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Abstract

The Widom-Rowlinson model (or the Area-interaction model) is a Gibbs point process in $\mathbb{R}^d$ with the formal Hamiltonian define as the volume of $\cup_{x\in\omega}B_1(x)$, where ω is a locally finite configuration of points and $B_1(x)$ denotes the unit closed ball centred at x. The model is also tuned by two other parameters: the activity $z>0$ related to the intensity of the process and the inverse temperature $\beta\geq 0$ related to the strength of the interaction. In the present paper we investigate the phase transition of the model in the point of view of percolation theory and the liquid-gas transition. First, considering the graph connecting points with distance smaller than $2r>0$, we show that for any $\beta\geq 0$, there exists $0<\widetilde{z}_c^a(\beta,r)<+\infty$ such that an exponential decay of connectivity at distance n occurs in the subcritical phase (i.e. $z<\widetilde{z}_c^a(\beta,r)$) and a linear lower bound of the connection at infinity holds in the supercritical case (i.e. $z>\widetilde{z}_c^a(\beta,r)$). These results are in the spirit of recent works using the theory of randomised tree algorithms [7, 8, 9]. Secondly we study a standard liquid-gas phase transition related to the uniqueness/non-uniqueness of Gibbs states depending on the parameters z,β. Old results [22, 24] claim that a non-uniqueness regime occurs for $z=\beta$ large enough and it is conjectured that the uniqueness should hold outside such an half line ($z=\beta\geq\beta_c>0$). We solve partially this conjecture in any dimension by showing that for β large enough the non-uniqueness holds if and only if $z=\beta$. We show also that this critical value $z=\beta$ corresponds to the percolation threshold $\widetilde{z}_c^a(\beta,r)=\beta$ for β large enough, providing a straight connection between these two notions of phase transition.

Key words: Gibbs point process, DLR equations, Boolean model, continuum percolation, random cluster model, Fortuin-Kasteleyn representation, randomised tree algorithm, OSSS inequality.

1 Introduction

The Widom-Rowlinson model (or the Area-interaction model) is a Gibbs point process in $\mathbb{R}^d$ with the formal Hamiltonian given by the volume of the union of

balls with radii 1 and centred at the points of the process;

$$H(\omega) = \text{Volume}(\cup_{x\in\omega} B_1(x)).$$

By changing the scale, any other value of the radius can be considered as well. Two other parameters, the activity $z > 0$, related to the intensity of the process, and the inverse temperature $\beta \geq 0$, related to the strength of the interaction, parametrize the distribution of the process following the standard Boltzmann-Gibbs formalism. In the finite volume regime, the Gibbs measure is absolutely continuous with respect to the Poisson point process with the unnormalized density

$$f(\omega) \sim z^{\#\omega} e^{-\beta H(\omega)},$$

where $\#\omega$ denotes the number of points in ω.

The popularity of this model is due to old results [2, 22, 24] which prove that the Gibbs measures are not unique in the infinite volume regime for $z = \beta$ large enough. This beautiful result is a consequence of a representation of the model via the bi-color Widom-Rowlinson model which identifies the parameters z and β as the activities of a two-species model of particles. The non-uniqueness of the bi-color Widom-Rowlinson model is proved using Peierls argument, and by symmetry the phase transition is obtained for $z = \beta$ [22]. An alternative proof via the random cluster model and a Fortuin-Kasteleyn representation has been obtain later in [2]. A generalization is proved recently in the case of random unbounded radii [16]. As far as we know, this model and another model with a particular Kac type potential treated in [17] are the only models, in the continuum setting without spin, for which a non-uniqueness result is proved. Note also that the Area-interaction have been abundantly studied by researchers from different communities in statistical physics, probability theory or spatial statistics [1, 5, 13, 19].

In the present paper we investigate percolation and liquid-gas transition questions for this Area-interaction model. These two notions are different but are related and relevant to each other. Our first result claims that the Area-interaction model exhibits a sharp phase transition of percolation for the graph connecting points with distance smaller than $2r > 0$. Precisely, for any $\beta > 0$, there exists a non-trivial threshold $0 < \widetilde{z}_c^a(\beta, r) < +\infty$ such that an exponential decay of connectivity at distance n occurs in the subcritical phase. It means that for $z < \widetilde{z}_c^a(\beta, r)$ the probability (for any Area-interaction model with parameters z and β) that the point 0 is connected to the boundary of the sphere of radius n centred at 0 (i.e. $\partial B_n(0)$), decreases exponentially to zero when n goes to infinity. By standard Palm theory arguments, that provides the exponential decay of the size of the clusters in the process itself. Moreover a local linear lower bound of the connection at infinity holds in the supercritical case. It means that $z > \widetilde{z}_c^a(\beta, r)$ not too large, the probability (for any Area-interaction model with parameters z and β) that the point 0 is connected to infinity is larger than $c(z - \widetilde{z}_c^a(\beta, r))$ for a fixed positive constant $c > 0$. Again, by standard Palm theory arguments, that provides a sub-linear bound for the density of the infinite cluster in the process itself. The proofs of these results are in the spirit

of recent works using the theory of randomised tree algorithms [7, 8, 9]. Our main contribution is to proof an OSSS inequality for the Widom-Rowlinson model and to adapt the general strategy of randomised tree algorithms to the setting of interacting continuum particle systems. Complementary results show that the function $\beta \mapsto \widetilde{z}_c^a(\beta, r)$ is an non-decreasing Lipschitz map.

Let us now discuss the sharp liquid-gas phase transition. As recall above, the Area-interaction model exhibits a non-uniqueness regime for $z = \beta$ large enough. This result is called in the literature a liquid-gas phase transition since the pressure is continuous and non-differentiable at the critical point. The derivatives before and after the critical points gives the abrupt difference between the densities of particles in the liquid and gas phases. Since these results in the seventies, it was conjectured that the phase transition is sharp which means that the non-uniqueness occurs if and only if $z = \beta$ larger than a threshold $\beta^* > 0$. The results mentioned above do not give any information when $z \neq \beta$, except the standard case where z or β are small enough and for which the uniqueness of Gibbs measures is known since long time. In the present paper, we solve partially the conjecture by showing that for β large enough (but it is not a threshold) the non-uniqueness holds if and only if $z = \beta$. For moderate values of β (not too small and not too large) we obtain uniqueness for z outside the interval $[\widetilde{z}_c^a(\beta, 1) \ ; \ \widetilde{z}_c^a(., 1)^{-1}(\beta)]$. See Figure 1 for a precise description of the phase diagram we obtain. Our main tool here is an extension of the disagreement percolation argument introduced in [15] for continuum models. Actually we show that the Gibbs measures are unique provided that the wired Area-interaction model does not percolate for $r = 1$. It means that the Gibbs measures are unique as soon as $z < \widetilde{z}_c^a(\beta, 1)$ and also for $z > \widetilde{z}_c^a(., 1)^{-1}(\beta)$ by duality. A last result claims that $\widetilde{z}_c^a(\beta, 1) = \widetilde{z}_c^a(., 1)^{-1}(\beta) = \beta$ for β large enough reducing the interval to the single point $\{\beta\}$.

Let us mention a similar sharp liquid-gas phase transition obtained for the 2D Widom-Rowlinson model on $\mathbb{Z}^2$ (Theorem 1.1 [14]). The proof is based on large circuit arguments and depends strongly on the lattice structure. It can not be adapted to the continuum setting developed here and moreover it involves only the dimension $d = 2$.

The paper is organized as follows. In Section 2 we introduce the Area-intercation model and the main required tools (stochastic domination, bicolor Widom-Rowlinson representation, duality). In Section 3, the results are presented and the proofs related to the sharp phase transition of percolation (respectively the liquid-gas phase transition) are given in Section 4 (respectively in Section 5). An annex Section contains some technical lemmas.

2 Preliminaries

2.1 Space

Let us consider the state space $S := \mathbb{R}^d$ with $d \geq 2$ being the dimension. Let Ω be the set of locally finite configurations ω on S. This means that $\#(\omega \cap \Lambda) < \infty$ for every bounded Borel set Λ of $\mathbb{R}^d$, with $\#\omega$ being the cardinal of the configuration ω. We write ω_Λ as a shorthand for $\omega \cap \Lambda$. The configuration space

is embedded with the usual σ-algebra $\mathcal{F}$ generated by the counting variables. To a configuration $\omega \in \Omega$ we associate the germ-grain structure

$$B_r(\omega) := \bigcup_{x \in \omega} B_r(x),$$

where $B_r(x)$ is the closed ball centred at x with radius $r > 0$.

2.2 Poisson point processes

Let π^z be the distribution on Ω of the homogeneous Poisson point process with intensity $z > 0$. Recall that it means

- for every bounded Borel set Λ, the distribution of the number of points in Λ under π^z is a Poisson distribution of mean $z\mathcal{L}^d(\Lambda)$, where $\mathcal{L}^d$ stands for the usual d-dimensional Lebesgue measure;
- given the number of points in a bounded Λ, the points are independent and uniformly distributed in Λ.

We refer to [3] for details on Poisson point processes.

For $\Lambda \subseteq \mathbb{R}^d$ bounded, we denote by π^z_Λ the restriction of π^z on Λ. For simplicity the special case of the Poisson point process of unit intensity (i.e. $z = 1$) is denoted by π, and its restriction by π_Λ.

2.3 Area-interaction measures

The area-interaction measures (or the one-color Widom-Rowlinson models) are defined through the standard Gibbs Dobrushin-Lanford-Ruelle formalism prescribing the conditional probabilities. For a bounded $\Lambda \subseteq \mathbb{R}^d$, we define the Λ-*Hamiltonian*

$$H_\Lambda(\omega) := \mathcal{L}^d(\ B_1(\omega_\Lambda) \setminus B_1(\omega_{\Lambda^c})\ ). \tag{2.1}$$

The *area specification* on a bounded $\Lambda \subseteq \mathbb{R}^d$ with boundary condition ω_{Λ^c} is defined by

$$\mathscr{P}^{z,\beta}_{\Lambda,\omega_{\Lambda^c}}(d\omega'_\Lambda) := \frac{z^{\#\omega'_\Lambda}\ e^{-\beta H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})}}{Z^{area}(z,\beta,\Lambda,\omega_{\Lambda^c})}\pi_\Lambda(d\omega'_\Lambda) \tag{2.2}$$

with the standard *partition function*

$$Z^{area}(z,\beta,\Lambda,\omega_{\Lambda^c}) := \int_\Omega z^{\#\omega'_\Lambda}\ e^{-\beta H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})}\pi_\Lambda(d\omega'_\Lambda) \tag{2.3}$$

which is always non-degenerate (i.e. $0 < Z^{area}(z,\beta,\Lambda,\omega_{\Lambda^c}) < +\infty$).

