand side of (10) is given by the first term on the right-hand side of (8). We turn to the second term on the right-hand side of (10). Let us define $\mathcal{F} = \sigma(T_i, i \geq 0)$ as well as the set of rvs

$$s_t(x, y) := \mathbb{E} \left(e^{\langle s, \tilde{Z}(t) \rangle} \mathbb{1}_{[X_{N_t} = y]} \mathbb{1}_{[N_t > 0]} \middle| X_0 = x, \mathcal{F} \right), \quad x, y \in \mathcal{S},$$

where $s = (s_1, \dots, s_k) \in \mathbb{R}^k$ is fixed throughout the proof. Using $\tilde{Z}(t) = e^{\delta t} Z(t; \delta)$ and (1), one obtains

$$s_t(x, y) = \mathbb{E} \left(\mathbb{1}_{[N_t > 0]} \prod_{i=1}^{N_t} \exp \left\{ \sum_{j=1}^k s_j X_{ij} e^{-\delta(L_{ij} - (t - T_i))} \mathbb{1}_{[L_{ij} > t - T_i]} \right\} \mathbb{1}_{[X_{N_t} = y]} \middle| X_0 = x, \mathcal{F} \right). \quad (11)$$

In order to compute $s_t(x, y)$, we use the fact that the Markov chain $(X_i)_{i \in \mathbb{N}}$ is independent from \mathcal{F} and $(L_{ij})_{i \in \mathbb{N}, j=1, \dots, k}$. Hence identifying $l = N_t + 1$, $S_i := X_{i-1}$, $f(S_1) = f(X_0) = 1$ and $f_i(S_i) = f_i(X_{i-1}) := \exp \left\{ \sum_{j=1}^k s_j X_{i-1, j} e^{-\delta(L_{(i-1), j} - (t - T_{i-1}))} \mathbb{1}_{[L_{(i-1), j} > t - T_{i-1}]} \right\}$ for $i = 2, \dots, N_t + 1$, in Corollary 2 yields from (11) that

$$[s_t(x, y)]_{(x, y) \in \mathcal{S}^2} = \mathbb{1}_{[N_t > 0]} I \cdot \prod_{i=2}^{N_t+1} \tilde{Q}(s, t - T_{i-1})' = \mathbb{1}_{[N_t > 0]} \prod_{i=1}^{N_t} \tilde{Q}(s, t - T_i)',$$

where we recall that I is the identity matrix and $\tilde{Q}(\cdot, \cdot)$ is defined in (3). Since $\mathbb{E} \left([s_t(x, y)]_{(x, y) \in \mathcal{S}^2} \right)$ is the second term in the right-hand side of (10), one thus obtains (8).

Finally, (9) is obtained by considering again $[N_t = 0] \iff [T_1 > t]$ and $[N_t > 0] \iff [T_1 \leq t]$ and conditioning with respect to T_1 . \square

It is known that a multidimensional integral equation such as (9) cannot be solved in general. One particular case is when arrivals occur according to a Poisson process, in which case one has the following result.

Proposition 4. *If $\tau \sim \mathcal{E}(\lambda)$ for $\lambda > 0$, then $\tilde{\psi}(s, t)$ is the unique solution to the first order linear (matrix) differential equation*

$$\partial_t \tilde{\psi}(s, t) = [-\lambda I + \lambda \tilde{Q}(s, t)'] \tilde{\psi}(s, t) = [\lambda(P - I) + \lambda P(\tilde{\pi}(s, t) - I)] \tilde{\psi}(s, t) \quad (12)$$

with the initial condition $\tilde{\psi}(s, 0) = I$ from (8).

Note that, even though (12) admits a unique solution, $\tilde{\psi}(s, t)$ is not explicit except in particular cases e.g. when $P = I$, i.e. when there is not Markov interference in the model, as indeed in that case the differential equation (12) can be solved componentwise. Also note that a similar differential equation was obtained when $\delta = 0$ in [9, Theorem 3.1] for the joint generating function, when interarrival is Matrix exponential distributed.

Proof. Setting $dF(y) = \lambda e^{-\lambda y} dy$ and $\bar{F}(t) = e^{-\lambda t}$ in (9) and differentiating with respect to t yields (12). \square

3.2. The first and second moments

We are now interested in the first two moments defined in (6). It is standard that $M_i(t)$ and $M_{ii'}(t)$ are linked to $\tilde{\psi}(s, t)$ by

$$M_i(t) = \partial_{s_i} \tilde{\psi}(s, t) \Big|_{s=\mathbf{0}}, \quad M_{ii'}(t) = \partial_{s_i} \partial_{s_{i'}} \tilde{\psi}(s, t) \Big|_{s=\mathbf{0}}, \quad i, i' = 1, \dots, k.$$

It requires to differentiate $\tilde{\pi}(s, r)$ with respect to s_i or s_i and $s_{i'}$ followed by putting $s = \mathbf{0}$. One obtains

$$\partial_{s_i} \tilde{\pi}(s, r) \Big|_{s=\mathbf{0}} = \mathbb{E} \left(e^{-\delta(L_i - r)} \mathbb{1}_{[L_i > r]} \right) \Delta_i, \quad (13)$$

$$\partial_{s_{i'}} \partial_{s_i} \tilde{\pi}(s, r) \Big|_{s=\mathbf{0}} = \mathbb{E} \left(e^{-\delta(L_i - r)} \mathbb{1}_{[L_i > r]} e^{-\delta(L_{i'} - r)} \mathbb{1}_{[L_{i'} > r]} \right) \Delta_i \Delta_{i'}. \quad (14)$$

Moreover, one also needs to compute $\tilde{\psi}(\mathbf{0}, r)$, namely from (8)

$$\tilde{\psi}(\mathbf{0}, r) = \mathbb{E} \left(\prod_{i=1}^{N_r} \tilde{Q}(\mathbf{0}, r - T_i)' \right) = \mathbb{E} (P^{N_r}), \quad r \geq 0, \quad (15)$$

as indeed $\tilde{Q}(\mathbf{0}, r) = P'$, again with the convention $P^0 = I$. We then need to introduce the following notation for all $i, i' = 1, \dots, k$:

$$\begin{aligned} b_i(t) &:= \int_0^t \left[\partial_{s_i} \tilde{\pi}(s, t - y) \Big|_{s=\mathbf{0}} \right] P \tilde{\psi}(\mathbf{0}, t - y) dF(y), \\ b_{ii'}(t) &:= \int_0^t \left[\partial_{s_{i'}} \partial_{s_i} \tilde{\pi}(s, t - y) \Big|_{s=\mathbf{0}} \right] P \tilde{\psi}(\mathbf{0}, t - y) dF(y) \\ &\quad + \int_0^t \left[\partial_{s_i} \tilde{\pi}(s, t - y) \Big|_{s=\mathbf{0}} \right] P M_{i'}(t - y) dF(y) + \int_0^t \left[\partial_{s_{i'}} \tilde{\pi}(s, t - y) \Big|_{s=\mathbf{0}} \right] P M_i(t - y) dF(y), \end{aligned} \quad (16)$$

$$(17)$$

where $\partial_{s_i} \tilde{\pi}(s, t - y)|_{s=\mathbf{0}}$, $\partial_{s_{i'}} \partial_{s_i} \tilde{\pi}(s, t - y)|_{s=\mathbf{0}}$, $\partial_{s_{i'}} \partial_{s_i} \tilde{\pi}(s, t - y)|_{s=\mathbf{0}}$ and $\tilde{\psi}(\mathbf{0}, t - y)$ are given by (13), (14) and (15). Note that $b_i(t)$ is not here explicit since (15) does not in general have a closed form expression, and so is $b_{i'}(t)$. We may then obtain general results concerning $M_i(t)$ and $M_{i'}(t)$. Following the notation in [2, Section 2], we define, for a $N \times N$ dimensional matrix of non decreasing right continuous functions $(t \mapsto F_{ij}(t))_{i,j=1,\dots,N}$ and a $N \times N$ dimensional matrix of bounded measurable functions $(t \mapsto H_{ij}(t))_{i,j=1,\dots,N}$ the convolution $t \mapsto F \star H(t) = (F \star H)_{i,j=1,\dots,N}(t)$ by

$$(F \star H)_{i,j}(t) := \sum_{h=1}^N \int_0^t H_{hj}(t-u) dF_{ih}(u) \quad i, j = 1, \dots, N, t \geq 0.$$

Then, differentiating (9) with respect to s_i or $s_{i'}$ and $s_{i'}$ followed by putting $s = \mathbf{0}$ results in the following result.

Proposition 5. *For $i, i' = 1, \dots, k$, $M_i(t)$ and $M_{i'}(t)$ satisfy the following multidimensional renewal equations*

$$M_i(t) = b_i(t) + (PF) \star M_i(t), \quad t \geq 0, \quad (18)$$

$$M_{i'}(t) = b_{i'}(t) + (PF) \star M_{i'}(t), \quad t \geq 0, \quad (19)$$

where $(PF)(t) := (p(x, y)F(t))_{(x,y) \in \mathcal{S}^2}$, and $b_i(t)$ and $b_{i'}(t)$ are given by (16) and (17).

Although the solution to (9) does not have a closed form expression, it turns out that (16) and (17) have solutions which can be expressed in terms of a multidimensional renewal function. Indeed, since interarrival times satisfy $\tau > 0$ a.s., one has that $(PF)(0)$ is the zero matrix, of which largest eigenvalue is thus 0. [2, Lemma 2.1] entails that

$$M_i(t) = U \star b_i(t), \quad M_{i'}(t) = U \star b_{i'}(t), \quad t \geq 0, \quad (20)$$

where $U(t)$ is the renewal function defined by $U(t) := \sum_{n=0}^{\infty} (PF)^{\star(n)}(t)$, an $\mathcal{S} \times \mathcal{S}$ matrix, see [2, Definition (2.3)]. At this point, solutions (20) are still not satisfactory because neither $U(t)$ nor $b_i(t)$ and $b_{i'}(t)$ are explicit. Some general results on the limiting behaviour of $M_i(t)$ and $M_{i'}(t)$ can be obtained and are given in the following result.

