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# DYNAMICAL PAIRS WITH AN ABSOLUTELY CONTINUOUS BIFURCATION MEASURE

by

Thomas Gauthier

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**Abstract.** — In this article, we study algebraic dynamical pairs  $(f, a)$  parametrized by an irreducible quasi-projective curve  $\Lambda$  having an absolutely continuous bifurcation measure. We prove that, if  $f$  is non-isotrivial and  $(f, a)$  is unstable, this is equivalent to the fact that  $f$  is a family of Lattès maps. To do so, we prove the density of transversely prerepelling parameters in the bifurcation locus of  $(f, a)$  and a similarity property, at any transversely prerepelling parameter  $\lambda_0$ , between the measure  $\mu_{f,a}$  and the maximal entropy measure of  $f_{\lambda_0}$ . We also establish an equivalent result for dynamical pairs of  $\mathbb{P}^k$ , under an additional assumption.

## Introduction

Let  $\Lambda$  be an irreducible quasi-projective complex curve. An algebraic *dynamical pair*  $(f, a)$  parametrized by  $\Lambda$  is an algebraic family  $f : \Lambda \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  of rational maps of degree  $d \geq 2$ , i.e.  $f$  is a morphism and  $f_\lambda$  is a degree  $d$  rational map for all  $\lambda \in \Lambda$ , together with a marked point  $a$ , i.e. a morphism  $a : \Lambda \rightarrow \mathbb{P}^1$ .

Recall that a dynamical pair  $(f, a)$  is *stable* if the sequence  $\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_{n \geq 1}$  is a normal family on  $\Lambda$ . Otherwise, we say that the pair  $(f, a)$  is *unstable*. Recall also that  $f$  is *isotrivial* if there exists a branched cover  $X \rightarrow \Lambda$  and an algebraic family of Möbius transformations  $M : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  so that  $M_\lambda \circ f_\lambda \circ M_\lambda^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is independent of the parameter  $\lambda$  and that the pair  $(f, a)$  is *isotrivial* if, in addition,  $M_\lambda(a(\lambda))$  is also independent of the parameter  $\lambda$ . A result of DeMarco [De] states that any stable algebraic pair is either isotrivial or preperiodic, i.e. there exists  $n > m \geq 0$  such that  $f_\lambda^n(a(\lambda)) = f_\lambda^m(a(\lambda))$  for all  $\lambda \in \Lambda$ .

When a dynamical pair  $(f, a)$  is unstable, the *stability locus*  $\text{Stab}(f, a)$  is the set of points  $\lambda_0 \in \Lambda$  admitting a neighborhood  $U$  on which the pair  $(f, a)$  the sequence  $\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_{n \geq 1}$  is a normal family. The *bifurcation locus*  $\text{Bif}(f, a)$  of the pair  $(f, a)$  is its complement  $\text{Bif}(f, a) := \Lambda \setminus \text{Stab}(f, a)$ . If  $a$  is the marking of a critical point, i.e.  $f'_\lambda(a(\lambda)) = 0$  for all  $\lambda \in \Lambda$ , it is classical that the bifurcation locus  $\text{Bif}(f, a)$  has empty interior, [MSS].

The bifurcation locus of a pair  $(f, a)$  is the support of natural a positive (finite) measure: the *bifurcation measure*  $\mu_{f,a}$  of the pair  $(f, a)$ , see Section 1 for a precise definition. The properties of this measure appear to be very important for studying arithmetic and dynamical properties of the pair  $(f, a)$ , see e.g. [BD1, BD2, De, DM, DMWY, FG1, FG2, FG3]. Note also that the entropy theory of dynamical pairs has been recently developed in [DGV]. In the present article, we study algebraic dynamical pairs having an absolutely continuous bifurcation measure.

Assume that for some parameter  $\lambda_0 \in \Lambda$ , the marked point  $a$  eventually lands on a repelling periodic point  $x$ , that is  $f_{\lambda_0}^n(a(\lambda_0)) = x$ . Let  $x(\lambda)$  is the (local) natural

continuation of  $x$  as a periodic point of  $f_\lambda$ . We say that  $a$  is *transversely prerepelling* at  $\lambda_0$  if the graphs of  $\lambda \mapsto f_\lambda^n(a(\lambda))$  and  $\lambda \mapsto x(\lambda)$  are transverse at  $\lambda_0$ .

Finally, recall that a rational map  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a *Lattès map* if there exists an elliptic curve  $E$ , an endomorphism  $L : E \rightarrow E$  and a finite branched cover  $p : E \rightarrow \mathbb{P}^1$  such that  $p \circ L = f \circ p$  on  $E$ . Such a map has an absolutely continuous maximal entropy measure, see [Z]. On the other hand, when  $f$  is a family of Lattès maps and the pair  $(f, a)$  is unstable, then  $\text{Bif}(f, a) = \Lambda$ , see e.g. [DM, §6].

Our main result is the following.

**Theorem A.** — *Let  $(f, a)$  be a dynamical pair of  $\mathbb{P}^1$  of degree  $d \geq 2$  parametrized by an irreducible quasi-projective curve  $\Lambda$ . Assume that  $f$  is non-isotrivial and that  $(f, a)$  is unstable. The following assertions are equivalent:*

1. *The bifurcation locus of the dynamical pair  $(f, a)$  is  $\text{Bif}(f, a) = \Lambda$ ,*
2. *Transversely prerepelling parameters are dense in  $\Lambda$ ,*
3. *The bifurcation measure  $\mu_{f,a}$  of the pair  $(f, a)$  is absolutely continuous,*
4. *The family  $f$  is a family of Lattès maps.*

Note that the hypothesis that  $f$  is not isotrivial is necessary to have the equivalence between 1. and 4. (see Proposition 4.3).

The first step of the proof consists in proving that transversely prerepelling parameters are dense in the support of  $\mu_{f,a}$ . Using properties of Polynomial-Like Maps in higher dimension and a transversality Theorem of Dujardin for laminar currents [Duj], under a mild assumption on Lyapunov exponents, we prove this property holds for the appropriate bifurcation current for any tuple  $(f, a_1, \dots, a_m)$ , where  $f : \Lambda \times \mathbb{P}^k \rightarrow \mathbb{P}^k$  is any holomorphic family of endomorphisms of  $\mathbb{P}^k$  and  $a_1, \dots, a_m : \Lambda \rightarrow \mathbb{P}^k$  are any marked points.

In a second time, we adapt the similarity argument of Tan Lei to show that, if  $\lambda_0$  is a transversely prerepelling parameter where the bifurcation measure is absolutely continuous, the maximal entropy measure  $\mu_{f_{\lambda_0}}$  of  $f_{\lambda_0}$  is also non-singular with respect to the Fubini-Study form on  $\mathbb{P}^1$ . As Zdunik [Z] has shown, this implies  $f_{\lambda_0}$  is a Lattès map.

We can see Theorem A as a partial parametric counterpart of Zdunik's result. However, the comparison with Zdunik's work ends there: Rational maps with  $\mathbb{P}^1$  as a Julia sets are, in general, not Lattès maps. Indeed, Lattès maps form a strict subvariety of the space of all degree  $d$  rational maps, and maps with  $J_f = \mathbb{P}^1$  form a set of positive volume by [R]. In a way, Theorem A is a stronger rigidity statement than the dynamical one.

Recall that an endomorphism of  $\mathbb{P}^k$  of degree  $d \geq 2$  has a unique maximal entropy measure  $\mu_f$  which Lyapunov exponents  $\chi_1, \dots, \chi_k$  all satisfy  $\chi_j \geq \frac{1}{2} \log d$ , by [BD4]. We say that the Lyapunov exponents of  $f$  *resonate* if there exists  $1 \leq i < j \leq k$  and an integer  $q \geq 2$  such that  $\chi_i = q\chi_j$ .

Recall also that, as in dimension 1, an endomorphism  $f$  of  $\mathbb{P}^k$  is a *Lattès map* if there exists an abelian variety  $A$ , a finite branched cover  $p : A \rightarrow \mathbb{P}^k$  and an isogeny  $I : A \rightarrow A$  such that  $p \circ I = f \circ p$  on  $A$ . Berteloot and Dupont [BD3] generalized Zdunik's work to endomorphisms of  $\mathbb{P}^k$ :  $f$  is a Lattès map of  $\mathbb{P}^k$  if and only if the measure  $\mu_f$  is not singular with respect to  $\omega_{\mathbb{P}^k}^k$ , see also [Dup].

As an important part of our arguments applies in any dimension, we have the following.

**Theorem B.** — *Fix integers  $d \geq 2$  and  $k \geq 1$  and let  $(f, a)$  be any holomorphic dynamical pair of degree  $d$  of  $\mathbb{P}^k$  parametrized by a Kähler manifold  $(M, \omega)$  of dimension  $k$ . Assume that for all  $\lambda \in \mathbb{B}$ , any  $J$ -repelling periodic point of  $f_\lambda$  is linearizable and that the Lyapunov exponents of  $f_\lambda$  do not resonate for all  $\lambda \in M$ . Assume in addition that  $\mu_{f,a} := T_{f,a}^k$  satisfies  $\text{supp}(\mu_{f,a}) = M$ .*

*Then  $\mu_{f,a}$  is absolutely continuous with respect to  $\omega^k$  if and only if  $f$  is a family of Lattès maps of  $\mathbb{P}^k$ .*

The paper is organised as follows. In section 1, we recall the construction of the bifurcation currents of marked points and properties of Polynomial-Like Maps. Section 2 is dedicated to proving the density of transversely prerepelling parameters. In section 3, we establish the similarity property for the bifurcation and maximal entropy measures. Finally, in section 4 we prove Theorem A and B and list related questions.

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## 1. Dynamical preliminaries

### 1.1. The bifurcation current of a dynamical tuple

For this section, we follow the presentation of [DF, Duj]. Even though everything is presented in the case  $k = 1$  and for marked *critical* points, the exact same arguments give what we present below.

Let  $\Lambda$  be a complex manifold and let  $f : \Lambda \times \mathbb{P}^k \rightarrow \mathbb{P}^k$  be a holomorphic family of endomorphisms of  $\mathbb{P}^k$  of algebraic degree  $d \geq 2$ :  $f$  is holomorphic and  $f_\lambda := f(\lambda, \cdot) : \mathbb{P}^k \rightarrow \mathbb{P}^k$  is an endomorphism of algebraic degree  $d$ .

