HAL open science

The complex WKB method for difference equations and Airy functions

Alexander Fedotov, Frédéric Klopp

To cite this version:

Alexander Fedotov, Frédéric Klopp. The complex WKB method for difference equations and Airy functions. 2019. hal-01892639v4

HAL Id: hal-01892639

https://hal.science/hal-01892639v4

Preprint submitted on 15 May 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE COMPLEX WKB METHOD FOR DIFFERENCE EQUATIONS AND AIRY FUNCTIONS

ALEXANDER FEDOTOV* AND FRÉDÉRIC KLOPP†

Abstract. We consider the difference Schrödinger equation $\psi(z+h)+\psi(z-h)+v(z)\psi(z)=0$ where z is a complex variable, $h>0$ is a parameter, and v is an analytic function. As $h\to 0$ analytic solutions to this equation have a simple WKB behavior near the points where $v(z)\neq\pm 2$. We study analytic solutions near the points z_0 satisfying $v(z_0)=\pm 2$ and $v'(z_0)\neq 0$. These points play the same role as simple turning points for the differential equation $-\psi''(z)+v(z)\psi(z)=0$. In an h-independent complex neighborhood of such a point, we derive uniform asymptotic expansions for analytic solutions to the difference equation.

Key words. Difference equations, WKB, turning point

AMS subject classifications. 34M60, 39A45

1. Introduction.

1.1. The problem. We study analytic solutions to the difference Schrödinger equation

$$\psi(z+h)+\psi(z-h)+v(z)\psi(z)=0 \tag{1.1}$$

where z is a complex variable, h is a positive parameter and v is an analytic function. We describe their asymptotics behavior for small h [1].

Note that the parameter h is a standard WKB or quasiclassical parameter. Indeed, formally, $\psi(z+h)=\sum_{l=0}^{\infty}\frac{h^l}{l!}\frac{d^l\psi}{dz^l}(z)=e^{h\frac{d}{dz}}\psi(z)$, and so, h can be regarded as a small parameter in front of the derivative.

It is worth comparing (1.1) with the equation

$$\phi_{k+1}+\phi_{k-1}+v(kh+\theta)\phi_k=0, \tag{1.2}$$

where $k\in\mathbb{Z}$ is an integer variable, and $\theta\in\mathbb{C}$ is a parameter. If ψ satisfies (1.1), then one can construct a solution to (1.2) (with a suitable θ) by the formula $\phi_k=\psi(kh+\theta)$. If h is small, then in (1.2) the coefficient $v(kh+\theta)$ slowly changes with k.

The semi-classical asymptotics of solutions to ordinary differential equations, e.g., the differential Schrödinger equation

$$-h^2\frac{d^2\psi}{dz^2}(z)+v(z)\psi(z)=0 \tag{1.3}$$

for small h, are described by means of the famous WKB (Wentzel, Kramers and Brillouin) method. There is a huge number of problems solved by this method. If the coefficients of the equation are analytic, one uses a powerful classical method often called the complex WKB method (see, e.g., chapter 3 in [8], chapter 10 in [20],

*St. Petersburg State University, 7/9 Universitetskaya nab., St.Petersburg, 199034, Russia (a.fedotov@spbu.ru).

†Sorbonne Université, Université Paris Diderot, CNRS, Institut de Mathématiques Jussieu - Paris Rive Gauche , F-75005, Paris, France (frederic.klopp@imj-prg.fr).

[1] The results were announced in the conference proceedings [12].

The work was supported by a PRC CNRS, France, and the Russian Foundation for Basic Research under the grant No 17-51-150008.

chapter 7 in [23]). It allows to study solutions to the equation on the complex plane. Even when the input problem does not require to go into the complex plane, the method is used to simplify the analysis: it allows to go around, say, singularities of solutions located on the real line, compute the Wronskians of solutions in domains where they are easily computable and so on. Moreover, the complex WKB method is very efficient for computing exponentially small quantities. One of the classical examples is the computation of the lengths of exponentially small gaps (lacunae) in the spectrum for the operator defined by the left hand side of (1.3) with analytic periodic v (see, e.g. section 9 of chapter 3 of [8]). In [8], the reader will find various interesting examples of problems solved by this method.
The asymptotic behavior of solutions to (1.3) changes crucially near turning points, i.e., the points where $v(z) = 0$. In the framework of the WKB method, the analysis of the solutions in a complex neighborhood of a turning point is one of the main topics (see, e.g., section 2 of chapter 4 in [8]).
A complex WKB method for difference equations has been initiated in [3, 14, 12]. In the present paper, we study analytic solutions to (1.1) near turning points.
Difference equations (1.1) on $\mathbb{R}$ or on $\mathbb{C}$ and (1.2) on $\mathbb{Z}$ with a small h arise in quantum physics. One encounters such equations, for example, when studying in various asymptotic situations an electron in a two-dimensional crystal submitted to a constant magnetic field (see, e.g., introduction sections in [24, 16] and references therein). The electron is described by a magnetic Schrödinger operator with a periodic electric potential, and, for example, in the semi-classical limit, its analysis reduces to analyzing an h-pseudo-differential operator with the symbol $H(x, p) = 2\cos p + 2\cos x$ (see [17], in particular, section 9.4 and page 106). Its eigenfunctions satisfy (1.1) with $v(z) = 2\cos z - E$ (here, E is the spectral parameter). The parameter h is proportional to the magnetic flux through the periodicity cell, and the case when h is small is a natural one. The other example is the case of a strong magnetic field. In this case the input equation can again be reduced to equations of the form (1.1); the parameter h is then proportional to the inverse of the magnetic flux and so is small (see, e.g., [6]). The reader can find more examples and references in [24, 16]. We add only that, for $v(z) = \cos(2\pi z) - E$, (1.2) is the famous almost Mathieu equation (see, e.g., [1]).

We note that, though in the present paper we concentrate on the analysis of (1.3), our method applies equally well to general second order linear difference equations or, equivalently, to the equations of the form

$$\phi(z + h) = M(z)\phi(z), \quad z \in U \subset \mathbb{C}, \quad \text{with} \quad M(z) \in SL(2, \mathbb{C}). \tag{1.4}$$

Such equations with analytic or meromorphic coefficients arise in many fields of mathematics and physics. In particular, they are standard in the study of the diffraction of classical waves by wedges (see, e.g., equation (2.1.2) in [18] or (10.13) in [2]) or in the theory of quasiperiodic equations (see, e.g., sections 4.2, 4.4 in [10], and the review [9] where, when studying various types of quasiperiodic equations, one systematically encounters difference equations with complex coefficients on $\mathbb{R}$ and/or $\mathbb{C}$, see, e.g., sections 2, 3.1.4, 4.1). A small shift parameter arises in the problems of diffraction by thin wedges (as, in these problems, the shift parameters are proportional to the angles of the wedges (see [18]). It can also appear in the case of quasiperiodic equations, e.g., in the case of one-dimensional quasiperiodic differential Schrödinger equations with two periods: in this case, one has to study a difference equation with complex coefficients where the shift parameter is proportional to the fractional part of the periods ratio and, thus, can be small (see [10]). We note that semiclassical

constructions are used to study the asymptotics of orthogonal polynomials as well as of formal orthogonal polynomials (see subsection 2.4).
When studying (1.2) on $\mathbb{Z}$, it is often very natural to pass to (1.1) on $\mathbb{R}$ or on $\mathbb{C}$ as, for this equation, one can use various analytic tools coming from the study of differential equations such as tools of the theory of pseudo-differential operators ([17]) or of the complex WKB method as in the present paper. If the coefficient v is periodic, for (1.1) one can use ideas of the Floquet theory for differential equations with periodic coefficients; this leads to a natural renormalization method (see review [9]).

In [24] heuristically, and in [17] rigorously, in the semiclassical limit, the authors describe the geometric structure of spectrum of the operator defined in $L_2(\mathbb{R})$ by the left-hand side of (1.1) with $v(z) = \cos z$. For irrational h, it coincides with the spectrum of the almost Mathieu operator and is a Cantor set. The authors obtain a description similar to the well-known inductive construction of the classical Cantor set.
To study the geometric properties of this spectrum, Buslaev and Fedotov introduced a renormalization approach based on ideas of the Floquet theory (see [9]). A crucial role in their analysis is played by the minimal entire solutions to (1.1) with $v(z) = \cos z - E$, E being a spectral parameter, and to kindred equations with complex coefficients (minimal entire solutions are the entire solutions having minimal possible growth as $|\operatorname{im} z| \to \infty$, see, section 2.3.2 in [9]). To study these solutions in the semiclassical limit, Buslaev and Fedotov developed a complex WKB method for difference equations (see [3, 4, 9]). While the analysis of the geometry of the spectrum does not require detailed asymptotics of the entire solutions along the real line, these are crucial to describe the eigenfunctions in the semiclassical limit. This provides a natural applications of the results developed in the present paper.

The present paper is devoted to uniform asymptotic formulas describing analytic solutions to (1.1) in h-independent complex neighborhoods of simple turning points (for the definitions, see subsection 2.1). To the best of our knowledge, for difference equations, this result is new and original. Its analogue for differential equations is well-known and derived in a completely different way: this is a consequence of the non-locality of (1.1).
There are many papers devoted to the asymptotic analysis of solutions to equations similar to (1.2) on $\mathbb{Z}$ with small h. The reader will find a short discussion of their results in subsection 2.4, i.e., after formulation of our results.
The question of the asymptotics of solutions to a difference equation in a complex neighborhood of a turning point is very natural. The techniques developed to get and justify these asymptotics are the main analytic innovation of the paper.

1.2. Notations and agreements. Below v is analytic on a disk $U \subset \mathbb{C}$.
A neighborhood is a δ-neighborhood.
Instead of saying that an asymptotic representation is valid for sufficiently small h, we write that it is valid as $h \to 0$.
The letter C denotes various positive constants independent of z and h. For two functions f and g defined on a domain $D \subset \mathbb{C}$, we write that $g(z) = O(f(z))$ in D if $|g(z)| \le C|f(z)|$ for all $z \in D$.

2. The main results. First, we define few objects needed to formulate our results.

2.1. The main analytic objects.

2.1.1. The complex momentum. One of the main analytic objects of the WKB methods is the *complex momentum* p. For (1.1) it is defined by the formula

$$2\cos p + v(z) = 0. \tag{2.1}$$

It is a multivalued analytic function on U. At its branching points one has $v(z) \in \{\pm 2\}$.
In analogy with the glossary of the complex WKB method for differential equations, the points z where $v(z) \in \{\pm 2\}$ are called *turning points*. A set $D \subset U$ is *regular* if $v(z) \neq \pm 2$ in D.

Remark 2.1. In the complex WKB theory for difference equations, see [3, 13, 14], one proves the existence of analytic solutions $\psi_\pm$ such that

$$\psi_\pm(z) = \frac{1}{\sqrt{\sin(p(z))}}\, e^{\pm \frac{i}{h}\int_{z_0}^{z} p(z)\,dz + o(1)}, \quad h \to 0. \tag{2.2}$$

For z, a turning point, one has $\cos p(z) = -v(z)/2 \in \{\pm 1\}$, and $\sin p(z) = 0$. Thus, near a turning point, the representation (2.2) cannot be valid.

Let z_0 be a turning point. If $v'(z_0) \neq 0$, we call this turning point *simple*. We note that, in this case, the complex momentum is analytic in $\tau = \sqrt{z - z_0}$ in a neighborhood of z_0 and

$$p(z) = p(z_0) + p_1\tau + O(\tau^2), \quad \tau \to 0, \tag{2.3}$$

where $p_1 \neq 0$ is a constant.
Let z_0 be the center of the disk U. From now on, we assume that the following hypotheses hold:

Hypothesis 2.2. $v(z) \notin \{\pm 2\}$ for $z \in U \setminus \{z_0\}$.

This means that the domain $U \setminus \{z_0\}$ is regular.

Hypothesis 2.3. $v(z_0) = 2$, and $v'(z_0) \neq 0$.

So, z_0 is a simple turning point. The condition $v(z_0) = 2$ implies that $p(z_0) = 0 \mod 2\pi$.
The case where $v(z_0) = -2$ reduces to the case we study. Indeed, let $v(z_0) = -2$. For ψ, a solution to (1.1), we set $\phi(z) = e^{i\pi z/h}\psi(z)$. Then, ϕ satisfies equation $\phi(z+h) + \phi(z-h) - v(z)\phi(z) = 0$, z_0 is again a turning point, and the coefficient in the front of $\phi(z)$ equals 2 at z_0.

2.1.2. Objects related to the complex momentum. Cut U from z_0 to a point of its boundary along a simple curve and denote the thus obtained domain by U'. In U', we fix an analytic branch p of the complex momentum so that $p(z_0) = 0$ (this is possible as at z_0 the complex momentum vanishes modulo 2π and as for p, a branch of the complex momentum, $p + 2\pi$ is also a branch of the complex momentum). In U' we consider the function

$$z \mapsto \left(\frac{3}{2i}\int_{z_0}^{z} p(z)\,dz\right)^{\frac{2}{3}}. \tag{2.4}$$

In view of (2.3), there are three branches of this function that are analytic in U near z_0. We denote by ζ one of them.
One has $\zeta(z_0) = 0$, and $\zeta'(z_0) \neq 0$.
Possibly reducing U, we can and do assume that

Hypothesis 2.4. The function ζ is a bi-analytic bijection of U onto its image.

Remark 2.5. The definition of ζ implies that it satisfies one of the two equations

$$\sqrt{\zeta(z)}\zeta'(z) = \pm ip(z), \qquad z \in U. \tag{2.5}$$

We let

$$g(z) := \sinh\left(\sqrt{\zeta(z)}\zeta'(z)\right) / \sqrt{\zeta(z)}, \qquad z \in U, \tag{2.6}$$

where the determination of the square roots in the denominator and the numerator are the same (the definition of g is independent of its choice). The function g is analytic in U and does not vanish there, see section 3.3.2. In U, we further define an analytic function

$$A_0(z) := 1 / \sqrt{g(z)}. \tag{2.7}$$

2.2. Asymptotic solutions. First, let us describe asymptotic solutions to equation (1.1). Therefore, we recall that Airy functions are solutions to the equation $y''(z) = y(z),\ \ z \in \mathbb{C}$ (see, e.g., [19], section 2.8). Presumably, the most famous Airy function is Ai. The reader can find its asymptotics at infinity in subsection 3.1. We shall use three other standard Airy functions (see, e.g., [25]) defined by the formulas

$$w_j(\zeta) = 2\pi i\omega^j \operatorname{Ai}(\omega^j\zeta), \quad \omega = e^{2\pi i/3}, \qquad \zeta \in \mathbb{C}. \tag{2.8}$$

Any two of them are linearly independent.
For a function f defined on U and $\{z-h, z, z+h\} \subset U$, we set

$$[H(f)](z) := f(z+h) + f(z-h) + v(z)f(z). \tag{2.9}$$

One has

THEOREM 2.6. *There exist functions $(A_{2l})_{l\in\mathbb{N}\cup\{0\}}$ and $(B_{2l+1})_{l\in\mathbb{N}\cup\{0\}}$ (A_0 being defined by (2.7)) that are analytic on U and such that, for any $L \in \mathbb{N}\cup\{0\}$, the following holds. Let w be one of the Airy functions $(w_j)_{j\in\mathbb{Z}_3}$ and set*

$$w_h(z) = w\left(\zeta(z)/h^{\frac{2}{3}}\right), \qquad w_h'(z) = w'\left(\zeta(z)/h^{\frac{2}{3}}\right), \tag{2.10}$$

and

$$W(z) = h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{L} h^{2l} A_{2l}(z) + h^{\frac{2}{3}} w_h'(z) \sum_{l=0}^{L-1} h^{2l+1} B_{2l+1}(z), \tag{2.11}$$

where the second sum vanishes if $L = 0$. Then, in any given compact set $K \subset U$, as $h \to 0$, we have

$$H(W) = O\left(h^{2L+2+\frac{1}{3}} w_h\right) + O\left(h^{2L+3+\frac{2}{3}} w_h'\right). \tag{2.12}$$

The formal expression

$$h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{\infty} h^{2l} A_{2l}(z) + h^{\frac{2}{3}} w_h'(z) \sum_{l=0}^{\infty} h^{2l+1} B_{2l+1}(z) \tag{2.13}$$

is called an *asymptotic solution* to (1.1).
Theorem 2.6 is proved in section 4; in its proof, we explicitly describe the computation of the coefficients $(A_l)_l$ and $(B_l)_l$. The computations being rather knotty, we describe the algorithm in Remark 4.3, Remark 4.4 and Remark 4.5. Let us underline that the complexity of these computations is a consequence of the non-locality of the difference equation (1.1).
Let us comment on the results of Theorem 2.6. First, we note that, for the differential equation $-h^2\psi''(z) + v(z)\psi(z) = 0$, in a neighborhood of a simple turning point (a point where $v(z) = 0$ and $v'(z) \neq 0$), there are asymptotic solutions of the form (2.13) (with different coefficients $(A_l)_l$, $(B_l)_l$ and a different function ζ).
To justify the Ansatz (2.13) for the difference equation, one has to derive asymptotic formulas of the form

$$w_h(z \pm h) = f(z)\, w_h(z) \pm h^{\frac{1}{3}}\, g(z)\, w_h'(z) + \dots\,, \tag{2.14}$$

where $f(z) = \cosh(\sqrt{\zeta(z)}\zeta'(z))$ and the dots denote smaller order terms. If one tries to prove this formula using Taylor expansions for the left hand side, one has to handle an infinite number of infinite subsequences of terms of the same order. So, an effective resummation of these sequences is required. As we shall show in this paper, to derive formulas analogous to (2.14), instead of resummation of Taylor series, it is very natural to use standard tools from complex analysis and, in particular, from the theory of integrals with two coalescing saddle points (see, e.g., [25]).
Formula (2.14) imply that

$$H\left(w_h\right)(z) = \Big(2\cosh(\sqrt{\zeta(z)}\zeta'(z)) + v(z)\Big)\, w_h(z) + \dots\,. \tag{2.15}$$

In view of (2.1) and (2.5), the leading term in the right-hand side of (2.15) vanishes. Finally, we note that if $h^{-\frac{2}{3}}|\zeta(z)|$ is large, then $w_h(z)$ and w_h' in (2.11) can be replaced by their asymptotic representations, see formula (3.2) in subsection 3.1 or [19] for more details. As a result, the leading terms in (2.11) turn into a linear combination of the leading terms from (2.2).