Definition 2.1. *A probability measure P on Ω is an* area-interaction measure *of activity z and inverse temperature β, written $P \in \mathcal{G}^{area}_{z,\beta}$, if for every bounded Borel set $\Lambda \subseteq \mathbb{R}^d$ and every bounded measurable function f,*

$$\int_\Omega f\ dP = \int_\Omega \int_\Omega f(\omega'_\Lambda \cup \omega_{\Lambda^c})\mathscr{P}^{z,\beta}_{\Lambda,\omega_{\Lambda^c}}(d\omega'_\Lambda)P(d\omega). \tag{2.4}$$

The equations (2.4)*, for all bounded* Λ*, are called* DLR equations*, after Dobrushin, Lanford and Ruelle. Those equations prescribe the conditional probabilities of a Gibbs measure.*

Remark 2.1. *There are several possible forms for the Hamiltonian* H_Λ*, all of which defining the same specification. The nice property about our definition of* H_Λ *is the additivity, in the sense that* $H_\Lambda(\omega)$ *can be seen as the sum of the contribution of each points, with respect to the already considered ones.*

2.4 Stochastic domination

Let us discuss stochastic domination, which is going to be a key element of several proofs of the paper. Recall that an event $E \in \mathcal{F}$ is said *increasing* if for $\omega' \in E$ and $\omega \supseteq \omega'$, we have $\omega \in E$. Finally if P and P' are two probability measures on Ω, the measure P is said to *stochastically dominate* the measure P', written $P' \preceq P$, if $P'(E) \leq P(E)$ for every increasing event $E \in \mathcal{F}$.

The following proposition is a direct application of the classical Georgii and Küneth stochastic domination result [12, Theorem 1.1] and gives standard stochastic dominations.

Proposition 2.1. *For every bounded* $\Lambda \subseteq \mathbb{R}^d$*,*

- *for every boundary condition* ω_{Λ^c} *and every* z, β *we have*

$$\pi_\Lambda^{ze^{-\beta v_d}} \preceq \mathscr{P}^{z,\beta}_{\Lambda,\omega_{\Lambda^c}}(d\omega'_\Lambda) \preceq \pi_\Lambda^z, \tag{2.5}$$

  where v_d *is the volume of the unit ball in dimension* d*.*

  This implies in particular that every $P \in \mathcal{G}^{area}_{z,\beta}$ *satisfies*

$$\pi^{ze^{-\beta v_d}} \preceq P \preceq \pi^z. \tag{2.6}$$

- *For every boundary conditions* $\omega^1_{\Lambda^c} \subseteq \omega^2_{\Lambda^c}$*, every* $z_1 \leq z_2$ *and every* $\beta_1 \geq \beta_2$ *we have*

$$\mathscr{P}^{z_1,\beta_1}_{\Lambda,\omega^1_{\Lambda^c}}(d\omega'_\Lambda) \preceq \mathscr{P}^{z_2,\beta_2}_{\Lambda,\omega^2_{\Lambda^c}}(d\omega'_\Lambda). \tag{2.7}$$

2.5 Free and wired measures

Two particular area-interaction measures are constructed as follows. Consider the increasing sequence $\Lambda_n :=]-n,n]^d$ and consider the *free and wired area-interaction measures* on the bounded box Λ_n, denoted by $P^{z,\beta}_{n,free}$ and $P^{z,\beta}_{n,wired}$ and defined as

$$P^{z,\beta}_{n,free}\left(d\omega'_{\Lambda_n}\right) := \frac{z^{\#\omega'_{\Lambda_n}}\ e^{-\beta\mathcal{L}^d(\ B_1(\omega'_{\Lambda_n})\ )}}{Z^{area}(z,\beta,n,free)}\ \pi_{\Lambda_n}(d\omega'_{\Lambda_n}); \tag{2.8}$$

$$P^{z,\beta}_{n,wired}\left(d\omega'_{\Lambda_n}\right) := \frac{z^{\#\omega'_{\Lambda_n}}\ e^{-\beta\mathcal{L}^d(\ B_1(\omega'_{\Lambda_n})\cap\Lambda_{n-1}\ )}}{Z^{area}(z,\beta,n,wired)}\ \pi_{\Lambda_n}(d\omega'_{\Lambda_n}); \tag{2.9}$$

where $Z^{area}(z,\beta,n,free)$ and $Z^{area}(z,\beta,n,wired)$ are the normalising constants. The measure $P^{z,\beta}_{n,free}$ is simply $\mathscr{P}^{z,\beta}_{\Lambda_n,\emptyset}$, whereas $P^{z,\beta}_{n,wired}$ is the limiting case where the boundary condition would be filled with points on the boundary of Λ_n.

From [12, Theorem 1.1] we get the following proposition.

Proposition 2.2. *For every n and every z,β we have*

- $P^{z,\beta}_{n,free} \preceq P^{z,\beta}_{n+1,free}$;
- $P^{z,\beta}_{n+1,wired_{|\Lambda_n}} \preceq P^{z,\beta}_{n,wired}$,

where $P^{z,\beta}_{n+1,wired_{|\Lambda_n}}$ stands for the measure $P^{z,\beta}_{n+1,wired}$ restricted to Λ_n.

From Proposition 2.2 and using Carathéodory's extension theorem we get the existence of $P^{z,\beta}_{free}$ and $P^{z,\beta}_{wired}$. Those probability measures are, thanks to [3, Theorem 11.1.VII], weak limits of the sequences $\left(P^{z,\beta}_{n,free}\right)_n$ and $\left(P^{z,\beta}_{n,wired}\right)_n$. They are also stationary (see [2] for details).

Proposition 2.3. *For every $z_1 \leq z_2$ and $\beta_1 \geq \beta_2$,*

- $P^{z_1,\beta_1}_{free} \in \mathcal{G}^{area}_{z_1,\beta_1}$ *and* $P^{z_1,\beta_1}_{wired} \in \mathcal{G}^{area}_{z_1,\beta_1}$;
- $P^{z_1,\beta_1}_{free} \preceq P^{z_2,\beta_2}_{free}$ *and* $P^{z_1,\beta_1}_{wired} \preceq P^{z_2,\beta_2}_{wired}$;
- $P^{z_1,\beta_1}_{free} \preceq P \preceq P^{z_1,\beta_1}_{wired}$ *for all* $P \in \mathcal{G}^{area}_{z_1,\beta_1}$.

As a consequence of the first item of Proposition 2.3, we know that the set of area-interaction measures $\mathcal{G}^{area}_{z,\beta}$ is never empty. From the last item of Proposition 2.3, the question of uniqueness of the area-interaction measure translates to the question of the equality of measures $P^{z_1,\beta_1}_{free} = P^{z_1,\beta_1}_{wired}$. The next Proposition is stating that this equality happens for a lot of parameters (z,β).

Proposition 2.4. *For all $\beta > 0$, the set $\{z > 0, P^{z,\beta}_{free} \neq P^{z,\beta}_{wired}\}$ is at most countable.*

The proof of this proposition is related to standard differentiability/convexity arguments of the pressure function. See for instance Theorem 3.34 in [10] for a proof in the case of Ising model or Theorem 4.2 in [14] for the lattice Widom-Rowlinson model. A direct adaptation for the continuum area-interaction measure is achievable and omitted here for brevity.

2.6 Bicolor Widom-Rowlinson representation of area-interaction measures

The bicolor Widom-Rowlinson model is simply defined as the reunion of two Poisson Boolean models (with deterministic radii equal to 0.5) conditioned on a hard-core non overlapping condition between the two Boolean models. A formal definition using standard DLR formalism is given below.

Definition 2.2. *Let $\boldsymbol{\omega} := (\omega^1, \omega^2)$ denotes a couple of configurations. Let $\mathcal{A} := \{(\omega^1, \omega^2) \in \Omega^2, B_{1/2}(\omega^1) \cap B_{1/2}(\omega^2) = \emptyset\}$ be the event of authorised (couple of) configurations. Let $\boldsymbol{\pi}^{z_1,z_2} := \pi^{z_1} \otimes \pi^{z_2}$.*

Then a probability measure $\boldsymbol{P}$ on Ω^2 is a Widom-Rowlinson measure *with parameters z_1, z_2, written $\boldsymbol{P} \in \mathcal{G}^{wr}_{z_1,z_2}$, if $\boldsymbol{P}(\mathcal{A}) = 1$ and if for every bounded $\Lambda \subseteq \mathbb{R}^d$ and every bounded measurable function f,*

$$\int_{\Omega^2} f d\boldsymbol{P} = \int_{\Omega^2}\int_{\Omega^2} f(\boldsymbol{\omega}'_\Lambda \cup \boldsymbol{\omega}_{\Lambda^c}) \frac{\mathbb{1}_{\mathcal{A}}(\boldsymbol{\omega}'_\Lambda \cup \boldsymbol{\omega}_{\Lambda^c})}{\boldsymbol{Z}^{wr}(\Lambda, z_1, z_2, \boldsymbol{\omega}_{\Lambda^c})} \boldsymbol{\pi}^{z_1,z_2}_\Lambda(d\boldsymbol{\omega}'_\Lambda)\boldsymbol{P}(d\boldsymbol{\omega}), \quad (2.10)$$

with $\boldsymbol{Z}^{wr}(\Lambda, z_1, z_2, \boldsymbol{\omega}_{\Lambda^c})$ being the standard partition function associated to the Widom-Rowlinson interaction.

The following identifications between the bicolor and the one-color Widom-Rowlinson models are standard; the proof is omitted and we refer to [2, 11, 24] for details.

Proposition 2.5.

- *Let $\boldsymbol{P} \in \mathcal{G}^{wr}_{z_1,z_2}$. Then the first marginal of $\boldsymbol{P}$ is an area-interaction measure of activity $z = z_1$ and inverse temperature $\beta = z_2$. The analogue is true for the second marginal.*
- *Let $P \in \mathcal{G}^{area}_{z,\beta}$. Then the measure $\boldsymbol{P} := \pi^{\beta}_{\mathbb{R}^d \setminus B_1(\omega^1)}(d\omega^2)P(d\omega^1)$ is a Widom-Rowlinson measure: $\boldsymbol{P} \in \mathcal{G}^{wr}(z_1 = z, z_2 = \beta)$.*

As a consequence of these two points, the sets $\mathcal{G}^{area}_{z,\beta}$ and $\mathcal{G}^{wr}(z_1 = z, z_2 = \beta)$ are in bijection. By symmetry of $\mathcal{G}^{wr}(z_1, z_2)$ with respect to z_1, z_2, the sets $\mathcal{G}^{area}_{z,\beta}$ and $\mathcal{G}^{area}_{\beta,z}$ are in bijection as well; this property is called the duality property.

2.7 Percolation

The theory of percolation studies the connectivity in random structures. Formally the percolation is defined as follows.

Definition 2.3. *Let $r > 0$;*

- *two sets $\Lambda_1, \Lambda_2 \subseteq \mathbb{R}^d$ are said to be r-connected in ω, written $\Lambda_1 \underset{B_r(\omega)}{\longleftrightarrow} \Lambda_2$ (or $\Lambda_1 \underset{r}{\longleftrightarrow} \Lambda_2$ when there is no possible confusion) if $B_r(\omega) \cup \Lambda_1 \cup \Lambda_2$ has a connected component containing both Λ_1 and Λ_2;*
- *a configuration ω is said to r-percolate if the germ-grain structure $B_r(\omega)$ has at least one unbounded connected component;*
- *a probability measure P on Ω is said to* r-percolate *(respectively* do not percolate*) if $P(\{\omega\ r\text{-percolates}\}) = 1$ (respectively $P(\{\omega\ r\text{-percolates}\}) = 0$).*

In the next proposition we state the standard percolation phase transition of the Poisson Boolean model. See for instance [9] for a modern proof.

