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RESOLVENT CONVERGENCE TO DIRAC OPERATORS ON PLANAR DOMAINS

JEAN-MARIE BARBAROUX, HORIA CORNEAN, LOÏC LE TREUST,
AND EDGARDO STOCKMEYER

ABSTRACT. Consider a Dirac operator defined on the whole plane with a mass term of size m supported outside a domain Ω . We give a simple proof for the norm resolvent convergence, as m goes to infinity, of this operator to a Dirac operator defined on Ω with infinite mass boundary conditions. The result is valid for bounded and unbounded domains and gives estimates on the speed of convergence. Moreover, the method easily extends when adding external matrix-valued potentials.

1. INTRODUCTION

On the plane $\mathbb{R}^2$ consider a smooth open set Ω , not necessarily bounded, with boundary $\partial\Omega$. For $m > 0$, let H_m be the standard two-dimensional Dirac operator, defined on the whole plane, with a mass term being zero on Ω and m on $\tilde{\Omega} := \mathbb{R}^2 \setminus \Omega$. Berry and Mondragon [8] realised that in the limit when $m \rightarrow \infty$ the operator H_m converges – in some sense – to an operator H_Ω , defined only on Ω , with certain prescribed conditions at the boundary (which were later known as infinite mass boundary conditions). Recently in [22] it was shown, assuming Ω to be bounded, that the spectral projections of H_m converge to those of H_Ω as $m \rightarrow \infty$. The strategy used in [22] was based on the mini-max principle, which is only appropriate for bounded domains. In this work we extend the results from [22] in several directions.

In recent years, Dirac operators have regain much attention since they provide an effective description for the dynamics of charge carriers in graphene [13] and in other novel materials that exhibit interesting physical properties (see e.g., [2]). In the physics literature, this type of spatially localised mass terms in the Dirac equation are used to model obstacles for the effective particles [1, 18, 21]. In addition, the limiting operator H_Ω is used to model particles confined to Ω , as it is the case for quantum dots [16] or for a periodically perforated sheet of graphene [12, 11]. It is therefore desirable to estimate the speed of convergence of H_m to H_Ω .

In this article we allow Ω to be unbounded and show that, as $m \rightarrow \infty$, the norm resolvent difference between H_m and H_Ω is of order $1/\sqrt{m}$, uniformly in the spectral parameter (see Theorem 1.1). Our proof is based on a novel resolvent identity (see Lemma 2.1), which

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we believe is interesting in its own right. The simplicity of the proof allows naturally the inclusion of matrix-valued external potentials (see Theorem 1.3).

1.1. Setting and main result. Let $\Omega \subset \mathbb{R}^2$ be an open set satisfying the following.

Hypothesis 1.

- (1) *Its boundary $\partial\Omega$ is $C^{2,1}$,*
- (2) *the curvature κ_Ω of $\partial\Omega$ is in $L^\infty(\partial\Omega)$,*
- (3) *there exist global tubular coordinates for a neighborhood of $\partial\Omega$. Namely, there exist $\delta_0 > 0$ and $C_0 > 0$ such that the mapping*

$$\begin{aligned} \Psi :] - \delta_0, \delta_0[\times \partial\Omega &\rightarrow \mathbb{R}^2 \\ (t, s) &\mapsto s - tn(s), \end{aligned}$$

satisfies

- (a) Ψ *is a C^1 -diffeomorphism on its range,*
- (b) $\partial\Omega = \Psi(\{0\} \times \partial\Omega)$,
- (c) $\Psi([0, \delta_0[\times \partial\Omega) = \Omega \cap \text{range}(\Psi)$,
- (d) $C_0 > |\det J_\Psi(s, t)| > C_0^{-1}$, *for all $(t, s) \in] - \delta_0, \delta_0[\times \partial\Omega$.*

This includes for instance the following cases:

- (i) **Bounded boundary case.** The set Ω or $\tilde{\Omega}$ is a bounded set, with $C^{2,1}$ boundary.
- (ii) **Deformed half-space.** The case in which Ω is a connected unbounded set given by $\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R} \text{ and } -\infty < x_1 < f(x_2)\}$ for some $f \in C^{2,1}(\mathbb{R})$.
- (iii) **Periodic holes.** Another typical situation in which we are interested in is the case where, for fixed $L > 0$, $\tilde{\Omega} = \bigcup_{\gamma \in L\mathbb{Z}^2} B_r(\gamma)$ consists of periodically distributed balls of radius $r < L$.

We work on the spaces of measurable square integrable functions $L^2(\Omega, \mathbb{C}^2)$, $L^2(\tilde{\Omega}, \mathbb{C}^2)$, $L^2(\partial\Omega, \mathbb{C}^2)$, and $L^2(\mathbb{R}^2, \mathbb{C}^2)$, with scalar products $\langle \cdot, \cdot \rangle_\Omega$, $\langle \cdot, \cdot \rangle_{\tilde{\Omega}}$, $\langle \cdot, \cdot \rangle_{\partial\Omega}$, and we set $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{\mathbb{R}^2}$. We indicate the set of integration in the same way for the corresponding norms $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$. The Euclidian scalar product on $\mathbb{C}^2$ is denoted by $(\cdot, \cdot)$. Thus, we have, for instance, for $f, g \in L^2(\Omega, \mathbb{C}^2)$

$$\langle f, g \rangle_\Omega = \int_\Omega (f(\mathbf{x}), g(\mathbf{x})) d\mathbf{x}.$$

Throughout this paper we use the identification

$$\begin{aligned} L^2(\mathbb{R}^2, \mathbb{C}^2) &\cong L^2(\Omega, \mathbb{C}^2) \oplus L^2(\tilde{\Omega}, \mathbb{C}^2) \\ \varphi &\mapsto \varphi \mathbb{1}_\Omega \oplus \varphi \mathbb{1}_{\tilde{\Omega}}, \end{aligned} \tag{1}$$

where $\mathbb{1}$ denotes the indicator function on the corresponding set.

For $\Omega' \in \{\Omega, \tilde{\Omega}\}$ we denote by $H^1(\Omega', \mathbb{C}^2)$ the Sobolev space of functions in $L^2(\Omega', \mathbb{C}^2)$ whose first-order distributional derivatives belong to $L^2(\Omega', \mathbb{C}^2)$. If it is clear from the context we may drop the reference to the spinor space $\mathbb{C}^2$ and simply write $L^2(\Omega')$, $H^1(\Omega')$,

etc. Due to the regularity of the boundary, there exist a continuous linear operator, the trace operator,

$$\text{tr}_{\Omega'} : H^1(\Omega', \mathbb{C}^2) \rightarrow L^2(\partial\Omega, \mathbb{C}^2),$$

such that $\text{tr}_{\Omega'}\varphi = \varphi$ on $\partial\Omega$ for all $\varphi \in H^1(\Omega', \mathbb{C}^2)$ continuous on $\overline{\Omega'}$ (see [20, Theorem 15.23] for a general statement valid in the case of unbounded boundaries).