**Definition 1.1.** — *Fix integers  $m \geq 1$ ,  $d \geq 2$  and let  $\Lambda$  be a complex manifold. A dynamical  $(m + 1)$ -tuple  $(f, a_1, \dots, a_m)$  of  $\mathbb{P}^k$  of degree  $d$  parametrized by  $\Lambda$  is a holomorphic family  $f$  of endomorphisms of  $\mathbb{P}^k$  of degree  $d$  parametrized by  $\Lambda$ , endowed with  $m$  holomorphic maps (marked points)  $a_1, \dots, a_m : \Lambda \rightarrow \mathbb{P}^k$ .*

Let  $\omega_{\mathbb{P}^k}$  be the standard Fubini-Study form on  $\mathbb{P}^k$  and  $\pi_\Lambda : \Lambda \times \mathbb{P}^k \rightarrow \Lambda$  and  $\pi_{\mathbb{P}^k} : \Lambda \times \mathbb{P}^k \rightarrow \mathbb{P}^k$  be the canonical projections. Finally, let  $\widehat{\omega} := (\pi_{\mathbb{P}^k})^* \omega_{\mathbb{P}^k}$ . A family  $f : \Lambda \times \mathbb{P}^k \rightarrow \mathbb{P}^k$  naturally induces a fibered dynamical system  $f : \Lambda \times \mathbb{P}^k \rightarrow \Lambda \times \mathbb{P}^k$ , given by  $\widehat{f}(\lambda, z) := (\lambda, f_\lambda(z))$ . It is known that the sequence  $d^{-n}(\widehat{f}^n)^* \widehat{\omega}$  converges to a closed positive  $(1, 1)$ -current  $\widehat{T}$  on  $\Lambda \times \mathbb{P}^k$  with continuous potential. Moreover, for any  $1 \leq j \leq k$ ,

$$\widehat{f}^* \widehat{T}^j = d^j \cdot \widehat{T}^j$$

and  $\widehat{T}^k|_{\{\lambda_0\} \times \mathbb{P}^1} = \mu_{\lambda_0}$  is the unique measure of maximal entropy  $k \log d$  of  $f_{\lambda_0}$  for all  $\lambda_0 \in \Lambda$ .

For any  $n \geq 1$ , we have  $\widehat{T} = d^{-n}(\widehat{f}^n)^* \widehat{\omega} + d^{-n} dd^c \widehat{u}_n$ , where  $(\widehat{u}_n)_n$  is a locally uniformly bounded sequence of continuous functions.

Pick now a dynamical  $(m + 1)$ -tuple  $(f, a_1, \dots, a_m)$  of degree  $d$  of  $\mathbb{P}^k$ . Let  $\Gamma_{a_j}$  be the graph of the map  $a_j$  and set

$$\mathbf{a} := (a_1, \dots, a_m).$$

**Definition 1.2.** — For  $1 \leq i \leq m$ , the bifurcation current  $T_{f,a_i}$  of the pair  $(f, a_i)$  is the closed positive  $(1, 1)$ -current on  $\Lambda$  defined by

$$T_{f,a_i} := (\pi_\Lambda)_* \left( \widehat{T} \wedge [\Gamma_{a_j}] \right)$$

and we define the bifurcation current  $T_{f,\mathbf{a}}$  of the  $(m+1)$ -tuple  $(f, a_1, \dots, a_m)$  as

$$T_{f,\mathbf{a}} := T_{f,a_1} + \dots + T_{f,a_k}.$$

For any  $\ell \geq 0$ , write

$$\mathbf{a}_\ell(\lambda) := \left( f_\lambda^\ell(a_1(\lambda)), \dots, f_\lambda^\ell(a_m(\lambda)) \right), \quad \lambda \in \Lambda.$$

Let now  $K \Subset \Lambda$  be a compact subset of  $\Lambda$  and let  $\Omega$  be some compact neighborhood of  $K$ , then  $(a_\ell)^*(\omega_{\mathbb{P}^k})$  is bounded in mass in  $\Omega$  by  $Cd^\ell$ , where  $C$  depends on  $\Omega$  but not on  $\ell$ .

Applying verbatim the proof of [DF, Proposition-Definition 3.1], we have the following.

**Lemma 1.3.** — For any  $1 \leq i \leq k$ , the support of  $T_{f,a_i}$  is the set of parameters  $\lambda_0 \in \Lambda$  such that the sequence  $\{\lambda \mapsto f_\lambda^n(a_i(\lambda))\}$  is not a normal family at  $\lambda_0$ .

Moreover, writing  $a_{i,\ell}(\lambda) := f_\lambda^\ell(a_i(\lambda))$ , there exists a locally uniformly bounded family  $(u_{i,\ell})$  of continuous functions on  $\Lambda$  such that

$$(a_{i,\ell})^*(\omega_{\mathbb{P}^q}) = d^\ell T_{f,a_i} + dd^c u_{i,\ell} \text{ on } \Lambda.$$

As a consequence, for all  $j \geq 1$ , we have

$$(a_{i,\ell})^*(\omega_{\mathbb{P}^k}^j) = d^{j\ell} T_{f,a_i}^j + dd^c O(d^{(j-1)\ell})$$

on compact subsets of  $\Lambda$ . In particular, one sees that

$$(1) \quad T_{f,a_i}^{k+1} = 0 \text{ on } \Lambda.$$

Let us still denote  $\pi_\Lambda : \Lambda \times (\mathbb{P}^k)^m \rightarrow \Lambda$  be the projection onto the first coordinate and for  $1 \leq i \leq k$ , let  $\pi_i : \Lambda \times (\mathbb{P}^k)^m \rightarrow \mathbb{P}^k$  be the projection onto the  $i$ -th factor of the product  $(\mathbb{P}^k)^m$ . Finally, we denote by  $\Gamma_{\mathbf{a}}$  the graph of  $\mathbf{a}$ :

$$\Gamma_{\mathbf{a}} := \{(\lambda, z_1, \dots, z_m), \forall j, z_j = a_j(\lambda)\} \subset \Lambda \times (\mathbb{P}^k)^m.$$

Following verbatim the proof of [AGMV, Lemma 2.6], we get

$$\frac{1}{(mk)!} T_{f,\mathbf{a}}^{mk} = \bigwedge_{\ell=1}^m T_{f,a_\ell}^k = (\pi_\Lambda)_* \left( \bigwedge_{i=1}^m \pi_i^* \left( \widehat{T}^k \right) \wedge [\Gamma_{\mathbf{a}}] \right).$$

## 1.2. Hyperbolic sets supporting a PLB ergodic measure

**Definition 1.4.** — Let  $W \subset \mathbb{C}^k$  be a bounded open set. We say that a positive measure  $\nu$  compactly supported on  $W$  is PLB if the psh functions on  $W$  are integrable with respect to  $\nu$ .

We aim here at proving the following lemma in the spirit of [Duj, Lemma 4.1]:

**Proposition 1.5.** — Pick an endomorphism  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  of degree  $d \geq 2$  which Lyapunov exponents don't resonate. There exists an integer  $m \geq 1$  a  $f^m$ -invariant compact set  $K$  contained in a small ball  $\mathbb{B} \subset \mathbb{P}^k$  and an integer  $N \geq 2$  such that

- $f^m|_K$  is uniformly expanding and repelling periodic points of  $f^m$  are dense in  $K$ ,

- there exists a unique probability measure  $\nu$  supported on  $K$  such that  $(f^m|_K)^*\nu = N\nu$  which is PLB.

To prove Proposition 1.5, we need the notion of *polynomial-like map*. We refer to [DS]. Given an complex manifold  $M$  and an open set  $V \subset M$ , we say that  $V$  is *S-convex* if there exists a continuous strictly plurisubharmonic function on  $V$ . In fact, this implies that there exists a smooth strictly psh function  $\psi$ , whence there exists a Kähler form  $\omega := dd^c\psi$  on  $V$ .

**Definition 1.6.** — *Given a connected S-convex open set and a relatively compact open set  $U \Subset V$ , a map  $f : U \rightarrow V$  is polynomial-like if  $f$  is holomorphic and proper.*

The *filled-Julia set* of  $f$  is the set

$$\mathcal{K}_f := \bigcap_{n \geq 0} f^{-n}(U).$$

The set  $\mathcal{K}_f$  is full, compact, non-empty and it is the largest totally invariant compact subset of  $V$ , i.e. such that  $f^{-1}(\mathcal{K}_f) = \mathcal{K}_f$ .

The topological degree  $d_t$  of  $f$  is the number of preimages of any  $z \in V$  by  $f$ , counted with multiplicity. Let  $k := \dim V$ . We define

$$d_{k-1}^* := \sup_{\varphi} \left\{ d_t \cdot \limsup_{n \rightarrow \infty} \|\Lambda^n dd^c \varphi\|_U^{1/n}; \varphi \text{ is psh on } V \right\},$$

where  $\Lambda := d_t^{-1}f_*$ . According to Theorem 3.2.1 and Theorem 3.9.5 of [DS], we have the following.

**Theorem 1.7 (Dinh-Sibony).** — *Let  $f : U \rightarrow V$  be a polynomial-like map of topological degree  $d_t \geq 2$ . There exists a unique probability measure  $\mu$  supported by  $\partial\mathcal{K}_f$  which is ergodic and such that*

1. for any volume form  $\Omega$  of mass 1 in  $L^2(V)$ , one has  $d_t^{-n}(f^n)^*\Omega \rightarrow \mu$  as  $n \rightarrow \infty$ ,
2. if  $d_{k-1}^* < d_t$ , the measure  $\mu$  is PLB and repelling periodic points are dense in  $\text{supp}(\mu)$ .

*Proof of Proposition 1.5.* — First, we fix an open set  $O \subset \mathbb{P}^k$  with  $\mu_f(O) > 0$ . According to [BB, Proposition 3.2], there exists a ball  $B \subset O$ , a constant  $C > 0$  depending only on  $O$  and  $n_0 \geq 1$  such that for all  $n \geq n_0$ ,  $f^n$  has  $M(n) \geq Cd^{mk}$  inverse branches  $g_1, \dots, g_{M(n)}$  defined on  $B$  with

- $g_i(B) \Subset B$  and  $g_i$  is uniformly contracting on  $B$  for all  $i$ ,
- $g_i(B) \cap g_j(B) = \emptyset$  for all  $i \neq j$ .

Fix  $m \geq n_0$  large enough so that  $Cd^{mk} > d^{(k-1)m} \geq 2$  and set

$$V := B, \quad U := \bigcup_{j=1}^{M(m)} g_j(B), \quad N := M(m) \quad \text{and} \quad g := f^m|_U.$$

The map  $g : U \rightarrow V$  is polynomial-like of topological degree  $N$ , whence its equilibrium measure  $\nu$  is the unique probability measure which satisfies  $g^*\nu = N\nu$  by the first part of Theorem 1.7. We let  $K := \text{supp}(\nu)$ . Since the  $g_i$ 's are uniformly contracting, the compact set  $K$  is  $f^m$ -hyperbolic.