Proposition 2.6. *For every $r > 0$, there exists $0 < z_c^p(r) < \infty$, called* r-percolation threshold *of the Poisson Boolean model, such that*

- *for every $z < z_c^p(r)$, the measure π^z does not r-percolate, and we have the existence of $c := c(r, z) > 0$ such that*

$$\pi^z \left(0 \underset{r}{\longleftrightarrow} \partial\Lambda_n \right) \leq \exp(-cn), \tag{2.11}$$

  where $\partial\Lambda_n$ is the boundary of the set $\Lambda_n =]-n, n]^d$;

- *For every $z > z_c^p(r)$, the measure π^z r-percolates, and we have the existence of $c' := c'(r) > 0$ such that for z in a neighbourhood of $z_c^p(r)$*

$$\pi^z \left(0 \underset{r}{\longleftrightarrow} \infty \right) \geq c'(z - z_c^p(r)). \tag{2.12}$$

Concerning area-interaction measures such a behaviour is not proven, and is one of the questions investigated in this paper. But as a consequence of the Propositions 2.1, 2.3, 2.4 and 2.6 we have the existence of a non degenerate percolation threshold, common to all area-interaction measures. This is stated in the following Proposition.

Proposition 2.7. *For all $\beta > 0$ and $r > 0$, there exists $0 < \widetilde{z}_c^a(\beta, r) < \infty$ such that*

- *for all $z < \widetilde{z}_c^a(\beta, r)$, any area-interaction measure $P \in \mathcal{G}_{z,\beta}^{area}$ almost never r-percolates, i.e*

$$P(\{\omega \ r\text{-}percolates\}) = 0;$$

- *for all $z > \widetilde{z}_c^a(\beta, r)$, any area-interaction measures $P \in \mathcal{G}_{z,\beta}^{area}$ almost surely r-percolates, i.e*

$$P(\{\omega \ r\text{-}percolates\}) = 1.$$

Proof. The fact that both the free and wired measures have the same threshold is a consequence of Proposition 2.3 and Proposition 2.4. The non-degeneracy of this threshold follows from the non-degeneracy of the Poisson Boolean model percolation threshold, see Proposition 2.6, and from the dominations of Proposition 2.1: for all $P \in \mathcal{G}_{z,\beta}^{area}$

$$\pi^{ze^{-\beta v_d}} \preceq P \preceq \pi^z. \tag{2.13}$$

□

Remark 2.2. *Let us first notice that by a scaling argument, the percolation thresholds of the Poisson Boolean model satisfies the well-known relation $z_c^p(r) = \frac{1}{r^d} z_c^p(1)$. Then, from the stochastic dominations (2.13), we have the following bound on the percolation threshold of the area-interaction measures: for all $r > 0$ and $\beta \geq 0$,*

$$z_c^p(r) \ \leq \ \widetilde{z}_c^a(\beta, r) \ \leq \ z_c^p(r) \exp(\beta v_d).$$

3 Results

Let us now present our results related to the phase transition of the area-interaction measures. The proofs are given in the following sections.

3.1 Sharp phase transition of percolation

The first result proves a *sharp phase transition* of percolation for the area-interaction measures in the spirit of Proposition 2.6 for the Boolean model. That means exponential decay of connectivity at distance n in the subcritical phase and a local linear lower bound of the connection at infinity in the supercritical case.

Theorem 1. *Let $\beta \geq 0$.*

1. *For all $z < \widetilde{z}_c^a(\beta, r)$, there exists $\alpha_1 = \alpha_1(z, \beta, d, r) > 0$ such that for all $P \in \mathcal{G}_{z,\beta}^{area}$ and all n,*

$$P\left(0 \underset{r}{\longleftrightarrow} \partial\Lambda_n\right) \leq \exp(-\alpha_1 n). \tag{3.1}$$

2. *There exists $\alpha_2 = \alpha_2(\beta, d, r)$ such that for all $z > \widetilde{z}_c^a(\beta, r)$ small enough and all $P \in \mathcal{G}_{z,\beta}^{area}$,*

$$P\left(0 \underset{r}{\longleftrightarrow} \infty\right) \geq \alpha_2(z - \widetilde{z}_c^a(\beta, r)). \tag{3.2}$$

The proof of this theorem relies on the theory of randomised algorithms developed by Duminil-Copin, Raoufi and Tassion in a series of papers [7, 8, 9]. The main ingredient, and our main contribution with respect to what was already done, is the proof of an OSSS-type inequality which gives a control of the variance of a function f by a bound depending on the influence of each point of the process. The proof of this inequality relies on a procedure, sampling an area-interaction configuration using a dominating Poisson configuration. This procedure is in some sense monotonic with respect to the dominating Poisson configuration. The proof is given in Section 4.

The next proposition gives some qualitative properties of the function $\beta \mapsto \widetilde{z}_c^a(\beta, r)$ and exact values for β large enough. The proof is given in Section 4 as well.

Proposition 3.1. *For every $r > 0$, the function $\beta \mapsto \widetilde{z}_c^a(\beta, r)$ is a non-decreasing Lipschitz map from $\mathbb{R}^+$ to $[z_c^p(r), +\infty)$. In particular, it is continuous. Moreover for every $r > 0$, there exists $0 < \tilde{\beta}_r < \infty$ such that for $\beta > \tilde{\beta}_r$, the equality $\widetilde{z}_c^a(\beta, r) = \beta$ holds.*

3.2 Sharp liquid-gas phase transition

The other question of interest is the *Sharp liquid-gas phase transition* for which there are several definitions based either on the regularity of the pressure or the uniqueness/non uniqueness of Gibbs measures. Here we say that a *sharp*

liquid-gas phase transition occurs at temperature $1/\beta$ if there exists only one value z such that the Gibbs measures are not unique. This phenomenon is conjectured for several models but there does no exist complete rigorous proof in the continuum. Here we improve existing results for the area-interaction measures.

3.2.1 Already known results

Several results are already known on this subject. First, it is well-known that the set of gibbs measures is generally reduced to a singleton when the parameters z or/and β are small enough (see for instance [21]). As a consequence of a recent disagreement percolation result [15], explicit bounds related to 1-percolation threshold of the Poisson Boolean model are given.

Proposition 3.2. *Recall that $z_c^p(r)$ is the percolation threshold of the Poisson Boolean model of constant radii r. Then for every $z < z_c^p(1)$ and every $\beta \geq 0$, there is an unique area-interaction measure. Moreover, by duality, for every $\beta < z_c^p(1)$ and every $z \geq 0$, the uniqueness occurs as well.*

In addition, a Fortuin-Kasteleyn representation and percolation properties of the Continuum Random Cluster Model allow to prove a non uniqueness result for the symmetric bicolor Widom-Rowlinson model [2]. This result translates directly, thanks to Proposition 2.5, to a non uniqueness result of the area-interaction measure.

Proposition 3.3. *There exists $0 < \widetilde{z}_{sym} < \infty$ such that for all $z > \widetilde{z}_{sym}$*

- *the measure $P^{z,z}_{wired}$ does 1/2-percolate;*
- *the measure $P^{z,z}_{free}$ does not 1/2-percolate;*

hence we have $P^{z,z}_{free} \neq P^{z,z}_{wired}$, and therefore $\#\mathcal{G}^{area}(z,z) > 1$.

So in the symmetric case $z = \beta$, a standard phase transition is already known, where uniqueness is obtain at low activity z and non-uniqueness at large activity. However it is not proved that there exists a threshold between both regimes. As far as we know, this conjecture is still open today.

In the non-symmetric case $z \neq \beta$, very few is known expect from Proposition 3.2. In particular, the sharp phase transition around the symmetric case $z = \beta$ was unknown.

3.2.2 New results about uniqueness

It is conjectured that the non-uniqueness holds if and only if $z = \beta$ larger than a certain threshold $\beta^* > 0$. We do not solve this conjecture here but we show in corollary 3.1 that for β large enough the non-uniqueness holds only for $z = \beta$. Actually we succeed to prove uniqueness in a larger domain drawn in Figure 1.

Our main Theorem, given below, ensures the uniqueness as soon as the area-measures do not 1-percolate.

Theorem 2. *For all $\beta \geq 0$ and $z < \widetilde{z}_c^a(\beta, 1)$, we have $P_{free}^{z,\beta} = P_{wired}^{z,\beta}$, and therefore there is uniqueness of the area-interacton measure. By duality the result holds also for all $z \geq 0$ and $\beta < \widetilde{z}_c^a(z, 1)$.*

The proof of this theorem relies on a generalization of the disagreement percolation technique, relying on the construction of a coupling, called *disagreement coupling* comparing the influence of the boundary condition to a domination Poisson point process. Using the monotonicity of the area interaction, see Proposition 2.1, a better dominating measure is the wired area-interaction measure. The dominating measure is not a Poisson point process and therefore the construction of the disagreement coupling is more elaborate, even though it still relies on the original idea of van den Berg and Maes [23]. The proof of Theorem 2 is done in Section 5.

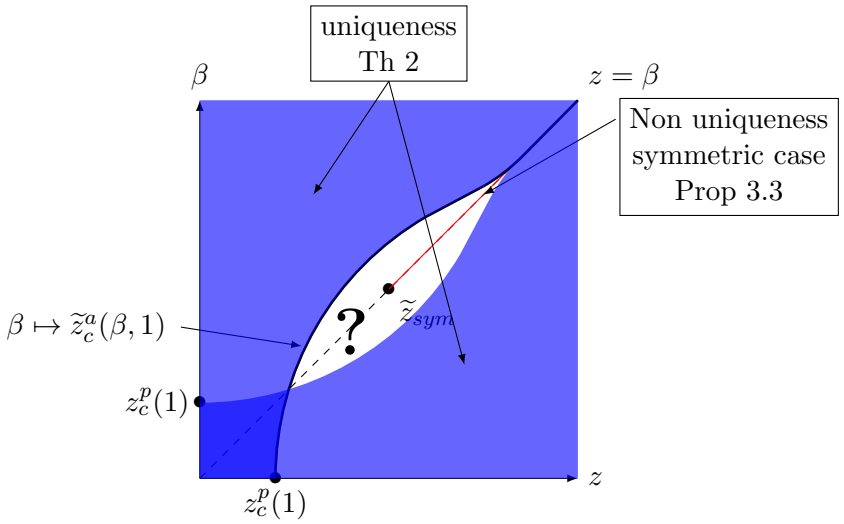

Figure 1: Uniqueness/non-uniqueness regimes for the area-interaction measures with parameters z, β.

Corollary 3.1. *For β larger than $\tilde{\beta}_1$, the area-interaction measures with parameters z, β are non-unique if and only if $z = \beta$. The sharp liquid-gas transition occurs.*

Proof. It is a direct consequence of Theorem 2, Proposition 3.1 and the duality property given in Proposition 2.5. □

4 Proofs related to percolation results

In this section we give the proofs of Theorem 1 and Proposition 3.1 involving the sharp phase transition of percolation.

4.1 Proof of Theorem 1

First let us note that it is enough to prove Theorem 1 for the wired area-interaction measure $P_{wired}^{z,\beta}$. Indeed, recall that from Proposition 2.3 we have

the sandwich domination: for all $P \in \mathcal{G}^{area}_{z,\beta}$, $P^{z,\beta}_{free} \preceq P \preceq P^{z,\beta}_{wired}$. Therefore the equation (3.1), i.e. the exponential decay of connectivity when $z < \widetilde{z}^a_c(\beta, r)$, translates directly from $P^{z,\beta}_{wired}$ to all $P \in \mathcal{G}^{area}_{z,\beta}$. For the equation (3.2), consider $z > z' > \widetilde{z}^a_c(\beta, r)$ such that $P^{z',\beta}_{free} = P^{z',\beta}_{wired}$. From Proposition 2.4 the parameter z' can be considered as close to z as we need. But once again from Proposition 2.3 we have

$$\begin{aligned} P\left(0 \underset{r}{\longleftrightarrow} \infty\right) &\geq P^{z,\beta}_{free}\left(0 \underset{r}{\longleftrightarrow} \infty\right) \\ &\geq P^{z',\beta}_{free}\left(0 \underset{r}{\longleftrightarrow} \infty\right) \\ &= P^{z',\beta}_{wired}\left(0 \underset{r}{\longleftrightarrow} \infty\right) \\ &\geq \alpha_2(z' - \widetilde{z}^a_c) \underset{z' \to z}{\longrightarrow} \alpha_2(z - \widetilde{z}^a_c), \end{aligned}$$

and Equation (3.2) is proved.