Lemma 6. *Let us suppose that τ is non lattice, then one has the following*

$$M_i(t) \longrightarrow \frac{1}{\mathbb{E}(\tau)} \mathbf{1} \pi \int_0^{\infty} b_i(t) dt, \quad t \rightarrow \infty, \quad i = 1, \dots, k, \quad (21)$$

$$M_{i'}(t) \longrightarrow \frac{1}{\mathbb{E}(\tau)} \mathbf{1} \pi \int_0^{\infty} b_{i'}(t) dt, \quad t \rightarrow \infty, \quad i, i' = 1, \dots, k. \quad (22)$$

Proof. Since $(PF)(\infty) = P$ has a spectral radius equal to 1 and has (row vector) π and $\mathbf{1}$ (column vector) as left and right eigenvectors associated to the eigenvalue 1, renewal equation (18) satisfied by $M_i(t)$ and [2, Theorem 2.2 (iii)] yield (21). The same method applied to the renewal equation (19) satisfied by $M_{i'}(t)$ yields (22). \square

Note that the limit in (21) is not clearly available because integrating $b_i(t)$ requires an explicit expression for $\mathbb{E}(P^{Nr})$ (see (16) with $\tilde{\psi}(\mathbf{0}, t - y)$ given by (15)). Likewise, the limit in (22) is not explicit either as it requires analytic expressions for $M_i(t)$ and $M_{i'}(t)$ in (17). However, the limiting first moment is explicitly shown below:

Proposition 7. *Assuming that τ is non lattice, the expectation $M_i(t)\mathbf{1} = [\mathbb{E}(\tilde{Z}_i(t) | X_1 = x)]'_{x \in \mathcal{S}}$ asymptotically behaves as*

$$M_i(t)\mathbf{1} \longrightarrow \frac{\mathbb{E}(X_i)}{\mathbb{E}(\tau)} \left[\frac{1 - \mathcal{L}^{L_i}(\delta)}{\delta} \right] \mathbf{1}, \quad t \rightarrow \infty, \quad i = 1, \dots, k. \quad (23)$$

Proof. Let us prove that one obtains (23) by post multiplying (21) by $\mathbf{1}$. First note that, as $P\mathbf{1} = \mathbf{1}$, $b_i(t)\mathbf{1}$ reduces thanks to (16), (13) and (15) to

$$\begin{aligned} b_i(t)\mathbf{1} &= \int_0^t \mathbb{E} \left(e^{-\delta(L_i - (t-y))} \mathbb{1}_{[L_i > t-y]} \right) \Delta_i \cdot P \mathbb{E} (P^{N_{t-y}} \mathbf{1} dF(y)) \\ &= \int_0^t \mathbb{E} \left(e^{-\delta(L_i - (t-y))} \mathbb{1}_{[L_i > t-y]} \right) dF(y) \cdot \Delta_i \mathbf{1} \\ &= \mathbb{E} \left(e^{-\delta(L_i - (t-\tau))} \mathbb{1}_{[0 \leq t-\tau < L_i]} \right) \Delta_i \mathbf{1}. \end{aligned} \quad (24)$$

In order to compute $\int_0^\infty b_i(t)\mathbf{1} dt$, one then calculates

$$\begin{aligned} \int_0^\infty \mathbb{E} \left(e^{-\delta(L_i - (t-\tau))} \mathbb{1}_{[L_i > t-\tau \geq 0]} \right) dt &= \int_0^\infty \mathbb{E} \left(e^{-\delta(L_i - (t-\tau))} \mathbb{1}_{[\tau \leq t < L_i + \tau]} \right) dt \\ &= \mathbb{E} \left(\int_\tau^{L_i + \tau} e^{-\delta(L_i - (t-\tau))} dt \right) = \mathbb{E} \left(\frac{1}{\delta} [1 - e^{-\delta L_i}] \right) = \frac{1 - \mathcal{L}^{L_i}(\delta)}{\delta}, \end{aligned}$$

so that one obtains from (24) and (21) that, as $t \rightarrow \infty$,

$$M_i(t)\mathbf{1} \longrightarrow \frac{1}{\mathbb{E}(\tau)} \mathbf{1} \pi \int_0^\infty b_i(t)\mathbf{1} dt = \frac{1}{\mathbb{E}(\tau)} \left[\frac{1 - \mathcal{L}^{L_i}(\delta)}{\delta} \right] \mathbf{1} \pi \Delta_i \mathbf{1}.$$

One checks easily that $\pi \Delta_i \mathbf{1} = \mathbb{E}(X_i)$ where Δ_i is given in (4), yielding (23). \square

3.3. The workload

We now turn to the workload of the queues jointly to the state of X_{N_t} . Following [11, Section 5.2], we recall that the workload for each queue $i = 1, \dots, k$ is defined as

$$D_i(t) := \sum_{j=1}^{N_t} (T_j + L_{ij} - t) \mathbb{1}_{[t < T_j + L_{ij}]}, \quad t \geq 0.$$

and that one has for all $t \geq 0$ and $i = 1, \dots, k$

$$D_i(t) = - \frac{\partial}{\partial \delta} \tilde{Z}(t; \delta) \Big|_{\delta=0}.$$

Here we define the joint expectation of the workload and the state of X_{N_t} given the initial state of X_0 as

$$\begin{aligned} W_i(t) &:= \left[\mathbb{E} \left(D_i(t) \mathbb{1}_{[X_{N_t} = y]} \Big| X_0 = x \right) \right]_{(x,y) \in \mathcal{S}^2} \\ &= - \frac{\partial}{\partial \delta} M_i(t) \Big|_{\delta=0} = \left[- \frac{\partial}{\partial \delta} \left[\partial_{s_i} \tilde{\psi}(s, t; \delta) \right]_{s=\mathbf{0}} \right]_{\delta=0}. \end{aligned} \quad (25)$$

The following results the analogs of Proposition 5, Lemma 6 and Proposition 7. First, let us define and compute

$$\ell_i(t) := \int_0^t \left[-\frac{\partial}{\partial \delta} [\partial_{s_i} \tilde{\pi}(s, t-y)]_{s=\mathbf{0}} \right]_{\delta=0} P \tilde{\psi}(\mathbf{0}, t-y) dF(y), \quad (26)$$

where it follows from (13) that

$$\left[-\frac{\partial}{\partial \delta} [\partial_{s_i} \tilde{\pi}(s, r)]_{s=\mathbf{0}} \right]_{\delta=0} = \mathbb{E}((L_i - r) \mathbb{1}_{[L_i > r]}) \Delta_i. \quad (27)$$

Consequently, the following proposition is provided.

Proposition 8. *The joint expectation of the workload and the state of Markov chain at N_t satisfies*

$$W_i(t) = \ell_i(t) + (PF) \star W_i(t), \quad t \geq 0, \quad (28)$$

of which asymptotic expression is given by

$$W_i(t) \longrightarrow \frac{1}{\mathbb{E}(\tau)} \mathbf{1} \pi \int_0^\infty \ell_i(t) dt, \quad t \rightarrow \infty, \quad i = 1, \dots, k. \quad (29)$$

Moreover, the asymptotic expected workload $W_i(t) \mathbf{1} = [\mathbb{E}(D_i(t) | X_0 = x)]'_{x \in \mathcal{S}}$ is given by

$$W_i(t) \mathbf{1} \longrightarrow \left[\frac{\mathbb{E}(L_i^2)}{2\mathbb{E}(\tau)} + \frac{\mathbb{E}(\tau^2)}{2\mathbb{E}(\tau)} + \mathbb{E}(L_i) \right] \mathbb{E}(X_i) \mathbf{1}, \quad t \rightarrow \infty, \quad i = 1, \dots, k. \quad (30)$$

Proof. Since the proof of (28) and (29) is analogous to the one of (18) and (21), our focus is on (30). As in the proof of Proposition 7 similar to (24), using $P\mathbf{1} = \mathbf{1}$ one finds

$$\ell_i(t) \mathbf{1} = \mathbb{E}((L_i + \tau - t) \mathbb{1}_{[L_i + \tau > t]}) \Delta_i \mathbf{1},$$

with $\int_0^\infty \mathbb{E}((L_i + \tau - t) \mathbb{1}_{[L_i + \tau > t]}) dt = \mathbb{E}\left(\int_0^{L_i + \tau} (L_i + \tau - t) dt\right) = \mathbb{E}((L_i + \tau)^2/2)$. Using independence of L_i and τ in the last expectation as well as $\pi \Delta_i \mathbf{1} = \mathbb{E}(X_i)$ yields (30) by post multiplying (29) by $\mathbf{1}$. \square

4. Special cases

The results given in Proposition 3, Lemma 6 and Proposition 8 hold under general assumptions on the service times (L_j) and interarrival times (τ). We present here some particular cases where those results are more explicitly obtainable with a specific distributional assumption for L_j or τ . Namely, as is customary when studying infinite server queues, one expects reasonably to obtain more information when one of those two rvs are exponentially distributed, see [12, Chapter 3].