Moreover, for any $\varepsilon > 0$ there exists a constant C_ε such that for all $\varphi \in H^1(\Omega', \mathbb{C}^2)$

$$\|\text{tr}_{\Omega'}\varphi\|_{\partial\Omega}^2 \leq \varepsilon \|\nabla\varphi\|_{\Omega'}^2 + C_\varepsilon \|\varphi\|_{\Omega'}^2. \quad (2)$$

Remark 1. The above trace inequality is usually stated with $\varepsilon = 1$. To obtain the desired result (2), following the standard proof, it suffices to apply $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ (e.g. in [17, proof of theorem 1, § 5.5, Eq.(1), p272]).

We define the differential expression associated to the standard massless Dirac operator on the plane as

$$T := \frac{1}{i} \boldsymbol{\sigma} \cdot \nabla = \begin{pmatrix} 0 & -i\partial_1 - \partial_2 \\ -i\partial_1 + \partial_2 & 0 \end{pmatrix},$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ and the Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the anticommutation relation

$$\{\sigma_j, \sigma_k\} := \sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}, \quad j, k \in \{1, 2, 3\}. \quad (3)$$

Let us now introduce the main operators of this work. For $m > 0$ we denote by H_m the self-adjoint operator in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ acting as

$$H_m = T + m\mathbb{1}_{\tilde{\Omega}}\sigma_3$$

with operator domain $\mathcal{D}(H_m) = H^1(\mathbb{R}^2, \mathbb{C}^2)$. For an element of its resolvent set $z \in \varrho(H_m)$ we write the resolvent of H_m as

$$R_m(z) := (H_m - z)^{-1}. \quad (4)$$

We are interested in the limit of R_m as $m \rightarrow \infty$; the limit operator acts non-trivially in $L^2(\Omega, \mathbb{C}^2)$. In order to describe its boundary conditions we set, for $\Omega' \in \{\Omega, \tilde{\Omega}\}$, $\mathbf{n}_{\Omega'} : \partial\Omega \rightarrow \mathbf{S}^1$ to be the normal vector to Ω' pointing outwards. For any $\mathbf{x} \in \partial\Omega$ we define the *boundary matrix* as

$$B_{\Omega'}(\mathbf{x}) := -i\sigma_3 \boldsymbol{\sigma} \cdot \mathbf{n}_{\Omega'}(\mathbf{x}),$$

which is self-adjoint with eigenvalues ± 1 . For the case $\Omega' = \Omega$ we write the corresponding eigenprojections as $P_\pm := (1 \pm B_\Omega)/2$. Due to the anticommutator relations (3) the following intertwining relation holds

$$\sigma_3 P_- = P_+ \sigma_3. \quad (5)$$

We define the self-adjoint operator (see e.g. [6])

$$H_\Omega \varphi = T\varphi, \quad \varphi \in \mathcal{D}_\Omega := \{\varphi \in H^1(\Omega, \mathbb{C}^2) : P_- \text{tr}_\Omega \varphi = 0\}. \quad (6)$$

In the physics literature conditions as $P_- \text{tr}_\Omega \varphi = 0$ or $P_+ \text{tr}_\Omega \varphi = 0$ are referred to as infinite-mass boundary conditions [1]. For further recent study of H_Ω from the mathematical point of view see e.g. [7, 19, 10].

Note that in dimension 3, this infinite mass limit is strongly related to the MIT bag model for quarks confinement. This model has been derived first by Bogolioubov [9] in 1968 and extended by physicists from MIT into a shape optimization problem [14] in 1974. We also refer to [3, 5, 4] for a study of these problems from the mathematical viewpoint and other physical references on the subject.

For $z \in \varrho(H_\Omega)$ we write

$$R_\Omega(z) := (H_\Omega - z)^{-1}. \quad (7)$$

Theorem 1.1 (Resolvent convergence - free case). *Let $K \subset \varrho(H_\Omega)$ be a compact set. Then, there exists $m_0 > 1$ such that for all $m > m_0$: $K \subset \varrho(H_m)$ and there is a constant $C_K \equiv C < \infty$ such that*

$$\sup_{z \in K} \|R_m(z) - R_\Omega(z) \oplus 0\| \leq \frac{C}{\sqrt{m}}. \quad (8)$$

Here the orthogonal sum is in the sense of the splitting (1).

Remark 2. Notice that Theorem 1.1 implies that the spectrum of H_m converges to that of H_Ω (see e.g., [15, Proposition 10.2.4]). Consequently, for $a, b \in \varrho(H_\Omega)$ with $a < b$, we have

$$\|\mathbb{1}_{(a,b)}(H_m) - \mathbb{1}_{(a,b)}(H_\Omega) \oplus 0\| = \mathcal{O}(m^{-1/2}), \quad m \gg 1,$$

where $\mathbb{1}_{(a,b)}$ is the indicator function on (a, b) and the spectral projections are defined through the functional calculus. This result is obtained by applying the estimate (8) to the Riesz' Formula $\mathbb{1}_{(a,b)}(H_m) - \mathbb{1}_{(a,b)}(H_\Omega) \oplus 0 = \frac{1}{2i\pi} \int_\gamma R_m(z) - R_\Omega(z) \oplus 0 \, dz$, where γ is a contour around (a, b) with distance $\nu > 0$ to (a, b) .

In particular, if λ is an isolated eigenvalue of H_Ω , there exists a sequence (λ_m) such that for all m large enough λ_m is an eigenvalue of H_m and $\lambda - \lambda_m = \mathcal{O}(m^{-1/2})$, $m \gg 1$.

Remark 3. It is easy to check that H_{-m} converges, as $m \rightarrow \infty$, to an operator defined as in (6) but with the boundary condition replaced by $P_+ \text{tr}_\Omega \varphi = 0$ (see [22, Remark 2]).

