



**HAL**  
open science

# The complex WKB method for difference equations and Airy functions

Alexander Fedotov, Frédéric Klopp

► **To cite this version:**

Alexander Fedotov, Frédéric Klopp. The complex WKB method for difference equations and Airy functions. 2019. hal-01892639v4

**HAL Id: hal-01892639**

**<https://hal.science/hal-01892639v4>**

Preprint submitted on 15 May 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THE COMPLEX WKB METHOD FOR DIFFERENCE EQUATIONS AND AIRY FUNCTIONS

ALEXANDER FEDOTOV\* AND FRÉDÉRIC KLOPP†

**Abstract.** We consider the difference Schrödinger equation  $\psi(z+h) + \psi(z-h) + v(z)\psi(z) = 0$  where  $z$  is a complex variable,  $h > 0$  is a parameter, and  $v$  is an analytic function. As  $h \rightarrow 0$  analytic solutions to this equation have a simple WKB behavior near the points where  $v(z) \neq \pm 2$ . We study analytic solutions near the points  $z_0$  satisfying  $v(z_0) = \pm 2$  and  $v'(z_0) \neq 0$ . These points play the same role as simple turning points for the differential equation  $-\psi''(z) + v(z)\psi(z) = 0$ . In an  $h$ -independent complex neighborhood of such a point, we derive uniform asymptotic expansions for analytic solutions to the difference equation.

**Key words.** Difference equations, WKB, turning point

**AMS subject classifications.** 34M60, 39A45

## 1. Introduction.

**1.1. The problem.** We study analytic solutions to the difference Schrödinger equation

$$(1.1) \quad \psi(z+h) + \psi(z-h) + v(z)\psi(z) = 0$$

where  $z$  is a complex variable,  $h$  is a positive parameter and  $v$  is an analytic function. We describe their asymptotic behavior for small  $h$ <sup>1</sup>.

Note that the parameter  $h$  is a standard WKB or quasiclassical parameter. Indeed, formally,  $\psi(z+h) = \sum_{l=0}^{\infty} \frac{h^l}{l!} \frac{d^l \psi}{dz^l}(z) = e^{h \frac{d}{dz}} \psi(z)$ , and so,  $h$  can be regarded as a small parameter in front of the derivative.

It is worth comparing (1.1) with the equation

$$(1.2) \quad \phi_{k+1} + \phi_{k-1} + v(kh + \theta)\phi_k = 0,$$

where  $k \in \mathbb{Z}$  is an integer variable, and  $\theta \in \mathbb{C}$  is a parameter. If  $\psi$  satisfies (1.1), then one can construct a solution to (1.2) (with a suitable  $\theta$ ) by the formula  $\phi_k = \psi(kh + \theta)$ . If  $h$  is small, then in (1.2) the coefficient  $v(kh + \theta)$  slowly changes with  $k$ .

The semi-classical asymptotics of solutions to ordinary differential equations, e.g., the differential Schrödinger equation

$$(1.3) \quad -h^2 \frac{d^2 \psi}{dz^2}(z) + v(z)\psi(z) = 0$$

for small  $h$ , are described by means of the famous WKB (Wentzel, Kramers and Brillouin) method. There is a huge number of problems solved by this method. If the coefficients of the equation are analytic, one uses a powerful classical method often called the complex WKB method (see, e.g., chapter 3 in [8], chapter 10 in [20],

---

\*St. Petersburg State University, 7/9 Universitetskaya nab., St.Petersburg, 199034, Russia ([a.fedotov@spbu.ru](mailto:a.fedotov@spbu.ru)).

†Sorbonne Université, Université Paris Diderot, CNRS, Institut de Mathématiques Jussieu - Paris Rive Gauche, F-75005, Paris, France ([frederic.klopp@imj-prg.fr](mailto:frederic.klopp@imj-prg.fr)).

<sup>1</sup>The results were announced in the conference proceedings [12].

The work was supported by a PRC CNRS, France, and the Russian Foundation for Basic Research under the grant No 17-51-150008.

chapter 7 in [23]). It allows to study solutions to the equation on the complex plane. Even when the input problem does not require to go into the complex plane, the method is used to simplify the analysis: it allows to go around, say, singularities of solutions located on the real line, compute the Wronskians of solutions in domains where they are easily computable and so on. Moreover, the complex WKB method is very efficient for computing exponentially small quantities. One of the classical examples is the computation of the lengths of exponentially small gaps (lacunae) in the spectrum for the operator defined by the left hand side of (1.3) with analytic periodic  $v$  (see, e.g. section 9 of chapter 3 of [8]). In [8], the reader will find various interesting examples of problems solved by this method.

The asymptotic behavior of solutions to (1.3) changes crucially near turning points, i.e., the points where  $v(z) = 0$ . In the framework of the WKB method, the analysis of the solutions in a complex neighborhood of a turning point is one of the main topics (see, e.g., section 2 of chapter 4 in [8]).

A complex WKB method for difference equations has been initiated in [3, 14, 12]. In the present paper, we study analytic solutions to (1.1) near turning points.

Difference equations (1.1) on  $\mathbb{R}$  or on  $\mathbb{C}$  and (1.2) on  $\mathbb{Z}$  with a small  $h$  arise in quantum physics. One encounters such equations, for example, when studying in various asymptotic situations an electron in a two-dimensional crystal submitted to a constant magnetic field (see, e.g., introduction sections in [24, 16] and references therein). The electron is described by a magnetic Schrödinger operator with a periodic electric potential, and, for example, in the semi-classical limit, its analysis reduces to analyzing an  $h$ -pseudo-differential operator with the symbol  $H(x, p) = 2 \cos p + 2 \cos x$  (see [17], in particular, section 9.4 and page 106). Its eigenfunctions satisfy (1.1) with  $v(z) = 2 \cos z - E$  (here,  $E$  is the spectral parameter). The parameter  $h$  is proportional to the magnetic flux through the periodicity cell, and the case when  $h$  is small is a natural one. The other example is the case of a strong magnetic field. In this case the input equation can again be reduced to equations of the form (1.1); the parameter  $h$  is then proportional to the inverse of the magnetic flux and so is small (see, e.g., [6]). The reader can find more examples and references in [24, 16]. We add only that, for  $v(z) = \cos(2\pi z) - E$ , (1.2) is the famous almost Mathieu equation (see, e.g., [1]).

We note that, though in the present paper we concentrate on the analysis of (1.3), our method applies equally well to general second order linear difference equations or, equivalently, to the equations of the form

$$(1.4) \quad \phi(z+h) = M(z)\phi(z), \quad z \in U \subset \mathbb{C}, \quad \text{with } M(z) \in SL(2, \mathbb{C}).$$

Such equations with analytic or meromorphic coefficients arise in many fields of mathematics and physics. In particular, they are standard in the study of the diffraction of classical waves by wedges (see, e.g., equation (2.1.2) in [18] or (10.13) in [2]) or in the theory of quasiperiodic equations (see, e.g., sections 4.2, 4.4 in [10], and the review [9] where, when studying various types of quasiperiodic equations, one systematically encounters difference equations with complex coefficients on  $\mathbb{R}$  and/or  $\mathbb{C}$ , see, e.g., sections 2, 3.1.4, 4.1). A small shift parameter arises in the problems of diffraction by thin wedges (as, in these problems, the shift parameters are proportional to the angles of the wedges (see [18])). It can also appear in the case of quasiperiodic equations, e.g., in the case of one-dimensional quasiperiodic differential Schrödinger equations with two periods: in this case, one has to study a difference equation with complex coefficients where the shift parameter is proportional to the fractional part of the periods ratio and, thus, can be small (see [10]). We note that semiclassical

constructions are used to study the asymptotics of orthogonal polynomials as well as of formal orthogonal polynomials (see [subsection 2.4](#)).

When studying (1.2) on  $\mathbb{Z}$ , it is often very natural to pass to (1.1) on  $\mathbb{R}$  or on  $\mathbb{C}$  as, for this equation, one can use various analytic tools coming from the study of differential equations such as tools of the theory of pseudo-differential operators ([17]) or of the complex WKB method as in the present paper. If the coefficient  $v$  is periodic, for (1.1) one can use ideas of the Floquet theory for differential equations with periodic coefficients; this leads to a natural renormalization method (see review [9]).

In [24] heuristically, and in [17] rigorously, in the semiclassical limit, the authors describe the geometric structure of spectrum of the operator defined in  $L_2(\mathbb{R})$  by the left-hand side of (1.1) with  $v(z) = \cos z$ . For irrational  $h$ , it coincides with the spectrum of the almost Mathieu operator and is a Cantor set. The authors obtain a description similar to the well-known inductive construction of the classical Cantor set.

To study the geometric properties of this spectrum, Buslaev and Fedotov introduced a renormalization approach based on ideas of the Floquet theory (see [9]). A crucial role in their analysis is played by the minimal entire solutions to (1.1) with  $v(z) = \cos z - E$ ,  $E$  being a spectral parameter, and to kindred equations with complex coefficients (minimal entire solutions are the entire solutions having minimal possible growth as  $|\operatorname{Im} z| \rightarrow \infty$ , see, section 2.3.2 in [9]). To study these solutions in the semiclassical limit, Buslaev and Fedotov developed a complex WKB method for difference equations (see [3, 4, 9]). While the analysis of the geometry of the spectrum does not require detailed asymptotics of the entire solutions along the real line, these are crucial to describe the eigenfunctions in the semiclassical limit. This provides a natural applications of the results developed in the present paper.

The present paper is devoted to uniform asymptotic formulas describing analytic solutions to (1.1) in  $h$ -independent complex neighborhoods of simple turning points (for the definitions, see [subsection 2.1](#)). To the best of our knowledge, for difference equations, this result is new and original. Its analogue for differential equations is well-known and derived in a completely different way: this is a consequence of the non-locality of (1.1).

There are many papers devoted to the asymptotic analysis of solutions to equations similar to (1.2) on  $\mathbb{Z}$  with small  $h$ . The reader will find a short discussion of their results in [subsection 2.4](#), i.e., after formulation of our results.

The question of the asymptotics of solutions to a difference equation in a complex neighborhood of a turning point is very natural. The techniques developed to get and justify these asymptotics are the main analytic innovation of the paper.

## 1.2. Notations and agreements.

Below  $v$  is analytic on a disk  $U \subset \mathbb{C}$ . A neighborhood is a  $\delta$ -neighborhood.

Instead of saying that an asymptotic representation is valid for sufficiently small  $h$ , we write that it is valid as  $h \rightarrow 0$ .

The letter  $C$  denotes various positive constants independent of  $z$  and  $h$ . For two functions  $f$  and  $g$  defined on a domain  $D \subset \mathbb{C}$ , we write that  $g(z) = O(f(z))$  in  $D$  if  $|g(z)| \leq C|f(z)|$  for all  $z \in D$ .

**2. The main results.** First, we define few objects needed to formulate our results.

### 2.1. The main analytic objects.

**2.1.1. The complex momentum.** One of the main analytic objects of the WKB methods is the *complex momentum*  $p$ . For (1.1) it is defined by the formula

$$(2.1) \quad 2 \cos p + v(z) = 0.$$

It is a multivalued analytic function on  $U$ . At its branching points one has  $v(z) \in \{\pm 2\}$ .

In analogy with the glossary of the complex WKB method for differential equations, the points  $z$  where  $v(z) \in \{\pm 2\}$  are called *turning points*. A set  $D \subset U$  is *regular* if  $v(z) \neq \pm 2$  in  $D$ .

*Remark 2.1.* In the complex WKB theory for difference equations, see [3, 13, 14], one proves the existence of analytic solutions  $\psi_{\pm}$  such that

$$(2.2) \quad \psi_{\pm}(z) = \frac{1}{\sqrt{\sin(p(z))}} e^{\pm \frac{i}{h} \int_{z_0}^z p(z) dz + o(1)}, \quad h \rightarrow 0.$$

For  $z$ , a turning point, one has  $\cos p(z) = -v(z)/2 \in \{\pm 1\}$ , and  $\sin p(z) = 0$ . Thus, near a turning point, the representation (2.2) cannot be valid.

Let  $z_0$  be a turning point. If  $v'(z_0) \neq 0$ , we call this turning point *simple*. We note that, in this case, the complex momentum is analytic in  $\tau = \sqrt{z - z_0}$  in a neighborhood of  $z_0$  and

$$(2.3) \quad p(z) = p(z_0) + p_1 \tau + O(\tau^2), \quad \tau \rightarrow 0,$$

where  $p_1 \neq 0$  is a constant.

Let  $z_0$  be the center of the disk  $U$ . From now on, we assume that the following hypotheses hold:

*Hypothesis 2.2.*  $v(z) \notin \{\pm 2\}$  for  $z \in U \setminus \{z_0\}$ .

This means that the domain  $U \setminus \{z_0\}$  is regular.

*Hypothesis 2.3.*  $v(z_0) = 2$ , and  $v'(z_0) \neq 0$ .

So,  $z_0$  is a simple turning point. The condition  $v(z_0) = 2$  implies that  $p(z_0) = 0 \pmod{2\pi}$ .

The case where  $v(z_0) = -2$  reduces to the case we study. Indeed, let  $v(z_0) = -2$ . For  $\psi$ , a solution to (1.1), we set  $\phi(z) = e^{i\pi z/h} \psi(z)$ . Then,  $\phi$  satisfies equation  $\phi(z+h) + \phi(z-h) - v(z)\phi(z) = 0$ ,  $z_0$  is again a turning point, and the coefficient in the front of  $\phi(z)$  equals 2 at  $z_0$ .

**2.1.2. Objects related to the complex momentum.** Cut  $U$  from  $z_0$  to a point of its boundary along a simple curve and denote the thus obtained domain by  $U'$ . In  $U'$ , we fix an analytic branch  $p$  of the complex momentum so that  $p(z_0) = 0$  (this is possible as at  $z_0$  the complex momentum vanishes modulo  $2\pi$  and as for  $p$ , a branch of the complex momentum,  $p+2\pi$  is also a branch of the complex momentum). In  $U'$  we consider the function

$$(2.4) \quad z \mapsto \left( \frac{3}{2i} \int_{z_0}^z p(z) dz \right)^{\frac{2}{3}}.$$

In view of (2.3), there are three branches of this function that are analytic in  $U$  near  $z_0$ . We denote by  $\zeta$  one of them.

One has  $\zeta(z_0) = 0$ , and  $\zeta'(z_0) \neq 0$ .

Possibly reducing  $U$ , we can and do assume that

*Hypothesis 2.4.* The function  $\zeta$  is a bi-analytic bijection of  $U$  onto its image.

*Remark 2.5.* The definition of  $\zeta$  implies that it satisfies one of the two equations

$$(2.5) \quad \sqrt{\zeta(z)}\zeta'(z) = \pm ip(z), \quad z \in U.$$

We let

$$(2.6) \quad g(z) := \sinh\left(\sqrt{\zeta(z)}\zeta'(z)\right) / \sqrt{\zeta(z)}, \quad z \in U,$$

where the determination of the square roots in the denominator and the numerator are the same (the definition of  $g$  is independent of its choice). The function  $g$  is analytic in  $U$  and does not vanish there, see [section 3.3.2](#). In  $U$ , we further define an analytic function

$$(2.7) \quad A_0(z) := 1 / \sqrt{g(z)}.$$

**2.2. Asymptotic solutions.** First, let us describe asymptotic solutions to equation (1.1). Therefore, we recall that Airy functions are solutions to the equation  $y''(z) = y(z)$ ,  $z \in \mathbb{C}$  (see, e.g., [19], section 2.8). Presumably, the most famous Airy function is  $\text{Ai}$ . The reader can find its asymptotics at infinity in [subsection 3.1](#). We shall use three other standard Airy functions (see, e.g., [25]) defined by the formulas

$$(2.8) \quad w_j(\zeta) = 2\pi i \omega^j \text{Ai}(\omega^j \zeta), \quad \omega = e^{2\pi i/3}, \quad \zeta \in \mathbb{C}.$$

Any two of them are linearly independent.

For a function  $f$  defined on  $U$  and  $\{z-h, z, z+h\} \subset U$ , we set

$$(2.9) \quad [H(f)](z) := f(z+h) + f(z-h) + v(z)f(z).$$

One has

**THEOREM 2.6.** *There exist functions  $(A_{2l})_{l \in \mathbb{N} \cup \{0\}}$  and  $(B_{2l+1})_{l \in \mathbb{N} \cup \{0\}}$  ( $A_0$  being defined by (2.7)) that are analytic on  $U$  and such that, for any  $L \in \mathbb{N} \cup \{0\}$ , the following holds. Let  $w$  be one of the Airy functions  $(w_j)_{j \in \mathbb{Z}_3}$  and set*

$$(2.10) \quad w_h(z) = w\left(\zeta(z)/h^{\frac{2}{3}}\right), \quad w'_h(z) = w'\left(\zeta(z)/h^{\frac{2}{3}}\right),$$

and

$$(2.11) \quad W(z) = h^{\frac{1}{3}} w_h(z) \sum_{l=0}^L h^{2l} A_{2l}(z) + h^{\frac{2}{3}} w'_h(z) \sum_{l=0}^{L-1} h^{2l+1} B_{2l+1}(z),$$

where the second sum vanishes if  $L = 0$ . Then, in any given compact set  $K \subset U$ , as  $h \rightarrow 0$ , we have

$$(2.12) \quad H(W) = O\left(h^{2L+2+\frac{1}{3}} w_h\right) + O\left(h^{2L+3+\frac{2}{3}} w'_h\right).$$

The formal expression

$$(2.13) \quad h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{\infty} h^{2l} A_{2l}(z) + h^{\frac{2}{3}} w'_h(z) \sum_{l=0}^{\infty} h^{2l+1} B_{2l+1}(z)$$

is called an *asymptotic solution* to (1.1).

Theorem 2.6 is proved in section 4; in its proof, we explicitly describe the computation of the coefficients  $(A_l)_l$  and  $(B_l)_l$ . The computations being rather knotty, we describe the algorithm in Remark 4.3, Remark 4.4 and Remark 4.5. Let us underline that the complexity of these computations is a consequence of the non-locality of the difference equation (1.1).