2.3. Solutions with standard asymptotic behavior. Our main result is

THEOREM 2.7. *There exists $\overset{\circ}{U} \subset U$, a neighborhood of z_0 and $h_0 > 0$ such that, for any $L \in \mathbb{N} \cup \{0\}$ and W, any of the functions constructed in Theorem 2.6 for the order L, for $h \in (0, h_0]$, there exists $\psi : \overset{\circ}{U} \to \mathbb{C}$, a solution to (1.1) that is analytic and admits the asymptotic representation*

$$\psi(z) = W(z) + O(w_h\, h^{2L+2+\frac{1}{3}}) + O(w_h'\, h^{2L+1+\frac{2}{3}}) \tag{2.16}$$

where w_h and w_h' are defined in (2.10).

Theorem 2.7 is proved in section 6 and section 7.
Let us briefly explain the ideas of the proof of Theorem 2.7. First, in section 6, using the approximate solutions constructed in Theorem 2.6, we construct a parametrix R, i.e., an operator such that, for suitable functions f, one has $HRf = f + Df$, where H is defined in (2.9) and D is a small operator. The operator D is a singular integral operator. We estimate its norm using natural geometric objects of the complex WKB method. This allows us to prove Theorem 2.7 on some special subdomains of U. In section 7, we study the thus constructed solutions on larger domains and complete the proof of Theorem 2.7.

2.4. Related results. The WKB asymptotics of solutions of difference equations on $\mathbb{Z}$ with "slowly varying" coefficients have been studied since the end of the 1960's. In [22] and [21], the authors essentially studied equations of the form

$$Y_{k+1} = M(hk)Y_k, \quad k \in \mathbb{Z}; \tag{2.17}$$

where h is small and positive and M is an $(n \times n)$-matrix valued function defined on $\mathbb{R}$. If

$$Y(x+h) = M(x)Y(x), \quad x \in \mathbb{R}, \tag{2.18}$$

then, the sequence $Y_k = Y(kh)$, $k \in \mathbb{Z}$, satisfies (2.17). We note that (1.1) restricted to $\mathbb{R}$ is equivalent to (2.18) with $M(x) = \begin{pmatrix} -v(x) & -1 \\ 1 & 0 \end{pmatrix}$, and that a turning point for (1.1) is a point x where the eigenvalues of the matrix $M(x)$ coincide.
The short note [22] is essentially devoted to the case where all the eigenvalues of the matrix M in (2.17) are distinct. In [21] the author constructed asymptotic solutions to (2.17) in a small h-dependent neighborhood of a point x where two eigenvalues of $M(x)$ coincide.
In [5], the authors considered difference equations of the form

$$\sum_{j=I}^{J} a_j(hk, h)\, y_{k+j} = 0, \quad k \in \mathbb{Z}. \tag{2.19}$$

We note that this class includes the difference Schrödinger equations

$$y_{k+1} + y_{k-1} + v(hk)y_k = 0, \quad k \in \mathbb{Z}.$$

In the limit $h \to 0$, [5] gives a description of the asymptotics of solutions to (2.19) for hk in a small h-dependent neighborhood of a point where $v(x) \in \{\pm 2\}$.
Let us also mention three (series of) papers motivated by problems originating in the theory of orthogonal polynomials.
First, there is a series of papers by J.S. Geronimo and co-authors (see [15] and references therein) devoted to the asymptotics of solutions to the equation

$$a_{k+1}\psi_{k+1} + b_k\psi_k + a_k\psi_{k-1} = \lambda\psi_k, \quad k \in \mathbb{Z},$$

where λ is the spectral parameter, and the coefficients a_k are positive and b_k are real and slowly depend on k. The authors describe asymptotic formulas uniform with respect to kh in an h-independent neighborhood of a turning point.
Next, R.Wong and coauthors devoted various works (see e.g., [26]) to the study of solutions to three terms recurrence relations with real coefficients for large values of the integer variable.
Finally, we mention [7] where the authors studied WKB asymptotics of solutions to a difference equation using the Maslov canonical operator.
There are more papers devoted to the subject. The reader can find more references in the papers that we mentioned above.
To the best of our knowledge, the present paper is the first where uniform asymptotics of analytic solutions to a difference equation on $\mathbb{C}$ in an h-independent neighborhood of a turning point are mathematically derived.
As we pointed out earlier, our method applies equally well to (1.4). Recalling the relation between solutions of difference equations on $\mathbb{R}$ or $\mathbb{C}$ and on $\mathbb{Z}$ (see our comments

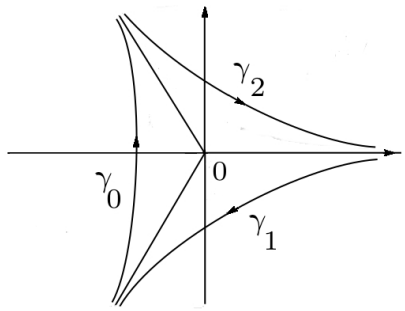

FIG. 1. *Integration paths*

to equation (1.2)) yields all the above results for second order difference equations on $\mathbb{Z}$. Moreover, it yields the asymptotics of solutions to such equations with complex coefficients. Such a result was not known and looks non-trivial for the difference equations studied only on $\mathbb{Z}$. In general the methods developed for the equations on $\mathbb{Z}$ do not yield results on analytic solutions on $\mathbb{C}$ at least without a non-trivial analysis.

3. Preliminaries.

3.1. Three Airy functions. Let $(\gamma_j)_{j\in\mathbb{Z}_3}$ be the curves shown in Figure 1; γ_0 is asymptotic to the half-lines $e^{\pm 2i\pi/3}\mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty) \subset \mathbb{R}$; for $j \in \mathbb{Z}_3$, rotating γ_0 around z_0 by $2j\pi/3$, one obtains γ_j. The functions defined by the formulas

$$w_j(\zeta) = \int_{\gamma_j} e^{-\left(\frac{s^3}{3}-\zeta s\right)}\, ds, \quad j \in \mathbb{Z}_3, \qquad \zeta \in \mathbb{C}, \tag{3.1}$$

are three Airy functions related to the standard Airy function Ai by (2.8) (see, e.g., [25]). Assume that $|\arg z| < 2\pi/3$. As $|z| \to \infty$ one has

$$\mathrm{Ai}\,(z) = \frac{\exp\left(-\frac{2}{3}z^{3/2} + o(1)\right)}{2\sqrt{\pi}\; z^{1/4}}, \;\; \mathrm{Ai}\,(-z) = \frac{\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4} + o(1)\right)}{\sqrt{\pi}\; z^{1/4}}\,(1+o(1)) \tag{3.2}$$

where we use the analytic branches of $z \to z^{3/2}$ and $z \to z^{1/4}$ that are positive for $z > 0$ (see [19], pp. 116, 118 and 392).

3.2. The space of solutions to (1.1). The observations that we discuss now are well-known in the theory of difference equations and are easily proved.

Let $c \in \mathbb{R}$ and $I = \{z \in U : \mathrm{Im} z = c\}$. We assume that the length of I is sufficiently large (with respect to h) and discuss the set S of solutions to (1.1) on I.

Let $\{f, g\} \subset S$. The expression

$$(f(z), g(z)) = f(z+h)g(z) - f(z)g(z+h), \quad \{z,\, z+h\} \subset I, \tag{3.3}$$

is called *the Wronskian* of f and g. It is h-periodic in z.
If the Wronskian of two solutions does not vanish, they form a basis in S, i.e, $\psi \in S$ if and only if

$$\psi(z) = a(z)f(z) + b(z)g(z), \quad \{z,\, z+h\} \subset I \tag{3.4}$$

where a and b are h-periodic complex coefficients. One has

$$a(z) = \frac{(\psi(z),\, g(z))}{(f(z),\, g(z))} \quad \text{and} \quad b(z) = \frac{(f(z),\, \psi\,(z))}{(f(z),\, g(z))}. \tag{3.5}$$

3.3. Remarks on the complex momentum and the function g.

3.3.1. Analytic branches of the complex momentum. Let p_0 be a branch of the complex momentum analytic in a regular simply connected domain D. Equation (2.1) implies that an analytic function $\tilde{p}: \ D \to \mathbb{C}$ is a branch of the complex momentum if and only if there exists $s \in \{\pm 1\}$ and $n \in \mathbb{Z}$ such that

$$\tilde{p}(z) = s p_0(z) + 2\pi n, \quad \forall z \in D. \tag{3.6}$$

3.3.2. Non-vanishing of the function g. Here, we prove that the function g defined by (2.6) does not vanish in U.
As $\zeta(z_0) = 0$ and $\zeta'(z_0) \neq 0$, g does not vanish at z_0. So, in view of (2.5), it suffices to check that $\sin p(z)$ does not vanish in $U \setminus \{z_0\}$. Therefore, we note that if $\sin p(z) = 0$, then $v(z) = -2\cos p(z) \in \{\pm 2\}$, i.e., z is a turning point. But, in U there is only one turning point, and it is z_0. This implies the needed.

4. Asymptotic solutions: the proof of Theorem 2.6.

4.1. The proof of Theorem 2.6 up to two propositions. We begin with two auxiliary statements. Below, we use the notations introduced in (2.10), and $K \subset U$ is a compact set.

Proposition 4.1. *Let A be analytic in U and pick $L \in \mathbb{N}$. Then, in K, one has*

$$\begin{aligned} H\left(A\, h^{\frac{1}{3}} w_h\right) = h^{\frac{1}{3}} w_h \left(\sum_{l=1}^{L} h^{2l} a_{2l}(A;z) + O(h^{2L+2})\right) \\ + h^{\frac{2}{3}} w_h' \left(\sum_{l=1}^{L} h^{2l-1} b_{2l-1}(A;z) + O(h^{2L+1})\right) \end{aligned} \tag{4.1}$$

as $h \to 0$. All the coefficients $(a_{2l})_{l\geq 1}$ and $(b_{2l-1})_{l\geq 1}$ are analytic in $z \in U$, independent of h and of the choice of w in (2.10), and

$$b_1 = Ag \frac{d}{dz} \log\left(A^2 g\right). \tag{4.2}$$

Proposition 4.2. *Let B be analytic in U and pick $L \in \mathbb{N}$. Then, in K, one has*

$$\begin{aligned} H\left(B\, h^{\frac{2}{3}} w_h'\right) = h^{\frac{1}{3}} w_h \left(\sum_{l=1}^{L} h^{2l-1} c_{2l-1}(B;z) + O(h^{2L+1})\right) \\ + h^{\frac{2}{3}} w_h' \left(\sum_{l=1}^{L} h^{2l} d_{2l}(B;z) + O(h^{2L+2})\right) \end{aligned} \tag{4.3}$$

as $h \to 0$. All the coefficients $(c_{2l-1})_{l\geq 1}$ and $(d_{2l})_{l\geq 1}$ are analytic in U, independent of h and of the choice of w in (2.10), and

$$c_1 = \zeta B g \frac{d}{dz} \log(\zeta B^2 g). \tag{4.4}$$

Before proving Propositions 4.1 and 4.2, we use them to prove Theorem 2.6.
Clearly, it suffices to consider $L \geq 2$ which we do. Assume that A_{2l} and B_{2l+1}, $l = 0, 1, 2, \ldots$ are analytic in U and pick $z \in K$. Define W by (2.11). Using Proposition 4.2

for L and Proposition 4.1 for $L+1$ instead of L, for sufficiently small h, we compute (below, the dependence on z is omitted)

$$HW = h^{\frac{1}{3}} w_h \mathcal{A} + h^{\frac{2}{3}} w_h' \mathcal{B},$$
$$\mathcal{A} = \sum_{l=0}^{L} \sum_{k=1}^{L+1} h^{2(l+k)} a_{2k}(A_{2l}) + \sum_{l=0}^{L-1} \sum_{k=1}^{L} h^{2(l+k)} c_{2k-1}(B_{2l+1}) + O(h^{2(L+1)}),$$
$$\mathcal{B} = \sum_{l=0}^{L} \sum_{k=1}^{L+1} h^{2(l+k)-1} b_{2k-1}(A_{2l}) + \sum_{l=0}^{L-1} \sum_{k=1}^{L} h^{2(l+k)+1} d_{2k}(B_{2l+1}) + O(h^{2L+3}).$$

Consider the expression for $\mathcal{A}$. Changing the summation index $l \to l-1$, we compute

$$\begin{aligned} \mathcal{A} &= \sum_{k,l\geq 1}^{k+l-1\leq L} h^{2(l+k-1)} \left(a_{2k}(A_{2(l-1)}) + c_{2k-1}(B_{2l-1}) \right) + O(h^{2(L+1)}) \\ &= \sum_{m=1}^{L} h^{2m} \sum_{l=1}^{m} \left(a_{2(m-l+1)}(A_{2(l-1)}) + c_{2(m-l+1)-1}(B_{2l-1}) \right) + O(h^{2(L+1)}). \end{aligned} \tag{4.5}$$

Let us turn to $\mathcal{B}$. Writing separately the term with $l=0$ from the first sum over l, changing the summation index $l \to l-1$ in the second one, we compute

$$\begin{aligned} \mathcal{B} &= \sum_{k=1}^{L+1} h^{2k-1} b_{2k-1}(A_0) + \sum_{k,l\geq 1}^{k+l\leq L+1} h^{2(l+k)-1} \left(b_{2k-1}(A_{2l}) + d_{2k}(B_{2l-1}) \right) + O(h^{2L+3}) \\ &= h b_1(A_0) + \sum_{m=2}^{L+1} h^{2m-1} \Bigg(b_{2m-1}(A_0) + \\ &\qquad + \sum_{l=1}^{m-1} \left(b_{2(m-l)-1}(A_{2l}) + d_{2(m-l)}(B_{2l-1}) \right) \Bigg) + O(h^{2L+3}). \end{aligned} \tag{4.6}$$

Let $A_0 = 1/\sqrt{g}$. By (4.2), one has $b_1(A_0) = 0$. Therefore, in view of (4.5) and (4.6), to prove Theorem 2.6, one has to choose $(A_{2l})_{l\geq 1}$ and $(B_{2l-1})_{l\geq 1}$ so that

$$\sum_{l=1}^{m} \left(a_{2(m-l+1)}(A_{2(l-1)}) + c_{2(m-l+1)-1}(B_{2l-1}) \right) = 0, \quad \forall m \geq 1,$$
$$b_{2m-1}(A_0) + \sum_{l=1}^{m-1} \left(b_{2(m-l)-1}(A_{2l}) + d_{2(m-l)}(B_{2l-1}) \right) = 0, \quad \forall m \geq 2.$$

Therefore, for $m = 1, 2, 3 \ldots$, it suffices to inductively solve the equations

$$c_1(B_{2m-1}) = -L_m, \qquad b_1(A_{2m}) = -M_m. \tag{4.7}$$

where

(4.8)
$$L_m = L_m(A_0, A_2, \dots A_{2m-2}, B_1, B_3, \dots, B_{2m-3}) = a_2(A_{2(m-1)}) + \sum_{l=1}^{m-1} \left(a_{2(m-l+1)}(A_{2(l-1)}) + c_{2(m-l+1)-1}(B_{2l-1})\right),$$

(4.9)
$$M_m = M_m(A_0, A_2, \dots, A_{2(m-1)}, B_1, B_2, \dots, B_{2m-1}) = b_{2m+1}(A_0) + \sum_{l=1}^{m-1} \left(b_{2(m-l)+1}(A_{2l}) + d_{2(m-l+1)}(B_{2l-1})\right) + d_2(B_{2m-1}),$$

and the sums $\sum_{l=1}^{m-1}(\dots)$ in (4.8) and (4.9) are equal to 0 if $m = 1$. In view of (4.2) and (4.4), equations (4.7) are equivalent to the equations

$$\zeta B_{2m-1}\, g \frac{d}{dz} \log\left(\zeta B_{2m-1}^2\, g\right) = -L_m, \quad \text{and} \quad A_{2m}\, g \frac{d}{dz} \log\left(A_{2m}^2\, g\right) = -M_m. \tag{4.10}$$

One constructs solutions to these equations by the formulas

(4.11)
$$A_{2m}(z) = -\frac{1}{2\sqrt{g(z)}} \int_{z_0}^{z} \frac{M_m\, dz}{\sqrt{g}}, \quad \text{and} \quad B_{2m-1}(z) = -\frac{1}{2\sqrt{\zeta(z) g(z)}} \int_{z_0}^{z} \frac{L_m\, dz}{\sqrt{\zeta g}}.$$

As g and ζ are analytic in U, g does not vanish in U, and ζ vanishes in U only at z_0 where it has a simple zero, the coefficients A_{2m} and B_{2m-1} are analytic in U.
The proof of Theorem 2.6 is complete.