In [22] it was shown that the spectral projections of H_m converge to those of H_Ω in operator norm. In their proof the authors used a combination of the mini-max principle (which is not suitable for unbounded domains) with certain apriori lower bound on the quadratic form $\varphi \mapsto \|H_m \varphi\|^2$. In this work we use the same lower bound (see [22, Lemma 4] and Proposition 1.2 below) together with certain novel resolvent identity stated in Lemma 2.1 below. The main point here is that we compare, through their resolvents, the operator H_m with $H_\Omega \oplus H_{\tilde{\Omega}}^{(m)}$, where $H_{\tilde{\Omega}}^{(m)} = T + m\sigma_3$ is an auxiliary operator having domain $H^1(\tilde{\Omega})$ with infinite-mass boundary conditions (see (10) below). We conclude the proof by estimating the corresponding resolvents and resolvent traces. The proof of Theorem 1.1 is given in Section 2.

The next theorem is an extension of the main theorem to the case of Dirac operators with matrix-valued potentials.

Hypothesis 2. Let V be a symmetric matrix-valued potential. We assume:

(i) There exists $a \in [0, 1)$ and $b \geq 0$ such that for all $\varphi \in H^1(\Omega, \mathbb{C}^2)$

$$\|V\varphi\|_{\Omega}^2 \leq a\|\nabla\varphi\|_{\Omega}^2 + b\|\varphi\|_{\Omega}^2.$$

(ii) The potential V is bounded in $\tilde{\Omega}$: $\|V\|_{L^\infty(\tilde{\Omega})} < \infty$.

We thus consider the following operators

$$H_m^V = H_m + V$$

and for $z \in \rho(H_m^V)$ we denote

$$R_m^V(z) = (H_m^V - z)^{-1}.$$

We also define

$$H_{\Omega}^V = H_{\Omega} + V \quad \text{on} \quad \{\varphi \in H^1(\Omega, \mathbb{C}^2) : P_- \text{tr}_{\Omega} \varphi = 0\}$$

and $R_{\Omega}^V(z) = (H_{\Omega}^V - z)^{-1}$ for $z \in \rho(H_{\Omega}^V)$.

Proposition 1.2. Assume Hypothesis 2 holds. Then H_m^V is self-adjoint on $\mathcal{D}(H_m^V) = H^1(\mathbb{R}^2, \mathbb{C}^2)$ and H_{Ω}^V is self-adjoint on $\mathcal{D}(H_{\Omega}^V) = \{\varphi \in H^1(\Omega, \mathbb{C}^2) : P_- \text{tr}_{\Omega} \varphi = 0\}$.

Theorem 1.3 (Resolvent convergence - potential case). Assume Hypothesis 2 holds. Let $K \subset \varrho(H_{\Omega})$ be a compact set. Then, there exists $m_0 > 1$ such that for all $m > m_0$: $K \subset \varrho(H_m)$ and there is a constant $C_K^V \equiv C < \infty$ such that

$$\sup_{z \in K} \|R_m^V(z) - R_{\Omega}^V(z) \oplus 0\| \leq \frac{C}{\sqrt{m}}. \quad (9)$$

The proofs of the above two results are postponed to Section 4.

2. PROOF OF THEOREM 1.1

For $m > 0$ we define the auxiliary self-adjoint operator acting in $L^2(\tilde{\Omega}, \mathbb{C}^2)$ as

$$H_{\tilde{\Omega}}^{(m)} \varphi = T\varphi + m\sigma_3\varphi, \quad \varphi \in \mathcal{D}_{\tilde{\Omega}} := \{\varphi \in H^1(\tilde{\Omega}, \mathbb{C}^2) : P_+ \text{tr}_{\tilde{\Omega}} \varphi = 0\}. \quad (10)$$

For $z \in \varrho(H_{\tilde{\Omega}}^{(m)})$ we write its resolvent

$$R_{\tilde{\Omega}}^{(m)}(z) := (H_{\tilde{\Omega}}^{(m)} - z)^{-1}. \quad (11)$$

Remark 4. Notice that, since $B_{\Omega} = -B_{\tilde{\Omega}}$, the boundary conditions stated in (6) and (10) can both be written in terms of the corresponding boundary matrix. More precisely, for $\Omega' \in \{\Omega, \tilde{\Omega}\}$, we have

$$\varphi \in \mathcal{D}_{\Omega'} \Leftrightarrow \varphi \in H^1(\Omega', \mathbb{C}^2) \quad \text{and} \quad B_{\Omega'} \text{tr}_{\Omega'} \varphi = \text{tr}_{\Omega'} \varphi.$$

Lemma 2.1 (Resolvent identity). Consider the resolvents (4), (7), and (11), evaluated at $z \in \varrho(H_m) \cap \varrho(H_{\Omega}) \cap \varrho(H_{\tilde{\Omega}}^{(m)})$. Then, for any $f, g \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ with $g = h \oplus \tilde{h} \in L^2(\Omega, \mathbb{C}^2) \oplus L^2(\tilde{\Omega}, \mathbb{C}^2)$, we have

$$\langle f, R_m g \rangle = \langle f, R_{\Omega} \oplus R_{\tilde{\Omega}}^{(m)} g \rangle + \langle P_- \text{tr}_{\Omega} R_m^* f, \sigma_3 \text{tr}_{\Omega} R_{\Omega} h \rangle_{\partial\Omega} + \langle P_+ \text{tr}_{\tilde{\Omega}} R_m^* f, \sigma_3 \text{tr}_{\tilde{\Omega}} R_{\tilde{\Omega}}^{(m)} \tilde{h} \rangle_{\partial\tilde{\Omega}}.$$

Proof. Let $\psi \in H^1(\mathbb{R}^2, \mathbb{C}^2)$ and $\varphi = \phi \oplus \tilde{\phi} \in \mathcal{D}_\Omega \oplus \mathcal{D}_{\tilde{\Omega}}$. Since the mass term of H_m is supported on $\tilde{\Omega}$ we have that

$$\langle H_m \psi, \varphi \rangle = \langle T\psi, \phi \rangle_\Omega + \langle H_m \psi, \tilde{\phi} \rangle_{\tilde{\Omega}} = \langle T\psi, \phi \rangle_\Omega + \langle T\psi, \tilde{\phi} \rangle_{\tilde{\Omega}} + \langle \psi, m\sigma_3 \tilde{\phi} \rangle_{\tilde{\Omega}}.$$