Let us comment on the results of Theorem 2.6. First, we note that, for the differential equation  $-h^2\psi''(z) + v(z)\psi(z) = 0$ , in a neighborhood of a simple turning point (a point where  $v(z) = 0$  and  $v'(z) \neq 0$ ), there are asymptotic solutions of the form (2.13) (with different coefficients  $(A_l)_l$ ,  $(B_l)_l$  and a different function  $\zeta$ ).

To justify the Ansatz (2.13) for the difference equation, one has to derive asymptotic formulas of the form

$$(2.14) \quad w_h(z \pm h) = f(z) w_h(z) \pm h^{\frac{1}{3}} g(z) w'_h(z) + \dots,$$

where  $f(z) = \cosh(\sqrt{\zeta(z)}\zeta'(z))$  and the dots denote smaller order terms. If one tries to prove this formula using Taylor expansions for the left hand side, one has to handle an infinite number of infinite subsequences of terms of the same order. So, an effective resummation of these sequences is required. As we shall show in this paper, to derive formulas analogous to (2.14), instead of resummation of Taylor series, it is very natural to use standard tools from complex analysis and, in particular, from the theory of integrals with two coalescing saddle points (see, e.g., [25]).

Formula (2.14) imply that

$$(2.15) \quad H(w_h)(z) = \left(2 \cosh(\sqrt{\zeta(z)}\zeta'(z)) + v(z)\right) w_h(z) + \dots$$

In view of (2.1) and (2.5), the leading term in the right-hand side of (2.15) vanishes. Finally, we note that if  $h^{-\frac{2}{3}}|\zeta(z)|$  is large, then  $w_h(z)$  and  $w'_h$  in (2.11) can be replaced by their asymptotic representations, see formula (3.2) in subsection 3.1 or [19] for more details. As a result, the leading terms in (2.11) turn into a linear combination of the leading terms from (2.2).

### 2.3. Solutions with standard asymptotic behavior.

Our main result is

**THEOREM 2.7.** *There exists  $\overset{\circ}{U} \subset U$ , a neighborhood of  $z_0$  and  $h_0 > 0$  such that, for any  $L \in \mathbb{N} \cup \{0\}$  and  $W$ , any of the functions constructed in Theorem 2.6 for the order  $L$ , for  $h \in (0, h_0]$ , there exists  $\psi : \overset{\circ}{U} \rightarrow \mathbb{C}$ , a solution to (1.1) that is analytic and admits the asymptotic representation*

$$(2.16) \quad \psi(z) = W(z) + O(w_h h^{2L+2+\frac{1}{3}}) + O(w'_h h^{2L+1+\frac{2}{3}})$$

where  $w_h$  and  $w'_h$  are defined in (2.10).

Theorem 2.7 is proved in section 6 and section 7.

Let us briefly explain the ideas of the proof of Theorem 2.7. First, in section 6, using the approximate solutions constructed in Theorem 2.6, we construct a parametrix  $R$ , i.e., an operator such that, for suitable functions  $f$ , one has  $HRf = f + Df$ , where  $H$  is defined in (2.9) and  $D$  is a small operator. The operator  $D$  is a singular integral operator. We estimate its norm using natural geometric objects of the complex WKB method. This allows us to prove Theorem 2.7 on some special subdomains of  $U$ . In section 7, we study the thus constructed solutions on larger domains and complete the proof of Theorem 2.7.

**2.4. Related results.** The WKB asymptotics of solutions of difference equations on  $\mathbb{Z}$  with “slowly varying” coefficients have been studied since the end of the 1960’s. In [22] and [21], the authors essentially studied equations of the form

$$(2.17) \quad Y_{k+1} = M(hk)Y_k, \quad k \in \mathbb{Z};$$

where  $h$  is small and positive and  $M$  is an  $(n \times n)$ -matrix valued function defined on  $\mathbb{R}$ . If

$$(2.18) \quad Y(x+h) = M(x)Y(x), \quad x \in \mathbb{R},$$

then, the sequence  $Y_k = Y(kh)$ ,  $k \in \mathbb{Z}$ , satisfies (2.17). We note that (1.1) restricted to  $\mathbb{R}$  is equivalent to (2.18) with  $M(x) = \begin{pmatrix} -v(x) & -1 \\ 1 & 0 \end{pmatrix}$ , and that a turning point for (1.1) is a point  $x$  where the eigenvalues of the matrix  $M(x)$  coincide.

The short note [22] is essentially devoted to the case where all the eigenvalues of the matrix  $M$  in (2.17) are distinct. In [21] the author constructed asymptotic solutions to (2.17) in a small  $h$ -dependent neighborhood of a point  $x$  where two eigenvalues of  $M(x)$  coincide.

In [5], the authors considered difference equations of the form

$$(2.19) \quad \sum_{j=I}^J a_j(hk, h) y_{k+j} = 0, \quad k \in \mathbb{Z}.$$

We note that this class includes the difference Schrödinger equations

$$y_{k+1} + y_{k-1} + v(hk)y_k = 0, \quad k \in \mathbb{Z}.$$

In the limit  $h \rightarrow 0$ , [5] gives a description of the asymptotics of solutions to (2.19) for  $hk$  in a small  $h$ -dependent neighborhood of a point where  $v(x) \in \{\pm 2\}$ .

Let us also mention three (series of) papers motivated by problems originating in the theory of orthogonal polynomials.

First, there is a series of papers by J.S. Geronimo and co-authors (see [15] and references therein) devoted to the asymptotics of solutions to the equation

$$a_{k+1}\psi_{k+1} + b_k\psi_k + a_k\psi_{k-1} = \lambda\psi_k, \quad k \in \mathbb{Z},$$

where  $\lambda$  is the spectral parameter, and the coefficients  $a_k$  are positive and  $b_k$  are real and slowly depend on  $k$ . The authors describe asymptotic formulas uniform with respect to  $kh$  in an  $h$ -independent neighborhood of a turning point.

Next, R.Wong and coauthors devoted various works (see e.g., [26]) to the study of solutions to three terms recurrence relations with real coefficients for large values of the integer variable.

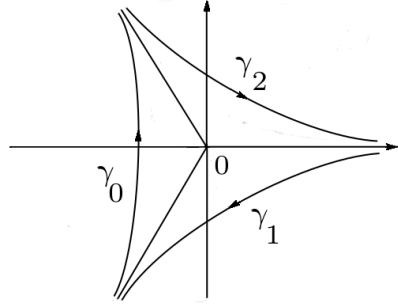
Finally, we mention [7] where the authors studied WKB asymptotics of solutions to a difference equation using the Maslov canonical operator.

There are more papers devoted to the subject. The reader can find more references in the papers that we mentioned above.

To the best of our knowledge, the present paper is the first where uniform asymptotics of analytic solutions to a difference equation on  $\mathbb{C}$  in an  $h$ -independent neighborhood of a turning point are mathematically derived.

As we pointed out earlier, our method applies equally well to (1.4). Recalling the relation between solutions of difference equations on  $\mathbb{R}$  or  $\mathbb{C}$  and on  $\mathbb{Z}$  (see our comments



FIG. 1. *Integration paths*

to equation (1.2)) yields all the above results for second order difference equations on  $\mathbb{Z}$ . Moreover, it yields the asymptotics of solutions to such equations with complex coefficients. Such a result was not known and looks non-trivial for the difference equations studied only on  $\mathbb{Z}$ . In general the methods developed for the equations on  $\mathbb{Z}$  do not yield results on analytic solutions on  $\mathbb{C}$  at least without a non-trivial analysis.

### 3. Preliminaries.

**3.1. Three Airy functions.** Let  $(\gamma_j)_{j \in \mathbb{Z}_3}$  be the curves shown in Figure 1;  $\gamma_0$  is asymptotic to the half-lines  $e^{\pm 2i\pi/3}\mathbb{R}_+$ ,  $\mathbb{R}_+ = [0, +\infty) \subset \mathbb{R}$ ; for  $j \in \mathbb{Z}_3$ , rotating  $\gamma_0$  around  $z_0$  by  $2j\pi/3$ , one obtains  $\gamma_j$ . The functions defined by the formulas

$$(3.1) \quad w_j(\zeta) = \int_{\gamma_j} e^{-\left(\frac{s^3}{3} - \zeta s\right)} ds, \quad j \in \mathbb{Z}_3, \quad \zeta \in \mathbb{C},$$

are three Airy functions related to the standard Airy function  $\text{Ai}$  by (2.8) (see, e.g., [25]). Assume that  $|\arg z| < 2\pi/3$ . As  $|z| \rightarrow \infty$  one has

$$(3.2) \quad \text{Ai}(z) = \frac{\exp\left(-\frac{2}{3}z^{3/2} + o(1)\right)}{2\sqrt{\pi} z^{1/4}}, \quad \text{Ai}(-z) = \frac{\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4} + o(1)\right)}{\sqrt{\pi} z^{1/4}} (1 + o(1))$$

where we use the analytic branches of  $z \rightarrow z^{3/2}$  and  $z \rightarrow z^{1/4}$  that are positive for  $z > 0$  (see [19], pp. 116, 118 and 392).

**3.2. The space of solutions to (1.1).** The observations that we discuss now are well-known in the theory of difference equations and are easily proved.

Let  $c \in \mathbb{R}$  and  $I = \{z \in U : \text{Im}z = c\}$ . We assume that the length of  $I$  is sufficiently large (with respect to  $h$ ) and discuss the set  $S$  of solutions to (1.1) on  $I$ .

Let  $\{f, g\} \subset S$ . The expression

$$(3.3) \quad (f(z), g(z)) = f(z+h)g(z) - f(z)g(z+h), \quad \{z, z+h\} \subset I,$$

is called *the Wronskian* of  $f$  and  $g$ . It is  $h$ -periodic in  $z$ .

If the Wronskian of two solutions does not vanish, they form a basis in  $S$ , i.e.,  $\psi \in S$  if and only if

$$(3.4) \quad \psi(z) = a(z)f(z) + b(z)g(z), \quad \{z, z+h\} \subset I$$

where  $a$  and  $b$  are  $h$ -periodic complex coefficients. One has

$$(3.5) \quad a(z) = \frac{(\psi(z), g(z))}{(f(z), g(z))} \quad \text{and} \quad b(z) = \frac{(f(z), \psi(z))}{(f(z), g(z))}.$$

### 3.3. Remarks on the complex momentum and the function $g$ .

**3.3.1. Analytic branches of the complex momentum.** Let  $p_0$  be a branch of the complex momentum analytic in a regular simply connected domain  $D$ . Equation (2.1) implies that an analytic function  $\tilde{p} : D \rightarrow \mathbb{C}$  is a branch of the complex momentum if and only if there exists  $s \in \{\pm 1\}$  and  $n \in \mathbb{Z}$  such that

$$(3.6) \quad \tilde{p}(z) = sp_0(z) + 2\pi n, \quad \forall z \in D.$$

**3.3.2. Non-vanishing of the function  $g$ .** Here, we prove that the function  $g$  defined by (2.6) does not vanish in  $U$ .

As  $\zeta(z_0) = 0$  and  $\zeta'(z_0) \neq 0$ ,  $g$  does not vanish at  $z_0$ . So, in view of (2.5), it suffices to check that  $\sin p(z)$  does not vanish in  $U \setminus \{z_0\}$ . Therefore, we note that if  $\sin p(z) = 0$ , then  $v(z) = -2 \cos p(z) \in \{\pm 2\}$ , i.e.,  $z$  is a turning point. But, in  $U$  there is only one turning point, and it is  $z_0$ . This implies the needed.

## 4. Asymptotic solutions: the proof of Theorem 2.6.

**4.1. The proof of Theorem 2.6 up to two propositions.** We begin with two auxiliary statements. Below, we use the notations introduced in (2.10), and  $K \subset U$  is a compact set.

PROPOSITION 4.1. *Let  $A$  be analytic in  $U$  and pick  $L \in \mathbb{N}$ . Then, in  $K$ , one has*

$$(4.1) \quad \begin{aligned} H \left( A h^{\frac{1}{3}} w_h \right) &= h^{\frac{1}{3}} w_h \left( \sum_{l=1}^L h^{2l} a_{2l}(A; z) + O(h^{2L+2}) \right) \\ &\quad + h^{\frac{2}{3}} w'_h \left( \sum_{l=1}^L h^{2l-1} b_{2l-1}(A; z) + O(h^{2L+1}) \right) \end{aligned}$$

as  $h \rightarrow 0$ . All the coefficients  $(a_{2l})_{l \geq 1}$  and  $(b_{2l-1})_{l \geq 1}$  are analytic in  $z \in U$ , independent of  $h$  and of the choice of  $w$  in (2.10), and

$$(4.2) \quad b_1 = Ag \frac{d}{dz} \log(A^2 g).$$

PROPOSITION 4.2. *Let  $B$  be analytic in  $U$  and pick  $L \in \mathbb{N}$ . Then, in  $K$ , one has*

$$(4.3) \quad \begin{aligned} H \left( B h^{\frac{2}{3}} w'_h \right) &= h^{\frac{1}{3}} w_h \left( \sum_{l=1}^L h^{2l-1} c_{2l-1}(B; z) + O(h^{2L+1}) \right) \\ &\quad + h^{\frac{2}{3}} w'_h \left( \sum_{l=1}^L h^{2l} d_{2l}(B; z) + O(h^{2L+2}) \right) \end{aligned}$$

as  $h \rightarrow 0$ . All the coefficients  $(c_{2l-1})_{l \geq 1}$  and  $(d_{2l})_{l \geq 1}$  are analytic in  $U$ , independent of  $h$  and of the choice of  $w$  in (2.10), and

$$(4.4) \quad c_1 = \zeta Bg \frac{d}{dz} \log(\zeta B^2 g).$$

Before proving Propositions 4.1 and 4.2, we use them to prove Theorem 2.6.

Clearly, it suffices to consider  $L \geq 2$  which we do. Assume that  $A_{2l}$  and  $B_{2l+1}$ ,  $l = 0, 1, 2, \dots$  are analytic in  $U$  and pick  $z \in K$ . Define  $W$  by (2.11). Using Proposition 4.2

for  $L$  and [Proposition 4.1](#) for  $L + 1$  instead of  $L$ , for sufficiently small  $h$ , we compute (below, the dependence on  $z$  is omitted)

$$\begin{aligned} HW &= h^{\frac{1}{3}} w_h \mathcal{A} + h^{\frac{2}{3}} w'_h \mathcal{B}, \\ \mathcal{A} &= \sum_{l=0}^L \sum_{k=1}^{L+1} h^{2(l+k)} a_{2k}(A_{2l}) + \sum_{l=0}^{L-1} \sum_{k=1}^L h^{2(l+k)} c_{2k-1}(B_{2l+1}) + O(h^{2(L+1)}), \\ \mathcal{B} &= \sum_{l=0}^L \sum_{k=1}^{L+1} h^{2(l+k)-1} b_{2k-1}(A_{2l}) + \sum_{l=0}^{L-1} \sum_{k=1}^L h^{2(l+k)+1} d_{2k}(B_{2l+1}) + O(h^{2L+3}). \end{aligned}$$

Consider the expression for  $\mathcal{A}$ . Changing the summation index  $l \rightarrow l - 1$ , we compute

$$\begin{aligned} (4.5) \quad \mathcal{A} &= \sum_{\substack{k+l-1 \leq L \\ k, l \geq 1}} h^{2(l+k-1)} (a_{2k}(A_{2(l-1)}) + c_{2k-1}(B_{2l-1})) + O(h^{2(L+1)}) \\ &= \sum_{m=1}^L h^{2m} \sum_{l=1}^m (a_{2(m-l+1)}(A_{2(l-1)}) + c_{2(m-l+1)-1}(B_{2l-1})) + O(h^{2(L+1)}). \end{aligned}$$

Let us turn to  $\mathcal{B}$ . Writing separately the term with  $l = 0$  from the first sum over  $l$ , changing the summation index  $l \rightarrow l - 1$  in the second one, we compute

$$\begin{aligned} (4.6) \quad \mathcal{B} &= \sum_{k=1}^{L+1} h^{2k-1} b_{2k-1}(A_0) + \sum_{\substack{k+l \leq L+1 \\ k, l \geq 1}} h^{2(l+k)-1} (b_{2k-1}(A_{2l}) + d_{2k}(B_{2l-1})) + O(h^{2L+3}) \\ &= hb_1(A_0) + \sum_{m=2}^{L+1} h^{2m-1} \left( b_{2m-1}(A_0) + \sum_{l=1}^{m-1} (b_{2(m-l)-1}(A_{2l}) + d_{2(m-l)}(B_{2l-1})) \right) + O(h^{2L+3}). \end{aligned}$$

Let  $A_0 = 1/\sqrt{g}$ . By [\(4.2\)](#), one has  $b_1(A_0) = 0$ . Therefore, in view of [\(4.5\)](#) and [\(4.6\)](#), to prove [Theorem 2.6](#), one has to choose  $(A_{2l})_{l \geq 1}$  and  $(B_{2l-1})_{l \geq 1}$  so that

$$\begin{aligned} \sum_{l=1}^m (a_{2(m-l+1)}(A_{2(l-1)}) + c_{2(m-l+1)-1}(B_{2l-1})) &= 0, \quad \forall m \geq 1, \\ b_{2m-1}(A_0) + \sum_{l=1}^{m-1} (b_{2(m-l)-1}(A_{2l}) + d_{2(m-l)}(B_{2l-1})) &= 0, \quad \forall m \geq 2. \end{aligned}$$

Therefore, for  $m = 1, 2, 3, \dots$ , it suffices to inductively solve the equations

$$(4.7) \quad c_1(B_{2m-1}) = -L_m, \quad b_1(A_{2m}) = -M_m.$$

where

(4.8)

$$\begin{aligned} L_m &= L_m(A_0, A_2, \dots, A_{2m-2}, B_1, B_3, \dots, B_{2m-3}) \\ &= a_2(A_{2(m-1)}) + \sum_{l=1}^{m-1} (a_{2(m-l+1)}(A_{2(l-1)}) + c_{2(m-l+1)-1}(B_{2l-1})), \end{aligned}$$

(4.9)

$$\begin{aligned} M_m &= M_m(A_0, A_2, \dots, A_{2(m-1)}, B_1, B_2, \dots, B_{2m-1}) \\ &= b_{2m+1}(A_0) + \sum_{l=1}^{m-1} (b_{2(m-l)+1}(A_{2l}) + d_{2(m-l+1)}(B_{2l-1})) + d_2(B_{2m-1}), \end{aligned}$$

and the sums  $\sum_{l=1}^{m-1}(\dots)$  in (4.8) and (4.9) are equal to 0 if  $m = 1$ . In view of (4.2) and (4.4), equations (4.7) are equivalent to the equations

$$(4.10) \quad \zeta B_{2m-1} g \frac{d}{dz} \log(\zeta B_{2m-1}^2 g) = -L_m, \quad \text{and} \quad A_{2m} g \frac{d}{dz} \log(A_{2m}^2 g) = -M_m.$$

One constructs solutions to these equations by the formulas

(4.11)

$$A_{2m}(z) = -\frac{1}{2\sqrt{g(z)}} \int_{z_0}^z \frac{M_m dz}{\sqrt{g}}, \quad \text{and} \quad B_{2m-1}(z) = -\frac{1}{2\sqrt{\zeta(z)g(z)}} \int_{z_0}^z \frac{L_m dz}{\sqrt{\zeta g}}.$$

As  $g$  and  $\zeta$  are analytic in  $U$ ,  $g$  does not vanish in  $U$ , and  $\zeta$  vanishes in  $U$  only at  $z_0$  where it has a simple zero, the coefficients  $A_{2m}$  and  $B_{2m-1}$  are analytic in  $U$ .