4.1. Exponentially distributed service times

First, it is assumed that service time L_j for class j customer or claim is exponentially distributed with parameter $\mu_j > 0$. To provide explicit expressions for the limits of $M_i(t)$, $M_{ii'}(t)$

and $W_i(t)$ as $t \rightarrow \infty$ in (21), (22) and (29), we define the Laplace transforms for $\tilde{\psi}(s, t)$, $M_i(t)$, and $b_i(t)$ by

$$\begin{aligned}\hat{\psi}(s, h) &= \int_0^\infty \tilde{\psi}(s, t)e^{-ht}dt, \quad s \in \mathbb{R}^k, \\ \hat{M}_i(h) &= \int_0^\infty M_i(t)e^{-ht}dt, \\ \hat{b}_i(h) &= \int_0^\infty b_i(t)e^{-ht}dt,\end{aligned}\tag{31}$$

respectively for all $h > 0$. Next, some relations between the above quantities are first given.

Proposition 9. *The Laplace transforms verify for all $h > 0$*

$$\hat{\psi}(\mathbf{0}, h) = \frac{1 - \mathcal{L}^\tau(h)}{h} (I - \mathcal{L}^\tau(h)P)^{-1},\tag{32}$$

$$\hat{M}_i(h) = (I - \mathcal{L}^\tau(h)P)^{-1} \hat{b}_i(h), \quad i = 1, \dots, k.\tag{33}$$

Proof. Recalling that $\tilde{\pi}(\mathbf{0}, r) = I$ from (2), (9) with $s = \mathbf{0}$ becomes the renewal equation

$$\tilde{\psi}(\mathbf{0}, t) = \bar{F}(t)I + (PF) \star \tilde{\psi}(\mathbf{0}, \cdot)(t),$$

which, upon taking Laplace transforms, yields

$$\hat{\psi}(\mathbf{0}, h) = \frac{1 - \mathcal{L}^\tau(h)}{h} I + P\mathcal{L}^\tau(h)\hat{\psi}(\mathbf{0}, h).$$

Then (32) is obtained by noting that, since $\mathcal{L}^\tau(h) < 1$ and P is a stochastic matrix, the matrix $\mathcal{L}^\tau(h)P$ has spectral radius less than 1 hence $I - \mathcal{L}^\tau(h)P$ is invertible. Similarly, (33) is obtained by taking Laplace transforms in renewal equation (18). \square

Theorem 10. *The asymptotic result for the first moment jointly to the state of X_{N_t} in (21) can be precisely expressed as*

$$M_i(t) \longrightarrow \frac{\mu_i}{(\mu_i + \delta)^2} \left[\frac{1 - \mathcal{L}^\tau(\mu_i + \delta)}{\mathbb{E}(\tau)} \right] \mathbf{1}\pi \Delta_i P (I - \mathcal{L}^\tau(\mu_i + \delta)P)^{-1}\tag{34}$$

as $t \rightarrow \infty$.

Proof. When $L_i \sim \mathcal{E}(\mu_i)$, one computes

$$\mathbb{E} \left(e^{-\delta(L_i - r)} \mathbb{1}_{[L_i > r]} \right) = \frac{\mu_i}{\mu_i + \delta} e^{-(\mu_i + \delta)r}, \quad r \geq 0,\tag{35}$$

so that one has (16) from (13) and (32) that

$$\begin{aligned}b_i(t) &= \frac{\mu_i}{\mu_i + \delta} \Delta_i P \int_0^t e^{-(\mu_i + \delta)(t-y)} \tilde{\psi}(\mathbf{0}, t-y) dF(y) \\ &= \frac{\mu_i}{\mu_i + \delta} \Delta_i P \mathbb{E} \left[e^{-(\mu_i + \delta)(t-\tau)} \tilde{\psi}(\mathbf{0}, t-\tau) \mathbb{1}_{[t \geq \tau]} \right].\end{aligned}\tag{36}$$

The right-hand side of (21) is thus computed as

$$\begin{aligned}
& \frac{1}{\mathbb{E}(\tau)} \mathbf{1}\pi \int_0^\infty b_i(t) dt \\
&= \frac{1}{\mathbb{E}(\tau)} \mathbf{1}\pi \frac{\mu_i}{\mu_i + \delta} \Delta_i P \int_0^\infty \mathbb{E} \left[e^{-(\mu_i + \delta)(t - \tau)} \tilde{\psi}(\mathbf{0}, t - \tau) \mathbb{1}_{[t \geq \tau]} \right] dt \\
&= \frac{1}{\mathbb{E}(\tau)} \mathbf{1}\pi \frac{\mu_i}{\mu_i + \delta} \Delta_i P \mathbb{E} \left[\int_\tau^\infty e^{-(\mu_i + \delta)(t - \tau)} \tilde{\psi}(\mathbf{0}, t - \tau) dt \right] \\
&= \frac{1}{\mathbb{E}(\tau)} \mathbf{1}\pi \frac{\mu_i}{\mu_i + \delta} \Delta_i P \hat{\psi}(\mathbf{0}, \mu_i + \delta),
\end{aligned}$$

and in turn, (34) is obtained thanks to (32). \square

Let us note that the previous proof enables us to similarly obtain the expression of $\hat{b}_i(h)$ defined in (31) thanks to (36) as follows

$$\begin{aligned}
\hat{b}_i(h) &= \frac{\mu_i}{\mu_i + \delta} \Delta_i P \int_0^\infty e^{-ht} \mathbb{E} \left[e^{-(\mu_i + \delta)(t - \tau)} \tilde{\psi}(\mathbf{0}, t - \tau) \mathbb{1}_{[t \geq \tau]} \right] dt \\
&= \frac{\mu_i}{\mu_i + \delta} \Delta_i P \mathbb{E} \left[\int_\tau^\infty e^{-h(t - \tau)} e^{-(\mu_i + \delta)(t - \tau)} \tilde{\psi}(\mathbf{0}, t - \tau) dt \cdot e^{-h\tau} \right] \\
&= \frac{\mu_i}{\mu_i + \delta} \Delta_i P \hat{\psi}(\mathbf{0}, \mu_i + \delta + h) \cdot \mathcal{L}^\tau(h), \quad h > 0.
\end{aligned} \tag{37}$$

Theorem 11. *The asymptotic result for the second moment jointly to the state of X_{N_t} in (22) can be precisely expressed as*

$$\begin{aligned}
M_{ii'}(t) &\longrightarrow \frac{1}{\mathbb{E}(\tau)} \left(\frac{\mu_i}{\mu_i + \delta} \right) \left(\frac{\mu_{i'}}{\mu_{i'} + \delta} \right) \left[\frac{1 - \mathcal{L}^\tau(\mu_i + \mu_{i'} + 2\delta)}{\mu_i + \mu_{i'} + 2\delta} \right] \\
&\quad \cdot \mathbf{1}\pi \left\{ \Delta_i \Delta_{i'} + \mathcal{L}^\tau(\mu_i + \delta) \Delta_i P (I - \mathcal{L}^\tau(\mu_i + \delta)P)^{-1} \Delta_{i'} \right. \\
&\quad \left. + \mathcal{L}^\tau(\mu_{i'} + \delta) \Delta_{i'} P (I - \mathcal{L}^\tau(\mu_{i'} + \delta)P)^{-1} \Delta_i \right\} P (I - \mathcal{L}^\tau(\mu_{i'} + \mu_i + 2\delta)P)^{-1}
\end{aligned} \tag{38}$$

as $t \rightarrow \infty$, when $i, i' = 1, \dots, k$, $i \neq i'$, and

$$\begin{aligned}
M_{ii}(t) &\longrightarrow \frac{1}{\mathbb{E}(\tau)} \left(\frac{\mu_i}{\mu_i + 2\delta} \right) \left[\frac{1 - \mathcal{L}^\tau(\mu_i + 2\delta)}{\mu_i + 2\delta} \right] \mathbf{1}\pi \Delta_i^2 P (I - \mathcal{L}^\tau(\mu_i + 2\delta)P)^{-1} \\
&\quad + \frac{2\mathcal{L}^\tau(\mu_i + \delta)}{\mathbb{E}(\tau)} \frac{1 - \mathcal{L}^\tau(2\mu_i + 2\delta)}{2\mu_i + 2\delta} \left(\frac{\mu_i}{\mu_i + \delta} \right)^2 \mathbf{1}\pi \Delta_i P (I - \mathcal{L}^\tau(\mu_i + \delta)P)^{-1} \\
&\quad \cdot \Delta_i P (I - \mathcal{L}^\tau(2\mu_i + 2\delta)P)^{-1}
\end{aligned} \tag{39}$$

as $t \rightarrow \infty$, when $i = 1, \dots, k$. We remark that (38) and (39) still hold when μ_i or $\mu_{i'}$ is infinite, i.e. when the corresponding delays L_i or $L_{i'}$ are 0.

