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Trapezoidal rule and sampling designs for the nonparametric estimation of the regression function in models with correlated errors

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Abstract: The problem of estimating the regression function in a fixed design models with correlated observations is considered. Such observations are obtained from several experimental units, each of them forms a time series. Based on the trapezoidal rule, we propose a simple kernel estimator and we derive the asymptotic expression of its integrated mean squared error IMSE and its asymptotic normality. The problems of the optimal bandwidth and the optimal design with respect to the asymptotic IMSE are also investigated. Finally, a simulation study is conducted to study the performance of the new estimator and to compare it with the classical estimator of Gasser and Müller in a finite sample set.

Key words: *Nonparametric regression, optimal design, autocovariance function, trapezoidal rule, asymptotic normality.*

1 Introduction

A classical problem in Statistics is the nonparametric estimation of the regression function of a response variable Y given an explanatory variable X , i.e, estimating the function g defined by $g(t) = \mathbb{E}(Y|X = t)$, based on the observations of $(X_i, Y_i)_{1 \leq i \leq n}$ which are independent copies of (X, Y) . These observations are often modeled as follows: $Y_i = g(t_i) + \varepsilon_i$ where g is the unknown regression function to be estimated, the $\{t_i, i = 1, \dots, n\}$ is the sampling design and $\{\varepsilon_i, i = 1, \dots, n\}$ are centered errors. Typically when $(\varepsilon_i)_i$ are i.i.d. the estimation of g has been extensively investigated by several authors. We mention, among others, the work of Priestly and Chao [1], Benedetti [2] and Gasser and Müller [3]. However, considering that the observations are independent is not always a realistic assumption. In pharmacokinetics for instance, one wishes to estimate the concentration-time of some injected medicine in the organism, based on the observation of blood tests over a period of time. It is clear that the observations provided from the same individual are correlated. For this, we shall investigate in this paper the nonparametric regression estimation problem where the observations are correlated.

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We consider the so-called fixed design regression model with repeated measurements, i.e.,

$$Y_j(t_i) = g(t_i) + \varepsilon_j(t_i) \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m, \quad (1)$$

where $\{\varepsilon_j, j = 1, \dots, m\}$ is a sequence of i.i.d. centered error processes with the same distribution as a process ε . Such models are well known in growth curve analysis and dose response curves. They can be obtained, as noted by Azzalini [4], from m individual being observed on a period of time. Generally, observations between different individuals will be uncorrelated. Hence, it is of interest to relax the assumption of correlation between the experimental units.

Müller [5] considered Model (1) for $m = 1$ (observations on one experimental unit) and he supposed that, for $s \neq t$, the covariance $\text{Cov}(\varepsilon_j(t), \varepsilon_j(s))$ tends to 0 as n tends to infinity, which is not a realistic assumption, as indicated by Hart and Wherly [6], in the growth curve problems. They investigated the estimation of g in Model (1) with a stationary error process. They used the estimator proposed by Gasser and Müller [3], and they showed that, in order to obtain the consistency of the kernel estimator in the presence of correlations, it is necessary to take m experimental units and let m tends to infinity.

The stationarity assumption is however restrictive, for instance, in the previous pharmacokinetics example, it is clear that the concentration of the medicine will be high at the beginning then decreases with time. For this, we shall investigate the estimation of g in Model (1) where ε is a nonstationary error process. This case was partially investigated by Ferreira et al. [7] and Benhenni and Rachdi [8], where the Gasser and Müller estimator was used.

In this paper, we propose a new estimator for the regression function g as an approximation of the kernel estimator based on continuous observations in the whole interval $[0, 1]$ constructed through a stochastic integral. See, for instance, Blanke and Bosq [9], Didi and Louani [10]. When only discrete observations are available, we use the "best" approximation of the stochastic integral, which is obtained by using the trapezoidal rule based on discrete observations at appropriate n sampling points generated by a sampling density in the interval $[0, 1]$.

This estimator has a relatively simpler expression than the kernel estimator proposed by Gasser and Müller [3]. Moreover, since this last one depends on n integrals of a kernel at middle samples; and may be subject to numerical (computational) instability, for instance when a Gaussian kernel is used, whereas the proposed estimator depends only on the observations and the values of the kernel at the sampling points.

In addition to its simple expression, the proposed estimator allows to bring an answer to another important and open statistical problem under correlated errors, which is the optimal design problem. For instance, in the previous pharmacokinetic example, one wishes to find the best moments for the blood testing to be made in order to have a better estimate of the concentration curve.

The optimal design problem has been extensively studied in parametric regression. We mention the work of Sacks and Ylvisaker [11], Belouni and Benhenni [12] and more recently Dette *et al.* [13] among others. In the nonparametric case, Müller [5] introduced the optimal design points when the errors are asymptotically independent. He used a regular design sequence generated by a density function f , i.e, $t_i = F^{-1}(\frac{i}{n})$, where F is the distribution function associated to f . He derived the optimal design generated by a density that minimizes the asymptotic Integrated Mean Squared Error (IMSE). To the best of our knowledge, there exists no result concerning the problem of optimal design for

nonparametric regression estimation in models under more general class of error processes.

We also investigate the problem of the asymptotic optimal bandwidth. We mention, for the nonparametric case, the work of Hart and Wherly [6] and Benhenni and Rachdi [8]. For results on the break down of some data based methods for bandwidth selection in the presence of correlation, for instance the cross validation, and other alternative methods, the reader is referred to Chiu [14], Altman [15], Hart [16], and Hart [17] among others.

This article is organized as follows. In Section 2, we present the new estimator of the regression function g in Model (1) where ε is a centered error process. In Section 3, we give the asymptotic expressions of the bias, the variance and the IMSE. We then derive the asymptotic optimal bandwidth with respect to the asymptotic IMSE. In addition, we obtain the optimal design density with respect to the asymptotic IMSE. We also prove the asymptotic normality of the proposed estimator. In Section 4, we conduct a simulation study to investigate the performance of the new estimator and then to compare it with that of Gasser and Müller [3]. Since the classical cross validation criteria turned out to be inefficient in the presence of correlation, we use the bandwidth that minimizes the exact IMSE, the comparison is performed for different numbers of experimental units and different numbers of design points. Finally, Section 5 is dedicated to the proofs of our theoretical results.

2 Model and estimator

We consider m experimental units, each of them having n different measurements of the response (say $0 \leq t_1 < t_2 < \dots < t_n \leq 1$). The so-called fixed design regression model is defined as follows:

$$Y_j(t_i) = g(t_i) + \varepsilon_j(t_i) \text{ where } j = 1, \dots, m \text{ and } i = 1, \dots, n, \quad (2)$$

where g is the unknown regression function on $[0, 1]$ and $\{\varepsilon_j(t), t \in [0, 1]\}_j$ is a sequence of error processes.

We assume that $g \in C^2([0, 1])$ and that $(\varepsilon_j)_j$ are i.i.d. processes with the same distribution as a centered second order process ε . We denote by R its autocovariance function.

2.1 Simple estimator and sampling design

In order to motivate the construction of our new estimator, we consider the regression model using m continuous experimental units, i.e,

$$Y_j(t) = g(t) + \varepsilon_j(t) \text{ for } t \in [0, 1] \text{ and } j = 1, \dots, m. \quad (3)$$

A continuous kernel estimator of g in Model (3) is given for any $x \in [0, 1]$ by,

$$\hat{g}_{[0,1]}(x) = \int_0^1 \varphi_{x,h}(t) \bar{Y}(t) dt \quad \text{with} \quad \bar{Y}(t) = \frac{1}{m} \sum_{j=1}^m Y_j(t), \quad (4)$$

where $\varphi_{x,h}(t) = \frac{1}{h} K\left(\frac{x-t}{h}\right)$ for a kernel K and a bandwidth h . For details on the Kernel estimation of the regression function based on continuous observations see Blanke and Bosq [9] and Didi and Louani [10].

In the practical case where we only have access to discrete observations, we apply the trapezoidal rule to approximate the continuous Kernel estimator given by (4). We construct then a new simple estimator of the regression function that we shall call the trapezoidal estimator.

Before introducing the proposed estimator, we begin with defining a sequence of designs which will be used in its construction. This class of designs was considered by Sacks and Ylvisaker [18].

Definition 1 *Let F be a distribution function of some density f satisfying $\inf_{t \in [0,1]} f(t) > 0$ and $\sup_{t \in [0,1]} f(t) < \infty$. The so-called regular sequence of designs generated by a density f is defined by,*

$$T_n = \left\{ t_{i,n} = F^{-1}\left(\frac{i}{n}\right), i = 1, \dots, n. \right\} \text{ for } n \geq 1.$$

Such a sequence of designs verifies the next useful lemma.

Lemma 1 *For $n \geq 1$ let $T_n = (t_{i,n})_{i=1, \dots, n}$ be a regular sequence of designs generated by some density function. Let $x \in]0, 1[$ and $h > 0$ and note by $N_{T_n} \triangleq \text{Card}(T_n \cap [x-h, x+h])$. Suppose that $N_{T_n} \neq 0$ and that $nh \geq 1$. Then,*

$$\sup_{0 \leq j \leq n} (t_{j+1,n} - t_{j,n}) = O\left(\frac{1}{n}\right) \quad \text{and} \quad N_{T_n} = O(nh). \quad (5)$$

We shall now give the definition of the trapezoidal estimator, obtained from a discrete approximation of the continuous estimator $\hat{g}_{[0,1]}$ given by (4).

Definition 2 *The trapezoidal estimator of the regression function g based on the observations $(t_{i,n}, Y_j(t_{i,n}))_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq m}}$, where $T_n = (t_{i,n})_{1 \leq i \leq n}$ is a regular sequence of designs generated by a density function f of support intersecting $[x-h, x+h]$ is given, for any $x \in [0, 1]$, by,*

$$\hat{g}_n^{\text{trap}}(x) = \frac{1}{2n} \sum_{k=1}^{N_{T_n}-1} \left\{ \left(\frac{\varphi_{x,h} \bar{Y}}{f} \right) (t_{x,k}) + \left(\frac{\varphi_{x,h} \bar{Y}}{f} \right) (t_{x,k+1}) \right\}, \quad (6)$$

where $t_{x,1} < \dots < t_{x,N_{T_n}}$ are the points of T_n in $[x-h, x+h]$, $\varphi_{x,h}(t) = \frac{1}{h} K\left(\frac{x-t}{h}\right)$, \bar{Y} is given in (4), K is a kernel of support $[-1, 1]$ and $h = h(n, m)$ is a bandwidth with $0 < h < 1$.

In order to derive our asymptotic results, the following assumptions on the autocovariance function R and the kernel K are required.

2.2 Assumptions

- (A) The autocovariance function R exists and is continuous on the square $[0, 1]^2$.
- (B) At the diagonal (when $t = s$ in the unit square), R has continuous left and right first-order derivatives, that is:

$$R^{(0,1)}(t, t^-) = \lim_{s \uparrow t} \frac{\partial R(t, s)}{\partial s} \quad \text{and} \quad R^{(0,1)}(t, t^+) = \lim_{s \downarrow t} \frac{\partial R(t, s)}{\partial s}.$$

The jump function along the diagonal $\alpha(t) \triangleq R^{(0,1)}(t, t^-) - R^{(0,1)}(t, t^+)$ is assumed to be continuous and not identically equal to zero.

(C) Off the diagonal (when $t \neq s$ in the unit square), R is assumed to have continuous mixed partial derivatives up to order two and,

$$A^{(i,j)} \triangleq \sup_{0 \leq t \neq s \leq 1} |R^{(i,j)}(t, s)| < \infty \text{ for } i, j \text{ such that } 0 \leq i + j \leq 2.$$

(D) The Kernel K is even at least in $C^2([-1, 1])$ and K'' is Lipschitz on $[-1, 1]$.

Examples of processes with autocovariances satisfying Assumptions (A), (B) and (C) are given as follows.

Example 1

1. The Wiener process with autocovariance function $R(s, t) = \sigma^2 \min(s, t)$, has a constant jump function $\alpha(t) = \sigma^2$ and $R^{(i,j)}(s, t) = 0$ for all i, j such that $i + j = 2$ and $s \neq t$.
2. The Ornstein-Uhlenbeck process with a stationary autocovariance $R(s, t) = \sigma^2 \exp(-\lambda|s-t|)$ for $\sigma > 0$ and $\lambda > 0$. For this process $\alpha(t) = 2\sigma^2\lambda$ and $R^{(0,2)}(s, t) = \sigma^2\lambda^2 \exp(-\lambda|s-t|)$.
3. Sacks and Ylvisaker [11] gave another general class of convex stationary autocovariance functions of the form,

$$R(s, t) = \int_0^{1/|t-s|} (1 - \mu|t-s|)p(\mu) d\mu,$$

where p is a probability density and p' its derivative are such that,

$$\lim_{\mu \rightarrow \infty} \mu^3 p(\mu) < \infty, \quad \text{and} \quad \int_a^\infty (\mu p'(\mu) + 3p(\mu))^2 \mu^6 d\mu < \infty,$$

for some finite constant a . For this autocovariance function, $\alpha(t) = 2 \int_0^\infty \mu p(\mu) d\mu$ for all t .

The following kernels satisfy Assumption (D).

Example 2

1. The Quadratic kernel defined by $K(u) = \frac{15}{16}(1 - u^2)^2 \mathbf{1}_{\{|u| \leq 1\}}$.
2. The Triweight kernel defined by $K(u) = \frac{35}{32}(1 - u^2)^3 \mathbf{1}_{\{|u| \leq 1\}}$.

3 Asymptotic results

The following propositions give the asymptotic expressions of the bias and the variance of the trapezoidal estimator as defined by (6).

Proposition 1 *Suppose that Assumption (D) is satisfied. Moreover assume that $f \in C^2([0, 1])$ and f'', g'' are Lipschitz functions on $[0, 1]$. If $\lim_{n \rightarrow \infty} h = 0$ and $\lim_{n \rightarrow \infty} nh = \infty$ then for any $x \in]0, 1[$,*

$$\text{Bias}(\hat{g}_n^{\text{trap}}(x)) = \frac{1}{2}h^2 g''(x)B + o(h^2) + O\left(\frac{1}{n^3 h^3}\right),$$

where $B = \int_{-1}^1 t^2 K(t) dt$.