The proof of [Theorem 2.6](#) is complete.

*Remark 4.3.* As is seen from the above proof, for  $m \geq 1$ , the coefficients  $A_{2m}$  and  $B_{2m-1}$  are constructed inductively by formulas (4.11) where  $L_m$  and  $M_m$  are given by (4.8) and (4.9) and, for  $l \geq 1$ ,  $a_{2l}$ ,  $b_{2l-1}$ ,  $c_{2l-1}$  and  $d_{2l}$  are defined in [Proposition 4.1](#) and [Proposition 4.2](#). The algorithms for their computation are described in [Remark 4.4](#) and [Remark 4.5](#).

**4.2. The proof of [Proposition 4.1](#).** Consider  $(w_j)_{j \in \mathbb{Z}_3}$ , the three Airy functions defined by (3.1). Let  $w$  be  $w_j$  for some  $j \in \mathbb{Z}_3$  and let  $\gamma$  be the corresponding integration path  $\gamma_j$  in (3.1). We note that

$$(4.12) \quad h^{\frac{1}{3}} w(h^{-\frac{2}{3}} \zeta) = \int_{\gamma} e^{-\frac{1}{h} \left( \frac{t^3}{3} - t\zeta \right)} dt, \quad h^{\frac{2}{3}} w'(h^{-\frac{2}{3}} \zeta) = \int_{\gamma} e^{-\frac{1}{h} \left( \frac{t^3}{3} - t\zeta \right)} t dt.$$

Below, we use the notations from (2.10). Let  $K \subset U$  be a closed disk centered at  $z_0$  and independent of  $h$ . Below, we assume that  $z \in K$  and that  $h > 0$  is sufficiently small. The proof of [Proposition 4.1](#) is broken into several steps.

1. In view of (4.12), we get

$$(4.13) \quad H \left( Ah^{\frac{1}{3}} w \right) = \int_{\gamma} e^{-\frac{1}{h} \left( \frac{t^3}{3} - t\zeta(z) \right)} F_0(t, z, h) dt,$$

$$(4.14) \quad F_0(t, z, h) = A(z+h) e^{\frac{t}{h}(\zeta(z+h) - \zeta(z))} + A(z-h) e^{\frac{t}{h}(\zeta(z-h) - \zeta(z))} + v(z)A(z).$$

We note that the function  $(t, z, h) \mapsto F_0(t, z, h)$  is analytic in  $\mathbb{C} \times K \times V$  provided that  $V$  is a sufficiently small neighborhood of 0.

2. Let  $N \in \mathbb{N}$ . By means of Taylor's theorem, we get

(4.15)

$$H(Ah^{\frac{1}{3}}w) = \sum_{n=0}^N h^n \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} F_{0,n}(t, z) dt + h^{N+1} \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} O(e^{C|t|}) dt,$$

$$(4.16) \quad F_{0,n}(t, z) = \frac{1}{n!} \frac{\partial^n F_0}{dh^n}(t, z, 0).$$

3. To get the asymptotics of the first  $N + 1$  integrals in (4.15), we apply Bleistein's method developed to study integrals with two coalescing saddle points (see, e.g., section 4 of chapter VII of [25]). We fix  $n \in \mathbb{N} \cup \{0\}$  and for  $m = 0, 1, 2, \dots$ , we inductively define

$$(4.17) \quad a_{m,n}(z) = \frac{1}{2} \left( F_{m,n}(\sqrt{\zeta(z)}, z) + F_{m,n}(-\sqrt{\zeta(z)}, z) \right),$$

$$(4.18) \quad b_{m,n}(z) = \frac{1}{2\sqrt{\zeta(z)}} \left( F_{m,n}(\sqrt{\zeta(z)}, z) - F_{m,n}(-\sqrt{\zeta(z)}, z) \right),$$

$$(4.19) \quad F_{m+1,n}(t, z) = \frac{\partial f_{m,n}}{\partial t}(t, z), \quad f_{m,n}(t, z) = \frac{F_{m,n}(t, z) - a_{m,n}(z) - b_{m,n}(z)t}{t^2 - \zeta(z)},$$

where we use one and the same branch of  $\sqrt{\zeta(z)}$ .

The functions  $(F_{m,n})_{m,n}$  are analytic in  $(t, z) \in \mathbb{C} \times K$ , and  $a_{m,n}$  and  $b_{m,n}$  are analytic in  $z \in K$  (we remove the removable singularities).

4. Note that  $F_0(-t, z, -h) = F_0(t, z, h)$ . Therefore,  $F_{0,n}(-t, z) = (-1)^n F_{0,n}(t, z)$  for  $n \geq 0$ . By induction on  $m \geq 0$ , one easily checks that

$$(4.20) \quad F_{m,n}(-t, z) = (-1)^{m+n} F_{m,n}(t, z),$$

$$(4.21) \quad a_{m,n} = 0 \text{ if } m+n \text{ is odd, and } b_{m,n} = 0 \text{ if } m+n \text{ is even.}$$

5. Let  $(m, n) \in (\mathbb{N} \cup \{0\})^2$ . In view of (4.19),  $F_{m,n}(t, z) = a_{m,n}(z) + b_{m,n}(z)t + (t^2 - \zeta(z))f_{m,n}(t, z)$ . Using this formula, representations (4.12) and integrating by parts, we get

(4.22)

$$\begin{aligned} & \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} F_{0,n}(t, z) dt \\ &= h^{\frac{1}{3}} w_h a_{0,n} + h^{\frac{2}{3}} w'_h b_{0,n} + \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} (t^2 - \zeta(z)) f_{0,n}(t, z) dt \\ &= h^{\frac{1}{3}} w_h a_{0,n} + h^{\frac{2}{3}} w'_h b_{0,n} + h \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} F_{1,n}(t, z) dt \\ & \quad \dots \dots \dots \\ &= h^{\frac{1}{3}} w_h \sum_{m=0}^N h^m a_{m,n} + h^{\frac{2}{3}} w'_h \sum_{m=0}^N h^m b_{m,n} + h^{N+1} \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} F_{N+1,n}(t, z) dt. \end{aligned}$$

6. To estimate the integral in the last line of (4.22), we show that

$$(4.23) \quad F_{m,n}(t, z) = O\left(e^{C|t|}\right), \quad m, n \in \mathbb{N} \cup \{0\}.$$

Here and below, the constants  $C$  are independent of  $z$ ,  $h$  and  $t$ , and the symbol  $O(\cdot)$  is used for estimates uniform in  $z$ ,  $t$  and  $h$ .

Fix  $n \geq 0$ . Estimate (4.23) being obvious for  $F_{0,n}$ , it suffices to assume that it is valid for some  $m = m_0$  and to prove it for  $m = m_0 + 1$ . Clearly,  $\zeta(z)$  is bounded on  $K$ . In view of the definitions of  $(a_{m,n})_{m,n \geq 0}$ ,  $(b_{m,n})_{m,n \geq 0}$  and the induction hypothesis, we have  $a_{m_0,n}(z) = O(1)$  and  $b_{m_0,n}(z) = O(1)$ . These observations, (4.19) and the induction hypothesis imply that there exists  $R > 0$  independent of  $h$  such that, for all  $|t| \geq R$ ,  $f_{m_0,n}(t, z) = O(e^{C|t|})$ . By the maximum principle, this implies that  $f_{m_0,n}$  satisfies this estimate for all  $t \in \mathbb{C}$ . Now the Cauchy estimates for the derivatives of the analytic functions imply (4.23) for  $m = m_0 + 1$ .

7. To estimate the last integral in (4.15) and the integral term in the last line of (4.22), we consider an analytic function  $(t, z) \mapsto g(z, t)$  such that  $g(z, t) = O(e^{C|t|})$ , and prove the estimate

$$(4.24) \quad \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} g(z, t) dt = O\left(h^{\frac{1}{3}} w_h\right) + O\left(h^{\frac{1}{3}} w'_h\right).$$

Therefore, we essentially repeat the reasoning made in Section 4, Chapter VII of [25]. So, we omit some details.

If  $Z = h^{-\frac{2}{3}}\zeta(z)$  is bounded by a constant, setting  $T = h^{-\frac{1}{3}}t$ , we change variable in (4.24) and get

$$\int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} g(z, t) dt = h^{\frac{1}{3}} \int_{\gamma} e^{-\left(\frac{T^3}{3} - TZ\right)} O(e^{Ch^{\frac{1}{3}}|T|}) dT = O(h^{\frac{1}{3}}).$$

This yields (4.24) as  $w$  and  $w'$  have no common zero.

If  $Z = h^{-\frac{2}{3}}\zeta(z)$  is large, we estimate the integral using the method of steepest descent.

We deform the integration path to a path of steepest descent for  $e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)}$  exactly as when computing the asymptotics of the Airy function  $w$ , i.e., the asymptotics of the integral  $\int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} dt$ . The saddle points  $\pm\sqrt{\zeta(z)}$  are uniformly bounded when  $z \in K$ . Let  $r > 0$  be sufficiently large for the saddle points to be inside the disk of radius  $r$  centered at 0. We compute the asymptotics of the integral over  $\gamma \cap \{|t| \leq r\}$  directly by means of the method of steepest descents and, comparing the answer with the asymptotics of the Airy function  $w(Z)$  as  $Z \rightarrow \infty$ , we find that this integral is bounded by  $O(h^{\frac{1}{3}} w_h) + O(h^{\frac{2}{3}} w'_h)$ . The integral over the remaining part of  $\gamma$  quickly tends to 0 as  $h \rightarrow 0$ : actually, it is exponentially small with respect to  $O(h^{\frac{1}{3}} w_h) + O(h^{\frac{2}{3}} w'_h)$ . This yields (4.24).

8. Formula (4.15), the representation obtained in the last line of (4.22) and the estimates (4.23) and (4.24) imply

(4.25)

$$H\left(A h^{\frac{1}{3}} w_h\right) = h^{\frac{1}{3}} w_h \left( \sum_{l=0}^N h^l a_l + O(h^{N+1}) \right) + h^{\frac{2}{3}} w'_h \left( \sum_{l=0}^N h^l b_l + O(h^{N+\frac{2}{3}}) \right),$$

$$(4.26) \quad a_l = \sum_{m+n=l} a_{m,n}, \quad b_l = \sum_{m+n=l} b_{m,n}.$$

In view of (4.21), we see that  $a_l = 0$  for odd  $l$ , and  $b_l = 0$  for even  $l$ . This observation and the fact that  $N$  is arbitrary yields that  $\forall L \in \mathbb{N}$

$$(4.27) \quad H(Ah^{\frac{1}{3}}w_h) = h^{\frac{1}{3}}w_h \left( \sum_{l=0}^L h^{2l} a_{2l} + O(h^{2L+2}) \right) + h^{\frac{2}{3}}w'_h \left( \sum_{l=1}^L h^{2l-1} b_{2l-1} + O(h^{2L+1}) \right).$$

To complete the proof of Proposition 4.1, it remains to prove that  $a_0 = 0$  and to compute  $b_1$ .

**9.** Clearly,  $a_0(z) = a_{0,0}(z) = F_{0,0}(\sqrt{\zeta(z)}, z) = F_0(\sqrt{\zeta(z)}, z, 0)$ . Therefore,

$$a_0(z) = A(z) \left( 2 \cosh(\sqrt{\zeta(z)}\zeta'(z)) + v(z) \right).$$

Recall that the complex momentum  $p$  is defined in (2.1). Therefore, in view of (2.5) we get  $2 \cosh(\sqrt{\zeta(z)}\zeta'(z)) + v(z) = 0$ . So,  $a_0 = 0$ . This and (4.27) imply (4.1).

**10.** Let us prove (4.2). One has  $b_1 = b_{0,1} + b_{1,0}$ . Clearly,

$$(4.28) \quad \begin{aligned} b_{0,1}(z) &= \frac{F_{0,1}(\sqrt{\zeta(z)}, z)}{\sqrt{\zeta(z)}} = \frac{1}{\sqrt{\zeta(z)}} \frac{\partial F_0}{\partial h}(\sqrt{\zeta(z)}, z, 0) \\ &= 2A'(z) \frac{\sinh(\sqrt{\zeta(z)}\zeta'(z))}{\sqrt{\zeta(z)}} + A(z)\zeta''(z) \cosh(\sqrt{\zeta(z)}\zeta'(z)). \end{aligned}$$

Furthermore,

$$(4.29) \quad b_{1,0}(z) = \frac{F_{1,0}(\sqrt{\zeta(z)}, z)}{\sqrt{\zeta(z)}} = \frac{1}{\sqrt{\zeta(z)}} \frac{\partial f_{0,0}}{\partial t}(\sqrt{\zeta(z)}, z).$$

As  $a_{0,0} = b_{0,0} = 0$ , one has  $f_{0,0}(t, z) = F_{0,0}(t, z)/(t^2 - \zeta(z))$ . Substituting this representation into (4.29), after elementary calculations, one obtains

$$(4.30) \quad b_{1,0}(z) = \frac{A\zeta'}{2\zeta} \left( \zeta'(z) \cosh(\sqrt{\zeta}\zeta') - \frac{\sinh(\sqrt{\zeta}\zeta')}{\sqrt{\zeta}} \right), \quad \zeta = \zeta(z).$$

Substituting representations (4.28) and (4.30) into the formula  $b_1 = b_{0,1} + b_{1,0}$  yields (4.2). This completes the proof of Proposition 4.1.

*Remark 4.4.* As is seen from the above proof, to compute the coefficients  $a_{2l} = a_{2l}(A; z)$  and  $b_{2l-1} = b_{2l-1}(A; z)$  from Proposition 4.1 for  $l \in \mathbb{N}$ , one proceed as follows. First, in terms of  $A$ , one defines a function  $(t, z, h) \mapsto F_0(t, z, h)$  by formula (4.14). Next, one computes the coefficients  $(F_{0,n}(t, z))_{n \geq 0}$  of the Taylor series of  $h \mapsto F_0(t, z, h)$  at  $h = 0$  (see (4.16)). Then, for each  $n \geq 0$ , in terms of  $F_{0,n}(t, z)$ , one computes two sequences  $(a_{m,n})_{m \geq 0}$  and  $(b_{m,n})_{m \geq 0}$  using the inductive formulas (4.17)–(4.19). When doing this, one takes into account relations (4.21). Finally, one expresses  $a_{2l}$  and  $b_{2l-1}$  in terms of  $a_{m,n}$  and  $b_{m,n}$  by means of formulas (4.26).

**4.3. The proof of Proposition 4.2.** The proof of Proposition 4.2 being parallel to that of Proposition 4.1, we concentrate only on the differences and omit details. We assume that  $B$  is analytic in  $U$ . Let  $w, \gamma, K$  be as in the proof of Proposition 4.1. We use the notations from (2.10). We assume that  $(t, z) \in \mathbb{C} \times K$  and that  $h > 0$  is sufficiently small. The derivation of the asymptotics of  $H(Bh^{\frac{2}{3}}w')$  is split into several

steps.

1. First, we note that

$$(4.31) \quad H\left(Bh^{\frac{2}{3}}w'_h\right) = \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} G_0(t, z, h) dt,$$

$$(4.32) \quad G_0(t, z, h) = t\left(B(z+h)e^{\frac{\zeta(z+h)-\zeta(z)}{h}t} + B(z-h)e^{\frac{\zeta(z-h)-\zeta(z)}{h}t} + v(z)B(z)\right).$$

2. Let

$$(4.33) \quad G_{0,n}(t, z) = \frac{1}{n!} \frac{\partial^n G_0}{dh^n}(t, z, 0).$$

We fix  $n \in \mathbb{N} \cup \{0\}$ . For  $m \geq 0$ , we inductively define

$$(4.34) \quad c_{m,n}(z) = \frac{1}{2} \left( G_{m,n}(\sqrt{\zeta(z)}, z) + G_{m,n}(-\sqrt{\zeta(z)}, z) \right),$$

$$(4.35) \quad d_{m,n}(z) = \frac{1}{2\sqrt{\zeta(z)}} \left( G_{m,n}(\sqrt{\zeta(z)}, z) - G_{m,n}(-\sqrt{\zeta(z)}, z) \right),$$

$$(4.36) \quad G_{m+1,n} = \frac{\partial g_{m,n}}{\partial t}, \quad g_{m,n}(t, z) = \frac{G_{m,n}(t, z) - c_{m,n}(z) - d_{m,n}(z)t}{t^2 - \zeta(z)},$$

where we use one and the same branch of  $\sqrt{\zeta(z)}$ .

