

Total variation distance for discretely observed Lévy processes: a Gaussian approximation of the small jumps

Alexandra Carpentier*
Céline Duval[†] and Ester Mariucci[‡]

Abstract

It is common practice to treat small jumps of Lévy processes as Wiener noise and to approximate its marginals by a Gaussian distribution. However, results that allow to quantify the goodness of this approximation according to a given metric are rare. In this paper, we clarify what happens when the chosen metric is the total variation distance. Such a choice is motivated by its statistical interpretation; if the total variation distance between two statistical models converges to zero, then no test can be constructed to distinguish the two models. We elaborate a fine analysis of a Gaussian approximation for the small jumps of Lévy processes in total variation distance. Non asymptotic bounds for the total variation distance between n discrete observations of small jumps of a Lévy process and the corresponding Gaussian distribution is presented and extensively discussed. As a byproduct, new upper bounds for the total variation distance between discrete observations of Lévy processes are provided. The theory is illustrated by concrete examples.

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1 Introduction

Although Lévy processes, or equivalently infinite divisible distributions, are mathematical objects introduced almost a century ago and even though a good knowledge of their basic properties has since long been achieved, they have recently enjoyed renewed interest. This is mainly due to the numerous applications in various fields. To name some examples, Lévy processes or Lévy-type processes (time changed Lévy processes, Lévy driven SDE, etc...) play a central role in mathematical finance, insurance, telecommunications, biology, neurology, telecommunications, seismology, meteorology and extreme value theory. Examples of applications may be found in the textbooks [2] and [6] whereas the manuscripts [4] and [17] provide a comprehensive presentation of the properties of these processes.

The transition from the purely theoretical study of Lévy processes to the need to understand Lévy driven models used in real life applications has led to new challenges. For instance, the questions of how to simulate the trajectories of Lévy processes and how to make inference (prediction, testing, estimation, etc...) for this class of stochastic processes have become a key issue. The literature concerning these two aspects is already quite large; without any claim of completeness, we quote [1], Chapter VI in [2], [3], [5] and Part II in [6]. We specifically focus on statistics and simulation for Lévy processes, because our paper aims to give an exhaustive answer to a recurring question in these areas: When are the small jumps of Lévy processes morally Gaussian?

Before entering into details, we take a step back and see where this question comes from. Thanks to the Lévy-Itô decomposition, the structure of the paths of any Lévy process is well understood and

*Otto von Guericke Universität Magdeburg, Germany. E-mail: alexandra.carpentier@ovgu.de.

[†]Université Paris Descartes, Laboratoire MAP5, UMR CNRS 8145, France. E-mail: celine.duval@parisdescartes.fr.

[‡]Universität Potsdam, Germany. E-mail: mariucci@uni-potsdam.de.

it is well known that any Lévy process X can be decomposed into the sum of three independent Lévy processes: a Brownian motion with drift, a centered martingale M associated with the small jumps of X and a compound Poisson process describing the big jumps of X (see the decomposition (2) in Section 1 below). If the properties of continuously or discretely observed compound Poisson processes and of Gaussian processes are well understood, the same cannot be said for the small jumps M . As usual in mathematics, when one faces a complex object a natural reflection is whether the problem can be simplified by replacing the difficult part with an easier but, in a sense, equivalent one. There are various notions of equivalence ranging from the weakest, convergence in law, to the stronger convergence in total variation.

For some time now, many authors have noticed that marginal laws of small jumps of Lévy processes with infinite Lévy measures resemble to Gaussian random variables, see Figure 1 and 2. This remark

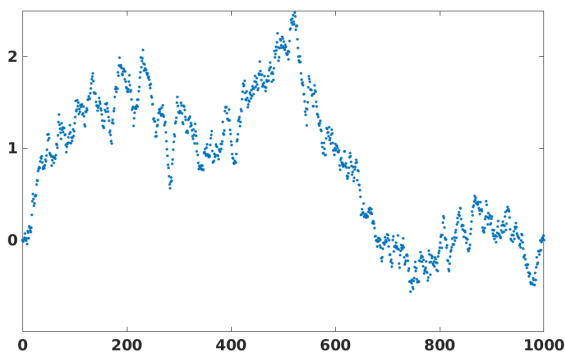


Figure 1: Discretized trajectory of a Lévy process $(0, 0, \frac{1_{(0,\varepsilon]}(x)}{x^{1+\beta}})$ for $(n = 10^3, \Delta = 1, \varepsilon = 10^{-2}, \beta = 0.9)$.

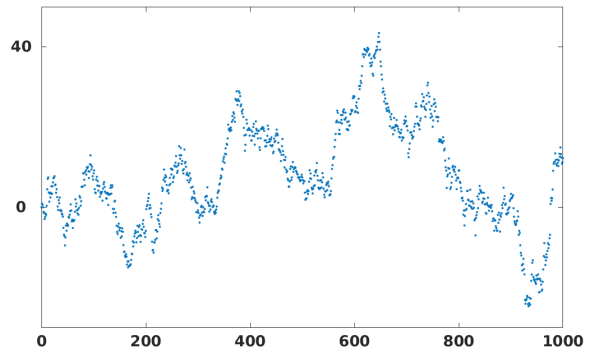


Figure 2: Discretized trajectory of a Lévy process $(0, 0, \frac{1_{[-\varepsilon,\varepsilon] \setminus \{0\}}(x)}{x^{1+\beta}})$ for $(n = 10^3, \Delta = 1, \varepsilon = 0.5, \beta = 1.8)$.

has led to propose algorithms of simulation of trajectories of Lévy processes based on a Gaussian approximation of the small jumps, see e.g. [5] or [6], Chapter 6. Regarding estimation procedures, a Gaussian approximation of the small jumps has, to the best of our knowledge, not been exploited yet. A fine control of the total variation distance between these two quantities could open the way of new statistical procedures. The choice of this distance is justified by its statistical interpretation: if the total variation distance between the law of the small jumps and the corresponding Gaussian component converges to zero then no statistical test can be built to distinguish between the two models. In terms of information theory, this means that the two models are asymptotically equally informative.

Investigating the goodness of a Gaussian approximation of the small jumps of a Lévy process in total variation distance makes sense only if one deals with discrete observations. From the continuous observation of a Lévy process, the problem of separating the continuous part from the jumping part does not arise: the jumps are observed. The measure corresponding to the continuous observation of a continuous Lévy process is orthogonal to the measure corresponding to the continuous observation of a Lévy process with non trivial jump part, see e.g. [11]. However, the situation changes when dealing with discrete observations. The matter of disentangling continuous and discontinuous part of the processes is much more complex. Intuitively, fine techniques are needed to understand whether, between two observations X_{t_0} and X_{t_1} , there has been a chaotic continuous behavior, many small jumps, one single bigger jump, or a mixture of these.

A criterion for the weak convergence for marginals of Lévy processes is given by Gnedenko and Kolmogorov [9]:

Theorem 1 (Gnedenko, Kolmogorov). *Marginals of Lévy processes $X^n = (X_t^n)_{t \geq 0}$ with Lévy triplets (b_n, σ_n^2, ν_n) converge weakly to marginals of a Lévy process $X = (X_t)_{t \geq 0}$ with Lévy triplet (b, σ^2, ν) if and only if*

$$b_n \rightarrow b \text{ and } \sigma_n^2 \delta_0 + (x^2 \wedge 1) \nu_n(dx) \xrightarrow{w} \sigma^2 \delta_0 + (x^2 \wedge 1) \nu(dx),$$

where δ_0 is the Dirac measure in 0 and \xrightarrow{w} denotes weak convergence of finite measures.

A remarkable fact in the previous statement is the non-separation between the continuous and discontinuous parts of the processes: the law at time t of a pure jumps Lévy process can weakly converge to that of a continuous Lévy process. In particular, if X is a Lévy process with Lévy measure ν then, for any $\varepsilon > 0$ and $t > 0$, the law of the centered jumps of X_t with magnitude less than ε converges weakly to a centered Gaussian distribution with variance $t\sigma^2(\varepsilon) := t \int_{|x| < \varepsilon} x^2 \nu(dx)$ as $\varepsilon \rightarrow 0$. We aim to understand this phenomenon, using a notion of closeness stronger than the weak convergence, providing a quantitative translation of the result of Gnedenko and Kolmogorov in total variation distance.

There exist already several results for distances between Lévy processes. Most of them (see for example [8], [11] and [12]) are distances on the Skorohod space, distances between the continuous observation of the processes, and thus out of the scope of this paper. Concerning discretely observed Lévy processes we mention the results in [13] and [14]. Liese [13] proved the following upper bound in total variation for marginals of Lévy processes $X^j \sim (b_j, \Sigma_j^2, \nu_j)$, $j = 1, 2$: for any $t > 0$

$$\|\mathcal{L}(X_t^1) - \mathcal{L}(X_t^2)\|_{TV} \leq 2 \sqrt{1 - \left(1 - \frac{H^2(\mathcal{N}(\tilde{b}_1, t\Sigma_1^2), \mathcal{N}(\tilde{b}_2, t\Sigma_2^2))}{2}\right)^2 \exp(-tH^2(\nu_1, \nu_2))}$$

with $\tilde{b}_1 = b_1 - \int_{-1}^1 x \nu_1(dx)$, $\tilde{b}_2 = b_2 - \int_{-1}^1 x \nu_2(dx)$ and H denotes the Hellinger distance. This result is the analogous in discrete time of the result of Mémin and Shyryaev [15] for continuously observed Lévy processes. There is a clear separation between the continuous and discontinuous parts of the processes, which is unavoidable on the Skorohod space but that can be relaxed when dealing with the marginals. Clearly, from this kind of upper bounds it is not possible to deduce a Gaussian approximation for the small jumps in total variation: the bound is actually trivial whenever $tH^2(\nu_1, \nu_2) > 1$.

Such an approximation may hold in total variation as proved in [14], where the convolution structure of Lévy processes with non-zero diffusion coefficients is exploited to transfer results from Wasserstein distances to total variation distance.

In the present paper we complete the work started in [14], providing a comprehensive answer to the question: Under which asymptotics does a Gaussian approximation capture the behavior of the small jumps of a discretely observed Lévy process adequately so that the two corresponding statistical models are equally informative? Differently from [14] which deals with marginals, we also establish sharp bounds for the distance between n given increments of the small jumps. This is an essential novelty. Even though from a bound in total variation between marginals one can always deduce a bound for the n sample using $\|P^{\otimes n} - Q^{\otimes n}\|_{TV} \leq \sqrt{2n}\|P - Q\|_{TV}$, this kind of control is in general sub-optimal as in many situations, $\|P^{\otimes n} - Q^{\otimes n}\|_{TV}$ is of order of $\sqrt{n}\|P - Q\|_{TV}$, as n goes to infinity. This faster rate can be obtained using the methods developed in the present work.

Our main results are presented below in a simplified context. First, we established an upper bound for the Gaussian approximation of the small jumps in total variation.

Main Result 1. *Let $\varepsilon > 0$ and $M(\varepsilon)$ be a Lévy process with Lévy triplet $(0, 0, \nu_\varepsilon)$ where ν_ε is a Lévy measure support in $[-\varepsilon, \varepsilon]$. Set $\sigma^2(\varepsilon) = \int_{-\varepsilon}^\varepsilon x^2 \nu_\varepsilon(dx)$, $\mu_3(\varepsilon) = \int_{-\varepsilon}^\varepsilon x^3 \nu_\varepsilon(dx)$ and $\mu_4(\varepsilon) = \int_{-\varepsilon}^\varepsilon x^4 \nu_\varepsilon(dx)$.*

Fix $b \in \mathbb{R}$ and $\Sigma \geq 0$ as well as $\Delta > 0$, $n \geq 1$. Under the assumptions of Theorem 2 below, there exists a constant $C > 0$ that depends only on the constants in the assumptions of Theorem 2 (and

therefore independent of $b, \Sigma, \nu, \Delta, n$ and ε) and such that

$$\begin{aligned} & \|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n}\|_{TV} \leq \\ & C \left[\sqrt{\frac{n\mu_4^2(\varepsilon)}{\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^4} + \frac{n\mu_3^2(\varepsilon)}{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^3} + \frac{1}{n}} \right]. \end{aligned}$$

Main Result 1 is non-asymptotic, which allows to quantify just how “small” the small jumps must be, in terms of the number of observations n and their frequency Δ , in order for it to be close in total variation to the corresponding Gaussian distribution.

More precisely, fix n and Δ , provided $\mu_4^2(\varepsilon)/(\sigma^2(\varepsilon))^4 \rightarrow 0$ and $\mu_3^2(\varepsilon)/(\sigma^2(\varepsilon))^3 \rightarrow 0$ as $\varepsilon \rightarrow 0$, the total variation in Main Result 1 for $\Sigma = 0$, that we write $\tilde{r}_{n,\Delta}(\varepsilon)$, is bounded under the assumptions of Main Result 1 by a quantity of order

$$\sqrt{n}r_\Delta(\varepsilon) + 1/n := \sqrt{n} \left(\frac{\mu_4(\varepsilon)}{\Delta(\sigma^2(\varepsilon))^2} + \frac{|\mu_3(\varepsilon)|}{\sqrt{\Delta}(\sigma^2(\varepsilon))^{3/2}} \right) + \frac{1}{n}. \quad (1)$$

A sufficient condition to ensure $\tilde{r}_{n,\Delta}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ is that $\varepsilon/\sigma(\varepsilon) \rightarrow_{\varepsilon \rightarrow 0} 0$ - since we have, taking $N = \max(n, 1/r_\Delta(\varepsilon)^{2/3}) \geq n$, that $\tilde{r}_{n,\Delta}(\varepsilon) \leq \tilde{r}_{N,\Delta}(\varepsilon)$. It is straightforward to see that if $\mu_3(\varepsilon) \neq 0$ then $\tilde{r}_{n,\Delta}(\varepsilon) \rightarrow_{\varepsilon \rightarrow 0} 0$ as soon as $\sqrt{n}\varepsilon/\sqrt{\Delta}\sigma(\varepsilon) \rightarrow_{\varepsilon \rightarrow 0} 0$. When $\mu_3(\varepsilon) = 0$, this can be further improved to the condition $\sqrt{n}\varepsilon^2/\Delta\sigma^2(\varepsilon) \rightarrow_{\varepsilon \rightarrow 0} 0$. To exemplify, consider the small jumps of symmetric β -stable processes with Lévy measure $\nu_\varepsilon = \mathbf{1}_{|x| \leq \varepsilon}/|x|^{1+\beta}$, $\beta \in (0, 2)$. Then, a sufficient condition for $\tilde{r}_{n,\Delta}(\varepsilon) \rightarrow_{\varepsilon \rightarrow 0} 0$ is $\sqrt{n}\varepsilon^\beta/\Delta \rightarrow_{\varepsilon \rightarrow 0} 0$ - see Theorem 3 and Proposition 2.

An interesting byproduct of Main Result 1 is Theorem 6 in Section 3 which provides a new upper bound for the total variation distance between n given increments of two Lévy processes. A peculiarity of the result is the non-separation between the continuous and discontinuous part of the processes. Then, Theorem 6 is close in spirit to Theorem 1 although it holds for the stronger notion of total variation distance.

Main Result 1 can be sharpened by considering separately large and rare jumps, see Theorem 3 - this has an impact in the case where the jumps of size of order ε are very rare. It is optimal, in the sense that whenever the jumps of size of order ε are not “too rare” and whenever the above quantity r in (1) is larger than a constant, then the upper bound in Main Result 1 is trivial, but the total variation distance can be bounded away from 0 as shown in Main Result 2 below.

Main Result 2. *Let ν be a Lévy measure, $b \in \mathbb{R}$ and $\Sigma \geq 0$. For any $\varepsilon > 0$, $\Delta > 0$ and $n \geq 1$ such that $\varepsilon \leq \sqrt{(\Sigma^2 + \sigma^2(\varepsilon))\Delta \log(e \vee n)^2}$ and $\int_{-\varepsilon}^{\varepsilon} d\nu \geq \Delta^{-1} \vee (\log(e \vee n)/(n\Delta))$, there exist an absolute sequence $\alpha_n \rightarrow 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta$) and an absolute constant $C > 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta, n$) such that the following holds:*

$$\begin{aligned} & \|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n}\|_{TV} \geq \\ & \left\{ 1 - C \left[\frac{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^3}{n\mu_3(\varepsilon)^2} \wedge \frac{\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^4}{n\mu_4(\varepsilon)^2} \right] - \alpha_n \right\}, \end{aligned}$$

where $M(\varepsilon)$ is a Lévy process with Lévy triplet $(0, 0, \nu \mathbf{1}_{[-\varepsilon, \varepsilon]})$.

A more technical and general lower bound, that holds without condition on the rarity of the jumps of size of order ε , is also available, see Theorem 5. This lower bound matches in order the general upper bound of Theorem 3 - implying optimality without conditions of our results. The proof of the lower bound for the total variation is based on the construction of a sharp Gaussian test for Lévy processes. This test combines three ideas, (i) the detection of extreme values that are “too large” for being produced by a Brownian motion, (ii) the detection of asymmetries around the drift in the third moment, and (iii) the detection of too heavy tails in the fourth moment for a Brownian motion. It

can be of independent interest as it does not rely on the knowledge of the Lévy triplet of the process and detects optimally the presence of jumps. It uses classical ideas from testing through moments and extreme values [10], and it adapts them to the specific structure of Lévy processes. The closest related work is [16]. We improve on the test proposed there as we go beyond testing based on the fourth moment only, and we tighten the results regarding the presence of rare and large jumps.

