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► **To cite this version:**

Cécile Ouvrier-Bufferet. Exploring Mathematical Definition Construction Processes. *Educational Studies in Mathematics*, 2006, 63 (3), pp.259 - 282. 10.1007/s10649-005-9011-3 . hal-01893657

HAL Id: hal-01893657

<https://hal.science/hal-01893657>

Submitted on 22 Aug 2024

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EXPLORING MATHEMATICAL DEFINITION CONSTRUCTION PROCESSES

CECILE OUVRIER-BUFFET

Reference : OUVRIER-BUFFET, C. (2006). Exploring Mathematical Definition Construction Processes. *Educational Studies in Mathematics*, 63(3), 259-282.

ABSTRACT. The definition of 'definition' cannot be taken for granted. The problem has been treated from various angles in different journals. Among other questions raised on the subject we find: the notions of *concept definition* and *concept image*, conceptions of mathematical definitions, redefinitions, and from a more axiomatic point of view, how to construct definitions. This paper will deal with 'definition construction processes' and aims more specifically at proposing a new approach to the study of the formation of mathematical concepts. I shall demonstrate that the study of the defining and concept formation processes demands the setting up of a general theoretical framework. I shall propose such a tool characterizing classical points of view of mathematical definitions as well as analyzing the dialectic involving definition construction and concept formation. In that perspective, a didactical exemplification will also be presented.

KEY WORDS. Concept formation, definition, discrete straight line, processes of definition construction.

1. INTRODUCTION

There is a far cry between the situation of the student striving to grasp a new mathematical concept and the neat definition produced by a mature professional mathematician. To understand how concept formation works implies exploring the wide field of mathematical definitions considered as concept holders. How can we reconcile rigour and clarity with the universe of trial and error, misdirected moves jostling with sudden insights in which the teacher and students labour? It is precisely in those arduous moments though, that our students need our help most.

So as to map the terrain provisionally with definitions serving as temporary markers for concept formation, we have, therefore to work out a theoretical framework through empirical research. Although definition construction has admittedly a place in mathematical research, precious little has been written on the subject in specialised journals. However, several features of definition are commonly accepted as crucial (Zaslavsky-Shir, 2005)¹. A frequently used approach to definitions leads us to consider that a *clear* definition is a part of a theory (Mariotti-Fischbein, 1997).

Bearing in mind the aforesaid situation, I note: Lakatos (1961) worked simultaneously on concept formation and definition construction: “A definitional procedure is a procedure of concept formation” (p. 54). Lakatos contributed to the debate on formulating a model of mathematical discovery while integrating both the social and the conceptual aspects.

Freudenthal (1973), Mariotti & Fischbein (1997) and Borasi (1992) have pointed to some didactical situations involving defining processes in geometry. The theoretical tools mobilized by them did not focus on the definitional procedure itself. Moreover, classification and redefining tasks are actually only the tip of the iceberg consisting of *Situation(s) of Definition Construction* (called from now on SDC(s)).

The proposed theoretical framework will extract tools from existing didactical and cognitive theories (Vergnaud and the theory of *conceptual fields*, 1991; Vinner and the *concept image* for instance, 1991). This will lead me to a characterisation of Schoenfeld’s “problem-solving heuristics” (1987, p. 18). I shall adopt a pragmatic position about “problem” and “problem-solving”, and focus on the characterisation of situations so as to diagnose students’ heuristics. This approach will be boosted by an analysis of historical and present uses of mathematics (e.g. D’Ambrosio, 1993; Lave, 1988; Nuñez et al., 1983; Sierpiska 1989; Thurston, 1994).

Tall (2004, p. 287) gains “an overview of the full range of mathematical cognitive development” by scanning a whole range of theories. A global vision of mathematical growth then emerges, making room for three worlds of thinking: the “embodied world”, the “proceptual world” and the “formal world”. In this way, a more coherent view of cognitive development may be obtained. Endorsing this point of view, I will question the place of definitions in such a theory. “Formal definitions” admittedly belong to Tall’s “formal world”. What happened before the “smooth” definitions were arrived at? What were the heuristic processes involved? Although the apprehension of new mathematical concepts began in the “embodied world” through perception, I still assume that the “proceptual world” is not always adequate to characterize a concept which is being constructed. So how are we going to grasp the dialectic between concept

formation and definition construction within this theoretical range? I think we can safely assume that there is another world, different from the “embodied”, “proceptual” and “formal” worlds, which is both transversal and complementary, fostering the characterisation of mathematical growth through definition construction processes in particular.

2 – ELEMENTS OF A THEORETICAL FRAMEWORK FOR THE STUDY OF DEFINITION

2.1. *Concept and definition are didactically far apart*

A definition is commonly identified with its attendant concept. Now Vergnaud (1991, p. 135) reminds us that a definition on its own will not enable a learner to apprehend and comprehend a concept: “it is through situations and problems to be solved that a concept acquires meaning”. I shall extend a little Vergnaud’s meaning of the word *concept* considering that “*apprehending* a concept” implies taking simultaneously all the following elements into consideration: examples and non-examples of the concept allowing a first apprehension of the concept, the definition(s) of this concept and the proof of their equivalence, several representations of this concept, and above all, the situations which allow the emergence of the concept and preserve its meaning. Of course, a situation can never be analysed through one concept. A well-analysed *conceptual field* (in Vergnaud’s sense) offers a mine of situations to be proposed to students. Let me take an example in geometry. Consider the circle, a figure which conveys an impression of perfect transparency and the knowledge of which is often associated with its perceptive apprehension, so much so that teachers often confuse the knowledge of the geometrical object and the perceptual recognition of that object, a confusion reinforced by the existence of a direct construction tool, the pair of compasses.

Let me now take up the idea of adopting different points of view on that object and of studying the situations in which the concept of ‘circle’ will emerge. Traditionally, the mathematical definition of a circle is that of a set of points at fixed distance (radius) from a fixed point (centre). Artigue and Robinet (1982) have shown that as far as children in the 8 to 9 age

range are concerned the prevailing conception is that of a regular curvature associated with the term 'round' in the ordinary meaning of the word. In that case it is obvious that tactile associations prevail. From a mathematical point of view, however, defining a curvature and, a fortiori, a regular curvature cannot be taken for granted. So as to bypass the notion of curvature we could develop an axiom set in which the curvature constitutes a primitive term. That would induce consistency in mathematical culture and practise.