Proof. One first computes that

$$\mathbb{E} \left(e^{-\delta(L_i - r)} \mathbb{1}_{[L_i > r]} e^{-\delta(L_{i'} - r)} \mathbb{1}_{[L_{i'} > r]} \right) = \begin{cases} \frac{\mu_i}{\mu_i + \delta} \frac{\mu_{i'}}{\mu_{i'} + \delta} e^{-(\mu_i + \delta)r} e^{-(\mu_{i'} + \delta)r} & \text{if } i \neq i', \\ \frac{\mu_i}{\mu_i + 2\delta} e^{-(\mu_i + 2\delta)r} & \text{if } i = i' \end{cases} \tag{40}$$

for $r \geq 0$. To evaluate the integral in (22) with (17), we thus need to compute the following integrals:

$$\int_0^\infty \int_0^t \partial_{s_{i'}} \partial_{s_i} \tilde{\pi}(s, t-y)|_{s=\mathbf{0}} P \tilde{\psi}(\mathbf{0}, t-y) dF(y), \quad (41)$$

$$\int_0^\infty \int_0^t \partial_{s_i} \tilde{\pi}(s, t-y)|_{s=\mathbf{0}} P M_{i'}(t-y) dF(y), \quad (42)$$

for $i, i' = 1, \dots, k$. When $i \neq i'$, using (14) with (40) followed by applying (32), (41) may be expressed as

$$\begin{aligned} & \left(\frac{\mu_i}{\mu_i + \delta} \right) \left(\frac{\mu_{i'}}{\mu_{i'} + \delta} \right) \Delta_i \Delta_{i'} P \int_0^\infty \mathbb{E} \left[e^{-(\mu_i + \mu_{i'} + 2\delta)(t-\tau)} \tilde{\psi}(\mathbf{0}, t-\tau) \mathbb{1}_{[t \geq \tau]} \right] dt \\ &= \left(\frac{\mu_i}{\mu_i + \delta} \right) \left(\frac{\mu_{i'}}{\mu_{i'} + \delta} \right) \Delta_i \Delta_{i'} P \hat{\psi}(\mathbf{0}, \mu_i + \mu_{i'} + 2\delta) \\ &= \left(\frac{\mu_i}{\mu_i + \delta} \right) \left(\frac{\mu_{i'}}{\mu_{i'} + \delta} \right) \left[\frac{1 - \mathcal{L}^\tau(\mu_i + \mu_{i'} + 2\delta)}{\mu_i + \mu_{i'} + 2\delta} \right] \Delta_i \Delta_{i'} P (I - \mathcal{L}^\tau(\mu_i + \mu_{i'} + 2\delta)P)^{-1}. \end{aligned} \quad (43)$$

When $i = i'$, similar computation yields that (41) is expressed as

$$\left(\frac{\mu_i}{\mu_i + 2\delta} \right) \left[\frac{1 - \mathcal{L}^\tau(\mu_i + 2\delta)}{\mu_i + 2\delta} \right] \Delta_i^2 P (I - \mathcal{L}^\tau(\mu_i + 2\delta)P)^{-1}, \quad (44)$$

where $\Delta_i^2 = \text{diag} [x_i^2, x = (x_1, \dots, x_k) \in \mathcal{S}]$ for $i = 1, \dots, k$. Turning to (42), replacing (13) with (35) followed by using (33) and (37) with (32) yields

$$\begin{aligned} & \frac{\mu_i}{\mu_i + \delta} \Delta_i P \int_0^\infty \mathbb{E} \left[e^{-(\mu_i + \delta)(t-\tau)} M_{i'}(t-\tau) \mathbb{1}_{[t \geq \tau]} \right] dt = \frac{\mu_i}{\mu_i + \delta} \Delta_i P \hat{M}_{i'}(\mu_i + \delta) \\ &= \frac{\mu_i}{\mu_i + \delta} \Delta_i P (I - \mathcal{L}^\tau(\mu_i + \delta)P)^{-1} \hat{b}_{i'}(\mu_i + \delta) \\ &= \mathcal{L}^\tau(\mu_i + \delta) \frac{\mu_i}{\mu_i + \delta} \frac{\mu_{i'}}{\mu_{i'} + \delta} \Delta_i P (I - \mathcal{L}^\tau(\mu_i + \delta)P)^{-1} \Delta_{i'} P \hat{\psi}(\mathbf{0}, \mu_{i'} + \mu_i + 2\delta) \\ &= \mathcal{L}^\tau(\mu_i + \delta) \frac{1 - \mathcal{L}^\tau(\mu_{i'} + \mu_i + 2\delta)}{\mu_{i'} + \mu_i + 2\delta} \frac{\mu_i}{\mu_i + \delta} \frac{\mu_{i'}}{\mu_{i'} + \delta} \Delta_i P \\ & \quad \cdot (I - \mathcal{L}^\tau(\mu_i + \delta)P)^{-1} \Delta_{i'} P (I - \mathcal{L}^\tau(\mu_{i'} + \mu_i + 2\delta)P)^{-1}. \end{aligned} \quad (45)$$

Then, gathering expressions (43) and (45) for (41) and (42) yields (38). Also, (39) is obtained with the help of (44) and (45). \square

Theorem 12. *The asymptotic result for the expectation of the workload jointly to the state of X_{N_t} in (29) can be precisely expressed as*

$$W_i(t) \longrightarrow \frac{1}{\mu_i^2} \left[\frac{1 - \mathcal{L}^\tau(\mu_i)}{\mathbb{E}(\tau)} \right] \mathbf{1} \pi \Delta_i P (I - \mathcal{L}^\tau(\mu_i)P)^{-1}, \quad t \rightarrow \infty. \quad (46)$$

Proof. When $L_i \sim \mathcal{E}(\mu_i)$, one straightforward verifies that $\mathbb{E}((L_i - r) \mathbb{1}_{[L_i > r]}) = e^{-\mu_i r} / \mu_i$. Hence One has from (26) and (27) that

$$\ell_i(t) = \frac{1}{\mu_i} \Delta_i P \mathbb{E} \left(e^{-\mu_i(t-\tau)} \tilde{\psi}(\mathbf{0}, t-\tau) \mathbb{1}_{[t \geq \tau]} \right),$$

from which the computation of $\frac{1}{\mathbb{E}(\tau)} \mathbf{1}\pi \int_0^\infty \ell_i(t)dt$ in (29) is led similarly to that of $\frac{1}{\mathbb{E}(\tau)} \mathbf{1}\pi \int_0^\infty b_i(t)dt$ in Theorem 10. Hence, the result (46) follows by using (32). \square

Example 13. This example illustrates numerically convergences of (34) and (39) for the first and second joint moments. This was done by simulating $(Z(t), X_{N_t})$ in (1) for large $t = 100$ and estimating the left-hand side of (34) and (39) thanks to the Law of Large Numbers (Monte Carlo) through $n = 500$ iterations. The right-hand side of (34) and (39) were computed explicitly by considering for the interarrival τ a Gamma distribution with shape a and rate b with the LT $\mathcal{L}^\tau(u) = \frac{1}{(1+u/b)^a}$ and $\mathbb{E}(\tau) = \frac{a}{b}$. We consider two cases of (a, b) choosing $(1, 10)$ (i.e. $\tau \sim \mathcal{E}(10)$), so that $\mathbb{E}(\tau) = 0.1$, and $(0.75, 15)$ with $\mathbb{E}(\tau) = 0.05$. Suppose that $k = 1$, i.e. a one dimensional process $\{Z(t), t \geq 0\}$ and a Markov chain $(X_i)_{i \in \mathbb{N}}$ with state space $\{0, 1\}$ (i.e. $K = 1$). Assume that the transition matrix given by $P = \begin{pmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{pmatrix}$ and the stationary distribution $\pi = (0.4, 0.6)$. All simulations and computations were done with Scilab. We finally suppose that all delays have same distribution $L \sim \mathcal{E}(1)$.

	$(a, b) = (1, 10)$	$(a, b) = (0.75, 15)$
Exact	$\begin{pmatrix} 2.44 & 3.56 \\ 2.44 & 3.56 \end{pmatrix}$	$\begin{pmatrix} 4.85 & 7.15 \\ 4.85 & 7.15 \end{pmatrix}$
Monte Carlo	$\begin{pmatrix} 2.50 & 3.53 \\ 2.12 & 3.85 \end{pmatrix}$	$\begin{pmatrix} 4.94 & 7.17 \\ 4.24 & 7.68 \end{pmatrix}$

TABLE 1
First order joint moments $M_1(t)$

	$(a, b) = (1, 10)$	$(a, b) = (0.75, 15)$
Exact	$\begin{pmatrix} 17.16 & 24.40 \\ 17.16 & 24.40 \end{pmatrix}$	$\begin{pmatrix} 63.60 & 92.65 \\ 63.60 & 92.65 \end{pmatrix}$
Monte Carlo	$\begin{pmatrix} 18.09 & 24.65 \\ 16.36 & 25.69 \end{pmatrix}$	$\begin{pmatrix} 60.22 & 97.50 \\ 60.30 & 90.90 \end{pmatrix}$

TABLE 2
Second order joint moments $M_{11}(t)$

Both tables above illustrate quite nicely how (34) and (39) are calculated. Also note that this simple example gives us some idea of potential applications of the model. For example, it would be interesting to utilize the model for an insurance company facing a situation where claims occurring at time T_i are either immediately reported when $X_i = 0$ or reported with delay L_i when $X_i = 1$. From P , some interesting feature appears as if a claim is not reported (resp. reported) at time T_i then the next one is reported (resp. not reported) at time T_{i+1} with probability 0.5 (resp. with probability 0.75). This model could reflect the policyholder's certain type of behaviour, e.g. after immediately reporting a claim at time T_i , the policyholder prefers to delay reporting of the next claim at time $L_{i+1} + T_{i+1}$ with probability 0.75 to avoid the increase of premium when the policyholder renews the insurance. On the other hand, it can also explain the opposite situation with a different transition matrix P . For example, a worker understands that if the reporting delay of workplace injury is longer, then it is harder to prove the injury is work related under the workers compensation claim. In this case, the transition probability from $X_i = 0$ to $X_{i+1} = 1$ is much lower, that is, the policyholder prefers to report the claim immediately. Furthermore, from a queueing point of view, the numerical model described here explains some infinite server queue system where, if an arriving customer

is not admitted in the queue at time T_i (i.e. $X_i = 0$) then the next one arriving at time T_{i+1} is accepted with high probability 0.75; this is especially interesting in a congestion regulation context, where one may choose to accept incoming customers more easily when the previous ones were rejected with high probability.