The functions  $(G_{m,n})_{m,n}$  are analytic in  $(t, z) \in \mathbb{C} \times K$ , and the functions  $(c_{m,n})_{m,n}$  and  $(d_{m,n})_{m,n}$  are analytic in  $z \in K$  (all the removable singularities are removed).

3. One has  $G_0(-t, z, -h) = -G_0(t, z, h)$ . This implies that  $\forall m, n \geq 0$

$$(4.37) \quad G_{m,n}(-t, z) = -(-1)^{m+n} G_{m,n}(t, z),$$

$$(4.38) \quad c_{m,n} = 0 \text{ if } m+n \text{ is even,} \quad d_{m,n} = 0 \text{ if } m+n \text{ is odd.}$$

4. We set

$$(4.39) \quad c_l = \sum_{m+n=l} c_{m,n}, \quad d_l = \sum_{m+n=l} d_{m,n}.$$

Reasoning as in the proof of [Proposition 4.1](#), for  $L \in \mathbb{N}$ , instead of [\(4.27\)](#), we get

$$(4.40) \quad H(Bh^{\frac{1}{3}}w'_h) = h^{\frac{1}{3}}w_h \left( \sum_{l=1}^L h^{2l-1} c_{2l-1} + O(h^{2L+1}) \right) + h^{\frac{2}{3}}w'_h \left( \sum_{l=0}^L h^{2l} d_{2l} + O(h^{2L+2}) \right).$$

5. One has  $d_0 = 0$  (for the same reason as  $a_0 = 0$  in the proof of [Proposition 4.2](#)). This already implies [\(4.3\)](#).

6. Finally, we obtain the formulas

$$\begin{aligned} c_1 &= c_{01} + c_{10}, \\ c_{0,1} &= \frac{\partial G_0}{\partial h}(\sqrt{\zeta}, z, 0) = 2B'(z)\sqrt{\zeta} \sinh(\sqrt{\zeta}\zeta') + B(z)\zeta\zeta'' \cosh(\sqrt{\zeta}\zeta'), \\ c_{1,0} &= \left. \frac{\partial G_0(t, z, 0)}{\partial t} \right|_{t=\sqrt{\zeta}} = \frac{B\zeta'}{2} \left( \zeta' \cosh(\sqrt{\zeta}\zeta') + \frac{\sinh(\sqrt{\zeta}\zeta')}{\sqrt{\zeta}} \right), \end{aligned}$$

where  $\zeta = \zeta(z)$ . They imply [\(4.4\)](#). This completes the proof of [Proposition 4.2](#).



*Remark 4.5.* As is seen from the above proof, to compute the coefficients  $c_{2l-1} = c_{2l-1}(B; z)$  and  $d_{2l} = d_{2l}(B; z)$  from [Proposition 4.2](#) for  $l \in \mathbb{N}$ , we proceed as follows. First, in terms of  $B$ , one defines a function  $(t, z, h) \mapsto G_0(t, z, h)$  by formula [\(4.32\)](#). Next, one computes the coefficients  $(G_{0,n}(t, z))$  of the Taylor series of  $h \mapsto G_0(t, z, h)$  at  $h = 0$  (see [\(4.33\)](#)). Then, for each  $n = 0, 1, 2, \dots$ , in terms of  $G_{0,n}(t, z)$ , one computes two sequences  $(c_{m,n})_{m \geq 0}$  and  $(d_{m,n})_{m \geq 0}$  using the induction formulas [\(4.34\)](#)–[\(4.36\)](#). When doing this, one takes into account the symmetry relations [\(4.37\)](#) and [\(4.38\)](#). Finally, one expresses  $c_{2l-1}$  and  $d_{2l}$  in terms of  $c_{m,n}$  and  $d_{m,n}$  by means of formulas [\(4.39\)](#).

**5. Properties of the asymptotic solutions.** We now study basic properties of the asymptotic solutions. More precisely, we fix an integer  $L \geq 0$  and study the functions  $(W_j)_{j \in \mathbb{Z}_3}$ , i.e., the functions  $W$  from [Theorem 2.6](#) corresponding to  $L$  and to the Airy functions  $w = w_j$ ,  $j \in \mathbb{Z}_3$ .

**5.1. Functional relations.** We recall that the function  $(W_j)_{j \in \mathbb{Z}_3}$  are defined in a domain  $U$  satisfying [Hypotheses 2.2–2.4](#).

LEMMA 5.1. *One has*

$$(5.1) \quad W_0(z) + W_1(z) + W_2(z) = 0, \quad \forall z \in U.$$

*Proof.* Formula [\(3.1\)](#) and the definitions of the integration paths  $(\gamma_j)_{j \in \mathbb{Z}_3}$  (see [Figure 1](#)) imply that

$$(5.2) \quad w_0(\zeta) + w_1(\zeta) + w_2(\zeta) = 0, \quad \zeta \in \mathbb{C}.$$

As the function  $\zeta$  and all the coefficients  $(A_{2l})_{l \in \mathbb{N} \cup \{0\}}$  and  $(B_{2l+1})_{l \in \mathbb{N} \cup \{0\}}$  in representations [\(2.10\)](#)–[\(2.11\)](#) are independent of the choice of  $w$ , the solution of the Airy equation in this representation, the relation [\(5.2\)](#) implies [\(5.1\)](#).  $\square$

Relation [\(5.1\)](#) implies that

$$(5.3) \quad (W_0(z), W_1(z)) = (W_1(z), W_2(z)) = (W_2(z), W_0(z)), \quad \forall \{z, z+h\} \subset U,$$

where  $(f(z), g(z))$  is defined in [\(3.3\)](#).

**5.2. Estimates on  $W_j$ .** To prove the existence of analytic solutions that admit asymptotic expansions of the form [\(2.13\)](#), we need rough estimates of  $(W_j)_{j \in \mathbb{Z}_3}$  in  $U$ . We first introduce some tools used in their derivation.

**5.2.1. Geometry.** We recall that the function  $\zeta$  defined in [\(2.4\)](#) is analytic in  $U$  and bijectively maps  $U$  onto  $V = \zeta(U)$ ,  $\zeta(z_0) = 0$ .

We put

$$(5.4) \quad \sigma_j = \zeta^{-1}(V \cap a_j), \quad a_j = e^{-2\pi i j/3} \mathbb{R}_-, \quad j \in \mathbb{Z}_3,$$

where  $\mathbb{R}_- = (-\infty, 0]$ . The curves  $(\sigma_j)_{j \in \mathbb{Z}_3}$  are analytic and all begin at  $z_0$ . Any two of them do not intersect except at  $z_0$ . The angles between these curves at  $z_0$  are equal to  $2\pi/3$ .

The curves  $(\sigma_j)_{j \in \mathbb{Z}_3}$  cut the domain  $U$  (a disk centered at  $z_0$ ) into three simply connected subdomains that we call *sectors*. We denote them by  $S_0$ ,  $S_1$  and  $S_2$  so that the sector  $S_0$  is bounded by  $\sigma_1$  and  $\sigma_2$ ,  $S_1$  is bounded by  $\sigma_2$  and  $\sigma_0$ , and  $S_2$  is bounded by  $\sigma_0$  and  $\sigma_1$  (see [Figure 2](#)). Let

$$(5.5) \quad U_j = U \setminus \sigma_j, \quad j \in \mathbb{Z}_3.$$

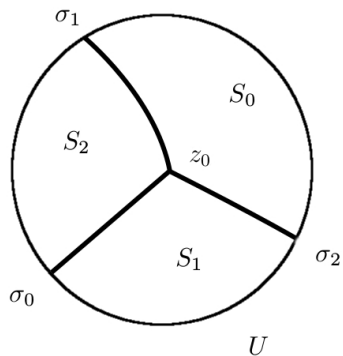


FIG. 2. Stokes lines and sectors

These domains do not contain branch points of the complex momentum  $p$  as the only branch point of  $p$  in  $U$  is  $z = z_0$ . We shall use

LEMMA 5.2. *For  $j \in \mathbb{Z}_3$ , there exists a branch of the complex momentum, say,  $p_j$  that is analytic in  $U_j$  and such that  $p_j(z_0) = 0$  and*

1.  $\text{Im} \int_{z_0}^z p_j(z) dz > 0$  inside  $S_j$ ;
2.  $\text{Im} \int_{z_0}^z p_j(z) dz < 0$  inside the two other sectors;
3.  $\text{Im} \int_{z_0}^z p_j(z) dz = 0$  along the curves  $\sigma_1, \sigma_2$  and  $\sigma_0$  (in the case of  $\sigma_j$ , we mean the boundary values);

Moreover, one has

$$(5.6) \quad p_1 = -p_0 \text{ in } \sigma_0 \cup S_1 \cup \sigma_2 \cup S_0 \cup \sigma_1, \quad p_2 = -p_0 \text{ in } \sigma_0 \cup S_2 \cup \sigma_1 \cup S_0 \cup \sigma_2.$$

In the WKB method, the curves  $(\sigma_j)_{j \in \mathbb{Z}_3}$  are called *Stokes lines*.

*Proof.* Let us check the first three points of Lemma 5.2 for  $j = 0$ . We recall that  $\zeta$  is an analytic branch of the function (2.4). We can assume that  $p$  is a branch of the complex momentum analytic in  $U_0$  and such that  $p(z_0) = 0$ .

Formulas (5.4) and the definition of  $\zeta$  imply that  $\text{Im} \int_{z_0}^z p(z) dz = 0$  on any of the Stokes lines. We note that

$$(5.7) \quad \zeta(S_j) = \{v \in V : v \neq 0, \arg v \in -2\pi j/3 + (-\pi/3, \pi/3)\}, \quad j \in \mathbb{Z}_3.$$

This and the definition of  $\zeta$  imply that  $\text{Im} \int_{z_0}^z p(z) dz \neq 0$  in each of the sectors.

In view of the analysis made in section 3.3.1, in  $U_0$ , we can choose an analytic branch  $p_0$  of the complex momentum so that  $\text{Im} \int_{z_0}^z p_0(z) dz > 0$  in  $S_0$ . For  $p_0$ , the statements 1 and 3 of Lemma 5.2 are obviously valid.

To prove point 2, it suffices to check that  $\text{Im} \int_{z_0}^z p_0(z) dz < 0$  in the sectors  $S_1$  and  $S_2$ . Therefore, we note that as  $z \neq z_0$ ,  $z \sim z_0$ , crosses  $\sigma_2$  moving from  $S_0$  to  $S_1$  the argument of  $\zeta(z)$  decreases ( $\zeta$  vanishes only at  $z = z_0$ ) as does the argument of  $\int_{z_0}^z p_0(z) dz$ . Therefore, point 2 of Lemma 5.2 follows from points 1 and 3.

To complete the proof of Lemma 5.2, we choose  $p_1$  in the following way. First, in  $S_1$  we choose  $p_1 = -p_0$  and continue  $p_1$  analytically from  $S_1$  to  $U_1$ . For this  $p_1$ , we have

$$\text{Im} \int_{z_0}^z p_1(z) dz = -\text{Im} \int_{z_0}^z p_0(z) dz > 0, \quad z \in S_1.$$

This proves point 1 for  $p_1$ . Points 2 and 3 for  $p_1$  are proved as for  $p_0$  and  $p$ .

Finally, in  $S_2$ , we set  $p_2 = -p_0$  and continue  $p_2$  analytically from  $S_2$  to  $U_2$ . To complete the proof of [Lemma 5.2](#) for  $p_2$ , we reason as for  $p_1$ . We omit further details.  $\square$

**5.2.2. Estimates.** For  $j \in \mathbb{Z}_3$  and  $z \in U_j$ , we set

$$(5.8) \quad \rho_j(z) = e^{\frac{i}{h} \int_{z_0}^z p_j(z') dz'}.$$

We note that  $\rho_j$  is continuous up to the cut along  $\sigma_j$ , and the boundary values of its absolute value  $|\rho_j|$  on both the sides of the cut equal one. So, below, we consider  $|\rho_j|$  as a continuous function in  $U$ .

Let us recall that  $H$  is defined by [\(2.9\)](#). We set

$$(5.9) \quad \delta_j(z) = [H(W_j)](z), \quad z \in U, \quad j \in \mathbb{Z}_3.$$

**PROPOSITION 5.3.** *For each  $j \in \mathbb{Z}_3$ , one has*

$$(5.10) \quad |W_j(z)| \leq Ch^{1/3} |\rho_j(z)|, \quad z \in U,$$

$$(5.11) \quad |\delta_j(z)| \leq Ch^{2L+2+1/3} |\rho_j(z)|, \quad \{z, z+h, z-h\} \subset U,$$

where  $L$  is the order entering the definition of  $W_j$  (see [\(2.11\)](#)).

[Proposition 5.3](#) immediately follows from formulas [\(2.10\)](#)–[\(2.12\)](#) with  $w = w_j$  and

**LEMMA 5.4.** *For  $j \in \mathbb{Z}_3$ , one has*

$$(5.12) \quad |w_j(h^{-\frac{2}{3}}\zeta(z))| \leq C|\rho_j(z)|, \quad |w'_j(h^{-\frac{2}{3}}\zeta(z))| \leq Ch^{-\frac{1}{6}}|\rho_j(z)|, \quad z \in U.$$

*Proof.* We prove [\(5.12\)](#) only for  $j = 0$ . The other cases are treated similarly. We recall that  $w_0 = \text{Ai}$ , that  $\zeta$  bijectively maps  $U$  onto its image and that  $\zeta(z_0) = 0$  (see [\(2.4\)](#)). Clearly,

$$(5.13) \quad w_0(h^{-\frac{2}{3}}\zeta(z)) = O(1) \text{ and } w'_0(h^{-\frac{2}{3}}\zeta(z)) = O(1) \text{ if } |\zeta(z)| \leq h^{\frac{2}{3}}.$$

Now we turn to the case where  $|\zeta(z)| \geq h^{\frac{2}{3}}$ . It suffices to prove [\(5.12\)](#) in  $U_0$ .

The asymptotic formulas [\(3.2\)](#) imply that, for  $Z \in \{Z \in \mathbb{C} \setminus \mathbb{R}_- : |Z| \geq 1\}$ , one has

$$(5.14) \quad |w_0(Z)| \leq C|Z|^{-\frac{1}{4}} \left| e^{-\frac{2}{3}Z^{\frac{3}{2}}} \right| \quad \text{and} \quad |w'_0(Z)| \leq C|Z|^{\frac{1}{4}} \left| e^{-\frac{2}{3}Z^{\frac{3}{2}}} \right|,$$

where the  $Z \mapsto Z^{\frac{3}{2}}$  is analytic in  $\mathbb{C} \setminus \mathbb{R}_-$  and positive when  $Z > 0$ .

Estimate [\(5.14\)](#) and [\(5.5\)](#), the definition of  $U_0$ , imply that, for  $z \in U_0$  such that  $|\zeta(z)| \geq h^{\frac{2}{3}}$ , one has

$$(5.15) \quad |w_0(h^{-\frac{2}{3}}\zeta(z))| \leq C \left| e^{-\frac{2}{3h}\zeta(z)^{\frac{3}{2}}} \right| \quad \text{and} \quad |w'_0(h^{-\frac{2}{3}}\zeta(z))| \leq Ch^{-\frac{1}{6}} \left| e^{-\frac{2}{3h}\zeta(z)^{\frac{3}{2}}} \right|,$$

where  $z \rightarrow \zeta(z)^{3/2}$  is analytic in  $U_0$  and positive along  $\alpha_0 = \zeta^{-1}((0, \infty))$ .

In view of the analysis made in [section 3.3.1](#),

$$\zeta(z)^{\frac{3}{2}} = \pm \frac{3i}{2} \int_{z_0}^z p_0(z') dz', \quad z \in U_0.$$

As  $\alpha_0 = \zeta^{-1}((0, \infty)) \subset S_0$ , along  $\alpha_0$  one has  $\text{Im} \int_{z_0}^z p_0(z') dz' > 0$ . Therefore, in  $U_0$   $\zeta(z)^{\frac{3}{2}} = -\frac{3i}{2} \int_{z_0}^z p_0(z') dz'$ , and  $\left| e^{-\frac{2}{3h}\zeta(z)^{\frac{3}{2}}} \right| = |\rho_0(z)|$ . This and [\(5.15\)](#) imply [\(5.12\)](#) for  $|\zeta(z)| \geq h^{-2/3}$ . This and [\(5.13\)](#) imply [Lemma 5.4](#).  $\square$

**5.3. Wronskians.** Below  $\mathcal{C} \subset U$  is a closed  $h$ -independent disk centered at  $z_0$ . We prove

LEMMA 5.5. *For  $z \in \mathcal{C}$ , as  $h \rightarrow 0$  one has*

$$(W_0(z), W_1(z)) = h(w'_0(z)w_1(z) - w'_1(z)w_0(z)) + O(h^{\frac{5}{3}}).$$

Before proving Lemma 5.5, we check

LEMMA 5.6. *Pick  $j \in \mathbb{Z}_3$  and set  $w = w_j$ . For  $z \in \mathcal{C}$ , as  $h \rightarrow 0$ , one has*

$$h^{\frac{1}{3}}w_h \Big|_{z+h} = h^{\frac{1}{3}}w_h(z) \left( \cosh(\sqrt{\zeta(z)}\zeta'(z)) + O(h) \right) + h^{\frac{2}{3}}w'_h(z) (g(z) + O(h)),$$

where we use the notations from (2.10) and  $g$  is defined in (2.6).

*Proof of Lemma 5.6.* We proceed as in the proof of Proposition 4.1. Thus, we omit some details and concentrate on the new computations.

Let  $\gamma = \gamma_j$  be the integration path in (3.1). Also, below we assume that  $z \in \mathcal{C}$  and that  $h$  is sufficiently small. The proof is broken into several steps.