Let me then ask the question differently. Would it be at all possible to define the concept of the circle without resorting to the concept of distance? The circle can be conceived as the locus of all possible positions of the vertex joining the two other sides of the right triangle with a fixed hypotenuse: the concept of distance does not intervene but that of orthogonality becomes necessary. From a different perspective, the concept of circle can be grasped through a problem of optimization: that has to do with the ancient problem of isoperimetry (Zenodorus 2nd century BC; Jacques Bernouilli in the 17th century). The circle is the only curve with a given length delimiting a maximal area.

The example of the circle shows us how important the ability to recognize a concept in different situations can be for the learner.

I have thereby shown that the construction of a definition can be at work in various fields, such as optimization, a geometrical characterization in which the perceptive approach plays an important part. An axiomatic shift may even turn out to be profitable (going out of the Euclidian transformations group to build up a new axiomatic system in which curvature is a primitive term). Each of those points of view throws a specific light on the concept of circle. Thus different definition construction processes are emerging from this example, the elements of which even when still not characterized clearly belong to different orders ranging from perception to axiomatic elements. The ambit of my study of concept formation and definition building dialectics rests therefore on the designing of situations implying a definition

construction process. So I have to set up a theoretical framework enabling me effectively to characterise those processes as well as the situations in which they can develop.

2.2. *A dynamic framework for the study of definition construction*

When confronted with a problem, a learner brings into play operational invariants, known as *schemes*. A scheme may or may not have been brought into play previously by the learner. What really matters in the determination of those schemes are the operational *invariants* which support the action of students. They can consist in a rule for action, a *theorem-in-action* (Vergnaud, 1991), etc. In any case there is a first emergence of the target concept envisaged exclusively as a tool and not as an object. Vergnaud, in a psychological and didactical perspective, characterizes the concept through a set of situations which serves as a referent for its different properties; the set of the schemes (i.e. ‘known’ or ‘unknown’ strategies) to be implemented by subjects engaged in such situations; and the linguistic and non-linguistic forms allowing the symbolic representation of the concept, its properties and the procedures of the treatment of it. Vergnaud fails to take on board the control processes brought into play by students but Balacheff tried to fill that gap (Balacheff & Gaudin, 2002). The aim of the epistemological study of the concept of definition I am presenting is twofold. It enables me to apprehend a defining process and therefore gives me a chance to capture the finer points and the different takes we can have on a concept; and to learn how to behave towards a student who has to define. I therefore propose to study definition construction processes through the following theoretical methods:

- 1) Finding a set of situations or, failing that a ‘fundamental situation’ (in Brousseau’s 1997 sense) suitable for the concept, that is to say yielding meaning for the concept. The work thus assigned to the students implies the construction of the concept and its definitions. I could also speak of “problems”, adopting a *pragmatic* position (as said above). I concentrate on the notion of *fundamental situation* marking out the specificity of my research.

2) To characterize mathematically (through an epistemological study of the concept) the nature of the definition construction processes in question, highlighting the successive defining utterances which emerge as the situation evolves. This of course implies that the definition should be considered as provisional. I have to characterize the *operators* at work along with the *control* processes. These terms will be taken in the natural sense, however I could also define them as follows: an *operator* is a tool for action, it can allow the transformation of the problem. This fact is attested by the observation of students' productions and behaviour. It can be for instance a syntactic rule applied to transform an algebraic expression, or a specific treatment of a graphical representation. That type of *operator* obviously depends upon the *representation system*.

In the wake of Pólya and a long tradition of research on metacognition, Schoenfeld (1985, p. 97-143) has shown the crucial role of **control** in problem solving. The *control structure* allows the observer to describe how the subject judges the adequacy and validity of an action, as well as the criteria of the *milieu* (in Brousseau's 1997 sense) for selecting a feedback. Thus the *control structure* is constituted by all the means needed in order to make choices, to make decisions, as well as to express judgment. From the theoretical point of view, the *control structure* ensures the non-contradictory nature of the *conception* (in Balacheff's sense): among the control instruments we must find **decision tools** legitimizing the **use** of an *operator* or the **state** of the ongoing problem (the problem should be declared solved or not). But *control structures* are often implicit. The distinction between *operators* and *control structures* is not absolute.

3) The *systems of representation* will be mentioned at the relevant moments during the explanation of the *operators* and *controls*. I will not be focussing on *representation systems* however. I shall complement the aforesaid elements with Vinner's notions of *concept image* and *concept definition* (Vinner, 1991): all the more so since the object I will consider next (discrete

straight line) by its very nature calls for the concept of real line, a concept which will elicit a particular *concept image* and *concept definition* from the students.

2.3. *Token definition construction processes serving as an epistemological reference*

To achieve this, we must have a first rough idea of what such definition construction situation and such defining processes could be. That is precisely the aim of my epistemological study of the concept of definition. The thought processes of a mathematician creating new concepts can never be fully accounted for; as Burton (1999) has shown many features remain implicit. Burton's study exposes a model for 'knowing mathematics' but more interestingly in my opinion, underlines three main points: 'you know when...', 'you know if it works' and 'knowing is helped by making connections'. We cannot overstress the importance of peer exchanges, and active connections have to be underlined in defining processes.

I consider Lakatos's works relevant because they seem to bring out the heuristic processes in the scientific method. In point of fact Lakatos (1961) proposes different types of definitions which could serve to characterize markers in the concept formation progression. Those proposals obviously call for special attention and demand an effort at formalisation. Lakatos himself did not distinguish those different types of definition. He was content to exhibit them in two historic examples. I must begin where Lakatos left off. I shall try and bring out the specific insights Lakatos brought to the question of *definitions*. We know that he was inspired by Popper, as well by the essentialists and the nominalists. Let me therefore disentangle his own contribution to the *definitions* from that of his aforesaid predecessors.

2.3.1. *Popper in Lakatos: conjectures and refutations*

Davis and Hersch pointed out that:

Lakatos argues that dogmatic philosophies of mathematics (logician or formalist) are unacceptable, and he shows that a Popperian philosophy of mathematics is possible. However, he does not actually carry out the program of reconstructing the philosophy of mathematics with a fallibilist epistemology (Davis & Hersch, 1981, p. 348).