Through the remainder of this section the parameters $\beta > 0$ and $r > 0$ are fixed and might be omitted from notations and we consider only the wired case. Let $\mu^z_n := P^{z,\beta}_{3n+2,wired}$. We are considering the connection probability

$$\theta_n(z) = \mu^z_n(0 \underset{r}{\longleftrightarrow} \partial\Lambda_n), \tag{4.1}$$

where $\partial\Lambda_n$ is the boundary of $\Lambda_n =]-n, n]^d$.

Remark 4.1. *The term $3n + 2$ was chosen for several reasons. First the term "+2" is there to ensure that the wired measure μ^z_n is well defined, even for $n = 0$. Second the factor 3 is there to ensure a good inclusion of boxes in* (4.15).

Lemma 4.1. *For each z, the sequence $(\theta_n(z))$ converges and we have*

$$\theta(z) := \lim_{n\to\infty} \theta_n(z) \leq P^{z,\beta}_{wired}\left(0 \underset{r}{\longleftrightarrow} \infty\right). \tag{4.2}$$

Proof. The event $\{0 \underset{r}{\longleftrightarrow} \partial\Lambda_n\}$ depends only on the points inside Λ_{n+r}, which is included in Λ_{3n+2} as soon as $n \geq r/2$. Therefore, using Proposition 2.2 we have for such n:

$$\begin{aligned} \theta_n(z) &= P^{z,\beta}_{3n+2,wired}(0 \underset{r}{\longleftrightarrow} \partial\Lambda_n) \\ &\geq P^{z,\beta}_{3(n+1)+2,wired}(0 \underset{r}{\longleftrightarrow} \partial\Lambda_n) \\ &\geq P^{z,\beta}_{3(n+1)+2,wired}(0 \underset{r}{\longleftrightarrow} \partial\Lambda_{n+1}) = \theta_{n+1}(z). \end{aligned}$$

Hence the sequence is decreasing for n large enough and the convergence follows. For the inequality, just notice that for any k

$$\theta(z) = \lim_{n\to\infty} \theta_n(z) \leq \lim_{n\to\infty} P^{z,\beta}_{n,wired}(0 \underset{r}{\longleftrightarrow} \partial\Lambda_k) = P^{z,\beta}_{wired}(0 \underset{r}{\longleftrightarrow} \partial\Lambda_k).$$

Letting k go to infinity yields the result. □

Recall that $\widetilde{z}_c^a(\beta, r) \in]0, \infty[$ is the percolation threshold of the wired (and free as well) area-interaction measure, defined in Proposition 2.7.

Theorem 3. *Let $\beta > 0$ and $r > 0$.*

1. *For all $z < \widetilde{z}_c^a(\beta, r)$, there exists $\alpha_1 = \alpha_1(z, \beta, d, r) > 0$ such that, for all n,*

$$\theta_n(z) \leq \exp(-\alpha_1 n). \tag{4.3}$$

2. *There exists $\alpha_2 = \alpha_2(\beta, d, r)$ such that for all $z > \widetilde{z}_c^a(\beta, r)$ small enough,*

$$\theta(z) \geq \alpha_2(z - \widetilde{z}_c^a). \tag{4.4}$$

Before proving this theorem, let us see quickly how it leads to the proof of Theorem 1. The equation (4.4) together with Lemma 4.1 implies the equation (3.2), while (3.1) is a consequence of (4.3) and Proposition 2.2.

The proof of Theorem 3 relies on the theory of randomised algorithms popularized in [7, 8, 9]. First we are going to prove in Section 4.2 a generalized version of the OSSS inequality satisfied by $\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}$ for every cube Λ and every boundary condition $\hat{\omega}_{\Lambda^c}$.

This inequality, valid in particular for the wired measure $P^{z,\beta}_{3n+2,wired}$, is then applied in Section 4.3 in order to get the result.

4.2 OSSS inequality

4.2.1 Introduction of the formalism

In this section Λ is a fixed cube of side length $c > 0$. Let $\varepsilon > 0$ such that c/ε is a positive integer. We are dividing the box Λ into small cubes of size ε. Let $t := (c/\varepsilon)^d$ be the total number of such cubes. Therefore a configuration ω in Λ can be written as the collection $(\omega_e)_{e \in E}$ where $E := \Lambda \cap \varepsilon(\mathbb{Z} + 1/2)^d$ and $\omega_e := \omega_{\Delta_e^\varepsilon}$, where $\Delta_e^\varepsilon = e \oplus \varepsilon]-1/2, 1/2]^d$. For an enumeration $(e_1, \ldots, e_t)$ of the cubes, we wrote $e_{[i]} = (e_1, \ldots, e_i)$ and $\omega_{e_{[i]}} = (\omega_{e_1}, \ldots, \omega_{e_i})$.

Consider a Boolean function $f : \Omega \to \{0, 1\}$ and consider a decision tree T determining the value $f(\omega)$. A decision tree queries the configuration ω one cube after the other. Hence to a random configuration is associated a random ordering of the cubes $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_t)$. It starts from a deterministic cube $\mathbf{e}_1$ and looks at the configuration $\omega_{\mathbf{e}_1}$. Then it chooses a second cube $\mathbf{e}_2$ depending on $\mathbf{e}_1$ and $\omega_{\mathbf{e}_1}$ and carries out. At step $i > 1$ the cubes $\mathbf{e}_{[i-1]}$ have been visited and the configuration $\omega_{\mathbf{e}_{[i-1]}}$ is known. The next cube $\mathbf{e}_i$ to be explored is then expressed as a deterministic function of what have been already explored, i.e.

$$\mathbf{e}_i = \phi_i(\mathbf{e}_{[i-1]}, \omega_{\mathbf{e}_{[i-1]}}). \tag{4.5}$$

We then define the (random) time τ that the algorithm takes to determine the function f, meaning

$$\tau(\omega) = \min\{i \geq 1, \forall \omega', \omega'_{\mathbf{e}_{[i]}} = \omega_{\mathbf{e}_{[i]}} \Rightarrow f(\omega) = f(\omega')\}. \tag{4.6}$$

Theorem 4 (OSSS inequality)**.** *If the function f is increasing, then*

$$Var_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f) \leq 2\sum_{e\in E}\delta(e,T)\,Cov_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f,\#\omega_e) + \mathcal{O}(\varepsilon^d), \tag{4.7}$$

where $\delta(e,T) := \mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}(\exists i \leq \tau, \mathbf{e}_i = e)$ is called the revealment *of e and is the probability that the cube e is needed to determine the value of the function f.*

4.2.2 Proof of Theorem 4

The original proof of the OSSS inequality, see [20] or [6] for more probabilistic version, uses the product structure of the space considered (the Bernoulli percolation model). But $\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}$ is not a product measure and we need a more elaborate method. For this we will generalize the idea from [8, Lemma 2.1] which sampled a finite family of dependant random variables, one after the other in a random order, using independent uniform variables.

In the continuum setting of point processes this simple idea is much harder to implement. We will use ideas from the theory of stochastic domination. Indeed the stochastic domination $\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}} \preceq \pi^z_\Lambda$ from Proposition 2.1 implies, using Strassen's Theorem, that a configuration $\omega \sim \mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}$ can be obtained from a dominating Poisson configuration $\omega^D \sim \pi^z_\Lambda$ by a random thinning of the dominating configuration, deciding for each $x \in \omega^D$ if it belongs to the thinned configuration ω. We will use an explicit form of the thinning probability proved in [15].

To formalize the thinning decision we are adding to each point of the configuration ω^D an independent uniform mark between 0 and 1. The marked configuration is denoted by $\omega^{D,U}$ and its law is simply a marked Poisson point process.

Even if $\omega^{D,U}$ is a marked configuration on Λ, the sampling procedure we will construct below needs as many dominating configurations $\omega^{D,U}$ as there is cubes e_i in Λ, meaning t. Let us write $\omega^{D,U,\otimes} = (\omega^{D,U,1}, \dots, \omega^{D,U,t})$. The law of $\omega^{D,U,\otimes}$ is a product of marked Poisson point processes.

Definition 4.1. *For two marked configurations $\omega^{D,U}, \widetilde{\omega}^{D,U}$, we say that $\omega^{D,U}$ is smaller than $\widetilde{\omega}^{D,U}$, writing $\omega^{D,U} \leq \widetilde{\omega}^{D,U}$, if for each $(x,u) \in \omega^{D,U}$ we have $(x,u) \in \widetilde{\omega}^{D,U}$.*

We write $\omega^{D,U,\otimes} \leq \widetilde{\omega}^{D,U,\otimes}$ if for all i we have $\omega^{D,U,i} \leq \widetilde{\omega}^{D,U,i}$

Proposition 4.1. *Let $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_t)$ be a random sequence such that for all i, $\omega^{D,U,i}$ is independent of $\mathbf{e}_{[i]}$ (in particular the construction (4.5) follows this property). Then there exists an increasing function $F_{\mathbf{e}}$ such that*

$$\omega = F_{\mathbf{e}}(\omega^{D,U,\otimes}) \sim \mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}.$$

Proof. The configuration ω is sampled in each cube $\mathbf{e}_i$ one by one. At each step i the configuration $\omega_{\mathbf{e}_{[i-1]}}$ is already sampled and we are sampling $\omega_{\mathbf{e}_i}$. The sampling procedure is taken from [15, Proposition 4.1] which gives explicitly the thinning probability for sampling a Gibbs point process dominated by a Poisson point process.

We are going to sample a configuration (which we wrote ω') on

$$\tilde{\Lambda}_i := \Lambda \setminus (\Delta^{\varepsilon}_{\mathbf{e}_1} \cup \cdots \cup \Delta^{\varepsilon}_{\mathbf{e}_{i-1}})$$

according to the specification $\mathscr{P}^{z,\beta}_{\tilde{\Lambda}_i, \hat{\omega}_{\Lambda^c} \cup \omega_{\mathbf{e}_{[i-1]}}}$ and then only keep the points inside $\Delta^{\varepsilon}_{\mathbf{e}_i}$ by setting $\omega_{\mathbf{e}_i} = \omega'_{\mathbf{e}_i}$.