Remark 4.3. As is seen from the above proof, for $m \geq 1$, the coefficients A_{2m} and B_{2m-1} are constructed inductively by formulas (4.11) where L_m and M_m are given by (4.8) and (4.9) and, for $l \geq 1$, a_{2l}, b_{2l-1}, c_{2l-1} and d_{2l} are defined in Proposition 4.1 and Proposition 4.2. The algorithms for their computation are described in Remark 4.4 and Remark 4.5.

4.2. The proof of Proposition 4.1. Consider $(w_j)_{j\in\mathbb{Z}_3}$, the three Airy functions defined by (3.1). Let w be w_j for some $j \in \mathbb{Z}_3$ and let γ be the corresponding integration path γ_j in (3.1). We note that

$$h^{\frac{1}{3}} w(h^{-\frac{2}{3}}\zeta) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta\right)}\, dt, \qquad h^{\frac{2}{3}} w'(h^{-\frac{2}{3}}\zeta) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta\right)} t\, dt. \tag{4.12}$$

Below, we use the notations from (2.10). Let $K \subset U$ be a closed disk centered at z_0 and independent of h. Below, we assume that $z \in K$ and that $h > 0$ is sufficiently small. The proof of Proposition 4.1 is broken into several steps.

1. In view of (4.12), we get

$$H\left(Ah^{\frac{1}{3}} w\right) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} F_0(t, z, h)\, dt, \tag{4.13}$$

$$F_0(t, z, h) = A(z+h) e^{\frac{t}{h}(\zeta(z+h) - \zeta(z))} + A(z-h) e^{\frac{t}{h}(\zeta(z-h) - \zeta(z))} + v(z) A(z). \tag{4.14}$$

We note that the function $(t, z, h) \mapsto F_0(t, z, h)$ is analytic in $\mathbb{C} \times K \times V$ provided that V is a sufficiently small neighborhood of 0.

2. Let $N \in \mathbb{N}$. By means of Taylor's theorem, we get

(4.15)
$$H(Ah^{\frac{1}{3}}w) = \sum_{n=0}^{N} h^n \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} F_{0,n}(t,z)dt + h^{N+1}\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} O(e^{C|t|})dt,$$

$$F_{0,n}(t,z) = \frac{1}{n!}\frac{\partial^n F_0}{dh^n}(t,z,0). \tag{4.16}$$

3. To get the asymptotics of the first $N+1$ integrals in (4.15), we apply Bleistein's method developed to study integrals with two coalescing saddle points (see, e.g., section 4 of chapter VII of [25]). We fix $n \in \mathbb{N}\cup\{0\}$ and for $m = 0, 1, 2\dots$, we inductively define

$$a_{m,n}(z) = \frac{1}{2}\left(F_{m,n}(\sqrt{\zeta(z)},z) + F_{m,n}(-\sqrt{\zeta(z)},z)\right), \tag{4.17}$$

$$b_{m,n}(z) = \frac{1}{2\sqrt{\zeta(z)}}\left(F_{m,n}(\sqrt{\zeta(z)},z) - F_{m,n}(-\sqrt{\zeta(z)},z)\right), \tag{4.18}$$

$$F_{m+1,n}(t,z) = \frac{\partial f_{m,n}}{\partial t}(t,z), \quad f_{m,n}(t,z) = \frac{F_{m,n}(t,z) - a_{m,n}(z) - b_{m,n}(z)t}{t^2-\zeta(z)}, \tag{4.19}$$

where we use one and the same branch of $\sqrt{\zeta(z)}$.
The functions $(F_{m,n})_{m,n}$ are analytic in $(t,z) \in \mathbb{C}\times K$, and $a_{m,n}$ and $b_{m,n}$ are analytic in $z \in K$ (we remove the removable singularities).
4. Note that $F_0(-t,z,-h) = F_0(t,z,h)$. Therefore, $F_{0,n}(-t,z) = (-1)^n F_{0,n}(t,z)$ for $n \geq 0$. By induction on $m \geq 0$, one easily checks that

$$F_{m,n}(-t,z) = (-1)^{m+n}F_{m,n}(t,z), \tag{4.20}$$

$$a_{m,n} = 0 \text{ if } m+n \text{ is odd}, \quad \text{and} \quad b_{m,n} = 0 \text{ if } m+n \text{ is even}. \tag{4.21}$$

5. Let $(m,n) \in (\mathbb{N}\cup\{0\})^2$. In view of (4.19), $F_{m,n}(t,z) = a_{m,n}(z) + b_{m,n}(z)t + (t^2 - \zeta(z))f_{m,n}(t,z)$. Using this formula, representations (4.12) and integrating by parts, we get

(4.22)
$$\begin{aligned}
&\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} F_{0,n}(t,z)\,dt \\
&\quad = h^{\frac{1}{3}} w_h\, a_{0,n} + h^{\frac{2}{3}} w_h'\, b_{0,n} + \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)}(t^2-\zeta(z))f_{0,n}(t,z)\,dt \\
&\quad = h^{\frac{1}{3}} w_h\, a_{0,n} + h^{\frac{2}{3}} w_h'\, b_{0,n} + h\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} F_{1,n}(t,z)\,dt \\
&\qquad\qquad \dots\dots\dots\dots\dots \\
&\quad = h^{\frac{1}{3}} w_h \sum_{m=0}^{N} h^m a_{m,n} + h^{\frac{2}{3}} w_h' \sum_{m=0}^{N} h^m b_{m,n} + h^{N+1}\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} F_{N+1,n}(t,z)dt.
\end{aligned}$$

6. To estimate the integral in the last line of (4.22), we show that

$$F_{m,n}(t,z) = O\left(e^{C|t|}\right), \quad m,n \in \mathbb{N}\cup\{0\}. \tag{4.23}$$

Here and below, the constants C are independent of z, h and t, and the symbol $O(\cdot)$ is used for estimates uniform in z, t and h.
Fix $n \geq 0$. Estimate (4.23) being obvious for $F_{0,n}$, it suffices to assume that it is valid for some $m = m_0$ and to prove it for $m = m_0 + 1$. Clearly, $\zeta(z)$ is bounded on K. In view of the definitions of $(a_{m,n})_{m,n\geq 0}$, $(b_{m,n})_{m,n\geq 0}$ and the induction hypothesis, we have $a_{m_0,n}(z) = O(1)$ and $b_{m_0,n}(z) = O(1)$. These observations, (4.19) and the induction hypothesis imply that there exists $R > 0$ independent of h such that, for all $|t| \geq R$, $f_{m_0,n}(t,z) = O(e^{C|t|})$. By the maximum principle, this implies that $f_{m_0,n}$ satisfies this estimate for all $t \in \mathbb{C}$. Now the Cauchy estimates for the derivatives of the analytic functions imply (4.23) for $m = m_0 + 1$.
7. To estimate the last integral in (4.15) and the integral term in the last line of (4.22), we consider an analytic function $(t,z) \mapsto g(z,t)$ such that $g(z,t) = O(e^{C|t|})$, and prove the estimate

$$\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} g(z,t)\, dt = O\left(h^{\frac{1}{3}} w_h\right) + O\left(h^{\frac{1}{3}} w_h'\right). \tag{4.24}$$

Therefore, we essentially repeat the reasoning made in Section 4, Chapter VII of [25]. So, we omit some details.
If $Z = h^{-\frac{2}{3}}\zeta(z)$ is bounded by a constant, setting $T = h^{-\frac{1}{3}} t$, we change variable in (4.24) and get

$$\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} g(z,t)\, dt = h^{\frac{1}{3}} \int_\gamma e^{-\left(\frac{T^3}{3} - TZ\right)} O(e^{Ch^{\frac{1}{3}}|T|})\, dT = O(h^{\frac{1}{3}}).$$

This yields (4.24) as w and w' have no common zero.
If $Z = h^{-\frac{2}{3}}\zeta(z)$ is large, we estimate the integral using the method of steepest descent. We deform the integration path to a path of steepest descent for $e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)}$ exactly as when computing the asymptotics of the Airy function w, i.e., the asymptotics of the integral $\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)}\, dt$. The saddle points $\pm\sqrt{\zeta(z)}$ are uniformly bounded when $z \in K$. Let $r > 0$ be sufficiently large for the saddle points to be inside the disk of radius r centered at 0. We compute the asymptotics of the integral over $\gamma \cap \{|t| \leq r\}$ directly by means of the method of steepest descents and, comparing the answer with the asymptotics of the Airy function $w(Z)$ as $Z \to \infty$, we find that this integral is bounded by $O(h^{\frac{1}{3}} w_h) + O(h^{\frac{2}{3}} w_h')$. The integral over the remaining part of γ quickly tends to 0 as $h \to 0$: actually, it is exponentially small with respect to $O(h^{\frac{1}{3}} w_h) + O(h^{\frac{2}{3}} w_h')$. This yields (4.24).
8. Formula (4.15), the representation obtained in the last line of (4.22) and the estimates (4.23) and (4.24) imply

$$H\left(A\, h^{\frac{1}{3}} w_h\right) = h^{\frac{1}{3}} w_h \left(\sum_{l=0}^{N} h^l a_l + O(h^{N+1})\right) + h^{\frac{2}{3}} w_h' \left(\sum_{l=0}^{N} h^l b_l + O(h^{N+\frac{2}{3}})\right), \tag{4.25}$$

$$a_l = \sum_{m+n=l} a_{m,n}, \qquad b_l = \sum_{m+n=l} b_{m,n}. \tag{4.26}$$

In view of (4.21), we see that $a_l = 0$ for odd l, and $b_l = 0$ for even l. This observation and the fact that N is arbitrary yields that $\forall L \in \mathbb{N}$

(4.27)
$$H(Ah^{\frac{1}{3}}w_h) = h^{\frac{1}{3}}w_h\left(\sum_{l=0}^{L} h^{2l}a_{2l} + O(h^{2L+2})\right) + h^{\frac{2}{3}}w_h'\left(\sum_{l=1}^{L} h^{2l-1}b_{2l-1} + O(h^{2L+1})\right).$$

To complete the proof of Proposition 4.1, it remains to prove that $a_0 = 0$ and to compute b_1.

9. Clearly, $a_0(z) = a_{0,0}(z) = F_{0,0}(\sqrt{\zeta(z)}, z) = F_0(\sqrt{\zeta(z)}, z, 0)$. Therefore,

$$a_0(z) = A(z)\left(2\cosh(\sqrt{\zeta(z)}\zeta'(z)) + v(z)\right).$$

Recall that the complex momentum p is defined in (2.1). Therefore, in view of (2.5) we get $2\cosh\left(\sqrt{\zeta(z)}\zeta'(z)\right) + v(z) = 0$. So, $a_0 = 0$. This and (4.27) imply (4.1).

10. Let us prove (4.2). One has $b_1 = b_{0,1} + b_{1,0}$. Clearly,

$$\begin{aligned} b_{0,1}(z) = \frac{F_{0,1}(\sqrt{\zeta(z)},z)}{\sqrt{\zeta(z)}} &= \frac{1}{\sqrt{\zeta(z)}}\frac{\partial F_0}{\partial h}(\sqrt{\zeta(z)}, z, 0) \\ &= 2A'(z)\frac{\sinh\left(\sqrt{\zeta(z)}\zeta'(z)\right)}{\sqrt{\zeta(z)}} + A(z)\zeta''(z)\cosh\left(\sqrt{\zeta(z)}\zeta'(z)\right). \end{aligned} \tag{4.28}$$

Furthermore,

$$b_{1,0}(z) = \frac{F_{1,0}(\sqrt{\zeta(z)}, z)}{\sqrt{\zeta(z)}} = \frac{1}{\sqrt{\zeta(z)}}\frac{\partial f_{0,0}}{\partial t}(\sqrt{\zeta(z)}, z). \tag{4.29}$$

As $a_{0,0} = b_{0,0} = 0$, one has $f_{0,0}(t,z) = F_{0,0}(t,z)/(t^2 - \zeta(z))$. Substituting this representation into (4.29), after elementary calculations, one obtains

$$b_{1,0}(z) = \frac{A\zeta'}{2\zeta}\left(\zeta'(z)\cosh\left(\sqrt{\zeta}\zeta'\right) - \frac{\sinh\left(\sqrt{\zeta}\zeta'\right)}{\sqrt{\zeta}}\right), \quad \zeta = \zeta(z). \tag{4.30}$$

Substituting representations (4.28) and (4.30) into the formula $b_1 = b_{0,1} + b_{1,0}$ yields (4.2). This completes the proof of Proposition 4.1.

Remark 4.4. As is seen from the above proof, to compute the coefficients $a_{2l} = a_{2l}(A; z)$ and $b_{2l-1} = b_{2l-1}(A; z)$ from Proposition 4.1 for $l \in \mathbb{N}$, one proceed as follows. First, in terms of A, one defines a function $(t, z, h) \mapsto F_0(t, z, h)$ by formula (4.14). Next, one computes the coefficients $(F_{0,n}(t, z))_{n\geq 0}$ of the Taylor series of $h \mapsto F_0(t, z, h)$ at $h = 0$ (see (4.16)). Then, for each $n \geq 0$, in terms of $F_{0,n}(t, z)$, one computes two sequences $(a_{m,n})_{m\geq 0}$ and $(b_{m,n})_{m\geq 0}$ using the inductive formulas (4.17)–(4.19). When doing this, one takes into account relations (4.21). Finally, one expresses a_{2l} and b_{2l-1} in terms of $a_{m,n}$ and $b_{m,n}$ by means of formulas (4.26).