To conclude, it is sufficient to verify that  $N > d_{k-1}^*$ . Fix  $n \geq 1$  and  $\varphi$  psh on  $V$ . Let  $\omega$  be the (normalized) restriction to  $V$  of the Fubini-Study form of  $\mathbb{P}^k$ . Then

$$\begin{aligned} \|\Lambda^n(dd^c\varphi)\|_U &= \int_U (\Lambda^n(dd^c\varphi)) \wedge \omega^{k-1} = \int_U \frac{1}{N^n} ((g^n)_*(dd^c\varphi)) \wedge \omega^{k-1} \\ &= \frac{1}{N^n} \int_U dd^c\varphi \wedge (g^n)^*\omega^{k-1} = \frac{1}{N^n} \int_U dd^c\varphi \wedge (d^{mn}\omega + dd^c u_{nm})^{k-1} \end{aligned}$$

where  $(u_n)_n$  is a uniformly bounded sequence of continuous functions on  $\mathbb{P}^k$ . In particular, by the Chern-Levine-Niremberg inequality, if  $U \Subset W \Subset V$ , there exists a constant  $C' > 0$  depending only on  $W$  such that

$$\begin{aligned} \|\Lambda^n(dd^c\varphi)\|_U &= \left(\frac{d^{(k-1)m}}{N}\right)^n \int_U dd^c\varphi \wedge (\omega + d^{-nm}dd^c u_{nm})^{k-1} \\ &\leq \left(\frac{d^{(k-1)m}}{N}\right)^n C' \|dd^c\varphi\|_W. \end{aligned}$$

Taking the  $n$ -th root and passing to the limit, we get

$$\frac{d_{k-1}^*}{N} \leq \frac{d^{(k-1)m}}{N} < 1$$

by assumption. The second part of Theorem 1.7 allows us to conclude.  $\square$

## 2. The support of bifurcation currents

Pick a complex manifold  $\Lambda$  and let  $m, k \geq 1$  be so that  $\dim \Lambda \geq km$ . Let  $(f, a_1, \dots, a_m)$  be a dynamical  $(m+1)$ -tuple of  $\mathbb{P}^k$  of degree  $d$  parametrized by  $\Lambda$ .

**Definition 2.1.** — *We say that  $a_1, \dots, a_m$  are transversely  $J$ -prerepelling (resp. properly  $J$ -prerepelling) at a parameter  $\lambda_0$  if there exists integers  $n_1, \dots, n_m \geq 1$  such that  $f_{\lambda_0}^{n_j}(a_j(\lambda_0)) = z_j$  is a repelling periodic point of  $f_{\lambda_0}$  and, if  $z_j(\lambda)$  is the natural continuation of  $z_j$  as a repelling periodic point of  $f_\lambda$  in a neighborhood  $U$  of  $\lambda_0$ , such that*

1.  $z_j(\lambda) \in J_\lambda$  for all  $\lambda \in U$  and all  $1 \leq j \leq m$ ,
2. the graphs of  $A : \lambda \mapsto (f_\lambda^{q_1}(a_1(\lambda)), \dots, f_\lambda^{q_m}(a_m(\lambda)))$  and of  $Z : \lambda \mapsto (z_1(\lambda), \dots, z_m(\lambda))$  intersect transversely (resp. properly) at  $\lambda_0$ .

Recall that Lyapunov exponents of an endomorphism  $f$  resonate if there exists  $1 \leq i < j \leq k$  and an integer  $q \geq 2$  such that  $\chi_i = q\chi_j$ , see 1.2. In this section, we prove

**Theorem 2.2.** — *Let  $(f, a_1, \dots, a_m)$  be a dynamical  $(m+1)$ -tuple of  $\mathbb{P}^k$  of degree  $d$  parametrized by  $\Lambda$  with  $km \leq \dim \Lambda$ . Assume that the Lyapunov exponent of  $f_\lambda$  don't resonate for all  $\lambda \in \Lambda$ .*

*Then the support of  $T_{f, a_1}^k \wedge \dots \wedge T_{f, a_m}^k$  coincides with the closure of the set of parameters at which  $a_1, \dots, a_m$  are transversely  $J$ -prerepelling.*

**Remark.** — The hypothesis on the Lyapunov exponents is used only to prove the density of transversely prerepelling parameters and we think it is only a technical artefact.

## 2.1. Properly prerepelling marked points bifurcate

In a first time, we give a quick proof of the fact that properly  $J$ -prerepelling parameters belong to the support of  $T_{f,a_1}^k \wedge \cdots \wedge T_{f,a_m}^k$ , without any additional assumption.

**Theorem 2.3.** — *Let  $(f, a_1, \dots, a_m)$  be a dynamical  $(m + 1)$ -tuple of  $\mathbb{P}^k$  of degree  $d$  parametrized by  $\Lambda$  with  $km \leq \dim \Lambda$ . Pick any parameter  $\lambda_0 \in \Lambda$  such that  $a_1, \dots, a_m$  are properly  $J$ -prerepelling at  $\lambda_0$ . Then  $\lambda_0 \in \text{supp} \left( T_{f,a_1}^k \wedge \cdots \wedge T_{f,a_m}^k \right)$ .*

The proof of this result is an adaptation of the strategy of Buff and Epstein [BE] and the strategy of Berteloot, Bianchi and Dupont [BBD], see also [G, AGMV]. Since it follows closely that of [AGMV, Theorem B], we shorten some parts of the proof.

Before giving the proof of Theorem 2.3, remark that our properness assumption is equivalent to saying that the local hypersurfaces

$$X_j := \{ \lambda \in \Lambda ; f_\lambda^{q_j}(a_j(\lambda)) = z_j(\lambda) \}$$

intersecting at  $\lambda_0$  satisfy  $\text{codim} \left( \bigcap_j X_j \right) = km$ .

*Proof of Theorem 2.3.* — According to [G, Lemma 6.3], we can reduce to the case when  $\Lambda$  is an open set of  $\mathbb{C}^{km}$ . Take a small ball  $B$  centered at  $\lambda_0$  in  $\Lambda$ . Up to reducing  $B$ , we can assume  $z_j(\lambda)$  can be followed as a repelling periodic point of  $f_\lambda$  for all  $\lambda \in B$ . Up to reducing  $B$ , our assumption is equivalent to the fact that  $\bigcap_j X_j = \{ \lambda_0 \}$ .

We let  $\mu := T_{f,a_1}^k \wedge \cdots \wedge T_{f,a_m}^k$ . Our aim here is to exhibit a basis of neighborhood  $\{ \Omega_n \}_n$  of  $\lambda_0$  in  $\mathbb{B}$  with  $\mu(\Omega_n) > 0$  for all  $n$ . For any  $m$ -tuple  $\underline{n} := (n_1, \dots, n_m) \in (\mathbb{N}^*)^m$ , we let

$$\begin{aligned} F_{\underline{n}} : \Lambda \times (\mathbb{P}^k)^m &\longrightarrow \Lambda \times (\mathbb{P}^k)^m \\ (\lambda, z_1, \dots, z_m) &\longmapsto (\lambda, f_\lambda^{n_1}(z_1), \dots, f_\lambda^{n_m}(z_m)) . \end{aligned}$$

For a  $m$ -tuple  $\underline{n} = (n_1, \dots, n_m)$  of positive integers, we set

$$|\underline{n}| := n_1 + \cdots + n_m .$$

We also denote

$$\mathfrak{A}_{\underline{n}}(\lambda) := (f_\lambda^{n_1}(a_1(\lambda)), \dots, f_\lambda^{n_m}(a_m(\lambda))), \lambda \in \Lambda .$$

As in [AGMV], we have the following.

**Lemma 2.4.** — *For any  $m$ -tuple  $\underline{n} = (n_1, \dots, n_m)$  of positive integers, we let  $\Gamma_{\underline{n}}$  be the graph in  $\Lambda \times (\mathbb{P}^k)^m$  of  $\mathfrak{A}_{\underline{n}}$ . Then, for any Borel set  $B \subset \Lambda$ , we have*

$$\mu(B) = d^{-k \cdot |\underline{n}|} \int_{B \times (\mathbb{P}^k)^m} \left( \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) \right) \wedge [\Gamma_{\underline{n}}] .$$

Suppose that the point  $z_j$  is  $r_j$ -periodic. For the sake of simplicity, we let in the sequel  $\mathfrak{A}_{\underline{n}} := \mathfrak{A}_{\underline{q} + n\underline{r}}$ , where  $\underline{q} = (q_1, \dots, q_m)$ ,  $\underline{r} = (r_1, \dots, r_m)$  are given as above and  $\underline{q} + n\underline{r} = (q_1 + nr_1, \dots, q_m + nr_m)$ . Again as above, we let  $\Gamma_n$  be the graph of  $\mathfrak{A}_n$ .



Let  $z := (z_1, \dots, z_m)$  and fix any small open neighborhood  $\Omega$  of  $\lambda_0$  in  $\Lambda$ . Set

$$I_n := \int_{\Omega \times (\mathbb{P}^k)^m} \left( \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) \right) \wedge [\Gamma_n],$$

and let  $\delta$  be given by the expansion condition as above. Let  $S_n$  be the connected component of  $\Gamma_n \cap \Lambda \times \mathbb{B}_\delta^m(z)$  containing  $(\lambda_0, z)$ . Since  $z_j(\lambda)$  is repelling and periodic for  $f_\lambda$  for all  $\lambda \in B$  (if  $B$  has been chosen small enough), there exists a constant  $K > 1$  such that

$$d_{\mathbb{P}^k}(f_\lambda^{r_j}(z), f_\lambda^{r_j}(w)) \geq K \cdot d_{\mathbb{P}^k}(z, w)$$

for all  $(z, w) \in \mathbb{B}(z_j(\lambda_0), \epsilon)$  and all  $\lambda \in B$  for some given  $\epsilon > 0$ . In particular, the current  $[S_n]$  is vertical-like in  $\Lambda \times \mathbb{B}_\delta^m(z)$  and there exists  $n_0 \geq 1$  and a basis of neighborhood  $\Omega_n$  of  $\lambda_0$  in  $\Lambda$  such that

$$\text{supp}([S_n]) = S_n \subset \Omega_n \times \mathbb{B}_\delta^m(z),$$

for all  $n \geq n_0$ .

Let  $S$  be any weak limit of the sequence  $[S_n]/\|[S_n]\|$ . Then  $S$  is a closed positive  $(mk, mk)$ -current of mass 1 in  $B \times \mathbb{B}_\delta^m(z)$  with  $\text{supp}(S) \subset \{\lambda_0\} \times \mathbb{B}_\delta^m(z)$ . Hence  $S = M \cdot [\{\lambda_0\} \times \mathbb{B}_\delta^m(z)]$ , where  $M^{-1} > 0$  is the volume of  $\mathbb{B}_\delta^m(z)$  for the volume form  $\bigwedge_j (\omega_j^k)$ , where  $\omega_j = (p_j)^* \omega_{\mathbb{P}^k}$  and  $p_j : (\mathbb{P}^k)^m \rightarrow \mathbb{P}^k$  is the projection on the  $j$ -th coordinate.