1. We compute

$$(5.16) \quad h^{\frac{1}{3}}w_h \Big|_{z+h} = \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} E(t, z, h) dt, \quad E(t, z, h) = e^{\frac{t}{h}(\zeta(z+h) - \zeta(z))}.$$

2. We represent  $E(t, z, 0)$  in the form

$$(5.17) \quad E(t, z, 0) = \alpha(z) + \beta(z)t + (t^2 - \zeta(z))\phi(t, z)$$

with

$$\alpha(z) = \frac{E(\sqrt{\zeta(z)}, z, 0) + E(-\sqrt{\zeta(z)}, z, 0)}{2}, \quad \beta(z) = \frac{E(\sqrt{\zeta(z)}, z, 0) - E(-\sqrt{\zeta(z)}, z, 0)}{2\sqrt{\zeta(z)}}.$$

3. Clearly,  $E(t, z, 0) = e^{t\zeta'(z)}$ , and, therefore, one has

$$(5.18) \quad \alpha(z) = \cosh(\sqrt{\zeta}\zeta'), \quad \beta(z) = g(z).$$

4. Using representation (5.17) and integrating by parts, we get

$$\int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} E(t, z, 0) dt = \alpha h^{\frac{1}{3}}w_h + \beta h^{\frac{2}{3}}w'_h + h \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} \frac{\partial \phi}{\partial t}(t, z) dt.$$

5. Reasoning as when proving Proposition 4.1, we check that

$$h^{\frac{1}{3}}w_h \Big|_{z+h} = \alpha h^{\frac{1}{3}}w_h + \beta h^{\frac{2}{3}}w'_h + O(h^{\frac{4}{3}}w) + O(h^{\frac{5}{3}}w').$$

Lemma 5.6 follows from this estimate and (5.18).  $\square$

Now, we turn to the proof of Lemma 5.5.

*Proof of Lemma 5.5.* We assume that  $z \in \mathcal{C}$  and that  $h$  is sufficiently small. Using (2.11) and Lemma 5.6, we compute

$$\begin{aligned} (W_0(z), W_1(z)) &= A_0^2(z) \\ &\times \left( (h^{\frac{1}{3}} \cosh(\sqrt{\zeta} \zeta') w_0 + gh^{\frac{2}{3}} w'_0 + O(h^{\frac{4}{3}} w_0) + O(h^{\frac{5}{3}} w'_0)) \right. \\ &\quad \cdot (h^{\frac{1}{3}} w_1 + O(h^{\frac{7}{3}} w_1) + O(h^{\frac{5}{3}} w'_1)) \\ &\quad - (h^{\frac{1}{3}} \cosh(\sqrt{\zeta} \zeta') w_1 + gh^{\frac{2}{3}} w'_1 + O(h^{\frac{4}{3}} w_1) + O(h^{\frac{5}{3}} w'_1)) \\ &\quad \left. \cdot (h^{\frac{1}{3}} w_0 + O(h^{\frac{7}{3}} w_0) + O(h^{\frac{5}{3}} w'_0)) \right). \end{aligned}$$

Here,  $w_j = w_j(h^{-\frac{2}{3}} \zeta(z))$ ,  $j \in \mathbb{Z}_3$ ,  $\zeta = \zeta(z)$ , and  $g = g(z)$ .

Now, we assume that  $z \in \sigma_0 \cup S_1 \cup \sigma_2 \cup S_0 \cup \sigma_1$ . Then, by (5.12) and (5.6)

$$(5.19) \quad |w_0 w_1| \leq C, \quad |w'_0 w_1| \leq Ch^{-\frac{1}{6}}, \quad |w_0 w'_1| \leq Ch^{-\frac{1}{6}}, \quad |w'_0 w'_1| \leq Ch^{-\frac{1}{3}},$$

and we get

$$(W_0(z), W_1(z)) = hg(z) A_0^2(z) (w'_0 w_1 - w_0 w'_1) + O(h^{\frac{5}{3}}).$$

In view of (2.7), Lemma 5.5 is proved for  $z \in \sigma_0 \cup S_1 \cup \sigma_2 \cup S_0 \cup \sigma_1$ .

When  $z \in \sigma_0 \cup S_2 \cup \sigma_1 \cup S_0 \cup \sigma_2$ , we similarly get

$$(5.20) \quad (W_0(z), W_2(z)) = hg(z) A_0^2(z) (w'_0 w_2 - w'_2 w_0) + O(h^{\frac{5}{3}}).$$

In view of (5.3),  $(W_0, W_2) = -(W_0, W_1)$  and relation (5.2) imply that  $w'_0 w_2 - w_0 w'_2 = -(w'_0 w_1 - w_0 w'_1)$ . Therefore, Lemma 5.5 for  $z \in S_2$  follows from (5.20). This completes the proof of Lemma 5.5.  $\square$

**6. Solutions to (1.1) on precanonical domains.** Fix  $L \geq 0$  and  $j \in \mathbb{Z}_3$ . The aim of this section is to construct a solution to equation (1.1), say,  $\psi = \psi_j$  that admits representation (2.16) with  $W$ , the function from Theorem 2.6 corresponding to the Airy function  $w_j$ . The results of this section is preliminary as we only construct the solutions  $(\psi_j)_{j \in \mathbb{Z}_3}$  on some subdomains of  $U$ .

### 6.1. The result of this section.

**6.1.1. Notations and some definitions.** First, to formulate the results of this section, we introduce some notations and recall some definitions related to the complex WKB method for difference equations, see, for example, [13].

For  $z \in \mathbb{C}$ , we let  $x = \operatorname{Re} z$  and  $y = \operatorname{Im} z$ .

A curve  $\gamma \subset \mathbb{C}$  is called *vertical* if  $z$  is a piecewise continuously differentiable function of  $y$  along  $\gamma$ .

Let  $\gamma \subset U$  be a regular vertical curve parameterized by  $z = z(y)$ . Let  $p$  be a branch of the complex momentum continuous on  $\gamma$ . The curve  $\gamma$  is *precanonical* with respect to  $p$ , if the function  $y \mapsto \operatorname{Im} \int_{z_0}^{z(y)} p(z) dz$  is non decreasing and the function  $y \mapsto \operatorname{Im} \int_{z_0}^{z(y)} (p(z) - \pi) dz$  is non increasing.

Let  $d > 0$ . For  $M \subset \mathcal{C}$  we define the *horizontal  $d$ -neighborhood* of  $M$  to be the set  $M^d := M + [-d, d]$  and  $M^{-d} := (M^d - d) \cap M \cap (M^d + d)$ .

We recall that, for  $j \in \mathbb{Z}_3$ , the sector  $S_j$  and the Stokes line  $\sigma_j$  are shown in Figure 2. For  $j \in \mathbb{Z}_3$ , we denote by  $S_{j,j+1}$  the closure of the domain  $S_j \cup S_{j+1}$

without the boundary of  $U$ . For example, one has

$$S_{1,2} = \sigma_1 \cup S_2 \cup \sigma_0 \cup S_1 \cup \sigma_2.$$

We also note that relations (5.6) imply that

$$(6.1) \quad |\rho_j(z)\rho_{j+1}(z)| = 1, \quad z \in S_{j,j+1}, \quad j \in \mathbb{Z}_3.$$

Let  $r_1 < r_2$ . We set  $S(r_1, r_2) = \{z \in \mathbb{C} : r_1 < \text{Im}z < r_2\}$ .

**6.1.2. The main result of the section.** One has

**THEOREM 6.1.** *Let  $j \in \mathbb{Z}_3$ ,  $L \in \mathbb{N} \cup \{0\}$ ,  $c \in (1, 2)$  and  $r > 0$ . Let  $K \subset S_{j,j+1}$  be a regular simply connected domain bounded by two curves having common endpoints  $z_1$  and  $z_2$  and both precanonical with respect to either the branch  $p_j$  or  $p_{j+1}$ .*

*Then, for sufficiently small  $h$ , there exist two solutions  $\psi_j$  and  $\psi_{j+1}$  to (1.1) that are analytic in  $K^{ch}$  and that, in  $K^{ch} \cap S(\text{Im}z_1 + rh, \text{Im}z_2 - rh)$  admit the asymptotic representations*

$$(6.2) \quad \psi_l(z) = W_l(z) + O(|\rho_l|h^{2L+1+\frac{1}{2}}), \quad l \in \{j, j+1\},$$

where  $W_l$  is the function described in Theorem 2.6 for  $w_l$  and the order  $L$ .

Let us fix  $j \in \mathbb{Z}_3$  and discuss the solutions  $\psi_j$  and  $\psi_{j+1}$  described in Theorem 6.1.

**COROLLARY 6.2.** *In the case of Theorem 6.1, the solutions  $\psi_j$  and  $\psi_{j+1}$  can be analytically continued to  $U \cap S(\text{Im}z_1, \text{Im}z_2)$ . Let  $r > 0$ . As  $h \rightarrow 0$ , one has*

$$(6.3) \quad (\psi_j(z), \psi_{j+1}(z)) = (W_j(z), W_{j+1}(z)) + O(h^{2L+1+\frac{5}{8}})$$

for  $z, z+h \in K^{ch} \cap S(\text{Im}z_1 + rh, \text{Im}z_2 - rh)$ .

*Proof.* The solutions being analytic in  $K^{ch}$  with  $c > 1$ , they can be analytically continued to  $U \cap S(\text{Im}z_1, \text{Im}z_2)$  just by means of equation (1.1).

We fix  $l \in \mathbb{Z}_3$  and note that, for all  $z$  in a compact set  $\mathcal{C} \subset U$ , for sufficiently small  $h$ , one has  $|\rho_l(z+h)|/|\rho_l(z)| \leq C$ . For  $z \in K^{ch} \cap S(\text{Im}z_1 + rh, \text{Im}z_2 - rh)$ , the representation (6.3) follows from this observation and from (5.10), (6.2) and (6.1).  $\square$

The remainder of this section is devoted to the proof of Theorem 6.1.

For the sake of definiteness, when proving Theorem 6.1, we assume that  $j = 0$  and that the two curves from Theorem 6.1 are precanonical with respect to the branch  $p_0$ . The other cases are treated similarly.

Below,  $K$  is as in Theorem 6.1 (for  $j = 0$ ): it is bounded by the precanonical curves  $\gamma_1$  and  $\gamma_2$ , and their common endpoints satisfy  $\text{Im}z_1 < \text{Im}z_2$ . Finally,  $h$  is supposed to be sufficiently small.

**6.2. The ideas of the proof.** In the present section, we describe the construction of the solution  $\psi_0$ . The solution  $\psi_1$  is constructed similarly.

Let us assume that  $\psi_0$  is a solution to (1.1) analytic in  $K^{ch}$  that we expect to be close to  $W_0$ . Let us recall that  $\delta_0 = HW_0$ . Clearly,  $\Delta_0 := W_0 - \psi_0$  satisfies the equation

$$(6.4) \quad H(\Delta_0)(z) = \delta_0(z), \quad \{z-h, z, z+h\} \subset K^{ch}.$$

For  $z \in K^{ch}$ , let  $\gamma(z)$  denote a vertical curve in  $K^{ch}$  that contains  $z$  and connects  $z_1$  and  $z_2$ . We construct a solution to (6.4) in the form

$$(6.5) \quad \Delta_0 = R_0 g_0 \quad \text{where} \quad R_0 g_0(z) := \int_{\gamma(z)} r_0(z, \zeta) g_0(\zeta) d\zeta,$$

$$(6.6) \quad r_0(z, \zeta) = \frac{1}{2ih} \frac{W_0(z)W_1(\zeta) - W_0(\zeta)W_1(z)}{(W_0(\zeta), W_1(\zeta))} \theta_0 \left( \frac{\zeta - z}{h} \right), \quad \theta_0(t) = \cot(\pi t) - i$$

where  $(W_0(\zeta), W_1(\zeta))$  is the Wronskian of  $W_0$  and  $W_1$ . The choice of Ansatz (6.5) is explained by

LEMMA 6.3. *Let  $0 < \beta < 1$ . Let  $f$  be a function defined and analytic in  $U \cap S(\text{Im}z_1, \text{Im}z_2)$  and such that the expression*

$$(6.7) \quad f_\beta(z) = (z - z_1)^\beta (z - z_2)^\beta f(z)$$

is bounded. Then, for  $\{z - h, z, z + h\} \subset U \cap S(\text{Im}z_1, \text{Im}z_2)$ , one has

$$(6.8) \quad HR_0f(z) = f(z) + D_0f(z), \quad D_0f(z) = \int_{\gamma(z)} d_0(z, \zeta) f(\zeta) d\zeta$$

where

$$(6.9) \quad d_0(z, \zeta) = \frac{1}{2ih} \frac{\delta_0(z)W_1(\zeta) - W_0(\zeta)\delta_1(z)}{(W_0(\zeta), W_1(\zeta))} \theta_0 \left( \frac{\zeta - z - 0}{h} \right),$$

and  $\delta_j := HW_j$  are the ‘‘error’’ terms estimated in (5.11). The function  $D_0f$  is analytic in  $U \cap S(\text{Im}z_1, \text{Im}z_2)$ .

*Proof.* The analyticity of  $f$  and the boundedness of  $f_\beta$  imply that  $D_0f$  is well defined and analytic in  $U \cap S(\text{Im}z_1, \text{Im}z_2)$ . The relation  $HR_0f = f + D_0f$  follows from the residue theorem. We omit further details.  $\square$

We note that an operator similar to  $R_0$  was introduced in [13], but, was not studied for small  $h$ .

In view of Lemma 6.3 and the formulas  $\Delta_0 = R_0g_0$  and  $H(\Delta_0) = \delta_0$ , in  $K^{ch}$ , we can expect that

$$(6.10) \quad g_0 + D_0g_0 = \delta_0.$$

Roughly, to prove Theorem 6.1, we consider (6.10) as an equation on a vertical curve  $\gamma$ . It appears that if  $\gamma$  is precanonical, the operator  $D_0$  is small. This enables us to construct a solution  $\psi_0$  to (6.10) on  $\gamma_1$ . Next, we check that it is analytic in  $K^{ch}$ , satisfies (1.1) and admits the asymptotic representation (6.2) with  $l = 0$ . The solution  $\psi_1$  is constructed similarly.

**6.3. The integral operator  $D_0$ .** The aim of this section is to estimate the operator norm of  $D_0$  in a suitable functional space.

Let  $\gamma$  be either  $\gamma_1$  or  $\gamma_2$ . We fix  $\alpha \in (0, 1)$  and define the strip

$$\Pi_{\gamma, \alpha} = \gamma \setminus \{z_1, z_2\} + [-\alpha h, \alpha h].$$

We recall that  $z \rightarrow |\rho_0(z)|$  defined in  $U_0$  is a continuous function in  $U$ . We fix  $0 < \beta < 1$  and let  $H_{\gamma, \alpha, \beta}$  be the linear space of functions analytic in  $\Pi_{\gamma, \alpha}$  and having finite norm

$$(6.11) \quad \|f\| = \sup_{z \in \Pi_{\gamma, \alpha}} \frac{|f_\beta(z)|}{|\rho_0(z)|}$$

$f_\beta$  being defined in (6.7). Clearly, endowed with this norm,  $H_{\gamma, \alpha, \beta}$  is a Banach space.

For  $f \in H_{\gamma, \alpha, \beta}$ , we define  $D_0f$  by the formula in (6.8) where  $\gamma(z)$  is a vertical curve that connects the points  $z_1$  and  $z_2$  in  $\Pi_{\gamma, \alpha}$  and passes through  $z$ . The function  $D_0f$  is then analytic in  $\Pi_{\gamma, \alpha}$ . One has

PROPOSITION 6.4. *For sufficiently small  $h$*

$$\|D_0\|_{H_{\gamma,\alpha,\beta} \rightarrow H_{\gamma,\alpha,\beta}} \leq Ch^{2L+\frac{2}{3}}.$$

The remainder of this subsection is devoted to the proof of [Proposition 6.4](#). Therefore, for  $f \in H_{\gamma,\alpha,\beta}$ , we estimate  $Rf(z)$ . Up to the end of this subsection, we assume that  $\{z, \zeta\} \subset \Pi_{\gamma,\alpha}$  and that  $h$  is sufficiently small.

**6.3.1. An auxiliary lemma.** When estimating  $Rf(z)$ , we use

LEMMA 6.5. *For  $q > 0$ , there exists  $C > 0$  such that*

$$(6.12) \quad \sup_{\substack{\{z, \zeta\} \subset \Pi_{\gamma,\alpha} \\ \min_{k \in \mathbb{Z}} |\zeta - z - kh| \geq qh}} \left| \frac{\rho_0(\zeta)}{\rho_0(z)} d_0(z, \zeta) \right| \leq Ch^{2L+\frac{2}{3}}.$$

*Proof.* We proceed in several steps.