4.3. The proof of Proposition 4.2. The proof of Proposition 4.2 being parallel to that of Proposition 4.1, we concentrate only on the differences and omit details. We assume that B is analytic in U. Let w, γ, K be as in the proof of Proposition 4.1. We use the notations from (2.10). We assume that $(t, z) \in \mathbb{C} \times K$ and that $h > 0$ is sufficiently small. The derivation of the asymptotics of $H(Bh^{\frac{2}{3}}w')$ is split into several

steps.
1. First, we note that

$$H\left(Bh^{\frac{2}{3}}w_h'\right) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} G_0(t,z,h)\,dt, \tag{4.31}$$

$$G_0(t,z,h) = t\left(B(z+h)e^{\frac{\zeta(z+h)-\zeta(z)}{h}t} + B(z-h)e^{\frac{\zeta(z-h)-\zeta(z)}{h}t} + v(z)B(z)\right). \tag{4.32}$$

2. Let

$$G_{0,n}(t,z) = \frac{1}{n!}\frac{\partial^n G_0}{dh^n}(t,z,0). \tag{4.33}$$

We fix $n \in \mathbb{N}\cup\{0\}$. For $m \geq 0$, we inductively define

$$c_{m,n}(z) = \frac{1}{2}\left(G_{m,n}(\sqrt{\zeta(z)},z) + G_{m,n}(-\sqrt{\zeta(z)},z)\right), \tag{4.34}$$

$$d_{m,n}(z) = \frac{1}{2\sqrt{\zeta(z)}}\left(G_{m,n}(\sqrt{\zeta(z)},z) - G_{m,n}(-\sqrt{\zeta(z)},z)\right), \tag{4.35}$$

$$G_{m+1,n} = \frac{\partial g_{m,n}}{\partial t}, \quad g_{m,n}(t,z) = \frac{G_{m,n}(t,z) - c_{m,n}(z) - d_{m,n}(z)t}{t^2-\zeta(z)}, \tag{4.36}$$

where we use one and the same branch of $\sqrt{\zeta(z)}$.
The functions $(G_{m,n})_{m,n}$ are analytic in $(t,z) \in \mathbb{C}\times K$, and the functions $(c_{m,n})_{m,n}$ and $(d_{m,n})_{m,n}$ are analytic in $z \in K$ (all the removable singularities are removed).
3. One has $G_0(-t,z,-h) = -G_0(t,z,h)$. This implies that $\forall m,n \geq 0$

$$G_{m,n}(-t,z) = -(-1)^{m+n}G_{m,n}(t,z), \tag{4.37}$$

$$c_{m,n} = 0 \text{ if } m+n \text{ is even}, \qquad d_{m,n} = 0 \text{ if } m+n \text{ is odd}. \tag{4.38}$$

4. We set

$$c_l = \sum_{m+n=l} c_{m,n}, \qquad d_l = \sum_{m+n=l} d_{m,n}. \tag{4.39}$$

Reasoning as in the proof of Proposition 4.1, for $L \in \mathbb{N}$, instead of (4.27), we get

$$H(Bh^{\frac{1}{3}}w_h') = h^{\frac{1}{3}}w_h\left(\sum_{l=1}^{L} h^{2l-1}c_{2l-1} + O(h^{2L+1})\right) + h^{\frac{2}{3}}w_h'\left(\sum_{l=0}^{L} h^{2l}d_{2l} + O(h^{2L+2})\right). \tag{4.40}$$

5. One has $d_0 = 0$ (for the same reason as $a_0 = 0$ in the proof of Proposition 4.2). This already implies (4.3).
6. Finally, we obtain the formulas

$$c_1 = c_{01} + c_{10},$$
$$c_{0,1} = \tfrac{\partial G_0}{\partial h}(\sqrt{\zeta},z,0) = 2B'(z)\sqrt{\zeta}\sinh(\sqrt{\zeta}\zeta') + B(z)\zeta\zeta''\cosh(\sqrt{\zeta}\zeta'),$$
$$c_{1,0} = \tfrac{\partial}{\partial t}\tfrac{G_0(t,z,0)}{t^2-\zeta}\Big|_{t=\sqrt{\zeta}} = \tfrac{B\zeta'}{2}\left(\zeta'\cosh\left(\sqrt{\zeta}\zeta'\right) + \tfrac{\sinh\left(\sqrt{\zeta}\zeta'\right)}{\sqrt{\zeta}}\right),$$

where $\zeta = \zeta(z)$. They imply (4.4). This completes the proof of Proposition 4.2.

Remark 4.5. As is seen from the above proof, to compute the coefficients $c_{2l-1} = c_{2l-1}(B;z)$ and $d_{2l} = d_{2l}(B;z)$ from Proposition 4.2 for $l \in \mathbb{N}$, we proceed as follows. First, in terms of B, one defines a function $(t,z,h) \mapsto G_0(t,z,h)$ by formula (4.32). Next, one computes the coefficients $(G_{0,n}(t,z))$ of the Taylor series of $h \mapsto G_0(t,z,h)$ at $h=0$ (see (4.33)). Then, for each $n = 0,1,2,\dots$, in terms of $G_{0,n}(t,z)$, one computes two sequences $(c_{m,n})_{m\geq 0}$ and $(d_{m,n})_{m\geq 0}$ using the induction formulas (4.34)–(4.36). When doing this, one takes into account the symmetry relations (4.37) and (4.38). Finally, one expresses c_{2l-1} and d_{2l} in terms of $c_{m,n}$ and $d_{m,n}$ by means of formulas (4.39).

5. Properties of the asymptotic solutions. We now study basic properties of the asymptotic solutions. More precisely, we fix an integer $L \geq 0$ and study the functions $(W_j)_{j\in\mathbb{Z}_3}$, i.e., the functions W from Theorem 2.6 corresponding to L and to the Airy functions $w = w_j$, $j \in \mathbb{Z}_3$.

5.1. Functional relations. We recall that the function $(W_j)_{j\in\mathbb{Z}_3}$ are defined in a domain U satisfying Hypotheses 2.2–2.4.

Lemma 5.1. *One has*

$$W_0(z) + W_1(z) + W_2(z) = 0, \qquad \forall z \in U. \tag{5.1}$$

Proof. Formula (3.1) and the definitions of the integration paths $(\gamma_j)_{j\in\mathbb{Z}_3}$ (see Figure 1) imply that

$$w_0(\zeta) + w_1(\zeta) + w_2(\zeta) = 0, \qquad \zeta \in \mathbb{C}. \tag{5.2}$$

As the function ζ and all the coefficients $(A_{2l})_{l\in\mathbb{N}\cup\{0\}}$ and $(B_{2l+1})_{l\in\mathbb{N}\cup\{0\}}$ in representations (2.10)– (2.11) are independent of the choice of w, the solution of the Airy equation in this representation, the relation (5.2) implies (5.1). ∎

Relation (5.1) implies that

$$(W_0(z), W_1(z)) = (W_1(z), W_2(z)) = (W_2(z), W_0(z)), \qquad \forall \{z, z+h\} \subset U, \tag{5.3}$$

where $(f(z),\, g(z))$ is defined in (3.3).

5.2. Estimates on W_j. To prove the existence of analytic solutions that admit asymptotic expansions of the form (2.13), we need rough estimates of $(W_j)_{j\in\mathbb{Z}_3}$ in U. We first introduce some tools used in their derivation.

5.2.1. Geometry. We recall that the function ζ defined in (2.4) is analytic in U and bijectively maps U onto $V = \zeta(U)$, $\zeta(z_0) = 0$.

We put

$$\sigma_j = \zeta^{-1}(V \cap a_j), \quad a_j = e^{-2\pi i j/3}\,\mathbb{R}_-, \quad j \in \mathbb{Z}_3, \tag{5.4}$$

where $\mathbb{R}_- = (-\infty, 0]$. The curves $(\sigma_j)_{j\in\mathbb{Z}_3}$ are analytic and all begin at z_0. Any two of them do not intersect except at z_0. The angles between these curves at z_0 are equal to $2\pi/3$.

The curves $(\sigma_j)_{j\in\mathbb{Z}_3}$ cut the domain U (a disk centered at z_0) into three simply connected subdomains that we call *sectors*. We denote them by S_0, S_1 and S_2 so that the sector S_0 is bounded by σ_1 and σ_2, S_1 is bounded by σ_2 and σ_0, and S_2 is bounded by σ_0 and σ_1 (see Figure 2). Let

$$U_j = U \setminus \sigma_j, \quad j \in \mathbb{Z}_3. \tag{5.5}$$

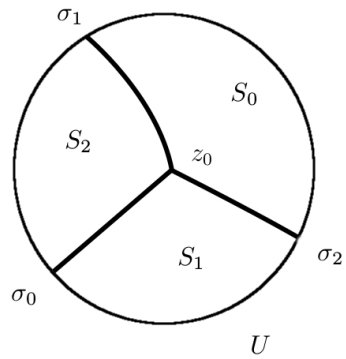

FIG. 2. *Stokes lines and sectors*

These domains do not contain branch points of the complex momentum p as the only branch point of p in U is $z = z_0$. We shall use

LEMMA 5.2. *For $j \in \mathbb{Z}_3$, there exists a branch of the complex momentum, say, p_j that is analytic in U_j and such that $p_j(z_0) = 0$ and*

1. $\operatorname{Im} \int_{z_0}^{z} p_j(z)\, dz > 0$ *inside S_j;*
2. $\operatorname{Im} \int_{z_0}^{z} p_j(z)\, dz < 0$ *inside the two other sectors;*
3. $\operatorname{Im} \int_{z_0}^{z} p_j(z)\, dz = 0$ *along the curves σ_1, σ_2 and σ_0 (in the case of σ_j, we mean the boundary values);*

Moreover, one has

$$p_1 = -p_0 \text{ in } \sigma_0 \cup S_1 \cup \sigma_2 \cup S_0 \cup \sigma_1, \quad p_2 = -p_0 \text{ in } \sigma_0 \cup S_2 \cup \sigma_1 \cup S_0 \cup \sigma_2. \tag{5.6}$$

In the WKB method, the curves $(\sigma_j)_{j\in\mathbb{Z}_3}$ are called *Stokes lines.*

Proof. Let us check the first three points of Lemma 5.2 for $j = 0$. We recall that ζ is an analytic branch of the function (2.4). We can assume that p is a branch of the complex momentum analytic in U_0 and such that $p(z_0) = 0$.

Formulas (5.4) and the definition of ζ imply that $\operatorname{Im} \int_{z_0}^{z} p(z)\, dz = 0$ on any of the Stokes lines. We note that

$$\zeta(S_j) = \{v \in V \ : \ v \neq 0, \ \arg v \in -2\pi j/3 + (-\pi/3, \pi/3)\}, \quad j \in \mathbb{Z}_3. \tag{5.7}$$

This and the definition of ζ imply that $\operatorname{Im} \int_{z_0}^{z} p(z)\, dz \neq 0$ in each of the sectors.

In view of the analysis made in section 3.3.1, in U_0, we can choose an analytic branch p_0 of the complex momentum so that $\operatorname{Im} \int_{z_0}^{z} p_0(z)\, dz > 0$ in S_0. For p_0, the statements 1 and 3 of Lemma 5.2 are obviously valid.

To prove point 2, it suffices to check that $\operatorname{Im} \int_{z_0}^{z} p_0(z)\, dz < 0$ in the sectors S_1 and S_2. Therefore, we note that as $z \neq z_0$, $z \sim z_0$, crosses σ_2 moving from S_0 to S_1 the argument of $\zeta(z)$ decreases (ζ vanishes only at $z = z_0$) as does the argument of $\int_{z_0}^{z} p_0(z)\, dz$. Therefore, point 2 of Lemma 5.2 follows from points 1 and 3.

To complete the proof of Lemma 5.2, we choose p_1 in the following way. First, in S_1 we choose $p_1 = -p_0$ and continue p_1 analytically from S_1 to U_1. For this p_1, we have

$$\operatorname{Im} \int_{z_0}^{z} p_1(z)\, dz = -\operatorname{Im} \int_{z_0}^{z} p_0(z)\, dz > 0, \quad z \in S_1.$$

This proves point 1 for p_1. Points 2 and 3 for p_1 are proved as for p_0 and p.

Finally, in S_2, we set $p_2 = -p_0$ and continue p_2 analytically from S_2 to U_2. To complete the proof of Lemma 5.2 for p_2, we reason as for p_1. We omit further details.□

5.2.2. Estimates. For $j \in \mathbb{Z}_3$ and $z \in U_j$, we set

$$\rho_j(z) = e^{\frac{i}{h}\int_{z_0}^{z} p_j(z')\,dz'}. \tag{5.8}$$

We note that ρ_j is continuous up to the cut along σ_j, and the boundary values of its absolute value $|\rho_j|$ on both the sides of the cut equal one. So, below, we consider $|\rho_j|$ as a continuous function in U.

Let us recall that H is defined by (2.9). We set

$$\delta_j(z) = [H(W_j)](z), \quad z \in U, \qquad j \in Z_3. \tag{5.9}$$

Proposition 5.3. *For each $j \in \mathbb{Z}_3$, one has*

$$|W_j(z)| \le Ch^{1/3}|\rho_j(z)|, \quad z \in U, \tag{5.10}$$

$$|\delta_j(z)| \le Ch^{2L+2+1/3}\,|\rho_j(z)|, \quad \{z, z+h, z-h\} \subset U, \tag{5.11}$$

where L is the order entering the definition of W_j (see (2.11)).

Proposition 5.3 immediately follows from formulas (2.10)–(2.12) with $w = w_j$ and

Lemma 5.4. *For $j \in \mathbb{Z}_3$, one has*

$$|w_j(h^{-\frac{2}{3}}\zeta(z))| \le C|\rho_j(z)|, \quad |w_j'(h^{-\frac{2}{3}}\zeta(z))| \le Ch^{-\frac{1}{6}}|\rho_j(z)|, \qquad z \in U. \tag{5.12}$$

Proof. We prove (5.12) only for $j = 0$. The other cases are treated similarly. We recall that $w_0 = \mathrm{Ai}$, that ζ bijectively maps U onto its image and that $\zeta(z_0) = 0$ (see (2.4)). Clearly,

$$w_0(h^{-\frac{2}{3}}\zeta(z)) = O(1) \text{ and } w_0'(h^{-\frac{2}{3}}\zeta(z)) = O(1) \text{ if } |\zeta(z)| \le h^{\frac{2}{3}}. \tag{5.13}$$

Now we turn to the case where $|\zeta(z)| \ge h^{\frac{2}{3}}$. It suffices to prove (5.12) in U_0.

The asymptotic formulas (3.2) imply that, for $Z \in \{Z \in \mathbb{C} \setminus \mathbb{R}_- \,:\, |Z| \ge 1\}$, one has

$$|w_0(Z)| \le C|Z|^{-\frac{1}{4}} \left|e^{-\frac{2}{3}Z^{\frac{3}{2}}}\right| \quad \text{and} \quad |w_0'(Z)| \le C|Z|^{\frac{1}{4}} \left|e^{-\frac{2}{3}Z^{\frac{3}{2}}}\right|, \tag{5.14}$$

where the $Z \mapsto Z^{\frac{3}{2}}$ is analytic in $\mathbb{C} \setminus \mathbb{R}_-$ and positive when $Z > 0$.

Estimate (5.14) and (5.5), the definition of U_0, imply that, for $z \in U_0$ such that $|\zeta(z)| \ge h^{\frac{2}{3}}$, one has

$$|w_0(h^{-\frac{2}{3}}\zeta(z))| \le C\left|e^{-\frac{2}{3h}\zeta(z)^{\frac{3}{2}}}\right| \text{ and } |w_0'(h^{-\frac{2}{3}}\zeta(z))| \le Ch^{-\frac{1}{6}}\left|e^{-\frac{2}{3h}\zeta(z)^{\frac{3}{2}}}\right|, \tag{5.15}$$

where $z \to \zeta(z)^{3/2}$ is analytic in U_0 and positive along $\alpha_0 = \zeta^{-1}((0,\infty))$.

In view of the analysis made in section 3.3.1,

$$\zeta(z)^{\frac{3}{2}} = \pm\frac{3i}{2}\int_{z_0}^{z} p_0(z')\,dz', \quad z \in U_0.$$

As $\alpha_0 = \zeta^{-1}((0,\infty)) \subset S_0$, along α_0 one has $\mathrm{Im}\int_{z_0}^{z} p_0(z')\,dz' > 0$. Therefore, in U_0 $\zeta(z)^{\frac{3}{2}} = -\frac{3i}{2}\int_{z_0}^{z} p_0(z')\,dz'$, and $\left|e^{-\frac{2}{3h}\zeta(z)^{\frac{3}{2}}}\right| = |\rho_0(z)|$. This and (5.15) imply (5.12) for $|\zeta(z)| \ge h^{-2/3}$. This and (5.13) imply Lemma 5.4. □

5.3. Wronskians. Below $\mathcal{C} \subset U$ is a closed h-independent disk centered at z_0. We prove

LEMMA 5.5. *For $z \in \mathcal{C}$, as $h \to 0$ one has*

$$(W_0(z), W_1(z)) = h(w_0'(z)w_1(z) - w_0'(z)w_1(z)) + O(h^{\frac{5}{3}}).$$

Before proving Lemma 5.5, we check

LEMMA 5.6. *Pick $j \in \mathbb{Z}_3$ and set $w = w_j$. For $z \in \mathcal{C}$, as $h \to 0$, one has*

$$h^{\frac{1}{3}} w_h \Big|_{z+h} = h^{\frac{1}{3}} w_h(z) \left(\cosh(\sqrt{\zeta(z)}\zeta'(z)) + O(h)\right) + h^{\frac{2}{3}}\, w_h'(z)\, \left(g(z) + O(h)\right),$$

where we use the notations from (2.10) and g is defined in (2.6).