The paper is organized as follows. In the remaining of this Section we fix notations. Section 2 is devoted to the analysis of the Gaussian approximation of the small jumps of Lévy processes. More precisely, in Section 2.1 we present upper bounds in total variation distance whereas in Section 2.2 we provide lower bounds proving the optimality of our findings. In Section 3 new upper bounds for the total variation distance between n given increments of two general Lévy processes are derived. Most of the proofs are postponed to Section 4. The paper also contains two Appendices. In Appendix A technical results can be found. In Appendix B we recall some results about total variation distance and present some general, and probably not new, results about the total variation distance between Gaussian distributions and discrete observations of compound Poisson processes.

Statistical setting and notation

For X a Lévy process with Lévy triplet (b, Σ^2, ν) (we write $X \sim (b, \Sigma^2, \nu)$), where $b \in \mathbb{R}$, $\Sigma \geq 0$ and ν satisfies $\int (x^2 \wedge 1) \nu(dx) < \infty$, the Lévy-Itô decomposition gives a decomposition of X into the sum of three independent Lévy processes: a Brownian motion with drift b , a centered martingale associated with the small jumps of X and a compound Poisson process associated with the large jumps of X . More precisely, for any $\varepsilon > 0$, $X \sim (b, \Sigma^2, \nu)$ can be decomposed as

$$\begin{aligned} X_t &= b(\varepsilon)t + \Sigma W_t + \lim_{\eta \rightarrow 0} \left(\sum_{s \leq t} \Delta X_s \mathbf{1}_{\eta < |\Delta X_s| \leq \varepsilon} - t \int_{\eta < |x| \leq \varepsilon} x \nu(dx) \right) + \sum_{s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| > \varepsilon}, \\ &:= b(\varepsilon)t + \Sigma W_t + M_t(\varepsilon) + Z_t(\varepsilon), \quad \forall t \geq 0 \end{aligned} \quad (2)$$

where $\Delta X_t = X_t - \lim_{s \uparrow t} X_s$ denotes the jump at time t of X and

- $b(\varepsilon) := b + \begin{cases} - \int_{\varepsilon < |x| \leq 1} x \nu(dx) & \text{if } \varepsilon \leq 1 \\ \int_{1 < |x| \leq \varepsilon} x \nu(dx) & \text{if } \varepsilon > 1. \end{cases}$;
- $W = (W_t)_{t \geq 0}$ is a standard Brownian motion;
- $M(\varepsilon) = (M_t(\varepsilon))_{t \geq 0}$ is a centered Lévy process (and a martingale) with a Lévy measure $\nu_\varepsilon := \nu \mathbf{1}_{[-\varepsilon, \varepsilon]}$ with support in $[-\varepsilon, \varepsilon]$ i.e. it is the Lévy process associated to the jumps of X smaller than ε . We write $\sigma^2(\varepsilon) = \int x^2 \nu_\varepsilon(dx)$ for the variance at time 1 of $M(\varepsilon)$.
- $Z(\varepsilon) = (Z_t(\varepsilon))_{t \geq 0}$ is a Lévy process with a Lévy measure concentrated on $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$ i.e. it is a compound Poisson process of the form $Z_t(\varepsilon) := \sum_{i=1}^{P_t} Z_i$ with intensity $\Lambda_\varepsilon := \nu(\mathbb{R} \setminus [-\varepsilon, \varepsilon])$ and jump measure $\mathbb{P}(Z_1 \in B) = \frac{1}{\Lambda_\varepsilon} \int_B \mathbf{1}_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(x) \nu(dx)$;
- W , $M(\varepsilon)$ and $Z(\varepsilon)$ are independent.

Note that in the present article we only focus on W and $M(\varepsilon)$ and $Z(\varepsilon)$ will not be taken into account.

Finally, we recall that the total variation distance between two probability measures P_1 and P_2 defined on the same σ -field \mathcal{B} is defined as

$$\|P_1 - P_2\|_{TV} := \sup_{B \in \mathcal{B}} |P_1(B) - P_2(B)| = \frac{1}{2} \int \left| \frac{dP_1}{d\mu}(x) - \frac{dP_2}{d\mu}(x) \right| \mu(dx),$$

where μ is a common dominating measure for P_1 and P_2 . In order to ease the reading, if X and Y are random variables with densities f_X and f_Y with respect to a same dominating measure, we sometimes write $\|X - Y\|_{TV}$ or $\|f_X - f_Y\|_{TV}$ instead of $\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV}$.

2 Gaussian approximation for the Lévy process in total variation distance

2.1 Upper bound results

We investigate under which conditions on Δ , n and ε , a Gaussian approximation of the small jumps is valid, provided that ν is sufficiently active. Define

$$\lambda_{\eta,\varepsilon} := \int_{\eta < |x| < \varepsilon} \nu(dx), \quad 0 \leq \eta < \varepsilon, \quad \text{and} \quad \lambda_{0,\varepsilon} = \lim_{\eta \rightarrow 0} \lambda_{\eta,\varepsilon}, \quad (3)$$

where $\lambda_{0,\varepsilon} = +\infty$ if ν is an infinite Lévy measure. These quantities account for the activity of the Lévy measure ν .

Theorem 2. *Let ν be Lévy measure, fix $0 < \varepsilon$, $\Delta > 0$, $n \geq 1$, $b \in \mathbb{R}$ and $\Sigma \geq 0$. Assume that $\lambda_{0,\varepsilon} \geq 24 \log(e \vee n)/\Delta$ and that there exist two constants $c > 1, C' > 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta$ and n) such that for all integers $k \in [0, 401 \log(e \vee n)]$ it holds*

$$\int_{c \log(e \vee n)}^{+\infty} |\Psi_\varepsilon^{(k)}(t)|^2 dt \leq C' 2^k k! n^{-2}, \quad \text{where } \Psi_\varepsilon(t) := \mathbb{E}[e^{it\tilde{X}_\Delta(\varepsilon)}] - e^{-\Delta \lambda_{0,\varepsilon}} \mathbf{1}_{\{\Sigma=0\}} \quad (\mathcal{H}_\Psi(\varepsilon))$$

$$\text{and } \tilde{X}_\Delta(\varepsilon) := \frac{\Sigma W_\Delta + M_\Delta(\varepsilon)}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}}.$$

Finally, assume that there exists $\tilde{c} \in (0, 1]$ such that

$$c\tilde{c} \leq \sqrt{\log(e \vee n)}/4 \quad \text{and} \quad \varepsilon \leq \tilde{c} \sqrt{(\sigma^2(\varepsilon) + \Sigma^2)\Delta} / \sqrt{\log(e \vee n)}.$$

Then there exists a constant $C > 0$ that depends only on c, C' and such that

$$\|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n}\|_{TV} \leq C \left[\sqrt{\frac{n\mu_4^2(\varepsilon)}{\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^4} + \frac{n\mu_3^2(\varepsilon)}{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^3} + \frac{1}{n}} \right]. \quad (4)$$

Remark 1. In $(\mathcal{H}_\Psi(\varepsilon))$, Ψ_ε is the characteristic function of the rescaled increment $\tilde{X}_\Delta(\varepsilon)$ restricted to the event “the process jumps at least once in the case of a finite pure jump Lévy processes.” Otherwise, if the Lévy measure is infinite and/or if it has a Brownian component, it is simply the characteristic function of $\tilde{X}_\Delta(\varepsilon)$. The addition of the indicator function in Ψ_ε permits to keep (some) compound Poisson processes in the study, e.g. compound Poisson processes with continuous jump density, for which $\lim_{|t| \rightarrow +\infty} \mathbb{E}[\exp(it\tilde{X}_\Delta(\varepsilon))] \rightarrow e^{-\Delta \lambda_{0,\varepsilon}}$. However these compound Poisson processes should also satisfy $\lambda_{0,\varepsilon} \geq 24 \log(e \vee n)/\Delta$, i.e. their intensity cannot be too small. The latter assumption implies that the probability of observing no jump in any of our n observations converges to 0, which is a necessary assumption if $\Sigma = 0$ in order to avoid getting a trivial bound, as explained below in Subsection 2.1.1.

If $\Sigma = 0$, we immediately get the following Corollary using that

$$\mu_4(\varepsilon) \leq \varepsilon^2 \sigma^2(\varepsilon), \quad \mu_3(\varepsilon) \leq \varepsilon \sigma^2(\varepsilon). \quad (5)$$

Corollary 1. *Let ν be a Lévy measure satisfying the assumptions of Theorem 2, and consider the same notations. Then, it holds*

- If ν is symmetric, then,

$$\|M_\Delta(\varepsilon)^{\otimes n} - \mathcal{N}(0, \Delta\sigma^2(\varepsilon))^{\otimes n}\|_{TV} \leq C \left[\sqrt{n \frac{\varepsilon^4}{\Delta^2\sigma^4(\varepsilon)}} + \frac{1}{n} \right]. \quad (6)$$

- If ν is not symmetric, then,

$$\|M_\Delta(\varepsilon)^{\otimes n} - \mathcal{N}(0, \Delta\sigma^2(\varepsilon))^{\otimes n}\|_{TV} \leq C \left[\sqrt{n \frac{\varepsilon^2}{\Delta\sigma^2(\varepsilon)}} + \frac{1}{n} \right]. \quad (7)$$

Theorem 2 can be improved as follows. We provide a control of the distance between the increments of a Lévy process $(b, \Sigma^2, \nu_\varepsilon)$, where ν_ε has support $[-\varepsilon, \varepsilon]$ and the closest Gaussian distribution, which may not be the one with average $b\Delta$ and variance $\Delta(\Sigma^2 + \sigma^2(\varepsilon))$. It permits to weaken the assumption $\varepsilon \lesssim \sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))/\log n}$ through the introduction of the following quantities. Set $u^+ := u_{\nu_\varepsilon}^+$ for the largest positive real number such that

$$\lambda_{u^+, \varepsilon} n \Delta \geq \log(e \vee n).$$

Note that such a u^+ exists and is unique whenever ν_ε is a Lévy measure such that $\lambda_{0, \varepsilon} \geq \frac{\log(e \vee n)}{n \Delta}$, which holds under the assumptions of Theorem 2. Consider the quantity

$$\tilde{u}^* = \sup \left\{ u : u \in [u^+, \varepsilon], u \leq \tilde{c} \sqrt{(\sigma(u)^2 + \Sigma^2) \Delta} / \sqrt{\log(e \vee n)} \right\} \vee u^+.$$

The introduction of the latter quantity permits to remove the assumption on ε in Theorem 2.

Theorem 3. Let ν be a Lévy measure and fix $0 < \varepsilon, \Delta > 0, n \geq 1, b \in \mathbb{R}$ and $\Sigma \geq 0$. Assume that $\lambda_{0, \varepsilon} \geq 24 \log(e \vee n) / \Delta$. Assume that there exist two constants $c > 1, C' > 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta$ and n) such that Assumption $(\mathcal{H}_\Psi(\varepsilon))$ is satisfied at point \tilde{u}^* , i.e. for all integers k such that $401 \log(e \vee n) \geq k \geq 0$

$$\int_{c \log(e \vee n)}^{+\infty} |\Psi_{\tilde{u}^*}^{(k)}(t)|^2 dt \leq C' 2^k k! n^{-2}, \quad \text{where } \Psi_{\tilde{u}^*}(t) := \mathbb{E}[e^{it \tilde{X}_\Delta(\tilde{u}^*)}] - e^{-\Delta \lambda_{0, \tilde{u}^*}} \mathbf{1}_{\{\Sigma=0\}} \quad (\mathcal{H}_\Psi(\tilde{u}^*))$$

$$\text{and } \tilde{X}_\Delta(\tilde{u}^*) := \frac{\Sigma W_\Delta + M_\Delta(\tilde{u}^*)}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))}}.$$

Assume that $1 \geq \tilde{c} > 0$ and $c\tilde{c} \leq \sqrt{\log(e \vee n)}/4$, in the definition of \tilde{u}^* . Then, there exists a constant $C > 0$ that depends only on c, C' such that

$$\min_{B \in \mathbb{R}, S^2 \geq 0} \left\| (\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(B\Delta, \Delta S^2)^{\otimes n} \right\|_{TV} \leq \left[e^{-\lambda_{\tilde{u}^*, \varepsilon} n \Delta} C \left[\sqrt{\frac{n\mu_4^2(\tilde{u}^*)}{\Delta^2(\Sigma^2 + \sigma^2(\tilde{u}^*))^4} + \frac{n\mu_3^2(\tilde{u}^*)}{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^3} + 1/n} \right] + [1 - e^{-\lambda_{\tilde{u}^*, \varepsilon} \Delta n}] \right]. \quad (8)$$

2.1.1 Comments

Discussion on the rates of Theorem 2 and Corollary 1. The results are non asymptotic and we stress that (4), (6) (7) and (8) hold without assuming to work with high or low frequency observations. An explicit relation between ε, Σ, n and Δ depending on ν via $\sigma^2(\varepsilon), \mu_3(\varepsilon)$ and $\mu_4(\varepsilon)$ is given. More precisely, we derive from Theorem 2 and under its assumptions that the total variation distance in (4) is bounded by

$$C \left(\sqrt{n} \left(\frac{\mu_4(\varepsilon)}{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^2} + \frac{\mu_3(\varepsilon)}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^3}} \right) + \frac{1}{n} \right). \quad (9)$$

As highlighted in Corollary 1, a sufficient condition (under the assumptions of Theorem 2) for the total variation distance to be small is given by $\varepsilon/\sigma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, with a rate depending on n, Δ . Unsurprisingly, we observe that the rate of convergence for a Gaussian approximation of the small jumps is faster when the Lévy measure is symmetric.

The assumption of Theorem 2 that $\varepsilon \lesssim \sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}/\sqrt{\log n}$ imposes a constraint on the cumulants of ν . It is restrictive, but intuitively meaningful, it means that the jumps cannot take values that are too extreme with respect to the standard deviation of the increments. These extreme values would indeed enable to differentiate it easily from a Gaussian distribution. Theorem 3 allows to get rid of this assumption.

Finally, the remainder term $1/n$ in (4) is a consequence of the strategy of the proof and can be improved, at the expense of modifying some details in the proof.

Discussion on Theorem 3. A restrictive assumption in Theorem 2 and Corollary 1 is that $\varepsilon \leq \tilde{c}\sqrt{(\sigma^2(\varepsilon) + \Sigma^2)\Delta}/\sqrt{\log(e \vee n)}$, *i.e.* ε smaller than the standard deviation $\sqrt{\Delta(\sigma^2(\varepsilon) + \Sigma^2)}$ of the increments (up to a multiplicative constant and a logarithmic term). This assumption is avoided in Theorem 3, which follows directly from Theorem 2 by dividing the total variation on two events, where respectively all jumps are smaller than \tilde{u}^* , or where there is at least one jump larger than \tilde{u}^* . The idea behind this subdivision is that the Gaussian distribution that is the closest to $\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n}$ is not $\mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n}$, but rather $\mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\tilde{u}^*)))^{\otimes n}$. Indeed all jumps that are larger than \tilde{u}^* are very rare and large enough to be recognized as non-Gaussian. The upper bound on the total variation distance is then composed of two terms, the first one that appears in (9), but expressed in \tilde{u}^* and not ε , and the second one is the probability of observing a jump larger than \tilde{u}^* , namely $\exp(-n\Delta\lambda_{\tilde{u}^*, \varepsilon})$.

Remark on the assumption on $\lambda_{0, \varepsilon}$. What one needs for establishing Theorems 2 and 3 and Corollary 1, is that $\lambda_{0, \varepsilon} \geq 24 \log(e \vee n)/\Delta$. For establishing Theorem 4 (lower bound on the total variation distance in Section 2.2) we only need that $\lambda_{0, \varepsilon} \geq \Delta^{-1} \vee (\log(e \vee n)/(n\Delta))$. Indeed, if this is not the case, the asymptotic total variation is either 0 or 1.

- **Case 1 :** $a_n \Delta^{-1} n^{-1} \geq \lambda_{0, \varepsilon}$ **where** $a_n \rightarrow 0$. In this case one does never observe any jump with probability going to 1. So the total variation distance goes to 0 as $n \rightarrow \infty$.
- **Case 2 (only in the noiseless case $b = 0, \Sigma = 0$):** $a_n \Delta^{-1} n^{-1} \leq \lambda_{0, \varepsilon} \leq A_n \log(e \vee n) \Delta^{-1}$ **where** $a_n \rightarrow \infty$ **and** $A_n \rightarrow 0$. In this case the probability of observing at least a time step where one, and only one, jump occurs goes to 1, as well as the probability of having at least one time step where no jump occur goes to 1 as $n \rightarrow \infty$. We conclude therefore that the total variation distance goes to 1: such a process is very different from a Gaussian process.