As a consequence,  $[S_n]/\|[S_n]\|$  converges weakly to  $S$  as  $n \rightarrow \infty$  and, since the  $(mk, mk)$ -current  $\bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k)$  is the wedge product of  $(1, 1)$ -currents with continuous potentials, we have

$$\bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) \wedge \frac{[S_n]}{\|[S_n]\|} \longrightarrow \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) \wedge S$$

as  $n \rightarrow +\infty$ . Whence

$$\liminf_{n \rightarrow \infty} (\|[S_n]\|^{-1} \cdot I_n) \geq \liminf_{n \rightarrow \infty} \int \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) \wedge \frac{[S_n]}{\|[S_n]\|} \geq \int \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) \wedge S$$

By the above, this gives

$$\liminf_{k \rightarrow \infty} (\|[S_n]\|^{-1} \cdot I_n) \geq M \cdot \int [\{\lambda_0\} \times \mathbb{B}_\delta^m(z)] \wedge \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k),$$

In particular, there exists  $n_2 \geq n_1$  such that for all  $n \geq n_2$ ,

$$\|[S_n]\|^{-1} \cdot I_n \geq \frac{M}{2} \cdot \int [\{\lambda_0\} \times \mathbb{B}_\delta^m(z)] \wedge \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k).$$

Finally, since  $[S_n]$  is a vertical current, up to reducing  $\delta > 0$ , Fubini Theorem gives

$$\liminf_{n \rightarrow \infty} \|[S_n]\| \geq \prod_{j=1}^m \int_{\mathbb{B}(z_j, \delta)} \omega_{\text{FS}}^k \geq (c \cdot \delta^{2k})^m > 0.$$

Up to increasing  $n_0$ , we may assume  $\|[S_n]\| \geq (c\delta^{2k})^m/2$  for all  $n \geq n_0$ . Letting  $\alpha = M(c\delta^{2k})^m/4 > 0$ , we find

$$\int_{\Omega \times (\mathbb{P}^k)^m} \left( \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) \right) \wedge [\Gamma_n] \geq \alpha \int [\{\lambda_0\} \times \mathbb{B}_\delta^m(z)] \wedge \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k).$$

To conclude the proof of Theorem 2.3, we rely on the following purely dynamical result, which is an immediate adaptation of [AGMV, Lemma 3.5].

**Lemma 2.5.** — *For any  $\delta > 0$ , and any  $x = (x_1, \dots, x_m) \in (\text{supp}(\mu_{\lambda_0}))^m$ , we have*

$$\int [\{\lambda_0\} \times \mathbb{B}_\delta^m(x)] \wedge \bigwedge_{j=1}^m (\pi_j)^* (\widehat{T}^k) = \prod_{j=1}^m \mu_{\lambda_0}(\mathbb{B}(x_j, \delta)) > 0.$$

We can now conclude the proof of Theorem 2.3. Pick any open neighborhood  $\Omega$  of  $\lambda_0$  in  $\Lambda$ . By the above and Lemma 2.5, we have an integer  $n_0 \geq 1$  and constants  $\alpha, \delta > 0$  such that for all  $n \geq n_0$ ,

$$\mu(\Omega) \geq \alpha \cdot d^{-k(|q|+n|r|)} \prod_{j=1}^m \mu_f(\mathbb{D}(z_j, \delta)) > 0.$$

In particular, this yields  $\mu(\Omega) > 0$ . By assumption, this holds for a basis of neighborhoods of  $\lambda_0$  in  $\Lambda$ , whence we have  $\lambda_0 \in \text{supp}(\mu)$ .  $\square$

## 2.2. Density of transversely prerepelling parameters

To finish the proof of Theorem 2.2, it is sufficient to prove that any point of the support of  $T_{f,a_1}^k \wedge \dots \wedge T_{f,a_m}^k$  can be approximated by transversely  $J$ -prerepelling parameters. We follow the strategy of the proof of Theorem 0.1 of [Duj] to establish this approximation property. Precisely, we prove here the following.

**Theorem 2.6.** — *Let  $(f, a_1, \dots, a_m)$  be a dynamical  $(m+1)$ -tuple of  $\mathbb{P}^k$  of degree  $d$  parametrized by  $\Lambda$  with  $km \leq \dim \Lambda$ . Assume that the Lyapunov exponent of  $f_\lambda$  don't resonate for all  $\lambda \in \Lambda$ .*

*Then, any parameter  $\lambda \in \Lambda$  lying in the support of the current  $T_{f,a_1}^k \wedge \dots \wedge T_{f,a_m}^k$  can be approximated by parameters at which  $a_1, \dots, a_m$  are transversely  $J$ -prerepelling.*

We rely on the following property of PLB measures (see [DS]):

**Lemma 2.7.** — *Let  $\nu$  be PLB with compact support in a bounded open set  $W \subset \mathbb{C}^k$  and let  $\psi$  be a psh function on  $\mathbb{C}^k$ . The function  $G_\psi$  defined by*

$$G_\psi(z) := \int \psi(z-w) d\nu(w), \quad z \in \mathbb{C}^k,$$

*is psh and locally bounded on  $\mathbb{C}^k$ .*

*Proof of Theorem 2.6.* — We follow the strategy of the proof of [Duj, Theorem 0.1]. Write  $\mu := T_{f,a_1}^k \wedge \dots \wedge T_{f,a_m}^k$  and pick  $\lambda_0 \in \text{supp}(\mu)$ .

According to Proposition 1.5, there exists an integer  $m \geq 1$  and a  $f_{\lambda_0}^m$ -compact set  $K \subset \mathbb{P}^k$  contained in a ball and  $N \geq 2$  such that

- $f_{\lambda_0}^m|_K$  is uniformly hyperbolic and repelling periodic points of  $f_{\lambda_0}^m$  are dense in  $K$ ,
- there exists a unique probability measure  $\nu$  supported on  $K$  such that  $(f_{\lambda_0}^m|_K)^* \nu = N\nu$  which is PLB.

Since  $K$  is hyperbolic, there exists  $\epsilon > 0$  and a unique holomorphic motion  $h : \mathbb{B}(\lambda_0, \epsilon) \times K \rightarrow \mathbb{P}^k$  which conjugates the dynamics, i.e.  $h$  is continuous and such that

- for all  $\lambda \in \mathbb{B}(\lambda_0, \epsilon)$ , the map  $h_\lambda := h(\lambda, \cdot) : K \rightarrow \mathbb{P}^k$  is injective and  $h_{\lambda_0} = \text{id}_K$ ,
- for all  $z \in K$ , the map  $\lambda \in \mathbb{B}(\lambda_0, \epsilon) \mapsto h_\lambda(z) \in \mathbb{P}^k$  is holomorphic, and

– for all  $(\lambda, z) \in \mathbb{B}(\lambda_0, \epsilon) \times K$ , we have  $h_\lambda \circ f_{\lambda_0}^m(z) = f_\lambda^m \circ h_\lambda(z)$ ,  
 see e.g. [dMvS, Theorem 2.3 p. 255]. For all  $z := (z_1, \dots, z_m) \in K^m$ , we denote by  $\Gamma_z$   
 the graph of the holomorphic map  $\lambda \mapsto (h_\lambda(z_1), \dots, h_\lambda(z_m))$ .

We define a closed positive  $(km, km)$ -current on  $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$  by letting

$$\hat{\nu} := \int_{K^m} [\Gamma_z] d\nu^{\otimes m}(z),$$

where  $\Gamma_z = \{(\lambda, h_\lambda(z_1), \dots, h_\lambda(z_m))\}; \lambda \in \mathbb{B}(\lambda_0, \epsilon)\}$  for all  $z = (z_1, \dots, z_m) \in K^m$ .

**Claim.** — *There exists a  $(km-1, km-1)$ -current  $V$  on  $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$  which is locally bounded and such that  $\hat{\nu} = dd^c V$ .*

Recall that we have set  $\mathbf{a}_n(\lambda) := (f_\lambda^n(a_1(\lambda)), \dots, f_\lambda^n(a_m(\lambda)))$ . We define  $\mathbf{a}_n^* \hat{\nu}$  by

$$\mathbf{a}_n^* \hat{\nu} := (\pi_1)_* (\hat{\nu} \wedge [\Gamma_{\mathbf{a}_n}]),$$

where  $\pi_1 : \mathbb{B}(t_0, \epsilon) \times (\mathbb{P}^k)^m \rightarrow \mathbb{B}(t_0, \epsilon)$  is the canonical projection onto the first coordinate. According to the claim, locally we have  $\hat{\nu} = dd^c V$ , for some bounded  $(km-1, km-1)$ -current  $V$ . In particular, we get  $\mathbf{a}_n^* \hat{\nu} = \mathbf{a}_n^*(dd^c V)$ , as expected.

Let  $\omega$  be the Fubini-Study form of  $\mathbb{P}^k$  and  $\hat{\Omega} := (\pi_2)^*(\omega^k \otimes \dots \otimes \omega^k)$ , where  $\pi_2 : \mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m \rightarrow (\mathbb{P}^k)^m$  is the canonical projection onto the second coordinate. Then

$$\hat{\nu} - \hat{\Omega} = dd^c V$$

where  $V$  is bounded on  $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$ , hence

$$d^{-kmn} \mathbf{a}_n^*(\hat{\nu}) - d^{-kmn} \mathbf{a}_n^*(\hat{\Omega}) = d^{-kmn} \mathbf{a}_n^*(dd^c V).$$

On the other hand, we have  $\frac{1}{d^{km}} \hat{f}^*(\hat{\Omega}) = \hat{\Omega} + dd^c W$ , where  $W$  is bounded on  $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$ , hence  $\frac{1}{d^{kmn}} (\hat{f}^n)^*(\hat{\Omega}) = \hat{\omega} + dd^c W_n$ , where  $W_n - W_{n+1} = O(d^{-n})$ . In particular,  $\frac{1}{d^n} (\hat{f}^n)^*(\hat{\Omega}) \wedge [\Gamma_{\mathbf{a}_n}] = d^{-n} (\hat{\Omega} \wedge [\Gamma_{\mathbf{a}_n}]) + dd^c O(d^{-n})$ , hence  $\mu = \lim_n d^{-kmn} \mathbf{a}_n^*(\hat{\Omega})$ . This yields

$$\lim_{n \rightarrow \infty} d^{-nkm} (\pi_1)_* (\hat{\nu} \wedge [\Gamma_{\mathbf{a}_n}]) = \mu.$$