What is actually the Popperian contribution to Lakatos's epistemology? Popper (1945 & 1974) characterises scientific progress through the analysis of competing theories, dismissing the Aristotelian notions of essentialism. Definitions thus appear as mere labels introduced for brevity's sake. Popper insists that a theory's responsiveness to tests is a direct measure of its interest. In other words, the main Popperian *operator* is the **generation of refutations through counter-examples**. As far as meta-scientific knowledge is concerned, resistance to refutations proves to be the main *control structure*. A professional mathematician surely evinces such a knowledge which is definitely akin to Vinner's *mathematical maturity* and Lakatos's *common scientific wisdom* (Feyerabend, 1975). Lakatos reconsiders the Popperian refutational stance while retaining the importance of definitions in the research process, filling the gap left by the nominalist Popperian view. I will then turn back to the Lakatosian point of view later.

2.3.2. Essentialist and nominalist inspirations in Lakatos's works

Lakatos (1961) speaks of 'problem-situations' and actually refers explicitly to classification activities. He follows the Aristotelian pattern (Aristotle, 1994 & 1997), keeping the method for defining by *genus and differentia*². From a mathematical point of view, it consists in searching the invariants starting from a given representation of an object. In my view of Aristotelian thought, the defined objects pre-exist their definition.

What are the *operators* and *controls* in a defining Aristotelian process? Linguistic and logical *operators* appear in order to elaborate a definitional statement: homonyms, redundancies and metaphors are banned as detrimental to the understanding of the mathematical concept. 'Vicious circles', and then the infinite regression, have to be dodged as well; hence the importance of the prime elements in an axiomatic construction. In the end, the aforesaid linguistic and logical *operators* may be also mobilised as *controls*.

2.3.3. Lakatos's contribution: a heuristic approach to definitions

Lakatos's (1961) thought concentrated mainly on the polyhedra and that does not make the generalisation of his heuristic processes any easier. However, working on this single example I shall attempt to bring out the formalisation of a dialectical process involving concept formation and definition construction. I shall present the different Lakatosian kinds of definitions as well as the *operators* and *control structures* mobilised when we are engaged in working on such definitions. Lakatos is actually steeped in the Popperian vision of definitions, and his stress on the notion of 'problem' is akin to the notion of *set of situations* included in Vergnaud's (1991) idea of *concept*. That becomes conspicuous when Lakatos writes:

There are other ways of communicating meaning than definitions. I, for one, shall initiate my pupils into the problem-situation which I am dealing with not by definitions, but by showing them a cube, an octahedron and showing that for these $V-E+F=2$. Then I shall ask for the domain of validity of this formula (Lakatos, 1961, p. 69).

In his attempt to map out concept formation by definitions different in nature, Lakatos explains that we can do nothing with a naive concept, because we cannot extend it, unless we are able to fit it into a theory; but then the considered concept will not be naive any more! So, he focuses his attention on what he called *zero-definitions*. He is not aiming at developing a linear defining process through *n-definitions* starting from a *0-definition*. So, what I will now ask is: how are we to recognize *zero-definitions*? According to what pattern(s) does it develop? I am inclined to hypothesize that the naming process plays an important part in that respect (according to Pimm's 1995 remark ³). *Zero-definitions*, echoing Popper's opinion about sense of words and precision, can assume a **denominative function**, and are used for practical purposes at the beginning of the research process before the proper mathematical terms are defined. When a mathematical concept has an obvious heuristic potential, *zero-definitions* may come naturally, according to a guiding principle of "expanding zero-concepts if expansion will be followed by a deeper proof-idea" (Lakatos, 1961, rule 4.2.2). This raises the crucial question of the *expansion*. In this context, a *zero-definition* can either just come to naught or develop into a *proof-*

generated definition. The guiding Lakatosian principle is as follows: during a scientific research process, there is a progression from *zero-definitions* to *proof-generated definitions* (which originate in the search for a proof).

And what about *proof-generated definitions*? A *proof-generated definition* originates from a proof while stemming from the development of the potential of a *zero-definition*, and may appear as the end product of *zero-definition*. Now a forceful unexplicated dialectic involving the considered proof and the catalysis of *proof-generated definition* is at work here. The Lakatosian notion that a definitional process tends to delimit a concept while preserving a proof or extending it in a particularly efficient way is relevant here.

I shall illustrate the great difficulties I experience while trying to design mathematical situations with a *proof-generated definition* potential by the following quotation from Pimm:

(This notion) seems particularly problematic in terms of teaching mathematics, because of needing to perceive the definition as a tool custom-made to do a particular job that cannot be known by those trying to learn it, certainly not with an order of presentation that seems to require definitions to come first (Pimm, 1993, p. 272).

So, through what kinds of *operators* can we tease out a *zero-definition* and how can we *control* a *proof-generated definition*? The most important Lakatosian *operator* to be found during a research process is certainly the use and the generation of examples and counter-examples. What Lakatos really shows us is the part played by the peer debate in the generation of examples and counter-examples. He is keen to remind us that scientific research starts and ends with problems; the expansion of concepts has to stop when the fertilizer becomes a positive weedkiller. Seen in this light, the above quotation testifying to the implicit presence of *operators* and *controls* in a definition construction process actually points to the notions of scientific wisdom and mathematical maturity. Anyway, formulating *zero-definitions* and searching for counter-examples for refuting propositions included in potential definitions, leads to re-formulation, exclusion or inclusion of counter-examples (Lakatos talked of '*monster-barring definitions, capitulation and exception-barring definitions*', when he dealt with the problem of '*real*' counter-examples on the one hand and exceptions on the other). Moreover, the assigned

function of a definition generates *operators*. The proof and the *proof-generated definitions* work together, and the denomination process produces *zero-definitions*. Hence, the attendant *operators* have to do with the test of the validity of a proof. These *operators* act on *zero-definitions*, while the other *operators* consist in mobilizing linguistic and logical demands.

Determining the *control structures* proves to be more difficult. Two levels of determination appear. The first concerns the catalysis of the proof:

The discovery of global counterexamples catalyses proof-analysis (...) Proof-analysis may catalyse the discovery of global counterexamples (...) This second catalysis is very important indeed as it shows the role of proofs in formation of concepts and growth of theories (Lakatos, 1961, p. 44-45).