Discussion on Assumption $(\mathcal{H}_\Psi(\varepsilon))$. Assumption $(\mathcal{H}_\Psi(\varepsilon))$ is technical, but does not seem to restrict drastically the class of Lévy processes one can consider. It holds for instance for the large class of alpha-stable processes - see Proposition 2 in Section 2.3 - which describes well the behavior of many Lévy densities around 0. It also holds as soon as Σ is large, *i.e.* whenever Σ is larger than $\sigma(\varepsilon)\sqrt{\log(e \vee n)}$ - see Proposition 1 in Section 2.3. Most usual compound Poisson processes with Lebesgue continuous jump density and intensity $\lambda_{0, \varepsilon}$ large enough also seem to satisfy it. It imposes a condition on the decay of the derivatives of the characteristic function of the rescaled increment, on the event where at least a jump is observed. A condition related to Assumption $(\mathcal{H}_\Psi(\varepsilon))$ in the particular case where $k = 0$ has already been investigated (see *e.g.* Trabs [18]), but the results therein do not apply to infinite Lévy densities.

This assumption is not straightforward to interpret, but we report the following observation. A necessary condition for it to be satisfied is that the characteristic function Ψ_ε of the rescaled increment - on the event where at least a jump is observed - goes to 0. Examples for which $(\mathcal{H}_\Psi(\varepsilon))$ does not

hold are for instance Lévy processes such that $\Sigma = 0$ and ν contains a finite number of Dirac masses. This is coherent, observations of a pure jump process with jump law taking finitely many values are perfectly detectable from Gaussian observations, *i.e.* the total variation is 1. However, if the law of the increments contains Dirac masses but if the total probability of these masses is much smaller than $1/n$, this in principle does not disturb the total variation bound. This is why, our analysis allows to consider compound Poisson processes whenever $\lambda_{0,\varepsilon} \geq 24 \log(e \vee n)/\Delta$ (and whenever $\lambda_{0,\varepsilon} \lesssim \log(e \vee n)/\Delta$, the bound on the total variation becomes trivial for $\Sigma = 0$ as noted above).

2.2 Lower bound

Theorem 2 is optimal in the following sense. If the upper bound of Theorem 2 does not converge, then the total variation distance between the random vector associated with the increments of the small jumps -eventually convoluted with Gaussian distributions- and the corresponding Gaussian random vector does not converge to 0.

Theorem 4. *Let ν be a Lévy measure, let $b \in \mathbb{R}$ and $\Sigma \geq 0$. Assume that $\lambda_{0,\varepsilon} \geq \Delta^{-1} \vee (\log(e \vee n)/(n\Delta))$. For any $\varepsilon > 0$ such that $\varepsilon \leq \sqrt{(\sigma^2(\varepsilon) + \Sigma^2)\Delta \log(e \vee n)^2}$, there exists an absolute sequence $\alpha_n \rightarrow 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta$) and an absolute constant $C > 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta$ and n) such that the following holds:*

$$\begin{aligned} & \left\| (\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n} \right\|_{TV} \geq \\ & \left\{ 1 - C \left[\frac{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^3}{n\mu_3(\varepsilon)^2} \wedge \frac{\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^4}{n\mu_4(\varepsilon)^2} \right] - \alpha_n \right\}. \end{aligned}$$

To establish Theorem 4 we construct an appropriate statistical test and use the following fact.

Lemma 1. *Let \mathbb{P} and \mathbb{Q} be two probability distributions and Φ a test of level $\alpha_0 \in (0, 1)$ that separates \mathbb{P} from \mathbb{Q} from n i.i.d. observation with power larger than $1 - \alpha_1$. Then, $\|\mathbb{P}^{\otimes n} - \mathbb{Q}^{\otimes n}\|_{TV} \geq 1 - \alpha_0 - \alpha_1$.*

Let $X \sim (b, \Sigma^2, \nu)$, with b, Σ^2 and ν possibly unknown and consider the problem of testing whether the process contains jumps of size smaller than ε or not, *i.e.* whether $\nu_\varepsilon = 0$ or not, recall that $\nu_\varepsilon = \nu \mathbf{1}_{[-\varepsilon, \varepsilon]}$ and that we defined u^+ for the largest positive real number such that $\lambda_{u^+, \varepsilon} n\Delta \geq \log(e \vee n)$. Write now u^* for the largest $u \in [u^+, \varepsilon]$ such that

$$u^* = \sup\{u, u \in [u^+, \varepsilon], u \leq \sqrt{\Delta(\Sigma^2 + \sigma^2(u)) \log(e \vee n)^2}\} \vee u^+,$$

where $\sup \emptyset = -\infty$.

We prove the following result, of which Theorem 4 is an immediate corollary.

Theorem 5. *Suppose that $\varepsilon > 0$, ν is a Lévy measure, $b \in \mathbb{R}$ and $\Sigma \geq 0$. Assume that $\lambda_{0,\varepsilon} \geq \Delta^{-1} \vee (\log(e \vee n)/(n\Delta))$. There exists an absolute sequence $\alpha_n \rightarrow 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta$) and an absolute constant $C > 0$ (independent of $b, \nu, \varepsilon, \Sigma, \Delta$ and n) such that the following holds:*

$$\begin{aligned} & \min_{B \in \mathbb{R}, \Sigma^2 \geq 0} \left\| (\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(B\Delta, \Delta\Sigma^2)^{\otimes n} \right\|_{TV} \geq \\ & \left\{ 1 - C \left[\frac{\Delta(\Sigma^2 + \sigma^2(u^*))^3}{n\mu_3(u^*)^2} \wedge \frac{\Delta^2(\Sigma^2 + \sigma^2(u^*))^4}{n\mu_4(u^*)^2} \right] - \alpha_n \right\} \vee \left(1 - \exp(-\lambda_{u^*, \varepsilon} n\Delta) - \alpha_n \right). \end{aligned}$$

The construction of the test we use to derive Theorem 5 is actually quite involved, we refer to Section 4.3.1 for more details. Here, we only illustrate the main ideas.

First, the intuitions behind the quantities u^+ and u^* are the following. The quantity u^+ is chosen such that with probability going to 1, there is (i) at least one jump larger than u^+ but (ii) not too

many of such jumps, *i.e.* less than $2\log(e \vee n)$, and finally (iii) at most one jump larger than u^+ per time increment Δ . Therefore, the discretized process of jumps larger than u^+ and smaller than ε (in absolute value), does not look Gaussian at all. It is composed of many null entries and a few larger than u^+ . Now u^* is the largest quantity (larger than u^+) such that u^* is smaller than a number slightly larger than the standard deviation of the increment $X_{i\Delta} - X_{(i-1)\Delta}$ *conditional to the event* there are no jumps larger than u^* in $((i-1)\Delta, i\Delta]$. In other words, any increment having a jump larger than u^* is going to be quite visible.

Second, the idea behind the test, is to build an event that occurs with high probability if $\nu_\varepsilon = 0$ and with small probability otherwise. This would then allow to bound the total variation distance between the two discretized processes using Lemma 1. The sketch of the proof is the following (all results mentioned below are stated and proved in Section 4 and Appendix A):

- First, we show that u^+ defined as above satisfies (i)-(ii)-(iii) with probability going to 1 and we bound the deviations of the difference of some¹ of the increments $X_{2i\Delta} - X_{(2i-1)\Delta} - (X_{2i-1\Delta} - X_{2(i-1)\Delta})$ (Lemma 6).
- Second, we build an estimator of the standard deviation of the increments of the Lévy process (b, Σ, ν_{u^+}) . In order to do so, we use a robust estimator of the mean which drops large increments, and thus the ones larger than u^+ (Lemma 7).
- From these preliminary steps, we prove that a test comparing the largest entry and the expected standard deviation if $\nu_\varepsilon = 0$ detects if there is a jump larger than u^* in the sample (Proposition 3). In the following steps, we focus on tests *conditional to the event there is no jump larger than u^* in the sample* - otherwise they are eliminated by the latter test. Two cases remain to be studied.
- If the dominant quantity in Theorem 2 is $\Delta\mu_3(u^*)$: we first construct a test for detecting if $\Delta\mu_6(u^*)$ is larger than a constant times $[\Delta(\Sigma^2 + \sigma^2(u^*))]^3$, to remove distributions that are too skewed (Proposition 4). Then, we build a test comparing the (estimated) third moment of the increments to the expected behavior if $\nu_\varepsilon = 0$ (Proposition 5).
- If the dominant quantity in Theorem 2 is $\Delta\mu_4(u^*)$: we build a test comparing the (estimated) fourth moment of the increments to the expected behavior if $\nu_\varepsilon = 0$ (Proposition 6).

2.2.1 Comments

Tightness of the lower bound on the total variation distance. The bounds on the total variation we establish in Theorems 3 and 5 are tight, up to a $\log(e \vee n)$ factor due to the differences in the definitions² of \tilde{u}^* and u^* , in the following sense. Whenever $(\lambda_{u^*, \varepsilon} n \Delta) \vee \frac{n\mu_4^2(u^*)}{\Delta^2(\Sigma^2 + \sigma^2(u^*))^4} \vee \frac{n\mu_3^2(u^*)}{\Delta(\Sigma^2 + \sigma^2(u^*))^3}$ does not converge to 0 with n , the total variation distance does not converge to 0 with n . And if it converges to $+\infty$ with n , the total variation converges to 1. Moreover, if

$$(\lambda_{\tilde{u}^*, \varepsilon} n \Delta) \vee \frac{n\mu_4^2(\tilde{u}^*)}{\Delta^2(\Sigma^2 + \sigma^2(\tilde{u}^*))^4} \vee \frac{n\mu_3^2(\tilde{u}^*)}{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^3}$$

converges to 0 with n , then the total variation converges to 0 by Theorem 2. Another implication of these bounds is that the Gaussian random variable closest to $(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n}$ is not necessarily $\mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon))^{\otimes n})$, in particular when rare and large jumps are present, a tighter Gaussian approximation is provided by $\mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(u^*))^{\otimes n})$, as pointed out in Section 2.1.1.

¹Of all increments when $\nu_\varepsilon = 0$ and of those where a jump larger than u^* occurs otherwise.

²We remind that u^* is the largest u larger than u^+ such that $u \leq \sqrt{\Delta(\Sigma^2 + \sigma^2(u)) \log(e \vee n)^2}$, and \tilde{u}^* is the largest u larger than u^+ such that $u \leq \tilde{c} \sqrt{\Delta(\Sigma^2 + \sigma^2(u)) / \sqrt{\log(e \vee n)}}$ where \tilde{c} is a constant.

The lower bound on the total variation distance is a jump detection test. The proof of Theorem 5 is based on the construction of a test of Gaussianity, adapted to Lévy processes, that detects whether the discrete observations we have at our disposal are purely Gaussian, or whether they are realizations of a Lévy process with non trivial Lévy measure. More precisely, (see the proof of Theorem 5 for details) we build a uniformly consistent test for the testing problem

$$H_0 : \nu_\varepsilon = 0, \quad \text{against} \quad H_1 : \lim_{\eta \rightarrow 0} \lambda_{\eta, \varepsilon} = +\infty \quad \text{and} \quad E$$

where

$$E = \left[\mu_3(u^*)^2 \geq \frac{C\Delta(\Sigma^2 + \sigma^2(u^*))^3}{n} \quad \text{or} \quad \mu_4(u^*)^2 \geq \frac{C\Delta^2(\Sigma^2 + \sigma^2(u^*))^4}{n} \right. \\ \left. \text{or a jump larger than } u^* \text{ occurs} \right].$$

This test is of interest in itself: it does not rely on the knowledge of the Lévy triplet.

Remark on the assumptions. Theorem 4 requires $\varepsilon \leq \sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))} \log(e \vee n)^2$, *i.e.* that ε is smaller (up to a multiplicative $\log(e \vee n)^2$ term) than the standard deviation of the increment. It implies that all moments of order k of the increment can be efficiently bounded -up to a constant depending on k - by $(\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))} \log(e \vee n)^2)^k$, which is helpful for bounding the deviations of the test statistics. This assumption is restrictive and is removed in Theorem 5 by introducing u^* and considering, two different types of tests in the construction of the lower bound : a test for the third and fourth moments and a test for extreme values. This latter test allows to detect -with very high probability- when a jump larger than u^* occurred.

Therefore, both theorems only rely on the assumption that $\lambda_{0, \varepsilon} \geq \Delta^{-1} \vee (\log(e \vee n)/(n\Delta))$. This bound is larger than $\log(e \vee n)/\Delta$ (see Theorems 2 and 3). As explained in Section 2.1.1, whenever $\lambda_{0, \varepsilon}$ is smaller than $\frac{\log(e \vee n)}{\Delta}$ (up to a multiplicative constant) simple arguments enable to bound the total variation distance when $\Sigma = 0$. In this sense, assumption $\lambda_{0, \varepsilon} \geq \Delta^{-1} \vee (\log(e \vee n)/(n\Delta))$ is not constraining as it permits to treat all relevant cases.

Improvement of Theorem 4 for mixtures. An immediate corollary of Theorem 4 (see its proof) is a lower bound on the total variation distance between any two mixture of Gaussian random variables and mixture of Lévy measures concentrated in $[-\varepsilon, \varepsilon]$. More precisely, let $d\Lambda(b, \Sigma, \nu)$ et $d\Lambda'(b, \Sigma)$ be two priors on Lévy processes and Brownian motions, respectively. Assume that the support of $d\Lambda(b, \Sigma, \nu)$ is included in a set \mathcal{A} , and that for any $(b, \Sigma, \nu) \in \mathcal{A}$, we have $\varepsilon \leq \sqrt{(\sigma^2(\varepsilon) + \Sigma^2)\Delta} \log(e \vee n)^2$. Then, it holds that

$$\left\| \int (\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta^{(\nu)}(\varepsilon))^{\otimes n} d\Lambda(b, \Sigma, \nu) - \int \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n} d\Lambda'(b, \Sigma) \right\|_{TV} \geq \\ \min_{(b, \Sigma, \nu) \in \mathcal{A}} \left[1 - C \left[\frac{\Delta(\Sigma^2 + \sigma_\nu^2(\varepsilon))^3}{n(\mu_3^{(\nu)}(\varepsilon))^2} \wedge \frac{\Delta^2(\Sigma^2 + \sigma_\nu^2(\varepsilon))^4}{n(\mu_4^{(\nu)}(\varepsilon))^2} \right] - \alpha_n \right],$$

whereby $M_\Delta^{(\nu)}, \sigma_\nu^2(\varepsilon), \mu_3^{(\nu)}(\varepsilon)$ and $\mu_4^{(\nu)}(\varepsilon)$ correspond to $M_\Delta, \sigma^2(\varepsilon), \mu_3(\varepsilon)$ and $\mu_4(\varepsilon)$ for the Lévy measure ν . A related result can be achieved for Theorem 5. Note that the corresponding lower bound on the total variation distance is a direct corollary of Theorem 2. The lower bound displayed above is not trivial, it holds because the test that we construct in the proof of Theorem 5 does not depend on the parameters of the Gaussian random variable nor on the Lévy triplet.

2.3 Examples

2.3.1 Preliminaries on Assumption $\mathcal{H}_\Psi(\varepsilon)$

Before displaying the results implied by Theorems 2 and 3 on the class of alpha-stable processes, we provide two contexts in which Assumptions $\mathcal{H}_\Psi(\varepsilon)$ and $\mathcal{H}_\Psi(\tilde{u}^*)$ are fulfilled.

Assumption $\mathcal{H}_\Psi(\varepsilon)$ when Σ is large enough. We first present the following proposition which proves that Assumption $\mathcal{H}_\Psi(\varepsilon)$ is satisfied, whenever Σ is large enough - namely, $\sigma(\varepsilon)\sqrt{\log(e \vee n)} \lesssim \Sigma$.

Proposition 1. *Let $\varepsilon > 0$ and consider a Lévy measure ν . Assume that Σ is such that $\Sigma \geq c_\Sigma \sigma(\varepsilon)\sqrt{\log(e \vee n)}$ where $c_\Sigma > 0$. If $c > 0$ is taken large enough depending only on c_{\sup}, c_Σ , then it holds that for any $k \leq K = 401 \log(e \vee n)$*

$$\int |\Psi_\varepsilon^{(k)}(t)|^2 dt \leq k! n^{-4} \quad \text{with} \quad \Psi_\varepsilon(t) := \mathbb{E}[e^{iu\tilde{X}_\Delta(\varepsilon)}], \quad \tilde{X}_\Delta(\varepsilon) = \frac{X_\Delta(\varepsilon) - b\Delta}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}}.$$

In this case we may apply directly Theorem 2 and Theorem 3, provided that we apply the previous proposition at \tilde{u}^* instead of ε .

Assumption $\mathcal{H}_\Psi(\varepsilon)$ in the case when ν is polynomially controlled at 0. The following result, whose proof can be found in Appendix A.4, implies that whenever ν satisfies Assumption (10) below then Assumption $(\mathcal{H}_\Psi(\varepsilon))$ is fulfilled. Note that Assumption (10) describes a class of functions that contains any Lévy measure that is regularly varying at 0.

Proposition 2. *Let $b \in \mathbb{R}$, $\Sigma^2 \geq 0$, $\Delta > 0$, $\varepsilon > 0$, $n \geq 1$ and let ν be a Lévy measure absolutely continuous with respect to the Lebesgue measure. Suppose that there exists two positive constants $c_+ > c_- > 0$ such that, $\forall x \in [-\varepsilon, \varepsilon] \setminus \{0\}$,*

$$\frac{c_-}{|x|^{\beta+1}} dx \leq d\nu(x) \leq \frac{c_+}{|x|^{\beta+1}} dx, \quad \beta \in (0, 2). \quad (10)$$

Assume that there exists $c_{\max} \geq 0$ such that $n^{c_{\max}}\Delta \geq 1$ and $\log(\Sigma^2 + \sigma^2(\varepsilon))/\log(e \vee n) \leq c_{\max}$.