We now use [Duj, Theorem 3.1]: as  $(2km, 2km)$ -currents on  $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{P}^k)^m$ ,

$$\hat{\nu} \wedge [\Gamma_{\mathbf{p}_n}] = \int_{K^m} [\Gamma_z] \wedge [\Gamma_{\mathbf{p}_n}] d\nu^{\otimes m}(z)$$

and only the geometrically transverse intersections are taken into account. In particular, this means there exists a sequence of parameters  $\lambda_n \rightarrow \lambda_0$  and  $z_n \in K^m$  such that the graph of  $\mathbf{a}_n$  and  $\Gamma_{z_n}$  intersect transversely at  $\lambda_n$ . Now, since repelling periodic points of  $f_{\lambda_0}^m$  are dense in  $K$ , there exists  $z_{n,j} \rightarrow z_n$  as  $j \rightarrow \infty$ , where  $z_{j,n} \in K^m$  and  $(f_{\lambda_0}^m, \dots, f_{\lambda_0}^m)$ -periodic repelling. Since  $z_{j,n}(\lambda) := (h_\lambda, \dots, h_\lambda)(z_{j,n})$  remains in  $(h_\lambda, \dots, h_\lambda)(K^m)$  and remains periodic, it remains repelling for all  $\lambda \in \mathbb{B}(\lambda_0, \epsilon)$ . By persistence of transverse intersections, for  $j$  large enough, there exists  $\lambda_{j,n}$  where  $\Gamma_{\mathbf{a}_n}$  and  $\Gamma_{z_{j,n}}$  intersect transversely and  $\lambda_{j,n} \rightarrow \lambda_n$  as  $j \rightarrow \infty$  and the proof is complete.  $\square$

To finish this section, we prove the Claim.

*Proof of the Claim.* — Since the compact set  $K$  is contained in a ball, we can choose an affine chart  $\mathbb{C}^k$  such that  $K \Subset \mathbb{C}^k$  and, up to reducing  $\epsilon > 0$ , we can assume  $K_\lambda = h_\lambda(K) \Subset \mathbb{C}^k$  for all  $\lambda \in \mathbb{B}(\lambda_0, \epsilon)$ . Let  $(x_1^1, \dots, x_k^1, \dots, x_1^m, \dots, x_k^m) = (x^1, \dots, x^m)$  be the coordinates of  $(\mathbb{C}^k)^m$  and let  $h_{\lambda,i}$  be the  $i$ -th coordinate of the function  $h_\lambda$ .

For all  $1 \leq i \leq k$  and  $1 \leq j \leq m$ , we define a psh function  $\Psi_i^j$  on  $\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{C}^k)^m$  by letting

$$\Psi_i^j(t, w) := \int_{K^m} \log |w_i^j - h_{t,i}(z^j)| d\nu^{\otimes m}(z).$$

According to Lemma 2.7 and Proposition 1.5, we have  $\Psi_i^j \in L_{\text{loc}}^\infty(\mathbb{B}(\lambda_0, \epsilon) \times (\mathbb{C}^k)^m)$ . Moreover, according to [Duj, Theorem 3.1], we have

$$\hat{\nu} = \bigwedge_{i,j} dd^c \Psi_i^j = dd^c \left( \Psi_1^1 \cdot \bigwedge_{i,j>1} dd^c \Psi_i^j \right).$$

Since the functions  $\Psi_i^j$  are locally bounded, this ends the proof.  $\square$

### 3. Local properties of bifurcation measures

#### 3.1. A renormalization procedure

Pick  $k, m \geq 1$  and let  $\mathbb{B}(0, \epsilon)$  be the open ball centered at 0 of radius  $\epsilon$  in  $\mathbb{C}^{km}$  and let  $(f, a_1, \dots, a_m)$  be a dynamical  $(m+1)$ -tuple of degree  $d$  of  $\mathbb{P}^k$  parametrized by  $\mathbb{B}(0, \epsilon)$ .

Assume there exists  $m$  holomorphically moving  $J$ -repelling periodic points  $z_1, \dots, z_m : \mathbb{B}(0, \epsilon) \rightarrow \mathbb{P}^k$  of respective period  $q_j \geq 1$  with  $f_0^{n_j}(a_j(0)) = z_j(0)$ . We also assume that  $(a_1, \dots, a_m)$  are transversely prerepelling at 0 and that  $z_j(\lambda)$  is linearizable for all  $\lambda \in \mathbb{B}(0, \epsilon)$  for all  $j$ . Let  $q := \text{lcm}(q_1, \dots, q_m)$  and

$$\Lambda_\lambda := (D_{z_1(\lambda)}(f_\lambda^q), \dots, D_{z_m(\lambda)}(f_\lambda^q)) : \bigoplus_{j=1}^m T_{z_j(\lambda)} \mathbb{P}^k \longrightarrow \bigoplus_{j=1}^m T_{z_j(\lambda)} \mathbb{P}^k$$

and denote by  $\phi_\lambda = (\phi_{\lambda,1}, \dots, \phi_{\lambda,m}) : (\mathbb{C}^k, 0) \rightarrow ((\mathbb{P}^k)^m, (z_1(\lambda), \dots, z_m(\lambda)))$ , where  $\phi_{\lambda,j}$  is the linearizing coordinate of  $f_\lambda^q$  at  $z_j(\lambda)$ .

Denote by  $\pi_j : (\mathbb{P}^k)^m \rightarrow \mathbb{P}^k$  the projection onto the  $j$ -th factor. Up to reducing  $\epsilon > 0$ , we can also assume there exists  $r_j > 0$  independent of  $\lambda$  such that

$$f_\lambda^q \circ \phi_{\lambda,j}(z) = \phi_{\lambda,j} \circ D_{z_j(\lambda)}(f_\lambda^{q_j})(z), \quad z \in \mathbb{B}(0, r_j),$$

and  $D_0 \phi_{\lambda,j} : \mathbb{C}^k \rightarrow T_{z_j(\lambda)} \mathbb{P}^k$  is an invertible linear map. Up to reducing again  $\epsilon$ , we can also assume  $f_\lambda^{n_j}(a_j(\lambda))$  always lies in the range of  $\phi_{\lambda,j}$  for all  $1 \leq j \leq m$ . Recall that we denoted  $\mathbf{a}_{\underline{n}}(\lambda) = (f_\lambda^{n_1}(a_1(\lambda)), \dots, f_\lambda^{n_m}(a_m(\lambda)))$ , where  $\underline{n} = (n_1, \dots, n_m)$  and let

$$h(\lambda) := \phi_\lambda^{-1} \circ \mathbf{a}_{\underline{n}}(\lambda) = \left( \phi_{\lambda,1}^{-1}(f_\lambda^{n_1}(a_1(\lambda))), \dots, \phi_{\lambda,m}^{-1}(f_\lambda^{n_m}(a_m(\lambda))) \right), \quad \lambda \in \mathbb{B}(0, \epsilon).$$

**Lemma 3.1.** — *The map  $h : \mathbb{B}(0, \epsilon) \rightarrow (\mathbb{C}^{km}, 0)$  is a local biholomorphism at 0.*

*Proof.* — Recall that  $f_0^{n_j}(a_j(0)) = z_j(0)$ . Write  $h = (h_1, \dots, h_m)$  with  $h_j : \mathbb{B}(0, \epsilon) \rightarrow (\mathbb{C}^k, 0)$  and let  $b_j(\lambda) := f_\lambda^{n_j}(a_j(\lambda))$  for all  $\lambda \in \mathbb{B}(0, \epsilon)$  so that  $b_j(\lambda) = \phi_{\lambda,j} \circ h_j(\lambda)$  for all  $\lambda \in \mathbb{B}(0, \epsilon)$ . Since  $\phi_{\lambda,j}(0) = z_j(\lambda)$ , differentiating and evaluating at  $\lambda = 0$ , we find

$$D_0 b_j = D_0 z_j + D_0 \phi_{0,j} \circ D_0 h_j.$$

Now our transversality assumption implies that

$$L := ((D_0 b_1 - D_0 z_1), \dots, (D_0 b_m - D_0 z_m)) : \mathbb{C}^{km} \rightarrow \bigoplus_{j=1}^m T_{z_j(0)} \mathbb{P}^k$$

is invertible. As a consequence, the linear map

$$D_0h = (D_0h_1, \dots, D_0h_m) = -(D_0\phi_0)^{-1} \circ L : \mathbb{C}^{km} \rightarrow \mathbb{C}^{km}$$

is invertible, ending the proof.  $\square$

Up to reducing again  $\epsilon$ , we assume  $h$  is a biholomorphism onto its image and let  $r := h^{-1} : h(\mathbb{B}(0, \epsilon)) \rightarrow \mathbb{B}(0, \epsilon)$ . Fix  $\delta_1, \dots, \delta_m > 0$  so that  $\mathbb{B}_{\mathbb{C}^k}(0, \delta_1) \times \dots \times \mathbb{B}_{\mathbb{C}^k}(0, \delta_m) \subset h(\mathbb{B}(0, \epsilon))$ .

Finally, let  $\Omega := \mathbb{B}_{\mathbb{C}^k}(0, \delta_1) \times \dots \times \mathbb{B}_{\mathbb{C}^k}(0, \delta_m)$  and, for any  $n \geq 1$ , let

$$r_n(x) := r \circ \Lambda_0^{-n}(x), \quad x \in \mathbb{B}_{\mathbb{C}^k}(0, \delta_1) \times \dots \times \mathbb{B}_{\mathbb{C}^k}(0, \delta_m).$$

The main goal of this paragraph is the following.

**Proposition 3.2.** — *In the weak sense of measures on  $\Omega$ , we have*

$$\prod_{j=1}^m d^{k(n_j+nq)} \cdot (r_n)^* \left( T_{f, a_1}^k \wedge \dots \wedge T_{f, a_m}^k \right) \xrightarrow{n \rightarrow \infty} (\phi_0)^* \left( \bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0} \right).$$

To simplify notations, we let

$$\mathbf{a}_{(n)} := \mathbf{a}_{\underline{n}+nq}, \quad \text{with } \underline{n} + nq = (n_1 + nq, \dots, n_m + nq).$$

**Lemma 3.3.** — *The sequence  $(\mathbf{a}_{(n)} \circ r_n)_{n \geq 1}$  converges uniformly to  $\phi_0$  on  $\Omega$ .*

*Proof.* — Note first that

$$\mathbf{a}_{(0)} \circ r(x) = \left( f_{r(x)}^{n_1}(a_1(r(x))), \dots, f_{r(x)}^{n_m}(a_m(r(x))) \right) = \phi_{r(x)}(x), \quad x \in \Omega,$$

by definition of  $r$ .