1. [Proposition 5.3](#) and [Lemma 5.5](#) imply that

$$(6.13) \quad \left| \frac{\rho_0(\zeta)}{\rho_0(z)} d_0(z, \zeta) \right| \leq Ch^{2L+\frac{2}{3}} \left( |\rho_1(\zeta)\rho_0(\zeta)| + \frac{|\rho_1(z)\rho_0^2(\zeta)|}{|\rho_0(z)|} \right) \left| \theta_0 \left( \frac{\zeta - z - 0}{h} \right) \right|.$$

2. Recall that in  $S_{0,1}$  one has  $\rho_0(z)\rho_1(z) = 1$  (see [\(6.1\)](#)). As  $\gamma_1 \cup \gamma_2 \subset S_{0,1}$ , one has  $|\rho_0(z)||\rho_1(z)| \leq C$  for  $z \in \Pi_{\gamma,\alpha}$ . Therefore,

$$(6.14) \quad \left| \frac{\rho_0(\zeta)}{\rho_0(z)} d_0(z, \zeta) \right| \leq Ch^{2L+\frac{2}{3}} (1 + e(z, \zeta)) \left| \theta_0 \left( \frac{\zeta - z - 0}{h} \right) \right|, \quad e(z, \zeta) = \left| \frac{\rho_0(\zeta)}{\rho_0(z)} \right|^2.$$

3. For  $z \in \Pi_{\gamma,\alpha}$ , we define  $z_\perp \in \gamma$  so that  $\text{Im}z_\perp = \text{Im}z$ . We have

$$|e(z, \zeta)| \leq C \left| \exp \left( \frac{2i}{h} \int_{z_\perp}^{\zeta_\perp} p_0(z') dz' \right) \right|.$$

4. On the complex plane outside a fixed neighborhood of  $\mathbb{Z}$ , we have the estimate

$$|\theta_0(z)| = |\cot(\pi z) - i| \leq C \begin{cases} 1, & \text{Im}z \geq 0; \\ e^{2\pi \text{Im}z}, & \text{Im}z \leq 0. \end{cases}$$

Therefore, for  $\zeta$  outside the  $(qh)$ -neighborhood of  $z + h\mathbb{Z}$ , we get

$$\left| \theta_0 \left( \frac{\zeta - z - 0}{h} \right) \right| \leq C \quad \text{and} \\ \left| e(z, \zeta) \theta_0 \left( \frac{\zeta - z - 0}{h} \right) \right| \leq C \begin{cases} e^{-\frac{2}{h} \text{Im} \int_{z_\perp}^{\zeta_\perp} p dz} & \text{if } \text{Im}(\zeta - z) \geq 0; \\ e^{\frac{2}{h} \text{Im} \int_{z_\perp}^{\zeta_\perp} (p - \pi) dz} & \text{if } \text{Im}(\zeta - z) \leq 0. \end{cases}$$

5. As  $\gamma$  is a precanonical curve, we finally get

$$(6.15) \quad \left| \theta_0 \left( \frac{\zeta - z - 0}{h} \right) \right| \leq C, \quad \text{and} \quad \left| e(z, \zeta) \theta_0 \left( \frac{\zeta - z - 0}{h} \right) \right| \leq C.$$

This and [\(6.14\)](#) imply [\(6.12\)](#). □



**6.3.2. Estimates in the strip**  $S(\text{Im}z_1 + h/2, \text{Im}z_2 - h/2)$ . When  $z \in S(\text{Im}z_1 + h/2, \text{Im}z_2 - h/2)$ , we prove

$$(6.16) \quad |\rho_0(z)^{-1} D_0 f(z)| \leq C h^{2L + \frac{2}{3}} \|f\|.$$

First, we assume that  $z$  is between the curves  $\gamma + \alpha h/2$  and  $\gamma + \alpha h$ . Then, in (6.8), one can deform the integration path  $\gamma(z)$  to  $\gamma$ . The distance between the poles of  $d_0$  and  $\gamma$  is larger than  $Ch$ . This, (6.11) and (6.12) imply (6.16).

Next, we assume that  $z$  is either between the curves  $\gamma$  and  $\gamma + ah/2$  or on one of them. In this case, in (6.8) we can replace the integration path  $\gamma$  by  $\tilde{\gamma}$  where

- $\tilde{\gamma}$  is a continuous curve that connects  $z_1$  to  $z_2$ ,
- $\tilde{\gamma}$  coincides with  $\gamma - ah/2$  in the strip  $\{\text{Im}z_1 + h/2 \leq \text{Im}z \leq \text{Im}z_2 - h/2\}$ ,
- outside this strip,  $\tilde{\gamma}$  consists of two segments of straight lines.

Reasoning as above on this new integral, we again obtain (6.16).

Let us assume now that  $z$  is to the left of  $\gamma$ . We note that, by the Residue Theorem, the integral in (6.8) decomposes as the sum of

$$-\frac{\delta_0(z)W_1(z) - W_0(z)\delta_1(z)}{(W_0(z), W_1(z))} f(z) = O(h^{2L+1+\frac{2}{3}}) f(z)$$

and the integral defined by (6.8)–(6.9) with  $\theta((\zeta - (z + 0))/h)$  replaced with  $\theta((\zeta - (z - 0))/h)$ .

This new integral for  $z$  to the left of  $\gamma$  is analyzed as above. This completes the proof of (6.16).

**6.3.3. Estimates in  $S(\text{Im}z_1, \text{Im}z_1 + h/2)$  and  $S(\text{Im}z_2 - h/2, \text{Im}z_2)$ .** Both domains are treated similarly. So, we detail only the analysis for the first one. We prove that

$$(6.17) \quad |\rho_0(z)^{-1} (D_0 f)_\beta(z)| \leq C h^{2L + \frac{2}{3}} \|f\|.$$

For  $z$  between  $\gamma + \alpha h/2$  and  $\gamma + \alpha h$ , reasoning as in section 6.3.2, one obtains (6.16) which implies (6.17).

For  $z$  between  $\gamma$  and  $\gamma + \alpha h/2$ , by contour deformation, the integration path in (6.8) is replaced with  $\tilde{\gamma}$  defined in section 6.3.2. We, thus, write  $D_0 f$  as the sum of an integral, say  $A$ , over the part of  $\tilde{\gamma} \cap \{\text{Im}\zeta \leq z_1 + h/2\}$  and an integral, say  $B$ , over the part of  $\tilde{\gamma} \cap \{\text{Im}z_1 + h/2 \leq \text{Im}\zeta\}$ .

Reasoning as in section 6.3.2, we estimate  $B$  and obtain

$$(6.18) \quad |\rho_0(z)^{-1} B| \leq C h^{2L + \frac{2}{3}} \|f\|.$$

Let us turn to  $A$ . We again use (6.13). Now, both  $|z - z_1|$  and  $|\zeta - z_1|$  are bounded by  $Ch$ ; thus,  $|\rho_0(z)/\rho_0(\zeta)| \leq C$ . Furthermore, for such  $z$ , only one pole of the integrand, the pole at the point  $z$ , can approach the integration path in  $A$ ; the other poles stay at a distance greater than  $Ch$  from it. Therefore, we get

$$|\rho_0(z)^{-1} A| \leq C h^{2L+1+\frac{2}{3}} \|f\| \int_{\text{Im}z \leq \text{Im}z_1 + h/2} \frac{|d\zeta|}{|z - \zeta| |\zeta - z_1|^\beta},$$

where we integrate along  $\tilde{\gamma}$ . Changing variable  $t = (\zeta - z_1)/|z - z_1|$ , one checks that the last integral is bounded by  $C/|z - z_1|^\beta$ . Thus,

$$|\rho_0(z)^{-1} A| \leq C h^{2L+1+\frac{2}{3}} \|f\| / |z - z_1|^\beta.$$

This and (6.18) yields (6.17).

We omit further details and note only that, to prove (6.17) when  $z$  is to the left of  $\gamma$ , we first transform the integral from (6.8) as when doing the estimations in the strip  $S(\text{Im}z_1 + h/2, \text{Im}z_2 - h/2)$  (see the end of the section section 6.3.2).

**6.3.4. Completing the proof of Proposition 6.4.** Proposition 6.4 follows from estimates (6.16) and (6.17).

**6.4. Solutions to the integral equation (6.10).** Consider the integral equation (6.10) in  $H_{\gamma, \alpha, \beta}$ . Proposition 6.4 and the estimate for  $\delta_0$  from (5.11) imply

LEMMA 6.6. *For sufficiently small  $h$ , the (6.10) has a unique solution  $g_0$  in  $H_{\gamma, \alpha, \beta}$ . It satisfies*

$$(6.19) \quad \|g_0(z)\| = O(h^{2L+2+\frac{1}{3}}).$$

Moreover, one has

LEMMA 6.7. *The solution  $g_0$ , constructed in Lemma 6.6 for the curve  $\gamma = \gamma_1$ , can be analytically continued to the domain  $K^{\alpha h}$ . It then satisfies (6.10) and in  $K^{\alpha h}$*

$$(6.20) \quad |(z_1 - z)(z_2 - z)|^\beta \frac{|g_0(z)|}{|\rho_0(z)|} \leq Ch^{2L+2+\frac{1}{3}}.$$

*Proof.* The proof is divided into four parts.

1. As  $g_0$  is analytic in  $\Pi_{\gamma_1, \alpha}$ , it suffices to continue it to the right of  $\gamma_1$ . The function  $\zeta \rightarrow \theta_0\left(\frac{\zeta - z - 0}{h}\right)$  has all its poles in  $z + 0 + h\mathbb{Z}$ . Hence, for  $z$  between  $\gamma_1$  and  $\gamma_1 + h$ , we can define  $D_0g_0$  by means of (6.8) with  $\gamma(z) = \gamma_1$ , and  $D_0g_0$  is analytic between  $\gamma_1$  and  $\gamma_1 + h$ .

As  $g_0$  is analytic between  $\gamma$  and  $\gamma + \alpha h$ , to define  $D_0g_0$  for  $z$  between  $\gamma_1 + \alpha h$  and  $\gamma_1 + (1 + \alpha)h$ , we can deform the path of the integral in (6.8) to a vertical curve connecting  $z_1$  to  $z_2$  and staying between  $\gamma_1$  and  $\gamma_1 + \alpha h$ . Thus, (6.8) implies that  $D_0g_0$  is analytic in  $z$  between  $\gamma_1$  and  $\gamma_1 + \alpha h + h$ . In view of equation (6.10), this implies that  $g_0$  itself is analytic to the left of  $\gamma + (\alpha + 1)h$ . Reasoning in this way inductively, one shows that  $g_0$  and  $D_0g_0$  are analytic between  $\gamma$  and  $\gamma + (\alpha + 2)h$ , between  $\gamma$  and  $\gamma + (\alpha + 3)h$  and so on. As a result,  $g_0$  and  $D_0g_0$  are analytic in  $K^{\alpha h}$  to the right of  $\gamma$  and satisfy (6.10) for all  $z \in K^{\alpha h}$ .

2. Clearly,  $g_0$  is analytic in  $\Pi_{\gamma_2, \alpha}$ , the expression  $|(z_1 - z)(z_2 - z)|^\beta |g_0(z)|$  stays bounded in  $\Pi_{\gamma_2, \alpha}$  (as  $\gamma_1$  and  $\gamma_2$  have common ends), and  $g_0$  satisfies (6.10) along  $\gamma = \gamma_2$ . By Lemma 6.6, for sufficiently small  $h$ , this equation has a unique solution in  $H_{\gamma_2, \alpha, \beta}$  which, thus, coincides with  $g_0$ . Hence,  $g_0$  satisfies (6.19) with the norm of  $H_{\gamma_2, \alpha, \beta}$ .

3. In view of the previous step,  $g_0$  satisfies (6.20) in  $\Pi_{\gamma_1, \alpha} \cup \Pi_{\gamma_2, \alpha}$ . This and the maximum principle for analytic functions imply that  $g_0$  satisfies (6.20) also in the domain bounded by  $\gamma_1$  and  $\gamma_2$ , i.e., in  $K$ .

The proof of Lemma 6.7 is complete.  $\square$

**6.5. The solution to the difference equation.** We define  $\Delta_0$  by (6.5) in terms of  $g_0$  constructed in subsection 6.4. One has

LEMMA 6.8. *The function  $\Delta_0$  can be analytically continued to  $K^{(1+\alpha)h}$  where it satisfies (6.4).*

*Let  $0 < c < 1 + \alpha$  and  $r > 0$ . In  $K^{ch} \cap S(\text{Im}z_1 + rh, \text{Im}z_2 - rh)$ , one has*

$$(6.21) \quad |\Delta_0(z)| \leq C|\rho_0(z)h^{2L+1}|.$$

*Proof.* By (6.20) the function  $z \mapsto |(z_1 - z)(z_2 - z)|^\beta |g_0(z)|$  is bounded in  $K^{\alpha h}$ . For a given  $z$ , the poles of the kernel in (6.5) are contained in  $z + h(\mathbb{Z} \setminus \{0\})$ . Thus, the function  $\Delta_0$  is analytic in  $(K^{\alpha h})^h = K^{(1+\alpha)h}$ .

By means of the Residue Theorem, one checks that  $H\Delta_0 = HR_0g_0$  is equal to  $g_0 + D_0g_0$  if  $\{z, z \pm h\} \subset K^{(1+\alpha)h}$ . As  $g_0$  satisfies (6.10) in  $K^{\alpha h}$ , we obtain (6.4) if  $\{z, z \pm h\} \subset K^{(1+\alpha)h}$ .

To prove (6.21), we estimate  $R_0g_0$  in the same way as in section 6.3.2 we estimated  $D_0f$ . So, we omit further details and only note that

1. outside  $(Ch)$ -neighborhood of the set  $z + h\mathbb{Z}$ , instead of (6.12) we obtain

$$\left| \frac{\rho_0(\zeta)}{\rho_0(z)} r_0(z, \zeta) \right| \leq Ch^{-\frac{4}{3}};$$

2. on the diagonal  $\{\zeta = z\}$ ,  $r_0$ , the kernel of  $R_0$ , is analytic whereas  $d_0$ , the kernel of  $D_0$ , has a pole. This simplifies the estimates of  $(R_0g_0)(z)$  to the left of  $\gamma_1$ .  $\square$

Having constructed  $\Delta_0$ , we construct a solution  $\psi_0$  to equation (1.1) setting  $\psi_0 = W_0 - \Delta_0$ , see (6.4). Let  $c \in (0, 2)$ . In view of (6.21), one has

$$(6.22) \quad \psi_0(z) = W_0(z) + O(|\rho_0(z)|h^{2L+1}), \quad z \in K^{ch} \cap S(\text{Im } z_1 + rh, \text{Im } z_2 - rh).$$

In view of (5.12), estimate (6.22) implies (6.2) with  $l = 0$  and  $L$  replaced with  $L - 1$ . As we could choose a larger  $L$ , this actually completes the proof of the statement of Theorem 6.1 on the solution  $\psi_0$ .

**6.6. The second solution.** Mutatis mutandis, the construction of the solution  $\psi_1$  repeats that of  $\psi_0$ . We omit further details and mention only that, in this case:

- we set  $\psi_1 = W_1 - R_1g_1$  where  $R_1$  is an integral operator with the kernel

$$r_1(z, \zeta) = \frac{1}{2ih} \frac{W_0(z)W_1(\zeta) - W_0(\zeta)W_1(z)}{(W_0(\zeta), W_1(\zeta))} \theta_1 \left( \frac{\zeta - z}{h} \right), \quad \theta_1(t) = \cot(\pi t) + i;$$

- instead of (6.11), we use the norm  $\|f\| = \sup_{z \in \Pi_{\gamma, \alpha}} \frac{|f_\beta(z)|}{|\rho_1(z)|}$ .

**7. The proof of the main theorem.** We now finally prove Theorem 2.7. We recall that in  $U$  there are three Stokes lines beginning at  $z_0$ . They are analytic curves and the angle between any two of them at  $z_0$  is equal to  $2\pi/3$ . So, possibly reducing  $U$  somewhat, we can assume that at least two of them form a vertical curve. We prove the theorem assuming that these curves are  $\sigma_1$  and  $\sigma_2$ , and that  $\sigma_1$  goes upwards from  $z_0$  (i.e., the vector tangent to  $\sigma_1$  at  $z_0$  is directed in the upper half-plane). Mutatis mutandis, the other cases are treated in the same way. Moreover, we assume that the tangent vector to  $\sigma_0$  is either directed in the lower half-plane or is parallel to the real line and directed to the left. Then the curves  $(\sigma_j)_{j \in \mathbb{Z}_3}$  are as in Figure 2. The complementary case is studied similarly.

Below we assume that  $h$  is sufficiently small.

**7.1. Two geometric lemmas.** To prove Theorem 2.7, we shall use the following two lemmas.

LEMMA 7.1. *There exist two curves in  $S_{1,2}$  precanonical with respect to  $p_2$  and having common endpoints, and  $\overset{\circ}{U}_1 \subset U$ , a neighborhood of  $z_0$ , such that*

- the domain  $K_1$  bounded by the two curves is simply connected,
- $K_1 \cap \mathring{U}_1 = S_{1,2} \cap \mathring{U}_1$ .

and

LEMMA 7.2. *There exist two curves in  $\sigma_2 \cup S_0 \cup \sigma_1$  precanonical with respect to  $p_0$  and having common endpoints, and  $\mathring{U}_0 \subset U$ , a neighborhood of  $z_0$ , such that*

- the domain  $K_0$  bounded by the two curves is simply connected,
- $K_0 \cap \mathring{U}_0 = (\sigma_2 \cup S_0 \cup \sigma_1) \cap \mathring{U}_0$ .

We prove these two lemmas in [subsection 8.1](#). We define  $\mathring{U} = \mathring{U}_0 \cap \mathring{U}_1$ .

**7.1.1. The solution  $\psi_1$ .** We denote by  $\psi_{0,0}$  and  $\psi_{1,0}$  the solutions  $\psi_0$  and  $\psi_1$  constructed by [Theorem 6.1](#) for the domain  $K_0$ , and consider the solution  $\psi_1$  constructed in [Theorem 6.1](#) for the domain  $K_1$ . In view of [Corollary 6.2](#), in  $\mathring{U}$  (possibly reduced somewhat), all the three solutions are analytic, the Wronskian of  $\psi_{0,0}$  and  $\psi_{1,0}$  does not vanish (see also [Lemma 5.5](#)), and one has

$$(7.1) \quad \psi_1 = a\psi_{1,0} + b\psi_{0,0},$$

where  $a$  and  $b$  are  $h$ -periodic coefficients (see section [subsection 3.2](#)). We prove

LEMMA 7.3. *One can reduce  $\mathring{U}$  so that, for  $z \in \mathring{U}$ , one has*

$$(7.2) \quad a(z) = 1 + O(h^{2L+\frac{5}{6}}), \quad b(z) = O(h^{2L+\frac{5}{6}}), \quad h \rightarrow 0.$$

*Proof.* In  $\mathring{U}$  (possibly reduced somewhat), the coefficients  $a$  and  $b$  are described by [\(3.5\)](#) with  $\psi = \psi_1$ ,  $f = \psi_{1,0}$  and  $g = \psi_{0,0}$ .

Let  $\gamma_{12} = (\sigma_1 \cup \sigma_2) \cap \mathring{U}$ . By [Lemma 7.1](#) and [Lemma 7.2](#) one has  $\gamma_{12} \subset K_0$  and  $\gamma_{12} \subset K_1$ .

First, we fix  $c \in (1, 2)$  and assume that  $\{z, z+h\} \subset (\gamma_{12})^{ch}$ .