To sample ω' we consider the dominating configuration $\omega^{D,U,i}_{\tilde{\Lambda}_i}$ restricted to the region where we sample the configuration. Consider on $\tilde{\Lambda}_i$ a lexicographic order, which orders the points $(x,u) \in \omega^{D,U,i}_{\tilde{\Lambda}_i}$. The marks play no role in the ordering of the configuration. The configuration ω' is then constructed inductively the following way:

- at the beginning of the induction we set $\omega' = \emptyset$.
- Then we consider each point $(x,u) \in \omega^{D,U,i}_{\tilde{\Lambda}_i}$ one after the other with respect to the lexicographic order, and if

  $$p_i\left(x, \omega' \cup \omega_{\mathbf{e}_{[i-1]}} \cup \hat{\omega}_{\Lambda^c}\right) \leq u$$

  we add x to the configuration ω', i.e. $\omega' \leftarrow \omega' \cup \{x\}$. The function p_i, whose expression comes from [15, Proposition 4.1], is defined as

  $$p_i\left(x, \omega\right) = e^{-\beta H_{\{x\}}(x \cup \omega)} \times \frac{Z^{area}(z, \beta, \tilde{\Lambda}_i \cap]x, \infty[, \omega \cup x)}{Z^{area}(z, \beta, \tilde{\Lambda}_i \cap]x, \infty[, \omega)}, \tag{4.8}$$

  where the interval $]x, \infty[$ is defined with respect to the lexicographic ordering on $\tilde{\Lambda}_i$, and where ω is a configuration such that $\omega \cap \tilde{\Lambda}_i \cap]x, \infty[= \emptyset$.

And finally at the end we set $\omega_{\mathbf{e}_i} = \omega'_{\mathbf{e}_i}$.

From the DLR equations (2.4), the assumption on $\mathbf{e}$ and the Proposition 4.1 from [15], the sampled configuration has the law of $\mathscr{P}^{z,\beta}_{\tilde{\Lambda}_i, \hat{\omega}_{\Lambda^c} \cup \omega_{\mathbf{e}_{[i-1]}}}$.

In order to prove that $F_{\mathbf{e}}$ is increasing, It remains to prove that $p_i\left(x, \omega\right)$ is increasing in ω. Let us write $\tilde{\Lambda}^x_i := \tilde{\Lambda}_i \cap]x, \infty[$. The function p_i can be rewritten as

$$\begin{aligned} p_i\left(x, \omega\right) &= \frac{\int z^{\#\gamma} e^{-\beta H_{\tilde{\Lambda}^x_i}(\gamma \cup \omega)} e^{-\beta H_{\{x\}}(x \cup \gamma \cup \omega)} \pi_{\tilde{\Lambda}^x_i}(d\gamma)}{Z^{area}(z, \beta, \tilde{\Lambda}^x_i, \omega)} \\ &= \int e^{-\beta H_{\{x\}}(x \cup \gamma \cup \omega)} \mathscr{P}^{z,\beta}_{\tilde{\Lambda}^x_i, \omega}(d\gamma). \end{aligned}$$

Now using the fact that the integrated function is increasing in ω with the stochastic domination from Proposition 2.1, we have that the function p_i is increasing in ω. □

Remark 4.2. *The Proposition 4.1 is the main improvement from the theory of randomized algorithm from Duminil-Copin, Raoufi and Tassion [7, 8, 9]. Considering the assumption on* $\mathbf{e}$*, the Proposition 4.1 applies in particular when* $\mathbf{e}$ *is independent of* $\omega^{D,U,\otimes}$ *or when* $\mathbf{e}$ *is constructed from* $\omega^{D,U,\otimes}$ *as in* (4.5).

The proof of Proposition 4.1 relies only on the fact that the function $H_{\{x\}}(x \cup \omega)$*, often called* local energy*, is*

- *uniformly bounded from below;*
- *decreasing with respect to ω.*

So Proposition 4.1, and more generally Theorem 3 would trivially generalised to every Gibbs measure whose interaction satisfies those two properties. While the first property is a standard property satisfied by most interactions considered in the literature, the second property, related to the monotony *of the Gibbs specification, is less common. To the best of our knowledge the area-interaction is the only interaction considered in the literature which satisfies this property.*

Now consider two independent configurations $\omega^{D,U,\otimes}$ and $\widetilde{\omega}^{D,U,\otimes}$. The random ordering of cubes $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_t)$ consider starting now is constructed from $\omega^{D,U,\otimes}$ with (4.5). We write $\omega = F_{\mathbf{e}}(\omega^{D,U,\otimes})$ and $\widetilde{\omega} = F_{\mathbf{e}}(\widetilde{\omega}^{D,U,\otimes})$. Thanks to Proposition 4.1, those are two realisations of $\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}$, which are independent even though they are constructed from the same $\mathbf{e}$.

Now write for $i \leq \tau = \tau(\omega)$

$$\gamma^i = F_{\mathbf{e}}(\widetilde{\omega}^{D,U,1}, \dots, \widetilde{\omega}^{D,U,i}, \omega^{D,U,i+1}, \dots, \omega^{D,U,\tau}, \widetilde{\omega}^{D,U,\tau+1}, \dots, \widetilde{\omega}^{D,U,t}).$$

Then we have

$$\mathrm{Var}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}[f] \leq \mathbb{E}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}[\ |f - \mathbb{E}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}[f]|\ ] = \mathbb{E}[\ |f(\gamma^0) - \mathbb{E}[f(\gamma^\tau)]|\ ],$$

where in the right hand side the expectation are with respect to the two independent marked Poisson realisations $\omega^{D,U,\otimes}$ and $\widetilde{\omega}^{D,U,\otimes}$ from which the γ^i are constructed. Then

$$\begin{aligned}
\mathrm{Var}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}[f] &\leq \mathbb{E}[\ |\mathbb{E}\left[f(\gamma^0)|\omega^{D,U,\otimes}\right] - \mathbb{E}\left[f(\gamma^\tau)|\omega^{D,U,\otimes}\right]|\ ] \\
&\leq \mathbb{E}[\ |f(\gamma^0) - f(\gamma^\tau)|\ ] \\
&\leq \sum_{i=1\dots t} \mathbb{E}[\ |f(\gamma^i) - f(\gamma^{i-1})|\ \mathbb{1}_{i\leq\tau}] \\
&= \sum_{i=1\dots t} \sum_{e\in E} \mathbb{E}[\mathbb{E}[|f(\gamma^i) - f(\gamma^{i-1})|\ |\ \omega^{D,U,1\dots i-1}]\ \mathbb{1}_{i\leq\tau}\ \mathbb{1}_{\mathbf{e}_i=e}],
\end{aligned}$$

where $\omega^{D,U,1\dots i-1}$ is a short-hand for $(\omega^{D,U,1}, \dots, \omega^{D,U,i-1})$.

Lemma 4.2. *On the events $\{i \leq \tau\}$ and $\{\mathbf{e}_i = e\}$ we have*

$$\mathbb{E}[|f(\gamma^i) - f(\gamma^{i-1})|\ |\ \omega^{D,U,1\dots i-1}] \leq 2\,Cov_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f, \#\omega_e) + \mathcal{O}(\varepsilon^{2d}).$$

Using the Lemma 4.2 we get

$$\begin{aligned}
\mathrm{Var}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}[f] &\leq \sum_{i=1\dots t} \sum_{e\in E} (\ 2\mathrm{Cov}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f, \#\omega_e) + \mathcal{O}(\varepsilon^{2d})\ )\mathbb{E}[\ \mathbb{1}_{i\leq\tau}\ \mathbb{1}_{\mathbf{e}_i=e}] \\
&= \sum_{e\in E} (\ 2\mathrm{Cov}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f, \#\omega_e) + \mathcal{O}(\varepsilon^{2d})\ )\delta(e, T) \\
&= 2\left(\sum_{e\in E} \mathrm{Cov}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f, \#\omega_e)\delta(e, T)\right) + \mathcal{O}(\varepsilon^d),
\end{aligned}$$

where the last equality uses that the cardinal of E is of order $1/\varepsilon^d$.

Proof of Lemma 4.2. Since f is a positive increasing function bounded by one, we have

$$|f(\gamma^i) - f(\gamma^{i-1})| \leq (f(\gamma^i) - f(\gamma^{i-1})) \left(\mathbb{1}_{\#\gamma_e^i \geq 1} - \mathbb{1}_{\#\gamma_e^{i-1} \geq 1} \right) + \mathbb{1}_{\#(\gamma_e^i \cup \gamma_e^{i-1}) > 1},$$

where we used the monotonicity of $F_{\mathbf{e}}$ claimed in Proposition 4.1 and the fact that if $\#\gamma_e^{i-1} = \#\gamma_e^i = 0$, then $\gamma^{i-1} = \gamma^i$.

But using the stochastic domination we have the following easy bound

$$\mathbb{E}\left[\mathbb{1}_{\#(\gamma_e^i \cup \gamma_e^{i-1}) > 1} \mid \omega^{D,U,1\ldots i-1} \right] \leq \pi^{2z}(\#\omega_e > 1) = \mathcal{O}(\varepsilon^{2d}).$$

Using Proposition 4.1 we have

$$\begin{aligned} \mathbb{E}\left[f(\gamma^{i-1}) \mathbb{1}_{\#\gamma_e^{i-1} \geq 1} \mid \omega^{D,U,1\ldots i-1} \right] &= \mathbb{E}\left[f(\gamma^i) \mathbb{1}_{\#\gamma_e^i \geq 1} \mid \omega^{D,U,1\ldots i-1} \right] \\ &= \mathbb{E}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}} \left[f(\omega) \mathbb{1}_{\omega_e \geq 1} \right]. \end{aligned}$$

Now since the functions $f(\gamma^{i-1})$ and $\mathbb{1}_{\gamma_e^i \geq 1}$ are increasing, we are using the second part of Proposition 4.1 and the FKG inequality to obtain

$$\begin{aligned} \mathbb{E}\left[f(\gamma^{i-1}) \mathbb{1}_{\#\gamma_e^i \geq 1} \mid \omega^{D,U,1\ldots i-1} \right] &\geq \mathbb{E}\left[f(\gamma^{i-1}) \mid \omega^{D,U,1\ldots i-1} \right] \mathbb{E}\left[\mathbb{1}_{\#\gamma_e^i \geq 1} \mid \omega^{D,U,1\ldots i-1} \right] \\ &= \mathbb{E}_{P_n^{z,\beta}}\left[f(\omega) \right] \mathbb{E}_{P_n^{z,\beta}}\left[\mathbb{1}_{\omega_e \geq 1} \right]. \end{aligned}$$

The same is true for

$$\mathbb{E}\left[f(\gamma^i) \mathbb{1}_{\#\gamma_e^{i-1} \geq 1} \mid \omega^{D,U}_{\mathbf{e}_{[i-1]}} \right] \geq \mathbb{E}_{P_n^{z,\beta}}\left[f(\omega) \right] \mathbb{E}_{P_n^{z,\beta}}\left[\mathbb{1}_{\omega_e \geq 1} \right],$$

and therefore

$$\begin{aligned} \mathbb{E}[|f(\gamma^i) - f(\gamma^{i-1})| \mid \omega^{D,U,1\ldots i-1}] &\leq 2\mathrm{Cov}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f, \mathbb{1}_{\omega_e \geq 1}) + \mathcal{O}(\varepsilon^{2d}) \\ &\leq 2\mathrm{Cov}_{\mathscr{P}^{z,\beta}_{\Lambda,\hat{\omega}_{\Lambda^c}}}(f, \#\omega_e) + \mathcal{O}(\varepsilon^{2d}), \end{aligned}$$

where the last inequality coming from the use of the FKG inequality applied to f and $\#\omega_e - \mathbb{1}_{\omega_e \geq 1}$. The result is proved. □

4.3 Proof of Theorem 3

We are now going to prove Theorem 3 by applying Theorem 4. We need the following classical lemma in the theory of randomised algorithms (see Lemma 3 in [7] for instance).