Then, for any $c > 0$ large enough depending only on $\beta, c_+, c_-, c_{\max}$, and for any $(\tilde{c})^{-1} > 0$ large enough depending only on $\beta, c_+, c_-, c_{\max}, c$ such that $\varepsilon \leq \tilde{c}\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}/\sqrt{\log(e \vee n)}$, it holds

$$\int_{t \geq c \log(e \vee n)} |\Psi_\varepsilon^{(k)}(t)|^2 dt \leq 3k! n^{-4}, \quad \forall k \in [0, 401 \log(e \vee n)],$$

where $\Psi_\varepsilon(t) := \mathbb{E}[e^{iu\tilde{X}_\Delta(\varepsilon)}]$ and $\tilde{X}_\Delta(\varepsilon) = \frac{X_\Delta(\varepsilon) - b\Delta}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}}$.

Remark 2. *Whenever there exists $\kappa > 0$ a constant that depends only on β, c_+, c_- such that $n^\kappa \Delta \geq 1$, and $(\Sigma^2 + \varepsilon^{2-\beta})n^{-\kappa} \leq 1$, then c_{\max} is an absolute constant and the dependence on c_{\max} in the proposition is not constraining. Moreover, the condition on ε is the same condition as in Theorems 2 and 3. Finally, in Theorems 2 and 3 the constraints on c, \tilde{c} are $c > 1$, $\tilde{c} \leq 1$ and $c\tilde{c} \leq \sqrt{\log(e \vee n)}/4$, that are easy to satisfy provided that \tilde{c} can be chosen small enough. As $\Sigma^2 + \sigma^2(\varepsilon)$ is of order $(\Sigma^2 + \varepsilon^{2-\beta})$, even in the most constraining case $\Sigma = 0$, \tilde{c} can be chosen small enough provided that $\varepsilon^\beta \leq \tilde{c}'\Delta/\log(e \vee n)$, where \tilde{c}' is chosen small enough (depending on $\tilde{c}, \beta, c_+, c_-, c_{\max}$).*

The main condition $\varepsilon \leq \tilde{c}\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}/\sqrt{\log(e \vee n)}$ is naturally satisfied for \tilde{u}^* , and the following corollary holds. It shows that Assumption $(\mathcal{H}_\Psi(\tilde{u}^*))$ holds for processes which fulfill (10).

Corollary 2. *Let $\Delta > 0$, $\varepsilon > 0$, $n \geq 1$ and let ν be a Lévy measure having a density with respect to the Lebesgue measure such that there exist two positive constants $c_+ > c_- > 0$, such that for any $x \in [-\varepsilon, \varepsilon] \setminus \{0\}$, Equation (10) holds. Assume that there exists $\kappa > 0$ a constant depending only on β, c_+, c_- such that $n^\kappa \Delta \geq 1$, and $(\Sigma^2 + \varepsilon^{2-\beta})n^{-\kappa} \leq 1$. Then for any $c > 1$ large enough depending only on β, c_+, c_- , and for any $(\tilde{c})^{-1} > 1$ large enough depending only on β, c_+, c_-, c (where \tilde{c} appears in the definition of \tilde{u}^*), we have for all $k \leq 401 \log(e \vee n)$*

$$\int_{t \geq c \log(e \vee n)} |\Psi_{\tilde{u}^*}^{(k)}(t)|^2 dt \leq 3k!n^{-4}, \quad c\tilde{c} \leq \sqrt{\log(e \vee n)}/4, \quad c > 1, \quad \tilde{c} \leq 1.$$

2.3.2 Stable processes

In this Section we illustrate the implications of Theorem 2 and Theorem 3 on the class of infinite stable processes. It is possible to extend the results valid for this example to other types of Lévy processes (e.g. inverse Gaussian processes, tempered stable distributions, etc...) as, around 0, stable measures well approximate many Lévy measures. Let $\beta \in (0, 2)$, $c_+, c_- \geq 0$, ($c_+ = c_-$ if $\beta = 1$) and assume that ν has a density with respect to the Lebesgue measure of the form

$$\nu(dx) = \frac{c_+}{x^{1+\beta}} \mathbf{1}_{(0, +\infty)}(x) dx + \frac{c_-}{|x|^{1+\beta}} \mathbf{1}_{(-\infty, 0)}(x) dx, \quad \forall x \in [-\varepsilon, \varepsilon].$$

These processes satisfy Equation (10). In accordance with the notations already used in the paper, $M(\varepsilon)$ will be a Lévy process with Lévy triplet $(0, 0, \nu_\varepsilon)$ where $\nu_\varepsilon := \nu \mathbf{1}_{|x| \leq \varepsilon}$. Let $b > 0$, $\Sigma^2 \geq 0$, $\Delta > 0$, $\varepsilon > 0$, $n \geq 1$. In what follows, we will consider the Lévy triplet $(b, \Sigma^2, \nu_\varepsilon)$. For $\varepsilon > 0$, all quantities u^+, u^*, \tilde{u}^* described above are written respective to this ε . Moreover, in this section, we use the symbols \approx , \lesssim , and $o(1)$ defined as follows. For $a, b \in \mathbb{R}$, $a \approx b$ if there exists $c > 0$ depending only on β, c_+, c_- such that $a = cb$ and $a \lesssim b$ if there exists $c > 0$ depending only on β, c_+, c_- such that $a \leq cb$. For a sequence $(a_n)_n$ in \mathbb{R}^+ , we have that $a_n = o(1)$ if $\lim_{n \rightarrow +\infty} a_n = 0$. In what follows, we allow $\varepsilon, \Delta, \Sigma, b, \varepsilon^*$ to depend on n .

We are interested in the question: “Given n and Δ , what is the largest $\varepsilon^* \geq 0$ such that it is not possible to distinguish between n independent realizations of $\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon^*)$ and the closest i.i.d. Gaussian vector?”

The answer to this question is provided by Theorems 3 and 5. Let $\varepsilon \geq 0$ be such that

$$\frac{\sqrt{n}\mu_3(\tilde{u}^*)}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^3}} \quad \text{and} \quad \frac{\sqrt{n}\mu_4(\tilde{u}^*)}{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^2} \quad \text{and} \quad n\Delta\lambda_{\tilde{u}^*, \varepsilon}, \quad \text{are small,}$$

then the sample of n independent realisations of $\mathcal{N}(0, \Delta\Sigma^2) * M_\Delta(\varepsilon)$ is almost undistinguishable from n independent realisations of $\mathcal{N}(0, \Delta(\sigma^2(\tilde{u}^*) + \Sigma^2))$. Conversely, if either

$$\frac{\sqrt{n}\mu_3(u^*)}{\sqrt{\Delta(\Sigma^2 + \sigma^2(u^*))^3}} \quad \text{or} \quad \frac{\sqrt{n}\mu_4(u^*)}{\Delta(\Sigma^2 + \sigma^2(u^*))^2} \quad \text{or} \quad n\Delta\lambda_{u^*, \varepsilon}, \quad \text{is large,}$$

then it is possible to test with low error that the sample of n realizations of the process is not an i.i.d. Gaussian vector.

Note that $\sqrt{n}\mu_4(u) \lesssim \Delta(\Sigma^2 + \sigma^2(u))^2$ is equivalent to

$$u \lesssim \left(\frac{\Delta}{\sqrt{n}}\right)^{1/\beta} \vee \left(\frac{\Delta\Sigma^4}{\sqrt{n}}\right)^{\frac{1}{4-\beta}}. \quad (11)$$

In the same way, for a very non symmetric process $\sqrt{n}\mu_3(u) \lesssim \sqrt{\Delta(\Sigma^2 + \sigma^2(u))^3}$ is equivalent to

$$u \lesssim \left(\frac{\Delta}{n}\right)^{1/\beta} \vee \left(\frac{\sqrt{\Delta}\Sigma^3}{\sqrt{n}}\right)^{\frac{1}{3-\beta}}. \quad (12)$$

For any $\varepsilon \geq u \geq 0$, $2\frac{c_-}{\beta}(u^{-\beta} - \varepsilon^{-\beta}) \leq \lambda_{u,\varepsilon} \leq 2\frac{c_+}{\beta}(u^{-\beta} - \varepsilon^{-\beta})$, so it holds $u^+ \approx \varepsilon \wedge (n\Delta/\log(e \vee n))^{1/\beta}$, which is much larger than the bound on u given by (11) and (12) that describes the distinguishability frontier in the third or fourth moments. For β -stable Lévy processes, $\beta \in (0, 2)$, the distinguishability frontier does only depend on the third and fourth moments.

The following two Tables summarize these findings and give the value of ε^* answering the initial question above. We distinguish four scenarios (depending on whether the process is symmetric or not, and on whether Σ is large with respect to $\sigma^2(\varepsilon^*)$ or not) and provide for each the optimal ε^* such that (i) if $\varepsilon/\varepsilon^* = o(1)$, then

$$\inf_{B \in \mathbb{R}, S \geq 0} \left\| \left(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon) \right)^{\otimes n} - \left(\mathcal{N}(B\Delta, \Delta S^2\Delta) \right)^{\otimes n} \right\|_{TV} \rightarrow 0,$$

and (ii) else if $\varepsilon^*/\varepsilon = o(1)$, then

$$\inf_{B \in \mathbb{R}, S \geq 0} \left\| \left(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon) \right)^{\otimes n} - \left(\mathcal{N}(B\Delta, \Delta S^2\Delta) \right)^{\otimes n} \right\|_{TV} \rightarrow 1.$$

In all cases we require, additionally to ν being the Lévy measure of a β -stable process, that there exists $\kappa > 0$ a constant that depends only on β, c_+, c_- such that $n^\kappa \Delta \geq 1$, and $(\Sigma^2 + \varepsilon^{2-\beta})n^{-\kappa} \leq 1$.

ν is symmetric		ν is non symmetric	
$\Sigma^2 \geq \left(\frac{\Delta}{n}\right)^{\frac{2-\beta}{\beta}}$	$\varepsilon^* \approx \left(\frac{\Delta\Sigma^4}{\sqrt{n}}\right)^{\frac{1}{4-\beta}}$	$\Sigma^2 \geq \left(\frac{\Delta}{\sqrt{n}}\right)^{\frac{2-\beta}{\beta}}$	$\varepsilon^* \approx \left(\frac{\sqrt{\Delta}\Sigma^3}{\sqrt{n}}\right)^{\frac{1}{3-\beta}}$
$\Sigma^2 \leq \left(\frac{\Delta}{n}\right)^{\frac{2-\beta}{\beta}}$	$\varepsilon^* \approx \left(\frac{\Delta}{\sqrt{n}}\right)^{\frac{1}{\beta}}$	$\Sigma^2 \leq \left(\frac{\Delta}{\sqrt{n}}\right)^{\frac{2-\beta}{\beta}}$	$\varepsilon^* \approx \left(\frac{\Delta}{n}\right)^{\frac{1}{\beta}}$

3 Total variation distance between Lévy processes

In this Section, let $X^i \sim (b_i, \Sigma_i^2, \nu_i)$, $i = 1, 2$, be two distinct Lévy processes. We shall use the notation introduced in Section 1 properly modified to take into account the dependencies on X^1 and X^2 . For instance, $\mu_3(\varepsilon)$ and $\mu_4(\varepsilon)$ become

$$\mu_{j,i}(\varepsilon) = \int_{|x| \leq \varepsilon} x^j \nu_i(dx), \quad i = 1, 2, \quad j = 3, 4,$$

where $\mu_{j,1}(\varepsilon)$ (resp. $\mu_{j,2}(\varepsilon)$), $j = 3, 4$, denote the 3rd and 4th moment of ν_1 (resp. ν_2) restricted on $\{x : |x| \leq \varepsilon\}$.

By means of the Lévy-Itô decomposition recalled in Section 1, for any $t > 0$ and $\varepsilon > 0$ we have that the law of X_t^i , $i = 1, 2$, is the convolution between a Gaussian distribution and the law of the marginal at time t of the processes $M^i(\varepsilon)$ and $Z^i(\varepsilon)$, i.e.

$$X_t^i(\varepsilon) = \mathcal{N}(b_i(\varepsilon)t, t\Sigma_i^2) * M_t^i(\varepsilon) * Z_t^i(\varepsilon), \quad i = 1, 2.$$

By subadditivity of the total variation distance, see Lemma 23 in Appendix B, we have:

$$\|X_t^1 - X_t^2\|_{TV} \leq \|\mathcal{N}(b_1(\varepsilon)t, t\Sigma_1^2) * M_t^1(\varepsilon) - \mathcal{N}(b_2(\varepsilon)t, t\Sigma_2^2) * M_t^2(\varepsilon)\|_{TV} + \|Z_t^1(\varepsilon) - Z_t^2(\varepsilon)\|_{TV}.$$

By triangular inequality first and subadditivity of the total variation distance together with Lemma 21 in Appendix B then, we obtain

$$\begin{aligned}
& \|\mathcal{N}(b_1(\varepsilon)t, t\Sigma_1^2) * M_t^1(\varepsilon) - \mathcal{N}(b_2(\varepsilon)t, t\Sigma_2^2) * M_t^2(\varepsilon)\|_{TV} \\
& \leq \|\mathcal{N}(b_1(\varepsilon)t, t\Sigma_1^2) * M_t^1(\varepsilon) - \mathcal{N}(b_1(\varepsilon)t, t\Sigma_1^2) * \mathcal{N}(0, t\sigma_1^2(\varepsilon))\|_{TV} \\
& \quad + \|\mathcal{N}(b_2(\varepsilon)t, t\Sigma_2^2) * M_t^2(\varepsilon) - \mathcal{N}(b_2(\varepsilon)t, t\Sigma_2^2) * \mathcal{N}(0, t\sigma_2^2(\varepsilon))\|_{TV} \\
& \quad + \|\mathcal{N}(b_1(\varepsilon)t, t\Sigma_1^2) * \mathcal{N}(0, t\sigma_1^2(\varepsilon)) - \mathcal{N}(b_2(\varepsilon)t, t\Sigma_2^2) * \mathcal{N}(0, t\sigma_2^2(\varepsilon))\|_{TV} \\
& \leq \sum_{i=1}^2 \|\mathcal{N}(b_i(\varepsilon)t, t\Sigma_i^2) * M_t^i(\varepsilon) - \mathcal{N}(b_i(\varepsilon)t, t(\sigma_i^2(\varepsilon) + \Sigma_i^2))\|_{TV} \\
& \quad + \frac{\sqrt{\frac{t}{2\pi}}|b_1(\varepsilon) - b_2(\varepsilon)| + \left| \sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)} - \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)} \right|}{\sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)} \vee \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)}}.
\end{aligned}$$

The terms $\|\mathcal{N}(b_i(\varepsilon)t, t\Sigma_i^2) * M_t^i(\varepsilon) - \mathcal{N}(b_i(\varepsilon)t, t(\Sigma_i^2 + \sigma_i^2(\varepsilon)))\|_{TV}$, $i = 1, 2$ can be bounded by means of Theorem 2 whereas for $\|Z_t^1(\varepsilon) - Z_t^2(\varepsilon)\|_{TV}$ we can use Lemma 22 taking $n = 1$:

$$\|Z_t^1(\varepsilon) - Z_t^2(\varepsilon)\|_{TV} \leq t|\Lambda_1(\varepsilon) - \Lambda_2(\varepsilon)| + t(\Lambda_1(\varepsilon) \wedge \Lambda_2(\varepsilon)) \left\| \frac{\nu_1^\varepsilon}{\Lambda_1(\varepsilon)} - \frac{\nu_2^\varepsilon}{\Lambda_2(\varepsilon)} \right\|_{TV},$$

with $\nu_j^\varepsilon = \nu_j(\cdot \cap (\mathbb{R} \setminus (-\varepsilon, \varepsilon)))$ and $\Lambda_j(\varepsilon) = \nu_j^\varepsilon(\mathbb{R})$. Applying Lemma 21 for $n = 1$, we thus obtain the following upper bounds for the total variation distance between marginals of Lévy processes:

$$\begin{aligned}
\|X_t^1 - X_t^2\|_{TV} & \leq \frac{\sqrt{\frac{t}{2\pi}}|b_1(\varepsilon) - b_2(\varepsilon)| + \left| \sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)} - \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)} \right|}{\sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)} \vee \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)}} \\
& \quad + C \sum_{i=1}^2 \sqrt{\frac{\mu_{4,i}^2(\varepsilon)}{t^2(\sigma_i^2(\varepsilon) + \Sigma_i^2)^4} + \frac{\mu_{3,i}^2(\varepsilon)}{t(\Sigma_i^2 + \sigma_i^2(\varepsilon))^3}} + 2C \\
& \quad + t|\Lambda_1(\varepsilon) - \Lambda_2(\varepsilon)| + t(\Lambda_1(\varepsilon) \wedge \Lambda_2(\varepsilon)) \left\| \frac{\nu_1^\varepsilon}{\Lambda_1(\varepsilon)} - \frac{\nu_2^\varepsilon}{\Lambda_2(\varepsilon)} \right\|_{TV}.
\end{aligned}$$

Extending these arguments to the case of discrete observations of Lévy processes we obtain the following result.