Using partial integration (see Lemma 2.2 below) we get for any $z \in \mathbb{C}$

$$\begin{aligned} \langle (H_m - \bar{z})\psi, \varphi \rangle &= \langle \psi, (H_\Omega \oplus H_{\tilde{\Omega}}^{(m)} - z)\varphi \rangle - \langle \text{tr}_\Omega \psi, \sigma_3 B_\Omega \text{tr}_\Omega \phi \rangle_{\partial\Omega} - \langle \text{tr}_{\tilde{\Omega}} \psi, \sigma_3 B_{\tilde{\Omega}} \text{tr}_{\tilde{\Omega}} \tilde{\phi} \rangle_{\partial\Omega} \\ &= \langle \psi, (H_\Omega - z)\phi \rangle + \langle \psi, (H_{\tilde{\Omega}}^{(m)} - z)\tilde{\phi} \rangle - \langle \text{tr}_\Omega \psi, \sigma_3 \text{tr}_\Omega \phi \rangle_{\partial\Omega} - \langle \text{tr}_{\tilde{\Omega}} \psi, \sigma_3 \text{tr}_{\tilde{\Omega}} \tilde{\phi} \rangle_{\partial\Omega}, \end{aligned}$$

where in the last step we used the boundary conditions for ϕ and $\tilde{\phi}$ (see Remark 4). Since $\text{tr}_\Omega \phi = P_+ \text{tr}_\Omega \phi$ and $\text{tr}_{\tilde{\Omega}} \tilde{\phi} = P_- \text{tr}_{\tilde{\Omega}} \tilde{\phi}$ we may write, in view of (5), the two boundary terms above as $\langle P_- \text{tr}_\Omega \psi, \sigma_3 \text{tr}_\Omega \phi \rangle_{\partial\Omega}$ and $\langle P_+ \text{tr}_{\tilde{\Omega}} \psi, \sigma_3 \text{tr}_{\tilde{\Omega}} \tilde{\phi} \rangle_{\partial\Omega}$, respectively.

Let $f, g, h, \tilde{h}$, and z be given as in the statement of the lemma. The claim follows from the above computation since it holds for $\psi := R_m^* f = R_m(\bar{z})f$, $\phi := R_\Omega(z)h$, and $\tilde{\phi} := R_{\tilde{\Omega}}^{(m)}(z)\tilde{h}$. $\square$

Proof of Theorem 1.1. Let $f, g \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ with $g = h \oplus \tilde{h} \in L^2(\Omega, \mathbb{C}^2) \oplus L^2(\tilde{\Omega}, \mathbb{C}^2)$. For fixed constants $\mu_0 > 0$ and $\rho \geq 0$ define the set

$$S_{\mu_0, \rho} := \{\xi \in \mathbb{C} : |\text{Im}(\xi)| \geq \mu_0 \text{ and } |\text{Re}(\xi)| \leq \rho\}. \quad (12)$$

According to Lemma A.1 below it suffices to show, for some $C' < \infty$ and $m' < \infty$,

$$\sup_{\xi \in S_{\mu_0, \rho}} |\langle f, (R_m(\xi) - R_\Omega(\xi) \oplus 0)g \rangle| \leq \frac{C'}{\sqrt{m}} \|f\| \|g\|, \quad \text{for } m \geq m'.$$

From the resolvent identity in Lemma 2.1 we get, using Cauchy-Schwarz Inequality,

$$\begin{aligned} |\langle f, (R_m(\xi) - R_\Omega(\xi) \oplus 0)g \rangle| &\leq \|R_{\tilde{\Omega}}^{(m)}(\xi)\tilde{h}\| \|f\| + \|P_- \text{tr}_\Omega R_m(\bar{\xi})f\|_{\partial\Omega} \|\text{tr}_\Omega R_\Omega(\xi)h\|_{\partial\Omega} \\ &\quad + \|\text{tr}_{\tilde{\Omega}} R_m(\bar{\xi})f\|_{\partial\Omega} \|\text{tr}_{\tilde{\Omega}} R_{\tilde{\Omega}}^{(m)}(\xi)\tilde{h}\|_{\partial\Omega}, \end{aligned}$$

for any $\xi \in S_{\mu_0, \rho}$. The theorem now follows from lemmas 3.2, 3.3, and Corollary 3.5 from Section 3 below. $\square$

The next is an elementary result from partial integration.

Lemma 2.2. *Let $\Omega' \in \{\Omega, \tilde{\Omega}\}$. For any $\phi, \psi \in H^1(\Omega', \mathbb{C}^2)$ we have*

$$\langle T\phi, \psi \rangle_{\Omega'} = \langle \phi, T\psi \rangle_{\Omega'} - \langle \text{tr}_{\Omega'} \phi, \sigma_3 B_{\Omega'} \text{tr}_{\Omega'} \psi \rangle_{\partial\Omega'}. \quad (13)$$

Proof. Using divergence theorem for the Sobolev functions $\phi, \psi \in H^1(\Omega', \mathbb{C}^2)$ we compute

$$\begin{aligned} \langle T\phi, \psi \rangle_{\Omega'} &= \int_{\Omega'} (-i\sigma \cdot \nabla \phi(\mathbf{x}), \psi(\mathbf{x})) d\mathbf{x} = \langle \phi, T\psi \rangle_{\Omega'} + \int_{\Omega'} \nabla \cdot (\phi(\mathbf{x}), i\sigma \psi(\mathbf{x})) d\mathbf{x} \\ &= \langle \phi, T\psi \rangle_{\Omega'} + \int_{\partial\Omega'} (\text{tr}_{\Omega'} \phi(\mathbf{x}), i(\sigma \cdot \mathbf{n}_{\Omega'}) \text{tr}_{\Omega'} \psi(\mathbf{x})) d\mathbf{x}. \end{aligned}$$

The lemma follows now since $i\sigma \cdot \mathbf{n}_{\Omega'} = -\sigma_3 B_{\Omega'}$. $\square$

3. RESOLVENT ESTIMATES

Let us start by introducing some additional geometrical objects for $\Omega' \in \{\Omega, \tilde{\Omega}\}$. Given $\mathbf{n}_{\Omega'} = (n_1, n_2)$ we define the tangent vector $\mathbf{t}_{\Omega'} := (-n_2, n_1)$. Normal and tangent vectors are related to the curvature $\kappa_{\Omega'}$ through the formula

$$(\mathbf{t}_{\Omega'} \cdot \nabla) \mathbf{t}_{\Omega'} = -\kappa_{\Omega'} \mathbf{n}_{\Omega'}. \quad (14)$$

Due to Hypothesis 1 on $\partial\Omega$ we have $\|\kappa_{\Omega'}\|_{\infty} \equiv \|\kappa\|_{\infty} < 0$. The following lemma is well known. We give its proof for completeness at the end of this section.