Proposition 2 *Suppose that Assumptions (A), (B), (C) and (D) are satisfied. Moreover assume that $f \in C^2([0, 1])$ and for any $t \in [0, 1]$, f'' and $R^{(0,2)}(t, \cdot)$ are all Lipschitz on $[0, 1]$. If $\lim_{n \rightarrow \infty} h = 0$ and $\lim_{n \rightarrow \infty} nh = \infty$ then for any $x \in]0, 1[$,*

$$\begin{aligned} \text{Var}(\hat{g}_n^{\text{trap}}(x)) &= \frac{1}{m} \left(R(x, x) - \frac{h}{2} C_K \alpha(x) \right) + \frac{V}{12mn^2h} \frac{\alpha(x)}{f^2(x)} \\ &\quad + o\left(\frac{h}{m}\right) + O\left(\frac{1}{mn^2} + \frac{1}{mn^3h^3}\right), \end{aligned}$$

where $V = \int_{-1}^1 K^2(t) dt$ and $C_K = \int_{-1}^1 \int_{-1}^1 |u - v| K(u) K(v) dudv$.

Propositions 1 and 2 allow to derive the asymptotic expression of the mean squared error (MSE) of the Trapezoidal estimator (6). The integrated mean squared error (IMSE) is then obtained by integrating the MSE with respect to some weight function w . The results are announced, without proof, in the following theorem.

Theorem 1 *If all the assumptions of Propositions 1 and 2 are satisfied then for any $x \in]0, 1[$,*

$$\begin{aligned} \text{MSE}(\hat{g}_n^{\text{trap}}(x)) &= \frac{1}{m} \left(R(x, x) - \frac{h}{2} \alpha(x) C_K \right) + \frac{V}{12mn^2h} \frac{\alpha(x)}{f^2(x)} + \frac{1}{4} h^4 [g''(x)]^2 B^2 \\ &\quad + o\left(h^4 + \frac{h}{m}\right) + O\left(\frac{1}{n^3h} + \frac{1}{mn^3h^3} + \frac{1}{mn^2} + \frac{1}{n^6h^6}\right), \end{aligned}$$

$$\begin{aligned} \text{IMSE}(\hat{g}_n^{\text{trap}}) &= \frac{1}{m} \int_0^1 \left(R(x, x) - \frac{h}{2} \alpha(x) C_K \right) w(x) dx + \frac{V}{12mn^2h} \int_0^1 \frac{\alpha(x)}{f^2(x)} w(x) dx \\ &\quad + \frac{1}{4} h^4 B^2 \int_0^1 [g''(x)]^2 w(x) dx + o\left(h^4 + \frac{h}{m}\right) \\ &\quad + O\left(\frac{1}{n^3h} + \frac{1}{mn^3h^3} + \frac{1}{mn^2} + \frac{1}{n^6h^6}\right), \end{aligned} \tag{7}$$

where w is a continuous density function, V , B and C_K are given in Propositions 1, 2.

The previous Theorem shows, the efficiency of the Trapezoidal estimator, since the IMSE tends to 0 when $m \rightarrow \infty$, $h \rightarrow 0$ and $nh \rightarrow \infty$ as $n \rightarrow \infty$.

The asymptotic optimal bandwidth is obtained by minimizing the asymptotic IMSE as given by the following proposition.

Proposition 3 (Optimal bandwidth) *Suppose that the assumptions of Theorem 1 are satisfied. Moreover assume that $\frac{m}{n} = O(1)$ as $n, m \rightarrow \infty$. Denote by $\text{IMSE}(h)$ the IMSE of the trapezoidal estimator when the bandwidth h is used. Then the bandwidth,*

$$h^* = \left(\frac{C_K \int_0^1 \alpha(x) w(x) dx}{2B^2 \int_0^1 [g''(x)]^2 w(x) dx} \right)^{1/3} m^{-1/3}, \tag{8}$$

is optimal in the sense that,

$$\overline{\lim}_{n, m \rightarrow \infty} \frac{\text{IMSE}(h^*)}{\text{IMSE}(h_{n, m})} \leq 1,$$

for any sequence of bandwidths $h_{n, m}$ verifying:

$$\lim_{n, m \rightarrow \infty} h_{n, m} = 0 \quad \text{and} \quad \overline{\lim}_{n, m \rightarrow \infty} m h_{n, m}^3 < +\infty.$$

where B and C_K are given in Propositions 1 and 2.

We are interested now in finding the optimal design density, i.e, f^* according to the criteria $f^* = \underset{f}{\operatorname{argmin}} \operatorname{IMSE}$, where the minimum is taken with respect to the class of positive densities defined on $[0, 1]$. In view of Theorem 1, the asymptotic optimal design density verifies,

$$f^* = \underset{f>0, \int_0^1 f(x)dx=1}{\operatorname{argmin}} \int_0^1 \frac{\alpha(x)}{f^2(x)} w(x) dx.$$

This optimization problem is solved in the following corollary.

Corollary 1 (Optimal design) *Suppose that the assumptions of Theorem 1 are satisfied. If $\lim_{n \rightarrow \infty} nh^2 = \infty$ and $\lim_{n, m \rightarrow \infty} \frac{n}{m} = \infty$, then the optimal sampling density with respect to the asymptotic IMSE is given by,*

$$f^*(t) = \frac{\{\alpha(t)w(t)\}^{1/3}}{\int_0^1 \{\alpha(s)w(s)\}^{1/3} ds} 1_{[0,1]}(t). \quad (9)$$

Let \hat{g}_{n, f^*}^{trap} be the Trapezoidal estimator (6) with $f = f^*$ defined by (9). We have,

$$\begin{aligned} \operatorname{IMSE}(\hat{g}_{n, f^*}^{trap}) &= \frac{1}{m} \int_0^1 \left(R(x, x) - \frac{1}{2} \alpha(x) C_K h \right) w(x) dx \\ &\quad + \frac{V}{12mn^2h} \left(\int_0^1 (\alpha(x)w(x))^{1/3} dx \right)^3 + \frac{1}{4} h^4 B^2 \int_0^1 [g''(x)]^2 w(x) dx \\ &\quad + o\left(h^4 + \frac{h}{m}\right) + O\left(\frac{1}{n^3h} + \frac{1}{mn^3h^3} + \frac{1}{mn^2} + \frac{1}{n^6h^6}\right), \end{aligned}$$

Remark 1 *Let $\hat{g}_{n, unif}^{trap}$ be the Trapezoidal estimator (6) with a uniform density, i.e, $f = f_{unif}$. The asymptotic IMSE of $\hat{g}_{n, unif}^{trap}$ is given by,*

$$\begin{aligned} \operatorname{IMSE}(\hat{g}_{n, unif}^{trap}) &= \frac{1}{m} \int_0^1 \left(R(x, x) - \frac{1}{2} \alpha(x) C_K h \right) w(x) dx + \frac{V}{12mn^2h} \int_0^1 \alpha(x) w(x) dx \\ &\quad + \frac{1}{4} h^4 B^2 \int_0^1 [g''(x)]^2 w(x) dx + o\left(h^4 + \frac{h}{m}\right) \\ &\quad + O\left(\frac{1}{n^3h} + \frac{1}{mn^3h^3} + \frac{1}{mn^2} + \frac{1}{n^6h^6}\right), \end{aligned}$$

The reduction of the residual IMSE, $\overline{\operatorname{IMSE}} \triangleq \operatorname{IMSE} - \sigma_{x,h}^2/m$, by using the asymptotic optimal design over the uniform design is then,

$$r\operatorname{IMSE} = \frac{\overline{\operatorname{IMSE}}(\hat{g}_{n, unif}^{trap}) - \overline{\operatorname{IMSE}}(\hat{g}_{n, f^*}^{trap})}{\overline{\operatorname{IMSE}}(\hat{g}_{n, unif}^{trap})} \sim 1 - \frac{\left(\int_0^1 (\alpha(x)w(x))^{1/3} dx\right)^3}{\int_0^1 \alpha(x)w(x) dx}.$$

For instance, if $R(s, t) = st \min(s, t)$ then $\alpha(t) = t^2$. Taking $w \equiv 1$ gives $r\operatorname{IMSE} \sim 35\%$.

Finally, the next theorem gives the asymptotic normality of the Trapezoidal estimator (6).

Theorem 2 (Asymptotic normality) *Suppose that the assumptions of Theorem 1 are satisfied. If $\lim_{m \rightarrow \infty} \sqrt{mh^2} = 0$ and $\lim_{n \rightarrow \infty} nh^2 = \infty$ then for any $x \in]0, 1[$,*

$$\sqrt{m} \left(\hat{g}_n^{trap}(x) - g(x) \right) \xrightarrow{\mathcal{D}} Z, \quad \text{with } Z \sim \mathcal{N}(0, R(x, x)),$$

where \mathcal{D} denotes the convergence in distribution and \mathcal{N} is the normal distribution.

4 Simulation study

In this section, we investigate the performance of our estimator (6) in a finite sample set. We shall use the cubic growth curve, used by Benhenni and Rachdi [8] and Hart and Wherly [6],

$$g(x) = 10x^3 - 15x^4 + 6x^5 \quad \text{for } 0 < x < 1. \quad (10)$$

This function was mainly used due to its similarity to the logistic function which is frequently found in growth curve analysis. The sampling points are taken to be:

$$t_i = (i - 0.5)/n \quad \text{for } i = 1, \dots, n. \quad (11)$$

The error process ε is taken to be the Wiener error process with autocovariance function $R(s, t) = \sigma^2 \min(s, t)$. The Kernel used here is the quadratic kernel given by $K(u) = (15/16)(1 - u^2)^2 I_{[-1,1]}(u)$. The bandwidth used in this study is the optimal bandwidth with respect to the exact IMSE.

We consider the mean of all estimators obtained from 100 simulations. We take $\sigma^2 = 0.5$ simulations for other values of σ^2 gave similar results. The results are given in Figure 1 for a fixed number of observations $n = 100$ and three different values of experimental units $m = 5, 20, 100$.

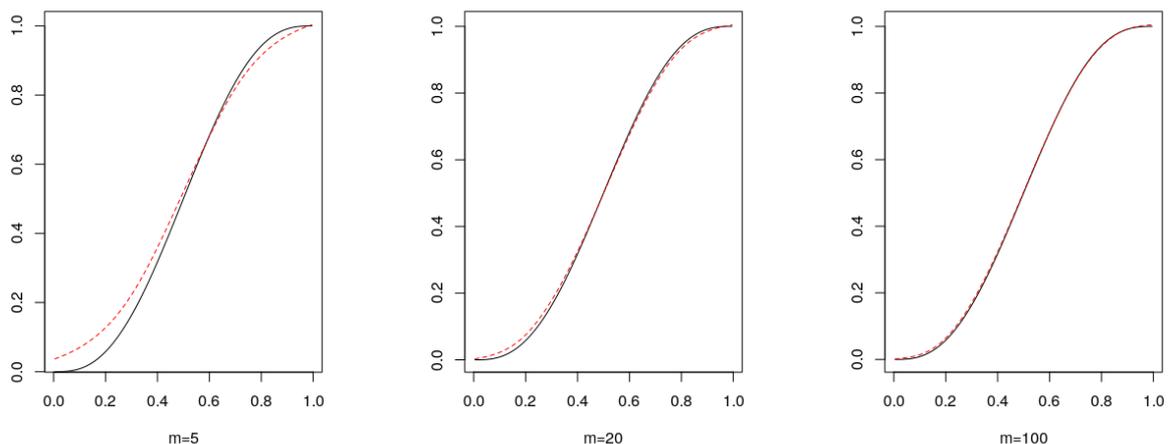


Figure 1: Cubic regression function is in plain line and the trapezoidal estimator is in dashed one.

It is clear that, the performance of the trapezoidal estimator gets better as m increases.

Our aim now is to compare the trapezoidal estimator to that of Gasser and Müller [3] (referred by GM estimator), given for any $x \in]0, 1[$ by,

$$\hat{g}_n^{GM}(x) = \sum_{i=1}^n \int_{m_{i-1}}^{m_i} \varphi_{x,h}(t) dt \bar{Y}(t_i), \quad (12)$$

where $m_0 = 0$, $m_n = 1$ and $m_i = (t_i + t_{i+1})/2$ for $i = 2, \dots, n - 1$, $\varphi_{x,h}(t) = (1/h)K((x - t)/h)$ and $\bar{Y}(t_i) = (1/m) \sum_{j=1}^m Y_j(t_i)$.

This comparison is conducted with respect to the non-asymptotic IMSE and under different types of correlation errors. We consider again the cubic regression function,

the design given by (11) and the quadratic kernel. The two error processes considered here are the stationary Ornstein-Uhlenbeck process with $R(s, t) = \exp(-\lambda|s - t|)$, and the nonstationary Wiener process with $R(s, t) = \sigma^2 \min(s, t)$. We investigate various "amount" of correlation by taking different values of both σ^2 and λ .

We take the weight density w to be the uniform on $[0, 1]$, and we compare the optimal non-asymptotic IMSE of the two estimators, i.e., $\inf_{0 < h < 1} \text{IMSE}(h)$. The bandwidth h is chosen over a grid from 0.09 to 0.5. The results are given in Tables 1-6 for $n = 30$ and for different values of m . The tables present the integrated bias squared denoted by $Ibias^2$, integrated variance denoted by $Ivar$ and the IMSE together with the optimal bandwidth associated to the smallest non-asymptotic IMSE for each estimator.

It can be seen that the optimal bandwidth is the same for both estimators, in addition, as expected, it decreases as m increases.

Consider first the case of strong correlated errors, i.e, for a large σ^2 and a small λ . In Table 1, for the Wiener process with $\sigma^2 = 1$, it appears that the G-M estimator has a slightly smaller $Ibias^2$ while the trapezoidal estimator has a slightly smaller $Ivar$ and since the $Ibias^2$ is too small compared to the $Ivar$ then the trapezoidal estimator has a slightly smaller IMSE. For the Ornstein-Uhlenbeck with $\lambda = 1$ (c.f. Table 2) it can be seen that the trapezoidal estimator has a slightly better IMSE due to a better $Ibias^2$ and a better $Ivar$.

Consider now the case of moderate correlated errors. In Table 3 (for the Wiener process with $\sigma^2 = 0.5$) it seems that the G-M estimator has a slightly smaller $Ibias^2$ while the trapezoidal estimator has a slightly smaller $Ivar$ and smaller IMSE. While for the Ornstein-Uhlenbeck process with $\lambda = 25$, presented in table 4, the G-M estimator has slightly better IMSE due to a better $Ibias^2$ and better $Ivar$.

Finally, consider the weakly correlated errors, i.e, for a small value of σ^2 and a large value of λ . In table 5, for the Wiener process with $\sigma^2 = 0.06$. it appears that the G-M estimator has a slightly smaller $Ibias^2$ while the trapezoidal estimator has a smaller $Ivar$ and smaller IMSE. However, for the Ornstein-Uhlenbeck process with $\lambda = 50$ (c.f. Table 6) the trapezoidal estimator has a slightly smaller $Ibias^2$ while the G-M estimator has a slightly smaller $Ivar$ and IMSE.