In view of [Lemma 5.2](#), one has  $|\rho_1| = |\rho_2| = 1$  on  $\gamma_{12}$ . This and the definitions of  $|\rho_1|$  and  $|\rho_2|$  in [section 5.2.2](#) imply that there exists  $C > 0$  such that  $|\rho_1(z)|, |\rho_2(z)| \leq C$  in  $(\gamma_{12})^{ch}$ .

As  $(\gamma_{12})^{ch}$  is a subset of both  $K_0^{ch}$  and  $K_1^{ch}$ , by means of [\(6.2\)](#) and [\(5.10\)](#), we get

$$a = \frac{(\psi_1, \psi_{0,0})}{(\psi_{1,0}, \psi_{0,0})} = \frac{(W_1 + O(h^{2L+\frac{3}{2}}), W_0 + O(h^{2L+\frac{3}{2}}))}{(W_1 + O(h^{2L+\frac{3}{2}}), W_0 + O(h^{2L+\frac{3}{2}}))} = \frac{(W_1, W_0) + O(h^{2L+1+\frac{5}{6}})}{(W_1, W_0) + O(h^{2L+1+\frac{5}{6}})}.$$

[Lemma 5.5](#), then, yields the asymptotic representation for  $a$  in [\(7.2\)](#). Reasoning similarly, we get

$$b = \frac{(\psi_{1,0}, \psi_1)}{(\psi_{1,0}, \psi_{0,0})} = \frac{(W_1, W_1) + O(h^{2L+1+\frac{5}{6}})}{(W_1, W_0) + O(h^{2L+1+\frac{5}{6}})} = O(h^{2L+\frac{5}{6}}).$$

This is the estimate for  $b$  in [\(7.2\)](#).

Let  $c_1$  and  $c_2$  correspond to the minimal strip  $S(c_1, c_2)$  containing  $(\gamma_{12})^{ch}$ . We proved estimates [\(7.2\)](#) for  $a(z)$  and  $b(z)$  in the case where  $\{z, z+h\} \subset (\gamma_{12})^{ch}$ . As  $c > 1$  and as  $a$  and  $b$  are  $h$ -periodic, these estimates remain valid in  $S(c_1, c_2)$ . This implies [Lemma 7.3](#).  $\square$

In view of [Lemma 7.1](#), the solution  $\psi_1$  admits representation [\(6.2\)](#) with  $l = 1$  in  $S_{1,2} \cap \overset{\circ}{U}$ . Let us prove that it admits this representation in  $S_0 \cap \overset{\circ}{U}$ .

In view of [Lemma 7.2](#), the solutions  $\psi_{0,0}$  and  $\psi_{1,0}$  admit representations [\(6.2\)](#) with  $l = 0$  and  $l = 1$  in  $S_0 \cap \overset{\circ}{U}$ . Substituting [\(7.2\)](#) and these representations into [\(7.1\)](#) and using [\(5.10\)](#), for  $z \in S_0 \cap \overset{\circ}{U}$ , we compute

$$\begin{aligned} \psi_1(z) &= (1 + O(h^{2L+\frac{5}{6}}))(W_1(z) + O(h^{2L+1+\frac{1}{2}}\rho_1(z))) \\ &\quad + O(h^{2L+\frac{5}{6}})(W_0(z) + O(h^{2L+1+\frac{1}{2}}\rho_0(z))) \\ &= W_1(z) + O(h^{2L+\frac{7}{6}}\rho_1(z)) + O(h^{2L+\frac{7}{6}}\rho_0(z)). \end{aligned}$$

In view of [Lemma 5.2](#), in  $S_0$  one has  $|\rho_0(z)| \leq |\rho_1(z)|$ . For  $\psi_1$  in  $S_0 \cap \overset{\circ}{U}$ , this implies representation [\(6.2\)](#) with  $l = 1$  and  $L$  replaced with  $L - 1$ . As  $L$  is arbitrary, we have proved [\(6.2\)](#) for  $\psi_1$  in the whole domain  $\overset{\circ}{U}$ .

Now, we note that

$$(7.3) \quad h^{\frac{1}{2}}|\rho_0(z)| \leq C|h^{\frac{1}{3}}w_0(h^{-\frac{2}{3}}\zeta(z))| + C|h^{\frac{1}{3}}w'_0(h^{-\frac{2}{3}}\zeta(z))|, \quad z \in U.$$

For sufficiently large values of  $h^{-\frac{2}{3}}|\zeta(z)|$ , this estimate follows from the definition of  $\rho_0$  and the asymptotic formulas [\(3.2\)](#). For bounded  $h^{-\frac{2}{3}}|\zeta(z)|$ , it follows from the fact that  $w$  and  $w'$  do not have common zeros.

Estimates [\(6.2\)](#) and [\(7.3\)](#) imply [\(2.16\)](#) with  $L$  replaced with  $L - 1$ . As  $L$  is arbitrary, this completes the proof of the statement of [Theorem 2.7](#) on the solution  $\psi_1$  in the case that we consider.

**7.1.2. The solution  $\psi_0$ .** Let  $\psi_{1,1}$  and  $\psi_{2,1}$  be the solutions  $\psi_1$  and  $\psi_2$  constructed by [Theorem 6.1](#) for the domain  $K_1$ , and let  $\psi_0$  be the solution constructed by [Theorem 6.1](#) for the domain  $K_0$ . For  $z \in \overset{\circ}{U}$  (possibly reduced somewhat) one has

$$(7.4) \quad \psi_0 = a\psi_{1,1} + b\psi_{2,1},$$

where  $a$  and  $b$  are  $h$ -periodic. One proves

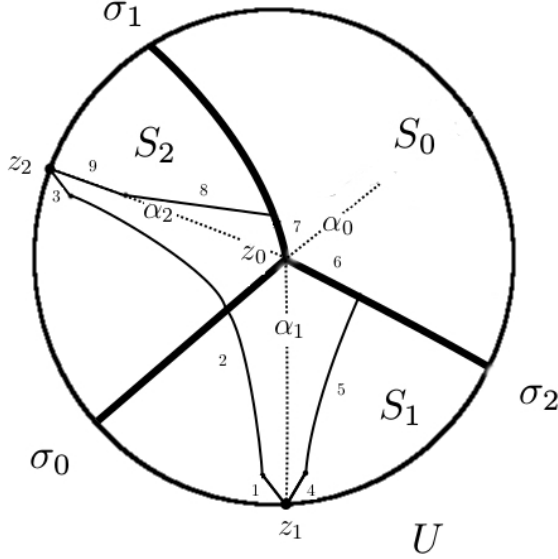
LEMMA 7.4. *One can reduce  $\overset{\circ}{U}$  so that, for  $z \in \overset{\circ}{U}$ , one has*

$$(7.5) \quad a(z) = -1 + O(h^{2L+\frac{5}{6}}), \quad b(z) = -1 + O(h^{2L+\frac{5}{6}}), \quad h \rightarrow 0.$$

*Proof.* We omit details explained in the course of the proof of [Lemma 7.3](#). We fix  $c \in (1, 2)$  and assume that  $\{z, z + h\} \subset (\gamma_{12})^{ch}$ . For the coefficient  $a$  from [\(7.4\)](#), we get

$$a = \frac{(\psi_0, \psi_{2,1})}{(\psi_{1,1}, \psi_{2,1})} = \frac{(W_0, W_2) + O(h^{2L+1+\frac{5}{6}})}{(W_1, W_2) + O(h^{2L+1+\frac{5}{6}})} = \frac{(-W_1 - W_2, W_2) + O(h^{2L+1+\frac{5}{6}})}{(W_1, W_2) + O(h^{2L+1+\frac{5}{6}})},$$

where, in the last step, we used relation [\(5.1\)](#). Continuing, we get  $a = -1 + O(h^{2L+\frac{5}{6}})$ . Similarly one proves that  $b = -1 + O(h^{2L+\frac{5}{6}})$ . So, [\(7.5\)](#) is proved when  $\{z, z + h\} \subset (\gamma_{12})^{ch}$ . Reasoning as in the completion of the proof of [Lemma 7.3](#), we complete the proof of [Lemma 7.4](#).  $\square$


 FIG. 3. Domain  $K_1$ 

By [Theorem 6.1](#) and [Lemma 7.2](#), the solution  $\psi_0$  admits representation [\(6.2\)](#) with  $l = 0$  in  $(\sigma_1 \cup S_0 \cup \sigma_2) \cap \overset{\circ}{U}$ . Estimates [\(7.5\)](#) and [\(5.10\)](#) imply that in  $S_{1,2} \cap \overset{\circ}{U}$  one has

$$\psi_0 = -W_1 - W_2 + O((|\rho_1| + |\rho_2|)h^{2L+7/6}) = W_0 + O((|\rho_1| + |\rho_2|)h^{2L+7/6}).$$

In view of [Lemma 5.2](#) and the definitions of  $|\rho_j|$ , in  $S_{1,2}$  one has  $|\rho_1| + |\rho_2| \leq C|\rho_0|$  which implies [\(6.2\)](#) with  $l = 0$  in  $S_2 \cap \overset{\circ}{U}$ . Reasoning as in the completion of [section 7.1.1](#), we complete the proof of [Theorem 2.7](#) for  $\psi_0$  in the case that we consider.

**7.1.3. The solution  $\psi_2$ .** As the main theorem for  $\psi_2$  is proved using the same techniques as for  $\psi_0$  and  $\psi_1$ , we omit the details and note only that, in  $S_0$ , one represents  $\psi_2$  as a linear combination of  $\psi_{1,0}$  and  $\psi_{0,0}$ , and computes the coefficients in this linear combination as in the case of  $\psi_0$ .

## 8. The proofs of the geometric lemmas.

**8.1. The proof of [Lemma 7.1](#).** This is done in several steps.

Below, all the precanonical lines are precanonical with respect to the branch  $p_2$  of the complex momentum. We recall that  $p_2$  is defined and analytic in the domain  $U_2$  and continuous up to its boundary.

**8.1.1. Anti-Stokes lines.** We recall that the Stokes lines  $\sigma_j$  are defined by [\(5.4\)](#). The *Anti-Stokes lines*,  $(\alpha_j)_{j \in \mathbb{Z}_3}$ , are defined as

$$(8.1) \quad \alpha_j := \zeta^{-1}(V \cap e^{-2\pi ij/3} [0, +\infty)).$$

For  $j \in \mathbb{Z}_3$ ,  $\sigma_j \cap \alpha_j = \{z_0\}$  and the curve  $\sigma_j \cup \alpha_j$  is analytic. The angles between any two of the Anti-Stokes lines at  $z_0$  equal  $2\pi/3$ .

In the case we study, the Stokes and Anti-Stokes lines are pictured in [Figure 3](#); the Anti-Stokes lines are represented by dotted lines. In particular,  $\alpha_2$  goes up from  $z_0$ , and  $\alpha_1$  goes down from  $z_0$ .

Reducing  $U$  if necessary, we assume that the Anti-Stokes lines  $\alpha_1$  and  $\alpha_2$  are vertical in  $U$ . As in [Figure 3](#), let  $z_1$  be the lower end of  $\alpha_1$  and  $z_2$  the upper end of  $\alpha_2$ .

One has

**LEMMA 8.1.** *Along the Anti-Stokes lines  $\alpha_0$ ,  $\alpha_1$  and  $\alpha_2$ , one has  $\operatorname{Re} \int_{z_0}^z p_2 dz = 0$ . The vector field  $z \mapsto v(z) = \nabla \operatorname{Im} \int_{z_0}^z p_2 dz$  vanishes only at  $z = z_0$ . The Anti-Stokes lines are tangent to this vector field at  $z \neq z_0$ . As  $z$  moves away from  $z_0$ ,  $\operatorname{Im} \int_{z_0}^z p_2 dz$  monotonously increases along  $\alpha_2$  and monotonously decreases along  $\alpha_1$  and  $\alpha_0$ .*

*Proof.* The statement on  $\operatorname{Re} \int_{z_0}^z p_2 dz$  follows directly from the definitions of the function  $\zeta$  and of the Anti-Stokes lines. We note that  $\|v(z)\| = |p_2(z)|$  and that  $p_2(z)$  vanishes only at  $z_0$  (modulo  $\pi$ , the complex momentum vanishes only at turning points and  $z_0$  is the only turning point in  $U$ ). Therefore, the vector field  $v$  vanishes only at  $z = z_0$ . The statement on  $\operatorname{Re} \int_{z_0}^z p_2 dz$  and the Cauchy-Riemann equations imply that the Anti-Stokes lines are tangent to the vector field  $v$  at points where it does not vanish. This and the first two points of [Lemma 5.2](#) imply the statements of [Lemma 8.1](#) on  $\operatorname{Im} \int_{z_0}^z p_2 dz$ .  $\square$

We also use

**LEMMA 8.2.** *There exists  $\tilde{U} \subset U$ , a neighborhood of  $z_0$ , such that the lines  $\alpha_1 \cap \tilde{U}$  and  $\alpha_2 \cap \tilde{U}$  are precanonical.*

Let us parameterize  $(\alpha_1 \cup \alpha_2) \cap \tilde{U}$  by  $y = \operatorname{Im} z$ ,  $z = z(y) = x(y) + iy$ . Then, if  $y \neq \operatorname{Im} z_0$ , one has

$$(8.2) \quad \frac{d}{dy} \operatorname{Im} \int_{z_0}^{z(y)} p_2(z) dz > 0,$$

$$(8.3) \quad \frac{d}{dy} \operatorname{Im} \int_{z_0}^{z(y)} (p_2(z) - \pi) dz < 0.$$

*Proof.* As  $\alpha_1$  and  $\alpha_2$  are vertical, inequality (8.2) follows from [Lemma 8.1](#). Furthermore, one has

$$\frac{d}{dy} \operatorname{Im} \int_{z_0}^{z(y)} (p_2 - \pi) dz = \operatorname{Im}(z'(y)p_2(z)) - \pi.$$

Therefore, as  $p_2(z_0) = 0$ , reducing  $U$  somewhat if necessary, we ensure (8.3).

Since  $\alpha_1 \cup \alpha_2$  is vertical, (8.2) and (8.3) imply that the curve  $\alpha_1 \cup \alpha_2$  is precanonical.  $\square$

Below, we assume that  $\tilde{U} = U$  (if necessary we reduce  $U$  somewhat).

**8.1.2. The precanonical line  $\gamma_1$ .** We now construct a precanonical line  $\gamma_1 \subset S_{1,2}$ . It consists of three segments 1, 2 and 3 shown in [Figure 3](#). Let us describe them. *The segments 1 and 3.* To construct these segments, we use

**LEMMA 8.3.** *Let  $\gamma$  be a compact vertical  $C^1$ -curve parameterized by  $y = \operatorname{Im} z$ ,  $z = z(y) = x(y) + iy$ . We assume that (8.2)–(8.3) hold along  $\gamma$ . Then, any compact  $C^1$ -curve sufficiently close in  $C^1$ -topology to  $\gamma$  is precanonical.*

This statement immediately follows from the definition of the precanonical curves.

*The segment 1.* It is a segment of a compact precanonical  $C^1$ -curve  $c_1 \subset S_{1,2}$  that begins at  $z_1$  and above  $z_1$  goes to the left of  $\alpha_1$ . When choosing  $c_1$ , we take an internal point of  $\alpha_1$  as  $\tilde{z}_1$ , and, as  $c_1$ , we take a  $C^1$ -curve close enough in  $C^1$ -topology to  $\alpha_1$  between  $z_1$  and  $\tilde{z}_1$ . [Lemma 8.2](#) and [Lemma 8.3](#) guarantee that  $c_1$  is a precanonical line.

*The segment 3.* Similarly, the segment 3 is a segment of a compact precanonical  $C^1$ -curve  $c_3 \subset S_{1,2}$ , having the upper end at  $z_2$  and going to the left of  $\alpha_2$  below the point  $z_2$ .

*The segment 2.* We note that  $\alpha_1 \cup \alpha_2$  is a level curve of the harmonic function  $z \rightarrow \operatorname{Re} \int_{z_0}^z p_2(z) dz$  in  $S_{1,2}$ . The segment 2 is a segment of another level curve  $c_2$  of this function in  $S_{1,2}$ . This curve is located to the left of  $\alpha_1 \cup \alpha_2$ . As it does not contain the point  $z_0$ , the only point in  $S_{1,2}$  where  $p_2$  vanishes,  $c_2$  is smooth. We choose  $c_2$  sufficiently close to  $\alpha_1 \cup \alpha_2$  to ensure that

- $c_2$  is vertical (as  $\alpha_1$  and  $\alpha_2$  are);
- one has [\(8.2\)](#) along  $c_2$  (the vector field  $\nabla \operatorname{Im} \int_{z_0}^z p_2(z) dz$  does not vanish along  $c_2$  and is tangent to  $c_2$ );
- [\(8.3\)](#) holds along  $c_2$  (as it holds along  $\alpha_1 \cup \alpha_2$ );
- $c_2$  intersects both  $c_1$  and  $c_3$ .

Clearly,  $c_2$  is precanonical.

*The curve  $\gamma_1$ .* The segment 1 is the segment of  $c_1$  between  $z_1$  and the point of intersection of  $c_1$  and  $c_2$ , the segment 2 is the segment of  $c_2$  between the segment 1 and the point of intersection of  $c_2$  and  $c_3$ , and the segment 3 is the segment of  $c_3$  connecting the segment 2 with  $z_2$ . Clearly, the curve  $\gamma_1$  made of segments 1–3 is precanonical.

**8.1.3. The sign of  $\operatorname{Imp}_2$  in  $S_2$ .** The only place where we use our assumption on the direction of the tangent vector to  $\sigma_0$  at  $z_0$  is the proof of

**LEMMA 8.4.** *Both in  $S_2$  between the curves  $\alpha_2$  and  $\sigma_1$  and on these curves, near  $z_0$ , one has  $\operatorname{Imp}_2(z) < 0$  when  $z \neq z_0$ .*

*Proof.* Below we assume that either  $z$  is in  $S_2$  between the curves  $\alpha_2$  and  $\sigma_1$  or on one of these curves. In view of [\(2.3\)](#), we can write

$$(8.4) \quad p_2(z) = k_1 \tau (1 + O(\tau)), \quad \int_{z_0}^z p_2(z) dz = \frac{2}{3} k_1 \tau^3 (1 + O(\tau)), \quad z \rightarrow z_0,$$

where  $k_1 \neq 0$  and  $\tau$  is the branch of  $\sqrt{z - z_0}$  analytic in  $U_2$  and positive if  $z > z_0$ .