Lemma 3.1. *Let $\Omega' \in \{\Omega, \tilde{\Omega}\}$. Then, for any $\varphi \in \mathcal{D}_{\Omega'}$, we have*

$$\|T\varphi\|_{\Omega'}^2 = \|\nabla\varphi\|_{\Omega'}^2 + \frac{1}{2} \langle \text{tr}_{\Omega'} \varphi, \kappa_{\Omega'} \text{tr}_{\Omega'} \varphi \rangle_{\partial\Omega}. \quad (15)$$

Let $\mu_0 \in (0, \infty)$ and $\rho \in [0, \infty)$ be fixed constants. In this section we will evaluate the resolvents on the set $S_{\mu_0, \rho}$ defined in (12).

Lemma 3.2. *There exist a constant $c < \infty$ such that for all $\xi \in S_{\mu_0, \rho}$ and all $h \in L^2(\Omega, \mathbb{C}^2)$*

$$\|R_{\Omega}(\xi)h\|_{\Omega} \leq \frac{1}{\mu_0} \|h\|_{\Omega}, \quad (16)$$

$$\|\text{tr}_{\Omega} R_{\Omega}(\xi)h\|_{\partial\Omega} \leq c \|h\|_{\Omega}. \quad (17)$$

Proof. The first inequality above is straightforward. In order to show (17) observe first that, for all $\varphi \in \mathcal{D}_{\Omega}$ and $\xi = \eta + i\mu \in S_{\mu_0, \rho}$,

$$\|(H_{\Omega} - \xi)\varphi\|^2 = \|(T - \eta)\varphi\|_{\Omega}^2 + \mu^2 \|\varphi\|_{\Omega}^2 \geq \frac{1}{2} \|T\varphi\|_{\Omega}^2 + (\mu_0^2 - 2\rho^2) \|\varphi\|_{\Omega}^2. \quad (18)$$

Recall that by the Trace Inequality (2) for every $\varepsilon > 0$ there is a $C_{\varepsilon} < \infty$ such that

$$\|\nabla\varphi\|^2 \geq \frac{1}{\varepsilon} \|\text{tr}_{\Omega}\varphi\|_{\partial\Omega}^2 - \frac{C_{\varepsilon}}{\varepsilon} \|\varphi\|_{\Omega}^2, \quad \varphi \in H^1(\Omega). \quad (19)$$

Thus, using the previous two inequalities and estimating equation (15) we get, for all $\varphi \in \mathcal{D}_{\Omega}$ and $\xi \in S_{\mu_0, \rho}$,

$$\begin{aligned} 2\|(H_{\Omega} - \xi)\varphi\|^2 &\geq \left(\frac{1}{\varepsilon} - \frac{\|\kappa\|_{\infty}}{2}\right) \|\text{tr}_{\Omega}\varphi\|_{\partial\Omega}^2 + \left(2\mu_0^2 - 4\rho^2 - \frac{C_{\varepsilon}}{\varepsilon}\right) \|\varphi\|_{\Omega}^2 \\ &\geq \|\text{tr}_{\Omega}\varphi\|_{\partial\Omega}^2 - \tilde{C}_{\varepsilon} \|\varphi\|_{\Omega}^2, \end{aligned} \quad (20)$$

where in the last step we choose $\varepsilon^{-1} = 1 + \|\kappa\|_{\infty}/2$ and set $\tilde{C}_{\varepsilon} := |2\mu_0^2 - 4\rho^2 - C_{\varepsilon}/\varepsilon|$. Let $h \in L^2(\Omega, \mathbb{C}^2)$. Replacing φ by $R_{\Omega}(\xi)h \in \mathcal{D}_{\Omega}$ in (20) we get that

$$\|\text{tr}_{\Omega} R_{\Omega}(\xi)h\|_{\partial\Omega}^2 \leq 2\|h\|_{\Omega}^2 + \tilde{C}_{\varepsilon} \|R_{\Omega}(\xi)h\|_{\Omega}^2 \leq (2 + \tilde{C}_{\varepsilon}/\mu_0^2) \|h\|_{\Omega}^2.$$

This finishes the proof of the lemma. $\square$

Lemma 3.3. *For all $\xi \in S_{\mu_0, \rho}$, $h \in L^2(\tilde{\Omega}, \mathbb{C}^2)$, and all $m > 2\sqrt{|\mu_0^2 - 2\rho^2|}$ the following inequalities hold*

$$\|R_{\tilde{\Omega}}^{(m)}(\xi)h\|_{\tilde{\Omega}} \leq \frac{2}{m} \|h\|_{\tilde{\Omega}}, \quad (21)$$

$$\|\text{tr}_{\tilde{\Omega}} R_{\tilde{\Omega}}^{(m)}(\xi)h\|_{\partial\tilde{\Omega}}^2 \leq \frac{2}{m} \|h\|_{\tilde{\Omega}}^2. \quad (22)$$

Proof. First observe that, for $\varphi \in \mathcal{D}_{\tilde{\Omega}}$,

$$\begin{aligned} \|H_{\tilde{\Omega}}^{(m)}\varphi\|_{\tilde{\Omega}}^2 &= \|T\varphi\|_{\tilde{\Omega}}^2 + m^2\|\varphi\|_{\tilde{\Omega}}^2 + 2\operatorname{Re}\langle T\varphi, m\sigma_3\varphi \rangle_{\tilde{\Omega}} \\ &= \|T\varphi\|_{\tilde{\Omega}}^2 + m^2\|\varphi\|_{\tilde{\Omega}}^2 + m\|\operatorname{tr}_{\tilde{\Omega}}\varphi\|_{\partial\tilde{\Omega}}^2. \end{aligned} \quad (23)$$

where in the last step we used the boundary conditions (see Remark 4) and that Green's identity gives, for $\varphi \in H^1(\tilde{\Omega})$,

$$2\operatorname{Re}\langle T\varphi, \sigma_3\varphi \rangle_{\tilde{\Omega}} = i(\langle \sigma \cdot \nabla \varphi, \sigma_3\varphi \rangle_{\tilde{\Omega}} + \langle \varphi, \sigma \cdot \nabla \sigma_3\varphi \rangle_{\tilde{\Omega}}) = \langle \operatorname{tr}_{\tilde{\Omega}}\varphi, B_{\tilde{\Omega}} \operatorname{tr}_{\tilde{\Omega}}\varphi \rangle_{\partial\tilde{\Omega}}.$$