4.2. Exponentially distributed interarrival times

We now suppose in this subsection that $\tau \sim \mathcal{E}(\lambda)$, i.e. that arrivals occur according to a Poisson process with intensity $\lambda > 0$. From Proposition 4, it has been shown that the transient mgf $\tilde{\psi}(s, t)$ is the unique solution to an ordinary differential equation. Under this Poisson arrival setting, we shall derive closed-form expressions for the the transient behavior of the first, second orders moments as well as the expectation of the workload. To begin, a closed form expression for $b_i(t)$ in (16) is obtained. In this case, one finds that (15) becomes

$$\tilde{\psi}(\mathbf{0}, r) = \mathbb{E}(P^{Nr}) = e^{\lambda r(P-I)}, \quad r > 0, \quad (47)$$

whence $b_i(t)$ in (16) with (13) may be expressed as

$$\begin{aligned} b_i(t) &= \Delta_i \int_0^t \mathbb{E} \left(e^{-\delta(L_i - (t-y))} \mathbb{1}_{[L_i > t-y]} \right) P e^{\lambda(t-y)(P-I)} \lambda e^{-\lambda y} dy \\ &= \Delta_i e^{-\lambda t} \int_0^t \mathbb{E} \left(e^{-\delta(L_i - y)} \mathbb{1}_{[L_i > y]} \right) P e^{\lambda y P} dy. \end{aligned}$$

Furthermore, one checks easily that for all $t \geq 0$

$$b_i'(t) + \lambda b_i(t) = \lambda \mathbb{E} \left(e^{-\delta(L_i - t)} \mathbb{1}_{[L_i > t]} \right) \Delta_i P e^{\lambda t(P-I)}. \quad (48)$$

Theorem 14. *One has the exact expression for the first joint moment given by*

$$M_i(t) = \lambda e^{\lambda t(P-I)} \int_0^t \mathbb{E} \left(e^{-\delta(L_i - v)} \mathbb{1}_{[L_i > v]} \right) e^{-\lambda v(P-I)} \Delta_i P e^{\lambda v(P-I)} dv, \quad t \geq 0. \quad (49)$$

Proof. We aim at obtaining a differential equation satisfied by $M_i(t)$. Remember from (18) that it satisfies the renewal matrix equation, with $dF(y) = \lambda e^{-\lambda y} dy$ as well as a change of variable $t - y := y$,

$$M_i(t) = b_i(t) + P \int_0^t M_i(t-y) \lambda e^{-\lambda y} dy = b_i(t) + e^{-\lambda t} P \int_0^t M_i(y) \lambda e^{\lambda y} dy,$$

which, upon differentiation, and thanks to (48), leads to the matrix first order differential equation

$$\begin{aligned} M_i'(t) &= b_i'(t) + \lambda b_i(t) - \lambda M_i(t) + \lambda P M_i(t) \\ &= \lambda \mathbb{E} \left(e^{-\delta(L_i - t)} \mathbb{1}_{[L_i > t]} \right) \Delta_i P e^{\lambda t(P-I)} + \lambda(P-I) M_i(t) \end{aligned} \quad (50)$$

with initial condition $M_i(0) = 0$. The solution to (50) is given by (49). \square

Next, using (47), (17) with (13) and (14) in this case is given by

$$\begin{aligned} b_{ii'}(t) &= \Delta_i \Delta_{i'} \int_0^t \mathbb{E} \left(e^{-\delta(L_i - (t-y))} \mathbb{1}_{[L_i > t-y]} e^{-\delta(L_{i'} - (t-y))} \mathbb{1}_{[L_{i'} > t-y]} \right) P e^{\lambda(t-y)(P-I)} \lambda e^{-\lambda y} dy \\ &\quad + \Delta_i \int_0^t \mathbb{E} \left(e^{-\delta(L_i - (t-y))} \mathbb{1}_{[L_i > t-y]} \right) P M_{i'}(t-y) \lambda e^{-\lambda y} dy \\ &\quad + \Delta_{i'} \int_0^t \mathbb{E} \left(e^{-\delta(L_{i'} - (t-y))} \mathbb{1}_{[L_{i'} > t-y]} \right) P M_i(t-y) \lambda e^{-\lambda y} dy, \end{aligned}$$

and thus one finds the following relation

$$\begin{aligned} b'_{ii'}(t) + \lambda b_{ii'}(t) &= \lambda \mathbb{E} \left(e^{-\delta(L_i - t)} \mathbb{1}_{[L_i > t]} e^{-\delta(L_{i'} - t)} \mathbb{1}_{[L_{i'} > t]} \right) \Delta_i \Delta_{i'} P e^{\lambda t(P-I)} \\ &\quad + \lambda \mathbb{E} \left(e^{-\delta(L_i - t)} \mathbb{1}_{[L_i > t]} \right) \Delta_i P M_{i'}(t) + \lambda \mathbb{E} \left(e^{-\delta(L_{i'} - t)} \mathbb{1}_{[L_{i'} > t]} \right) \Delta_{i'} P M_i(t), \quad (51) \end{aligned}$$

where $M_i(t)$ is given in (49).

Theorem 15. *One has the exact expression for the second moment given by*

$$\begin{aligned} M_{ii'}(t) &= \lambda e^{\lambda t(P-I)} \int_0^t e^{-\lambda v(P-I)} \left\{ \mathbb{E} \left(e^{-\delta(L_i - v)} \mathbb{1}_{[L_i > v]} e^{-\delta(L_{i'} - v)} \mathbb{1}_{[L_{i'} > v]} \right) \Delta_i \Delta_{i'} P e^{\lambda v(P-I)} \right. \\ &\quad \left. + \mathbb{E} \left(e^{-\delta(L_i - v)} \mathbb{1}_{[L_i > v]} \right) \Delta_i P M_{i'}(v) + \mathbb{E} \left(e^{-\delta(L_{i'} - v)} \mathbb{1}_{[L_{i'} > v]} \right) \Delta_{i'} P M_i(v) \right\} dv, \quad t \geq 0, \quad (52) \end{aligned}$$

for $i, i' = 1, \dots, k$, where $M_i(v)$ and $M_{i'}(v)$ are given by (49) in Theorem 14.

Let us note that the structure of the expression of $M_{ii'}(t)$ is different according to whether $i = i'$ or $i \neq i'$, as $\mathbb{E} \left(e^{-\delta(L_i - v)} \mathbb{1}_{[L_i > v]} e^{-\delta(L_{i'} - v)} \mathbb{1}_{[L_{i'} > v]} \right)$ is equal to $\mathbb{E} \left(e^{-2\delta(L_i - v)} \mathbb{1}_{[L_i > v]} \right)$ if $i = i'$, or $\mathbb{E} \left(e^{-\delta(L_i - v)} \mathbb{1}_{[L_i > v]} \right) \mathbb{E} \left(e^{-\delta(L_{i'} - v)} \mathbb{1}_{[L_{i'} > v]} \right)$ when $i \neq i'$, by independence.

Proof. Similar to the proof of Theorem 14, we write the renewal equation (19) satisfied by $M_{ii'}(t)$ as $M_{ii'}(t) = b_{ii'}(t) + P \int_0^t M_{ii'}(t-y) \lambda e^{-\lambda y} dy$, $t \geq 0$. The same first order differential equation analysis with the expression of $b'_{ii'}(t) + \lambda b_{ii'}(t)$ given in (51) yields thus the explicit expression (52). \square

Finally, the transient workload has the same structure as the first moment, and the following result is given without proof:

Theorem 16. *One has the exact expression for the expectation of the workload given by*

$$W_i(t) = \lambda e^{\lambda t(P-I)} \int_0^t \mathbb{E} \left((L_i - v) \mathbb{1}_{[L_i > v]} \right) e^{-\lambda v(P-I)} \Delta_i P e^{\lambda v(P-I)} dv, \quad t \geq 0, \quad i = 1, \dots, k.$$

5. Moment generating function for deterministic interarrival times

So far, it has been shown that the (transient or limiting) distribution of process $\tilde{Z}(t)$ is hard to study explicitly in general, except for the Poisson arrivals. Hence we shall consider a specific distribution for the interarrival times being deterministic, and equal to 1 without loss of generality to obtain some results on the mgf $\tilde{\psi}(s, t)$.

Theorem 17. *Suppose that $\tau = 1$ a.s., then $\tilde{Z}(t) = \tilde{Z}(t; \delta)$ has a closed form expressions for the mgf given by*

$$\tilde{\psi}(s, t) = \tilde{\psi}(s, t; \delta) = \left[\prod_{m=0}^{t-1} \tilde{Q}(s, m) \right]' = \prod_{m=1}^t \tilde{Q}(s, t-m)', \quad t \in \mathbb{N}, \quad (53)$$

where $\tilde{\pi}(s, t)$ and $\tilde{Q}(s, t)$ are given in (2) and (3) respectively. Besides, when $\mathbb{E}(L_j)$ is finite for all $j = 1, \dots, k$ then $\lim_{t \rightarrow \infty} \prod_{m=0}^t \tilde{Q}(s, m) = \prod_{m=0}^{\infty} \tilde{Q}(s, m)$ exists, and $(\tilde{Z}(t), X_{N_t})$ converges in distribution as $t \rightarrow \infty$ given $X_0 = x$ towards $(Z_\infty, X_\infty) \in \mathbb{R}^k \times \mathcal{S}$ with joint mgf given by

$$\tilde{\psi}_\infty(s) = \tilde{\psi}_\infty(s; \delta) = [\mathbb{E}(e^{\langle s, Z_\infty \rangle} \mathbb{1}_{[X_\infty^x = y]})]_{(x, y) \in \mathcal{S}^2} = \left[\prod_{m=0}^{\infty} \tilde{Q}(s, m) \right]', \quad s \in \mathbb{R}^k. \quad (54)$$

Proof. Since $T_m = m \in \mathbb{N}$ and $N_t = t \in \mathbb{N}$, (53) is a straightforward application of (8) in Proposition 3.