The second is more theoretical and concerns the axiomatic point of view: I shall not develop this aspect because it is specific to the above-characterised Popperian elements and because Lakatos explicitly and deliberately puts aside axiomatic considerations in order to focus on the heuristic process. The synthesised table below situates the different kinds of definition in the concept formation perspective.

Functions of a definition	Reference thought	Strictures on a definition (operators and/or controls)	Level
<i>To communicate, to denominate, to prove, to recognize</i>	Popperian view	- Non redundancy, minimality - Derivative - Non-contradictory - Non-paradoxical <i>(formal definitions)</i>	Meta-level (inscription of a definition within a theory, construction of a theory)
Transposition			
<i>To prove, to delimit</i>	Aristotelian tools and Lakatosian heuristics	- Logical exigencies (equivalence between definitions, existence, uniqueness of the defined object) Catalysis with the situation and proof (<i>proof-generated definition</i>) ----- Temporary denomination (<i>zero-definition</i>) ----- Reflective process	Mathematical defining process Definition by genus and differentia Heuristic level

Situations: Generalisation, modelling, relations, classification, problem-situation characterisation, construction etc.

Table 1 – Situation of the epistemological formalisations within a defining process

2.3.4. *Different types of Situations of Definition Construction (SDCs)*

The defining process is threefold: a dialectical process involving definitional procedure and concept formation (Lakatos & *Problem-situation*), a linguistic and logical dimension (Aristotle & taxonomy tasks), and finally an axiomatic dimension (Popper & construction of theories). This three dimensional approach has prompted me to present a typology of SDCs. The first kind of SDCs includes all the problems of ‘Classification’. The next paragraph presents an analysis and an implementation of such a SDC. Both Popper’s and Lakatos’s influences will be felt in the second and the third types. The set of situations which deal only with ‘theoretisation’ as proposed by Popper and his attendant notion of scientific progress cannot be used adequately to define a type of SDC: implementing Popperian inspired SDCs in a classroom is simply too ambitious in my perspective. I would characterise the set of the Lakatosian problems as intra-mathematical problems of research of the validity domain of a conjecture. Those problems start from a mathematical object accessible by its representation and questionings about a proof. It means in particular that the resolution of the Lakatosian problems implies the construction of a concept and of its definition. I will then consider from now on that such Lakatosian ‘Problem-Situations’ constitute a kind of SDC. I shall also introduce a third type of SDC, called ‘Mathematisation/Modelling’: the characterisation of this type of SDC flows naturally from the name it has been given. Let me take an example: “what is the mathematical structure common to the quasi-totality of vegetals?”. What is at stake here is to determine a unifying, simplifying structure relevant for all vegetals: thus and in so doing we perform a modelling task. The *tree* was the only mathematical structure which occurred to us in this perspective. It is also worth noting that the graph theory uses the vocabulary of nature (tree, rooted tree, forest, leaves and branches or edges and vertices, etc.). We can reach out to the tree through its representations and mentioning the vegetal tree enables us to import specific features while working on inductive definitions for instance: “when we cut down a leaf of a tree, what’s left is still a tree” (see Ouvrier-Bufferet, 2003 a & b).

3. A TYPICAL SDC: DISCRETE STRAIGHT LINES

We can safely hypothesize that some of the students' prior conceptions of mathematical definitions (what definitions consist of or should consist of, the aspect they should assume etc.) will play an unpredictable though crucial role during SDCs. According to the particular circumstances of a given case they may turn out to be an obstacle, or, on the contrary, a catalyst. Let me consider, for instance, a student who would labour under the illusion that there is only one *good* definition for a mathematical concept. The student in question would find it rather difficult to engage in the defining process. More importantly, a question should be asked about students' own attitudes towards their own efforts at definition. How aware and creative are they in those circumstances? The study reported here is part of a larger project aiming at modelling students' mathematical definition construction processes (freshman year, scientific section). The experimental SDCs were conducted according to the epistemological characterisation of the types of situations involving a defining process. The students' own defining processes are characterised through markers such as *operators* and *control structures*.

3.1. *Why discrete straight lines?*

The mathematical object concerned here is the *discrete straight line*. Bearing in mind the objectives I mentioned, I shall choose a mathematical object putting everybody (all the actors involved) on a level playing field. The *straight line* acts as a referent; *discrete straight line* is accessible through its representation, it is non-institutionalised. (An *institutionalised* concept is a "curriculum" concept i.e. a concept that has a place in the classic taught content.) Delimiting what a straight line can be in a discrete context proves to be quite a challenge. Professional researchers have several definitions of *discrete straight line* at their disposal, but the proof of the equivalence of these definitions remains worth considering. Research on axiomatic field is still in progress: that means that the question of a "good" definition of discrete straight line is actually an open and interesting problem. Even when a discrete object seems innocent, a great

complexity may lurk behind. From our point of view, the *discrete straight line* concept makes at least two types of SDC possible, considering how easily accessible it can be through its perception and the axiomatic prospects it opens up.

This is why I chose the following methodology: different groups of three or two students (freshman year scientific section) took part in the experiment, which lasted from 2 to 3 hours at least and was videotaped. An observer was present ready to recall the instructions if necessary. So as to describe how defining processes really work and explore in particular the influence of an explicit demand of a definition, two situations were conceived with two different starting points.

3.2. Presentation of the experiment

I focus my analysis on two groups of three and two students, called groupA and groupB respectively.

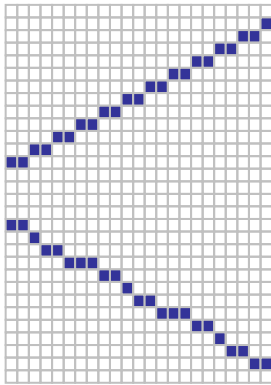
- For groupA, two SDCs were proposed, without explicit request of definition: a *Problem-situation* referring to an axiomatic *problématique*⁴ at first, and a *Classification* task second.

The *Problem-situation* was proposed to students as follows: “We have to colour squares on a regular grid map. Draw triangles (colouring squares). Explain your construction.”

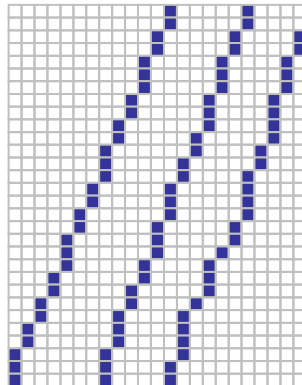
And the *Classification* situation starts from about ten examples and counter-examples of discrete straight lines non-identified as such (see Figure 1 below), and asks explicitly for delimitation between straight lines and non-straight lines.