Reasoning as in (18) and using (23) we obtain, for any $\varphi \in \mathcal{D}_{\tilde{\Omega}}$ and $\xi \in S_{\mu_0, \rho}$,

$$\begin{aligned} \|(H_{\tilde{\Omega}}^{(m)} - \xi)\varphi\|_{\tilde{\Omega}}^2 &\geq \frac{1}{2}\|H_{\tilde{\Omega}}^{(m)}\varphi\|_{\tilde{\Omega}}^2 + (\mu_0^2 - 2\rho^2)\|\varphi\|_{\tilde{\Omega}}^2 \\ &= \frac{1}{2}\|T\varphi\|_{\tilde{\Omega}}^2 + \frac{m}{2}\|\operatorname{tr}_{\tilde{\Omega}}\varphi\|_{\partial\tilde{\Omega}}^2 + (\frac{m^2}{2} + \mu_0^2 - 2\rho^2)\|\varphi\|_{\tilde{\Omega}}^2 \\ &\geq \frac{m}{2}\|\operatorname{tr}_{\tilde{\Omega}}\varphi\|_{\partial\tilde{\Omega}}^2 + \frac{m^2}{4}\|\varphi\|_{\tilde{\Omega}}^2, \end{aligned}$$

where in the last step we used the hypothesis on m . We find the bounds (21) and (22) after inserting $\varphi = R_{\tilde{\Omega}}^{(m)}(\xi)h$ in the last inequality. $\square$

The next statement follows from [22, Lemma 4].

Proposition 3.4 (A priori lower bound [22]). *There exist constants $c < \infty$ and $m'_0 > 1$ such that for all $\varphi \in H^1(\mathbb{R}^2, \mathbb{C}^2)$*

$$\|H_m\varphi\|^2 \geq \|\nabla\varphi\|_{\Omega}^2 + m\|P_- \operatorname{tr}_{\Omega}\varphi\|_{\partial\Omega}^2 - c\|\operatorname{tr}_{\Omega}\varphi\|_{\partial\Omega}^2, \quad m > m'_0.$$

Remark 5. This result is stated in [22, Lemma 4] for a bounded and regular domain Ω . However, its proof extends easily to unbounded domains. It also extends to the case in which different components of the boundary are at a certain minimal distance from each other and when we can choose global tubular coordinates with a globally bounded curvature. Hypothesis 1 on the boundary guarantees just that.

For completeness let us quickly review the argument: In view of Lemma 2.2 one can check that, for any $\varphi \in H^1(\mathbb{R}^2, \mathbb{C}^2)$,

$$\begin{aligned} \|H_m\varphi\|^2 &= \|T\varphi\|^2 + m^2\|\varphi\|_{\tilde{\Omega}}^2 + 2m\operatorname{Re}\langle T\varphi, \sigma_3\varphi \rangle_{\tilde{\Omega}} \\ &= \|\nabla\varphi\|^2 + m^2\|\varphi\|_{\tilde{\Omega}}^2 + m(\|P_- \operatorname{tr}\varphi\|_{\partial\Omega}^2 - \|P_+ \operatorname{tr}\varphi\|_{\partial\Omega}^2). \end{aligned}$$

Thus, it suffices to show that, for m large enough,

$$\|\nabla\varphi\|_{\tilde{\Omega}}^2 + m^2\|\varphi\|_{\tilde{\Omega}}^2 \geq (m - c)\|\operatorname{tr}\varphi\|_{\partial\Omega}^2. \quad (24)$$

Recall the definition of δ_0 given in Hypothesis 1. The next step in the proof given in [22] uses a smooth partition of the unity u, v on the set $Q_{\delta_0} = \{x \in \tilde{\Omega} : \operatorname{dist}(x, \partial\Omega) < \delta_0\}$ such that $u^2 + v^2 = 1$ on $\tilde{\Omega}$ and u is supported in Q_{δ_0} with $u(x) = 1$ when $\operatorname{dist}(x, \partial\Omega) < \delta_0/2$. Using the IMS localisation formula one gets, for some $c > 0$,

$$\|\nabla\varphi\|_{\tilde{\Omega}}^2 + m^2\|\varphi\|_{\tilde{\Omega}}^2 \geq \|\nabla(u\varphi)\|^2 + (m^2 - c^2)\|u\varphi\|_{\tilde{\Omega}}^2.$$

The latter expression, locates the problem only on Q_{δ_0} . One can then obtain (24) by expressing the integrals in tubular coordinates and using [22, Lemma 4].

Corollary 3.5. *There exist a constant $c' < \infty$ such that for all $\xi \in S_{\mu_0, \rho}$, $f \in L^2(\mathbb{R}^2, \mathbb{C}^2)$, and all $m > m'_0$*

$$\|P_- \text{tr}_\Omega R_m(\xi) f\|_{\partial\Omega} \leq \frac{c'}{\sqrt{m}} \|f\| \quad (25)$$

$$\|\text{tr}_\Omega R_m(\xi) f\|_{\partial\Omega} \leq c' \|f\|. \quad (26)$$

Proof. We follow the same strategy as in the proof of Lemma 3.2. By Proposition 3.4 and (19) we have, for $\varepsilon^{-1} \geq 1 + c$,

$$\|H_m \varphi\|^2 \geq \|\text{tr}_\Omega \varphi\|_{\partial\Omega}^2 + m \|P_- \text{tr}_\Omega \varphi\|_{\partial\Omega}^2 - \frac{C_\varepsilon}{\varepsilon} \|\varphi\|_\Omega^2.$$

Therefore, $2\|(H_m - \xi)\varphi\|^2 \geq \|\text{tr}_\Omega \varphi\|_{\partial\Omega}^2 + m \|P_- \text{tr}_\Omega \varphi\|_{\partial\Omega}^2 - \tilde{C}_\varepsilon \|\varphi\|^2$, where $\tilde{C}_\varepsilon := |2\mu_0^2 - 4\rho^2 - C_\varepsilon/\varepsilon|$. Thus, with the substitution $\varphi = R_m(\xi)f$, we get that

$$\|\text{tr}_\Omega R_m(\xi) f\|_{\partial\Omega}^2 + m \|P_- \text{tr}_\Omega R_m(\xi) f\|_{\partial\Omega}^2 \leq (2 + \tilde{C}_\varepsilon/\mu_0^2) \|f\|^2,$$

from which follows the claim. $\square$

Proof of Lemma 3.1. Let $\varphi \in C^1(\overline{\Omega'}, \mathbb{C}^2) \cap \mathcal{D}(\Omega')$ be a continuous differentiable spinor. Then, a direct computation yields

$$\|T\varphi\|_{\Omega'}^2 = \|\nabla\varphi\|_{\Omega'}^2 + i \int_{\partial\Omega} (\varphi(\mathbf{x}), \sigma_3(\mathbf{t}_{\Omega'} \cdot \nabla) \varphi(\mathbf{x})) d\omega_{\mathbf{x}}.$$