Overall, the two estimators, i.e, the trapezoidal estimator and the Gasser and Müller estimator, have "approximately" the same performance. Hence, the proposed estimator, which has a simpler expression, is as efficient as the classical Gasser and Müller estimator.

Another aspect we looked into in this simulation study was the use of the asymptotic optimal design in a finite sample set. We consider the autocovariance function $R(s, t) = st \min(s, t)$ for which $\alpha(t) = t^2$. We compare, for $m = 5, 30$ and $h = 0.123$ for instance, the non-asymptotic IMSE (taking $w \equiv 1$) of the Trapezoidal estimator (6), using both the uniform design (11), i.e., $f \equiv 1$ and the optimal design generated by f^* given in (9), i.e.,

$$f^*(t) = \frac{5}{3} t^{2/3} 1_{[0,1]}(t) \quad \text{and} \quad t_i^* = \left(\frac{i}{n}\right)^{2/3}.$$

The results are given in Tables 7 and 8, where the reduction in the IMSE by taking the optimal design instead of the uniform design is given by,

$$rIMSE = \frac{\text{IMSE}(\hat{g}_{n,unif}^{trap}) - \text{IMSE}(\hat{g}_{n,f^*}^{trap})}{\text{IMSE}(\hat{g}_{n,unif}^{trap})}.$$

It can be seen in Tables 7 and 8 that there exists a significant reduction of the IMSE of the Trapezoidal estimator when using the optimal design, for a small values of the sampling

size n . and this reduction gets smaller when n gets bigger due to the convergence of the IMSE to $\sigma_{x,h}^2/m$. When m gets very large, simulations yielded reductions up to 30%.

In all the previous cases, it appears that $Ibias^2$ is always smaller than $Ivar$. It should be noted here that, both of the estimators have boundary problems. A modified kernel at the edges, as suggested by Hart and Wherly [6], was used in this simulation.

Table 1: The integrated squared bias, Integrated variance, IMSE and the optimal bandwidth in terms of m under the Wiener error process with $\sigma^2 = 1$, for the GM and the trapezoidal estimator.

$n = 20$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
<i>GM</i>	5	2.8832×10^{-3}	8.4967×10^{-2}	8.7850×10^{-2}	0.411
<i>Trap</i>		2.8833×10^{-3}	8.4959×10^{-2}	8.7843×10^{-2}	0.411
<i>GM</i>	15	1.04816×10^{-3}	2.9293×10^{-2}	3.0341×10^{-2}	0.322
<i>Trap</i>		1.04856×10^{-3}	2.9276×10^{-2}	3.0325×10^{-2}	0.322
<i>GM</i>	30	2.7691×10^{-4}	1.5169×10^{-2}	1.5446×10^{-2}	0.233
<i>Trap</i>		2.8535×10^{-4}	1.5124×10^{-2}	1.5409×10^{-2}	0.233

Table 2: The integrated squared bias, Integrated variance, IMSE and the optimal bandwidth in terms of m under the Ornstein-Uhlenbeck error process with $\lambda = 1$ for the GM and the trapezoidal estimator.

$n = 20$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
<i>GM</i>	5	4.57002×10^{-3}	1.70570×10^{-1}	1.75140×10^{-1}	0.46
<i>Trap</i>		4.57001×10^{-3}	1.70565×10^{-1}	1.75135×10^{-1}	0.46
<i>GM</i>	15	1.31050×10^{-3}	5.8884×10^{-2}	6.0194×10^{-2}	0.34
<i>Trap</i>		1.30997×10^{-3}	5.8857×10^{-2}	6.0167×10^{-2}	0.34
<i>GM</i>	30	7.7889×10^{-4}	2.9818×10^{-2}	3.0597×10^{-2}	0.30
<i>Trap</i>		7.7828×10^{-4}	2.9791×10^{-2}	3.0569×10^{-2}	0.30

Table 3: The integrated squared bias, Integrated variance, IMSE and the optimal bandwidth in terms of m under the Wiener error process with $\sigma^2 = 0.5$ for the GM and the trapezoidal estimator.

$n = 20$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
<i>GM</i>	5	1.0481×10^{-3}	4.3939×10^{-2}	4.4988×10^{-2}	0.322
<i>Trap</i>		1.0485×10^{-3}	4.3915×10^{-2}	4.4963×10^{-2}	0.322
<i>GM</i>	15	2.7691×10^{-4}	1.5169×10^{-2}	1.5446×10^{-2}	0.233
<i>Trap</i>		2.8535×10^{-4}	1.5124×10^{-2}	1.5409×10^{-2}	0.233
<i>GM</i>	30	1.1792×10^{-4}	7.7228×10^{-3}	7.8407×10^{-3}	0.188
<i>Trap</i>		1.4175×10^{-4}	7.6733×10^{-3}	7.8150×10^{-3}	0.188

Table 4: The integrated squared bias, Integrated variance, IMSE and the optimal bandwidth in terms of m under the Ornstein-Uhlenbeck error process with $\lambda = 25$ for the GM and the trapezoidal estimator.

$n = 20$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
<i>GM</i>	5	4.3931×10^{-3}	2.7163×10^{-2}	3.1556×10^{-2}	0.455
<i>Trap</i>		4.3930×10^{-3}	2.7165×10^{-2}	3.1558×10^{-2}	0.455
<i>GM</i>	15	1.7942×10^{-3}	1.2819×10^{-2}	1.4613×10^{-2}	0.366
<i>Trap</i>		1.7935×10^{-3}	1.2824×10^{-2}	1.4618×10^{-2}	0.366
<i>GM</i>	30	1.0481×10^{-3}	7.0808×10^{-3}	8.1290×10^{-3}	0.322
<i>Trap</i>		1.0485×10^{-3}	7.0855×10^{-3}	8.1341×10^{-3}	0.322

Table 5: The integrated squared bias, Integrated variance, IMSE and the optimal bandwidth in terms of m under the Wiener error process with $\sigma^2 = 0.06$ for the GM and the trapezoidal estimator.

$n = 20$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
<i>GM</i>	5	9.9714×10^{-5}	5.5781×10^{-3}	5.6778×10^{-3}	0.181
<i>Trap</i>		1.2841×10^{-4}	5.5373×10^{-3}	5.6657×10^{-3}	0.181
<i>GM</i>	15	9.9714×10^{-5}	4.6484×10^{-3}	4.7481×10^{-3}	0.181
<i>Trap</i>		1.2841×10^{-4}	4.6145×10^{-3}	4.7429×10^{-3}	0.181
<i>GM</i>	30	9.9714×10^{-4}	3.9844×10^{-3}	4.0841×10^{-3}	0.181
<i>Trap</i>		1.2841×10^{-4}	3.9552×10^{-3}	4.0836×10^{-3}	0.181

Table 6: The integrated squared bias, Integrated variance, IMSE and the optimal bandwidth in terms of m under the Ornstein-Uhlenbeck error process with $\lambda = 50$ for the GM and the trapezoidal estimator.

$n = 20$	m	$Ibias^2$	$Ivar$	IMSE	h_{opt}
<i>GM</i>	5	4.3496×10^{-3}	1.9905×10^{-2}	2.4255×10^{-2}	0.454
<i>Trap</i>		4.3494×10^{-3}	1.9907×10^{-2}	2.4257×10^{-2}	0.454
<i>GM</i>	15	2.8194×10^{-3}	1.8049×10^{-2}	2.0868×10^{-2}	0.408
<i>Trap</i>		2.8192×10^{-3}	1.8053×10^{-2}	2.0872×10^{-2}	0.408
<i>GM</i>	30	2.8194×10^{-3}	1.5470×10^{-2}	1.8290×10^{-2}	0.408
<i>Trap</i>		2.8192×10^{-3}	1.5474×10^{-2}	1.8293×10^{-2}	0.408

Table 7: The IMSE and the reduction rIMSE of \hat{g}_n^{trap} using the uniform design or optimal design when $R(s, t) = st \min(s, t)$ and $m = 5$.

n	$Trap_{unif}$	$Trap_{opt}$	rIMSE
5	0.363	0.313	13.72%
10	0.170	0.157	7.55%
15	0.133	0.129	3.74%
20	0.124	0.120	2.79%

Table 8: The IMSE and the reduction rIMSE of \hat{g}_n^{trap} using the uniform design or optimal design when $R(s, t) = st \min(s, t)$ and $m = 30$.

n	$Trap_{unif}$	$Trap_{opt}$	rIMSE
5	0.345	0.292	15.30%
10	0.143	0.125	12.41%
15	0.099	0.092	7.12%
20	0.086	0.082	5.38%

5 Proofs

5.1 Proof of Lemma 1.

For the sake of clarity, we omit the n in $t_{i,n}$. For $i = 1, \dots, n-1$ the Mean Value Theorem (m.v.t) yields that there exists $\eta_i \in]t_i, t_{i+1}[$ such that,

$$t_{i+1} - t_i = F^{-1}\left(\frac{i+1}{n}\right) - F^{-1}\left(\frac{i}{n}\right) = \frac{1}{nf(\eta_i)}.$$

Since $\inf_{0 \leq t \leq 1} f(t) > 0$ then $t_{i+1} - t_i = O(\frac{1}{n})$. We shall now prove the second part of the Lemma. Since $T_n \cap [x-h, x+h] \neq \emptyset$, there exist i_1, i_N indexes in $\{1, \dots, n\}$ such that,

$$N_{T_n} \leq i_N - i_1 + 1.$$

From the definition of the regular sequence we have for all $i = 1, \dots, n$,

$$t_i = F^{-1}\left(\frac{i}{n}\right) \quad \text{thus} \quad i = nF(t_i).$$

Using this and the m.v.t we obtain for some $\epsilon_x \in]t_{i_1}, t_{i_N}[$,

$$N_{T_n} \leq n(F(t_{i_N}) - F(t_{i_1})) + 1 = n(t_{i_N} - t_{i_1})f(\epsilon_x) + 1,$$

The boundedness of f and the fact that $t_{i_N} - t_{i_1} \leq 2h$ yield,

$$N_{T_n} \leq (2 \sup_{0 \leq t \leq 1} f(t)) nh + 1.$$

This concludes the proof of the second part of Lemma 1 since $1 \leq nh$. \square

5.2 Proof of Proposition 1.

For h small enough and since $T_n \cap [x - h, x + h] \neq \emptyset$ we take $t_{x,1} < t_{x,2} < \dots < t_{x,N_{T_n}}$ the points of T_n in $[x - h, x + h]$. Since $\mathbb{E}(\bar{Y}(t_i)) = g(t_i)$ for all $i = 1, \dots, n$ we have,

$$\mathbb{E}(\hat{g}_n^{\text{trap}}(x)) = \frac{1}{2n} \left\{ \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)(t_{x,k}) + \left(\frac{\varphi_{x,h}}{f} g \right)(t_{x,k+1}) \right\}.$$

From the definition of the regular sequence of designs we have for $k = 1, \dots, N_{T_n} - 1$,

$$F(t_{x,k+1}) - F(t_{x,k}) = \frac{1}{n} \iff \int_{t_{x,k}}^{t_{x,k+1}} f(t) dt = \frac{1}{n}. \quad (13)$$

Thus,

$$\mathbb{E}(\hat{g}_n^{\text{trap}}(x)) = \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} \left\{ \left(\frac{\varphi_{x,h}}{f} g \right)(t_{x,k}) + \left(\frac{\varphi_{x,h}}{f} g \right)(t_{x,k+1}) \right\} f(t) dt.$$

Let,

$$\begin{aligned} I_h(x) &= \int_{x-h}^{x+h} \varphi_{x,h}(t) g(t) dt \\ &= \sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} \varphi_{x,h}(t) g(t) dt + \int_{x-h}^{t_{x,1}} \varphi_{x,h}(t) g(t) dt + \int_{t_{x,N_{T_n}}}^{x+h} \varphi_{x,h}(t) g(t) dt, \end{aligned}$$

and write,

$$\mathbb{E}(\hat{g}_n^{\text{trap}}(x)) = \mathbb{E}(\hat{g}_n^{\text{trap}}(x)) - I_h(x) + I_h(x) \stackrel{\Delta}{=} \Delta_{x,h} + I_h(x). \quad (14)$$

We first control $\Delta_{x,h}$. Let,

$$\Delta_{x,h} = \Delta_{x,h}^1 + \Delta_{x,h}^2, \quad (15)$$

where,

$$\begin{aligned} \Delta_{x,h}^1 &= \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} \left(\left(\frac{\varphi_{x,h}}{f} g \right)(t_{x,k}) f(t) - \varphi_{x,h}(t) g(t) \right) dt \\ &\quad - \frac{1}{2} \int_{x-h}^{t_{x,1}} \varphi_{x,h}(t) g(t) dt - \frac{1}{2} \int_{t_{x,N_{T_n}}}^{x+h} \varphi_{x,h}(t) g(t) dt. \\ \Delta_{x,h}^2 &= \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} \left(\left(\frac{\varphi_{x,h}}{f} g \right)(t_{x,k+1}) f(t) - \varphi_{x,h}(t) g(t) \right) dt \\ &\quad - \frac{1}{2} \int_{x-h}^{t_{x,1}} \varphi_{x,h}(t) g(t) dt - \frac{1}{2} \int_{t_{x,N_{T_n}}}^{x+h} \varphi_{x,h}(t) g(t) dt. \end{aligned}$$

For $t \in [x - h, t_{x,1}]$, Taylor expansion of $\varphi_{x,h}$ around $(x - h)$ yields,

$$\varphi_{x,h}(t) = \varphi_{x,h}(x - h) + (t - (x - h)) \varphi'_{x,h}(x - h) + \frac{1}{2} (t - (x - h))^2 \varphi''_{x,h}(\theta_{x,h}), \quad (16)$$

for some $\theta_{x,h} \in]x - h, t_{x,1}[$. Recall that by definition of $\varphi_{x,h}$ we have,

$$\sup_{0 \leq t \leq 1} |\varphi_{x,h}^{(j)}(t)| \leq \frac{c_j}{h^{j+1}} \quad \text{for } j = 0, 1, 2, \quad (17)$$

for some appropriate constants c_j where $j = 0, 1, 2$. In addition, since $\varphi_{x,h}$ is in C^2 and of support $[x - h, x + h]$ then,

$$\varphi_{x,h}(x - h) = \varphi_{x,h}(x + h) = \varphi'_{x,h}(x - h) = \varphi'_{x,h}(x + h) = 0. \quad (18)$$