By definition, the sequence  $(r_n)_{n \geq 1}$  converges uniformly and exponentially fast to 0 on  $\Omega$ , since we assumed  $z_1(0), \dots, z_m(0)$  are repelling periodic points and since  $r(0) = 0$ . Moreover,  $\Lambda_{r_n} \rightarrow \Lambda_0$  and  $\phi_{r_n(x)} \rightarrow \phi_0$  exponentially fast. In particular,

$$\lim_{n \rightarrow \infty} \Lambda_{r_n(x)}^n \circ \Lambda_0^{-n}(x) = x$$

and the convergence is uniform on  $\Omega$ . Fix  $x \in \Omega$ . Then

$$\begin{aligned} \mathbf{a}_{(n)} \circ r_n(x) &= (f_{r_n(x)}^{q n_1}, \dots, f_{r_n(x)}^{q n_m}) (\mathbf{a}_{(0)} \circ r \circ \Lambda_0^{-n}(x)) \\ &= (f_{r_n(x)}^{q n_1}, \dots, f_{r_n(x)}^{q n_m}) \circ \phi_{r_n(x)} (\Lambda_0^{-n}(x)) \\ &= \phi_{r_n(x)} \left( \Lambda_{r_n(x)}^n \circ \Lambda_0^{-n}(x) \right) \end{aligned}$$

and the conclusion follows.  $\square$

*Proof of Proposition 3.2.* — Recall that we can assume there exists a holomorphic family of non-degenerate homogeneous polynomial maps  $F_\lambda : \mathbb{C}^{k+1} \rightarrow \mathbb{C}^{k+1}$  of degree  $d$  such that, if  $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$  is the canonical projection, then

$$\pi \circ F_\lambda = f_\lambda \circ \pi \text{ on } \mathbb{C}^{k+1} \setminus \{0\}.$$

For  $1 \leq j \leq m$ , let  $\tilde{a}_j : \mathbb{B}(0, \epsilon) \rightarrow \mathbb{C}^{k+1} \setminus \{0\}$  be a lift of  $a_j$ , i.e.  $a_j = \pi \circ \tilde{a}_j$ . Recall that

$$\bigwedge_{j=1}^m T_{a_j}^k = \bigwedge_{j=1}^m (dd^c G_\lambda(\tilde{a}_j(\lambda)))^k.$$

For  $1 \leq j \leq m$ , pick a open set  $U_j \subset \mathbb{P}^k$  such that  $\phi_{0,j}(B_{\mathbb{C}^k}(0, \delta_j)) \Subset U_j$  and such that there exists a section  $\sigma_j : U_j \rightarrow \mathbb{C}^{k+1} \setminus \{0\}$  of  $\pi$  on  $U_j$ . Let  $U := U_1 \times \cdots \times U_m$  and  $\sigma := (\sigma_1, \dots, \sigma_m) : U \rightarrow (\mathbb{C}^{k+1} \setminus \{0\})^m$  so that  $\phi_0(\Omega) \Subset U$ . According to Lemma 3.3, there exists  $n_0 \geq 1$  such that

$$\mathbf{a}_{(n)} \circ r_n(\Omega) \Subset U.$$

In other words, for any  $x \in \Omega$ , any  $1 \leq j \leq m$  and any  $n \geq n_0$ ,

$$a^{n,j}(x) := f_{r_n(x)}^{n_j+nq}(a_j \circ r_n(x)) \in U_j.$$

Moreover, for all  $x \in \Omega$ , we have

$$\begin{aligned} \pi \circ F_{r_n(x)}^{n_j+nq}(\tilde{a}_j \circ r_n(x)) &= f_{r_n(x)}^{n_j+nq} \circ \pi(\tilde{a}_j \circ r_n(x)) = f_{r_n(x)}^{n_j+nq}(a_j \circ r_n(x)) \\ &= \pi \circ \sigma_j(a^{n,j}(x)). \end{aligned}$$

In particular, there exists a holomorphic function  $u_{n,j} : \Omega \rightarrow \mathbb{C}^*$  such that

$$F_{r_n(x)}^{n_j+nq}(\tilde{a}_j \circ r_n(x)) = u_{n,j}(x) \cdot \sigma_j \circ a^{n,j}(x)$$

and

$$\begin{aligned} d^{nq+n_j} G_{r_n(x)}(\tilde{a}_j \circ r_n(x)) &= G_{r_n(x)} \left( F_{r_n(x)}^{n_j+nq}(\tilde{a}_j \circ r_n(x)) \right) \\ &= G_{r_n(x)}(\sigma_j \circ a^{n,j}(x)) + \log |u_{n,j}(x)|, \end{aligned}$$

for all  $x \in \Omega$ . Since  $\log |u_{n,j}|$  is pluriharmonic on  $\Omega$ , the above gives

$$d^{nq+n_j}(r_n)^* T_{f,a_j} = dd^c G_{r_n(x)}(\sigma_j \circ a^{n,j}(x)),$$

so that

$$\mu_n := \prod_{j=1}^m d^{k(n_j+nq)} \cdot (r_n)^* \left( T_{f,a_1}^k \wedge \cdots \wedge T_{f,a_m}^k \right) = \bigwedge_{j=1}^m \left( dd^c G_{r_n(x)}(\sigma_j \circ a^{n,j}(x)) \right)^k.$$

Using again Lemma 3.3 gives

$$\mu_n \xrightarrow{n \rightarrow \infty} \bigwedge_{j=1}^m \left( dd^c G_0(\sigma_j \circ \phi_{0,j}(x)) \right)^k = \bigwedge_{j=1}^m (\phi_{0,j})^* \mu_{f_0}.$$

This ends the proof since  $\phi_{0,j} = \pi_j \circ \phi_0$  by definition of  $\phi_0$ . □

### 3.2. Families with an absolutely continuous bifurcation measure

Fix integers  $k, m \geq 1$  and  $d \geq 2$ . The following is a consequence of the above renormalization process.

**Proposition 3.4.** — *Let  $(f, a_1, \dots, a_m)$  be a dynamical  $(m+1)$ -tuple of degree  $d$  of  $\mathbb{P}^k$  parametrized by the unit Ball  $\mathbb{B}$  of  $\mathbb{C}^{km}$ . Assume that  $a_1, \dots, a_m$  are transversely  $J$ -prerepelling at 0 to a  $J$ -repelling cycle of  $f_0$  which moves holmorphically in  $\mathbb{B}$  as a  $J$ -repelling cycle of  $f_\lambda$  which is linearizable for all  $\lambda \in \mathbb{B}$ . Assume in addition that the measure  $\mu := T_{f,a_1}^k \wedge \cdots \wedge T_{f,a_m}^k$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{B}$ .*

*Then the measure  $\mu_{f_0}$  is non-singular with respect to  $\omega_{\mathbb{P}^k}^k$ .*

*Proof.* — By assumption, we can write  $\mu = h \cdot \text{Leb}$  where  $h : \mathbb{B} \rightarrow \mathbb{R}_+$  is a measurable function. Let  $\Omega := \mathbb{B}_{\mathbb{C}^k}(0, \delta_1) \times \cdots \times \mathbb{B}_{\mathbb{C}^k}(0, \delta_m)$ ,  $r_n$  and  $\phi_0$  be given as in Section 3.1. We can apply Proposition 3.2:

$$\prod_{j=1}^m d^{k(n_j+ng)} h \circ r_n \cdot (r_n)^* \text{Leb} = \prod_{j=1}^m d^{k(n_j+ng)} \cdot (r_n)^* \mu \xrightarrow{n \rightarrow \infty} (\phi_0)^* \left( \bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0} \right).$$

Since  $\phi_0(0) = (z_1(0), \dots, z_m(0)) \in (\text{supp}(\mu_{f_0}))^k$ , the measure

$$(\phi_0)^* \left( \bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0} \right)$$

has (finite) strictly positive mass in  $\Omega$ . In particular, the measure

$$d^{knqm} \cdot (r_n)^* (h \cdot \text{Leb}) = d^{knqm} \cdot (h \circ r_n) \cdot (\Lambda_0^{-n})^* (r^* \text{Leb})$$

converges to a non-zero finite mass positive measure on  $\Omega$ . As  $r$  is a local holomorphic diffeomorphism, there exists a neighborhood of 0 in  $\mathbb{B}$  such that we have  $r^* \text{Leb} = v \cdot \text{Leb}$  for some smooth function  $v > 0$ . Whence

$$d^{knqm} \cdot (r_n)^* \text{Leb} = d^{knqm} \cdot (h \circ r_n) \cdot (v \circ \Lambda_0^{-n}) (\Lambda_0^{-n})^* (\text{Leb}).$$

By the change of variable formula and Fubini,

$$(\Lambda_0^{-n})^* (\text{Leb}) = \prod_{j=1}^m |\det D_{z_j(0)}(f_0^q)|^{-2nk} \cdot \text{Leb}.$$

For all  $n$ , define a measurable function  $\alpha_n : \mathbb{B} \rightarrow \mathbb{R}_+$  by letting

$$\alpha_n(x) := d^{knqm} \prod_{j=1}^m |\det D_{z_j(0)}(f_0^q)|^{-2nk} \cdot (h \circ r_n(x)) \cdot (v \circ \Lambda_0^{-n}(x)) \in \mathbb{R}_+.$$

By assumption, the measure  $\alpha_n \cdot \text{Leb}$  converges weakly on  $\Omega$  to a non-zero finite positive measure, whence  $\alpha_n \rightarrow \alpha_\infty$ , as  $n \rightarrow \infty$ , where  $\alpha_\infty : \Omega \rightarrow \mathbb{R}_+$  is not identically zero. As a consequence,

$$(\phi_0)^* \left( \bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0} \right) = \alpha_\infty \cdot \text{Leb}.$$

Using again Fubini, on  $\Omega$ , we find

$$(\phi_0)^* \left( \bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0} \right) = \alpha_\infty \cdot \text{Leb}_{\mathbb{C}^k} \boxtimes \cdots \boxtimes \text{Leb}_{\mathbb{C}^k}.$$

Finally, since as positive measures on  $\phi_0(\Omega)$ , we have

$$\bigwedge_{j=1}^m (\pi_j)^* \mu_{f_0} = \mu_{f_0} \boxtimes \cdots \boxtimes \mu_{f_0},$$

the measure  $\mu_{f_0}$  is absolutely continuous with respect to  $\text{Leb}$  in an open set.  $\square$

We now want to deduce Theorem B from the above, using [Z] when  $k = 1$  and [BD3] when  $k > 1$ . In fact, they prove that  $f$  is a Lattès map if and only if the sum of its Lyapunov exponents  $L(f) = \int_{\mathbb{P}^k} \log |\det(Df)| \mu_f$  is equal to  $\frac{k}{2} \log d$ . We use this characterization to prove Theorem B.