Proof of Lemma 5.6. We proceed as in the proof of Proposition 4.1. Thus, we omit some details and concentrate on the new computations.

Let $\gamma = \gamma_j$ be the integration path in (3.1). Also, below we assume that $z \in \mathcal{C}$ and that h is sufficiently small. The proof is broken into several steps.

1. We compute

$$h^{\frac{1}{3}} w_h\Big|_{z+h} = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} E(t,z,h)\, dt, \quad E(t,z,h) = e^{\frac{t}{h}(\zeta(z+h) - \zeta(z))}. \tag{5.16}$$

2. We represent $E(t,z,0)$ in the form

$$E(t,z,0) = \alpha(z) + \beta(z)t + (t^2 - \zeta(z))\phi(t,z) \tag{5.17}$$

with

$$\alpha(z) = \frac{E(\sqrt{\zeta(z)}, z, 0) + E(-\sqrt{\zeta(z)}, z, 0)}{2},\ \beta(z) = \frac{E(\sqrt{\zeta(z)}, z, 0) - E(-\sqrt{\zeta(z)}, z, 0)}{2\sqrt{\zeta(z)}}.$$

3. Clearly, $E(t,z,0) = e^{t\zeta'(z)}$, and, therefore, one has

$$\alpha(z) = \cosh(\sqrt{\zeta}\zeta'), \qquad \beta(z) = g(z). \tag{5.18}$$

4. Using representation (5.17) and integrating by parts, we get

$$\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} E(t,z,0)\, dt = \alpha h^{\frac{1}{3}} w_h + \beta h^{\frac{2}{3}} w_h' + h \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} \frac{\partial \phi}{\partial t}(t,z)\, dt.$$

5. Reasoning as when proving Proposition 4.1, we check that

$$h^{\frac{1}{3}} w_h\Big|_{z+h} = \alpha h^{\frac{1}{3}} w_h + \beta h^{\frac{2}{3}} w_h' + O(h^{\frac{4}{3}} w) + O(h^{\frac{5}{3}} w').$$

Lemma 5.6 follows from this estimate and (5.18). ∎

Now, we turn to the proof of Lemma 5.5.

Proof of Lemma 5.5. We assume that $z \in \mathcal{C}$ and that h is sufficiently small. Using (2.11) and Lemma 5.6, we compute

$$\begin{aligned}(W_0(z),&W_1(z)) = A_0^2(z)\\ &\times \Big((h^{\frac{1}{3}} \cosh(\sqrt{\zeta}\zeta')\, w_0 + gh^{\frac{2}{3}}\, w_0' + O(h^{\frac{4}{3}} w_0) + O(h^{\frac{5}{3}} w_0'))\\ &\qquad\qquad \cdot (h^{\frac{1}{3}} w_1 + O(h^{\frac{7}{3}} w_1) + O(h^{\frac{5}{3}} w_1'))\\ &\quad - (h^{\frac{1}{3}} \cosh(\sqrt{\zeta}\zeta')\, w_1 + gh^{\frac{2}{3}}\, w_1' + O(h^{\frac{4}{3}} w_1) + O(h^{\frac{5}{3}} w_1'))\\ &\qquad\qquad \cdot (h^{\frac{1}{3}} w_0 + O(h^{\frac{7}{3}} w_0) + O(h^{\frac{5}{3}} w_0'))\Big).\end{aligned}$$

Here, $w_j = w_j(h^{-\frac{2}{3}}\zeta(z))$, $j \in \mathbb{Z}_3$, $\zeta = \zeta(z)$, and $g = g(z)$.

Now, we assume that $z \in \sigma_0 \cup S_1 \cup \sigma_2 \cup S_0 \cup \sigma_1$. Then, by (5.12) and (5.6)

$$|w_0 w_1| \le C, \quad |w_0' w_1| \le Ch^{-\frac{1}{6}}, \quad |w_0 w_1'| \le Ch^{-\frac{1}{6}}, \quad |w_0' w_1'| \le Ch^{-\frac{1}{3}}, \tag{5.19}$$

and we get

$$(W_0(z), W_1(z)) = hg(z)A_0^2(z)(w_0' w_1 - w_0 w_1') + O(h^{\frac{5}{3}}).$$

In view of (2.7), Lemma 5.5 is proved for $z \in \sigma_0 \cup S_1 \cup \sigma_2 \cup S_0 \cup \sigma_1$.

When $z \in \sigma_0 \cup S_2 \cup \sigma_1 \cup S_0 \cup \sigma_2$, we similarly get

$$(W_0(z), W_2(z)) = hg(z)A_0^2(z)(w_0' w_2 - w_2' w_0) + O(h^{\frac{5}{3}}). \tag{5.20}$$

In view of (5.3), $(W_0, W_2) = -(W_0, W_1)$ and relation (5.2) imply that $w_0' w_2 - w_0 w_2' = -(w_0' w_1 - w_0 w_1')$. Therefore, Lemma 5.5 for $z \in S_2$ follows from (5.20). This completes the proof of Lemma 5.5. ∎

6. Solutions to (1.1) on precanonical domains. Fix $L \ge 0$ and $j \in \mathbb{Z}_3$. The aim of this section is to construct a solution to equation (1.1), say, $\psi = \psi_j$ that admits representation (2.16) with W, the function from Theorem 2.6 corresponding to the Airy function w_j. The results of this section is preliminary as we only construct the solutions $(\psi_j)_{j\in\mathbb{Z}_3}$ on some subdomains of U.

6.1. The result of this section.

6.1.1. Notations and some definitions. First, to formulate the results of this section, we introduce some notations and recall some definitions related to the complex WKB method for difference equations, see, for example, [13].

For $z \in \mathbb{C}$, we let $x = \mathrm{Re} z$ and $y = \mathrm{Im} z$.

A curve $\gamma \subset \mathbb{C}$ is called *vertical* if z is a piecewise continuously differentiable function of y along γ.

Let $\gamma \subset U$ be a regular vertical curve parameterized by $z = z(y)$. Let p be a branch of the complex momentum continuous on γ. The curve γ is *precanonical* with respect to p, if the function $y \mapsto \mathrm{Im} \int_{z_0}^{z(y)} p(z)\, dz$ is non decreasing and the function $y \mapsto \mathrm{Im} \int_{z_0}^{z(y)} (p(z) - \pi)\, dz$ is non increasing.

Let $d > 0$. For $M \subset C$ we define the *horizontal d-neighborhood* of M to be the set $M^d := M + [-d, d]$ and $M^{-d} := (M^d - d) \cap M \cap (M^d + d)$.

We recall that, for $j \in \mathbb{Z}_3$, the sector S_j and the Stokes line σ_j are shown in Figure 2. For $j \in \mathbb{Z}_3$, we denote by $S_{j,j+1}$ the closure of the domain $S_j \cup S_{j+1}$

without the boundary of U. For example, one has

$$S_{1,2} = \sigma_1 \cup S_2 \cup \sigma_0 \cup S_1 \cup \sigma_2.$$

We also note that relations (5.6) imply that

$$|\rho_j(z)\rho_{j+1}(z)| = 1, \qquad z \in S_{j,j+1}, \quad j \in \mathbb{Z}_3. \tag{6.1}$$

Let $r_1 < r_2$. We set $S(r_1, r_2) = \{z \in \mathbb{C} : r_1 < \mathrm{Im} z < r_2\}$.

6.1.2. The main result of the section. One has

THEOREM 6.1. *Let $j \in \mathbb{Z}_3$, $L \in \mathbb{N} \cup \{0\}$, $c \in (1, 2)$ and $r > 0$. Let $K \subset S_{j,j+1}$ be a regular simply connected domain bounded by two curves having common endpoints z_1 and z_2 and both precanonical with respect to either the branch p_j or p_{j+1}.*

Then, for sufficiently small h, there exist two solutions ψ_j and ψ_{j+1} to (1.1) that are analytic in K^{ch} and that, in $K^{ch} \cap S(\mathrm{Im} z_1 + rh, \mathrm{Im} z_2 - rh)$ admit the asymptotic representations

$$\psi_l(z) = W_l(z) + O(|\rho_l| h^{2L+1+\frac{1}{2}}), \quad l \in \{j, j+1\}, \tag{6.2}$$

where W_l is the function described in Theorem 2.6 for w_l and the order L.

Let us fix $j \in \mathbb{Z}_3$ and discuss the solutions ψ_j and ψ_{j+1} described in Theorem 6.1.

COROLLARY 6.2. *In the case of Theorem 6.1, the solutions ψ_j and ψ_{j+1} can be analytically continued to $U \cap S(\mathrm{Im} z_1, \mathrm{Im} z_2)$. Let $r > 0$. As $h \to 0$, one has*

$$(\psi_j(z), \psi_{j+1}(z)) = (W_j(z), W_{j+1}(z)) + O(h^{2L+1+\frac{5}{6}}) \tag{6.3}$$

for $z, z + h \in K^{ch} \cap S(\mathrm{Im} z_1 + rh, \mathrm{Im} z_2 - rh)$.

Proof. The solutions being analytic in K^{ch} with $c > 1$, they can be analytically continued to $U \cap S(\mathrm{Im} z_1, \mathrm{Im} z_2)$ just be means of equation (1.1).

We fix $l \in \mathbb{Z}_3$ and note that, for all z in a compact set $\mathcal{C} \subset U$, for sufficiently small h, one has $|\rho_l(z + h)|/|\rho_l(z)| \le C$. For $z \in K^{ch} \cap S(\mathrm{Im} z_1 + rh, \mathrm{Im} z_2 - rh)$, the representation (6.3) follows from this observation and from (5.10), (6.2) and (6.1). ∎

The remainder of this section is devoted to the proof of Theorem 6.1.

For the sake of definiteness, when proving Theorem 6.1, we assume that $j = 0$ and that the two curves from Theorem 6.1 are precanonical with to respect to the branch p_0. The other cases are treated similarly.

Below, K is as in Theorem 6.1 (for $j = 0$): it is bounded by the precanonical curves γ_1 and γ_2, and their common endpoints satisfy $\mathrm{Im} z_1 < \mathrm{Im} z_2$. Finally, h is supposed to be sufficiently small.

6.2. The ideas of the proof. In the present section, we describe the construction of the solution ψ_0. The solution ψ_1 is constructed similarly.

Let us assume that ψ_0 is a solution to (1.1) analytic in K^{ch} that we expect to be close to W_0. Let us recall that $\delta_0 = HW_0$. Clearly, $\Delta_0 := W_0 - \psi_0$ satisfies the equation

$$H(\Delta_0)(z) = \delta_0(z), \quad \{z - h,\, z,\, z + h\} \subset K^{ch}. \tag{6.4}$$

For $z \in K^{ch}$, let $\gamma(z)$ denote a vertical curve in K^{ch} that contains z and connects z_1 and z_2. We construct a solution to (6.4) in the form

$$\Delta_0 = R_0\, g_0 \quad \text{where} \quad R_0 g_0\,(z) := \int_{\gamma(z)} r_0(z, \zeta) g_0(\zeta)\, d\zeta, \tag{6.5}$$

$$r_0(z,\zeta) = \frac{1}{2ih}\,\frac{W_0(z)W_1(\zeta) - W_0(\zeta)W_1(z)}{(W_0(\zeta),\, W_1(\zeta))}\,\theta_0\left(\frac{\zeta - z}{h}\right), \quad \theta_0(t) = \cot(\pi t) - i \tag{6.6}$$

where $(W_0(\zeta),\, W_1(\zeta))$ is the Wronskian of W_0 and W_1. The choice of Ansatz (6.5) is explained by

Lemma 6.3. *Let $0 < \beta < 1$. Let f be a function defined and analytic in $U \cap S(\mathrm{Im} z_1, \mathrm{Im} z_2)$ and such that the expression*

$$f_\beta(z) = (z - z_1)^\beta (z - z_2)^\beta f(z) \tag{6.7}$$

is bounded. Then, for $\{z - h, z, z + h\} \subset U \cap S(\mathrm{Im} z_1, \mathrm{Im} z_2)$, one has

$$HR_0 f(z) = f(z) + D_0 f(z), \quad D_0 f(z) = \int_{\gamma(z)} d_0(z,\zeta) f(\zeta)\, d\zeta \tag{6.8}$$

where

$$d_0(z,\zeta) = \frac{1}{2ih}\,\frac{\delta_0(z)W_1(\zeta) - W_0(\zeta)\delta_1(z)}{(W_0(\zeta),\, W_1(\zeta))}\,\theta_0\left(\frac{\zeta - z - 0}{h}\right), \tag{6.9}$$

and $\delta_j := HW_j$ are the "error" terms estimated in (5.11). The function $D_0 f$ is analytic in $U \cap S(\mathrm{Im} z_1, \mathrm{Im} z_2)$.

Proof. The analyticity of f and the boundedness of f_β imply that $D_0 f$ is well defined and analytic in $U \cap S(\mathrm{Im} z_1, \mathrm{Im} z_2)$. The relation $HR_0 f = f + D_0 f$ follows from the residue theorem. We omit further details. ∎

We note that an operator similar to R_0 was introduced in [13], but, was not studied for small h.

In view of Lemma 6.3 and the formulas $\Delta_0 = R_0 g_0$ and $H(\Delta_0) = \delta_0$, in K^{ch}, we can expect that

$$g_0 + D_0 g_0 = \delta_0. \tag{6.10}$$

Roughly, to prove Theorem 6.1, we consider (6.10) as an equation on a vertical curve γ. It appears that if γ is precanonical, the operator D_0 is small. This enables us to construct a solution ψ_0 to (6.10) on γ_1. Next, we check that it is analytic in K^{ch}, satisfies (1.1) and admits the asymptotic representation (6.2) with $l = 0$. The solution ψ_1 is constructed similarly.

6.3. The integral operator D_0. The aim of this section is to estimate the operator norm of D_0 in a suitable functional space.

Let γ be either γ_1 or γ_2. We fix $\alpha \in (0,1)$ and define the strip

$$\Pi_{\gamma,\alpha} = \gamma \setminus \{z_1, z_2\} + [-\alpha h, \alpha h].$$

We recall that $z \to |\rho_0(z)|$ defined in U_0 is a continuous function in U. We fix $0 < \beta < 1$ and let $H_{\gamma,\alpha,\beta}$ be the linear space of functions analytic in $\Pi_{\gamma,\alpha}$ and having finite norm

$$\|f\| = \sup_{z \in \Pi_{\gamma,\alpha}} \frac{|f_\beta(z)|}{|\rho_0(z)|} \tag{6.11}$$

f_β being defined in (6.7). Clearly, endowed with this norm, $H_{\gamma,\alpha,\beta}$ is a Banach space.

For $f \in H_{\gamma,\alpha,\beta}$, we define $D_0 f$ by the formula in (6.8) where $\gamma(z)$ is a vertical curve that connects the points z_1 and z_2 in $\Pi_{\gamma,\alpha}$ and passes through z. The function $D_0 f$ is then analytic in $\Pi_{\gamma,\alpha}$. One has

PROPOSITION 6.4. *For sufficiently small h*

$$\|D_0\|_{H_{\gamma,\alpha,\beta}\to H_{\gamma,\alpha,\beta}} \le Ch^{2L+\frac{2}{3}}.$$

The remainder of this subsection is devoted to the proof of Proposition 6.4. Therefore, for $f \in H_{\gamma,\alpha,\beta}$, we estimate $Rf(z)$. Up to the end of this subsection, we assume that $\{z,\zeta\} \subset \Pi_{\gamma,\alpha}$ and that h is sufficiently small.

6.3.1. An auxiliary lemma. When estimating $Rf(z)$, we use

LEMMA 6.5. *For $q > 0$, there exists $C > 0$ such that*

$$\sup_{\substack{\{z,\zeta\}\subset\Pi_{\gamma,\alpha}\\ \min\limits_{k\in\mathbb{Z}}|\zeta-z-kh|\ge qh}} \left|\frac{\rho_0(\zeta)}{\rho_0(z)}\, d_0(z,\zeta)\right| \le Ch^{2L+\frac{2}{3}}. \tag{6.12}$$

Proof. We proceed in several steps.