Using (18) and (17) in (16) and using Lemma (1) we obtain for $t \in [x - h, t_{x,1}]$,

$$\varphi_{x,h}(t) = \frac{1}{2}(t - (x - h))^2 \varphi''_{x,h}(\theta_{x,h}) = O\left(\frac{1}{n^2 h^3}\right), \quad (19)$$

Likewise, for $t \in [t_{x,N_{T_n}}, x + h]$ we have,

$$\varphi_{x,h}(t) = \frac{1}{2}(t - (x + h))^2 \varphi''_{x,h}(\theta'_{x,h}) = O\left(\frac{1}{n^2 h^3}\right), \quad (20)$$

where $\theta'_{x,h} \in]t_{x,N_{T_n}}, x + h[$. Hence,

$$\int_{x-h}^{t_{x,1}} \varphi_{x,h}(t)g(t) dt = O\left(\frac{1}{n^3 h^3}\right) \quad \text{and} \quad \int_{t_{x,N_{T_n}}}^{x+h} \varphi_{x,h}(t)g(t) dt = O\left(\frac{1}{n^3 h^3}\right).$$

Thus,

$$\Delta_{x,h}^1 = \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} \left(\left(\frac{\varphi_{x,h}}{f} g \right) (t_{x,k}) - \left(\frac{\varphi_{x,h}}{f} g \right) (t) \right) f(t) dt + O\left(\frac{1}{n^3 h^3}\right),$$

and,

$$\Delta_{x,h}^2 = \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} \left(\left(\frac{\varphi_{x,h}}{f} g \right) (t_{x,k+1}) - \left(\frac{\varphi_{x,h}}{f} g \right) (t) \right) f(t) dt + O\left(\frac{1}{n^3 h^3}\right).$$

Recall that $\varphi_{x,h}$ is in C^2 and $f, g \in C^2([0, 1])$, then for any $t \in]t_{x,k}, t_{x,k+1}[$ Taylor expansions of $\frac{\varphi_{x,h}}{f}g$ and f around $t_{x,k}$ give,

$$\begin{aligned} \Delta_{x,h}^1 &= \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)' (t_{x,k}) f(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t_{x,k} - t) dt \\ &\quad - \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)' (t_{x,k}) f'(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^2 dt \\ &\quad - \frac{1}{4} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)' (t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^3 f''(\eta_{x,k}) dt \\ &\quad - \frac{1}{4} \sum_{k=1}^{N_{T_n}-1} f(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^2 \left(\frac{\varphi_{x,h}}{f} g \right)'' (\theta_{x,k}) dt \\ &\quad - \frac{1}{4} \sum_{k=1}^{N_{T_n}-1} f'(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^3 \left(\frac{\varphi_{x,h}}{f} g \right)'' (\theta_{x,k}) dt \\ &\quad - \frac{1}{8} \sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^4 \left(\frac{\varphi_{x,h}}{f} g \right)'' (\theta_{x,k}) f''(\eta_{x,k}) dt + O\left(\frac{1}{n^3 h^3}\right), \end{aligned}$$

where $\theta_{x,k}$ and $\eta_{x,k}$ are in $]t_{x,k}, t[$. Recall that the functions $g^{(j)}, f^{(j)}$ for $j = 0, 1, 2$ are all bounded, then using (17) and Lemma 1 we get,

$$\sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^3 f''(\eta_{x,k}) dt = O\left(\frac{1}{n^3 h}\right). \quad (21)$$

$$\sum_{k=1}^{N_{T_n}-1} f'(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^3 \left(\frac{\varphi_{x,h}}{f} g \right)''(\theta_{x,k}) dt = O\left(\frac{1}{n^3 h^2}\right). \quad (22)$$

$$\sum_{k=1}^{N_{T_n}-1} \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^4 \left(\frac{\varphi_{x,h}}{f} g \right)''(\theta_{x,k}) f''(\eta_{x,k}) dt = O\left(\frac{1}{n^4 h^2}\right). \quad (23)$$

Note that, since $\varphi_{x,h}''$, g'' and f'' are all lipschitz then,

$$\begin{aligned} \left(\frac{\varphi_{x,h}}{f} g \right)''(\theta_{x,k}) &= \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) + \left[\left(\frac{\varphi_{x,h}}{f} g \right)''(\theta_{x,k}) - \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) \right] \\ &= \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) + O\left(\frac{1}{nh^4}\right). \end{aligned} \quad (24)$$

Injecting (21), (22), (23) and (24) in $\Delta_{x,h}^1$ we have,

$$\begin{aligned} \Delta_{x,h}^1 &= \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) f(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t_{x,k} - t) dt \\ &\quad - \frac{1}{2} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) f'(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^2 dt \\ &\quad - \frac{1}{4} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) f(t_{x,k}) \int_{t_{x,k}}^{t_{x,k+1}} (t - t_{x,k})^2 dt + O\left(\frac{1}{n^3 h^3}\right). \end{aligned}$$

Let $d_{x,k} = t_{x,k+1} - t_{x,k}$. We obtain by basic integration,

$$\begin{aligned} \Delta_{x,h}^1 &= -\frac{1}{4} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) f(t_{x,k}) d_{x,k}^2 - \frac{1}{6} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) f'(t_{x,k}) d_{x,k}^3 \\ &\quad - \frac{1}{12} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) f(t_{x,k}) d_{x,k}^3 + O\left(\frac{1}{n^3 h^3}\right). \end{aligned} \quad (25)$$

Similarly we verify that,

$$\begin{aligned} \Delta_{x,h}^2 &= \frac{1}{4} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k+1}) f(t_{x,k+1}) d_{x,k}^2 - \frac{1}{6} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k+1}) f'(t_{x,k+1}) d_{x,k}^3 \\ &\quad - \frac{1}{12} \sum_{k=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k+1}) f(t_{x,k+1}) d_{x,k}^3 + O\left(\frac{1}{n^3 h^3}\right), \end{aligned} \quad (26)$$

Summing (25) and (26) gives,

$$\begin{aligned}
\Delta_{x,h} &= \Delta_{x,h}^1 + \Delta_{x,h}^2 \\
&= \frac{1}{4} \sum_{k=1}^{N_{T_n}-1} d_{x,k}^2 \left[\left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k+1}) f(t_{x,k+1}) - \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) f(t_{x,k}) \right] \\
&\quad - \frac{1}{6} \sum_{k=1}^{N_{T_n}-1} d_{x,k}^3 \left[\left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k+1}) f'(t_{x,k+1}) + \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) f'(t_{x,k}) \right] \\
&\quad - \frac{1}{12} \sum_{k=1}^{N_{T_n}-1} d_{x,k}^3 \left[\left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k+1}) f(t_{x,k+1}) + \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) f(t_{x,k}) \right] + O\left(\frac{1}{n^3 h^3}\right).
\end{aligned}$$

Since $\varphi'_{x,h}$ is in C^1 and $g', f' \in C^1([0, 1])$, Taylor expansion of $\left(\frac{\varphi_{x,h}}{f} g\right)' f$ around $t_{x,k}$ yields,

$$\left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)(t_{x,k+1}) = \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)(t_{x,k}) + d_{x,k} \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)'(\nu_{x,k}),$$

where $\nu_{x,k} \in]t_{x,k}, t_{x,k+1}[$. We then have,

$$\begin{aligned}
\Delta_{x,h} &= \frac{1}{4} \sum_{k=1}^{N_{T_n}-1} d_{x,k}^3 \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)'(\nu_{x,k}) \\
&\quad - \frac{1}{6} \sum_{k=1}^{N_{T_n}-1} d_{x,k}^3 \left[\left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k+1}) f'(t_{x,k+1}) + \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) f'(t_{x,k}) \right] \\
&\quad - \frac{1}{12} \sum_{k=1}^{N_{T_n}-1} d_{x,k}^3 \left[\left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k+1}) f(t_{x,k+1}) + \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) f(t_{x,k}) \right] + O\left(\frac{1}{n^3 h^3}\right).
\end{aligned}$$

From the definition of the regular sequence of designs and using the m.v.t. we obtain for $k = 1, \dots, N_{T_n} - 1$,

$$\int_{t_{x,k}}^{t_{x,k+1}} f(t) dt = \frac{1}{n} \iff d_{x,k} = \frac{1}{nf(t_{x,k}^*)} \text{ for some } t_{x,k}^* \in]t_{x,k}, t_{x,k+1}[. \quad (27)$$

This equation yields,

$$\begin{aligned}
\Delta_{x,h} &= \frac{1}{4n^2} \sum_{k=1}^{N_{T_n}-1} d_{x,k} \frac{1}{f^2(t_{x,k}^*)} \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)'(\nu_{x,k}) \\
&\quad - \frac{1}{6n^2} \sum_{k=1}^{N_{T_n}-1} d_{x,k} \left[\left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k+1}) \frac{f'(t_{x,k+1})}{f^2(t_{x,k}^*)} + \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k}) \frac{f'(t_{x,k})}{f^2(t_{x,k}^*)} \right] \\
&\quad - \frac{1}{12n^2} \sum_{k=1}^{N_{T_n}-1} d_{x,k} \left[\left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k+1}) \frac{f(t_{x,k+1})}{f^2(t_{x,k}^*)} + \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k}) \frac{f(t_{x,k})}{f^2(t_{x,k}^*)} \right] \\
&\quad + O\left(\frac{1}{n^3 h^3}\right).
\end{aligned}$$

Using the Riemann integrability of $\varphi_{x,h}^{(j)}$, $f^{(j)}$ and $g^{(j)}$ for $j = 0, 1, 2$ and applying Lemma 2 in the Appendix with $u(t) = \frac{1}{f^2(t)}$ and $v(t) = \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)'(t)$ we obtain,

$$\sum_{k=1}^{N_{T_n}-1} d_{x,k} \frac{1}{f^2(t_{x,k}^*)} \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)'(\nu_{x,k}) = \int_{x-h}^{x+h} \frac{1}{f^2(t)} \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)'(t) dt + O\left(\frac{1}{nh^3}\right).$$

Similarly, taking $u(t) = \left(\frac{\varphi_{x,h}}{f} g \right)'(t)$ and $v(t) = \frac{f'(t)}{f^2(t)}$ in Lemma 2 we obtain,

$$\sum_{k=1}^{N_{T_n}-1} d_{x,k} \left(\frac{\varphi_{x,h}}{f} g \right)'(t_{x,k+1}) \frac{f'(t_{x,k+1})}{f^2(t_{x,k}^*)} = \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)'(t) \frac{f'(t)}{f^2(t)} dt + O\left(\frac{1}{nh^3}\right).$$

Again taking $u(t) = \left(\frac{\varphi_{x,h}}{f} g \right)''(t)$ and $v(t) = \frac{f(t)}{f^2(t)}$ we obtain,

$$\sum_{k=1}^{N_{T_n}-1} d_{x,k} \left(\frac{\varphi_{x,h}}{f} g \right)''(t_{x,k+1}) \frac{f(t_{x,k+1})}{f^2(t_{x,k}^*)} = \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)''(t) \frac{1}{f(t)} dt + O\left(\frac{1}{nh^3}\right).$$

Hence,

$$\begin{aligned} \Delta_{x,h} &= \frac{1}{4n^2} \int_{x-h}^{x+h} \frac{1}{f^2(t)} \left(\left(\frac{\varphi_{x,h}}{f} g \right)' f \right)'(t) dt - \frac{1}{3n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)'(t) \frac{f'(t)}{f^2(t)} dt \\ &\quad - \frac{1}{6n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)''(t) \frac{1}{f(t)} dt + O\left(\frac{1}{n^3 h^3}\right). \end{aligned}$$

Simple derivations yield,

$$\begin{aligned} \Delta_{x,h} &= \frac{1}{4n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)''(t) \frac{1}{f(t)} dt + \frac{1}{4n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)'(t) \frac{f'(t)}{f^2(t)} dt \\ &\quad - \frac{1}{3n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)'(t) \frac{f'(t)}{f^2(t)} dt - \frac{1}{6n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)''(t) \frac{1}{f(t)} dt + O\left(\frac{1}{n^3 h^3}\right) \\ &= \frac{1}{12n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)''(t) \frac{1}{f(t)} dt - \frac{1}{12n^2} \int_{x-h}^{x+h} \left(\frac{\varphi_{x,h}}{f} g \right)'(t) \frac{f'(t)}{f^2(t)} dt + O\left(\frac{1}{n^3 h^3}\right) \\ &= \frac{1}{12n^2} \int_{x-h}^{x+h} \left(\left(\frac{\varphi_{x,h}}{f} g \right)' \frac{1}{f} \right)'(t) dt + O\left(\frac{1}{n^3 h^3}\right). \end{aligned}$$

Finally,

$$\Delta_{x,h} = \frac{1}{12n^2} \left(\left(\frac{\varphi_{x,h}}{f} g \right)' \frac{1}{f} \right)'(x+h) - \left(\left(\frac{\varphi_{x,h}}{f} g \right)' \frac{1}{f} \right)'(x-h) + O\left(\frac{1}{n^3 h^3}\right).$$

The last equation together with (18) yield,

$$\Delta_{x,h} = O\left(\frac{1}{n^3 h^3}\right). \quad (28)$$

The control of $I_h(x)$ is classical and it can be seen from Gasser and Müller (1984) [19] that,

$$I_h(x) = g(x) + \frac{1}{2} h^2 g''(x) \int_{-1}^1 t^2 K(t) dt + o(h^2). \quad (29)$$

Finally, collecting (14), (28) and (29) gives,

$$\text{Bias}(\hat{g}_n^{\text{trap}}(x)) = \frac{1}{2}h^2 g''(x)B + o(h^2) + O\left(\frac{1}{n^3 h^3}\right),$$

where $B = \int_{-1}^1 t^2 K(t) dt$. This concludes the proof of Proposition 1. \square

5.3 Proof of Proposition 2.

The greatest lines of this proof are based on the work of Belouni and Benhenni [12]. For h small enough and since $T_n \cap [x - h, x + h] \neq \emptyset$ we have,

$$0 \leq t_1 < \dots < x - h \leq t_{x,1} < \dots < t_{x,N_{T_n}} \leq x + h < \dots < t_n \leq 1.$$

Let,

$$\Phi(t, s) = \left(\frac{\varphi_{x,h}}{f}\right)(t)R(t, s)\left(\frac{\varphi_{x,h}}{f}\right)(s),$$

and,

$$\sigma_{x,h}^2 = \int_{x-h}^{x+h} \int_{x-h}^{x+h} \varphi_{x,h}(t)R(t, s)\varphi_{x,h}(s) ds dt. \quad (30)$$