Lemma 4.3. *Consider a converging sequence of increasing differentiable functions $g_n :]0, z_{max}] \to]0, 1]$ satisfying for all $n \geq 1$*

$$g_n'(z) \geq \alpha_2 \frac{n}{\Sigma_n(z)} g_n(z), \tag{4.9}$$

where $\alpha_2 > 0$ is a positive constant and

$$\Sigma_n(z) = \sum_{i=0}^{n-1} g_i(z).$$

Then there exists $\tilde{z} \in [0, z_{max}]$ such that:

- *For every $z < \tilde{z}$, there exists $\alpha_1 := \alpha_1(z) > 0$ such that for all n,*

$$g_n(z) \leq \exp(-\alpha_1 n). \tag{4.10}$$

- *For every $z > \tilde{z}$,*

$$g(z) := \lim_{n\to\infty} g_n(z) \geq \alpha_2 \left(z - \tilde{z}\right). \tag{4.11}$$

Therefore in order to prove Theorem 3 it is sufficient to prove that the functions $g_n = \theta_n$ satisfies the assumptions of Lemma 4.3. By construction, the functions $z \mapsto \theta_n(z)$ are increasing. The following lemma proves the differentiability of the functions θ_n.

Lemma 4.4. *For all functions f and all $z > 0$,*

$$\frac{d}{dz} P^{z,\beta}_{n,wired}(f) = \frac{1}{z} Cov_{P^{z,\beta}_{n,wired}}(f, \#\omega). \tag{4.12}$$

The proof of this result is done in the annex Section 6. Therefore the only remaining task in order to prove Theorem 3 is to prove that the functions θ_n satisfy (4.9). This is done using the OSSS inequality (4.7) to the wired measure μ^z_n for the Boolean function $f(\omega) = \mathbb{1}_{0 \underset{r}{\longleftrightarrow} \partial\Lambda_n}(\omega)$ to the well chosen algorithms from [7, 9].

Proposition 4.2. *For $0 \leq s \leq n$, there exists an algorithm T_s such that we have $\delta(e, T) = 0$ if $e \notin \Lambda_{n+r+1}$ and otherwise*

$$\delta(e, T_s) \leq \mu^z_n \left(\Delta^\varepsilon_e \oplus B_r(0) \underset{r}{\longleftrightarrow} \partial\Lambda_s\right). \tag{4.13}$$

This proposition uses the now standard algorithms used in [6, 7, 8]. The proof of Proposition 4.2 is done in the Annex Section 6.

Using Theorem 4 and summing over s between 0 and $n-1$ we get

$$\begin{aligned}
\theta_n(z)(1-\theta_n(z)) &\leq \frac{2}{n} \sum_{e\in E} \sum_{s=0}^{n-1} \mu^z_n \left(\Delta^\varepsilon_e \oplus B_r(0) \underset{r}{\longleftrightarrow} \partial\Lambda_s\right) \mathrm{Cov}_{\mu^z_n}(f, \#\omega_e) + \mathcal{O}(\varepsilon^d) \\
&\leq \frac{2}{n} \sum_{e\in E} \sum_{s=0}^{n-1} \mu^z_n \left(B_{r+1}(e) \underset{r}{\longleftrightarrow} \partial\Lambda_s\right) \mathrm{Cov}_{\mu^z_n}(f, \#\omega_e) + \mathcal{O}(\varepsilon^d),
\end{aligned}$$

where the last inequality is valid by considering ε small enough. To see if $B_{r+1}(e)$ is r-connected to $\partial\Lambda_s$, it is enough to check if at least one point y, belonging to a finite collection of points Y^e_r, is r-connected to $\partial\Lambda_s$. Actually the set Y^e_r contains points close to the boundary of $B_{r+1}(e)$ (inside and outside) and also a point in $B_{r+1}(e) \cap \partial\Lambda_s$ if the intersection is non empty. It is easy to see also that the set Y^e_r can be chosen with cardinal α_r depending only on r and d.

Therefore

$$\theta_n(z)(1-\theta_n(z)) \leq \frac{2}{n} \sum_{e\in E} \sum_{y\in Y^e_r} \sum_{s=0}^{n-1} \mu^z_n \left(y \underset{r}{\longleftrightarrow} \partial\Lambda_s\right) \mathrm{Cov}_{\mu^z_n}(f, \#\omega_e) + \mathcal{O}(\varepsilon^d). \tag{4.14}$$

But we have

$$\begin{aligned}\sum_{s=0}^{n-1}\mu_n^z\left(y\underset{r}{\longleftrightarrow}\partial\Lambda_s\right) &\leq \sum_{s=0}^{n-1}\mu_n^z\left(y\underset{r}{\longleftrightarrow}\partial\Lambda_{\lfloor s-||y||\rfloor}(y)\right)\\ &\leq 2\sum_{s=0}^{n-1}\mu_n^z\left(y\underset{r}{\longleftrightarrow}\partial\Lambda_s(y)\right)\\ &\leq 4\sum_{s=0}^{n/2}\mu_n^z\left(y\underset{r}{\longleftrightarrow}\partial\Lambda_s(y)\right),\end{aligned}$$

where $\lfloor s\rfloor$ stands for the floor function of the absolute value of s, and where $\Lambda_s(y)=y\oplus\Lambda_s$ is the cube translated by the vector y. Now using the stochastic domination between wired measures $\mu_n^z\preceq\mu_s^z$ from Proposition 2.2, we finally obtain

$$\sum_{s=0}^{n-1}\mu_n^z\left(y\underset{r}{\longleftrightarrow}\partial\Lambda_s(y)\right)\leq 4\sum_{s=0}^{n/2}\mu_s^z\left(0\underset{r}{\longleftrightarrow}\partial\Lambda_s\right)\leq 4\Sigma_n(z). \tag{4.15}$$

Remark 4.3. *Equation* (4.15) *is the only step of the proof of Theorem 3 where we used the fact that μ_n^z is a wired area-interaction measure.*

The purpose of summing up to $n/2$ is to ensure that $\Lambda_{3s+2}(y)\subseteq\Lambda_{3n+2}$ and therefore $\mu_n^z\preceq\mu_s^z$. This is why we considered Λ_{3n+2} in the definition on μ_n^z.

Hence from (4.14), (4.15) and Lemma 4.4, we obtain

$$\begin{aligned}\theta_n(z)(1-\theta_n(z)) &\leq 8\alpha_r\frac{\Sigma_n(z)}{n}\sum_{e\in E}\mathrm{Cov}_{\mu_n^z}(f,\#\omega_e)+\mathcal{O}(\varepsilon^d)\\ &\leq 8\alpha_r\frac{\Sigma_n(z)}{n}\mathrm{Cov}_{\mu_n^z}(f,\#\omega)+\mathcal{O}(\varepsilon^d)\\ &\leq 8\alpha_r\frac{\Sigma_n(z)}{n}z\theta_n'(z)+\mathcal{O}(\varepsilon^d).\end{aligned}$$

Now consider $z_{max}>\widetilde{z}_c^a$. Then for $z\leq z_{max}$ we have that $1-\theta_n(z)\geq c>0$, where c can be chosen uniformly in n. Therefore by letting ε goes to 0 we obtain

$$\theta_n'(z)\geq\frac{c}{8z_{max}\alpha_r}\frac{n}{\Sigma_n(z)}\theta_n(z):=\alpha_2\ \frac{n}{\Sigma_n(z)}\theta_n(z)$$

and equation (4.9) is fulfilled. From Lemma 4.3 we get the existence of a threshold $\tilde{z}$ but from the conclusion of Lemma 4.3 this threshold has to be the percolation threshold $\widetilde{z}_c^a$. The proof of Theorem 3 is complete.

4.4 Proof of proposition 3.1

We are in this section proving proposition 3.1, meaning the regularity of the function $\beta\to\widetilde{z}_c^a(\beta,r)$ and that for all $r>0$ and β large enough (depending on r) we have $\widetilde{z}_c^a(\beta,r)=\beta$.

4.4.1 Explicit value for the threshold

To prove that $\widetilde{z}_c^a(\beta, r) = \beta$ for β large enough, we are going to use the well-known *Fortuin-Kasteleyn* representation of bicolor Widom-Rowlinson measures (and therefore area-interaction measures as well) by *Continuum Random Cluster measures*, defined as follows.

Definition 4.2. *A stationary measure P^{crc} on Ω is a Continuum Random Cluster measure of activity z, if for all bounded $\Lambda \subseteq \mathbb{R}^d$ and all bounded measurable function f,*

$$\int f \, dP^{crc} = \int \int f(\omega'_\Lambda \cup \omega_{\Lambda^c}) \frac{2^{N_{cc}^\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})}}{Z^{crc}(z, \Lambda, \omega_{\Lambda^c})} \pi_\Lambda^z(d\omega'_\Lambda) P^{crc}(d\omega), \tag{4.16}$$

where $N_{cc}^\Lambda(\omega) = \lim_{\Delta \to \mathbb{R}^d} N_{cc}(\omega_\Delta) - N_{cc}(\omega_{\Delta \setminus \Lambda})$ with $N_{cc}(\omega_\Delta)$ counting the number of connected components of $B_{1/2}(\omega_\Delta)$, and $Z^{crc}(z, \Lambda, \omega_{\Lambda^c})$ being the standard non-degenerate partition function.

The existence of Random Cluster measures for every activity z was proved in [4]. In [16] the author proves that for any $r > 0$ there exists $\widetilde{z}_{sym}^r$ such that for any $z > \widetilde{z}_{sym}^r$ every Continuum Random Cluster measures r-percolates, which as a consequence gives the non uniqueness of area-interaction measures in the symmetric case for $z = \beta > \widetilde{z}_{sym}^{1/2}$. Indeed the standard *Fortuin-Kasteleyn* representation claims that keeping each finite 1/2-connected component from a Continuum Random Cluster measure with probability 1/2 and keeping as you want the infinite 1/2-connected component, this construction produces an area-interaction measure with parameters z, $\beta = z$ (see [2, 16]). Therefore, as recalled in Proposition 3.3, one area-interaction measure 1/2-percolates and another one does not 1/2-percolate. Consequently $\widetilde{z}_c^a(\beta, 1/2) = \beta$ for $\beta > \widetilde{z}_{sym}^{1/2}$ and the result is proved for $r = 1/2$.

For $r < 1/2$, we notice that for $\beta > \widetilde{z}_{sym}^r$ every Continuum Random Cluster measure, with activity β, r-percolates and therefore at least one area-interaction measure with parameters $z = \beta$ and β, r-percolates. That implies that $\widetilde{z}_c^a(\beta, r) \leq \beta$ but by monotonicity $\widetilde{z}_c^a(\beta, r) \geq \widetilde{z}_c^a(\beta, 1/2) = \beta$ which proves the result for $r < 1/2$.

It remains the case $r > 1/2$ which is more delicate. From now let $z > \widetilde{z}_{sym}^{1/2}$ and let P^{crc} be a Continuum Random Cluster measure of activity z, which is therefore 1/2-percolating by the choice of z. Consider the measure P^{thin} obtained from P^{crc} by removing all points $x \in \omega$ belonging to the (unique) infinite component of $B_{1/2}(\omega)$. Then as a consequence of the Fortuin-Kasteleyn representation, we have the following domination:

$$P_{free}^{z,z} \preceq P^{thin}. \tag{4.17}$$

By construction P^{thin} does not 1/2-percolates, since the 1/2-infinite connected component was removed. We are proving in the following lemma that by considering z large enough, this removed infinite connected component prevents r-percolation in P^{thin}.