- For the second group (groupB), I proposed only the *Classification* situation, with an explicit request of definition. The following oral instruction was given to students, with a set of representations of discrete objects: “Here are objects which are straight lines, and others that are not straight lines. The question is: how to define **straight line** starting with these elements?”.

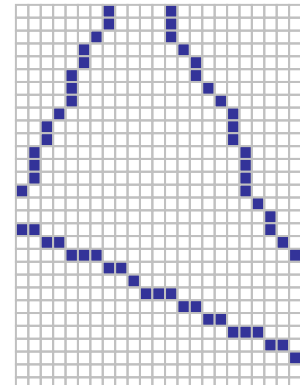
The word “discrete” is not introduced, and, in all these situations, students decide on their own accord when their research process of construction of definitions is finished.



Figures A



Figures B



Figures C

Figure 1 – SDC Classification, discrete straight lines

I shall analyze the *operators* and *controls* mobilised by students as they go about defining processes and specify the role and the place of the teacher who now acts as a *Manager-Observer* (coded MO). I shall also measure the impact of the explicit request of definition on the defining process.

3.3. Summary of mathematical problématiques about discrete straight lines

I shall present first the mathematical potentialities of the *discrete straight line* concept through the notion of *zero-definitions* and their evolution: *zero-definitions*, by their essence, underline the evolution potential of the defining process.

To consider a discrete straight line - on a regular grid map, while colouring pixels - can generate a reference to a real straight line and thus induce a use of properties of the latter in order to define the same object in a discrete context. It can be linked to the redefining situations proposed by Borasi (1992) for instance, but in my case, I assume that students are not in a position to work on definitions of *straight line*. What is arguable? From a didactical point of view, there is actually no available geometrical definition of *straight line*, only an analytical one in the present French curriculum, in which the perceptive aspect of geometry is predominant at the elementary level, and analytical geometry is taking pride of place at the secondary level.

A first approach to the *discrete straight line* concept consists in using the referent *real straight line* as a material support. If we draw a real straight line and choose some pixels crossed by it so as to form a discrete straight line, then some criteria of choice have to be elaborated (see table 2 below). Thus, different *zero-definitions* become conceivable: the words in bold type mark the orientation of the evolution potential of the *zero-definitions* (Zdef 1,2,3).

The second *problématique* called “regularity” consists in researching a regularity in the sequence of stages of pixels (see table 2 below: how can we modify a sequence in order to obtain a better regularity?). This *problématique* brings to us two potentially evolutionary *zero-definitions* coded Zdef 4,5.

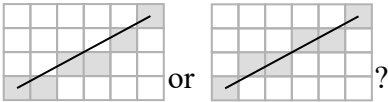
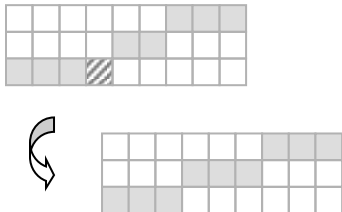
<i>Problématiques</i>	Figures	<i>Zero-definitions</i>
<p>“Real straight line” Function of def.: to build the object</p>		<p>Zdef1: set of the pixels crossed by a real line. Zdef2: set of the pixels “the nearest” to a real line. Zdef3: set of the pixels “inside” a band</p>
<p>“Regularity” Function of def.: to recognize, to build the object</p>		<p>Zdef4: sequence of stages of pixels with specific properties. Zdef5: sequence of pixels’ stages with a uniform repartition, non-improved from the regularity viewpoint.</p>

Table 2 - Two *problématiques* producing *zero-definitions* of *discrete straight line*

A third more complicated approach, axiomatic point of view, consists in questioning the mathematical object *discrete straight line*, bearing in mind our knowledge of Euclidian geometry. This approach could lead us to study the intersection of two discrete straight lines, the number of discrete straight lines by two given pixels etc. This axiomatic approach is at work in the SDC we proposed with *discrete triangles*. Questions of existence and uniqueness *problématiques* are implied here (see table 3 below).

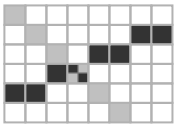
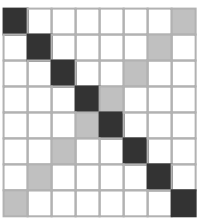
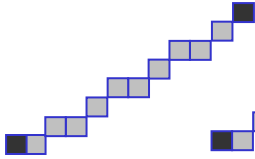
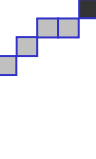
<i>Problématiques</i>	Figures	
Existence of the intersection		
Can we speak of the uniqueness of a discrete straight line?		

Table 3 - Questions of existence and uniqueness

Engaging in axiomatic *problématique* carries great difficulty. This approach deals with both the perceptive aspect of a straight line (real or discrete) and the axiomatic perspective: how can we build a discrete geometrical theory if we would remain close to the Euclidian geometry? We are being confronted with two markedly defining styles: a local one and a global-theoretical one, the latter mobilizing some implicit skills and knowledge (building a theory and choosing definitions among competing definitions for instance).

To identify the pixels “the nearest” to a real line, while searching for a property relating to the sequence of stages (called *chaincode string*), can lead us to a theorem. Through this theorem we can prove the equivalence between definitions derived from both Zdef2 and Zdef5. This approach to the discretization of a straight line by checking linearity conditions is directly related to number theoretical issues in the approximation of real numbers by rational numbers. These linearity conditions can be checked incrementally, leading to a decomposition of arbitrary strings into straight substrings (Wu, 1982).

3.4. *A priori overall didactical analysis of the defining situation*

From a mathematical point of view, in terms of definitional processes, the potentialities of these two SDCs are attested by the evolution potential of *zero-definitions*. The experiments I have

conducted will bring elements for discussing students' defining processes: what makes students able to take responsibilities and how do they go about it?