Theorem 6. *Let $X^i \sim (b_i, \Sigma_i^2, \nu_i)$ be any Lévy process with $b_i \in \mathbb{R}$, $\Sigma_i \geq 0$ and ν_i Lévy measures $i = 1, 2$. For all $\Delta > 0$, $\varepsilon > 0$ and $n \geq 1$ and under the Assumptions of Theorem 2, there exists a positive constant C such that*

$$\begin{aligned}
& \|(X_{k\Delta}^1 - X_{(k-1)\Delta}^1)_{k=1}^n - (X_{k\Delta}^2 - X_{(k-1)\Delta}^2)_{k=1}^n\|_{TV} \leq \frac{\sqrt{n\Delta}}{\sqrt{2\pi}} \frac{|b_1(\varepsilon) - b_2(\varepsilon)|}{\max(\sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)}, \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)})} \\
& \quad + 1 - \left(\frac{\min(\sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)}, \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)})}{\max(\sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)}, \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)})} \right)^n \\
& \quad + C \sum_{i=1}^2 \sqrt{\frac{n\mu_{4,i}^2(\varepsilon)}{\Delta^2(\sigma_i^2(\varepsilon) + \Sigma_i^2)^4} + \frac{n\mu_{3,i}^2(\varepsilon)}{\Delta(\sigma_i^2(\varepsilon) + \Sigma_i^2)^3}} + \frac{2C}{n} \\
& \quad + 1 - \exp(-n\Delta|\Lambda_1(\varepsilon) - \Lambda_2(\varepsilon)|) + n\Delta(\Lambda_1(\varepsilon) \wedge \Lambda_2(\varepsilon)) \left\| \frac{\nu_1^\varepsilon}{\Lambda_1(\varepsilon)} - \frac{\nu_2^\varepsilon}{\Lambda_2(\varepsilon)} \right\|_{TV}.
\end{aligned}$$

Proof. It directly follows from the Lévy-Itô decomposition together with Lemmas 23, 21, 22 and Theorem 2. \square

4 Proofs

4.1 Proof of Theorem 2

4.1.1 Assumptions and notations

Assume here that $n \geq 3$ - but note that the bound on the total variation distance for $n = 3$ is also a bound on the total variation distance for $n = 1$ or $n = 2$. First, we introduce some notations and reformulate the assumptions of Theorem 2. Denote by $s^2 := \Sigma^2 + \sigma^2(\varepsilon)$, where $\Sigma^2 \geq 0$, by \mathcal{I} the integration interval $\mathcal{I} := [-c_{sup}\sqrt{\log(n)}, c_{sup}\sqrt{\log(n)}]$, $c_{sup} \geq 2$. For a function g , we write $g_{\mathcal{I}} := g\mathbf{1}_{\{\mathcal{I}\}}$.

In what follows, we write μ for a measure that is the sum of the Lebesgue measure and of the sum of (countably) many Dirac masses, which dominates the measure associated to $\tilde{X}_{\Delta}(\varepsilon) := (X_{\Delta} - b\Delta)/\sqrt{\Delta(\sigma^2(\varepsilon) + \Sigma^2)}$. Let f be the density, with respect to the measure μ of the rescaled increment $\tilde{X}_{\Delta}(\varepsilon)$ and φ be the density with respect to the measure μ of the centered Gaussian random variable with unit variance. Whenever we write an integral involving f or φ the sequel, it is with respect to μ (or a corresponding product measure).

Recall that

$$\begin{aligned}\Psi(t) &:= \Psi_{\varepsilon}(t) = \mathbb{E}[e^{it\tilde{X}_{\Delta}(\varepsilon)}] - e^{(-\lambda_{0,\varepsilon}n\Delta)}\mathbf{1}_{\{\Sigma=0\}} \\ &= \exp\left(-\frac{\Sigma^2}{s^2}\frac{t^2}{2} + \Delta \int \left(\exp\left(\frac{iut}{s\sqrt{\Delta}}\right) - \frac{iut}{s\sqrt{\Delta}} - 1\right)d\nu(u)\right) - e^{(-\lambda_{0,\varepsilon}n\Delta)}\mathbf{1}_{\{\Sigma=0\}}.\end{aligned}$$

We establish the result under the following assumptions which are implied by the assumptions of Theorem 2.

- Set $K := c_{int}^2 \log(n)$, where $c_{int} > 2c_{sup}$, it holds for some constant $c > 0$ that

$$\int_{c\log(n)}^{+\infty} |\Psi^{(k)}|^2 \leq C' 2^k k! n^{-2}, \quad \forall 0 \leq k \leq K, \quad (\mathcal{H}_{\Psi})$$

where C' is a universal constant.

- For some constant $1/8 > c_p > 0$, it holds

$$\mathbb{P}_f(\mathcal{I}^c) \leq c_p/n. \quad (\mathcal{H}_0)$$

- For some small enough universal constant $1 \geq \tilde{c} > 0$, such that $\tilde{c}c \leq \sqrt{\log n}/4$, it holds

$$\varepsilon \leq \tilde{c}\sqrt{(\sigma(\varepsilon)^2 + \Sigma^2)\Delta/\log(n)} := \tilde{c}s\sqrt{\Delta}/\sqrt{\log(n)} := \tilde{c}_n s\sqrt{\Delta}. \quad (\mathcal{H}_{\varepsilon})$$

Note that this assumption permits to simply derive (\mathcal{H}_0) from the following lemma.

Lemma 2. *Assume that ν is a Lévy measure (potentially infinite) such that $\lambda_{0,\varepsilon} \geq 24\log(n)/\Delta$. Then, whenever $c_{sup} \geq 10$ and $\tilde{c}_n \leq 1$ and $(\mathcal{H}_{\varepsilon})$ holds, we get $\mathbb{P}_f(\mathcal{I}^c) \leq 3/n^3$.*

Lemma 2 implies that under the assumptions of Theorem 2, (\mathcal{H}_0) is satisfied with $c_p = 3/n^2$.

Remark 3. *For the proof of Theorem 2 Assumption $(\mathcal{H}_{\varepsilon})$ can be weakened in $\varepsilon \leq \tilde{c}\sqrt{(\sigma(\varepsilon)^2 + \Sigma^2)\Delta}$, the extra log is used to establish Lemma 2 related to (\mathcal{H}_0) .*

- For some constant $1/2 \geq c_m > 0$, it holds

$$M := \tilde{c}_n^{-4} \left[\frac{|\mu_3(\varepsilon)|}{\sqrt{\Delta}s^3} + \frac{\mu_4(\varepsilon)}{\Delta s^4} \right] \leq c_m/\sqrt{n}. \quad (\mathcal{H}_M)$$

Remark 4. *This Assumption will be used in the proof, it is not limiting as if (\mathcal{H}_M) is not satisfied, the upper bound of Theorem 2 is not small and is therefore irrelevant.*

In the sequel, C stands for a universal constant, whose value may change from line to line.

4.1.2 Proof of Theorem 2.

If $\Sigma = 0$, introduce the event ξ_M ensuring that there is at least one jump in each interval $[(i-1)\Delta, i\Delta]$

$$\xi_M = \{\Sigma = 0, \forall i \leq n, \lim_{\eta \rightarrow 0} (N_{i\Delta}(\eta, \varepsilon) - N_{(i-1)\Delta}(\eta, \varepsilon)) \geq 1\} \cup \{\Sigma > 0\},$$

where $N(\eta, \varepsilon)$ is the compound Poisson process associated to the jumps of $M(\varepsilon)$ that are larger than η . Define also the event, in dimension 1,

$$\tilde{\xi}_M = \{\Sigma = 0, \lim_{\eta \rightarrow 0} N_\Delta(\eta, \varepsilon) \geq 1\} \cup \{\Sigma > 0\}.$$

Note that, if $\Sigma > 0$ or if ν is infinite, then ξ_M and $\tilde{\xi}_M$ are the whole probability space.

First, a change of variable implies

$$\|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n}\|_{TV} = \|f^{\otimes n} - \varphi^{\otimes n}\|_{TV},$$

where we set

$$\|f^{\otimes n} - \varphi^{\otimes n}\|_{TV} := \|(\mathcal{N}(0, \frac{\Sigma^2}{\Sigma^2 + \sigma^2(\varepsilon)}) * \widetilde{M}_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(0, 1)^{\otimes n}\|_{TV},$$

where $\widetilde{M}_\Delta(\varepsilon) = \frac{M_\Delta(\varepsilon)}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}}$.

To bound the total variation distance we consider separately the sets \mathcal{I} and its complementary and ξ_M and its complementary set (recall that the integrals are with respect to $\mu^{\otimes n}$)

$$\|f^{\otimes n} - \varphi^{\otimes n}\|_{TV} = \|(f_{\mathcal{I}}^{\otimes n} - \varphi_{\mathcal{I}}^{\otimes n})\mathbf{1}_{\{\xi_M\}}\|_{TV} + \int_{(\mathcal{I}^n)^c} |f^{\otimes n} - \varphi^{\otimes n}| \mathbf{1}_{\{\xi_M\}} + \int |f^{\otimes n} - \varphi^{\otimes n}| \mathbf{1}_{\{\xi_M^c\}}. \quad (13)$$

Under (\mathcal{H}_0) and using that $\mathbb{P}_{\varphi^{\otimes n}}(\mathcal{I}^n)^c \leq n(\mathbb{P}_\varphi(\mathcal{I}^c)) \leq n \exp(-c_{sup}^2 \log(n)/2) \leq 1/n$ for $c_{sup} \geq 2$ the second term is bounded by $c_p + 1/n$. From now on, we focus on the first term. The third term is equal to $\mathbb{P}_{f^{\otimes n}}(\xi_M) = \exp(-\lambda_{0,\varepsilon} n \Delta)$, which is smaller by $1/n$ using that $\lambda_{0,\varepsilon} \geq 24 \log(n)/\Delta$.

Introduce a positive function $h > 0$ such that

$$\int h_{\mathcal{I}} \mathbf{1}_{\{\xi_M\}} < +\infty, \quad \int \frac{f_{\mathcal{I}}^2}{h_{\mathcal{I}}} \mathbf{1}_{\{\xi_M\}} < +\infty, \quad \int \frac{\varphi_{\mathcal{I}}^2}{h_{\mathcal{I}}} \mathbf{1}_{\{\xi_M\}} < +\infty. \quad (14)$$

It follows from the Cauchy Schwarz inequality that

$$\begin{aligned} \frac{\|(f_{\mathcal{I}}^{\otimes n} - \varphi_{\mathcal{I}}^{\otimes n})\mathbf{1}_{\{\xi_M\}}\|_{TV}^2}{(\int h_{\mathcal{I}} \mathbf{1}_{\{\xi_M\}})^n} &\leq \int \frac{(f_{\mathcal{I}}^{\otimes n} - \varphi_{\mathcal{I}}^{\otimes n})^2}{h_{\mathcal{I}}^{\otimes n}} \mathbf{1}_{\{\xi_M\}} = \int \frac{(f_{\mathcal{I}}^{\otimes n})^2 - 2f_{\mathcal{I}}^{\otimes n} \varphi_{\mathcal{I}}^{\otimes n} + (\varphi_{\mathcal{I}}^{\otimes n})^2}{h_{\mathcal{I}}^{\otimes n}} \mathbf{1}_{\{\xi_M\}} \\ &= \left[\int_{\tilde{\xi}_M} \frac{(f_{\mathcal{I}} - \varphi_{\mathcal{I}})^2 + 2\varphi_{\mathcal{I}}(f_{\mathcal{I}} - \varphi_{\mathcal{I}}) + \varphi_{\mathcal{I}}^2}{h_{\mathcal{I}}} \right]^n - 2 \left[\int_{\tilde{\xi}_M} \frac{(f_{\mathcal{I}} - \varphi_{\mathcal{I}})\varphi_{\mathcal{I}} + \varphi_{\mathcal{I}}^2}{h_{\mathcal{I}}} \right]^n + \left[\int_{\tilde{\xi}_M} \frac{\varphi_{\mathcal{I}}^2}{h_{\mathcal{I}}} \right]^n. \end{aligned}$$

For $K = c_{int}^2 \log(n)$, take

$$h_{\mathcal{I}}^{-1}(x) = \sqrt{2\pi} \mathbf{1}_{\{\mathcal{I}\}} \sum_{k \leq K} \frac{x^{2k}}{2^k k!},$$

and consider the quantities

$$A^2 := \int_{\tilde{\xi}_M} \frac{\varphi_{\mathcal{I}}^2}{h_{\mathcal{I}}}, \quad D^2 := \int_{\tilde{\xi}_M} \frac{(f_{\mathcal{I}} - \varphi_{\mathcal{I}})^2}{h_{\mathcal{I}}} \quad \text{and} \quad E := \int_{\tilde{\xi}_M} \frac{\varphi_{\mathcal{I}}}{h_{\mathcal{I}}} (f_{\mathcal{I}} - \varphi_{\mathcal{I}}).$$

It holds

$$\begin{aligned}
\frac{\|(f_{\mathcal{I}}^{\otimes n} - \varphi_{\mathcal{I}}^{\otimes n})\mathbf{1}_{\{\tilde{\xi}_M\}}\|_{TV}^2}{(\int_{\tilde{\xi}_M} h_{\mathcal{I}})^n} &\leq \left[D^2 + 2E + A^2\right]^n - 2\left[E + A^2\right]^n + A^{2n} \\
&\leq \sum_{2 \leq k \leq n} \binom{n}{k} \left[D^2 + 2E\right]^k A^{2(n-k)} + nD^2 A^{2(n-1)} - 2 \sum_{2 \leq k \leq n} \binom{n}{k} E^k A^{2(n-k)} \\
&\leq \sum_{2 \leq k \leq n} \binom{n}{k} 2^k D^{2k} A^{2(n-k)} + \sum_{2 \leq k \leq n} \binom{n}{k} 2^k [2|E|]^k A^{2(n-k)} + nD^2 A^{2(n-1)} \\
&\quad + 2 \sum_{2 \leq k \leq n} \binom{n}{k} |E|^k A^{2(n-k)}. \tag{15}
\end{aligned}$$

To bound this last term, consider the following Lemma.

Lemma 3. Assume (\mathcal{H}_0) and that $c_{int} \geq 2c_{sup} \vee 1$. It holds for $c_h > 0$ a universal constant that

$$0 \leq A^2 \leq 1 + c_h/n^2 \quad \text{and} \quad \int_{\tilde{\xi}_M} h_{\mathcal{I}} \leq 1 + c_h/n^2 \quad \text{and} \quad |E| \leq c_p/n + n^{-c_{sup}^2/2} + 2c_h/n^2,$$

where the constant c_p is defined in (\mathcal{H}_0) .

Using Lemma 3 and $\binom{n}{k} \leq n^k$ to bound (15) lead to

$$\begin{aligned}
\frac{\|(f_{\mathcal{I}}^{\otimes n} - \varphi_{\mathcal{I}}^{\otimes n})\mathbf{1}_{\{\xi_M\}}\|_{TV}^2}{(\int_{\tilde{\xi}_M} h_{\mathcal{I}})^n} &\leq \exp(c_h) \left[\sum_{2 \leq k \leq n} 2^k (nD^2)^k + \sum_{2 \leq k \leq n} 2^k [2n|E|]^k + nD^2 + 2 \sum_{2 \leq k \leq n} (n|E|)^k \right] \\
&\leq nD^2 \exp(c_h) \left[\sum_{2 \leq k \leq n} 2^k (nD^2)^{k-1} + 1 \right] + 3 \exp(c_h) \sum_{2 \leq k \leq n} 2^k [2n|E|]^k.
\end{aligned}$$

Moreover, we have thanks to Lemma 3 if $c_{sup} \geq 2$ and $c_p < \frac{1}{8}$

$$\sum_{2 \leq k \leq n} 2^k [2n|E|]^k \leq \sum_{2 \leq k \leq n} 4^k \left(c_p + \frac{2c_h + 1}{n}\right)^k \leq \left(4\left(c_p + \frac{2c_h + 1}{n}\right)\right)^2 \frac{1}{1 - 4\left(c_p + \frac{2c_h + 1}{n}\right)} \leq Cc_p^2,$$

where C is a universal constant.