*Proof of Theorem B.* — Assume first that  $\mu_{f,a}$  is absolutely continuous with respect to  $\omega^k$  and let  $T$  be the set of parameters  $\lambda \in M$  such that  $a$  is transversely prerepelling at  $\lambda$ . The set  $T$  is dense in  $M$  by Theorem 2.2. Applying Proposition 3.4 at all  $\lambda \in T$  gives that  $\mu_{f_\lambda}$  is non-singular with respect to  $\omega_{\mathbb{P}^k}^k$  for all  $\lambda \in T$ .

We then apply Zdunik or Berteloot-Dupont Theorem we have proven there exists a countable subset  $T$  which is dense in  $M$  such that the map  $f_\lambda$  is a Lattès map for all  $\lambda \in T$ . In particular,  $L(f_\lambda) = \frac{k}{2} \log d$  for all  $\lambda \in T$ . As the function  $\lambda \in M \mapsto L(f_\lambda)$  is continuous, this implies  $L(f_\lambda) = \frac{k}{2} \log d$  for all  $\lambda \in M$ , i.e.  $f_\lambda$  is a Lattès map for all  $\lambda \in M$ .

To conclude, we assume  $f$  is a family of Lattès maps and the pair  $(f, a)$  satisfies  $\mu_{f,a} > 0$ . Let  $\omega_{\mathbb{P}^k}$  be the Fubini-Study form on  $\mathbb{P}^k$ . For all  $\lambda \in M$ , there exists a function  $u_\lambda : \mathbb{P}^k \rightarrow \mathbb{R}_+$  such that

$$\mu_{f_\lambda} = u_\lambda \cdot \omega_{\mathbb{P}^k}^k.$$

Let  $u(\lambda, z) := u_\lambda(z)$  for all  $(\lambda, z) \in M \times \mathbb{P}^k$ . The above can be expressed as

$$\widehat{T} = u \cdot \widehat{\omega}^k,$$

where  $\widehat{\omega} = \pi_{\mathbb{P}^k}^*(\omega_{\mathbb{P}^k})$  and  $\pi_{\mathbb{P}^k} : M \times \mathbb{P}^k \rightarrow \mathbb{P}^k$  is the canonical projection. Pick a local chart  $U \subset M$  and a local chart  $V \subset \mathbb{P}^k$  so that  $a(U) \subset V$  and  $\omega_{\mathbb{P}^k} = dd^c v$  on  $V$  where  $v$  is smooth. In  $U \times V$ , the above gives

$$(\pi_\Lambda)_* \left( \widehat{T}^k \wedge [\Gamma_a] \right) = (\pi_\Lambda)_* \left( u \cdot (dd_{\lambda,z}^c v(z))^k \wedge [\Gamma_a] \right) = u(\lambda, a(\lambda)) (dd_\lambda^c (v \circ a(\lambda)))^k.$$

Letting  $h(\lambda) := u(\lambda, a(\lambda))$  and  $w(\lambda) := v \circ a(\lambda)$ , we find

$$\mu_{f,a} = h \cdot (dd^c w)^k \text{ on } U.$$

Since  $w$  is smooth, the conclusion follows.  $\square$

## 4. Proof of the main result and concluding remarks

### 4.1. $J$ -stability and bifurcation of dynamical pairs on $\mathbb{P}^1$

We say that a family  $f : \Lambda \times \mathbb{P}^1 \rightarrow \Lambda \times \mathbb{P}^1$  of degree  $d$  rational maps of  $\mathbb{P}^1$  is  $J$ -stable if all the repelling cycles can be followed holomorphically throughout the whole family  $\Lambda$ , i.e. if for all  $n \geq 1$ , there exists  $N \geq 0$  and holomorphic maps  $z_1, \dots, z_N : \Lambda \rightarrow \mathbb{P}^1$  such that  $\{z_1(\lambda), \dots, z_N(\lambda)\}$  is exactly the set of all repelling cycles of  $f_\lambda$  of exact period  $n$  for all  $\lambda \in \Lambda$ .

Recall that an endomorphism of  $\mathbb{P}^1$  has a unique measure of maximal entropy  $\mu_f$  and let  $L(f) := \int_{\mathbb{P}^1} \log |f'| \mu_f$  be the Lyapunov exponents of  $f$  with respect to  $\mu_f$ . By a classical result of Mañé, Sad and Sullivan [MSS], it is also locally equivalent to the existence of a unique holomorphic motion of the Julia set which is compatible with the dynamics, i.e. for  $\lambda_0 \in \Lambda$ , there exists  $h : \Lambda \times J_{f_{\lambda_0}} \rightarrow \Lambda \times \mathbb{P}^1$  such that

- for any  $\lambda \in \Lambda$ , the map  $h_\lambda := h(\lambda, \cdot) : J_{f_{\lambda_0}} \rightarrow \mathbb{P}^1$  is a homeomorphism which conjugates  $f_{\lambda_0}$  to  $f_\lambda$ , i.e.  $h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda$  on  $J_{f_{\lambda_0}}$ ,
- for any  $z \in J_{f_{\lambda_0}}$ , the map  $\lambda \mapsto h_\lambda(z)$  is holomorphic on  $\Lambda$ ,
- $h_{\lambda_0}$  is the identity on  $J_{f_{\lambda_0}}$ .



**Lemma 4.1.** — *Let  $(f, a)$  be any dynamical pair of  $\mathbb{P}^1$  of degree  $d \geq 2$  parametrized by the unit disk  $\mathbb{D}$ . If  $f$  is  $J$ -stable and  $\text{supp}(\mu_{f,a}) \neq \emptyset$ , we have*

$$\text{supp}(\mu_{f,a}) = \{\lambda \in \mathbb{D}; a(\lambda) \in J_{f_\lambda}\}.$$

*Proof.* — Since  $\text{Bif}(f, a) = \text{supp}(\mu_{f,a}) \neq \emptyset$ , the set  $D$  of parameters  $\lambda_0 \in \mathbb{D}$  such that  $a$  is transversely prerepelling at  $\lambda_0$  is a non-empty countable dense subset of  $\text{Bif}(f, a)$ . As  $J$ -repelling points of  $f_{\lambda_0}$  are contained in  $J_{f_{\lambda_0}}$ , this gives  $\text{Bif}(f, a) \subset \{\lambda \in \mathbb{D}; a(\lambda) \in J_{f_\lambda}\}$ .

Pick now  $\lambda_0 \in \{\lambda \in \mathbb{D}; a(\lambda) \in J_{f_\lambda}\}$  and assume  $\lambda_0 \notin \text{Bif}(f, a)$ . Set  $a_n(\lambda) := f_\lambda^n(a(\lambda))$  for all  $n \geq 0$  and all  $\lambda \in \mathbb{D}$ . Let  $h : \mathbb{D} \times J_{f_0} \rightarrow \mathbb{P}^1$  be the unique holomorphic motion of  $J_{f_0}$  parametrized by  $\mathbb{D}$  such that, if  $h_\lambda := h(\lambda, \cdot)$ , then

$$h_\lambda \circ f_0 = f_\lambda \circ h_\lambda \text{ on } J_{f_0}.$$

Note that for all  $z \in J_{f_0}$ , the sequence  $\{\lambda \mapsto h_\lambda(f_0^n(z))\}_n$  is a normal family on  $\mathbb{D}$ .

Beware that for all periodic point  $z \in J_{f_0}$  of  $f_0$ , the function  $z(\lambda) := h_\lambda(z)$  is a marking of  $z$  as a periodic point of  $f_\lambda$ . For all  $s \in \mathbb{D}$ , if we let  $h_\lambda^s := h_\lambda \circ h_s^{-1}$ . The family  $(h_\lambda^s)_\lambda$  is a holomorphic motion of  $J_{f_s}$  which satisfies

$$h_\lambda^s \circ f_s = f_\lambda \circ h_\lambda^s \text{ on } J_{f_s},$$

for all  $\lambda \in \mathbb{D}(s, 1 - |s|)$ . Since we assumed  $\lambda_0 \notin \text{Bif}(f, a)$ , there exists  $\epsilon > 0$  such that  $\mathbb{D}(\lambda_0, \epsilon) \cap \text{Bif}(f, a) = \emptyset$  and we can choose an affine chart of  $\mathbb{P}^1$  such that  $a_n(\lambda)$  and  $h_\lambda^{\lambda_0}(a_n(\lambda_0))$  lie in this chart for all  $n \geq 1$  and all  $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$ . For all  $n$ , set

$$s_n(\lambda) := a_n(\lambda) - h_\lambda^{\lambda_0}(a_n(\lambda_0)), \lambda \in \mathbb{D}(\lambda_0, \epsilon).$$

Assume first  $s_m \equiv 0$  on  $\mathbb{D}(\lambda_0, \epsilon)$  for some  $m \geq 0$ . This implies  $a_m(\lambda) = h_\lambda(a_m(0))$  for all  $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$ . By the Isolated Zero Theorem, we thus have

$$a_m(\lambda) = h_\lambda(a_m(0)) \text{ for all } \lambda \in \mathbb{D}.$$

As  $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ , this yields  $a_n(\lambda) \equiv h_\lambda(a_n(0))$  for all  $n \geq m$ , and  $(a_n)$  is a normal family on  $\mathbb{D}$ . This is a contradiction, since we assumed  $\text{Bif}(f, a) \neq \emptyset$ . We thus may assume  $s_m \not\equiv 0$  on  $\mathbb{D}(\lambda_0, \epsilon)$ . In particular, up to reducing  $\epsilon$ , we may assume  $s_m(\lambda) \neq 0$  for all  $\lambda \in \mathbb{D}(\lambda_0, \epsilon) \setminus \{\lambda_0\}$ . Let  $z_0 := a_m(\lambda_0)$ . By Rouché Theorem, there exists  $\eta > 0$  such that for any  $z \in \mathbb{D}(z_0, \eta) \cap J_{f_{\lambda_0}}$ , the function

$$s_{m,z}(\lambda) := a_m(\lambda) - h_\lambda^{\lambda_0}(z)$$

has finitely many isolated zeros in  $\mathbb{D}(\lambda_0, \epsilon)$ . As repelling periodic points are dense in  $J_{f_{\lambda_0}}$ , there exists  $z_1 \in \mathbb{D}(z_0, \eta) \cap J_{f_{\lambda_0}}$  which is  $f_{\lambda_0}$ -periodic and repelling. This implies there exists  $\lambda_1 \in \mathbb{D}(\lambda_0, \epsilon)$  such that  $a$  is properly prerepelling at  $\lambda_1$ . Finally, Theorem 2.3 (or simply Montel Theorem in this case) gives  $\lambda_1 \in \text{Bif}(f, a)$  ending the proof.  $\square$

Using Montel theorem, one can deduce the following.