We are now venturing on an uncharted territory. I shall limit my study to the part played by the observer and his range of action and I will then present the results of the experiments stressing the contribution of a specific theoretical framework to the characterisation of defining processes. We are now watching a break from the traditional didactical contract: definitions are traditionally given by the teacher. As far as SDCs is concerned, a new didactical contract is emerging with the teacher assuming a new role. And we should not forget that working in a discrete context is special. In SDCs, and more generally in RSCs (Research-Situations for Classroom) (Godot & Grenier, 2004), the MO (Manager Observer) has a specific place. He does not necessarily know "the" solution (the definition(s) in our case). That puts the teacher on a level playing field with the student.

My epistemological study brings in elements for characterising the teacher's guiding style during SDCs. Of course I still bear in mind that a defining process is based on four poles: one of these poles concerns the **construction of a theory** (Popper), another deals with **heuristics** (Lakatos), and the two other poles concern the **logical** as well as the **linguistic** aspects (Aristotle). In this context, the MO may interfere with students' defining processes through logical requests, linguistic or axiomatic demands or supplied counter-examples. The MO may also ask students to construct and/or recognize a straight line: that request is obviously related to the **function** of definition. For instance, the two following questions will induce students to engage in a new reflection of an axiomatic kind, implying thus a new movement in the defining process: "draw a discrete straight line crossing these two given pixels" and "does the definition help to recognise or construct the discrete object?".

The results below are presented in the light of the different kinds of definitions, mentioned in the epistemological study. I shall underline both the *operators* and *controls* taking part in a defining process. I shall also add a few remarks concerning the guidance of SDCs.

3.5. Presentation of the results

Two main *problématiques* were tackled by students: they brought several *zero-definitions*.

Group	<i>Zero-definitions</i> produced	Operators, controls	Final Statement
Group A (Problem-Situation; no explicit request for definition)	- Zdef 1,2,3 (abandoned because problem of the criteria of choice) - Zdef 5 (search of regularity by modification of stages)	Perceptive controls	<i>Arithmetical rule involving slope</i>
Group B (Classification; explicit request for definition)	- Zdef 1 (abandoned because of the non appropriate use of a ruler for a mathematical definition) - Zdef 4	Counter-examples, linguistic and logical operators; Perceptive controls	<i>Repetition of a sequence, in the same direction. The difference between two stages does not exceed 1.</i>

Table 4 – A summary of students’ statements and their evolution

3.5.1. A theoretical framework applied to an experiment: an analysis of groupA (Problem-situation and Classification task, no explicit definition was requested)

I identified in the products of groupA *zero-definitions* involving a real approach, attested by the following excerpts:

(z-def 1) *Nicolas*: I thought that a triangle was three lines (...) there you are! I’m drawing the three straight lines and I’m colouring the parts where the straight line moves up. I’m colouring the squares which are touched by the straight line.

(z-def 2) *Rémi*: It’s necessary to keep a square crossed (by the line); a square didn’t cross, forget it. And then, some squares aren’t really crossed, we have to decide whether we keep them or not ... perhaps we shouldn’t.

(z-def 3) *Nicolas*: You draw two parallel straight lines ... if there is a pixel outside this lane, forget it!

From the point of view of students at work, the *zero-definitions* I identified from an epistemological point of view boil down to *definitions-in-action* (Vergnaud, 1991). That means that the statements I called *zero-definitions* are not considered as objects for study by students but only as tools: they consist in operational rules, used for the construction of discrete straight lines. The students set aside the real approach and the attendant *zero-definitions* (Zdef 1,2,3), and understandably so: the absence of an explicit demand of definition entails the absence of

linguistic and logical *operators*, and students are labouring under the extra difficulty of having to choose a couple of points for drawing a real straight line.

The students focus their research on the following problem: how to draw a discrete straight line, two pixels being given. This research leads them to another *zero-definition* dealing with the regularity of the stages of a discrete object. By then they produce a statement which the MO can accept as some sort of definition but which is not recognized as such by the learners (they talk about an “operational fast rule, without much calculation”). The statement in question involves the notion of slope, which considers the number of squares horizontally and vertically. I noted a mistake, a square was forgotten in the counting:

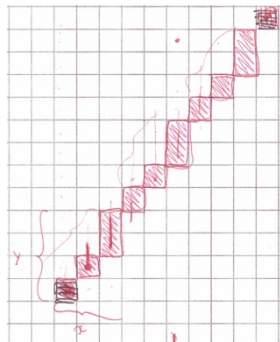


Figure 2 - The trace produced by groupA

(Rule) Rémi: 13 and 10. Here is our rule: I divide 13 by 10, I'm getting one. And 13 minus 10 times 1 is 3. So, I have to colour ten times one square and to distribute the three remaining squares in order to obtain something regular.

The MO proposed then to the students the Classification task, still without any explicit request for definition. This second situation followed up the work on discrete triangles: it gave students a chance to reapply the “rules” produced before, thereby ensuring a place for the validation. In this new situation, the MO could observe what kinds of *zero-definitions* were reinvented and their evolution: such an evolution is expected because the function of the definition is now to recognize a discrete straight line and not only to construct it.

In fact, two *zero-definitions* (still not considered as definition by students) were mobilized in the Classification task. The potentialities of their evolution were suggested by students but were not finalized. It concerned Zdef3 (lane) and Zdef5 (search of a regularity by modifications of stages). The operational rule used by students was as follows:

(z-def3) *Nicolas*: If a square is coloured and doesn't belong to the lane, then the set is not a straight line. And if a square is totally included in the lane and is not coloured, then the set is not a straight line.

As they did not ask themselves whether this lane is unique or not, they missed out the questions on the trace of this lane. The search for a regularity by modifications of stages of discrete object was included in the following example:

(z-def5) 3 – 1 – 3 – 1 ... it is not very good ... it is borderline ...but it could be a straight line. 3-2-1: this is not the best sequence ... 2-2-2 is the best!

We were left with an open question: how could we extend a discrete segment?

Rémi: Straight line ... we don't know if we have defined that. And if we add something to a "piece" of straight line, it can become something different from a straight line.

I call *zero-definition 5* a definition which is not verbalised as such by students: they use it as an action. The emergence of such questionings (modification of the stages and prolongation of discrete segment) does not confer on this *zero-definition 5* the status of operational rule.