On the one hand,

$$\begin{aligned} \text{Var}(\hat{g}_n^{\text{trap}}(x)) &= \frac{1}{4mn^2} \sum_{i=1}^{N_{T_n}-1} \sum_{j=1}^{N_{T_n}-1} \left\{ \Phi(t_{x,i}, t_{x,j}) + \Phi(t_{x,i}, t_{x,j+1}) + \Phi(t_{x,i+1}, t_{x,j}) \right. \\ &\quad \left. + \Phi(t_{x,i+1}, t_{x,j+1}) \right\} \end{aligned}$$

Using (13) one can write,

$$\begin{aligned} \text{Var}(\hat{g}_n^{\text{trap}}(x)) &= \frac{1}{4m} \sum_{i=1}^{N_{T_n}-1} \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} \left\{ \Phi(t_{x,i}, t_{x,j}) + \Phi(t_{x,i}, t_{x,j+1}) \right. \\ &\quad \left. + \Phi(t_{x,i+1}, t_{x,j}) + \Phi(t_{x,i+1}, t_{x,j+1}) \right\} f(s) f(t) ds dt. \end{aligned}$$

On the other hand we have,

$$\begin{aligned} \sigma_{x,h}^2 &= \sum_{i=1}^{N_{T_n}-1} \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} \Phi(t, s) f(t) f(s) ds dt \\ &\quad + 2 \int_{x-h}^{t_{x,1}} \int_{t_{x,N_{T_n}}}^{x+h} \Phi(t, s) f(t) f(s) ds dt + \int_{t_{x,N_{T_n}}}^{x+h} \int_{t_{x,N_{T_n}}}^{x+h} \Phi(t, s) f(t) f(s) ds dt \\ &\quad + \int_{x-h}^{t_{x,1}} \int_{x-h}^{t_{x,1}} \Phi(t, s) f(t) f(s) ds dt + 2 \sum_{j=1}^{N_{T_n}-1} \int_{x-h}^{t_{x,1}} \int_{t_{x,j}}^{t_{x,j+1}} \Phi(t, s) f(t) f(s) ds dt \\ &\quad + 2 \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,N_{T_n}}}^{x+h} \int_{t_{x,j}}^{t_{x,j+1}} \Phi(t, s) f(t) f(s) ds dt. \end{aligned}$$

Recall that Lemma 1 yields $N_{T_n} = O(nh)$ and $\sup_{1 \leq i \leq n} d_{x,i} = O(\frac{1}{n})$. Using (19) and (20) we have,

$$\sup_{(x-h) \leq t \leq t_{x,1}} |\varphi_{x,h}(t)| = O\left(\frac{1}{n^2 h^3}\right) \quad \text{and} \quad \sup_{t_{x,N_{T_n}} \leq t \leq (x+h)} |\varphi_{x,h}(t)| = O\left(\frac{1}{n^2 h^3}\right). \quad (31)$$

Since f and R are bounded, using (17) and (31) we obtain,

$$\begin{aligned} \int_{x-h}^{t_{x,1}} \int_{t_{x,N_{T_n}}}^{x+h} \Phi(t,s) f(t) f(s) ds dt &= O\left(\frac{1}{n^6 h^6}\right), \\ \int_{t_{x,N_{T_n}}}^{x+h} \int_{t_{x,N_{T_n}}}^{x+h} \Phi(t,s) f(t) f(s) ds dt &= O\left(\frac{1}{n^6 h^6}\right), \\ \int_{x-h}^{t_{x,1}} \int_{x-h}^{t_{x,1}} \Phi(t,s) f(t) f(s) ds dt &= O\left(\frac{1}{n^6 h^6}\right), \\ \sum_{j=1}^{N_{T_n}-1} \int_{x-h}^{t_{x,1}} \int_{t_{x,j}}^{t_{x,j+1}} \Phi(t,s) f(t) f(s) ds dt &= O\left(\frac{1}{n^3 h^3}\right), \\ \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,N_{T_n}}}^{x+h} \int_{t_{x,j}}^{t_{x,j+1}} \Phi(t,s) f(t) f(s) ds dt &= O\left(\frac{1}{n^3 h^3}\right). \end{aligned}$$

Thus,

$$\sigma_{x,h}^2 = \sum_{i=1}^{N_{T_n}-1} \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} \Phi(t,s) f(t) f(s) ds dt + O\left(\frac{1}{n^3 h^3}\right).$$

We shall control the residual variance $\text{Var}(\hat{g}_n^{trap}(x)) - \frac{\sigma_{x,h}^2}{m}$. For this, let,

$$N_{i,j}(t,s) = \Phi(t_{x,i}, t_{x,j}) + \Phi(t_{x,i+1}, t_{x,j}) + \Phi(t_{x,i}, t_{x,j+1}) + \Phi(t_{x,i+1}, t_{x,j+1}) - 4\Phi(t,s), \quad (32)$$

and put,

$$I_{i,j} = \frac{1}{4m} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} N_{i,j}(t,s) f(t) f(s) ds dt. \quad (33)$$

The residual variance can then be written as follows,

$$\text{Var}(\hat{g}_n^{trap}(x)) - \frac{\sigma_{x,h}^2}{m} = \sum_{i=1}^{N_{T_n}-1} I_{i,i} + \sum_{i \neq j=1}^{N_{T_n}-1} I_{i,j} + O\left(\frac{1}{mn^3 h^3}\right), \quad (34)$$

Starting with the diagonal terms $I_{i,i}$. Since for any $s, t \in [0, 1]$, we have $N_{i,i}(s, t) = N_{i,i}(t, s)$, then we can write,

$$I_{i,i} = \frac{1}{2m} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,i}}^t N_{i,i}(t,s) f(t) f(s) ds dt. \quad (35)$$

Because of Assumption (B), $N_{i,i}$ has left and right first order derivatives on the diagonal on $[0, 1]^2$. For any s, t such that $(t_{x,i} < s \leq t < t_{x,i+1})$, Taylor expansion of Φ around

$(t_{x,i}, t_{x,i})$ gives,

$$\begin{aligned}\Phi(t, s) &= \Phi(t, t_{x,i}) + (s - t_{x,i})\Phi^{(0,1)}(t, t_{x,i}) + \frac{1}{2}(s - t_{x,i})^2\Phi^{(0,2)}(t, \eta_{s,i}^{(1)}) \\ &= \Phi(t_{x,i}, t_{x,i}) + (t - t_{x,i})\Phi^{(1,0)}(\epsilon_{t,i}^{(1)}, t_{x,i}) + (s - t_{x,i})\Phi^{(0,1)}(\epsilon_i, t_{x,i}) \\ &\quad + (s - t_{x,i})(t - \epsilon_i)\Phi^{(1,1)}(\epsilon_{t,i}^{(2)}, t_{x,i}) + \frac{1}{2}(s - t_{x,i})^2\Phi^{(0,2)}(t, \eta_{s,i}^{(1)}),\end{aligned}$$

for some $\epsilon_i \in]t_{x,i}, t_{x,i+1}[$, some $\epsilon_{t,i}^{(1)}$ in $]t_{x,i}, t[$, some $\epsilon_{t,i}^{(2)}$ between t and ϵ_i and some $\eta_{t,i}^{(1)}$ in $]t_{x,i}, s[$. We have,

$$\begin{aligned}\Phi(t, s) &= \Phi(t_{x,i}, t_{x,i}) + (t - t_{x,i})\Phi^{(1,0)}(\epsilon_i, t_{x,i}) + (s - t_{x,i})\Phi^{(0,1)}(\epsilon_i, t_{x,i}) \\ &\quad + (t - t_{x,i})\left(\Phi^{(1,0)}(\epsilon_{t,i}^{(1)}, t_{x,i}) - \Phi^{(1,0)}(\epsilon_i, t_{x,i})\right) \\ &\quad + (s - t_{x,i})(t - \epsilon_i)\Phi^{(1,1)}(\epsilon_{t,i}^{(2)}, t_{x,i}) + \frac{1}{2}(s - t_{x,i})^2\Phi^{(0,2)}(t, \eta_{s,i}^{(1)}).\end{aligned}$$

For l and l' integers such that $l + l' \leq 2$, Assumption (C) yields,

$$\sup_{s \neq t} |\Phi^{(l,l')}(t, s)| = O\left(\frac{1}{h^{l+l'+2}}\right). \quad (36)$$

In addition, since $\varphi_{x,h}, \varphi'_{x,h}, \frac{1}{f}, R$ and $R(\cdot, t_{x,i})$ are all continuous on $]t_{x,i}, t_{x,i+1}[$, then for $s \neq t$ in $]t_{x,i}, t_{x,i+1}[$ we have,

$$\begin{aligned}\left|\Phi^{(1,0)}(s, t_{x,i}) - \Phi^{(1,0)}(t, t_{x,i})\right| &= \left|\frac{\varphi_{x,h}}{f}(t_{x,i})\right| \left|R(s, t_{x,i})\left(\frac{\varphi'_{x,h}}{f}(s) - \frac{\varphi'_{x,h}}{f}(t)\right)\right. \\ &\quad \left.+ R^{(1,0)}(s, t_{x,i})\left(\frac{\varphi_{x,h}}{f}(s) - \frac{\varphi_{x,h}}{f}(t)\right) + \frac{\varphi_{x,h}}{f}(t)\left(R^{(1,0)}(s, t_{x,i}) - R^{(1,0)}(t, t_{x,i})\right)\right. \\ &\quad \left.+ \frac{\varphi'_{x,h}}{f}(t)\left(R(s, t_{x,i}) - R(t, t_{x,i})\right)\right| = O\left(\frac{1}{nh^4}\right).\end{aligned}$$

Finally, using this equation together with Lemma 1 we obtain,

$$\Phi(t, s) = \Phi(t_{x,i}, t_{x,i}) + (t - t_{x,i})\Phi^{(1,0)}(\epsilon_i, t_{x,i}) + (s - t_{x,i})\Phi^{(0,1)}(\epsilon_i, t_{x,i}) + O\left(\frac{1}{n^2h^4}\right). \quad (37)$$

Similarly we verify that,

$$\Phi(t_{x,i+1}, t_{x,i+1}) = \Phi(t_{x,i}, t_{x,i}) + d_{x,i}\Phi^{(1,0)}(\epsilon_i, t_{x,i}) + d_{x,i}\Phi^{(0,1)}(\epsilon_i, t_{x,i}) + O\left(\frac{1}{n^2h^4}\right), \quad (38)$$

and that,

$$\Phi(t_{x,i+1}, t_{x,i}) = \Phi(t_{x,i}, t_{x,i}) + d_{x,i}\Phi^{(1,0)}(\epsilon_i, t_{x,i}) + O\left(\frac{1}{n^2h^4}\right). \quad (39)$$

Inserting (37), (38) and (39) in (32) for $i = j$ and using (36) and Lemma 1, we obtain,

$$\begin{aligned}N_{i,i}(t, s) &= 3d_{x,i}\Phi^{(1,0)}(\epsilon_i, t_{x,i}) - 4(t - t_{x,i})\Phi^{(1,0)}(\epsilon_i, t_{x,i}) \\ &\quad + d_{x,i}\Phi^{(0,1)}(\epsilon_i, t_{x,i}) - 4(s - t_{x,i})\Phi^{(0,1)}(\epsilon_i, t_{x,i}) + O\left(\frac{1}{n^2h^4}\right).\end{aligned}$$

Replacing this expression in (35), and using the boundedness of f and Lemma 1, we obtain,

$$\begin{aligned}
I_{i,i} &= \frac{1}{2m} \left(d_{x,i} \left(3\Phi^{(1,0)}(\epsilon_i, t_{x,i}) + \Phi^{(0,1)}(\epsilon_i, t_{x,i}) \right) \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,i}}^t f(t) f(s) ds dt \right. \\
&\quad - 4\Phi^{(1,0)}(\epsilon_i, t_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,i}}^t (t - t_{x,i}) f(t) f(s) ds dt \\
&\quad \left. - 4\Phi^{(0,1)}(\epsilon_i, t_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,i}}^t (s - t_{x,i}) f(t) f(s) ds dt \right) + O\left(\frac{1}{mn^4 h^4}\right). \quad (40)
\end{aligned}$$

Recall that f is in $C^2([0, 1])$ and that $d_{x,i} = O(\frac{1}{n})$ from Lemma 1. It can easily be verified that for any integers l and l' :

$$\int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,i}}^t (s - t_{x,i})^{l'} (s - t_{x,i})^l f(t) f(s) ds dt = \frac{f^2(t_{x,i}) d_{x,i}^{(l+l'+2)}}{(l'+1)(l+l'+2)} + O\left(\frac{1}{n^{l+l'+3}}\right).$$

Using this last Equation together with (36) in (40) above, and (27) we obtain,

$$\begin{aligned}
I_{i,i} &= \frac{1}{12m} \left(\Phi^{(1,0)}(\epsilon_i, t_{x,i}) - \Phi^{(0,1)}(\epsilon_i, t_{x,i}) \right) f^2(t_{x,i}) d_{x,i}^3 + O\left(\frac{1}{mn^4 h^4}\right) \\
&= \frac{1}{12mn^2} \left(\Phi^{(1,0)}(\epsilon_i, t_{x,i}) - \Phi^{(0,1)}(\epsilon_i, t_{x,i}) \right) \frac{f^2(t_{x,i})}{f^2(t_{x,i}^*)} d_{x,i} + O\left(\frac{1}{mn^4 h^4}\right).
\end{aligned}$$

Finally using Lemma 1, the integrability of $\varphi_{x,h}, \varphi'_{x,h}, f, f'$ and $R^{(0,1)}(., t)$ and applying Lemma 2 in the Appendix, we obtain,

$$\begin{aligned}
\sum_{i=1}^{N_{T_n}-1} I_{i,i} &= \frac{1}{12mn^2} \sum_{i=1}^{N_{T_n}-1} \left(\Phi^{(1,0)}(\epsilon_i, t_{x,i}) - \Phi^{(0,1)}(\epsilon_i, t_{x,i}) \right) \frac{f^2(t_{x,i})}{f^2(t_{x,i}^*)} d_{x,i} + O\left(\frac{1}{mn^3 h^3}\right) \\
&= \frac{1}{12mn^2} \int_{x-h}^{x+h} \left(\Phi^{(1,0)}(t^+, t) - \Phi^{(0,1)}(t^+, t) \right) dt + O\left(\frac{1}{mn^3 h^3}\right). \quad (41)
\end{aligned}$$

Since $\Phi^{(0,1)}(t^+, t) = \Phi^{(0,1)}(t, t^-) = \Phi^{(1,0)}(t^-, t)$, then,

$$\sum_{i=1}^{N_{T_n}-1} I_{i,i} = -\frac{1}{12mn^2} \int_{x-h}^{x+h} \left(\Phi^{(1,0)}(t^-, t) - \Phi^{(1,0)}(t^+, t) \right) dt + O\left(\frac{1}{mn^3 h^3}\right). \quad (42)$$