1. Proposition 5.3 and Lemma 5.5 imply that

$$\left|\frac{\rho_0(\zeta)}{\rho_0(z)}\, d_0(z,\zeta)\right| \le Ch^{2L+\frac{2}{3}} \left(|\rho_1(\zeta)\rho_0(\zeta)| + \frac{|\rho_1(z)\rho_0^2(\zeta)|}{|\rho_0(z)|}\right) \left|\theta_0\left(\frac{\zeta-z-0}{h}\right)\right|. \tag{6.13}$$

2. Recall that in $S_{0,1}$ one has $\rho_0(z)\rho_1(z) = 1$ (see (6.1)). As $\gamma_1 \cup \gamma_2 \subset S_{0,1}$, one has $|\rho_0(z)||\rho_1(z)| \le C$ for $z \in \Pi_{\gamma,\alpha}$. Therefore,

$$\left|\frac{\rho_0(\zeta)}{\rho_0(z)}\, d_0(z,\zeta)\right| \le Ch^{2L+\frac{2}{3}}\,(1+e(z,\zeta)) \left|\theta_0\left(\frac{\zeta-z-0}{h}\right)\right|, \quad e(z,\zeta) = \left|\frac{\rho_0(\zeta)}{\rho_0(z)}\right|^2. \tag{6.14}$$

3. For $z \in \Pi_{\gamma,\alpha}$, we define $z_\perp \in \gamma$ so that $\mathrm{Im} z_\perp = \mathrm{Im} z$. We have

$$|e(z,\zeta)| \le C \left|\exp\left(\frac{2i}{h}\int_{z_\perp}^{\zeta_\perp} p_0(z')\,dz'\right)\right|.$$

4. On the complex plane outside a fixed neighborhood of $\mathbb{Z}$, we have the estimate

$$|\theta_0(z)| = |\cot(\pi z) - i| \le C \begin{cases} 1, & \mathrm{Im} z \ge 0;\\ e^{2\pi \mathrm{Im} z}, & \mathrm{Im} z \le 0.\end{cases}$$

Therefore, for ζ outside the (qh)-neighborhood of $z + h\mathbb{Z}$, we get

$$\left|\theta_0\left(\frac{\zeta-z-0}{h}\right)\right| \le C \quad \text{and}$$

$$\left|e(z,\zeta)\,\theta_0\left(\frac{\zeta-z-0}{h}\right)\right| \le C \begin{cases} e^{-\frac{2}{h}\mathrm{Im}\int_{z_\perp}^{\zeta_\perp} p\,dz} & \text{if } \mathrm{Im}(\zeta-z) \ge 0;\\ e^{\frac{2}{h}\mathrm{Im}\int_{\zeta_\perp}^{z_\perp} (p-\pi)\,dz} & \text{if } \mathrm{Im}\,(\zeta-z) \le 0.\end{cases}$$

5. As γ is a precanonical curve, we finally get

$$\left|\theta_0\left(\frac{\zeta-z-0}{h}\right)\right| \le C, \quad \text{and} \quad \left|e(z,\zeta)\,\theta_0\left(\frac{\zeta-z-0}{h}\right)\right| \le C. \tag{6.15}$$

This and (6.14) imply (6.12). ∎

6.3.2. Estimates in the strip $S(\mathrm{Im} z_1 + h/2,\, \mathrm{Im} z_2 - h/2)$. When $z \in S(\mathrm{Im} z_1 + h/2,\, \mathrm{Im} z_2 - h/2)$, we prove

$$|\rho_0(z)^{-1} D_0 f(z)| \le C h^{2L+\frac{2}{3}} \|f\|. \tag{6.16}$$

First, we assume that z is between the curves $\gamma + \alpha h/2$ and $\gamma + \alpha h$. Then, in (6.8), one can deform the integration path $\gamma(z)$ to γ. The distance between the poles of d_0 and γ is larger than Ch. This, (6.11) and (6.12) imply (6.16).

Next, we assume that z is either between the curves γ and $\gamma + ah/2$ or on one of them. In this case, in (6.8) we can replace the integration path γ by $\tilde\gamma$ where

- $\tilde\gamma$ is a continuous curve that connects z_1 to z_2,
- $\tilde\gamma$ coincides with $\gamma - \alpha h/2$ in the strip $\{\mathrm{Im} z_1 + h/2 \le \mathrm{Im} z \le \mathrm{Im} z_2 - h/2\}$,
- outside this strip, $\tilde\gamma$ consists of two segments of straight lines.

Reasoning as above on this new integral, we again obtain (6.16).

Let us assume now that z is to the left of γ. We note that, by the Residue Theorem, the integral in (6.8) decomposes as the sum of

$$-\frac{\delta_0(z) W_1(z) - W_0(z)\delta_1(z)}{(W_0(z),\, W_1(z))} f(z) = O(h^{2L+1+\frac{2}{3}})\, f(z)$$

and the integral defined by (6.8)–(6.9) with $\theta((\zeta - (z+0))/h)$ replaced with $\theta((\zeta - (z-0))/h)$.

This new integral for z to the left of γ is analyzed as above. This completes the proof of (6.16).

6.3.3. Estimates in $S(\mathrm{Im} z_1,\, \mathrm{Im} z_1 + h/2)$ and $S(\mathrm{Im} z_2 - h/2,\, \mathrm{Im} z_2)$. Both domains are treated similarly. So, we detail only the analysis for the first one. We prove that

$$|\rho_0(z)^{-1} (D_0 f)_\beta(z)| \le C\, h^{2L+\frac{2}{3}}\, \|f\|. \tag{6.17}$$

For z between $\gamma + \alpha h/2$ and $\gamma + \alpha h$, reasoning as in section 6.3.2, one obtains (6.16) which implies (6.17).

For z between γ and $\gamma + \alpha h/2$, by contour deformation, the integration path in (6.8) is replaced with $\tilde\gamma$ defined in section 6.3.2. We, thus, write $D_0 f$ as the sum of an integral, say A, over the part of $\tilde\gamma \cap \{\mathrm{Im}\zeta \le z_1 + h/2\}$ and an integral, say B, over the part of $\tilde\gamma \cap \{\mathrm{Im} z_1 + h/2 \le \mathrm{Im}\,\zeta\}$.

Reasoning as in section 6.3.2, we estimate B and obtain

$$|\rho_0(z)^{-1} B| \le C h^{2L+\frac{2}{3}} \|f\|. \tag{6.18}$$

Let us turn to A. We again use (6.13). Now, both $|z - z_1|$ and $|\zeta - z_1|$ are bounded by Ch; thus, $|\rho_0(z)/\rho_0(\zeta)| \le C$. Furthermore, for such z, only one pole of the integrand, the pole at the point z, can approach the integration path in A; the other poles stay at a distance greater than Ch from it. Therefore, we get

$$|\rho_0(z)^{-1} A| \le C\, h^{2L+1+\frac{2}{3}}\, \|f\| \int_{\mathrm{Im} z \le \mathrm{Im} z_1 + h/2} \frac{|d\zeta|}{|z-\zeta|\, |\zeta - z_1|^\beta},$$

where we integrate along $\tilde\gamma$. Changing variable $t = (\zeta - z_1)/|z - z_1|$, one checks that the last integral is bounded by $C/|z - z_1|^\beta$. Thus,

$$|\rho_0(z)^{-1} A| \le C\, h^{2L+1+\frac{2}{3}}\, \|f\| / |z - z_1|^\beta.$$

This and (6.18) yields (6.17).

We omit further details and note only that, to prove (6.17) when z is to the left of γ, we first transform the integral from (6.8) as when doing the estimations in the strip $S(\mathrm{Im} z_1 + h/2, \mathrm{Im} z_2 - h/2)$ (see the end of the section section 6.3.2).

6.3.4. Completing the proof of Proposition 6.4. Proposition 6.4 follows from estimates (6.16) and (6.17).

6.4. Solutions to the integral equation (6.10). Consider the integral equation (6.10) in $H_{\gamma,\alpha,\beta}$. Proposition 6.4 and the estimate for δ_0 from (5.11) imply

LEMMA 6.6. *For sufficiently small h, the (6.10) has a unique solution g_0 in $H_{\gamma,\alpha,\beta}$. It satisfies*

$$\|g_0(z)\| = O(h^{2L+2+\frac{1}{3}}). \tag{6.19}$$

Moreover, one has

LEMMA 6.7. *The solution g_0, constructed in Lemma 6.6 for the curve $\gamma = \gamma_1$, can be analytically continued to the domain $K^{\alpha h}$. It then satisfies (6.10) and in $K^{\alpha h}$*

$$|(z_1 - z)(z_2 - z)|^\beta \frac{|g_0(z)|}{|\rho_0(z)|} \le C h^{2L+2+\frac{1}{3}}. \tag{6.20}$$

Proof. The proof is divided into four parts.

1. As g_0 is analytic in $\Pi_{\gamma_1,\alpha}$, it suffices to continue it to the right of γ_1. The function $\zeta \to \theta_0\left(\frac{\zeta - z - 0}{h}\right)$ has all its poles in $z + 0 + h\mathbb{Z}$. Hence, for z between γ_1 and $\gamma_1 + h$, we can define $D_0 g_0$ by means of (6.8) with $\gamma(z) = \gamma_1$, and $D_0 g_0$ is analytic between γ_1 and $\gamma_1 + h$.

As g_0 is analytic between γ and $\gamma + \alpha h$, to define $D_0 g_0$ for z between $\gamma_1 + \alpha h$ and $\gamma_1 + (1+\alpha)h$, we can deform the path of the integral in (6.8) to a vertical curve connecting z_1 to z_2 and staying between γ_1 and $\gamma_1 + \alpha h$. Thus, (6.8) implies that $D_0 g_0$ is analytic in z between γ_1 and $\gamma_1 + \alpha h + h$. In view of equation (6.10), this implies that g_0 itself is analytic to the left of $\gamma + (\alpha+1)h$. Reasoning in this way inductively, one shows that g_0 and $D_0 g_0$ are analytic between γ and $\gamma + (\alpha+2)h$, between γ and $\gamma + (\alpha+3)h$ and so on. As a result, g_0 and $D_0 g_0$ are analytic in $K^{\alpha h}$ to the right of γ and satisfy (6.10) for all $z \in K^{\alpha h}$.

2. Clearly, g_0 is analytic in $\Pi_{\gamma_2,\alpha}$, the expression $|(z_1 - z)(z_2 - z)|^\beta |g_0(z)|$ stays bounded in $\Pi_{\gamma_2,\alpha}$ (as γ_1 and γ_2 have common ends), and g_0 satisfies (6.10) along $\gamma = \gamma_2$. By Lemma 6.6, for sufficiently small h, this equation has a unique solution in $H_{\gamma_2,\alpha,\beta}$ which, thus, coincides with g_0. Hence, g_0 satisfies (6.19) with the norm of $H_{\gamma_2,\alpha,\beta}$.

3. In view of the previous step, g_0 satisfies (6.20) in $\Pi_{\gamma_1,\alpha} \cup \Pi_{\gamma_2,\alpha}$. This and the maximum principle for analytic functions imply that g_0 satisfies (6.20) also in the domain bounded by γ_1 and γ_2, i.e., in K.

The proof of Lemma 6.7 is complete. ∎

6.5. The solution to the difference equation. We define Δ_0 by (6.5) in terms of g_0 constructed in subsection 6.4. One has

LEMMA 6.8. *The function Δ_0 can be analytically continued to $K^{(1+\alpha)h}$ where it satisfies (6.4).*
Let $0 < c < 1 + \alpha$ and $r > 0$. In $K^{ch} \cap S(\mathrm{Im} z_1 + rh, \mathrm{Im} z_2 - rh)$, one has

$$|\Delta_0(z)| \le C|\rho_0(z) h^{2L+1}|. \tag{6.21}$$

Proof. By (6.20) the function $z \mapsto |(z_1 - z)(z_2 - z)|^\beta |g_0(z)|$ is bounded in $K^{\alpha h}$. For a given z, the poles of the kernel in (6.5) are contained in $z + h(\mathbb{Z} \setminus \{0\})$. Thus, the function Δ_0 is analytic in $\left(K^{\alpha h}\right)^h = K^{(1+\alpha)h}$.

By means of the Residue Theorem, one checks that $H\Delta_0 = HR_0 g_0$ is equal to $g_0 + D_0 g_0$ if $\{z, z \pm h\} \subset K^{(1+\alpha)h}$. As g_0 satisfies (6.10) in $K^{\alpha h}$, we obtain (6.4) if $\{z, z \pm h\} \subset K^{(1+\alpha)h}$.

To prove (6.21), we estimate $R_0 g_0$ in the same way as in section 6.3.2 we estimated $D_0 f$. So, we omit further details and only note that

1. outside (Ch)-neighborhood of the set $z + h\mathbb{Z}$, instead of (6.12) we obtain
$$\left|\frac{\rho_0(\zeta)}{\rho_0(z)}\, r_0(z,\zeta)\right| \le Ch^{-\frac{4}{3}};$$
2. on the diagonal $\{\zeta = z\}$, r_0, the kernel of R_0, is analytic whereas d_0, the kernel of D_0, has a pole. This simplifies the estimates of $(R_0 g_0)(z)$ to the left of γ_1. ∎

Having constructed Δ_0, we construct a solution ψ_0 to equation (1.1) setting $\psi_0 = W_0 - \Delta_0$, see (6.4). Let $c \in (0, 2)$. In view of (6.21), one has

$$\psi_0(z) = W_0(z) + O(|\rho_0(z)| h^{2L+1}), \qquad z \in K^{ch} \cap S(\operatorname{Im} z_1 + rh, \operatorname{Im} z_2 - rh). \tag{6.22}$$

In view of (5.12), estimate (6.22) implies (6.2) with $l = 0$ and L replaced with $L - 1$. As we could choose a larger L, this actually completes the proof of the statement of Theorem 6.1 on the solution ψ_0.

6.6. The second solution. Mutatis mutandis, the construction of the solution ψ_1 repeats that of ψ_0. We omit further details and mention only that, in this case:

- we set $\psi_1 = W_1 - R_1 g_1$ where R_1 is an integral operator with the kernel
$$r_1(z,\zeta) = \frac{1}{2ih}\, \frac{W_0(z)W_1(\zeta) - W_0(\zeta)W_1(z)}{(W_0(\zeta),\, W_1(\zeta))}\, \theta_1\left(\frac{\zeta - z}{h}\right), \quad \theta_1(t) = \cot(\pi t) + i;$$
- instead of (6.11), we use the norm $\|f\| = \sup\limits_{z \in \Pi_{\gamma,\alpha}} \dfrac{|f_\beta(z)|}{|\rho_1(z)|}$.

7. The proof of the main theorem. We now finally prove Theorem 2.7. We recall that in U there are three Stokes lines beginning at z_0. They are analytic curves and the angle between any two of them at z_0 is equal to $2\pi/3$. So, possibly reducing U somewhat, we can assume that at least two of them form a vertical curve. We prove the theorem assuming that these curves are σ_1 and σ_2, and that σ_1 goes upwards from z_0 (i.e., the vector tangent to σ_1 at z_0 is directed in the upper half-plane). Mutatis mutandis, the other cases are treated in the same way. Moreover, we assume that the tangent vector to σ_0 is either directed in the lower half-plane or is parallel to the real line and directed to the left. Then the curves $(\sigma_j)_{j \in \mathbb{Z}_3}$ are as in Figure 2. The complementary case is studied similarly.

Below we assume that h is sufficiently small.

7.1. Two geometric lemmas. To prove Theorem 2.7, we shall use the following two lemmas.

Lemma 7.1. *There exist two curves in $S_{1,2}$ precanonical with respect to p_2 and having common endpoints, and $\overset{\circ}{U}_1 \subset U$, a neighborhood of z_0, such that*

- *the domain K_1 bounded by the two curves is simply connected,*
- $K_1 \cap \overset{\circ}{U}_1 = S_{1,2} \cap \overset{\circ}{U}_1$.

and

LEMMA 7.2. *There exist two curves in $\sigma_2 \cup S_0 \cup \sigma_1$ precanonical with respect to p_0 and having common endpoints, and $\overset{\circ}{U}_0 \subset U$, a neighborhood of z_0, such that*

- *the domain K_0 bounded by the two curves is simply connected,*
- $K_0 \cap \overset{\circ}{U}_0 = (\sigma_2 \cup S_0 \cup \sigma_1) \cap \overset{\circ}{U}_0$.

We prove these two lemmas in subsection 8.1. We define $\overset{\circ}{U} = \overset{\circ}{U}_0 \cap \overset{\circ}{U}_1$.