I notice some *operators* like generation of examples in this group, but under no circumstances do the few examples and counter-examples built by students act as *controls* on potential *zero-definitions*. There is clearly a discrepancy between the epistemological point of view talking about *zero-definitions* and the students' real experience concerning *definitions-in-action*: I assume that the lack of an explicit request of definition is not neutral in this circumstance. Although the MO had obviously a part to play in students' research, students remain unaware of the potentialities of their *zero-definitions*. I note that all the *operators* in relation with linguistic and logical aspects were not mobilised and consequently the evolution of *zero-definitions* was blocked. In other words, students did not assume the responsibility of the construction of definitions and they did not mention it.

That leads me to present the main results of the second experiment in which an explicit request of definition of "straight line" was being asked.

3.5.2. *An analysis of group B (Classification task, explicit definition was requested)*

I began by introducing some key elements on the students' *concept image* about *straight line*. Some aspects of the *concept image* of straight line are as follows: perceptive regularity, slope and infinity of points. This *concept image* proves insufficient here in view of the difficulty surrounding the *discrete straight line* concept, but it is still integrated into the students' perceptive *controls*. The main apparent feature of the *concept image* of "straight line" has to do with a physical point of view, connected to the use of the ruler. I should stress here that the *concept definition* of straight line was not mobilised as such by the students, but that did not prevent them from recalling well-known situations involving real straight lines, such as statistical linear approximation. The role of perception was crucial in that perspective and led to a first classification of discrete given objects, with the use of the ruler for drawing a real straight line crossing given discrete objects. The ruler was also used for drawing a box enclosing discrete objects when possible.

The accurate study of the passage between the discrete and the continuous points of view, and vice versa, led students to give up the use of the ruler: "A definition is something precise, we can't refer to the use of the ruler, we can't use the trace of a real straight line of a lane in particular" (quoting students). Their research was then focused on the characterisation of properties of the stages constituting a discrete straight line. The search for regularity by modifications of stages of a discrete straight line (corresponding to *zero-definition 5*) was still present. But students' judgement was only a perceptive one and students were well aware of this fact. They changed their point of view of the mathematical object and thus abandoned the external referent "real straight line". The *zero-definition* finally formulated happens to be very close to a present-day mathematical definition. I recognize *zero-definition 4* in students' statements, in accordance with our previous mathematical study: "A straight line is the repetition of a pattern, and the difference between the lengths of two consecutive stages is less than or equal to one".

I noted that the main *operators* also act as *controls*. They consist in generating examples and counter-examples, and this excerpt points to an entirely conscious process:

Clara: Now, we didn't manage to counter-prove (...) I need other counter-examples; (to the MO:) could you please give me some more counter-examples? We have to test our definition.

The explicit request for a definition induced students to undertake a formulation process: it generated a form of reflexivity in their efforts at "definition" which triggered off an evolution of their definitions. Students are particularly careful about the choice of words, and even suggest establishing a glossary so as to define precisely commonly used words such as "stage" and "pattern". Their "classical" conception of mathematical definitions now clearly emerges: definitions are to be precise and minimal. The role of the explicit request for definition assumes crucial dimension here. Unlike what happened in the previous group, in this case linguistic and logical *operators* lead to conscious work on *zero-definitions*, while leaving the *definitions-in-action* stage. When *zero-definitions* were effectively formulated, students could work on them, and the MO could also interact more with students' processes. It is noteworthy that when *operators*, *controls* and *zero-definitions* are known in advance, acting on the defining process becomes possible because we can identify the relevant elements. This is best shown by an example. Let me take groupB: the research began with the use of the ruler, the trace of a box enclosing discrete objects. Those techniques were abandoned by students because they were felt to be inadequate for a mathematical definition: the students found the ruler to be too imprecise. Then they specified some characteristic features of a discrete straight line. It should have a "pattern", it should go through a "cycle" (the students actually express the same idea in different ways almost as if they had to do with two distinct ideas). And there should be no isolated square: students insisted on there being no stage with a length equal to one.

The MO asked then to trace the straight line going through two given pixels. The students resorted to their own perception of the straightness of a line for drawing:

MO: You can try to choose two squares. How will it work in this case? How will you join these two squares? From what you said previously, can you find tools for drawing the straight line which crosses these squares?

Clara: If I draw a box ... it is difficult to see ... I can draw a large straight line, and perhaps, after I shall make it thinner? We will have to go back on this idea later ... Here I've just drawn something which allows me to reach this square. But it answers none of the criteria we worked out a moment ago: it has no pattern, no cycle, the length of some of the stages is one. Both of us reach the conclusion that is a straight line!

MO: And if you have to prolong this segment, how would you do it?

The use of perception is potentially promising but obviously students cannot remain at that stage. GroupB became aware of how difficult it was to use the box (uniqueness of the trace of the box). Starting from here, students will work at the characterisation of the regularity of stages. The importance of the role of the MO becomes apparent: his or her experience of epistemological *operators* and *controls* equips him with operational levers, which will act effectively on students' defining processes. By asking for the prolongation of a segment, the MO introduces the problem of uniqueness of a discrete straight line going through two pixels: a means of introducing a viewpoint of the validation of the *zero-definition* has been found.

4. CONCLUSION - DEFINING PROCESSES AND CONCEPTS FORMATION

A theoretical framework, based upon an epistemological and philosophical ground, has been proposed in this paper. This formalisation is a multi-purpose one. Firstly, it provides for dynamic characterization of recurrent ways of thinking of definitions in mathematics, steering clear of a linear presentation of concept formation from the operational level to abstract mathematical structures. Secondly, the theoretical framework in question brings an overall picture of the mathematical processes of defining, incorporating four main items: formulating, logic, heuristics, theorising.

Designing SDCs in the classroom requires a teacher able to elicit feedbacks from students if necessary (particularly if a difficult point has to be tackled). I indicate how the present theoretical approach can be used in order to observe and moderate the dialectic between concept formation and definition construction.

So, the main positive points of SDCs will be the following: SDC(s) give(s) us an opportunity to work on scientific processes (construction of definitions and proof in particular). Scientific processes are constituted by students' experiments with different cognitive attitudes: doubting, conjecturing, refuting, generating new counter-examples, testing etc. Such research activities with SDC(s) enrich and develop students' conception of the concept of definition in mathematics. The question had already been approached by Borasi (1992). And most importantly, SDCs are conducive to mathematical concept formation. This fact is attested more often by *zero-definitions* evolution than by *proof-generated definitions*. The challenge concerning concept formation and the nature of the understanding of a mathematical concept in a definition construction process is fundamental: I have now to design SDCs taking great care to construct a suitable methodology for an accurate evaluation of the impact of SDCs on the learning of new concepts in mathematics. Here are my main objectives. This paper tried to open up new avenues in that direction.