Now, it remains to handle the off diagonal term. Assumption (B) yields that $N_{i,j}$ for $i \neq j$ is twice differentiable off the diagonal on $[0, 1]^2$. Taylor expansion of $N_{i,j}$ around $(t_{x,i}, t_{x,j})$ for $i \neq j$ up to order 2 gives,

$$\begin{aligned}
\Phi(t, s) &= \Phi(t_{x,i}, t_{x,j}) + (t - t_{x,i})\Phi^{(1,0)}(t_{x,i}, t_{x,j}) + (s - t_{x,j})\Phi^{(0,1)}(t_{x,i}, t_{x,j}) \\
&\quad + \frac{1}{2}(t - t_{x,i})^2\Phi^{(2,0)}(\epsilon_{x,i}^{(1)}, t_{x,j}) + \frac{1}{2}(s - t_{x,j})^2\Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(1)}) \\
&\quad + (t - t_{x,i})(s - t_{x,j})\Phi^{(1,1)}(\epsilon_{x,i}^{(1)}, \eta_{x,j}^{(1)}), \quad (43)
\end{aligned}$$

for some $\epsilon_{x,i}^{(1)}$ between $t_{x,i}$ and t and some $\eta_{x,j}^{(1)}$ between $t_{x,j}$ and s . Taking $t = t_{x,i+1}$ and $s = t_{x,j}$ in (43), we obtain,

$$\Phi(t_{x,i+1}, t_{x,j}) = \Phi(t_{x,i}, t_{x,j}) + d_{x,i}\Phi^{(1,0)}(t_{x,i}, t_{x,j}) + \frac{1}{2}d_{x,i}^2\Phi^{(2,0)}(\epsilon_{x,i}^{(2)}, t_{x,j}), \quad (44)$$

for some $\epsilon_{x,i}^{(2)}$ in $]t_{x,i}, t_{x,i+1}[$. Taking $t = t_{x,i}$ and $s = t_{x,j+1}$ in (43), we obtain,

$$\Phi(t_{x,i}, t_{x,j+1}) = \Phi(t_{x,i}, t_{x,j}) + d_{x,j}\Phi^{(0,1)}(t_{x,i}, t_{x,j}) + \frac{1}{2}d_{x,j}^2\Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(2)}), \quad (45)$$

for some $\eta_{x,j}^{(2)}$ in $]t_{x,j}, t_{x,j+1}[$. Taking $t = t_{x,i+1}$ and $s = t_{x,j+1}$ in (43), we obtain,

$$\begin{aligned} \Phi(t_{x,i+1}, t_{x,j+1}) &= \Phi(t_{x,i}, t_{x,j}) + d_{x,i}\Phi^{(1,0)}(t_{x,i}, t_{x,j}) + d_{x,j}\Phi^{(0,1)}(t_{x,i}, t_{x,j}) \\ &\quad + \frac{1}{2}d_{x,i}^2\Phi^{(2,0)}(\epsilon_{x,i}^{(3)}, t_{x,j}) + \frac{1}{2}d_{x,j}^2\Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(3)}) \\ &\quad + d_{x,i}d_{x,j}\Phi^{(1,1)}(\epsilon_{x,i}^{(3)}, \eta_{x,j}^{(3)}), \end{aligned} \quad (46)$$

We obtain by inserting (43), (44), (45) and (46) in (32),

$$\begin{aligned} N_{i,j}(t, s) &= \Phi^{(1,0)}(t_{x,i}, t_{x,j})(2d_{x,i} - 4(t - t_{x,i})) + \Phi^{(0,1)}(t_{x,i}, t_{x,j})(2d_{x,j} - 4(s - t_{x,j})) \\ &\quad + \frac{1}{2}d_{x,i}^2(\Phi^{(2,0)}(\epsilon_{x,i}^{(2)}, t_{x,j}) + \Phi^{(2,0)}(\epsilon_{x,i}^{(3)}, t_{x,j})) - 2(t - t_{x,i})^2\Phi^{(2,0)}(\epsilon_{x,i}^{(1)}, t_{x,j}) \\ &\quad + \frac{1}{2}d_{x,j}^2(\Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(2)}) + \Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(3)})) - 2(s - t_{x,j})^2\Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(1)}) \\ &\quad + d_{x,i}d_{x,j}\Phi^{(1,1)}(\epsilon_{x,i}^{(3)}, \eta_{x,j}^{(3)}) - 4(t - t_{x,i})(s - t_{x,j})\Phi^{(1,1)}(\epsilon_{x,i}^{(1)}, \eta_{x,j}^{(1)}). \end{aligned}$$

We obtain inserting the last equation in (33),

$$I_{i,j} = \frac{1}{4m} \sum_{l=1}^5 I_{i,j}^{(l)}, \quad (47)$$

where,

$$\begin{aligned} I_{i,j}^{(1)} &= \Phi^{(1,0)}(t_{x,i}, t_{x,j}) \left(2d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} f(t)f(s) dt ds \right. \\ &\quad \left. - 4 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i}) f(t)f(s) dt ds \right). \\ I_{i,j}^{(2)} &= \Phi^{(0,1)}(t_{x,i}, t_{x,j}) \left(2d_{x,j} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} f(t)f(s) dt ds \right. \\ &\quad \left. - 4 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j}) f(t)f(s) dt ds \right). \\ I_{i,j}^{(3)} &= \frac{1}{2}d_{x,i}^2 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (\Phi^{(2,0)}(\epsilon_{x,i}^{(2)}, t_{x,j}) + \Phi^{(2,0)}(\epsilon_{x,i}^{(3)}, t_{x,j})) f(t)f(s) dt ds \\ &\quad - 2 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i})^2 \Phi^{(2,0)}(\epsilon_{x,i}^{(1)}, t_{x,j}) f(t)f(s) dt ds. \\ I_{i,j}^{(4)} &= \frac{1}{2}d_{x,j}^2 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (\Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(2)}) + \Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(3)})) f(t)f(s) dt ds \\ &\quad - 2 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})^2 \Phi^{(0,2)}(t_{x,i}, \eta_{x,j}^{(1)}) f(t)f(s) dt ds. \\ I_{i,j}^{(5)} &= d_{x,i}d_{x,j} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} \Phi^{(1,1)}(\epsilon_{x,i}^{(3)}, \eta_{x,j}^{(3)}) f(t)f(s) dt ds \\ &\quad - 4 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i})(s - t_{x,j}) \Phi^{(1,1)}(\epsilon_{x,i}^{(1)}, \eta_{x,j}^{(1)}) f(t)f(s) dt ds. \end{aligned}$$

We first consider the term $I_{i,j}^{(1)}$. For $l = 0, 1, 2$, let,

$$\omega_{i,l} = \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^l f(t) dt \quad (48)$$

The term $I_{i,j}^{(1)}$ can then be written as,

$$I_{i,j}^{(1)} = \Phi^{(1,0)}(t_{x,i}, t_{x,j}) \left(2d_{x,i}\omega_{i,0}\omega_{j,0} - 4\omega_{i,1}\omega_{j,0} \right). \quad (49)$$

Expanding f around $t_{x,i}$ yields,

$$\begin{aligned} \omega_{i,l} &= \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^l \left(f(t_{x,i}) + (t - t_{x,i})f'(t_{x,i}) + \frac{1}{2}(t - t_{x,i})^2 f''(\epsilon_{x,i}^{(4)}) \right) dt \\ &= \frac{d_{x,i}^{(l+1)}}{(l+1)} f(t_{x,i}) + \frac{d_{x,i}^{(l+2)}}{(l+2)} f'(t_{x,i}) + O\left(\frac{1}{n^{(l+3)}}\right), \end{aligned} \quad (50)$$

for some $\epsilon_{x,i}^{(4)}$ in $]t_{x,i}, t_{x,i+1}[$. Thus for $l = 0, 1, 2$,

$$\begin{aligned} I_{i,j}^{(1)} &= \Phi^{(1,0)}(t_{x,i}, t_{x,j}) \left(2d_{x,i} \left(d_{x,i} f(t_{x,i}) + \frac{d_{x,i}^2}{2} f'(t_{x,i}) + O\left(\frac{1}{n^3}\right) \right) \right. \\ &\quad \times \left(d_{x,j} f(t_{x,j}) + \frac{d_{x,j}^2}{2} f'(t_{x,i}) + O\left(\frac{1}{n^3}\right) \right) \\ &\quad \left. - 4 \left(\frac{d_{x,i}^2}{2} f(t_{x,i}) + \frac{d_{x,i}^3}{3} f'(t_{x,i}) + O\left(\frac{1}{n^4}\right) \right) \left(d_{x,j} f(t_{x,j}) + \frac{d_{x,j}^2}{2} f'(t_{x,i}) + O\left(\frac{1}{n^3}\right) \right) \right) \\ &= \Phi^{(1,0)}(t_{x,i}, t_{x,j}) \left(-\frac{1}{3} f'(t_{x,i}) f(t_{x,j}) d_{x,i}^3 d_{x,j} + O\left(\frac{1}{n^5}\right) \right). \end{aligned}$$

We obtain using Equations (36) and (27),

$$\begin{aligned} I_{i,j}^{(1)} &= -\frac{1}{3} \Phi^{(1,0)}(t_{x,i}, t_{x,j}) f'(t_{x,i}) f(t_{x,j}) d_{x,i}^3 d_{x,j} + O\left(\frac{1}{n^5 h^3}\right) \\ &= -\frac{1}{3n^2} \Phi^{(1,0)}(t_{x,i}, t_{x,j}) \frac{f'(t_{x,i})}{f^2(t_{x,i}^*)} f(t_{x,j}) d_{x,i} d_{x,j} + O\left(\frac{1}{n^5 h^3}\right), \end{aligned}$$

for some $t_{x,i}^*$ in $]t_{x,i}, t_{x,i+1}[$. Using Lemma 1 and the integrability of $\varphi_{x,h}, \varphi'_{x,h}, f$, and of $R^{(0,1)}(., t)$ and applying Lemma 2 twice, we obtain,

$$\begin{aligned} \sum_{i \neq j=1}^{N_{T_n}-1} \sum_{i \neq j=1}^{N_{T_n}-1} I_{i,j}^{(1)} &= -\frac{1}{3n^2} \sum_{i \neq j=1}^{N_{T_n}-1} \sum_{i \neq j=1}^{N_{T_n}-1} \Phi^{(1,0)}(t_{x,i}, t_{x,j}) \frac{f'(t_{x,i})}{f^2(t_{x,i}^*)} f(t_{x,j}) d_{x,i} d_{x,j} + O\left(\frac{1}{n^3 h}\right) \\ &= -\frac{1}{3n^2} \int_{x-h}^{x+h} \int_{x-h}^{x+h} \Phi^{(1,0)}(t, s) \frac{f'(t)}{f^2(t)} f(s) 1_{\{s \neq t\}} dt ds + O\left(\frac{1}{n^3 h^2}\right). \end{aligned} \quad (51)$$

Similarly we verify that,

$$\begin{aligned} \sum_{i \neq j=1}^{N_{T_n}-1} \sum_{i \neq j=1}^{N_{T_n}-1} I_{i,j}^{(2)} &= -\frac{1}{3n^2} \int_{x-h}^{x+h} \int_{x-h}^{x+h} \Phi^{(0,1)}(t, s) \frac{f'(s)}{f^2(s)} f(t) 1_{\{s \neq t\}} dt ds + O\left(\frac{1}{n^3 h^2}\right) \\ &= -\frac{1}{3n^2} \int_{x-h}^{x+h} \int_{x-h}^{x+h} \Phi^{(1,0)}(t, s) \frac{f'(t)}{f^2(t)} f(s) 1_{\{s \neq t\}} dt ds + O\left(\frac{1}{n^3 h^2}\right). \end{aligned} \quad (52)$$

We now control the term $I_{i,j}^3$. We have,

$$\begin{aligned}
I_{i,j}^{(3)} &= d_{x,i}^2 \Phi^{(2,0)}(t_{x,i}, t_{x,j}) \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} f(t) f(s) dt ds \\
&\quad - 2\Phi^{(2,0)}(t_{x,i}, t_{x,j}) \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i})^2 f(t) f(s) dt ds \\
&\quad + \frac{1}{2} d_{x,i}^2 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} \Phi^{(2,0)}(\epsilon_{x,i}^{(2)}, t_{x,j}) + \Phi^{(2,0)}(\epsilon_{x,i}^{(3)}, t_{x,j}) - 2\Phi^{(2,0)}(t_{x,i}, t_{x,j}) f(t) f(s) dt ds \\
&\quad - 2 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i})^2 (\Phi^{(2,0)}(\epsilon_{x,i}^{(1)}, t_{x,j}) - \Phi^{(2,0)}(t_{x,i}, t_{x,j})) f(t) f(s) dt ds.
\end{aligned}$$

Using (36), Lemma 1 and Equation (48) we get,

$$I_{i,j}^{(3)} = d_{x,i}^2 \Phi^{(2,0)}(t_{x,i}, t_{x,j}) \omega_{i,0} \omega_{j,0} - 2\Phi^{(2,0)}(t_{x,i}, t_{x,j}) \omega_{i,2} \omega_{i,0} + O\left(\frac{1}{n^5 h^5}\right).$$

Note first that, using (50) for $l = 0$ along with $l = 2$ and Lemma 1, we obtain,

$$\begin{aligned}
I_{i,j}^{(3)} &= \frac{1}{3} \Phi^{(2,0)}(t_{x,i}, t_{x,j}) d_{x,i}^3 d_{x,j} f(t_{x,i}) f(t_{x,j}) + O\left(\frac{1}{n^5 h^5}\right) \\
&= \frac{1}{3n^2} \Phi^{(2,0)}(t_{x,i}, t_{x,j}) \frac{f(t_{x,i})}{f^2(t_{x,i}^*)} f(t_{x,j}) d_{x,i} d_{x,j} + O\left(\frac{1}{n^5 h^5}\right),
\end{aligned}$$

Likewise, using Lemma 1 and the integrability of $\varphi_{x,h}^{(k)}, f^{(k)}$ for $k = 0, 1, 2$ we have,

$$\sum_{i \neq j=1}^{N_{T_n}-1} I_{i,j}^{(3)} = \frac{1}{3n^2} \int_{x-h}^{x+h} \int_{x-h}^{x+h} \Phi^{(2,0)}(t, s) \frac{f(s)}{f(t)} 1_{\{s \neq t\}} dt ds + O\left(\frac{1}{n^3 h^3}\right). \quad (53)$$