Remember that we mentioned shortly after Definition (5) that $s = (s_1, \dots, s_k)$ may belong to the set $\mathbb{S} := \{(s_1, \dots, s_k) \in \mathbb{C}^k \mid s_j \in i\mathbb{R}, j = 1, \dots, k\}$, in which case $\tilde{\psi}(s, t)$ is the characteristic function of $\tilde{Z}(t)$ jointly to X_{N_t} . Then, in order to prove the convergence in distribution of $(\tilde{Z}(t), X_{N_t})$ given $X_0 = x$, it suffices by Lévy's convergence theorem to prove that $\tilde{\psi}(s, t)$ in (53) converges towards $\tilde{\psi}_\infty(s)$ given in (54) for all $s \in \mathbb{S}$ and that $\tilde{\psi}_\infty(s)$ is continuous at $s = \mathbf{0}$. This part constitutes the main bulk of the proof of the theorem. Let $\|\cdot\|$ be a submultiplicative norm on $\mathcal{S} \times \mathcal{S}$ matrices, i.e. such that $\|MN\| \leq \|M\| \|N\|$ for all matrices M and N . We write from (3) that

$$\tilde{Q}(s, m) = P' + (\tilde{\pi}(s, m) - I)P'.$$

To apply the result given in [1], we first introduce the two following norms defined respectively on complex valued and matrices valued sequences (defined similarly as in [1])

$$\|(u_m)_{m \in \mathbb{N}}\|_E := \sum_{m=0}^{\infty} |u_m|, \quad \|(M_m)_{m \in \mathbb{N}}\|_E = \sum_{m=0}^{\infty} \|M_m\|,$$

where $u_m \in \mathbb{R}$ and M_m is an $\mathcal{S} \times \mathcal{S}$ matrix for all $m \in \mathbb{N}$. Let us set $A_m = A_m(s) := (\tilde{\pi}(s, m) - I)P'$ (so as to comply with the notation of the latter paper). Since $(P')^m$ converges towards $(\mathbf{1}\pi)'$ as $m \rightarrow \infty$, from [1, Theorem 2.1] it is sufficient to prove that $\|(A_m)_{m \in \mathbb{N}}\|_E = \sum_{m=0}^{\infty} \|A_m\| < +\infty$ for the existence of $\prod_{m=0}^{\infty} \tilde{Q}(s, m)$. One has

$$\|A_m\| \leq \|\tilde{\pi}(s, m) - I\| \|P'\| \quad (55)$$

with $\tilde{\pi}(s, m) - I$ a diagonal matrix of which the (x, x) th component, $x \in \mathcal{S}$, is given from (2) by $\mathbb{E}\left(\exp\left(\sum_{j=1}^k s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]}\right)\right) - 1$. Using the inequality $|e^u - 1| \leq e^{|u|} - 1$ for all $u \in \mathbb{C}$, and remembering that $x_j \in \{0, \dots, K\}$ is non negative for all $j = 1, \dots, k$, one finds

for all $m \in \mathbb{N}$ that

$$\begin{aligned}
\left| \mathbb{E} \left(\exp \left(\sum_{j=1}^k s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \right) \right) - 1 \right| &\leq \mathbb{E} \left(\left| \exp \left(\sum_{j=1}^k s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \right) - 1 \right| \right) \\
&\leq \mathbb{E} \left(\exp \left(\sum_{j=1}^k |s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \right) - 1 \right) \\
&\leq \mathbb{E} \left(\exp \left(\sum_{j=1}^k |s_j| x_j \mathbb{1}_{[L_j > m]} \right) \right) - 1 \\
&= \prod_{j=1}^k \left[1 + (e^{|s_j| x_j} - 1) \mathbb{P}(L_j > m) \right] - 1 \\
&= \sum_{I \subset \{1, \dots, k\}} \prod_{\ell \in I} \left[(e^{|s_\ell| x_\ell} - 1) \mathbb{P}(L_\ell > m) \right], \quad (56)
\end{aligned}$$

where the independence of L_1, \dots, L_k was used. Note now that for all $I \subset \{1, \dots, k\}$,

$$\begin{aligned}
\sum_{m=1}^{\infty} \prod_{\ell \in I} \left[(e^{|s_\ell| x_\ell} - 1) \mathbb{P}(L_\ell > m) \right] &\leq \left(e^{\max(|s_1|, \dots, |s_k|) \cdot K} - 1 \right)^k \sum_{m=1}^{\infty} \prod_{\ell \in I} \mathbb{P}(L_\ell > m) \\
&= \left(e^{\max(|s_1|, \dots, |s_k|) \cdot K} - 1 \right)^k \mathbb{E} \left(\max_{\ell \in I} L_\ell \right),
\end{aligned}$$

which is finite thanks to the assumption that $\mathbb{E}(L_j) < +\infty$ for all $j = 1, \dots, k$. We thus deduce from (56) that $\sum_{m=0}^{\infty} \|\tilde{\pi}(s, m) - I\| < +\infty$ and in turn, from (55) $\|(A_m)_{m \in \mathbb{N}}\|_E = \sum_{m=0}^{\infty} \|A_m\| < +\infty$.

Now it remains to prove that $\psi_\infty(s)$ in (54) is continuous at $s = \mathbf{0}$. Let us first recall the inequality $|e^u - 1| \leq e|u|$ for all $u \in \mathbb{C}$ such that $|u| \leq 1$. If $x = (x_1, \dots, x_k)$ is in \mathcal{S} , this entails that, for all $j = 1, \dots, k$ and $m \in \mathbb{N}$,

$$\left| \exp \left(s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \right) - 1 \right| \leq e |s_j| x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \quad (57)$$

for all s_j such that $|s_j| x_j \leq 1$, which is satisfied if $s = (s_1, \dots, s_k) \in [-i/K, i/K]^k \subset \mathbb{S}$. Letting $w_m^j(s_j) := \mathbb{E} \left(\exp \left(s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \right) \right)$, we deduce from (57) that

$$\begin{aligned}
|w_m^j(s_j) - 1| &\leq \mathbb{E}(e |s_j| x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]}) \leq e |s_j| x_j \mathbb{P}(L_j > m), \\
|w_m^j(s_j)| &\leq e |s_j| x_j \mathbb{P}(L_j > m) + 1 \leq e + 1 := M,
\end{aligned} \quad (58)$$

for all $m \in \mathbb{N}$, $s_j \in [-i/K, i/K]$ and $j = 1, \dots, k$. Then it follows from (58) that for all $m \in \mathbb{N}$

and $s = (s_1, \dots, s_k) \in [-i/K, i/K]^k$, again by independence of L_1, \dots, L_k ,

$$\begin{aligned} \left| \mathbb{E} \left(\exp \left(\sum_{j=1}^k s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \right) \right) - 1 \right| &= \left| \prod_{j=1}^k u_m^j(s_j) - 1 \right| \\ &= \left| \sum_{r=1}^k \left[\prod_{j=1}^{r-1} u_m^j(s_j) \right] [u_m^r(s_r) - 1] \right| \leq \sum_{r=1}^k \prod_{j=1}^{r-1} |u_m^j(s_j)| |u_m^r(s_r) - 1| \\ &\leq \sum_{r=1}^k M^{r-1} e^{|s_r| x_r} \mathbb{P}(L_r > m), \end{aligned}$$

which, summing from $m = 0$ to $+\infty$, yields the following bound for the $|\cdot|_E$ norm for all $x = (x_1, \dots, x_k) \in \mathcal{S}$

$$\left| \left(\mathbb{E} \left(\exp \left(\sum_{j=1}^k s_j x_j e^{-\delta(L_j - m)} \mathbb{1}_{[L_j > m]} \right) \right) - 1 \right)_{m \in \mathbb{N}} \right|_E \leq \sum_{r=1}^k M^{r-1} e^{|s_r| x_r} (\mathbb{E}(L_r) + 1)$$

and $s = (s_1, \dots, s_k) \in [-i/K, i/K]^k$. The right-hand side of the above inequality tends to 0 as $s \rightarrow \mathbf{0}$, $s \in [-i/K, i/K]^k \subset \mathbb{S}$, and is valid for all $x = (x_1, \dots, x_k) \in \mathcal{S}$. By the definition of matrices $\tilde{\pi}(s, m)$, $m \in \mathbb{N}$, this immediately implies that

$$\|(\tilde{\pi}(s, m) - I)_{m \in \mathbb{N}}\|_E \rightarrow 0, \quad s \rightarrow \mathbf{0}, \quad s \in \mathbb{S}.$$

One then deduces from [1, (2.20) in Corollary 2.1] that the infinite product $s \mapsto \tilde{\psi}_\infty(s) = \left[\prod_{m=0}^\infty \tilde{Q}(s, m) \right]' = \left[\prod_{m=0}^\infty (P' + A_m(s)) \right]'$ is continuous at $s = \mathbf{0}$, $s \in \mathbb{S}$. This completes the proof. \square

One interesting consequence of Theorem 17 is that the limiting mgf is expressed conveniently when the L_j 's are bounded by some constant M . In that case, one has from (2) and (3) that $\tilde{\pi}(s, r) = I$ and $\tilde{Q}(s, r) = P'$ when $r \geq M$, and we thus obtain the following result for this particular case:

Corollary 18. *Suppose that $\tau = 1$ a.s. and rvs L_j , $j = 1, \dots, k$, are all upper bounded such that $L_j \leq M$ a.s. for some deterministic $M \in \mathbb{N}^*$. Then the transient mgf in (53) simplifies as*

$$\tilde{\psi}(s, t) = \left[\prod_{m=0}^{M-1} \tilde{Q}(s, m) (P')^{t+1-M} \right]', \quad t \geq M - 1,$$

and the limiting mgf is given by

$$\tilde{\psi}_\infty(s) = \mathbf{1}\pi \left[\prod_{m=0}^{M-1} \tilde{Q}(s, m) \right]'$$

6. Application: Queue/IBNR process modulated by an external Semi-Markovian process

The model described in Section 2 is flexible enough to study the following process in queueing theory and actuarial science. We consider here a semi-Markov process $\{Y(t), t \geq 0\}$ with finite

state space $\{1, \dots, \kappa\}$, jump times $(T_i)_{i \in \mathbb{N}}$ such that $(T_i - T_{i-1})_{i \in \mathbb{N}^*}$ is iid distributed as τ with cdf F , and the embedded Markov chain $\{Y(T_n), n \in \mathbb{N}\}$ having transition matrix and stationary distribution denoted by $P_Y = (p_Y(\ell, m))_{\ell, m=1, \dots, \kappa}$ and $\pi_Y = (\pi_Y(\ell))_{\ell=1, \dots, \kappa}$ respectively. Let us suppose that $\{Y(t), t \geq 0\}$ models the arrival of customers or claims, such that the n th arriving customer has service time/delay $\mathcal{L}_{n, (Y(T_{n-1}), Y(T_n))}$, where $((\mathcal{L}_{n, (\ell, m)})_{\ell, m=1, \dots, \kappa})_{n \in \mathbb{N}}$ is an iid sequence, with independent $\mathcal{L}_{n, (\ell, m)}$'s, $n \in \mathbb{N}, \ell, m = 1, \dots, \kappa$. In other words, if N_t denotes the number of clients arrived by time t , the N_t th customer has service time which depends on both states of the semi Markov process at the switching time T_{N_t} and the state prior to this switching time (i.e. depending on both state $Y(T_{N_t})$ and $Y(T_{N_t}-)$), as illustrated in Figure 1. This model has potential applications in queueing theory where an incoming customer may inspect the state of the environment $Y(T_{N_t}-)$ before deciding to join the queue; in an actuarial setting, there are different reasons for the reporting delay when the IBNR claims are considered. In particular, in this case, this model allows random fluctuations in the underlying delay distribution influenced by external process. For example, policyholder may decide to delay the submission of claims under special circumstances such as the external environment process is in a particular state. We also let $(\mathcal{L}_{(\ell, m)})_{\ell, m=1, \dots, \kappa}$ a generic corresponding rv. Also note that some flexibility for this rv is available, for example one may have $\mathbb{P}(\mathcal{L}_{(\ell, m)} = 0) > 0$, implying that a customer finding the environment in state ℓ before it switches to state m decides not to join the queue with positive probability. Also, in the case of IBNR claims, some claims do not have any delays to be reported, that is, once they occur they are immediately reported to the insurers, see the discussion concerning the numerical application at the end of Section 4.1.

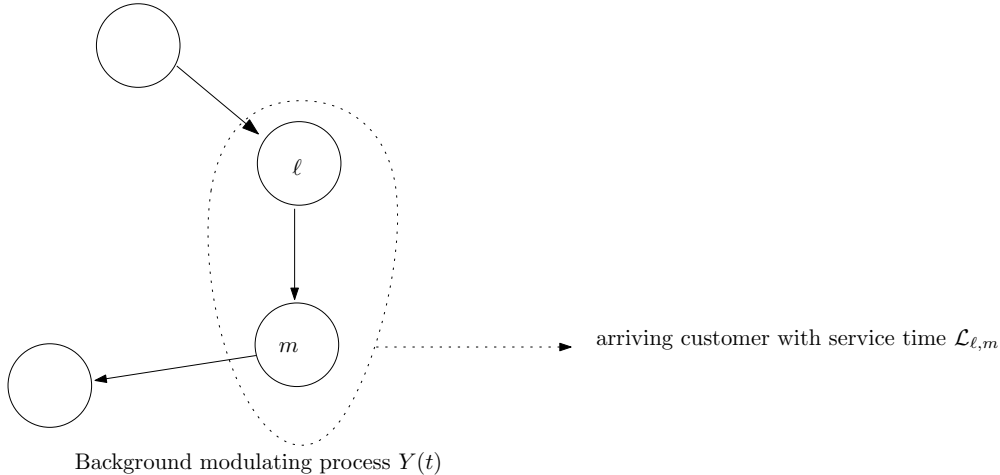


FIG 1. *Modulating semi-Markov process and service time.*

We then define the corresponding IBNR discounted processes $\{\mathcal{Z}(t), t \geq 0\}$ and $\{\tilde{\mathcal{Z}}(t), t \geq 0\}$ by

$$\mathcal{Z}(t) = \sum_{i=1}^{\infty} e^{-\delta(T_i + \mathcal{L}_{i, (Y(T_{i-1}), Y(T_i))})} \mathbb{1}_{[T_i \leq t < T_i + \mathcal{L}_{i, (Y(T_{i-1}), Y(T_i))}]}, \quad \tilde{\mathcal{Z}}(t) = e^{\delta t} \mathcal{Z}(t). \quad (59)$$

As such, the process defined in (59) is different from the one introduced in (1) because the arrival times and the service times are now modulated by some semi-Markov process. However in

the following, we shall show that (59) is actually embedded into (1), and this embedding procedure is essentially the central point of this section. Note in particular that this includes the particular case where $\{Y(t), t \geq 0\}$ is a continuous time Markov chain by considering $\tau \sim \mathcal{E}(\lambda)$ for some $\lambda > 0$, of which infinitesimal generating matrix is given by $Q_Y = (q_Y(\ell, m))_{\ell, m=1, \dots, \kappa}$ with $q_Y(\ell, m) = \lambda p_Y(\ell, m)$ if $\ell \neq m$ and $q_Y(\ell, \ell) = -\lambda \sum_{m \neq \ell} p_Y(\ell, m)$. In that case, inter-arrivals may not be identically distributed by considering the generic rv $(\mathcal{L}_{(\ell, m)})_{\ell, m=1, \dots, \kappa}$ to satisfy $\mathcal{L}_{(\ell, \ell)} = 0$ for all $\ell = 1, \dots, \kappa$, so that a new customer actually arrives exactly at each time when $Y(t)$ switches to a new state, with service time distributed as $\mathcal{L}_{(\ell, m)}$ when switching from state ℓ to state m .

So, we need to define the corresponding Markov chain $(X_i)_{i \in \mathbb{N}}$. Let us define for all ℓ and m in $\{1, \dots, \kappa\}$ the $\kappa \times \kappa$ matrix $e(\ell, m)$ of which the (j, j') th entry is $\delta_{(j, j'), (\ell, m)}$. We then define the state space of $(X_i)_{i \in \mathbb{N}}$ as

$$\mathcal{S} = \{e(\ell, m), (\ell, m) \in \{1, \dots, \kappa\}^2\} \subset \{0, 1\}^{\kappa \times \kappa}$$

so that one sets $k = \kappa^2$ and $K = 1$, sticking with the notation in Section 2. Then for all $i \in \mathbb{N}$, $X_i = (X_{i, (j, j')})_{(j, j') \in \{1, \dots, \kappa\}^2}$ is of the form $X_i = e(\ell, m)$ for some ℓ and m in $\{1, \dots, \kappa\}$, in which case one has

$$X_{i, (j, j')} = \delta_{(j, j'), (\ell, m)}, \quad \forall (j, j') \in \{1, \dots, \kappa\}^2.$$

The only difference here is in \mathcal{S} which is a strict subset of $\{0, 1\}^{\kappa \times \kappa}$, however this will not raise any additional technical difficulty in the following analysis. We then define the corresponding transition matrix as $P = (p(x, x'))_{(x, x') \in \mathcal{S}^2}$ with

$$p(e(\ell, m), e(\ell', m')) = \begin{cases} p_Y(\ell', m') & \text{if } m = \ell', \\ 0 & \text{otherwise,} \end{cases} \quad \ell, m, \ell', m' = 1, \dots, \kappa, \quad (60)$$

so that a transition from state $e(\ell, m)$ to state $e(\ell', m')$ of the Markov chain $(X_i)_{i \in \mathbb{N}}$ is only possible if $m = \ell' \iff e(\ell, m)e(\ell', m') = e(\ell, m')$. One checks that (60) really defines a transition matrix, i.e. the sum over each row is equal to 1, and $(X_i)_{i \in \mathbb{N}}$ is stationary and ergodic iff $\{Y(T_n), n \in \mathbb{N}\}$ is, with corresponding stationary distribution $(\pi(x))_{x \in \mathcal{S}}$ given by

$$\pi(x) = \pi(e(\ell, m)) = p_Y(\ell, m)\pi_Y(\ell), \quad \forall x = e(\ell, m) \in \mathcal{S}, \quad \ell, m = 1, \dots, \kappa. \quad (61)$$

Finally, we let $(L_{i, (j, j')})_{i \in \mathbb{N}, (j, j') \in \{1, \dots, \kappa\}^2}$ a sequence of independent rvs with corresponding distribution given by

$$L_{i, (j, j')} \sim \mathcal{L}_{i, (j, j')}, \quad \forall (j, j') \in \{1, \dots, \kappa\}^2.$$