7.1.1. The solution ψ_1. We denote by $\psi_{0,0}$ and $\psi_{1,0}$ the solutions ψ_0 and ψ_1 constructed by Theorem 6.1 for the domain K_0, and consider the solution ψ_1 constructed in Theorem 6.1 for the domain K_1. In view of Corollary 6.2, in $\overset{\circ}{U}$ (possibly reduced somewhat), all the three solutions are analytic, the Wronskian of $\psi_{0,0}$ and $\psi_{1,0}$ does not vanish (see also Lemma 5.5), and one has

$$\psi_1 = a\psi_{1,0} + b\psi_{0,0}, \tag{7.1}$$

where a and b are h-periodic coefficients (see section subsection 3.2). We prove

LEMMA 7.3. *One can reduce $\overset{\circ}{U}$ so that, for $z \in \overset{\circ}{U}$, one has*

$$a(z) = 1 + O(h^{2L+\frac{5}{6}}), \quad b(z) = O(h^{2L+\frac{5}{6}}), \qquad h \to 0. \tag{7.2}$$

Proof. In $\overset{\circ}{U}$ (possibly reduced somewhat), the coefficients a and b are described by (3.5) with $\psi = \psi_1$, $f = \psi_{1,0}$ and $g = \psi_{0,0}$.

Let $\gamma_{12} = (\sigma_1 \cup \sigma_2) \cap \overset{\circ}{U}$. By Lemma 7.1 and Lemma 7.2 one has $\gamma_{12} \subset K_0$ and $\gamma_{12} \subset K_1$.

First, we fix $c \in (1,2)$ and assume that $\{z, z+h\} \subset (\gamma_{12})^{ch}$.

In view of Lemma 5.2, one has $|\rho_1| = |\rho_2| = 1$ on γ_{12}. This and the definitions of $|\rho_1|$ and $|\rho_2|$ in section 5.2.2 imply that there exists $C > 0$ such that $|\rho_1(z)|, |\rho_2(z)| \le C$ in $(\gamma_{12})^{ch}$.

As $(\gamma_{12})^{ch}$ is a subset of both K_0^{ch} and K_1^{ch}, by means of (6.2) and (5.10), we get

$$a = \frac{(\psi_1,\, \psi_{0,0})}{(\psi_{1,0}, \psi_{0,0})} = \frac{(W_1 + O(h^{2L+\frac{3}{2}}),\, W_0 + O(h^{2L+\frac{3}{2}}))}{(W_1 + O(h^{2L+\frac{3}{2}}),\, W_0 + O(h^{2L+\frac{3}{2}}))} = \frac{(W_1, W_0) + O(h^{2L+1+\frac{5}{6}})}{(W_1, W_0) + O(h^{2L+1+\frac{5}{6}})}.$$

Lemma 5.5, then, yields the asymptotic representation for a in (7.2). Reasoning similarly, we get

$$b = \frac{(\psi_{1,0}, \psi_1)}{(\psi_{1,0}, \psi_{0,0})} = \frac{(W_1, W_1) + O(h^{2L+1+\frac{5}{6}})}{(W_1, W_0) + O(h^{2L+1+\frac{5}{6}})} = O(h^{2L+\frac{5}{6}}).$$

This is the estimate for b in (7.2).

Let c_1 and c_2 correspond to the minimal strip $S(c_1, c_2)$ containing $(\gamma_{12})^{ch}$. We proved estimates (7.2) for $a(z)$ and $b(z)$ in the case where $\{z, z+h\} \subset (\gamma_{12})^{ch}$. As $c > 1$ and as a and b are h-periodic, these estimates remain valid in $S(c_1, c_2)$. This implies Lemma 7.3. ∎

In view of Lemma 7.1, the solution ψ_1 admits representation (6.2) with $l = 1$ in $S_{1,2} \cap \overset{\circ}{U}$. Let us prove that it admits this representation in $S_0 \cap \overset{\circ}{U}$.
In view of Lemma 7.2, the solutions $\psi_{0,0}$ and $\psi_{1,0}$ admit representations (6.2) with $l = 0$ and $l = 1$ in $S_0 \cap \overset{\circ}{U}$. Substituting (7.2) and these representations into (7.1) and using (5.10), for $z \in S_0 \cap \overset{\circ}{U}$, we compute

$$\begin{aligned}\psi_1(z) &= (1 + O(h^{2L+\frac{5}{6}}))(W_1(z) + O(h^{2L+1+\frac{1}{2}}\rho_1(z))) \\ &\qquad\qquad + O(h^{2L+\frac{5}{6}})(W_0(z) + O(h^{2L+1+\frac{1}{2}}\rho_0(z))) \\ &= W_1(z) + O(h^{2L+\frac{7}{6}}\rho_1(z)) + O(h^{2L+\frac{7}{6}}\rho_0(z)).\end{aligned}$$

In view of Lemma 5.2, in S_0 one has $|\rho_0(z)| \le |\rho_1(z)|$. For ψ_1 in $S_0 \cap \overset{\circ}{U}$, this implies representation (6.2) with $l = 1$ and L replaced with $L - 1$. As L is arbitrary, we have proved (6.2) for ψ_1 in the whole domain $\overset{\circ}{U}$.

Now, we note that

$$h^{\frac{1}{2}}|\rho_0(z)| \le C|h^{\frac{1}{3}} w_0(h^{-\frac{2}{3}}\zeta(z))| + C|h^{\frac{1}{3}} w_0'(h^{-\frac{2}{3}}\zeta(z))|, \quad z \in U. \tag{7.3}$$

For sufficiently large values of $h^{-\frac{2}{3}}|\zeta(z)|$, this estimate follows from the definition of ρ_0 and the asymptotic formulas (3.2). For bounded $h^{-\frac{2}{3}}|\zeta(z)|$, it follows from the fact that w and w' do not have common zeros.

Estimates (6.2) and (7.3) imply (2.16) with L replaced with $L - 1$. As L is arbitrary, this completes the proof of the statement of Theorem 2.7 on the solution ψ_1 in the case that we consider.

7.1.2. The solution ψ_0. Let $\psi_{1,1}$ and $\psi_{2,1}$ be the solutions ψ_1 and ψ_2 constructed by Theorem 6.1 for the domain K_1, and let ψ_0 be the solution constructed by Theorem 6.1 for the domain K_0. For $z \in \overset{\circ}{U}$ (possibly reduced somewhat) one has

$$\psi_0 = a\psi_{1,1} + b\psi_{2,1}, \tag{7.4}$$

where a and b are h-periodic. One proves

Lemma 7.4. *One can reduce $\overset{\circ}{U}$ so that, for $z \in \overset{\circ}{U}$, one has*

$$a(z) = -1 + O(h^{2L+\frac{5}{6}}), \quad b(z) = -1 + O(h^{2L+\frac{5}{6}}), \quad h \to 0. \tag{7.5}$$

Proof. We omit details explained in the course of the proof of Lemma 7.3. We fix $c \in (1, 2)$ and assume that $\{z, z + h\} \subset (\gamma_{12})^{ch}$. For the coefficient a from (7.4), we get

$$a = \frac{(\psi_0,\, \psi_{2,1})}{(\psi_{1,1}, \psi_{2,1})} = \frac{(W_0,\, W_2) + O(h^{2L+1+\frac{5}{6}})}{(W_1,\, W_2) + O(h^{2L+1+\frac{5}{6}})} = \frac{(-W_1 - W_2,\, W_2) + O(h^{2L+1+\frac{5}{6}})}{(W_1,\, W_2) + O(h^{2L+1+\frac{5}{6}})},$$

where, in the last step, we used relation (5.1). Continuing, we get $a = -1 + O(h^{2L+\frac{5}{6}})$. Similarly one proves that $b = -1 + O(h^{2L+\frac{5}{6}})$. So, (7.5) is proved when $\{z, z + h\} \subset (\gamma_{12})^{ch}$. Reasoning as in the completion of the proof of Lemma 7.3, we complete the proof of Lemma 7.4. ∎

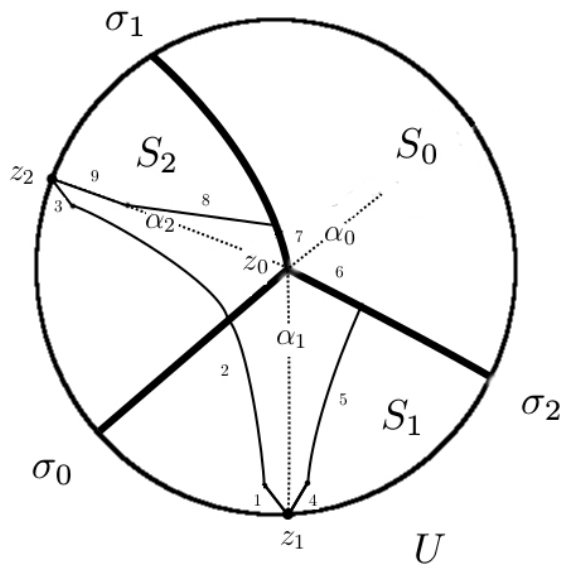

FIG. 3. *Domain* K_1

By Theorem 6.1 and Lemma 7.2, the solution ψ_0 admits representation (6.2) with $l = 0$ in $(\sigma_1 \cup S_0 \cup \sigma_2) \cap \overset{\circ}{U}$. Estimates (7.5) and (5.10) imply that in $S_{1,2} \cap \overset{\circ}{U}$ one has

$$\psi_0 = -W_1 - W_2 + O((|\rho_1| + |\rho_2|)h^{2L+7/6}) = W_0 + O((|\rho_1| + |\rho_2|)h^{2L+7/6}).$$

In view of Lemma 5.2 and the definitions of $|\rho_j|$, in $S_{1,2}$ one has $|\rho_1| + |\rho_2| \leq C|\rho_0|$ which implies (6.2) with $l = 0$ in $S_2 \cap \overset{\circ}{U}$. Reasoning as in the completion of section 7.1.1, we complete the proof of Theorem 2.7 for ψ_0 in the case that we consider.

7.1.3. The solution ψ_2. As the main theorem for ψ_2 is proved using the same techniques as for ψ_0 and ψ_1, we omit the details and note only that, in S_0, one represents ψ_2 as a linear combination of $\psi_{1,0}$ and $\psi_{0,0}$, and computes the coefficients in this linear combination as in the case of ψ_0.

8. The proofs of the geometric lemmas.

8.1. The proof of Lemma 7.1. This is done in several steps.

Below, all the precanonical lines are precanonical with respect to the branch p_2 of the complex momentum. We recall that p_2 is defined and analytic in the domain U_2 and continuous up to its boundary.

8.1.1. Anti-Stokes lines. We recall that the Stokes lines σ_j are defined by (5.4). The *Anti-Stokes lines*, $(\alpha_j)_{j\in\mathbb{Z}_3}$, are defined as

$$\alpha_j := \zeta^{-1}(V \cap e^{-2\pi i j/3}\,[0, +\infty)). \tag{8.1}$$

For $j \in \mathbb{Z}_3$, $\sigma_j \cap \alpha_j = \{z_0\}$ and the curve $\sigma_j \cup \alpha_j$ is analytic. The angles between any two of the Anti-Stokes lines at z_0 equal $2\pi/3$.

In the case we study, the Stokes and Anti-Stokes lines are pictured in Figure 3; the Anti-Stokes lines are represented by dotted lines. In particular, α_2 goes up from z_0, and α_1 goes down from z_0.

Reducing U if necessary, we assume that the Anti-Stokes lines α_1 and α_2 are vertical in U. As in Figure 3, let z_1 be the lower end of α_1 and z_2 the upper end of α_2.

One has

LEMMA 8.1. *Along the Anti-Stokes lines α_0, α_1 and α_2, one has* $\mathrm{Re}\int_{z_0}^{z} p_2\,dz = 0$. *The vector field $z \mapsto v(z) = \nabla \mathrm{Im}\int_{z_0}^{z} p_2\,dz$ vanishes only at $z = z_0$. The Anti-Stokes lines are tangent to this vector field at $z \neq z_0$. As z moves away from z_0,* $\mathrm{Im}\int_{z_0}^{z} p_2\,dz$ *monotonously increases along α_2 and monotonously decreases along α_1 and α_0.*

Proof. The statement on $\mathrm{Re}\int_{z_0}^{z} p_2\,dz$ follows directly from the definitions of the function ζ and of the Anti-Stokes lines. We note that $\|v(z)\| = |p_2(z)|$ and that $p_2(z)$ vanishes only at z_0 (modulo π, the complex momentum vanishes only at turning points and z_0 is the only turning point in U). Therefore, the vector field v vanishes only at $z = z_0$. The statement on $\mathrm{Re}\int_{z_0}^{z} p_2\,dz$ and the Cauchy-Riemann equations imply that the Anti-Stokes lines are tangent to the vector field v at points where it does not vanish. This and the first two points of Lemma 5.2 imply the statements of Lemma 8.1 on $\mathrm{Im}\int_{z_0}^{z} p_2\,dz$. ∎

We also use

LEMMA 8.2. *There exists $\tilde{U} \subset U$, a neighborhood of z_0, such that the lines $\alpha_1 \cap \tilde{U}$ and $\alpha_2 \cap \tilde{U}$ are precanonical.*

Let us parameterize $(\alpha_1 \cup \alpha_2) \cap \tilde{U}$ by $y = \mathrm{Im} z$, $z = z(y) = x(y) + iy$. Then, if $y \neq \mathrm{Im} z_0$, one has

$$\frac{d}{dy}\mathrm{Im}\int_{z_0}^{z(y)} p_2(z)\,dz > 0, \tag{8.2}$$

$$\frac{d}{dy}\mathrm{Im}\int_{z_0}^{z(y)} (p_2(z) - \pi)\,dz < 0. \tag{8.3}$$

Proof. As α_1 and α_2 are vertical, inequality (8.2) follows from Lemma 8.1. Furthermore, one has

$$\frac{d}{dy}\mathrm{Im}\int_{z_0}^{z(y)} (p_2 - \pi)\,dz = \mathrm{Im}(z'(y)p_2(z)) - \pi.$$

Therefore, as $p_2(z_0) = 0$, reducing U somewhat if necessary, we ensure (8.3).

Since $\alpha_1 \cup \alpha_2$ is vertical, (8.2) and (8.3) imply that the curve $\alpha_1 \cup \alpha_2$ is precanonical. ∎

Below, we assume that $\tilde{U} = U$ (if necessary we reduce U somewhat).

8.1.2. The precanonical line γ_1. We now construct a precanonical line $\gamma_1 \subset S_{1,2}$. It consists of three segments 1,2 and 3 shown in Figure 3. Let us describe them. *The segments 1 and 3.* To construct these segments, we use

LEMMA 8.3. *Let γ be a compact vertical C^1-curve parameterized by $y = \mathrm{Im} z$, $z = z(y) = x(y) + iy$. We assume that (8.2)– (8.3) hold along γ. Then, any compact C^1-curve sufficiently close in C^1-topology to γ is precanonical.*

This statement immediately follows from the definition of the precanonical curves.
The segment 1. It is a segment of a compact precanonical C^1-curve $c_1 \subset S_{1,2}$ that begins at z_1 and above z_1 goes to the left of α_1. When choosing c_1, we take an internal point of α_1 as $\tilde{z}_1$, and, as c_1, we take a C^1-curve close enough in C^1-topology to α_1 between z_1 and $\tilde{z}_1$. Lemma 8.2 and Lemma 8.3 guarantee that c_1 is a precanonical line.
The segment 3. Similarly, the segment 3 is a segment of a compact precanonical C^1-curve $c_3 \subset S_{1,2}$, having the upper end at z_2 and going to the left of α_2 below the point z_2.
The segment 2. We note that $\alpha_1 \cup \alpha_2$ is a level curve of the harmonic function $z \to \operatorname{Re} \int_{z_0}^{z} p_2(z)\, dz$ in $S_{1,2}$. The segment 2 is a segment of another level curve c_2 of this function in $S_{1,2}$. This curve is located to the left of $\alpha_1 \cup \alpha_2$. As it does not contain the point z_0, the only point in $S_{1,2}$ where p_2 vanishes, c_2 is smooth. We choose c_2 sufficiently close to $\alpha_1 \cup \alpha_2$ to ensure that

- c_2 is vertical (as α_1 and α_2 are);
- one has (8.2) along c_2 (the vector field $\nabla \operatorname{Im} \int_{z_0}^{z} p_2(z)\, dz$ does not vanish along c_2 and is tangent to c_2);
- (8.3) holds along c_2 (as it holds along $\alpha_1 \cup \alpha_2$);
- c_2 intersects both c_1 and c_3.

Clearly, c_2 is precanonical.
The curve γ_1. The segment 1 is the segment of c_1 between z_1 and the point of intersection of c_1 and c_2, the segment 2 is the segment of c_2 between the segment 1 and the point of intersection of c_2 and c_3, and the segment 3 is the segment of c_3 connecting the segment 2 with z_2. Clearly, the curve γ_1 made of segments 1–3 is precanonical.