Lemma 4.5. *There exists $\bar{z}_r \geq \widetilde{z}^{1/2}_{sym}$ such that for all $z > \bar{z}_r$, P^{thin} does not r-percolate.*

This lemma implies that $\widetilde{z}^a_c(\beta, r) \geq \beta$ for $\beta > \bar{z}_r$ and by monotonicity $\widetilde{z}^a_c(\beta, r) \leq \widetilde{z}^a_c(\beta, 1/2) = \beta$. That concludes the proof.

Proof of Lemma 4.5. We are divinding the space $\mathbb{R}^d$ into squares $C_y := y \oplus]-r, r]^d$, with $y \in 2r\mathbb{Z}^d$. We are going to prove that a lot of cubes are filled by balls from $B_{1/2}(\omega)$, where $\omega \sim P^{crc}$. By applying a well-known result from Liggett,Schonmann and Stacey [18], the infinite connected component of $B_{1/2}(\omega)$ will be (for large activities) very thick. This will prevent P^{thin} to r-percolate.

By stationarity of P^{crc} we are only considering the conditional probability

$$p_z(\omega_{\Lambda_0^c}) = P^{crc}(C_0 \subseteq B_{1/2}(\omega) | \omega_{\Lambda_0^c}),$$

where $\Lambda_0 :=]-r-1, r+1]^d$. For C_0 to be $1/2$-covered, it is sufficent that ω has enough nicely placed points. Therefore we are once again diving C_0 into smaller cubes $\tilde{C}_i$, $1 \leq i \leq k$, of side smaller than $\frac{1}{2\sqrt{d}}$. If all those small cubes contains a point then C_0 will be $1/2$-covered. Using the union bound we have

$$1 - p_z(\omega_{\Lambda_0^c}) \leq \sum_{i=1}^{k} P^{crc}(\#\omega_{\tilde{C}_i} = 0 | \omega_{\Lambda_0^c}).$$

But

$$\begin{aligned}
&P^{crc}(\#\omega_{\tilde{C}_i} = 0 | \omega_{\Lambda_0^c}) \\
&\quad = \int\int \mathbb{1}_{\#\omega''_{\tilde{C}_i}=0} \frac{2^{N_{cc}^{\tilde{C}_i}(\omega''_{\tilde{C}_i} \cup \omega'_{\Lambda_0 \setminus \tilde{C}_i} \cup \omega_{\Lambda_0^c})}}{Z^{crc}\left(z, \tilde{C}_i, \omega'_{\Lambda_0 \setminus \tilde{C}_i} \cup \omega_{\Lambda_0^c}\right)} \pi^z_{\tilde{C}_i}(d\omega''_{\tilde{C}_i}) P^{crc}(d\omega'_{\Lambda_0} | \omega_{\Lambda_0^c}) \\
&\quad = \int \frac{e^{-z\mathcal{L}^d(\tilde{C}_i)}}{Z^{crc}\left(z, \tilde{C}_i, \omega'_{\Lambda_0 \setminus \tilde{C}_i} \cup \omega_{\Lambda_0^c}\right)} P^{crc}(d\omega'_{\Lambda_0} | \omega_{\Lambda_0^c}),
\end{aligned}$$

where we used (4.16) and the fact that $N_{cc}^{\tilde{C}_i}\left(\emptyset \cup \omega'_{\Lambda_0 \setminus \tilde{C}_i} \cup \omega_{\Lambda_0^c}\right) = 0$. But we also have the following bound on $N_{cc}^{\tilde{C}_i}(\omega)$:

$$\#\omega_{\tilde{C}_i}(1 - c_d) \leq N_{cc}^{\tilde{C}_i}(\omega) \leq \#\omega_{\tilde{C}_i},$$

where c_d is the kissing number in dimension d, which is always larger than 2. This implies

$$P^{crc}(\#\omega_{\tilde{C}_i} = 0 | \omega_{\Lambda_0^c}) \leq e^{-z\mathcal{L}^d(\tilde{C}_i)} e^{z\mathcal{L}^d(\tilde{C}_i)(1-2^{1-c_d})} = e^{-2^{1-c_d} z \mathcal{L}^d(\tilde{C}_i)},$$

and therefore

$$p_z(\omega_{\Lambda_0^c}) \geq 1 - k e^{-2^{1-c_d} z \mathcal{L}^d(\tilde{C}_i)}. \tag{4.18}$$

The bound (4.18) is uniform in $\omega_{\Lambda_0^c}$ and goes to 1 as z goes to infinity. Therefore by applying the result of Liggett Schonmann and Stacey [18], we have for z large enough that the set of completely covered (by balls of radii $1/2$) boxes C_y stochastically dominates an independent and identically Bernoulli field with parameter $0 < p < 1$ as large as we want. Let $\mathcal{C}^{infinite}$ be the set of sites $y \in 2r\mathbb{Z}^d$ belonging to the infinite connected component.

Now consider the set $\mathcal{C}^{finite} = 2r\mathbb{Z}^d \backslash \mathcal{C}^{infinite}$ of sites $y \in 2r\mathbb{Z}^d$ not belonging to the infinite cluster of this site percolation. By considering z large enough we have that $\mathcal{C}^{finite}$ only contains bounded connected components, with respect to the site percolation.

Now consider the configuration ω^{finite} obtained from ω by removing the points x in the infinite connected component of $B_{1/2}(\omega)$. By construction we have that $B_{1/2}(\omega^{finite})$ is entirely contained in the cubes C_y for $y \in \mathcal{C}^{finite}$ and the configuration ω^{finite} cannot r-percolates, since otherwise the infinite connected component would have to crosses cubes $y \in \mathcal{C}^{infinite}$, which is not possible.

This implies that for z large enough the measure P^{thin} does not r-percolate. □

4.4.2 Regularity of the threshold

The regularity of the function $\beta \to \widetilde{z}_c^a(\beta, r)$ is a consequence of uniform properties of the Papangelou intensity of the area-interaction measure which is defined for every $x \in \mathbb{R}^d$ and $\omega \in \Omega$ by

$$\gamma_{z,\beta}(x, \omega) = z e^{-\beta H_{\{x\}}(\omega \cup x)} = z e^{-\beta \mathcal{L}^d(B_1(x) \backslash B_1(\omega))}.$$

Roughly speaking, this quantity is the quotient of densities of the process with and without the point x. We refer to [12] for rigorous definitions, interpretations and results on the topic. Let z, β be in a compact set and let u be in $[0, 1]$. By simple geometric considerations there exists a positive constant $c > 0$ such that

$$\gamma_{z+u,\beta+cu}(x, \omega') \geq \gamma_{z,\beta}(x, \omega),$$

for any x and $\omega \subset \omega'$. Then we deduce by Theorem 1.1 in [12] that the wired area-interaction measure with parameter $z+u$, $\beta+cu$ stochastically dominates the free area-interaction measure with parameter z, β: $P_{free}^{z,\beta} \preceq P_{wired}^{z+u,\beta+cu}$. Coming back now to the percolation threshold $\widetilde{z}_c^a(\beta, r)$, that implies that for $h > 0$ small enough

$$0 < \widetilde{z}_c^a(\beta + h, r) - \widetilde{z}_c^a(\beta, r) < 1/c.$$

That proves that the function $\beta \to \widetilde{z}_c^a(\beta + h, r)$ is locally a Lipschitz increasing map. Given that $\widetilde{z}_c^a(0, r) = z_c^p(r)$ and that for large β, $\widetilde{z}_c^a(\beta, r) = \beta$ it is enough to claim that the function $\beta \mapsto \widetilde{z}_c^a(\beta, r)$ is a non-decreasing Lipschitz map from $\mathbb{R}^+$ to $[z_c^p(r), +\infty)$.

5 Proof of Theorem 2

The proof relies on a disagreement coupling method. In this section we are considering $z < \widetilde{z}_c^a(\beta, 1)$, and we want to prove $P^{z,\beta}_{free} = P^{z,\beta}_{wired}$.

Definition 5.1. *A disagreement coupling family* $\left(\mathbb{P}^{\boldsymbol{dcf}}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}\right)$ *index by a bounded* $\Lambda \subseteq \mathbb{R}^d$ *and two configurations satisfying* $\omega^1_{\Lambda^c} \subseteq \omega^2_{\Lambda^c}$*, is a coupling of two marginals, with canonical variables* ξ^1 *and* ξ^2 *satisfying*

$$\forall 1 \leq i \leq 2: \quad \mathbb{P}^{\boldsymbol{dcf}}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}(\xi^i = d\omega') = \mathscr{P}^{z,\beta}_{\Lambda,\omega^i_{\Lambda^c}}(d\omega') \tag{5.1a}$$

$$\mathbb{P}^{\boldsymbol{dcf}}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}(\xi^1 \subseteq \xi^2) = 1 \tag{5.1b}$$

$$\mathbb{P}^{\boldsymbol{dcf}}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}\left(\forall x \in \xi^2 \setminus \xi^1 | B_1(x) \underset{B_1(\xi^2)}{\longleftrightarrow} B_1(\omega^2_{\Lambda^c})\right) = 1 \tag{5.1c}$$

Remark 5.1. *In the definition of a disagreement coupling family from [15], there is a third marginal dominating the two first which is a Poisson point process. But from the monotonicity property of area-interaction measures, see Proposition 2.1, one can only consider two marginals in the coupling.*

Proposition 5.1. *If there exists a disagreement coupling family, then* $P^{z,\beta}_{free} = P^{z,\beta}_{wired}$.

Proof. Let E be an event, that without loss of generality, only depends on the configurations inside a given bounded Λ. Then for $\Lambda \subseteq \Lambda_n$ we have

$$\begin{aligned}
&|P^{z,\beta}_{free}(E) - P^{z,\beta}_{wired}(E)| \\
&\quad \leq \int\int |\mathscr{P}^{z,\beta}_{\Lambda_n,\omega^1_{\Lambda_n^c}}(E) - \mathscr{P}^{z,\beta}_{\Lambda_n,\omega^2_{\Lambda_n^c}}(E)| P^{z,\beta}_{free}(d\omega^1) P^{z,\beta}_{wired}(d\omega^2) \\
&\quad = \int\int |\mathscr{P}^{z,\beta}_{\Lambda_n,\omega^1_{\Lambda_n^c}}(E) - \mathscr{P}^{z,\beta}_{\Lambda_n,\omega^2_{\Lambda_n^c}}(E)| P^{z,\beta}_{free|\Lambda_{n+2}}(d\omega^1) P^{z,\beta}_{wired|\Lambda_{n+2}}(d\omega^2),
\end{aligned}$$

where $P^{z,\beta}_{free|\Lambda_{n+2}}$ (respectively $P^{z,\beta}_{wired|\Lambda_{n+2}}$) is the restriction of $P^{z,\beta}_{free}$ (respectively $P^{z,\beta}_{wired}$) on Λ_{n+2}. Now by the stochastic domination $P^{z,\beta}_{free} \preceq P^{z,\beta}_{wired}$, we have from Strassen's theorem the existence of a thinning probability $\phi^1_{\omega^2}$ such that

$$\begin{aligned}
&|P^{z,\beta}_{free}(E) - P^{z,\beta}_{wired}(E)| \\
&\leq \int \sum_{\omega^1 \subseteq \omega^2} |\mathscr{P}^{z,\beta}_{\Lambda_n,\omega^1_{\Lambda_n^c}}(E) - \mathscr{P}^{z,\beta}_{\Lambda_n,\omega^2_{\Lambda_n^c}}(E)| \phi_{\omega^2}(\omega^1) P^{z,\beta}_{wired|\Lambda_{n+2}}(d\omega^2) \\
&\leq \int \sum_{\omega^1 \subseteq \omega^2} \mathbb{P}^{\mathbf{dcf}}_{\Lambda_n,\omega^1_{\Lambda_n^c},\omega^2_{\Lambda_n^c}}, \left(\Lambda \underset{B_1(\xi^2)}{\longleftrightarrow} \Lambda_{n-2}\right) \phi_{\omega^2}(\omega^1) P^{z,\beta}_{wired|\Lambda_{n+2}}(d\omega^2),
\end{aligned}$$

where the last inequality comes from the existence of the disagreement coupling

family and the property (5.1c). Therefore

$$|P^{z,\beta}_{free}(E) - P^{z,\beta}_{wired}(E)| \leq \int \mathscr{P}^{z,\beta}_{\Lambda_n,\omega_{\Lambda_n^c}} \left(\Lambda \underset{B_1(\xi)}{\longleftrightarrow} \Lambda_{n-2}\right) P^{z,\beta}_{wired|\Lambda_{n+2}}(d\omega)$$
$$= P^{z,\beta}_{wired} \left(\Lambda \underset{B_1(\xi)}{\longleftrightarrow} \Lambda_{n-2}\right) \underset{n\to\infty}{\longrightarrow} 0,$$

where the convergence is a consequence of $z < \widetilde{z}^a_c(\beta, r)$. □

It remains to prove the existence of the disagreement coupling family.