**Proposition 4.2.** — *Let  $(f, a)$  be a dynamical pair of degree  $d$  of  $\mathbb{P}^1$  parametrized by a one-dimensional complex manifold  $\Lambda$ . Assume that  $\text{Bif}(f, a) = \Lambda$ . Then  $f$  is  $J$ -stable and if  $f$  has a persistent attracting cycle, then it has period at most 2.*

*Proof.* — Assume first  $f$  has a persistent attracting cycle of period at  $p \geq 3$ . Pick a topological disk  $D \subset \Lambda$ . Then there exists holomorphic functions  $z_1, \dots, z_p : D \rightarrow \mathbb{P}^1$  which parameterize this attracting cycle. In particular,  $z_i(\lambda) \neq z_j(\lambda)$  for all  $i \neq j$  and all

$\lambda \in D$ . Since we assumed  $\text{Bif}(f, a) = \Lambda$ , the sequence  $\{\lambda \mapsto f_\lambda^n(a(\lambda))\}_{n \geq 1}$  is not a normal family on  $D$ . By Montel Theorem, there exists  $n \geq 1$ ,  $1 \leq i \leq p$  and  $\lambda_0 \in D$  such that

$$f_{\lambda_0}^n(a(\lambda_0)) = z_i(\lambda_0).$$

By Lemma 4.1, since  $\lambda_0 \in \text{Bif}(f, a)$  this implies  $z_i(\lambda_0) \in J_{f_{\lambda_0}}$ . This is a contradiction with the fact that  $z_i$  is attracting.

If  $f$  is not  $J$ -stable, by Montel Theorem, there exists a non-empty open set  $U$  of  $\Lambda$  such that  $(f_\lambda)_{\lambda \in U}$  is  $J$ -stable with an attracting periodic  $z_1, \dots, z_p$  of period  $p \geq 3$ , and we reduce to the previous case (see e.g. [DF, Proposition 2.4]).  $\square$

## 4.2. Proof of the main result

*Proof of Theorem A.* — First, remark that points 1. and 2. are equivalent by Theorem 2.2. Assume  $\text{Bif}(f, a) = \Lambda$ . By Proposition 4.2 the family  $f$  is  $J$ -stable. By [M, Theorem 2.4], since  $f$  is not isotrivial,  $f$  is a family of Lattès maps. Assume now  $f$  is a non-isotrivial family of Lattès and that  $\mu_{f,a}$  is non-zero. Recall that, since  $f$  is a family of Lattès maps, it is stable. We want to prove that  $\text{supp}(\mu_{f,a}) = \Lambda$ . Assume it is not the case, then there exists a non-empty open set  $U \subset \Lambda$  such that  $U \subset \Lambda \setminus \text{supp}(\mu_{f,a})$ . The pair  $(f, a)$  being stable in  $U$ ,  $a(\lambda)$  cannot be a repelling periodic point of  $f_\lambda$  for any  $\lambda \in U$ . From the uniqueness of the holomorphic motion, it follows that there exists  $z_0 \in \mathbb{P}^1$  such that  $a(\lambda) = h_\lambda(z_0)$  for all  $\lambda \in U$ . By analytic continuation, this gives  $a(\lambda) = h_\lambda(z_0)$  for all  $\lambda \in \Lambda$ . This contradicts the fact that  $\mu_{f,a}$  is non-zero.

As in the proof of Theorem 4.5, let  $T$  be the set of parameters  $\lambda \in \Lambda$  such that  $a$  is transversely prerepelling at  $\lambda$ . The set  $T$  is dense in  $\text{supp}(\mu_{f,a})$  by Theorem 2.2. Applying Proposition 3.4 at all  $\lambda \in T$  gives that  $\mu_{f_\lambda}$  is non-singular with respect to  $\omega_{\mathbb{P}^1}$  for all  $\lambda \in T$ . By [Z], the map  $f_\lambda$  is a Lattès map for all  $\lambda \in T$ . Consider now the morphism  $\varphi : \lambda \in \Lambda \mapsto f_\lambda \in \text{Rat}_d$  and let  $X \subset \text{Rat}_d$  be the set of Lattès maps of degree  $d$ .  $X$  is a strict subvariety of  $\text{Rat}_d$ . By the above, the algebraic curve  $\varphi(\Lambda)$  has an intersection with the subvariety  $X$  which admits accumulation points. By the Isolated Zeros Theorem, we have  $\varphi(\Lambda) \subset X$ , i.e.  $f$  is a family of Lattès maps.

The converse implication follows immediately from Theorem B.  $\square$

Recall that when  $f$  is isotrivial, either  $J_{f_\lambda} = \mathbb{P}^1$  for all  $\lambda$ , or  $J_{f_\lambda} \neq \mathbb{P}^1$  for all  $\lambda$ . We conclude this section with the following easy proposition, which clarifies the case when  $f$  is isotrivial.

**Proposition 4.3.** — *Let  $f$  be an isotrivial algebraic family parametrized by an irreducible quasiprojective curve  $\Lambda$  and let  $a : \Lambda \rightarrow \mathbb{P}^1$  be such that the pair  $(f, a)$  is unstable.*

1. *If  $J_{f_\lambda} = \mathbb{P}^1$  for all  $\lambda \in \Lambda$ , we have  $\text{Bif}(f, a) = \Lambda$ .*
2. *If  $\mu_{f,a}$  is absolutely continuous,  $f$  is an isotrivial family of Lattès maps.*

*Proof.* — Assume first  $J_{f_\lambda} = \mathbb{P}^1$  for all  $\lambda \in \Lambda$ . Applying Lemma 4.1 in local charts gives

$$\text{Bif}(f, a) = \{\lambda \in \Lambda ; a(\lambda) \in J_{f_\lambda}\}$$

and the conclusion follows. When  $\mu_{f,a}$  is absolutely continuous, the conclusion follows as in the proof of Theorem A.  $\square$

### 4.3. Concluding remarks and questions

**Dynamical pairs with a non-singular bifurcation measure.** — First, when  $k > 1$ , the statement of Theorem B holds only if the Lyapunov exponents of  $f$  don't resonate and if all repelling  $J$ -cycles are linearizable.

This results raises several questions:

**Question 4.4.** — *Can we generalize Theorem B to the cases when*

1. *At some parameter, Lyapunov exponents do resonate?*
2. *There exists  $J$ -repelling cycles that are non-linearizable?*
3.  *$T_a^k$  is just non-singular with respect to a smooth volume form?*

In fact, Zdunik [Z] completely classifies rational maps with a maximal entropy measure which is not singular with respect to a Hausdorff measure  $\mathcal{H}^\alpha$ : either  $\alpha = 1$  and the rational map is conjugated to a monomial map  $z^{\pm d}$  or to a Chebichev polynomial  $T_d$ , i.e. the only polynomial satisfying  $T_d(z + \frac{1}{z}) = z^d + \frac{1}{z^d}$  for all  $z \in \mathbb{C}$ , or  $\alpha = 2$  and the rational map is a Lattès map.

We expect the following complete parametric counterpart to [Z] to be true:

**Question 4.5.** — *Let  $(f, a)$  be any holomorphic dynamical pair of  $\mathbb{P}^1$  of degree  $d \geq 2$  parametrized by the unit disk  $\mathbb{D}$  of  $\mathbb{C}$ . Assume that  $(f, a)$  is unstable. Assume also there exists  $\alpha > 0$  and a function  $h : \mathbb{D} \rightarrow \mathbb{R}_+$  such that  $\mu_{f,a} = h \cdot \mathcal{H}^\alpha$  on  $\mathbb{D}$ . Can we prove that*

- *either  $\alpha = 2$  and  $f$  is a family of Lattès maps,*
- *or  $\alpha = 1$ ,  $f$  is isotrivial and all  $f_\lambda$ 's are conjugated to  $z^{\pm d}$  or a Chebichev polynomial?*

As in the case of families of Lattès maps, we can expect the proof to generalize to the case when  $k > 1$ . This raises the following question.

**Question 4.6.** — *Classify endomorphisms of  $\mathbb{P}^k$  which maximal entropy measure is not singular with respect to some Hausdorff measure  $\mathcal{H}^\alpha$  on  $\mathbb{P}^k$  (and possible values of  $\alpha$ ).*

As seen above, the case  $\alpha = 2k$  has been treated by Berteloot and Dupont [BD3]. Of course, there are also easy examples where  $\alpha = k$ : take  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  which maximal entropy measure has dimension 1, then the endomorphism  $F : \mathbb{P}^k \rightarrow \mathbb{P}^k$  making the following diagram commute

$$\begin{array}{ccc} (\mathbb{P}^1)^k & \xrightarrow{(f, \dots, f)} & (\mathbb{P}^1)^k \\ \eta_k \downarrow & & \downarrow \eta_k \\ \mathbb{P}^k & \xrightarrow{F} & \mathbb{P}^k \end{array}$$

where  $\eta_k$  is the quotient map of the action by permutation of coordinates of the symmetric group  $\mathfrak{S}_k$ , satisfies  $\dim(\mu_F) = k$  (see [GHK] for a study of symmetric products).

**$J$ -stability and dynamical pairs, when  $k \geq 2$ .** — We say that a family  $f : \Lambda \times \mathbb{P}^k \rightarrow \mathbb{P}^k$  of degree  $d \geq 2$  endomorphisms of  $\mathbb{P}^k$  is *weakly  $J$ -stable* if all the  $J$ -repelling cycles can be followed holomorphically throughout the whole family  $\Lambda$ , i.e. if for all  $n \geq 1$ , there exists  $N \geq 0$  and holomorphic maps  $z_1, \dots, z_N : \Lambda \rightarrow \mathbb{P}^k$  such that  $\{z_1(\lambda), \dots, z_N(\lambda)\}$  is exactly the set of all repelling  $J$ -cycles of  $f_\lambda$  of exact period  $n$  for all  $\lambda \in \Lambda$ .

For any endomorphism  $f$  of  $\mathbb{P}^k$ , let  $L(f) := \int_{\mathbb{P}^k} \log |\det Df| \mu_f$  be the sum of the Lyapunov exponents of  $f$  with respect to its Green measure  $\mu_f$ . By a beautiful result of Berteloot, Bianchi and Dupont [BBD],  $f$  is  $J$ -stable if and only if  $\lambda \mapsto L(f_\lambda)$  is pluriharmonic on  $\Lambda$ .

A natural question is then the following:

**Question 4.7.** — *Given any dynamical pair  $(f, a)$  of degree  $d$  of  $\mathbb{P}^k$  parametrized by the unit disk  $\mathbb{D}$  such that  $f$  is a weakly  $J$ -stable family, do we still have*

$$\text{Supp}(T_a^k) = \{\lambda \in \mathbb{D}; a(\lambda) \in J_{f_\lambda}\} ?$$

One of the difficulties is that the weak  $J$ -stability is equivalent to the existence of a branched holomorphic motion.

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