8.1.3. The sign of $\operatorname{Im} p_2$ in S_2. The only place where we use our assumption on the direction of the tangent vector to σ_0 at z_0 is the proof of

LEMMA 8.4. *Both in S_2 between the curves α_2 and σ_1 and on these curves, near z_0, one has $\operatorname{Im} p_2(z) < 0$ when $z \neq z_0$.*

Proof. Below we assume that either z is in S_2 between the curves α_2 and σ_1 or on one of these curves. In view of (2.3), we can write

$$p_2(z) = k_1 \tau (1 + O(\tau)), \quad \int_{z_0}^{z} p_2(z)\, dz = \frac{2}{3} k_1 \tau^3 (1 + O(\tau)), \qquad z \to z_0, \tag{8.4}$$

where $k_1 \neq 0$ and τ is the branch of $\sqrt{z - z_0}$ analytic in U_2 and positive if $z > z_0$.

Let $0 < \theta_2 < \pi$ be the angle at z_0 between the line $\{z \geq z_0\}$ and the curve α_2 . Note that the angle between σ_0 and α_2 equals $\pi/3$. Therefore, as the tangent vector to σ_0 at z_0 is either directed downwards or parallel to the real line and directed to the left, one has $2\pi/3 \leq \theta_2 < \pi$.

In view of Lemma 8.1, along α_2, $\operatorname{Re} \int_{z_0}^{z} p_2 dz = 0$ and $\operatorname{Im} \int_{z_0}^{z} p_2 dz$ is monotonously increasing. This and the second formula in (8.4) imply that

$$\arg k_1 + \frac{3}{2}\theta_2 = \frac{\pi}{2} \mod 2\pi. \tag{8.5}$$

Let $z - z_0 = |z - z_0| e^{i\theta}$. Using (8.5) and the first formula in (8.4), we get near z_0

$$\frac{\operatorname{Im} p_2(z)}{|p_2(z)|} = \sin\left(\arg k_1 + \frac{\theta}{2} + o(1)\right) = \cos\left(\theta_2 - \frac{\theta - \theta_2}{2} + o(1)\right). \tag{8.6}$$

Now, we note that, for z we consider, near z_0 one has $\theta_2 - \pi/3 + o(1) \le \theta \le \theta_2 + o(1)$. Therefore, for z sufficiently close to z_0, one has

$$\frac{2\pi}{3} + o(1) \le \theta_2 + o(1) \le \theta_2 - \frac{\theta - \theta_2}{2} \le \theta_2 + \frac{\pi}{6} + o(1) < \frac{7\pi}{6}.$$

This and (8.6) implies the statement of Lemma 8.4. ∎

8.1.4. The precanonical line γ_2. The precanonical line γ_2 is located in $S_{1,2}$ and consists of six segments 4–9 shown in Figure 3. Let us describe them.
The segments 4-5-6-7. The segment 4 is a segment of a compact precanonical C^1-curve $c_4 \subset S_{1,2}$. This curve begins at z_1 and above z_1 goes to the right of α_1. It is constructed as the curve c_1 containing the segment 1.

The segment 5 is a segment of a level curve c_5 of the function $z \to \mathrm{Re} \int_{z_0}^z p_2(z)\, dz$ in $S_{1,2}$. The construction of c_5 is similar that of c_2. The curve c_5 is located to the right of α_1. We choose c_5 sufficiently close to α_1. Then, c_5 is a precanonical curve and intersects both c_4 and the Stokes line σ_2.

The segment 4 is the segment of c_4 between z_1 and the point of intersection of c_4 and c_5. The segment 5 connects this point to a point of σ_2.

We prove

Lemma 8.5. *Let γ be a vertical curve, let $a \in \gamma$ and let p be a branch of the complex momentum continuous on γ. If either* $\mathrm{Im} \int_a^z p(z)\, dz = 0$ *or* $\mathrm{Im} \int_a^z (p(z) - \pi)\, dz = 0$ *are satisfied on γ, then γ is precanonical with respect to p.*

Proof. Assume that $\mathrm{Im} \int_a^z p(z)\, dz = 0$ on γ. Then, $z \mapsto \mathrm{Im} \int_a^z (p(z) - \pi)\, dz = -\pi \mathrm{Im}(z - a)$ is decreasing along γ when $\mathrm{Im} z$ increases. Thus, γ is precanonical.
If $\mathrm{Im} \int_a^z (p(z) - \pi)\, dz = 0$, then $z \mapsto \mathrm{Im} \int_a^z p(z)\, dz = \mathrm{Im} \int_a^z \pi\, dz = \mathrm{Im}(z - a)$ is increasing along γ when $\mathrm{Im} z$ increases. Thus, γ is precanonical. ∎

The segment 6 is the segment of $c_6 = \sigma_2$ between the upper end of the segment 5 and the point z_0. The segment 7 is the segment of $c_7 = \sigma_1$ between z_0 and an internal point a of σ_1. We describe this point later. Lemma 8.5 implies that the segments 6 and 7 are precanonical.
Segment 8. This segment is a segment of c_8, the level curve $\gamma(a)$ of the harmonic function $z \to \mathrm{Im} \int_{z_0}^z (p_2(z) - \pi)\, dz$ that contains $a \in \sigma_1$. To choose the segment 8, we check

Lemma 8.6. *If $a \in \sigma_1 \setminus \{z_0\}$ is sufficiently close to z_0, then $\gamma(a)$ intersects σ_1 transversally at a, enters S_2 at a going upwards, intersects α_2 and remains vertical up to this intersection.*

Proof of Lemma 8.6. Below, we identify vectors on $\mathbb{R}^2$ with complex numbers in the standard way and the bar denotes complex conjugation. The Stokes line σ_1 is tangent to the vector field $z \mapsto v_0(z) = \overline{p_2(z)}$ at $z \ne z_0$ ($p_2(z_0) = 0$). The curve $\gamma(a)$ is tangent to the vector field $z \mapsto v_\pi(z) = \overline{p_2(z)} - \pi$.

Let $a \in \sigma_1 \setminus z_0$ be sufficiently close to the point z_0. In view of Lemma 8.4, $\mathrm{Im} p_2(a) < 0$. Therefore, $\gamma(a)$ is vertical at a. Moreover, both the vectors $v_0(a)$ and $v_\pi(a)$ are directed upwards and $v_\pi(a)$ is directed to the left of $v_0(a)$. Therefore, at a, the curve $\gamma(a)$ intersects σ_1 transversally and enters S_2 going upwards.

Furthermore, in view of Lemma 8.4, as long as $\gamma(a)$ stays in S_2 near z_0 between the curves α_2 and σ_1 or on them, it remains vertical.

To complete the proof, it suffices to show that if a is sufficiently close to z_0, then $\gamma(a)$ intersects α_2 remaining vertical. Therefore, we note that $v_\pi(z_0) = -\pi$. So, at

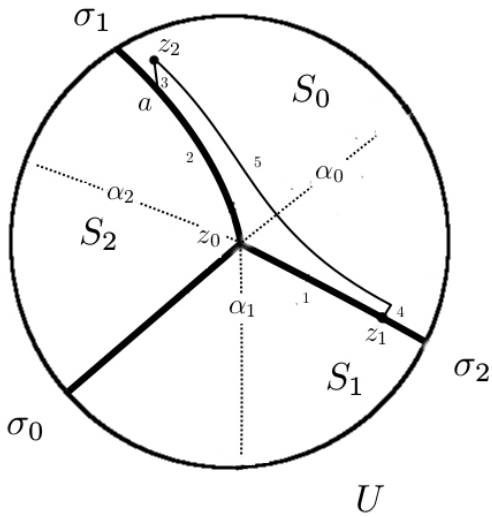

FIG. 4. *Domain K_0*

z_0 the vector tangent to $\gamma(z_0)$ is parallel to $\mathbb{R}$, and the curve $\gamma(z_0)$ intersects the analytic curve $\alpha_2 \cup \sigma_2$ transversally. Depending continuously on a, $\gamma(a)$ intersects this curve also for all a sufficiently close to z_0. But, if $\operatorname{Im} a > \operatorname{Im} z_0$ and a is sufficiently close to z_0, the curve $\gamma(a)$ goes upward from a. Therefore, for a sufficiently close to z_0, the curve $\gamma(a)$ intersects α_2 still remaining vertical. This completes the proof of Lemma 8.6. ∎

The segments 8 and 9. We choose the point a, the end of the segment 7 and the beginning of the segment 8, so that $c_8 = \gamma(a)$ intersects α_2 as described in Lemma 8.6. The end of the segment 8 is the point of intersection of c_8 and α_2. By Lemma 8.5, the segment 8 is precanonical. The segment 9 is the segment of α_2 connecting the upper end of the segment 8 to the point z_2. It precanonical by Lemma 8.2.

The domain K_1 bounded by γ_1 and γ_2 is the one described in Lemma 7.1, the proof of which is complete.

8.2. The proof of Lemma 7.2. The proof uses the same techniques as the proof of Lemma 7.1. Therefore, we omit some details. The construction of the curves γ_1 and γ_2 bounding the domain K_0 from Lemma 7.2 is illustrated by Figure 4. Below, all the precanonical lines are precanonical with respect to p_0.

8.2.1. The curve γ_1. This curve consists of segments 1–3 that we describe now.

Let z_1 be an internal point of σ_2 and fix a, an internal point of σ_1. The segment 1 is the segment of σ_2 between z_1 and z_0, and the segment 2 is the segment of σ_1 between z_0 and a.

To describe the segment 3, we consider $\gamma_0(a)$, the curve in $\sigma_2 \cup S_0 \cup \sigma_1$ described by the equation $\operatorname{Im} \int_a^z (p_0(z) - \pi)\, dz = 0$. We suppose that a is sufficiently close to z_0. Then, $\gamma_0(a)$ intersects σ_1 at a transversally, enters in S_0 going upwards and is vertical in a neighborhood a. To prove this, note that, near z_0 on σ_1, one has $\operatorname{Im} p_0(z) > 0$ (this is proved in the same way as Lemma 8.4). The segment 3 is a segment of $\gamma_0(a)$ connecting in this neighborhood a to a point $z_2 \in S_2$ that we choose later on.
Lemma 8.5 implies that the segments 1–3 are precanonical.

The points z_1 and z_2 are the ends of γ_1.

8.2.2. The curve γ_2. This curve consists of two segments, segments 4 and 5. The segment 4 is a segment of c_4, the level curve of the function $z \to \operatorname{Re} \int_{z_1}^{z} p_0(z)\,dz$ in $S_0 \cup \sigma_2$ that passes through z_1. The curve c_4 is orthogonal to σ_2 at z_1.

Let us note that, under our assumptions on σ_0 and σ_2 (see the very beginning of section 7), the angle at z_0 between σ_2 and the horizontal line $\{z \geq z_0\}$ belongs to $(0, \pi/3)$. Possibly reducing U somewhat, we assume that, at any point $\zeta \in \sigma_2$, the angle between σ_2 and the line $\{z \geq \zeta\}$ belongs to $(0, \pi/2)$. Then, c_4 is vertical at least in a neighborhood of the point z_1 and goes upward from z_1 into S_0.

The segment 5 is a segment of a level curve c_5 of the function $z \to \operatorname{Im} \int_{z_0}^{z} p_0(z)\,dz$ in S_0. It is located to the right of $\sigma_2 \cup \sigma_1$ (which is also a level curve of this function). We choose the curve c_5 sufficiently close to $\sigma_2 \cup \sigma_1$. Then, it is vertical, intersects $\gamma_0(a)$ and c_4, and the segments of these curves between $\sigma_2 \cup \sigma_1$ and the intersection points are vertical.

The point z_2 is the point of intersection of $\gamma_0(a)$ and c_5. The segment 4 is the segment of c_4 between $\sigma_2 \cup \sigma_1$ and c_5, and the segment 5 is the segment of c_5 connecting c_4 to z_2.

The segment 5 is precanonical in view of Lemma 8.5. Arguing as when proving Lemma 8.2 and reducing somewhat U if necessary, we check that the segment 4 is precanonical.

The domain K_0 bounded by the curves γ_1 and γ_2 is the domain described in Lemma 7.2, the proof is complete.

REFERENCES

[1] A. Avila, S. Jitomirskaya, *The Ten Martini Problem*, Annals Math., 170 (2009), pp. 303–342.

[2] V. Babich, M. Lyalinov, and V. Grikurov, *Diffraction theory: the Sommerfeld-Malyuzhinets technique*, Alpha Science, Oxford, 2008.

[3] V. Buslaev and A. Fedotov, *Complex WKB method for Harper equation*, St. Petersburg Math. J., 6 (1995), pp. 495–517.

[4] V. Buslaev and A. Fedotov, *The monodromization and Harper equation*, Séminaire sur les Équations aux Dérivées Partielles 1993-1994, Exp. no EXXI, Ãcole Polytech., Palaiseau, 1994, 23 pp.

[5] O. Costin and R. Costin, *Rigorous WKB for finite-order linear recurrence relations with smooth coefficients*, SIAM J. math. an., 27 (1996), pp. 110–134.

[6] A. Eckstein, *Unitary reduction for the two-dimensional Schrödinger operator with strong magnetic field*, Math. Nachr., 282(2009), pp. 504-525.

[7] S. Y. Dobrokhotov and A. V. Tsvetkova, *On lagrangian manifolds related to asymptotics of Hermite polynomials*, Math. notes, 110 (2018).

[8] M. Fedoryuk, *Asymptotic Analysis. Linear Ordinary Differential Equations*, Springer-Verlag, Berlin, Heidelberg, 2009.

[9] A. Fedotov, *Monodromization method in the theory of almost-periodic equations*, St. Petersburg Math. J., 25 (2014), pp. 303-325.

[10] A. Fedotov and F. Klopp, *Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case*, Commun. in Math. Phys., 227 (2002), pp. 1–92.

[11] A. Fedotov and F. Klopp, *On the absolutely continuous spectrum of an one-dimensional quasiperiodic Schrödinger operator in adiabatic limit*, Transactions of AMS, 357 (2005), pp. 4481–4516.

[12] A. Fedotov and F. Klopp, *Difference equations, uniform quasiclassical asymptotics and Airy functions*, in Proceeding of the conference "Days on Diffraction 2018", St. Petersburg, 2018, IEEE, pp. 98–101.

[13] A. Fedotov and E. Shchetka, *Complex WKB method for difference equations in bounded domains*, J. of Math. Sciences (New York), 224 (2017), pp. 157–169.

[14] A. Fedotov and E. Shchetka, *Complex WKB method for the difference Schrödinger equation with the potential being a trigonometric polynomial*, St. Petersburg Math. J., 29 (2018),

pp. 363–381.

[15] J. S. Geronimo, O. Bruno, and W. V. Assche, *WKB and turning point theory for second-order difference equations*, Operator theory: advances and app., 69 (1992), pp. 269–301.

[16] J. P. Guillement, B. Helffer, and P. Treton, *Walk inside Hofstadter's butterfly*, J. Phys. France, 50 (1989), pp. 2019–2058.

[17] B. Helffer, and J. Sjöstrand, *Analyse semi-classique pour l'équation de Harper (avec application à l'équationde Schrödinger avec champ magnétique)*, Mémoires de la SMF (nouvelle série), 34(1988), pp. 1-113.

[18] M. A. Lyalinov, and N. Y. Zhu, *A solution procedure for second-order difference equations and its application to electromagnetic-wave diffraction in a wedge-shaped region*, Proc. R. Soc. Lond. A, 459(2003), pp. 3159-3180.

[19] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[20] Y. Sibuya, *Global theory of a second order linear ordinary differential euqation with a polynomial coefficient*, North Holland/American Elsevier, 1975.

[21] G. A. Tsyganov, *Asymptotic behavior of solutions of a linear difference system with a small shift and a turning point*, Diff. equations (in Russian), 10 (1974), pp. 1312–1321.

[22] A. B. Vasilieva and E. Yuldashev, *Linear difference systems with a small shift*, Diff. equations (in Russian), 6 (1970), pp. 2267–2269.

[23] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Dover Publications, New York, 1987.

[24] M. Wilkinson, *An exact renormalization group for Bloch electrons in a magnetic field*, J. Phys. A: Math. Gen., 20 (1987), pp. 4337–4354.

[25] R. Wong, *Asymptotic approximations of integrals*, SIAM, Philadelphia, 2001.

[26] R. Wong and Z. Wang, *Asymptotic expansions for second-order linear difference equations with a turning point*, Numerische Mathematik, 94 (2003), pp. 147–194.