Proposition 5.2. *There exists a disagreement coupling family* $\left(\mathbb{P}^{\boldsymbol{dcf}}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}\right)$.

The construction of the disagreement coupling family is a generalisation of the one made in [15], where the dominating measure is a Poisson point process. The coupling is sampled starting from the balls close to the boundary of Λ, and going inductively inside Λ.

Proof. The coupling is constructed inductively. Recall that $\Lambda \subseteq \mathbb{R}^d$ is bounded and that $\omega^1_{\Lambda^c} \subseteq \omega^2_{\Lambda^c}$. Define the *disagreement zone*

$$\Gamma = \{x \in \Lambda, ||x; \omega^2_{\Lambda^c}|| \leq 2\}$$

as the region where a point x of the point process would be directly $1-$*connected* to the boundary condition $\omega^2_{\Lambda^c}$ (i.e. the ball $B_1(x)$ would overlap $B_1(y)$ for at least one $y \in \omega^2_{\Lambda^c}$).

The induction will be made with respect to the disagreement zone Γ in the following way.

- If $\Gamma \neq \emptyset$, let us first sample $\xi^2 \sim \mathscr{P}^{z,\beta}_{\Lambda,\omega^2_{\Lambda^c}}$. We are then sampling $\xi^1 \sim \mathscr{P}^{z,\beta}_{\Lambda,\omega^1_{\Lambda^c}}$ as a thinning of ξ^2.

  This procedure is possible, since the condition $\omega^1_{\Lambda^c} \subseteq \omega^2_{\Lambda^c}$ implies, thanks to Proposition 2.1, the following domination:

  $$\mathscr{P}^{z,\beta}_{\Lambda,\omega^1_{\Lambda^c}} \preceq \mathscr{P}^{z,\beta}_{\Lambda,\omega^2_{\Lambda^c}}.$$

  From ξ^1 and ξ^2 we are only keeping the points inside Γ. The induction then goes on with $\Lambda \leftarrow \Lambda \cap \Gamma^c$ with the new boundary conditions $\omega^1_{\Lambda^c \cup \Gamma} = \omega^1_{\Lambda^c} \cup \xi^1_\Gamma$ and $\omega^2_{\Lambda^c \cup \Gamma} = \omega^2_{\Lambda^c} \cup \xi^2_\Gamma$.

- If $\Gamma = \emptyset$. This is the terminal step of the induction. In this case we have $\mathscr{P}^{z,\beta}_{\Lambda,\omega^1_{\Lambda^c}} = \mathscr{P}^{z,\beta}_{\Lambda,\omega^2_{\Lambda^c}} = \mathscr{P}^{z,\beta}_{\Lambda,\emptyset}$. Therefore we simply sample $\xi^1 = \xi^2 \sim \mathscr{P}^{z,\beta}_{\Lambda,\emptyset}$.

It is easy to see that the induction terminates almost surely. Indeed if at one step the sampled configuration ξ^2 is empty (which happens with positive bounded from below probability) then at the following step we will have $\Gamma = \emptyset$. Therefore the number of steps is dominated by a geometric random variable, which is almost surely finite.

Finally the construction ensures that all properties of (5.1) are fullfilled. □

6 Annex

6.1 Proof of Proposition 4.2

Let us define the algorithm T_s which explores the r-connected components of $\partial\Lambda_s$.

Definition 6.1 (Definition of the algorithm)**.** *During the first i steps, the configuration inside the cubes $\mathbf{e}_1, \dots, \mathbf{e}_i$ have been explored, and we write $Z_i = B_r(\omega_{\mathbf{e}_{[i]}})$ for the region know to be covered by r-balls. Remark that Z_i might not be entirely connected (in Z_i) to $\partial\Lambda_s$, and wrote Z_i^s for the subregion of Z_i of points connected (in Z_i) to $\partial\Lambda_s$.*

At the step $i+1$ we take a cube $\mathbf{e}_{i+1} \in \varepsilon(\mathbb{Z}+1/2)^d \cap \Lambda_{n+r+1} \setminus \mathbf{e}_{[i]}$ such that

$$||\Delta^{\varepsilon}_{\mathbf{e}_{i+1}}; Z_i^s \cup \partial\Lambda_s|| \leq r.$$

If no such $\mathbf{e}_{i+1}$ exists, then the algorithm stops as the connected components of $\partial\Lambda_s$ have been entirely explored. If there is several $\mathbf{e}_{i+1}$ satisfying the condition, we choose one using a deterministic (but not important) rule.

We then look at the configuration inside the chosen cube $\mathbf{e}_{i+1}$ and set $Z_{i+1} = Z_i \cup B_r(\omega_{\mathbf{e}_{i+1}}) = B_r(\omega_{\mathbf{e}_{[i+1]}})$.

The bound (4.13) from Proposition 4.2 follows directly from the definition of the algorithms.

6.2 Proof of Lemma 4.4

We have $P^{z,\beta}_{n,wired}(f) = \int f(\omega)\frac{z^{\#\omega}h(\omega)}{Z^{area}(z,\beta,n,wired)}\pi_{\Lambda_n}(d\omega)$, where

$$h(\omega) := e^{-\beta\mathcal{L}^d(B(\omega_{\Lambda_n},1)\cap\Lambda_{n-1})}.$$

Using a standard derivative theorem we obtain

$$\frac{d}{dz}P^{z,\beta}_{n,wired}(f) = \frac{1}{z}P^{z,\beta}_{n,wired}(f\times\#) - \frac{\frac{d}{dz}Z^{area}(z,\beta,n,wired)}{Z^{area}(z,\beta,n,wired)}P^{z,\beta}_{n,wired}(f). \quad (6.1)$$

Taking $f = 1$ in (6.1) yields

$$0 = \frac{1}{z}P^{z,\beta}_{n,wired}(\#) - \frac{\frac{d}{dz}Z^{area}(z,\beta,n,wired)}{Z^{area}(z,\beta,n,wired)},$$

which transforms (6.1) into

$$\frac{d}{dz}P^{z,\beta}_{n,wired}(f) = \frac{1}{z}\left(P^{z,\beta}_{n,wired}(f\times\#) - P^{z,\beta}_{n,wired}(\#)P^{z,\beta}_{n,wired}(f)\right),$$

proving the result.

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References

[1] A. J. Baddeley and M. N. M. van Lieshout. Area-interaction point processes. *Ann. Inst. Statist. Math.*, 47(4):601–619, 1995.

[2] J. T. Chayes, L. Chayes, and R. Kotecký. The analysis of the widom-rowlinson model by stochastic geometric methods. *Comm. Math. Phys.*, 172(3):551–569, 1995.

[3] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes*. Springer Series in Statistics. Springer-Verlag, New York, 1988.

[4] D. Dereudre and P. Houdebert. Infinite volume continuum random cluster model. *Electron. J. Probab.*, 20:no. 125, 24, 2015.

[5] D. Dereudre and P. Houdebert. Phase transition for continuum Widom-Rowlinson model with random radii. *ArXiv e-prints*, December 2017.

[6] H. Duminil-Copin. Lectures on the Ising and Potts models on the hypercubic lattice. *ArXiv e-prints*, July 2017.

[7] H. Duminil-Copin, A. Raoufi, and V. Tassion. Exponential decay of connection probabilities for subcritical Voronoi percolation in $\mathbb{R}^d$. *ArXiv e-prints*, May 2017.

[8] H. Duminil-Copin, A. Raoufi, and V. Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees. *ArXiv e-prints*, May 2017.

[9] H. Duminil-Copin, A. Raoufi, and V. Tassion. Subcritical phase of d-dimensional Poisson-Boolean percolation and its vacant set. *ArXiv e-prints*, May 2018.

[10] S. Friedli and Y. Velenik. *Statistical mechanics of lattice systems*. Cambridge University Press, Cambridge, 2018. A concrete mathematical introduction.

[11] H.-O. Georgii, O. Häggström, and C. Maes. The random geometry of equilibrium phases. In *Phase transitions and critical phenomena, Vol. 18*, volume 18 of *Phase Transit. Crit. Phenom.*, pages 1–142. Academic Press, San Diego, CA, 2001.

[12] H.-O. Georgii and J. M. Küneth. Stochastic comparison of point random fields. *J. Appl. Probab.*, 34(4):868–881, 1997.

[13] O. Häggström, M.-C. N. M. van Lieshout, and J. Møller. Characterization results and Markov chain Monte Carlo algorithms including exact simulation for some spatial point processes. *Bernoulli*, 5(4):641–658, 1999.

[14] Yasunari Higuchi and Masato Takei. Some results on the phase structure of the two-dimensional Widom-Rowlinson model. *Osaka J. Math.*, 41(2):237–255, 2004.

[15] C. Hofer-Temmel and P. Houdebert. Disagreement percolation for marked Gibbs point processes. *ArXiv e-prints*, September 2017.

[16] P. Houdebert. Percolation results for the Continuum Random Cluster Model. *ArXiv e-prints*, October 2016.

[17] J. L. Lebowitz, A. Mazel, and E. Presutti. Liquid-vapor phase transitions for systems with finite-range interactions. *J. Statist. Phys.*, 94(5-6):955–1025, 1999.

[18] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25(1):71–95, 1997.

[19] A. Mazel, Y. Suhov, and I. Stuhl. A classical WR model with q particle types. *J. Stat. Phys.*, 159(5):1040–1086, 2015.

[20] R. O'Donnell, M. Saks, O. Schramm, and R. A. Servedio. Every decision tree has an influential variable. pages 31–39, Oct 2005.

[21] D. Ruelle. *Statistical mechanics: Rigorous results.* W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[22] D. Ruelle. Existence of a phase transition in a continuous classical system. *Phys. Rev. Lett.*, 27:1040–1041, Oct 1971.

[23] J. van den Berg and C. Maes. Disagreement percolation in the study of Markov fields. *Ann. Probab.*, 22(2):749–763, 1994.

[24] B. Widom and J.S Rowlinson. New model for the study of liquid-vapor phase transitions. *J. Chem. Phys.*, 52:1670–1684, 1970.