We now arrive at the embedding result. We let $\{Z(t) = Z(t; \delta) = (Z_{(j, j')}(t))_{(j, j') \in \{1, \dots, \kappa\}^2} \in \mathbb{R}^{\{1, \dots, \kappa\}^2}, t \geq 0\}$ and $\tilde{Z}(t) = e^{\delta t} Z(t)$ defined by (1), i.e.

$$Z_{(j, j')}(t) = \sum_{i=1}^{\infty} X_{i, (j, j')} e^{-\delta(L_{i, (j, j')} + T_i)} \mathbb{1}_{[T_i \leq t < T_i + L_{i, (j, j')}]}, \quad (j, j') \in \{1, \dots, \kappa\}^2.$$

Then, one checks immediately that the following relation between $\tilde{Z}(t)$ and $\tilde{\tilde{Z}}(t)$ defined in (59) holds

$$\left\{ \tilde{\tilde{Z}}_{(j, j')}(t), t \geq 0 \right\} \stackrel{\mathcal{D}}{=} \left\{ \tilde{Z}(t) \mathbb{1}_{[Y(T_{N_t^-})=j, Y(T_{N_t})=j']}, t \geq 0 \right\}, \quad \forall (j, j') \in \{1, \dots, \kappa\}^2. \quad (62)$$

We remark that the above relation is interesting as it enables us to study $(\tilde{Z}(t), Y(T_{N_t-}), Y(T_{N_t}))$ through $\tilde{Z}(t)$ i.e. through the analysis developed in Section 3. More precisely, one checks that

$$\begin{aligned}
& \left(\mathcal{M}_{(j_0, j_1), (j_2, j_3)}^{(1)}(t) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} \\
& := \left(\mathbb{E} \left(\tilde{Z}(t) \mathbb{1}_{[Y(T_{N_t-})=j_2, Y(T_{N_t})=j_3]} \middle| Y(T_{-1}) = j_0, Y(T_0) = j_1 \right) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} \\
& = \left(\mathbb{E} \left(\tilde{Z}_{(j_2, j_3)}(t) \middle| X_0 = e(j_0, j_1) \right) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} \\
& = \left(M_{(j_2, j_3)}(t) \mathbf{1} \right)_{(j_2, j_3) \in \{1, \dots, \kappa\}^2} \tag{63}
\end{aligned}$$

where $M_i(t)$ is defined in (6). Similarly, one can consider for all j_0, j_1, j_2 and j_3 the second moment $\mathcal{M}_{(j_0, j_1), (j_2, j_3)}^{(2)}(t) := \mathbb{E} \left(\tilde{Z}(t)^2 \mathbb{1}_{[Y(T_{N_t-})=j_2, Y(T_{N_t})=j_3]} \middle| Y(T_{-1}) = j_0, Y(T_0) = j_1 \right)$, which verifies

$$\left(\mathcal{M}_{(j_0, j_1), (j_2, j_3)}^{(2)}(t) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} = \left(M_{(j_2, j_3), (j_2, j_3)}(t) \mathbf{1} \right)_{(j_2, j_3) \in \{1, \dots, \kappa\}^2}, \tag{64}$$

where $M_{ii'}(t)$ is also defined in (6), as well as the expectation of the workload $\mathcal{D}(t)$ given by

$$\begin{aligned}
& \left(\mathcal{W}_{(j_0, j_1), (j_2, j_3)}(t) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} \\
& := \left(\mathbb{E} \left(\mathcal{D}(t) \mathbb{1}_{[Y(T_{N_t-})=j_2, Y(T_{N_t})=j_3]} \middle| Y(T_{-1}) = j_0, Y(T_0) = j_1 \right) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} \\
& = \left(W_{(j_2, j_3)}(t) \mathbf{1} \right)_{(j_2, j_3) \in \{1, \dots, \kappa\}^2},
\end{aligned}$$

where $W_i(t)$ is defined in (25).

Furthermore, one could find the following relation between the joint mgf of $\tilde{Z}(t)$ and distribution of $(Y(T_{N_t-}), Y(T_{N_t}))$ defined for all $z \in \mathbb{R}$ and $t \geq 0$ by

$$\left[\tilde{\Psi}(z, t) \right]_{(j_0, j_1), (j_2, j_3)} = \mathbb{E} \left(e^{z\tilde{Z}(t)} \mathbb{1}_{[Y(T_{N_t-})=j_2, Y(T_{N_t})=j_3]} \middle| Y(T_{-1}) = j_0, Y(T_0) = j_1 \right),$$

where j_0, j_1, j_2 and j_3 are in $\{1, \dots, \kappa\}$. Then one notices that the above mgf is linked to the joint mgf of $\tilde{\psi}(s, t)$ of $\left\{ (\tilde{Z}_{(j, j')}(t))_{j, j'=1, \dots, \kappa}, t \geq 0 \right\}$ thanks to (62) by the relation

$$\left[\tilde{\Psi}(z, t) \right]_{(j_0, j_1), (j_2, j_3)} = \left[\tilde{\psi}(z.e(j_2, j_3), t) \right]_{(j_0, j_1), (j_2, j_3)}, \tag{65}$$

where we recall that $e(j_2, j_3)$ is the $\{1, \dots, \kappa\} \times \{1, \dots, \kappa\}$ matrix of which the (j, j') th entry is $\delta_{(j, j'), (j_2, j_3)}$.

The embedding relations (63), (64) and (65) thus enable us to derive results on the present model with semi-Markovian input, in particular thanks to Sections 3.2 and 4. Let us present some of them in the following. For example, (63) together with Proposition 7 yields the asymptotic first moment given by

$$\begin{aligned}
& \left(\mathcal{M}_{(j_0, j_1), (j_2, j_3)}^{(1)}(t) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} \\
& \longrightarrow \left(\frac{1}{\mathbb{E}(\tau)} \frac{1 - \mathcal{L}^{\mathcal{L}_{(j_2, j_3)}}(\delta)}{\delta} \mathbb{E}(X_{(j_2, j_3)}) \mathbf{1} \right)_{(j_2, j_3) \in \{1, \dots, \kappa\}^2},
\end{aligned}$$

as $t \rightarrow \infty$, where $\mathbb{E}(X_{(j_2, j_3)}) = \pi(e(j_2, j_3)) = p_Y(j_2, j_3)\pi_Y(j_2)$ by (61). When service times are exponentially distributed with $\mathcal{L}_{(j_2, j_3)} \sim \mathcal{E}(\mu_{(j_2, j_3)})$, then (64) and Theorem 11 results in

$$\begin{aligned} & \left(\mathcal{M}_{(j_0, j_1), (j_2, j_3)}^{(2)}(t) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2, (j_2, j_3) \in \{1, \dots, \kappa\}^2} \longrightarrow \\ & \frac{1}{\mathbb{E}(\tau)} \frac{\mu_{(j_2, j_3)}}{\mu_{(j_2, j_3)} + 2\delta} \frac{1 - \mathcal{L}^\tau(\mu_{(j_2, j_3)} + 2\delta)}{\mu_{(j_2, j_3)} + 2\delta} \mathbf{1} \pi \Delta_{(j_2, j_3)}^2 P (I - \mathcal{L}^\tau(\mu_{(j_2, j_3)} + 2\delta)P)^{-1} \\ & + \frac{2}{\mathbb{E}(\tau)} \mathcal{L}^\tau(\mu_{(j_2, j_3)} + \delta) \frac{1 - \mathcal{L}^\tau(2\mu_{(j_2, j_3)} + 2\delta)}{2\mu_{(j_2, j_3)} + 2\delta} \left(\frac{\mu_{(j_2, j_3)}}{\mu_{(j_2, j_3)} + \delta} \right)^2 \mathbf{1} \pi \Delta_{(j_2, j_3)} P (I - \mathcal{L}^\tau(\mu_{(j_2, j_3)} + \delta)P)^{-1} \\ & \quad \cdot \Delta_{(j_2, j_3)} P (I - \mathcal{L}^\tau(2\mu_{(j_2, j_3)} + 2\delta)P)^{-1} \end{aligned}$$

as $t \rightarrow \infty$. When $\tau \sim \mathcal{E}(\lambda)$, i.e. when arrivals occur according to a Poisson process and the model is Markov modulated, the transient moment is explicit thanks to (49) in Theorem 14 and one computes easily for all j_2 and j_3 , using $PP'\Delta_\pi \mathbf{1} = \mathbf{1}$, $e^{\lambda v} \mathbf{1} = e^{\lambda v} \mathbf{1}$, that

$$\begin{aligned} & \left(\mathcal{M}_{(j_0, j_1), (j_2, j_3)}^{(1)}(t) \right)_{(j_0, j_1) \in \{1, \dots, \kappa\}^2} = M_{(j_2, j_3)}(t) \mathbf{1} \\ & = \lambda e^{\lambda t(P-I)} \int_0^t \mathbb{E} \left(e^{-\delta(L_i - v)} \mathbb{1}_{[L_i > v]} \right) e^{-\lambda v(P-I)} \Delta_i P e^{\lambda v(P-I)} dv \mathbf{1}, \quad t \geq 0. \end{aligned}$$

Further, when interarrival are deterministic equal to 1, then (65) and Theorem 17 entail that the limiting joint mgf is given for all $(j_0, j_1), (j_2, j_3)$ in $\{1, \dots, \kappa\}^2$, by

$$\lim_{t \rightarrow \infty} \left[\tilde{\Psi}(z, t) \right]_{(j_0, j_1), (j_2, j_3)} = \left[\prod_{i=0}^{\infty} \tilde{Q}(z.e(j_2, j_3), i) \right]_{(j_0, j_1), (j_2, j_3)}'.$$

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