To complete the proof, we need to control the order of D^2 , indeed using Lemma 3 to bound $\int h_{\mathcal{I}}$ we derive, for $c_{sup} \geq 2$ and $c_p < \frac{1}{8}$,

$$\|(f_{\mathcal{I}}^{\otimes n} - \varphi_{\mathcal{I}}^{\otimes n})\mathbf{1}_{\{\xi_M\}}\|_{TV}^2 \leq nD^2 \exp(c_h) \left[\sum_{2 \leq k \leq n} 2^k (nD^2)^{k-1} + 1 \right] + C \exp(c_h) c_p^2. \tag{16}$$

To control the order of nD^2 in (16), introduce $G(x) = (f(x) - \varphi(x))\mathbf{1}_{\{\tilde{\xi}_M\}}$, we have

$$D^2 = \int_{\tilde{\xi}_M} \frac{(f_{\mathcal{I}} - \varphi_{\mathcal{I}})^2}{h_{\mathcal{I}}} = \int \mathbf{1}_{\{\mathcal{I}\}} \frac{G^2}{h}.$$

Denote by $P_k(x) = x^k$, the Plancherel formula leads to

$$D^2 = \sqrt{2\pi} \int \mathbf{1}_{\{\mathcal{I}\}} G^2(x) \sum_{k \leq K} \frac{x^{2k}}{2^k k!} dx \leq \sqrt{2\pi} \sum_{k \leq K} \frac{1}{2^k k!} \|P_k G\|_2^2 = \frac{1}{\sqrt{2\pi}} \sum_{k \leq K} \frac{1}{2^k k!} \|\widehat{G}^{(k)}\|_2^2, \tag{17}$$

using that $\widehat{P}_k \widehat{G} = i^k \widehat{G}^{(k)} / \sqrt{2\pi}$. The quantity \widehat{G} appearing in the preceding formula is given by

$$\begin{aligned} \widehat{G}(t) &= \exp\left(-\frac{t^2}{2} - i\frac{\mu_3(\varepsilon)t^3}{6\sqrt{\Delta}s^3} + \sum_{m \geq 4} \Delta\mu_m(\varepsilon) \frac{(ti)^m}{(\Delta s^2)^{m/2} m!}\right) - \exp(-\lambda_{0,\varepsilon} n \Delta) \mathbf{1}_{\{\Sigma=0\}} - \exp(-t^2/2) \\ &= \exp(-t^2/2) \left[\exp\left(-\frac{it^3 \mu_3(\varepsilon)}{6\sqrt{\Delta}s^3} + \sum_{m \geq 4} \Delta\mu_m(\varepsilon) \frac{(ti)^m}{(\Delta s^2)^{m/2} m!}\right) - 1 \right] - \exp(-\lambda_{0,\varepsilon} n \Delta) \mathbf{1}_{\{\Sigma=0\}}. \end{aligned}$$

Assumption $(\mathcal{H}_\varepsilon)$, $\varepsilon \leq \widetilde{c}_n s \sqrt{\Delta}$, implies that $|\mu_m(\varepsilon)| \leq (\widetilde{c}_n)^{m-4} \mu_4(\varepsilon) s^{m-4} \Delta^{m/2-2}$ for any $m > 4$. Therefore,

$$\widehat{G}(t) = \exp(-t^2/2) \left[\exp\left(-it^3 \frac{\mu_3(\varepsilon)}{6\sqrt{\Delta}s^3} + \frac{\mu_4(\varepsilon)}{\Delta s^4} \sum_{m \geq 4} a_m \frac{(ti)^m}{m!}\right) - 1 \right] - \exp(-\lambda_{0,\varepsilon} n \Delta) \mathbf{1}_{\{\Sigma=0\}},$$

where $a_m = \frac{\Delta\mu_m(\varepsilon)}{(\Delta s^2)^{m/2}} \frac{\Delta s^4}{\mu_4(\varepsilon)}$ is such that $a_m \leq (\widetilde{c}_n)^{m-4}$.

We intend to bound D^2 by M^2 , then following (17) it suffices to bound $\|\widehat{G}^{(k)}\|_2^2$ by a quantity much smaller in k than $2^k k! M^2$ for any $k \leq K = c_{int}^2 \log(n)$ and where $M := \widetilde{c}_n^{-4} \left[\frac{|\mu_3(\varepsilon)|}{\sqrt{\Delta}s^3} + \frac{\mu_4(\varepsilon)}{\Delta s^4} \right]$. For illustration, consider first the term $k = 0$, it holds

$$|\widehat{G}(t)| \leq \exp(-t^2/2) \left[\exp\left(M e^{|t|\widetilde{c}_n}\right) - 1 \right] + \exp(-\lambda_{0,\varepsilon} n \Delta) \mathbf{1}_{\{\Sigma=0\}}. \quad (18)$$

Then, for any $c > 0$

$$\int_{-c \log(n)}^{c \log(n)} |\widehat{G}(t)|^2 dt \leq 4 \int_0^{c \log(n)} \exp(-t^2) \left[\exp\left(M e^{t\widetilde{c}_n}\right) - 1 \right]^2 dt + 4c \log(n) \exp(-2\lambda_{0,\varepsilon} n \Delta) \mathbf{1}_{\{\Sigma=0\}}.$$

Assumption (\mathcal{H}_M) ensures that for a small enough universal constant $c_m \geq 0$ we have $M \leq c_m / \sqrt{n}$. In this case, we use a Taylor expansion between $[0, c \log(n)]$ and get if $c\widetilde{c}_n \leq 1/2$, and since $\lambda_{0,\varepsilon} \geq 24 \log(n) / \Delta$

$$\int_{-c \log(n)}^{c \log(n)} |\widehat{G}(t)|^2 dt \leq C' \int_0^{c \log(n)} \exp(-t^2) \left[M e^{t\widetilde{c}_n} \right]^2 dt + 4c/n^2 \leq C'' M^2 + 2/n^2,$$

where C', C'' are two universal constants. This together with $(\mathcal{H}_\Psi(\varepsilon))$ permits to bound $\|\widehat{G}\|_2^2$ by M^2 . The derivatives k th derivatives of \widehat{G} are treated similarly, though the procedure is more cumbersome.

Write $\widehat{G} + \exp(-\lambda_{0,\varepsilon} n \Delta) \mathbf{1}_{\{\Sigma=0\}} = \phi V$, where $V = \exp(g) - 1$,

$$g(t) = -it^3 \frac{\mu_3(\varepsilon)}{6\sqrt{\Delta}s^3} + \frac{\mu_4(\varepsilon)}{\Delta s^4} \sum_{m \geq 4} a_m \frac{(ti)^m}{m!}$$

and $\phi(t) = \exp(-t^2/2)$. We have the following result.

Corollary 3. Suppose $(\mathcal{H}_\varepsilon)$ with $\widetilde{c}_n \leq 1$, $\widetilde{c}_n c \leq 1/4$ and (\mathcal{H}_M) with $c_m \leq 1/2$. There exists a constant $C_{c_{int}}$ that depends on c_{int} only such that we have for any $t \in \mathcal{I}$,

$$\begin{aligned} |\widehat{G}^{(k)}(t)|^2 &\leq C_{c_{int}} \left[k^2 M^2 \sup_{d \leq k-2} \left[2^{-8(k-d)} \binom{k}{d}^2 (k-d)^{k-d} |\phi^{(d)}(t)|^2 \right] \right. \\ &\quad \left. \vee \left[k^4 (\widetilde{c}_n M)^2 e^{2\widetilde{c}_n |t|} |H_{k-1}(t) \phi(t)|^2 \right] \right] \vee |H_k(t) \widehat{G}(t)|^2, \end{aligned}$$

where H_k is the Hermite polynomial of degree k .

We now provide the following two technical lemmas. The first one leads to a bound on the integral of the squared first term in Corollary 3 and the second one to bound the integral of the squared last two terms in Corollary 3.

Lemma 4. *For any $k \leq K$ it holds that*

$$\int_{-c \log(n)}^{c \log(n)} \left[k^2 M^2 \sup_{d \leq k-2} \left[2^{-8(k-d)} \binom{k}{d}^2 (k-d)^{k-d} |\phi^{(d)}(t)|^2 \right] \right] dt \leq 2k^2 k! M^2 \left[\left[\bar{C} 2^{k(1-\bar{c}/16)} \right] \vee 1 \right],$$

where $\bar{C}, \bar{c} > 0$ are strictly positive constants.

Lemma 5. *For any $k \leq K$ it holds that*

$$\int_{|t| \leq c \log n} \exp(-t^2) e^{2|t|\tilde{c}_n} |H_k(t)|^2 dt \leq \frac{4e^{\tilde{c}_n^2}}{\sqrt{2\pi}} k(k!)(1 + \tilde{c}_n^2)^k.$$

To complete the proof, we bound all the terms appearing in Corollary 3 which will lead to a bound on D^2 through (17) and $(\mathcal{H}_\Psi(\varepsilon))$, and finally to a bound on the total variation through (16) and (13). First, whenever $|t| \leq c \log(n)$ we have using (18)

$$|H_k(t)\hat{G}(t)| \leq C|H_k(t)| \exp(-t^2/2) M e^{\tilde{c}_n|t|}.$$

Lemma 5 implies that

$$\int_{|t| \leq c \log n} \left[k^4 (\tilde{c}_n M)^2 e^{2\tilde{c}_n|t|} |H_{k-1}(t)|^2 \exp(-t^2) \right] \vee |H_k(t)\hat{G}(t)|^2 dt \leq C M^2 k^5 (k!)(1 + \tilde{c}_n^2)^k, \quad (19)$$

where C is an absolute constant. Combining Lemma 4 with Equation (19) and Corollary 3, we have that for any $k \leq K$

$$\int_{-c \log(n)}^{c \log(n)} |\hat{G}^{(k)}(t)|^2 dt \leq C'_{c_{int}} k^5 k! M^2 2^{k-\bar{c}'},$$

where $\bar{c} > 0$ is a universal constant strictly positive and $C'_{c_{int}} > 0$ depends only on C_{int} . Injecting this together with the tail Assumption $(\mathcal{H}_\Psi(\varepsilon))$ in (17), we get

$$D^2 \leq M^2 \frac{C'_{c_{int}}}{\sqrt{2\pi}} \sum_{k \leq K} \frac{k^5}{2^{\bar{c}'k}} \leq C''_{c_{int}} M^2,$$

together with Equation (16), we finally obtain using (\mathcal{H}_M) and the definitions of M and s that,

$$\begin{aligned} \|f^{\otimes n} - \varphi^{\otimes n}\|_{TV} &\leq \sqrt{n D^2 \exp(c_h) \left[\sum_{2 \leq k \leq n} 2^k (n D^2)^{k-1} + 1 \right]} + C \exp(c_h) c_p^2 + c_p + \frac{3+2c}{n} \\ &\leq C \sqrt{n} \left[\frac{|\mu_3(\varepsilon)|}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^3}} + \frac{\mu_4(\varepsilon)}{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^2} \right] + \frac{4(c+1)}{n}, \end{aligned}$$

where C depends on \tilde{c}_n , c_m , c_{sup} and c_{int} and where we used Lemma 2 ensuring that under $(\mathcal{H}_\varepsilon)$, Assumption (\mathcal{H}_0) holds with $c_p = 1/n^2$. The proof of Theorem 2 is now complete.

4.2 Proof of Theorem 3

Fix $0 < u \leq \varepsilon$, it holds $M(\varepsilon) = M(u) + M(u, \varepsilon)$ where $M(u, \varepsilon)$ is a compound Poisson process with intensity $\lambda_{u, \varepsilon}$ independent of $M(u)$. Decomposing on the values of the Poisson process induced by $M(u, \varepsilon)$ at time $n\Delta$ and using that the total variation distance is bounded by 1, we have

$$\begin{aligned} & \|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(u)))^{\otimes n}\|_{TV} \\ & \leq \exp(-\lambda_{u, \varepsilon}n\Delta) \|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(u))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(u)))^{\otimes n}\|_{TV} + [1 - \exp(-\lambda_{u, \varepsilon}\Delta n)] \\ & \leq C \exp(-\lambda_{u, \varepsilon}n\Delta) \left[\sqrt{\frac{n\mu_4^2(\varepsilon)}{\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^4} + \frac{n\mu_3^2(\varepsilon)}{\Delta(\Sigma^2 + \sigma^2(\varepsilon))^3} + \frac{1}{n}} \right] + [1 - \exp(-\lambda_{u, \varepsilon}\Delta n)], \end{aligned} \quad (20)$$

applying Theorem 2 to get the last inequality. Finally, the result is obtained using that $\forall 0 < u \leq \varepsilon$

$$\begin{aligned} \min_{B \in \mathbb{R}, S^2 \geq 0} & \|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(B\Delta, \Delta S^2)^{\otimes n}\|_{TV} \leq \\ & \|(\mathcal{N}(b\Delta, \Delta\Sigma^2) * M_\Delta(\varepsilon))^{\otimes n} - \mathcal{N}(b\Delta, \Delta(\Sigma^2 + \sigma^2(u)))^{\otimes n}\|_{TV} \end{aligned}$$

and taking $u = \tilde{u}^*$ in (20).

4.3 Proof of Theorem 5

4.3.1 Preliminary: Four statistical tests

Let $X \sim (b, \Sigma^2, \nu_\varepsilon)$, in particular the increments $X_{i\Delta} - X_{(i-1)\Delta}$ are i.i.d. realizations of

$$X_\Delta = \Delta b + \Sigma W_\Delta + M_\Delta(\varepsilon), \quad \text{where } W_\Delta \sim \mathcal{N}(0, \Delta).$$

For any $n \in \mathbb{N}$, set $\tilde{n} = \lfloor n/2 \rfloor$ and define

$$\begin{aligned} Z_i &:= |(X_{2i\Delta} - X_{(2i-1)\Delta}) - (X_{(2i-1)\Delta} - X_{(2i-2)\Delta})|, \quad i = 1, \dots, \tilde{n} = \lfloor n/2 \rfloor \\ S_n &= \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}-2\log(n)} Z_{(i)}, \quad \text{and} \quad Z_{(\tilde{n})} = \max\{Z_i, 1 \leq i \leq \tilde{n}\}, \end{aligned}$$

where for any sequence a , the sequence $a_{(\cdot)}$ is a reordering of a by increasing order.

For any $0 < u \leq \varepsilon$, we write X as $X = \bar{X}(u) + M(u, \varepsilon)$, where

$$\bar{X}(u)_t = bt + \Sigma W_t + M_t(u)$$

is a Lévy process whose jumps are smaller (or equal) than u and $M(u, \varepsilon) = M_t(\varepsilon) - M_t(u)$ is a pure jumps Lévy process with jumps of size between u and ε . We write $N(u)$ for the number of jumps larger than u , that is, for any $t > 0$, $N_t(u)$ is a Poisson random variable with mean $t\lambda_{u, \varepsilon}$.

Furthermore, in order to present the test needed to prove Theorem 5, we introduce the following

notations:

$$\begin{aligned}
\bar{X}_{\Delta,n} &:= \frac{1}{n - \lfloor n/2 \rfloor} \sum_{i=\lfloor n/2 \rfloor + 1}^n (X_{i\Delta} - X_{(i-1)\Delta}) \\
\bar{Y}_{n,3} &:= \frac{1}{\lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n/2 \rfloor} ((X_{i\Delta} - X_{(i-1)\Delta}) - \bar{X}_{\Delta,n})^3 \\
\bar{Y}_{n,2} &:= \frac{1}{\lfloor n/4 \rfloor} \sum_{i=1}^{\lfloor n/4 \rfloor} Z_i^2, \quad \bar{Y}'_{n,2} := \frac{1}{\lfloor n/4 \rfloor} \sum_{i=\lfloor n/4 \rfloor + 1}^{\lfloor n/2 \rfloor} Z_i^2 \\
\bar{Y}_{n,4} &:= \frac{1}{\lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n/2 \rfloor} Z_i^4, \quad \bar{Y}_{n,6} := \frac{1}{\lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n/2 \rfloor} Z_i^6 \\
T_n^{(3)} &:= \frac{1}{1 - (n - \lfloor n/2 \rfloor)^{-2}} \bar{Y}_{n,3}, \quad T_n^{(4)} := 4^{-1} (\bar{Y}_{n,4} - 3\bar{Y}_{n,2}\bar{Y}'_{n,2}).
\end{aligned}$$

By definition, $\bar{X}_{\Delta,n}$ is the empirical version of $\mathbb{E}[X_\Delta]$ computed on the second half of the sample only and $\bar{Y}_{n,3}$ (resp. $\bar{Y}_{n,6}$) is an estimator of $\mathbb{E}[(X_\Delta - \Delta b)^3]$ (resp. of $8\mathbb{E}[(X_\Delta - \Delta b)^6]$) computed on the first half of the sample. Moreover, since $\mathbb{E}[(X_\Delta - b\Delta)^3] = \Delta\mu_3(\varepsilon)$, using Corollary 6 joined with the independence of X_Δ and $\bar{X}_{\Delta,n}$, we have that $T_n^{(3)}$ is an unbiased estimator of $\Delta\mu_3(\varepsilon)$. Instead $\bar{Y}_{n,4}$ is an estimator of $4\mathbb{E}[(X_\Delta - \Delta b)^4]$ while $\bar{Y}_{n,2}$ and $\bar{Y}'_{n,2}$ are two independent estimators of $2\mathbb{E}[(X_\Delta - \Delta b)^2]$ (see also Corollary 6). Using that $\mathbb{E}[Z_1^4] - 3(\mathbb{E}[Z_1^2])^2 = 4\Delta\mu_4(\varepsilon)$, it is easy to prove that $T_n^{(4)}$ is an unbiased estimator of $\Delta\mu_4(\varepsilon)$ (see, e.g. [7]).