Starting from where I leave off, it should now be possible to conceive the integration of further theoretical perspectives so as to arrive at a finer characterization of defining processes. A close observation of the students' defining processes raises the teacher's awareness of those aspects of a concept students are more likely to respond to. A gate is now open on students' potential difficulties during the process of learning to refine concepts and formulate definitions.

REFERENCES

- Ambrosio U. d': 1993, *Etnomatemática*. Editora Atica, São Paulo.
- Aristotle: 1994, *Posterior Analytics*, 2nd Ed (trans. J.Barnes). Oxford University Press.
- Aristotle: 1997, *Topics*, (trans. R. Smith). Oxford University Press.
- Artigue M., Robinet J.: 1982, 'Conceptions du cercle chez des enfants de l'école primaire', *Recherches en Didactique des Mathématiques*, Vol 3.1, 5-64.
- Balacheff N. & Gaudin N.: 2002, 'Students conceptions: an introduction to a formal characterization'. *Cahiers du Laboratoire Leibniz* n°65, Grenoble. (<http://www-leibniz.imag.fr/LesCahiers/>).
- Borasi, R.: 1992, *Learning mathematics through inquiry*, Heinemann, New Hampshire.
- Brousseau, G.: 1997, *Theory of Didactical Situations in Mathematics*, Kluwer, Dordrecht.
- Burton, L.: 1999, 'The Practices of Mathematicians: What Do They Tell Us about Coming to Know Mathematics?'. *Educational Studies in Mathematics* 37, 121-143.

- Carnap, R.: 1954, *The Logical Syntax of Language*, Routledge & Kegan Paul. London.
- Davis, P. J. & Hersh R.: 1981, *The Mathematical Experience*. Brighton: The Harvester Press.
- Feyerabend, P.: 1975, *Against Method – Outline of an Anarchistic Theory of Knowledge*, Publ. Humanities Press.
- Freudenthal, H.: 1973, *Mathematics as an educational task*, Reidel, Dordrecht.
- Godot K. & Grenier D.: 2004, ‘Theoretical and Experimental Study of “Research-Situations for the Classroom”’. *ICME 10, TSG 14*. To be published.
- Lakatos, I.: 1961, *Essays in the logic of mathematical discovery*, Thesis, Cambridge University Press.
- Lakatos, I.: 1976, *Proofs and refutations*, Cambridge University Press.
- Lave J.: 1988, *Cognition into practice*. Cambridge University Press.
- Mariotti M.A. & Fischbein E.: 1997, ‘Defining in classroom activities’, *Educational Studies in Mathematics*, 34, 219-248.
- Núñez T., Carraher D., Schliemann A.: 1983, *Mathematics in streets and schools*. Cambridge University Press.
- Ouvrier-Bufferet, C.: 2003a, *Construction de définitions / construction de concept : vers une situation fondamentale pour la construction de définitions en mathématiques*. Leibniz, Grenoble. Web available: <http://tel.ccsd.cnrs.fr/>
- Ouvrier-Bufferet, C.: 2003b, ‘Can the Aristotelian and Lakatosian Conceptions of Definition Constitute a Tool for the Analysis of a Definition Construction Process?’ *Mediterranean Journal for Research in Mathematics Education*, Vol2, n°2, 19-36.
- Pimm, D.: 1993, ‘Just a matter of definition’, *Educational Studies in Mathematics*, 25, 261-277.
- Pimm, D.: 1995, *Symbols and Meanings in School Mathematics*, Routledge, London.
- Popper, K.: 1945, *The Open Society and its Enemies*, Routledge & Kegan Paul, London.
- Popper, K.: 1974, *Conjectures and Refutations: the growth of scientific knowledge*, 5th ed. Routledge & Kegan Paul, London.
- Shoenfeld A.: 1985, *Mathematical Problem Solving*. Academic Press. Orlando.
- Shoenfeld A. (ed): 1987, *Cognitive Science and Mathematics Education*, Lawrence Erlbaum Associates, Hillsdale.
- Sierpinska, A.: 1989, ‘On 15-17 years old students’ conceptions of functions, iteration of functions and attractive fixed points’. *Institut de Mathématiques, preprint 454*. Varsovie: Académie des Sciences de Pologne.
- Tall, D.: 2004, ‘Thinking through Three World of Mathematics’. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*. Vol 4, 281-288. Bergen. Norway.
- Thurston, W. P.: 1994, ‘On proof and progress in mathematics’. *Bulletin of the American Mathematical Society* 30(2) 161-177.
- Vergnaud, G.: 1991, ‘La théorie des champs conceptuels’, *Recherches en Didactique des Mathématiques*, n°10/2.3, 133-169.
- Vinner, S.: 1991, ‘The role of definitions in teaching and learning of mathematics’. In Tall, D. (Ed.) *Advanced Mathematical Thinking*. Kluwer, Dardrecht.
- Wu, L.D: 1982, ‘On the chaincode of a line’, *IEEE Trans. on Pattern Anal. Machine Intell.* PAMI 4, 347-353.
- Zaslavsky, O. & Shir, K. (2005) ‘Students’ Conceptions of a Mathematical Definition’. *Journal for Research in Mathematics Education*. Vol36, Issue 4, 317-346.

¹ “The imperative features relate to the following requirements: a mathematical definition must be *non-contradicting* (i.e., all conditions of a definition should co-exist), and *unambiguous* (i.e., its meaning should be uniquely interpreted). In addition, there are some features of a mathematical definition that are imperative only when applicable: A mathematical definition must be *invariant* under change of representation; and it should also be *hierarchical*, that is, it should be based on basic or previously defined concepts, in a *non-circular* manner” (Zaslavsky & Shir, 2005, p. 319).

² That is to say: to indicate what specific object a word means, to identify a larger class within which that object falls, and then to try and see what distinguishes it from the rest of that class.

³ “Most importantly, naming is one of the fundamental activities of mathematics. It is far from passive; on the contrary, it allows mathematics to be done [and more]” (Pimm, 1995, p. xiv).

⁴ I use the French word “problématique” in the following sense: the problematic area of a subject (Oxford Shorter Dictionary).