Let  $0 < \theta_2 < \pi$  be the angle at  $z_0$  between the line  $\{z \geq z_0\}$  and the curve  $\alpha_2$ . Note that the angle between  $\sigma_0$  and  $\alpha_2$  equals  $\pi/3$ . Therefore, as the tangent vector to  $\sigma_0$  at  $z_0$  is either directed downwards or parallel to the real line and directed to the left, one has  $2\pi/3 \leq \theta_2 < \pi$ .

In view of [Lemma 8.1](#), along  $\alpha_2$ ,  $\operatorname{Re} \int_{z_0}^z p_2 dz = 0$  and  $\operatorname{Im} \int_{z_0}^z p_2 dz$  is monotonously increasing. This and the second formula in [\(8.4\)](#) imply that

$$(8.5) \quad \arg k_1 + \frac{3}{2} \theta_2 = \frac{\pi}{2} \pmod{2\pi}.$$

Let  $z - z_0 = |z - z_0| e^{i\theta}$ . Using [\(8.5\)](#) and the first formula in [\(8.4\)](#), we get near  $z_0$

$$(8.6) \quad \frac{\operatorname{Imp}_2(z)}{|p_2(z)|} = \sin \left( \arg k_1 + \frac{\theta}{2} + o(1) \right) = \cos \left( \theta_2 - \frac{\theta - \theta_2}{2} + o(1) \right).$$



Now, we note that, for  $z$  we consider, near  $z_0$  one has  $\theta_2 - \pi/3 + o(1) \leq \theta \leq \theta_2 + o(1)$ . Therefore, for  $z$  sufficiently close to  $z_0$ , one has

$$\frac{2\pi}{3} + o(1) \leq \theta_2 + o(1) \leq \theta_2 - \frac{\theta - \theta_2}{2} \leq \theta_2 + \frac{\pi}{6} + o(1) < \frac{7\pi}{6}.$$

This and (8.6) implies the statement of Lemma 8.4.  $\square$

**8.1.4. The precanonical line  $\gamma_2$ .** The precanonical line  $\gamma_2$  is located in  $S_{1,2}$  and consists of six segments 4–9 shown in Figure 3. Let us describe them.

*The segments 4-5-6-7.* The segment 4 is a segment of a compact precanonical  $C^1$ -curve  $c_4 \subset S_{1,2}$ . This curve begins at  $z_1$  and above  $z_1$  goes to the right of  $\alpha_1$ . It is constructed as the curve  $c_1$  containing the segment 1.

The segment 5 is a segment of a level curve  $c_5$  of the function  $z \rightarrow \operatorname{Re} \int_{z_0}^z p_2(z) dz$  in  $S_{1,2}$ . The construction of  $c_5$  is similar that of  $c_2$ . The curve  $c_5$  is located to the right of  $\alpha_1$ . We choose  $c_5$  sufficiently close to  $\alpha_1$ . Then,  $c_5$  is a precanonical curve and intersects both  $c_4$  and the Stokes line  $\sigma_2$ .

The segment 4 is the segment of  $c_4$  between  $z_1$  and the point of intersection of  $c_4$  and  $c_5$ . The segment 5 connects this point to a point of  $\sigma_2$ .

We prove

LEMMA 8.5. *Let  $\gamma$  be a vertical curve, let  $a \in \gamma$  and let  $p$  be a branch of the complex momentum continuous on  $\gamma$ . If either  $\operatorname{Im} \int_a^z p(z) dz = 0$  or  $\operatorname{Im} \int_a^z (p(z) - \pi) dz = 0$  are satisfied on  $\gamma$ , then  $\gamma$  is precanonical with respect to  $p$ .*

*Proof.* Assume that  $\operatorname{Im} \int_a^z p(z) dz = 0$  on  $\gamma$ . Then,  $z \mapsto \operatorname{Im} \int_a^z (p(z) - \pi) dz = -\pi \operatorname{Im}(z - a)$  is decreasing along  $\gamma$  when  $\operatorname{Im} z$  increases. Thus,  $\gamma$  is precanonical. If  $\operatorname{Im} \int_a^z (p(z) - \pi) dz = 0$ , then  $z \mapsto \operatorname{Im} \int_a^z p(z) dz = \operatorname{Im} \int_a^z \pi dz = \operatorname{Im}(z - a)$  is increasing along  $\gamma$  when  $\operatorname{Im} z$  increases. Thus,  $\gamma$  is precanonical.  $\square$

The segment 6 is the segment of  $c_6 = \sigma_2$  between the upper end of the segment 5 and the point  $z_0$ . The segment 7 is the segment of  $c_7 = \sigma_1$  between  $z_0$  and an internal point  $a$  of  $\sigma_1$ . We describe this point later. Lemma 8.5 implies that the segments 6 and 7 are precanonical.

*Segment 8.* This segment is a segment of  $c_8$ , the level curve  $\gamma(a)$  of the harmonic function  $z \rightarrow \operatorname{Im} \int_{z_0}^z (p_2(z) - \pi) dz$  that contains  $a \in \sigma_1$ . To choose the segment 8, we check

LEMMA 8.6. *If  $a \in \sigma_1 \setminus \{z_0\}$  is sufficiently close to  $z_0$ , then  $\gamma(a)$  intersects  $\sigma_1$  transversally at  $a$ , enters  $S_2$  at a going upwards, intersects  $\alpha_2$  and remains vertical up to this intersection.*

*Proof of Lemma 8.6.* Below, we identify vectors on  $\mathbb{R}^2$  with complex numbers in the standard way and the bar denotes complex conjugation. The Stokes line  $\sigma_1$  is tangent to the vector field  $z \mapsto v_0(z) = \overline{p_2(z)}$  at  $z \neq z_0$  ( $p_2(z_0) = 0$ ). The curve  $\gamma(a)$  is tangent to the vector field  $z \mapsto v_\pi(z) = p_2(z) - \pi$ .

Let  $a \in \sigma_1 \setminus z_0$  be sufficiently close to the point  $z_0$ . In view of Lemma 8.4,  $\operatorname{Im} p_2(a) < 0$ . Therefore,  $\gamma(a)$  is vertical at  $a$ . Moreover, both the vectors  $v_0(a)$  and  $v_\pi(a)$  are directed upwards and  $v_\pi(a)$  is directed to the left of  $v_0(a)$ . Therefore, at  $a$ , the curve  $\gamma(a)$  intersects  $\sigma_1$  transversally and enters  $S_2$  going upwards.

Furthermore, in view of Lemma 8.4, as long as  $\gamma(a)$  stays in  $S_2$  near  $z_0$  between the curves  $\alpha_2$  and  $\sigma_1$  or on them, it remains vertical.

To complete the proof, it suffices to show that if  $a$  is sufficiently close to  $z_0$ , then  $\gamma(a)$  intersects  $\alpha_2$  remaining vertical. Therefore, we note that  $v_\pi(z_0) = -\pi$ . So, at

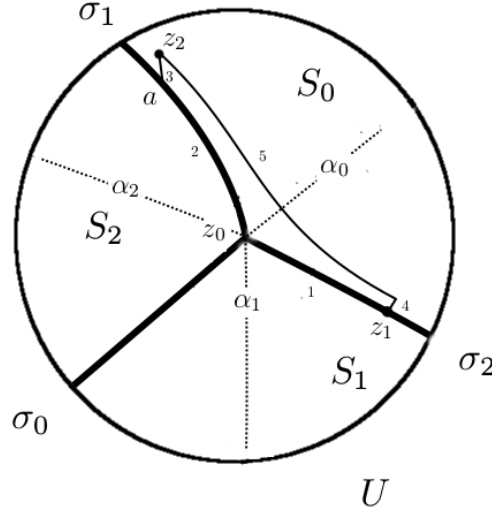


FIG. 4. Domain  $K_0$

$z_0$  the vector tangent to  $\gamma(z_0)$  is parallel to  $\mathbb{R}$ , and the curve  $\gamma(z_0)$  intersects the analytic curve  $\alpha_2 \cup \sigma_2$  transversally. Depending continuously on  $a$ ,  $\gamma(a)$  intersects this curve also for all  $a$  sufficiently close to  $z_0$ . But, if  $\text{Im } a > \text{Im } z_0$  and  $a$  is sufficiently close to  $z_0$ , the curve  $\gamma(a)$  goes upward from  $a$ . Therefore, for  $a$  sufficiently close to  $z_0$ , the curve  $\gamma(a)$  intersects  $\alpha_2$  still remaining vertical. This completes the proof of Lemma 8.6.  $\square$

*The segments 8 and 9.* We choose the point  $a$ , the end of the segment 7 and the beginning of the segment 8, so that  $c_8 = \gamma(a)$  intersects  $\alpha_2$  as described in Lemma 8.6. The end of the segment 8 is the point of intersection of  $c_8$  and  $\alpha_2$ . By Lemma 8.5, the segment 8 is precanonical. The segment 9 is the segment of  $\alpha_2$  connecting the upper end of the segment 8 to the point  $z_2$ . It precanonical by Lemma 8.2.

*The domain  $K_1$*  bounded by  $\gamma_1$  and  $\gamma_2$  is the one described in Lemma 7.1, the proof of which is complete.

**8.2. The proof of Lemma 7.2.** The proof uses the same techniques as the proof of Lemma 7.1. Therefore, we omit some details. The construction of the curves  $\gamma_1$  and  $\gamma_2$  bounding the domain  $K_0$  from Lemma 7.2 is illustrated by Figure 4. Below, all the precanonical lines are precanonical with respect to  $p_0$ .

**8.2.1. The curve  $\gamma_1$ .** This curve consists of segments 1–3 that we describe now.

Let  $z_1$  be an internal point of  $\sigma_2$  and fix  $a$ , an internal point of  $\sigma_1$ . The segment 1 is the segment of  $\sigma_2$  between  $z_1$  and  $z_0$ , and the segment 2 is the segment of  $\sigma_1$  between  $z_0$  and  $a$ .

To describe the segment 3, we consider  $\gamma_0(a)$ , the curve in  $\sigma_2 \cup S_0 \cup \sigma_1$  described by the equation  $\text{Im} \int_a^z (p_0(z) - \pi) dz = 0$ . We suppose that  $a$  is sufficiently close to  $z_0$ . Then,  $\gamma_0(a)$  intersects  $\sigma_1$  at  $a$  transversally, enters in  $S_0$  going upwards and is vertical in a neighborhood  $a$ . To prove this, note that, near  $z_0$  on  $\sigma_1$ , one has  $\text{Im} p_0(z) > 0$  (this is proved in the same way as Lemma 8.4). The segment 3 is a segment of  $\gamma_0(a)$  connecting in this neighborhood  $a$  to a point  $z_2 \in S_2$  that we choose later on.

Lemma 8.5 implies that the segments 1–3 are precanonical.

The points  $z_1$  and  $z_2$  are the ends of  $\gamma_1$ .

**8.2.2. The curve  $\gamma_2$ .** This curve consists of two segments, segments 4 and 5. The segment 4 is a segment of  $c_4$ , the level curve of the function  $z \rightarrow \operatorname{Re} \int_{z_1}^z p_0(z) dz$  in  $S_0 \cup \sigma_2$  that passes through  $z_1$ . The curve  $c_4$  is orthogonal to  $\sigma_2$  at  $z_1$ .

Let us note that, under our assumptions on  $\sigma_0$  and  $\sigma_2$  (see the very beginning of section 7), the angle at  $z_0$  between  $\sigma_2$  and the horizontal line  $\{z \geq z_0\}$  belongs to  $(0, \pi/3)$ . Possibly reducing  $U$  somewhat, we assume that, at any point  $\zeta \in \sigma_2$ , the angle between  $\sigma_2$  and the line  $\{z \geq \zeta\}$  belongs to  $(0, \pi/2)$ . Then,  $c_4$  is vertical at least in a neighborhood of the point  $z_1$  and goes upward from  $z_1$  into  $S_0$ .

The segment 5 is a segment of a level curve  $c_5$  of the function  $z \rightarrow \operatorname{Im} \int_{z_0}^z p_0(z) dz$  in  $S_0$ . It is located to the right of  $\sigma_2 \cup \sigma_1$  (which is also a level curve of this function). We choose the curve  $c_5$  sufficiently close to  $\sigma_2 \cup \sigma_1$ . Then, it is vertical, intersects  $\gamma_0(a)$  and  $c_4$ , and the segments of these curves between  $\sigma_2 \cup \sigma_1$  and the intersection points are vertical.

The point  $z_2$  is the point of intersection of  $\gamma_0(a)$  and  $c_5$ . The segment 4 is the segment of  $c_4$  between  $\sigma_2 \cup \sigma_1$  and  $c_5$ , and the segment 5 is the segment of  $c_5$  connecting  $c_4$  to  $z_2$ .

The segment 5 is precanonical in view of Lemma 8.5. Arguing as when proving Lemma 8.2 and reducing somewhat  $U$  if necessary, we check that the segment 4 is precanonical.

The domain  $K_0$  bounded by the curves  $\gamma_1$  and  $\gamma_2$  is the domain described in Lemma 7.2, the proof is complete.

#### REFERENCES

- [1] A. AVILA, S. JITOMIRSKAYA, *The Ten Martini Problem*, Annals Math., 170 (2009), pp. 303–342.
- [2] V. BABICH, M. LYALINOV, AND V. GRIKUROV, *Diffraction theory: the Sommerfeld-Malyuzhinets technique*, Alpha Science, Oxford, 2008.
- [3] V. BUSLAEV AND A. FEDOTOV, *Complex WKB method for Harper equation*, St. Petersburg Math. J., 6 (1995), pp. 495–517.
- [4] V. BUSLAEV AND A. FEDOTOV, *The monodromization and Harper equation*, Séminaire sur les Équations aux Dérivées Partielles 1993-1994, Exp. no EXXI, École Polytech., Palaiseau, 1994, 23 pp.
- [5] O. COSTIN AND R. COSTIN, *Rigorous WKB for finite-order linear recurrence relations with smooth coefficients*, SIAM J. math. an., 27 (1996), pp. 110–134.
- [6] A. ECKSTEIN, *Unitary reduction for the two-dimensional Schrödinger operator with strong magnetic field*, Math. Nachr., 282(2009), pp. 504-525.
- [7] S. Y. DOBROKHOTOV AND A. V. TSVETKOVA, *On lagrangian manifolds related to asymptotics of Hermite polynomials*, Math. notes, 110 (2018).
- [8] M. FEDORYUK, *Asymptotic Analysis. Linear Ordinary Differential Equations*, Springer-Verlag, Berlin, Heidelberg, 2009.
- [9] A. FEDOTOV, *Monodromization method in the theory of almost-periodic equations*, St. Petersburg Math. J., 25 (2014), pp. 303-325.
- [10] A. FEDOTOV AND F. KLOPP, *Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case*, Commun. in Math. Phys., 227 (2002), pp. 1–92.
- [11] A. FEDOTOV AND F. KLOPP, *On the absolutely continuous spectrum of an one-dimensional quasiperiodic Schrödinger operator in adiabatic limit*, Transactions of AMS, 357 (2005), pp. 4481–4516.
- [12] A. FEDOTOV AND F. KLOPP, *Difference equations, uniform quasiclassical asymptotics and Airy functions*, in Proceeding of the conference "Days on Diffraction 2018", St. Petersburg, 2018, IEEE, pp. 98–101.
- [13] A. FEDOTOV AND E. SHCHETKA, *Complex WKB method for difference equations in bounded domains*, J. of Math. Sciences (New York), 224 (2017), pp. 157–169.
- [14] A. FEDOTOV AND E. SHCHETKA, *Complex WKB method for the difference Schrödinger equation with the potential being a trigonometric polynomial*, St. Petersburg Math. J., 29 (2018),

- pp. 363–381.
- [15] J. S. GERONIMO, O. BRUNO, AND W. V. ASSCHE, *WKB and turning point theory for second-order difference equations*, Operator theory: advances and app., 69 (1992), pp. 269–301.
  - [16] J. P. GUILLEMENT, B. HELFFER, AND P. TRETON, *Walk inside Hofstadter’s butterfly*, J. Phys. France, 50 (1989), pp. 2019–2058.
  - [17] B. HELFFER, AND J. SJÖSTRAND, *Analyse semi-classique pour l’équation de Harper (avec application à l’équation de Schrödinger avec champ magnétique)*, Mémoires de la SMF (nouvelle série), 34(1988), pp. 1-113.
  - [18] M. A. LYALINOV, AND N. Y. ZHU, *A solution procedure for second-order difference equations and its application to electromagnetic-wave diffraction in a wedge-shaped region*, Proc. R. Soc. Lond. A, 459(2003), pp. 3159-3180.
  - [19] F. W. J. OLVER, *Asymptotics and Special Functions*, Academic Press, New York, 1974.
  - [20] Y. SIBUYA, *Global theory of a second order linear ordinary differential equation with a polynomial coefficient*, North Holland/American Elsevier, 1975.
  - [21] G. A. TSYGANOV, *Asymptotic behavior of solutions of a linear difference system with a small shift and a turning point*, Diff. equations (in Russian), 10 (1974), pp. 1312–1321.
  - [22] A. B. VASILIEVA AND E. YULDASHEV, *Linear difference systems with a small shift*, Diff. equations (in Russian), 6 (1970), pp. 2267–2269.
  - [23] W. WASOW, *Asymptotic expansions for ordinary differential equations*, Dover Publications, New York, 1987.
  - [24] M. WILKINSON, *An exact renormalization group for Bloch electrons in a magnetic field*, J. Phys. A: Math. Gen., 20 (1987), pp. 4337–4354.
  - [25] R. WONG, *Asymptotic approximations of integrals*, SIAM, Philadelphia, 2001.
  - [26] R. WONG AND Z. WANG, *Asymptotic expansions for second-order linear difference equations with a turning point*, Numerische Mathematik, 94 (2003), pp. 147–194.