Finally, let $C > 0$ be the absolute constant introduced in Lemma 8 below and consider the following events:

- If $\nu_\varepsilon = 0$, set

$$\xi_n := \{\forall i, Z_i \leq 4\sqrt{\Delta\Sigma^2 \log(n)}\}, \quad (21)$$

$$\xi'_n := \left\{ \frac{\sqrt{\Delta\Sigma^2}}{\sqrt{\pi}} \leq S_n \right\}, \quad (22)$$

$$\xi''_n := \{C(\sqrt{\Delta\Sigma^2})^6 \geq \bar{Y}_{n,6}\}. \quad (23)$$

- If $\nu_\varepsilon \neq 0$, set

$$\begin{aligned}
\xi_n &:= \{1 \leq N_{n\Delta}(u^+) \leq 2\log(n)\} \cap \{\forall i \leq n, N_{i\Delta}(u^+) - N_{(i-2)\Delta}(u^+) \leq 1\} \\
&\quad \cap \{\forall i \text{ s.t. } N_{i\Delta}(u^+) - N_{(i-2)\Delta}(u^+) \neq 0, \\
&\quad |\bar{X}_{i\Delta}(u^*) - \bar{X}_{(i-1)\Delta}(u^*) - (\bar{X}_{(i-1)\Delta}(u^*) - \bar{X}_{(i-2)\Delta}(u^*))| \leq 2\sqrt{(\Sigma^2 + \sigma^2(u^*))\Delta \log(n)}\}, \quad (24) \\
\xi'_n &:= \{S_n \leq 2\sqrt{2\Delta(\Sigma^2 + \sigma^2(u^+))}\}, \quad (25) \\
\xi''_n &:= \left\{ \bar{Y}_{n,6} \geq \frac{\Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3}{2} \right\}.
\end{aligned}$$

Lemma 6. *There exists a universal sequence $\alpha_n \rightarrow 0$ such that $\mathbb{P}(\xi_n) \geq 1 - \alpha_n$.*

Lemma 7. *There exists a universal sequence $\alpha_n \rightarrow 0$ such that $\mathbb{P}(\xi'_n) \geq 1 - \alpha_n$.*

Lemma 8. *There exist a universal sequence $\alpha_n \rightarrow 0$ and a universal constant $C > 0$ such that the following holds.*

Whenever $\nu_\varepsilon = 0$, with probability larger than $1 - \alpha_n$ we have

$$C(\sqrt{\Delta\Sigma^2})^6 \geq \bar{Y}_{n,6}.$$

In any case, with probability larger than $1 - \alpha_n$ and conditional on $N_{n\Delta}(u^*) = 0$, it holds

$$\bar{Y}_{n,6} \geq \frac{\Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3}{2}.$$

Observe that Lemmas 6, 7 and Lemma 8 joined with Equation (26), imply the existence of two absolute sequences $\alpha_n \rightarrow 0$ and $\beta_n \rightarrow 0$ such that

$$\begin{aligned} \mathbb{P}(\xi_n \cap \xi'_n) &\geq 1 - \alpha_n, \\ \mathbb{P}(\xi_n \cap \xi'_n \cap \xi''_n) &\geq 1 - \beta_n. \end{aligned} \tag{26}$$

We are finally ready to introduce the four tests we use to establish Theorem 5:

$$\begin{aligned} \Phi_n^{(\max)} &= \mathbf{1}\{Z_{(\tilde{n})} \geq \log(n)^{3/2} S_n\}, & \Phi_{n,c}^{(6)} &= \mathbf{1}\{\bar{Y}_{n,6} \geq cS_n^6\}, \\ \Phi_{n,c,\alpha}^{(3)} &= \mathbf{1}\{|T_n^{(3)}| \geq \frac{c}{\sqrt{\alpha}} \sqrt{\frac{S_n^6}{n}}\}, & \Phi_{n,c,\alpha}^{(4)} &= \mathbf{1}\{|T_n^{(4)}| \geq \frac{c}{\sqrt{\alpha}} S_n^4 \sqrt{\frac{1}{n}}\}. \end{aligned}$$

Their properties are investigated in Propositions 3, 4, 5 and 6 below.

Proposition 3. Under H_0 , for any $n > e^{4\sqrt{\pi}}$, it holds that $\xi_n \cap \xi'_n \subset \{\Phi_n^{(\max)} = 0\}$. Moreover, for any $n > e^2$, it holds $\xi_n \cap \xi'_n \cap \{N_{n\Delta}(u^*) \geq 1\} \subset \{N_{n\Delta}(u^*) \geq 1\} \cap \{\Phi_n^{(\max)} = 1\}$.

Proposition 4. There exist $c > 0$ a universal constant and C_c depending only on c such that the following holds, for n larger than a constant. Under H_0 , it holds that $\xi'_n \cap \xi''_n \subset \{\Phi_{n,c}^{(6)} = 0\}$. Moreover if $\Delta\mu_6(u^*) \geq C_c \Delta^3(\Sigma^2 + \sigma^2(u^*))^3$, then $\xi'_n \cap \xi''_n \cap \{N_{n\Delta}(u^*) = 0\} \subset \{N_{n\Delta}(u^*) = 0\} \cap \{\Phi_{n,c}^{(6)} = 1\}$.

Proposition 5. Let $\alpha > 2\log(n)^{-1}$. Let $c > 0$ and $c' > 0$ be large enough absolute constant and let $C_{c,c'} > 0$ be a large enough absolute constant depending only on c and c' . Then, the following holds.

Under H_0 , $\Phi_{n,c,\alpha}^{(3)} = 0$ with probability larger than $1 - \alpha - \mathbb{P}(\xi_n^c)$.

Under the hypothesis $H_{1,\rho_n}^{(3)} : \mu_3(u^*) > \rho_n^{(3)}$ and conditional to the event $N_{n\Delta}(u^*) = 0$, if

$$u^* > u^+, \quad \Delta\mu_6(u^*) \leq c' \Delta^3(\Sigma^2 + \sigma^2(u^*))^3, \quad \rho_n^{(3)} \geq \frac{C_{c,c'}}{\sqrt{\alpha}} \left\{ \frac{\sqrt{\Delta(\Sigma^2 + \sigma^2(u^*))^3}}{\sqrt{n}} \right\}, \tag{27}$$

it holds that $\Phi_{n,c,\alpha}^{(3)} = 1$ with probability larger than $1 - \alpha - \mathbb{P}(\xi_n^c)$.

Proposition 6. Let $\alpha > 2\log(n)^{-1}$. Let $c > 0$ and $c' > 0$ be large enough absolute constant and let $C_{c,c'} > 0$ be a large enough absolute constant depending only on c and c' . Then, the following holds.

Under H_0 , it holds that $\Phi_{n,c,\alpha}^{(4)} = 0$ with probability larger than $1 - \alpha - \mathbb{P}(\xi_n^c)$.

Under the hypothesis $H_{1,\rho_n}^{(4)} : \mu_4(u^*) > \rho_n^{(4)}$ and conditional to the event $N_{n\Delta}(u^*) = 0$, if

$$u^* > u^+, \quad \rho_n^{(4)} \geq \frac{C_c}{\sqrt{\alpha}} \left\{ \frac{\Delta(\Sigma^2 + \sigma^2(u^*))^2}{\sqrt{n}} \right\},$$

it holds that $\Phi_{n,c,\alpha}^{(4)} = 1$ with probability larger than $1 - \alpha - \mathbb{P}(\xi_n^c)$.

4.3.2 Completion on the proof of Theorem 5

Let $(\tilde{b}, \tilde{\Sigma}, \tilde{\nu}_\varepsilon)$ be a Lévy triplet where $\tilde{\nu}_\varepsilon$ is a Lévy measure with support in $[-\varepsilon, \varepsilon]$. Assume that we want to test

$$H_0 : \nu_\varepsilon = 0, \quad \text{against} \quad H_1 : (b, \Sigma, \nu_\varepsilon) = (\tilde{b}, \tilde{\Sigma}, \tilde{\nu}_\varepsilon).$$

We write $\tilde{\mu}_\cdot, \tilde{\lambda}_\cdot$ and \tilde{u}^* for all the quantities related to $(\tilde{b}, \tilde{\Sigma}, \tilde{\nu}_\varepsilon)$.

We can choose $c^{(3)}, c^{(4)}, c^{(6)} > 0$ large enough universal constants and $C^{(3)}, C^{(4)}, C^{(6)} > 0$ large enough depending only on $c^{(3)}, c^{(4)}, c^{(6)}$, and an absolute sequence α_n that converges to 0 such that Propositions 3, 4, 5 and 6 hold. Set

$$\alpha = \left\{ \left(\frac{C^{(3)}}{\tilde{\mu}_3(\tilde{u}^*)} \frac{\sqrt{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^3}}{\sqrt{n}} \right)^2 \wedge \left(\frac{C^{(4)}}{\tilde{\mu}_4(u^*)} \frac{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^2}{\sqrt{n}} \right)^2 \right\} \vee \alpha_n.$$

Write $i = 3$ if $\left(\frac{C^{(3)}}{\tilde{\mu}_3(\tilde{u}^*)} \frac{\sqrt{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^3}}{\sqrt{n}} \right)^2 \leq \left(\frac{C^{(4)}}{\tilde{\mu}_4(u^*)} \frac{\Delta(\Sigma^2 + \sigma^2(\tilde{u}^*))^2}{\sqrt{n}} \right)^2$ and $i = 4$ otherwise. In the remaining of the proof α_n denotes a vanishing sequence whose value may change from line to line.

Case 1 : $1 - \exp(-\tilde{\lambda}_{\tilde{u}^*, \varepsilon} n \Delta) \geq 1 - \alpha$. In this case, consider the test $\Phi_n = \Phi_n^{(\max)}$. If $X \sim (b, \Sigma^2, \nu_\varepsilon)$ is in H_0 (i.e. $\nu_\varepsilon = 0$), an application of Proposition 3 and Lemmas 6 and 7 yields $\mathbb{P}(\Phi_n = 0) \geq 1 - \alpha_n$. If, instead, X is such that $(b, \Sigma, \nu_\varepsilon) = (\tilde{b}, \tilde{\Sigma}, \tilde{\nu}_\varepsilon)$, by means of Proposition 3 and Lemmas 6, 7 we get

$$\mathbb{P}(\Phi_n = 1) \geq \mathbb{P}(\{N_{n\Delta}(\tilde{u}^*) \neq 0\} \cap \xi_n \cap \xi'_n) \geq 1 - \exp(-\tilde{\lambda}_{\tilde{u}^*, \varepsilon} n \Delta) - \alpha_n.$$

So by Lemma 1 it follows that the total variation between the observations of n increments of X at the sampling rate Δ and the closest Gaussian random variable is larger than $1 - \exp(-\tilde{\lambda}_{\tilde{u}^*, \varepsilon} n \Delta) - \alpha_n$.

Case 2 : $1 - \exp(-\tilde{\lambda}_{\tilde{u}^*, \varepsilon} n \Delta) \leq 1 - \alpha$. In this case consider the test

$$\Phi_{n, c^{(i)}, c^{(6)}, \alpha} = \Phi_n^{(\max)} \vee \Phi_{n, c^{(i)}, \alpha}^{(i)} \vee \Phi_{n, c^{(6)}}^{(6)}.$$

If X is in H_0 (i.e. $\nu_\varepsilon = 0$), by Propositions 3, 4, 5 and 6 we have that

$$\mathbb{P}(\Phi_{n, c^{(i)}, c^{(6)}, \alpha} = 0) \geq 1 - \alpha - \alpha_n.$$

If X is such that $(b, \Sigma, \nu_\varepsilon) = (\tilde{b}, \tilde{\Sigma}, \tilde{\nu}_\varepsilon)$, we distinguish two cases.

- If $\Delta \mu_6(u^*) \geq C^{(6)}(\Delta(\Sigma^2 + \sigma^2(u^*)))^3$: Propositions 3, 4 yield

$$\mathbb{P}(\Phi_{n, c_\alpha^{(i)}, c^{(6)}, \alpha} = 1) \geq \mathbb{P}(\{N_{n\Delta}(\tilde{u}^*) \neq 0\} \cap \xi_n \cap \xi'_n) + \mathbb{P}(\{N_{n\Delta}(\tilde{u}^*) = 0\})(1 - \alpha_n).$$

- If $\Delta \mu_6(u^*) < C^{(6)}(\Delta(\Sigma^2 + \sigma^2(u^*)))^3$: Propositions 3, 5, 6 joined with $\{u^* > u^+\}$ yield

$$\mathbb{P}(\Phi_{n, c_\alpha^{(i)}, c^{(6)}, \alpha} = 1) \geq \mathbb{P}(\{N_{n\Delta}(\tilde{u}^*) \neq 0\} \cap \xi_n \cap \xi'_n) + \mathbb{P}(\{N_{n\Delta}(\tilde{u}^*) = 0\})(1 - \alpha - \alpha_n).$$

In both cases we conclude that,

$$\begin{aligned} \mathbb{P}(\Phi_{n, c^{(i)}, c^{(6)}, \alpha} = 1) &\geq \mathbb{P}(\{N_{n\Delta}(\tilde{u}^*) \neq 0\}) + \mathbb{P}(\{N_{n\Delta}(\tilde{u}^*) = 0\})(1 - \alpha) - \alpha_n \\ &\geq 1 - \alpha \exp(-\tilde{\lambda}_{\tilde{u}^*, \varepsilon} n \Delta) - \alpha_n. \end{aligned}$$

By Lemma 1 we deduce that the total variation distance between the observations of n increments of X at the sampling rate Δ and the closest Gaussian random variable is larger than $1 - 2\alpha - \alpha_n$.

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A Technical results

A.1 Proofs of the auxiliary Lemmas used in the proof of Theorem 2

A.1.1 Proof of Lemma 2

We consider a compound Poisson approximation of the increment $\tilde{X}_\Delta(\varepsilon)$ to apply the Berstein inequality and first focus on its jump part. Let $0 < \eta < \varepsilon$, and define

$$\begin{aligned}
 \tilde{M}_\Delta(\eta, \varepsilon) &= \frac{\sum_{i=0}^{N_\Delta(\lambda, \varepsilon)} Y_i - \Delta \int_{\eta \leq |x| \leq \varepsilon} x d\nu}{\sqrt{\Delta s^2}} \\
 &= \frac{\sum_{i=0}^{N_\Delta(\lambda, \varepsilon)} (Y_i - \lambda_{\eta, \varepsilon}^{-1} \int_{\eta \leq |x| \leq \varepsilon} x d\nu) + N_\Delta(\lambda, \varepsilon) \lambda_{\eta, \varepsilon}^{-1} \int_{\eta \leq |x| \leq \varepsilon} x d\nu - \Delta \int_{\eta \leq |x| \leq \varepsilon} x d\nu}{\sqrt{\Delta s^2}} \\
 &= \bar{M}_\Delta(\eta, \varepsilon) + \frac{N_\Delta(\lambda, \varepsilon) \lambda_{\eta, \varepsilon}^{-1} \int_{\eta \leq |x| \leq \varepsilon} x d\nu - \Delta \int_{\eta \leq |x| \leq \varepsilon} x d\nu}{\sqrt{\Delta s^2}}, \tag{28}
 \end{aligned}$$

where

$$\bar{M}_\Delta(\eta, \varepsilon) = \frac{\sum_{i=0}^{N_\Delta(\lambda, \varepsilon)} (Y_i - \lambda_{\eta, \varepsilon}^{-1} \int_{\eta \leq |x| \leq \varepsilon} x d\nu)}{\sqrt{\Delta s^2}},$$

$N_\Delta(\lambda, \varepsilon)$ is a Poisson random variable of intensity $\Delta \int_{\eta \leq |x| \leq \varepsilon} d\nu := \Delta \lambda_{\eta, \varepsilon}$, and the Y_i are i.i.d. bounded random variables with density $\frac{1}{\lambda_{\eta, \varepsilon}} \frac{\nu(dx)}{dx} \mathbf{1}_{\eta \leq |x| \leq \varepsilon}$. Note that for any N , $\mathbb{E}[\bar{M}_\Delta(\eta, \varepsilon) | N_\Delta(\lambda, \varepsilon) = N] = 0$, and if $|N_\Delta(\lambda, \varepsilon) - \Delta \lambda_{\eta, \varepsilon}| \leq \Delta \lambda_{\eta, \varepsilon}/2$ we have $\lambda_{\eta, \varepsilon} \mathbb{V}(Y_i) = \int_{\eta \leq |x| \leq \varepsilon} x^2 d\nu \leq \sigma^2(\varepsilon) \leq s^2$ and

$$\mathbb{V}[\bar{M}_\Delta(\eta, \varepsilon) | N_\Delta(\lambda, \varepsilon) = N] \leq 2.$$