Similarly, we obtain,

$$\begin{aligned}
\sum_{i \neq j=1}^{N_{T_n}-1} I_{i,j}^{(4)} &= \frac{1}{3n^2} \int_{x-h}^{x+h} \int_{x-h}^{x+h} \Phi^{(0,2)}(t, s) \frac{f(t)}{f(s)} 1_{\{s \neq t\}} dt ds + O\left(\frac{1}{n^3 h^3}\right) \\
&= \frac{1}{3n^2} \int_{x-h}^{x+h} \int_{x-h}^{x+h} \Phi^{(2,0)}(t, s) \frac{f(s)}{f(t)} 1_{\{s \neq t\}} dt ds + O\left(\frac{1}{n^3 h^3}\right).
\end{aligned} \quad (54)$$

Finally, for the term $I_{i,j}^{(5)}$, we have,

$$\begin{aligned}
I_{i,j}^{(5)} &= d_{x,i}d_{x,j}\Phi^{(1,1)}(t_{x,i}, t_{x,j}) \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} f(t)f(s)dt ds \\
&- 4\Phi^{(1,1)}(t_{x,i}, t_{x,j}) \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i})(s - t_{x,j})f(t)f(s)dt ds \\
&+ d_{x,i}d_{x,j} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} \left(\Phi^{(1,1)}(\epsilon_{x,i}^{(3)}, \eta_{x,j}^{(3)}) - \Phi^{(1,1)}(t_{x,i}, t_{x,j}) \right) f(t)f(s)dt ds \\
&- 4 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i})(s - t_{x,j}) \left(\Phi^{(1,1)}(\epsilon_{x,i}^{(1)}, \eta_{x,j}^{(1)}) - \Phi^{(1,1)}(t_{x,i}, t_{x,j}) \right) f(t)f(s)dt ds \\
&= d_{x,i}d_{x,j}\Phi^{(1,1)}(t_{x,i}, t_{x,j})\omega_{i,0}\omega_{j,0} - 4\Phi^{(1,1)}(t_{x,i}, t_{x,j})\omega_{i,1}\omega_{j,1} \\
&+ d_{x,i}d_{x,j} \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} \left(\Phi^{(1,1)}(\epsilon_{x,i}^{(3)}, \eta_{x,j}^{(3)}) - \Phi^{(1,1)}(t_{x,i}, t_{x,j}) \right) f(t)f(s)dt ds \\
&- 4 \int_{t_{x,i}}^{t_{x,i+1}} \int_{t_{x,j}}^{t_{x,j+1}} (t - t_{x,i})(s - t_{x,j}) \left(\Phi^{(1,1)}(\epsilon_{x,i}^{(1)}, \eta_{x,j}^{(1)}) - \Phi^{(1,1)}(t_{x,i}, t_{x,j}) \right) f(t)f(s)dt ds.
\end{aligned}$$

Recall that $f, f', \frac{1}{f}$ are all bounded and using (36) and (50) with $l = l' = 1$ we obtain,

$$I_{i,j}^{(5)} = O\left(\frac{1}{n^5 h^5}\right).$$

Finally, since $N_{T_n} = O(nh)$ from Lemma 1, we obtain,

$$\sum_{i \neq j=1}^{N_{T_n}-1} \sum_{i,j} I_{i,j}^{(5)} = O\left(\frac{1}{n^3 h^3}\right). \quad (55)$$

Replacing (51), (52), (53), (54) and (55) in (47) we obtain,

$$\begin{aligned}
\sum_{i \neq j=1}^{N_{T_n}-1} \sum_{i,j} I_{i,j} &= \frac{1}{6mn^2} \int_{x-h}^{x+h} \int_{x-h}^{x+h} \left(\frac{\Phi^{(2,0)}(t, s)f(t) - \Phi^{(1,0)}(t, s)f'(t)}{f^2(t)} \right) 1_{\{s \neq t\}} f(s) dt ds \\
&+ O\left(\frac{1}{mn^3 h^3}\right) \\
&= \frac{1}{6mn^2} \int_{x-h}^{x+h} \left(\int_{x-h}^s \frac{\partial}{\partial s} \left(\frac{\Phi^{(1,0)}(t, s)}{f(t)} \right) \right) dt f(s) ds + O\left(\frac{1}{mn^3 h^3}\right) \\
&+ \frac{1}{6mn^2} \int_{x-h}^{x+h} \left(\int_s^{x+h} \frac{\partial}{\partial s} \left(\frac{\Phi^{(1,0)}(t, s)}{f(t)} \right) \right) dt f(s) ds + O\left(\frac{1}{mn^3 h^3}\right) \\
&= \frac{1}{6mn^2} \int_{x-h}^{x+h} (\Phi^{(1,0)}(s^-, s) - \Phi^{(1,0)}(s^+, s)) ds + O\left(\frac{1}{mn^3 h^3}\right) \\
&+ \frac{1}{6mn^2} \int_{x-h}^{x+h} \left(\frac{\Phi^{(1,0)}(x+h, s)}{f(x+h)} - \frac{\Phi^{(1,0)}(x-h, s)}{f(x-h)} \right) f(s) ds + O\left(\frac{1}{mn^3 h^3}\right).
\end{aligned}$$

Note that for $t \neq s$,

$$\Phi^{(1,0)}(t, s) = \left(\frac{\varphi'_{x,h}(t)f(t) - \varphi_{x,h}(t)f'(t)}{f^2(t)} R(t, s) + \frac{\varphi_{x,h}(t)}{f(t)} R^{(1,0)}(t, s) \right) \frac{\varphi_{x,h}(s)}{f(s)}. \quad (56)$$

It follows from (18) that,

$$\frac{\Phi^{(1,0)}(x+h, s)}{f(x+h)} = \frac{\Phi^{(1,0)}(x-h, s)}{f(x-h)} = 0 \quad \text{for all } s \in]x-h, x+h[.$$

Thus,

$$\sum_{i \neq j=1}^{N_{T_n}-1} I_{i,j} = \frac{1}{6mn^2} \int_{x-h}^{x+h} (\Phi^{(1,0)}(t^-, t) - \Phi^{(1,0)}(t^+, t)) dt + O\left(\frac{1}{mn^3h^3}\right). \quad (57)$$

Inserting (42) and (57) in (34), we obtain,

$$\begin{aligned} \text{Var}(\hat{g}_n^{trap}(x)) &= \frac{1}{m} \sigma_{x,h}^2 + \frac{1}{12mn^2} \int_{x-h}^{x+h} (\Phi^{(1,0)}(t^-, t) - \Phi^{(1,0)}(t^+, t)) dt \\ &\quad + O\left(\frac{1}{mn^3h^3}\right). \end{aligned} \quad (58)$$

Applying (56) it follows that,

$$\Phi^{(1,0)}(t^-, t) - \Phi^{(1,0)}(t^+, t) = \frac{\varphi_{x,h}^2(t)}{f^2(t)} \left(R^{(1,0)}(t^-, t) - R^{(1,0)}(t^+, t) \right) = \frac{\varphi_{x,h}^2(t)}{f^2(t)} \alpha(t). \quad (59)$$

Replacing (59) in (58) we obtain,

$$\text{Var}(\hat{g}_n^{trap}(x)) = \frac{1}{m} \sigma_{x,h}^2 + \frac{1}{12mn^2} \int_{x-h}^{x+h} \frac{\varphi_{x,h}^2(t)}{f^2(t)} \alpha(t) dt + O\left(\frac{1}{mn^3h^3}\right). \quad (60)$$

Since α and f are continuous on $[0, 1]$, then one can write,

$$\begin{aligned} \int_{x-h}^{x+h} \frac{\alpha(t)}{f^2(t)} \varphi_{x,h}^2(t) dt &= \frac{1}{h} \int_{-1}^1 \frac{\alpha(x-th)}{f^2(x-th)} K^2(t) dt \\ &= \frac{1}{h} \frac{\alpha(x)}{f^2(x)} \int_{-1}^1 K^2(t) dt + \frac{1}{h} \int_{-1}^1 \left(\frac{\alpha(x-th)}{f^2(x-th)} - \frac{\alpha(x)}{f^2(x)} \right) K^2(t) dt \\ &= \frac{1}{h} \frac{\alpha(x)}{f^2(x)} \int_{-1}^1 K^2(t) dt + O(1). \end{aligned} \quad (61)$$

Recall that for an even kernel, we have a simplified expression of $\sigma_{x,h}^2$ given by Benhenni and Rachdi [8] as follows,

$$\sigma_{x,h}^2 = R(x, x) - \frac{1}{2} \alpha(x) C_K h + o(h), \quad (62)$$

where $C_K = \int_{-1}^1 \int_{-1}^1 |u-v| K(u) K(v) du dv$.

Finally, using (61) and (62) in (60) yields,

$$\begin{aligned} \text{Var}(\hat{g}_n^{trap}(x)) &= \frac{1}{m} \left(R(x, x) - \frac{1}{2} \alpha(x) C_K h \right) + \frac{1}{12mn^2h} \frac{\alpha(x)}{f^2(x)} \int_{-1}^1 K^2(t) dt \\ &\quad + o\left(\frac{h}{m}\right) + O\left(\frac{1}{mn^2} + \frac{1}{mn^3h^3}\right). \end{aligned}$$

This concludes the proof of Proposition 2. \square

5.4 Proof of Proposition 3.

Let $I_1 = \int_0^1 R(x, x)w(x) dx$, $I_2 = \int_0^1 \frac{\alpha(x)}{f^2(x)}w(x) dx$ and put,

$$\Psi(h, m) = -\frac{C_K h}{2m} \int_0^1 \alpha(x)w(x) dx + \frac{1}{4}h^4 B^2 \int_0^1 [g''(x)]^2 w(x) dx.$$

We have from Equation (7) in Theorem 1,

$$\text{IMSE}(h) = \frac{I_1}{m} + \Psi(h, m) + \frac{VI_2}{12mn^2h} + o\left(h^4 + \frac{h}{m}\right) + O\left(\frac{1}{n^3h} + \frac{1}{mn^3h^3} + \frac{1}{mn^2} + \frac{1}{n^6h^6}\right).$$

Let h^* be as defined in (8). It is clear that $h^* = \underset{0 < h < 1}{\text{argmin}} \Psi(h, m)$ so that $\Psi(h, m) \geq \Psi(h^*, m)$ for every $0 < h < 1$. Let $h_{n,m}$ be as defined in Corollary 3. We have,

$$\begin{aligned} & \frac{\text{IMSE}(h^*)}{\overline{\text{IMSE}(h_{n,m})}} \\ &= \frac{\frac{I_1}{m} + \Psi(h^*, m) + \frac{VI_2}{12mn^2h^*} + o\left(h^{*4} + \frac{h^*}{m}\right) + O\left(\frac{1}{n^3h^*} + \frac{1}{mn^3h^{*3}} + \frac{1}{mn^2} + \frac{1}{n^6h^{*6}}\right)}{\frac{I_1}{m} + \Psi(h_{n,m}, m) + \frac{VI_2}{12mn^2h_{n,m}} + o\left(h_{n,m}^4 + \frac{h_{n,m}}{m}\right) + O\left(\frac{1}{n^3h_{n,m}} + \frac{1}{mn^3h_{n,m}^3} + \frac{1}{mn^2} + \frac{1}{n^6h_{n,m}^6}\right)} \\ &\leq \frac{I_1 + m\Psi(h_{n,m}, m) + \frac{VI_2}{12n^2h^*} + o\left(mh^{*4} + h^*\right) + O\left(\frac{m}{n^3h^*} + \frac{1}{n^3h^{*3}} + \frac{1}{n^2} + \frac{m}{n^6h^{*6}}\right)}{I_1 + m\Psi(h_{n,m}, m) + \frac{VI_2}{12n^2h_{n,m}} + o\left(mh_{n,m}^4 + h_{n,m}\right) + O\left(\frac{m}{n^3h_{n,m}} + \frac{1}{n^3h_{n,m}^3} + \frac{1}{n^2} + \frac{m}{n^6h_{n,m}^6}\right)}. \end{aligned}$$

Using the definition of h^* , $mh_{n,m}^3 = O(1)$, $\lim_{n,m \rightarrow \infty} h_{n,m} = 0$ and the assumption $\frac{m}{n} = O(1)$ as $n, m \rightarrow \infty$ we know that $m\Psi(h_{n,m}, m) = O(h_{n,m})$. Then,

$$\overline{\lim}_{n,m \rightarrow \infty} \frac{\text{IMSE}(h^*)}{\text{IMSE}(h_{n,m})} \leq 1.$$

This concludes the proof of Proposition 3. \square

5.5 Proof of Corollary 1.

Let f^* be as defined in (9). Let $D(f) = \int_0^1 \frac{\alpha(x)}{f^2(x)}w(x) dx$, then it is sufficient to prove that:

$$D(f^*) \leq D(f) \quad \text{for every positive density } f \text{ on } [0, 1].$$

Applying Hölder's inequality, we get,

$$\begin{aligned} D(f^*) &= \left(\int_0^1 \{\alpha(x)w(x)\}^{1/3} dx \right)^3 = \left(\int_0^1 \left(\frac{\alpha(x)w(x)}{f^2(x)} \right)^{1/3} f^{2/3}(x) dx \right)^3 \\ &\leq \left(\int_0^1 \frac{\alpha(x)w(x)}{f^2(x)} dx \right) \left(\int_0^1 f(x) dx \right)^2 = D(f). \end{aligned}$$

Hence,

$$\underset{\{f > 0 \text{ density on } [0,1]\}}{\text{argmin}} D(f) = f^*.$$

This completes the proof of Corollary 1. \square

5.6 Proof of Theorem 2.

Let $x \in]0, 1[$ be fixed. We have,

$$\sqrt{m} \left(\hat{g}_{n,m}^{trap}(x) - g(x) \right) = \sqrt{m} \left(\hat{g}_{n,m}^{trap}(x) - \mathbb{E}(\hat{g}_{n,m}^{trap}(x)) \right) + \sqrt{m} \text{Bias} \left(\hat{g}_{n,m}^{trap}(x) \right). \quad (63)$$

Since $\lim_{n,m \rightarrow \infty} \sqrt{mh^2} = 0$ and $\lim_{n,m \rightarrow \infty} nh^2 = \infty$ then Proposition 1 implies that,

$$\lim_{n,m \rightarrow \infty} \sqrt{m} \text{Bias} \left(\hat{g}_{n,m}^{trap}(x) \right) = 0. \quad (64)$$