Assume further that $\varphi \in \mathcal{D}_{\Omega'}$ and write $\partial_{\mathbf{t}} := (\mathbf{t} \cdot \nabla)$ (we skip the index reference to the set Ω'). Then, according to Remark 4, we find that on $\partial\Omega$

$$(\varphi, \sigma_3 \partial_{\mathbf{t}} \varphi) = (\varphi, \sigma_3 (\partial_{\mathbf{t}} B) \varphi) + (\varphi, \sigma_3 B \partial_{\mathbf{t}} \varphi) = (\varphi, \sigma_3 (\partial_{\mathbf{t}} B) \varphi) - (\varphi, \sigma_3 \partial_{\mathbf{t}} \varphi),$$

where in the last inequality we use that $\{B, \sigma_3\} = 0$. This implies that $(\varphi, \sigma_3 \partial_{\mathbf{t}} \varphi) = (\varphi, \sigma_3 (\partial_{\mathbf{t}} B) \varphi)/2$. Moreover, using (14), we get on $\partial\Omega$ that

$$i\sigma_3 \partial_{\mathbf{t}} B = \partial_{\mathbf{t}} \boldsymbol{\sigma} \cdot \mathbf{n} = \kappa B.$$

Thus, for $\varphi \in C^1(\overline{\Omega'}, \mathbb{C}^2) \cap \mathcal{D}_{\Omega'}$,

$$\|T\varphi\|_{\Omega'}^2 = \|\nabla\varphi\|_{\Omega'}^2 + \frac{1}{2} \int_{\partial\Omega} (\varphi(\mathbf{x}), \kappa \varphi(\mathbf{x})) d\omega_{\mathbf{x}}.$$

Using a density argument we can extend the above expression to $\mathcal{D}_{\Omega'}$ obtaining (15). $\square$

4. PROOF OF PROPOSITION 1.2 AND THEOREM 1.3

Proof of Proposition 1.2. We first prove self-adjointness of H_Ω^V . Lemma 3.1 and trace inequalities (2) give, for $\epsilon \in (0, 1)$ and all $\varphi \in \mathcal{D}_\Omega$, with C_ϵ given by (2)

$$\begin{aligned} \|T\varphi\|_\Omega^2 &= \|\nabla\varphi\|_\Omega^2 + \frac{1}{2} \langle \text{tr}_\Omega \varphi, \kappa_\Omega \text{tr}_\Omega \varphi \rangle_{\partial\Omega} \\ &\geq \left(1 - \epsilon \frac{\|\kappa_\Omega\|_\infty}{2}\right) \|\nabla\varphi\|^2 - C_\epsilon \frac{\|\kappa_\Omega\|_\infty}{2} \|\varphi\|_\Omega^2. \end{aligned} \quad (27)$$

Moreover, from Hypothesis 2(i), we have, for $\varphi \in \mathcal{D}_\Omega \subset H^1(\Omega, \mathbb{C}^2)$,

$$\|V\varphi\|_\Omega^2 \leq a\|\nabla\varphi\|_\Omega^2 + b\|\varphi\|_\Omega^2. \quad (28)$$

Picking ϵ small enough, (27)-(28) imply that there exists $a' \in (0, 1)$ and b' such that for all $\varphi \in \mathcal{D}_\Omega$

$$\|V\varphi\|_\Omega^2 \leq a'\|T\varphi\|_\Omega^2 + b'\|\varphi\|_\Omega^2, \quad (29)$$

which proves that H_Ω^V is self-adjoint on $\mathcal{D}_\Omega$.

The proof of self-adjointness of H_m^V is a direct consequence of the next lemma. $\square$

Lemma 4.1. *Assume Hypothesis 2 is fulfilled. Then, for m large enough, V is relatively bounded with respect to H_m , with relative bound less than one. Namely there exists $\tilde{a} \in (0, 1)$ and $\tilde{b} \geq 0$ such that for $m > m_0$ and $\varphi \in H^1(\mathbb{R}^2, \mathbb{C}^2)$,*

$$\|V\varphi\|^2 \leq \tilde{a}\|H_m\varphi\|^2 + \tilde{b}\|\varphi\|^2. \quad (30)$$

Proof. From Proposition 3.4, there exists $c > 0$ and $m_0 > 0$ such that for $m > m_0$ and $\varphi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$, $\|H_m\varphi\|^2 \geq \|\nabla\varphi\|_\Omega^2 + m\|P_- \text{tr}_\Omega \varphi\|_{\partial\Omega}^2 - c\|\text{tr}_\Omega \varphi\|_{\partial\Omega}^2$. Hence using trace inequalities (2), we obtain that for all $\epsilon \in (0, 1)$, there exists $c_\epsilon > 0$ such that for all $\varphi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$

$$\|H_m\varphi\|^2 \geq (1 - \epsilon)\|\nabla\varphi\|_\Omega^2 - c_\epsilon\|\varphi\|_\Omega^2.$$

This implies, from Hypothesis 2(i),

$$\|V\varphi\|_\Omega^2 \leq \frac{a}{1 - \epsilon}\|H_m\varphi\|^2 + \left(\frac{ac_\epsilon}{1 - \epsilon} + b\right)\|\varphi\|_\Omega^2. \quad (31)$$

Now using the Hypothesis 2(ii), implies

$$\|V\varphi\|^2 \leq \|V\varphi\|_\Omega^2 + \|V\|_{L^\infty(\tilde{\Omega})}^2\|\varphi\|^2. \quad (32)$$

In view of estimates (31) and (32), we conclude by picking ϵ small enough so that $\tilde{a} := a/(1 - \epsilon) < 1$. $\square$

We are now equipped to prove Theorem 1.3.

Proof of Theorem 1.3. As in the proof of Theorem 1.1, for fixed constants $\mu_0 > 0$ and $\rho \geq 0$, we consider the set $S_{\mu_0, \rho}$ defined by (12). Pick $\xi \in S_{\mu_0, \rho}$ and denote $R_m^V = R_m^V(\xi)$, $R_\Omega^V = R_\Omega^V(\xi)$, $R_\Omega = R_\Omega(\xi)$ and $R_m = R_m(\xi)$.