Consider now the first term of the right side of (63). Since $\bar{Y}(t_{x,i}) - \mathbb{E}(\bar{Y}(t_{x,i})) = \bar{\varepsilon}(t_{x,i})$, we have, as done by Fraïman and Pérez Iribarren [20],

$$\begin{aligned} \sqrt{m} \left(\hat{g}_{n,m}^{trap}(x) - \mathbb{E}(\hat{g}_{n,m}^{trap}(x)) \right) &= \frac{1}{\sqrt{m}} \left\{ \sum_{j=1}^m \frac{1}{2n} \sum_{i=1}^{N_{T_n}-1} \left(\left(\frac{\varphi_{x,h}}{f} \varepsilon_j \right)(t_{x,i}) + \left(\frac{\varphi_{x,h}}{f} \varepsilon_j \right)(t_{x,i+1}) \right) \right\} \\ &= \frac{1}{\sqrt{m}} \sum_{j=1}^m \frac{1}{2n} \sum_{i=1}^{N_{T_n}-1} \frac{\varphi_{x,h}}{f}(t_{x,i}) (\varepsilon_j(t_{x,i}) - \varepsilon_j(x)) \\ &\quad + \frac{1}{\sqrt{m}} \sum_{j=1}^m \frac{1}{2n} \sum_{i=1}^{N_{T_n}-1} \frac{\varphi_{x,h}}{f}(t_{x,i+1}) (\varepsilon_j(t_{x,i+1}) - \varepsilon_j(x)) \\ &\quad + \left(\frac{1}{2n} \sum_{i=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f}(t_{x,i}) + \frac{\varphi_{x,h}}{f}(t_{x,i+1}) \right) \right) \left(\frac{1}{\sqrt{m}} \sum_{j=1}^m \varepsilon_j(x) \right). \end{aligned} \quad (65)$$

We start by controlling the last term of this last equation. Recall that Equation (27) yields for some $t_{x,i}^* \in]t_{x,i}, t_{x,i+1}[$ that $\frac{1}{n} = (t_{x,i+1} - t_{x,i})f(t_{x,i}^*)$. From the Riemann integrability of $\varphi_{x,h}$ and f and Lemma 2 we obtain,

$$\begin{aligned} \frac{1}{2n} \sum_{i=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f}(t_{x,i}) + \frac{\varphi_{x,h}}{f}(t_{x,i+1}) \right) &= \\ \frac{1}{2} \sum_{i=1}^{N_{T_n}-1} \left(\frac{\varphi_{x,h}}{f}(t_{x,i}) + \frac{\varphi_{x,h}}{f}(t_{x,i+1}) \right) f(t_{x,i}^*) (t_{x,i+1} - t_{x,i}) &\xrightarrow{m,n \rightarrow \infty} \int_{-1}^1 K(t) dt = 1. \end{aligned}$$

where $d_{x,i} = t_{x,i+1} - t_{x,i}$ and $t_{x,i}^* \in]t_{x,i}, t_{x,i+1}[$. The Central Limit Theorem for i.i.d. variables yields,

$$\frac{1}{\sqrt{m}} \sum_{j=1}^m \varepsilon_j(x) \xrightarrow{m \rightarrow \infty} Z \quad \text{where } Z \sim \mathcal{N}(0, R(x, x)).$$

We shall prove now that the two first terms of Equation (65) tend to 0 in probability as n, m tends to infinity. We will only study the first term, the second one is treated analogously. Let,

$$A_{m,n}(x) = \frac{1}{\sqrt{m}} \sum_{j=1}^m \frac{1}{2n} \sum_{i=1}^{N_{T_n}-1} \frac{\varphi_{x,h}}{f}(t_{x,i}) (\varepsilon_j(t_{x,i}) - \varepsilon_j(x)) \triangleq \frac{1}{\sqrt{m}} \sum_{j=1}^m T_{n,j}(x).$$

From the Chebyshev inequality, it suffices to prove that $\lim_{n,m \rightarrow \infty} \mathbb{E}(A_{m,n}^2(x)) = 0$. We have for $j \neq l$, $\mathbb{E}(\varepsilon_j(x)\varepsilon_l(y)) = 0$ so $\mathbb{E}(T_{n,j}(x)T_{n,l}(x)) = 0$. Hence,

$$\mathbb{E}(A_{m,n}^2(x)) = \frac{1}{m} \sum_{j=1}^m \sum_{l=1}^m \mathbb{E}(T_{n,j}(x)T_{n,l}(x)) = \frac{1}{m} \sum_{j=1}^m \mathbb{E}(T_{n,j}^2(x)).$$

We have,

$$\begin{aligned} \mathbb{E}(T_{n,j}^2(x)) &= \\ \frac{1}{4n^2} \sum_{i=1}^{N_{T_n}-1} \sum_{k=1}^{N_{T_n}-1} \frac{\varphi_{x,h}(t_{x,i})}{f} \frac{\varphi_{x,h}(t_{x,k})}{f} \mathbb{E} \left((\varepsilon_j(t_{x,i}) - \varepsilon_j(x)) (\varepsilon_j(t_{x,k}) - \varepsilon_j(x)) \right) \\ &= \frac{1}{4n^2} \sum_{i=1}^{N_{T_n}-1} \sum_{k=1}^{N_{T_n}-1} \frac{\varphi_{x,h}(t_{x,i})}{f} \frac{\varphi_{x,h}(t_{x,k})}{f} \left(R(t_{x,i}, t_{x,k}) - R(t_{x,i}, x) - R(x, t_{x,k}) + R(x, x) \right). \end{aligned}$$

Since $\mathbb{E}(T_{n,j}^2(x))$ does not depend on j we get,

$$\begin{aligned} \mathbb{E}(A_{m,n}^2(x)) &= \\ \frac{1}{4n^2} \sum_{i=1}^{N_{T_n}-1} \sum_{k=1}^{N_{T_n}-1} \frac{\varphi_{x,h}(t_{x,i})}{f} \frac{\varphi_{x,h}(t_{x,k})}{f} \left(R(t_{x,i}, t_{x,k}) - R(t_{x,i}, x) - R(x, t_{x,k}) + R(x, x) \right) \\ &\triangleq \frac{1}{4} \left(B_{n,1}(x) - B_{n,2}(x) - B_{n,3}(x) + B_{n,4}(x) \right). \end{aligned} \quad (66)$$

We obtain using Equation (27) for $t_{x,i}^* \in]t_{x,i}, t_{x,i+1}[$,

$$B_{n,1}(x) = \sum_{i=1}^{N_{T_n}-1} \sum_{k=1}^{N_{T_n}-1} f(t_{x,i}^*) f(t_{x,k}^*) \frac{\varphi_{x,h}(t_{x,i})}{f} \frac{\varphi_{x,h}(t_{x,k})}{f} R(t_{x,i}, t_{x,k}) d_{x,i} d_{x,k}.$$

The use of Lemma 2 twice yields,

$$\begin{aligned} B_{n,1}(x) &= \sum_{i=1}^{N_{T_n}-1} f(t_{x,i}^*) \frac{\varphi_{x,h}(t_{x,i})}{f} d_{x,i} \left\{ \int_{x-h}^{x+h} \varphi_{x,h}(t) R(t_{x,i}, t) dt + O\left(\frac{1}{nh}\right) \right\} \\ &= \int_{x-h}^{x+h} \varphi_{x,h}(t) \left\{ \sum_{i=1}^{N_{T_n}-1} f(t_{x,i}^*) \frac{\varphi_{x,h}(t_{x,i})}{f} R(t_{x,i}, t) d_{x,i} \right\} dt + O\left(\frac{1}{nh}\right) \\ &= \int_{x-h}^{x+h} \int_{x-h}^{x+h} \varphi_{x,h}(s) \varphi_{x,h}(t) R(s, t) ds dt + O\left(\frac{1}{nh}\right) = \sigma_{x,h}^2 + O\left(\frac{1}{nh}\right). \end{aligned}$$

Using (62) we obtain,

$$B_{n,1}(x) = R(x, x) - \frac{1}{2} \alpha(x) C_K h + o(h) + O\left(\frac{1}{nh}\right).$$

where $C_K = \int_{-1}^1 \int_{-1}^1 |u-v| K(u) K(v) dudv$. Since $\lim_{n \rightarrow \infty} h = 0$ and $\lim_{n \rightarrow \infty} nh = \infty$. Thus,

$$\lim_{n \rightarrow \infty} B_{n,1}(x) = R(x, x). \quad (67)$$

Consider now the term $B_{n,2}(x)$. We obtain using Lemma 2 twice,

$$\begin{aligned}
B_{n,2}(x) &= \int_{x-h}^{x+h} \int_{x-h}^{x+h} \varphi_{x,h}(s) \varphi_{x,h}(t) R(s, x) \, ds \, dt + O\left(\frac{1}{nh}\right) \\
&= \int_{x-h}^{x+h} \varphi_{x,h}(s) R(s, x) \, ds + O\left(\frac{1}{nh}\right) \\
&= \int_{-1}^1 K(s) R(x - hs, x) \, ds + O\left(\frac{1}{nh}\right) \\
&= \int_{-1}^0 K(s) R(x - hs, x) \, ds + \int_0^1 K(s) R(x - hs, x) \, ds + O\left(\frac{1}{nh}\right).
\end{aligned}$$

For $s \in]-1, 0[$, Taylor expansion of $R(\cdot, x)$ around x yields,

$$R(s, x) = R(x - sh, x) - shR^{(1,0)}(x+, x) + o(h).$$

Similarly for $s \in]0, 1[$ we obtain,

$$R(x - sh, x) = R(x, x) - shR^{(1,0)}(x-, x) + o(h).$$

Thus,

$$\begin{aligned}
B_{n,2}(x) &= R(x, x) - hR^{(1,0)}(x+, x) \int_{-1}^0 s K(s) \, ds \\
&\quad - hR^{(1,0)}(x-, x) \int_0^1 s K(s) \, ds + o(h) + O\left(\frac{1}{nh}\right).
\end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} B_{n,2}(x) = R(x, x). \tag{68}$$

Similarly,

$$\lim_{n \rightarrow \infty} B_{n,3}(x) = R(x, x). \tag{69}$$

It is easy to see that,

$$\begin{aligned}
\lim_{n \rightarrow \infty} B_{n,4}(x) &= \lim_{n \rightarrow \infty} R(x, x) \sum_{i=1}^{N_{T_n}-1} \sum_{k=1}^{N_{T_n}-1} \frac{\varphi_{x,h}(t_{x,i}) \varphi_{x,h}(t_{x,k})}{f} \\
&= R(x, x) \left(\int_{-1}^1 K(t) \, dt \right)^2 = R(x, x).
\end{aligned} \tag{70}$$

Inserting (67), (68), (69) and (70) in (66) yields,

$$\lim_{n, m \rightarrow \infty} \mathbb{E}(A_{m,n}^2(x)) = 0.$$

This concludes the proof of Theorem 2. \square

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Appendix

Lemma 2 (Integral approximation of a sum) *Let u and v be two Lipschitz functions on $[x - h, x + h]$, i.e, there exists two positive numbers l_1 and l_2 such that,*

$$|u(s) - u(t)| \leq l_1 |s - t|, \quad |v(s) - v(t)| \leq l_2 |s - t|.$$

Let $t_{x,1} < \dots < t_{x,N_{T_n}}$ be points in $[x - h, x + h]$ and put $d_{x,i} = t_{x,i+1} - t_{x,i}$. Then,

$$\sum_{i=1}^{N_{T_n}-1} u(t_{x,i})v(t'_{x,i})d_{x,i} = \int_{x-h}^{x+h} u(t)v(t) dt + \Delta_{n,h},$$

for any $t'_{x,i} \in [t_{x,i}, t_{x,i+1}]$ for all $i = 1, \dots, n$ and for some appropriate positive constants c_1, c_2 and c_3 ,

$$|\Delta_{n,h}| \leq c_1 l_1 \frac{h}{n} \sup_{t \in [0,1]} |v(t)| + c_2 l_2 \frac{h}{n} \sup_{t \in [0,1]} |u(t)| + 2 \frac{c_3}{n} \sup_{\substack{t \in [x-h, t_{x,1}] \\ \cup [t_{x,N_{T_n}}, x+h]}} |v(t)u(t)|.$$

Proof of Lemma 2. In fact, let $\Delta_{x,h} = A - B$ where,

$$A = \sum_{i=1}^{N_{T_n}-1} u(t_{x,i})v(t'_{x,i})d_{x,i} \quad \text{and} \quad B = \int_{x-h}^{x+h} u(t)v(t) dt.$$

We have,

$$B = \sum_{i=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} u(t)v(t) dt + \int_{x-h}^{t_{x,1}} u(t)v(t) dt + \int_{t_{x,N_{T_n}}}^{x+h} u(t)v(t) dt \triangleq B_1 + B_2,$$

where $B_2 = \int_{x-h}^{t_{x,1}} u(t)v(t) dt + \int_{t_{x,N_{T_n}}}^{x+h} u(t)v(t) dt$. On the one hand, since $(t_{x,1} - (x - h)) \leq \sup_{1 \leq i \leq n} d_{x,i}$ and $(x + h - t_{x,N_{T_n}}) \leq \sup_{1 \leq i \leq n} d_{x,i}$ we have,

$$|B_2| \leq 2c_3 \sup_{\substack{t \in [x-h, t_{x,1}] \\ \cup [t_{x,N_{T_n}}, x+h]}} |v(t)u(t)| \sup_{1 \leq i \leq n} d_{x,i}.$$

On the other hand, we have,

$$\begin{aligned} A - B_1 &= \sum_{i=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} (u(t_{x,i})v(t'_{x,i}) - u(t)v(t)) dt \\ &= \sum_{i=1}^{N_{T_n}-1} v(t'_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (u(t_{x,i}) - u(t)) dt + \sum_{i=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} u(t)(v(t'_{x,i}) - v(t)) dt. \end{aligned}$$

Since u and v are Lipschitz continuous we obtain,

$$|A - B_1| \leq N_{T_n} \sup_{t \in [0,1]} |v(t)| l_1 \sup_{1 \leq i \leq n} d_{x,i}^2 + N_{T_n} \sup_{t \in [0,1]} |u(t)| l_2 \sup_{1 \leq i \leq n} d_{x,i}^2.$$

Since $nh \geq 1$, Lemma 1 yields that $\sup_{1 \leq i \leq n} d_{x,i} = O(\frac{1}{n})$ and $N_{T_n} = O(nh)$. Hence,

$$\begin{aligned} |\Delta_{n,h}| &= |A - B| \leq |A - B_1| + |B_2| \\ &\leq c_1 l_1 \frac{h}{n} \sup_{t \in [0,1]} |v(t)| + c_2 l_2 \frac{h}{n} \sup_{t \in [0,1]} |u(t)| + 2 \frac{c_3}{n} \sup_{\substack{t \in [x-h, t_{x,1}] \\ \cup [t_{x, N_{T_n}}, x+h]}} |v(t)u(t)|. \end{aligned}$$

This concludes the proof of Lemma 2. \square