From Hypothesis 2(i) and (29), we have, in view of [23, Satz 9], for μ_0 large enough

$$\|R_\Omega V\| < 1. \quad (33)$$

Similarly, with (30), we have for μ_0 large enough

$$\|VR_m\| < 1. \quad (34)$$

Hence, using resolvent formulae for R_m^V and $R_\Omega^V \oplus 0$, we obtain

$$R_m^V - R_\Omega^V \oplus 0 = (1 - R_\Omega V \oplus 0)^{-1}(R_m - R_\Omega \oplus 0)(1 + VR_m^V). \quad (35)$$

In view of Theorem 1.1, using (33)-(35) and the fact that

$$VR_m^V = VR_m(1 + VR_m)^{-1},$$

implies that there exists $\mu_0 > 0$, $m_0 > 1$ and $C > 0$ such that for all $m > m_0$,

$$\sup_{\xi \in S_{\mu_0, \rho}} \|R_m^V(\xi) - R_\Omega^V(\xi) \oplus 0\| \leq \frac{C}{\sqrt{m}}.$$

The result then follows from Lemma A.1 that remains true replacing H_m and H_Ω respectively by H_m^V and H_Ω^V and replacing in the statement Theorem 1.1 by Theorem 1.3. $\square$

APPENDIX A. SUFFICIENT CONDITION

For the next lemma consider the setting of Theorem 1.1. It gives a sufficient condition to control the supremum norm in (8).

Lemma A.1. *Let $K \subset \varrho(H_\Omega)$ be a compact set. For constants $\mu_0 \in (\max_{z \in K} |\operatorname{Im}(z)|, \infty)$ and $\rho \in (0, \max_{z \in K} |\operatorname{Re}(z)|)$ recall the definition of the set $S_{\mu_0, \rho}$ from (12). Assume that for some $C' < \infty$ and $m'_0 > 1$ we have that*

$$\sup_{z \in S_{\mu_0, \rho}} \|R_m(z) - R_\Omega(z) \oplus 0\| \leq \frac{C'}{\sqrt{m}}, \quad (36)$$

for all $m > m'_0$. Then, the statement of Theorem 1.1 holds for some $m_0 > m'_0$.

Proof. Assuming (36) we will show Theorem 1.1, separately, in the two compact regions $K^+ := \{z \in K : \operatorname{Im}(z) \geq 0\}$ and $K^- := \{z \in K : \operatorname{Im}(z) \leq 0\}$. We define the connection between the sets K^+ and $S_{\mu_0, \rho}$ through the map

$$K^+ \ni z \mapsto \xi_z := \operatorname{Re}(z) + i\mu_0 \in S_{\mu_0, \rho}.$$

We first show that $K^+ \subset \varrho(H_m)$ for sufficiently large $m > 1$. For $\varphi \in H^1(\mathbb{R}^2, \mathbb{C}^2)$ and $z \in K^+$ we have

$$(H_m - z)\varphi = (H_m - \xi_z)\varphi + (\xi_z - z)R_m(\xi_z)(H_m - \xi_z)\varphi \quad (37)$$

(above and along this proof we simply write R_Ω for $R_\Omega \oplus 0$).

It is easy to verify that the function

$$\|(\xi_z - z)R_\Omega(\xi_z)\| = \frac{\mu_0 - \operatorname{Im}(z)}{\operatorname{dist}(\xi_z, \operatorname{spec}(H_\Omega))}, \quad z \in K^+, \quad (38)$$

is continuous and strictly smaller than 1 (here $\operatorname{dist}(\cdot, \cdot)$ denotes the distance from a point to a set). Thus, since K^+ is compact, there is a $\delta \in (0, 1)$ such that

$$\sup_{z \in K^+} \|(z - \xi_z)R_\Omega(\xi_z)\| = 1 - \delta. \quad (39)$$

Writing $(\xi_z - z)R_m(\xi_z) = (\xi_z - z)R_\Omega(\xi_z) + (\xi_z - z)(R_m(\xi_z) - R_\Omega(\xi_z))$ we see that, using (39) and (36), we may pick m so large that

$$\|(\xi_z - z)R_m(\xi_z)\| \leq 1 - \delta + \frac{2\mu_0 C'}{\sqrt{m}} \leq 1 - \delta/2, \quad (40)$$

uniformly for $z \in K^+$. This, together with (37), implies that

$$\|(H_m - z)\varphi\| \geq \delta/2 \|(H_m - \xi_z)\varphi\| \geq \mu_0 \delta/2 \|\varphi\|. \quad (41)$$

Therefore, $z \in \varrho(H_m)$ for m sufficiently large. One can show analogously that $K^- \subset \varrho(H_m)$ for m large enough, but this time using the connection $K^- \ni z \mapsto \operatorname{Re}(z) - i\mu_0 \in S_{\mu_0, \rho}$.

Next, we show that the limit in (8) holds for K replaced by K^+ . Let m be so large that (40) holds. Then, we have that $K^+ \subset \varrho(H_m)$ and, for any $z \in K^+$, we may use the first resolvent identity to get

$$\begin{aligned} R_\Omega(z) - R_m(z) &= R_\Omega(\xi_z)(1 + (\xi_z - z)R_\Omega(\xi_z))^{-1} - R_m(\xi_z)(1 + (\xi_z - z)R_m(\xi_z))^{-1} \\ &=: R_\Omega(\xi_z)M_\Omega(z) - R_m(\xi_z)M_m(z). \end{aligned}$$

Notice that $M_m(z) - M_\Omega(z) = (\xi_z - z)M_m(z)(R_\Omega(\xi_z) - R_m(\xi_z))M_\Omega(z)$. Thus, in view of (39) and (40), we get that

$$\begin{aligned} \|R_\Omega(z) - R_m(z)\| &\leq \|R_m(\xi_z) - R_\Omega(\xi_z)\| \|M_m(z)\| + \|R_\Omega(\xi_z)\| \|M_m(z) - M_\Omega(z)\| \\ &\leq \frac{2C''}{\sqrt{m}\delta} + \frac{4C''}{\sqrt{m}\delta^2}. \end{aligned}$$

This proves the claim for K^+ . For K^- one proceeds analogously with the aforementioned connection. This finishes the proof of the lemma. $\square$

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JEAN-MARIE BARBAROUX, AIX MARSEILLE UNIV, UNIVERSITÉ DE TOULON, CNRS, CPT, MARSEILLE, FRANCE.

E-mail address: barbarou@univ-tln.fr

HORIA CORNEAN, DEPARTMENT OF MATHEMATICAL SCIENCES, AALBORG UNIVERSITY, FREDRIK BAJERS VEJ 7G, 9220 AALBORG Ø, DENMARK.

E-mail address: cornean@math.aau.dk

LOÏC LE TREUST, AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE.

E-mail address: loic.le-treust@univ-amu.fr

EDGARDO STOCKMEYER, INSTITUTO DE FÍSICA, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, VICUÑA MACKENNA 4860, SANTIAGO 7820436, CHILE.

E-mail address: stock@fis.puc.cl