Finally, the random variables $|Y_i|$ are bounded by ε . For any N such that $|N - \Delta \lambda_{\eta, \varepsilon}| \leq \Delta \lambda_{\eta, \varepsilon}/2$, the Bernstein’s inequality, conditional on $N_\Delta(\lambda, \varepsilon) = N$, leads to

$$\begin{aligned}
 \mathbb{P}(|\bar{M}_\Delta(\eta, \varepsilon)| > c_{sup} \sqrt{\log(n)/2} | N_\Delta(\lambda, \varepsilon) = N) &\leq 2 \exp \left(- \frac{1}{2} \frac{c_{sup}^2 \log n/4}{2 + \frac{1}{3} \frac{\varepsilon}{\sqrt{\Delta s^2}} c_{sup} \sqrt{\log(n)/2}} \right) \\
 &\leq 2 \exp \left(- \frac{1}{8} \frac{c_{sup}^2 \log n}{2 + \frac{1}{6} \tilde{c}_n c_{sup}} \right),
 \end{aligned}$$

where we used $(\mathcal{H}_\varepsilon)$. Therefore, it holds for any N such that $|N - \Delta\lambda_{\eta,\varepsilon}| \leq \Delta\lambda_{\eta,\varepsilon}/2$, that

$$\mathbb{P}(|\overline{M}_\Delta(\eta, \varepsilon)| > c_{sup} \sqrt{\log(n)}/2 | N_\Delta(\lambda, \varepsilon) = N) \leq n^{-3},$$

if $c_{sup} \geq 10$. Now by assumption on $\lambda_{0,\varepsilon}$, there exists $\bar{\eta} := \bar{\eta}_\delta > 0$ such that for any $\eta \leq \bar{\eta}$, we have $\Delta\lambda_{\eta,\varepsilon} \geq 1$ and $\frac{24\log(n)}{\Delta} \leq \lambda_{\eta,\varepsilon}$. Moreover, for $\eta \leq \bar{\eta}$, since $N_\Delta(\lambda, \varepsilon)$ is a Poisson random variable of parameter $\Delta\lambda_{\eta,\varepsilon} \geq 1$, we have for any $0 \leq x \leq \sqrt{\Delta\lambda_{\eta,\varepsilon}}$

$$\mathbb{P}(|N_\Delta(\lambda, \varepsilon) - \Delta\lambda_{\eta,\varepsilon}| \geq \sqrt{\Delta\lambda_{\eta,\varepsilon}x}/2) \leq \exp(-x^2/8).$$

This implies that for $x := \sqrt{24\log(n)} \leq \sqrt{\Delta\lambda_{\eta,\varepsilon}}$,

$$\mathbb{P}(|N_\Delta(\lambda, \varepsilon) - \Delta\lambda_{\eta,\varepsilon}| \geq \sqrt{6\Delta\lambda_{\eta,\varepsilon}\log(n)}) \leq n^{-3}.$$

Removing the conditioning on $N_\Delta(\lambda, \varepsilon)$ (noting that $\sqrt{6\Delta\lambda_{\eta,\varepsilon}\log(n)} \leq \Delta\lambda_{\eta,\varepsilon}/2$) we get

$$\mathbb{P}\left(\left|\overline{M}_\Delta(\eta, \varepsilon) + \frac{N_\Delta(\lambda, \varepsilon)}{\sqrt{\Delta s^2}} \lambda_{\eta,\varepsilon}^{-1} \int_{\eta \leq |x| \leq \varepsilon} x d\nu(x) - \frac{\Delta}{\sqrt{\Delta s^2}} \int_{\eta \leq |x| \leq \varepsilon} x d\nu(x)\right| > c_{sup} \sqrt{\log(n)}\right) \leq 2n^{-3},$$

using that $c_{sup} \geq 10$ and that by the Cauchy-Schwartz inequality $|\int_{\eta \leq |x| \leq \varepsilon} x d\nu(x)| \leq \sqrt{\lambda_{\eta,\varepsilon}\sigma(\varepsilon)}$. Taking the limit in $\eta \rightarrow 0$ leads to

$$\mathbb{P}(|\widetilde{M}_\Delta(0, \varepsilon)| > c_{sup} \sqrt{\log(n)}) \leq 2n^{-3}. \quad (29)$$

Adding the Gaussian part and the drift, by Gaussian concentration we get that

$$\mathbb{P}(|\widetilde{X}_\Delta| > c_{sup} \sqrt{\log(n)}) \leq 3n^{-3}.$$

This implies the result whenever $c_{sup} \geq 10$.

A.1.2 Proof of Lemma 3

In this proof, we assume that all integrals are computed on $\widetilde{\xi}_M$ only, for sake of readability we do not write the indicator everywhere.

Recall that Stirling's approximation gives for $n \geq 1$, $n! \geq (n/e)^n$. For any $x \in \mathcal{I}$, using that $\sqrt{K} \geq 2c_{sup}\sqrt{\log(n)}$ (since $c_{int} \geq 2c_{sup}$), we derive that

$$|\sqrt{2\pi} \exp(x^2/2) - h_{\mathcal{I}}^{-1}(x)| = \sqrt{2\pi} \mathbf{1}_{\{\mathcal{I}\}} \sum_{k=K+1}^{+\infty} \frac{x^{2k}}{2^{2k} k!} \leq \sqrt{2\pi} \mathbf{1}_{\{\mathcal{I}\}} \sum_{k=K+1}^{+\infty} \frac{(\sqrt{K}/2)^{2k} e^k}{2^{2k} k^k} \leq \frac{1}{1 - \frac{e}{4}} \frac{1}{2^K},$$

since $\sum_{k \geq K+1} \frac{(Ke)^k}{2^{2k} k^k} \leq \sum_{k \geq 0} \left(\frac{e}{4}\right)^k = \frac{1}{1 - e/4}$.

It follows that for $c_{int} \geq 1$ and $x \in \mathcal{I}$, that we have

$$|\varphi^{-1}(x) - h^{-1}(x)| = |\sqrt{2\pi} \exp(x^2/2) - h_{\mathcal{I}}^{-1}(x)| \leq \frac{1}{1 - e/4} \frac{1}{n^2}. \quad (30)$$

Equation (30) implies that

$$A = \int_{\widetilde{\xi}_M} \frac{\varphi_{\mathcal{I}}^2}{h_{\mathcal{I}}} = \int_{\widetilde{\xi}_M} \left[\varphi_{\mathcal{I}} + \varphi_{\mathcal{I}}^2 [h_{\mathcal{I}}^{-1} - \varphi_{\mathcal{I}}^{-1}] \right] \leq 1 + \frac{1}{1 - e/4} \frac{1}{n^2},$$

and

$$\left| \int h_{\mathcal{I}} - \int \varphi_{\mathcal{I}} \right| = \left| \int_{\mathcal{I}} h \varphi (\varphi^{-1} - h^{-1}) \right| \leq \int_{\mathcal{I}} h(x) \varphi(x) |\sqrt{2\pi} \exp(x^2/2) - h^{-1}(x)| \leq \frac{1}{1 - e/4} \frac{1}{n^2},$$

since, by definition, $h_{\mathcal{I}} \leq 1/\sqrt{2\pi}$. This together with $\mathbb{P}_{\varphi}(\mathcal{I}^c) \leq n^{-c_{sup}^2/2}$, the second inequality follows. Finally, using (30) we get

$$|E| = \left| \int_{\tilde{\xi}_M} \varphi_{\mathcal{I}} h_{\mathcal{I}}^{-1} (f_{\mathcal{I}} - \varphi_{\mathcal{I}}) - \int_{\tilde{\xi}_M} (f_{\mathcal{I}} - \varphi_{\mathcal{I}}) \right| = \left| \int_{\tilde{\xi}_M} \varphi_{\mathcal{I}} (h_{\mathcal{I}}^{-1} - \varphi_{\mathcal{I}}^{-1}) (f_{\mathcal{I}} - \varphi_{\mathcal{I}}) \right| \leq 2 \frac{1}{1-e/4} \frac{1}{n^2}.$$

By (\mathcal{H}_0) and $\mathbb{P}_{\varphi}(\mathcal{I}^c) \leq n^{-c_{sup}^2/2}$ we have

$$\left| \int_{\tilde{\xi}_M} (f_{\mathcal{I}} - \varphi_{\mathcal{I}}) \right| = |\mathbb{P}_f(\mathcal{I}^c) - \mathbb{P}_{\varphi}(\mathcal{I}^c)| \leq n^{-c_{sup}^2/2} + c_p/n,$$

and the bound on $|E|$ is established.

A.1.3 Proof of Corollary 3

We start with two preliminary Lemmas.

Lemma 9. Suppose $(\mathcal{H}_{\varepsilon})$ with $\tilde{c}_n \leq 1$, it holds that

$$|\widehat{G}^{(k)}(t)|^2 \leq \left[k^2 e^{2Me^{|t|\tilde{c}_n}} \sup_{d \leq k-1} \left[\binom{k}{d}^2 |\phi^{(d)}(t)|^2 \left[2^{2(k-d)} \max_{1 \leq u \leq k-d} u^{2(k-d-u)} (\tilde{c}_n M)^{2u} e^{2\tilde{c}_n |t|u} \right] \right] \right] \vee \left[|\phi^{(k)}|^2 V^2 \right].$$

Proof of Lemma 9. By the binomial formula we bound

$$|\widehat{G}^{(k)}| \leq k \sup_{d \leq k} C_k^d |\phi^{(d)}| |V^{(k-d)}|. \quad (31)$$

Moreover $V^{(m)} = (\exp(g) - 1)^{(m)}$, Lemma 10 below (with $(\mathcal{H}_{\varepsilon})$ and $\tilde{c}_n \leq 1$) leads to

$$|V^{(m)}| = |(\exp(g) - 1)^{(m)}| \leq 2^m e^{Me^{|t|\tilde{c}_n}} \max_{1 \leq u \leq m} u^{m-u} (\tilde{c}_n M)^u e^{\tilde{c}_n |t|u}.$$

Injecting this upper bound in (31) leads to the desired result. \square

Lemma 10. Suppose $(\mathcal{H}_{\varepsilon})$ with $\tilde{c}_n \leq 1$, it holds for $m \geq 1$

$$|V^{(m)}| = |(\exp(g) - 1)^{(m)}| \leq 2^m e^{Me^{|t|\tilde{c}_n}} \max_{1 \leq u \leq m} u^{m-u} (\tilde{c}_n M)^u e^{\tilde{c}_n |t|u}.$$

Proof of Lemma 10. First, note that for any $j \geq 1$, we have

$$g^{(j)}(t) = -i(t^3)^{(j)} \frac{\mu_3(\varepsilon)}{6\sqrt{\Delta}s^3} + \frac{\mu_4(\varepsilon)}{\Delta s^4} \sum_{m \geq (j \wedge 4)} a_m \frac{i^m t^{m-j}}{(m-j)!},$$

and

$$|g^{(j)}(t)| \leq \left[\frac{|\mu_3(\varepsilon)|}{\sqrt{\Delta}s^3} + \frac{\mu_4(\varepsilon)}{\Delta s^4} \right] \sum_{m \geq (j \wedge 3)} |a_m| \frac{|t|^{m-j}}{(m-j)!},$$

where $a_3 = 1$. And since for any $m \geq 4$ we have $a_m \leq \tilde{c}_n^{m-4}$, this implies since $\tilde{c}_n \leq 1$

$$|g^{(j)}(t)| \leq \left[\frac{|\mu_3(\varepsilon)|}{\sqrt{\Delta}s^3} + \frac{\mu_4(\varepsilon)}{\Delta s^4} \right] \sum_{m \geq (j \wedge 3)} \tilde{c}_n^{m-4} \frac{|t|^{m-j}}{(m-j)!} \leq \tilde{c}_n M e^{|t|\tilde{c}_n}. \quad (32)$$

Write

$$R_m = \frac{(\exp(g))^{(m)}}{\exp(g)}$$

and note that $R_{m+1} = R_m^{(1)} + g^{(1)}R_m$. So we have for any $d \geq 0$

$$|R_{m+1}^{(d)}| = |R_m^{(d+1)} + (g^{(1)}R_m)^{(d)}| \leq |R_m^{(d+1)}| + \sum_{j \leq d} C_d^j |g^{(d-j+1)}| |R_m^{(j)}|, \quad (33)$$

by the Leibniz formula. Write for $m \geq 1$ the induction assumption:

$$H(m) : \forall d \in \mathbb{N}, |R_m^{(d)}| \leq 2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m-u} u^d.$$

Assumption $H(1)$ hold since $R_1 = g^{(1)}$ and by (32) we obtain $|g^{(j)}| \leq M \tilde{c}_n e^{\tilde{c}_n |t|}$ for j larger than 1. Assume for some $m \geq 1$ that $H(m)$ holds. We have by (33)

$$|R_{m+1}^{(d)}| \leq 2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m-u} u^{d+1} + \left[M \tilde{c}_n \exp(\tilde{c}_n |t|) \right] \sum_{j \leq d} C_d^j \left[2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m-u} u^j \right],$$

by $H(m)$ and (32). It follows that

$$\begin{aligned} |R_{m+1}^{(d)}| &\leq 2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m+1-u} u^d + \sum_{j \leq d} C_d^j \left[2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^{u+1} e^{\tilde{c}_n |t| (u+1)} u^{m-u} u^j \right] \\ &= 2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m+1-u} u^d + \left[2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^{u+1} e^{\tilde{c}_n |t| (u+1)} u^{m-u} (1+u)^d \right] \\ &\leq 2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m+1-u} u^d + \left[2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^{u+1} e^{\tilde{c}_n |t| (u+1)} (u+1)^{m-u+d} \right], \end{aligned}$$

where we used the binomial formula for the second equation. Finally,

$$\begin{aligned} |R_{m+1}^{(d)}| &\leq 2^m \max_{u \in \{1, \dots, m\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m+1-u} u^d + \left[2^m \max_{u \in \{2, \dots, m+1\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m+1-u+d} \right] \\ &\leq 2^{m+1} \max_{u \in \{1, \dots, m+1\}} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u} u^{m+1-u}. \end{aligned}$$

Therefore, $H(m+1)$ holds and the induction is proven. In particular

$$|R_m| \leq 2^m \max_{u \in \{1, \dots, m\}} u^{m-u} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u},$$

and for any $m \geq 1$ we obtain

$$|V^{(m)}| = |(\exp(g) - 1)^{(m)}| \leq |R_m| |\exp(g)| \leq 2^m e^{Me|t|\tilde{c}_n} \max_{1 \leq u \leq m} u^{m-u} (\tilde{c}_n M)^u e^{\tilde{c}_n |t| u}.$$

□

Proof of Corollary 3. By Lemma 9 we have since $\tilde{c}_n \leq 1$

$$|\widehat{G}^{(k)}(t)|^2 \leq \left[k^2 e^{2Me|t|\tilde{c}_n} \sup_{d \leq k-1} \left[\binom{k}{d}^2 |\phi^{(d)}(t)|^2 \left[2^{2(k-d)} \max_{1 \leq u \leq k-d} u^{2(k-d-u)} (\tilde{c}_n M)^{2u} e^{2\tilde{c}_n |t| u} \right] \right] \right] \vee \left[|\phi^{(k)}|^2 V^2 \right].$$

The term $|\phi^{(k)}|^2 V^2$ lead to the last term of the corollary since $H_m \phi = \phi^{(m)}$ for any integer m and $|V| = |\phi^{-1} \widehat{G}|$. Then, the first term for $d = k-1$ leads to the second term of the corollary using (\mathcal{H}_M) and that $|t| \leq c \log(n)$, $e^{2Me\tilde{c}_n |t|} \leq e^{2c_m n \tilde{c}_n c - \frac{1}{2}} < e$ whenever $\tilde{c}_n c \leq \frac{1}{2}$, $c_m \leq \frac{1}{2}$.

Next, we control the remaining term using the decomposition $(\tilde{c}_n M)^{2u} = ((\tilde{c}_n M)^{\frac{u-1}{2}})^2 (\tilde{c}_n M)^{u-1} (\tilde{c}_n M)^2$. First, for any integer $u \geq 2$, t such that $|t| \leq c \log(n)$ if $\tilde{c}_n c \leq \frac{1}{4}$ and using (\mathcal{H}_M) with $c_m \leq \frac{1}{2}$ it holds

$$M^{u-1} \exp(2\tilde{c}_n |t| u) \leq 1 \quad \text{and} \quad e^{2Me|t|\tilde{c}_n} \leq 1.$$

We are left with the term, for any $u \geq 2$,

$$\begin{aligned} & 2^{2(k-d)} \max_{2 \leq u \leq k-d} u^{2(k-d-u)} (\tilde{c}_n M)^{2 \frac{(u-1)}{2}} \\ & \leq [M(k-d)^{k-d}] \vee \left[2^{2(k-d)} \max_{(k-d)^{1/4} \leq u \leq k-d} e^{2(k-d) \log(u) - \frac{1}{4} \log(n)(u-1)} M^{\frac{(u-1)}{2}} \right]. \end{aligned}$$

Since $k \leq K = c_{int}^2 \log(n)$, we know that $2(k-d) \log(u) - \log(n)(u-1)/4$ is negative whenever $4c_{int}^2 \log(u) \leq u$. Moreover, using [\(H_M\)](#), it follows that for $k \leq c_{int}^2 \log(n)$ there exists a constant, $C_{c_{int}}$ that depends only on c_{int} , such that

$$2^{2(k-d)} \max_{2 \leq u \leq k-d} u^{2(k-d-u)} (\tilde{c}_n M)^{(u-1)} \leq C_{c_{int}} 2^{-8(k-d)} (k-d)^{k-d}.$$

The proof of the corollary is complete.

A.1.4 Proof of Lemma 4

Recall the Stirling approximation

$$\sqrt{\frac{\pi k}{2}} \left(\frac{k}{e}\right)^k \leq k! \leq 2\sqrt{\pi k} \left(\frac{k}{e}\right)^k, \quad \forall k \geq 1, \quad (34)$$

we derive that if $Z \sim \mathcal{N}(0, \omega^2)$ then,

$$\mathbb{E}[Z^{2m}] \leq 4(2\omega^2)^m \frac{m^m}{e^m} \leq 4(2\omega^2)^m m! \quad \forall m \geq 1 \quad (35)$$

By Plancherel and [\(35\)](#) used with $\omega^2 = 1/2$, it holds for $m \geq 1$

$$\int |\phi^{(m)}(t)|^2 dt = \|P_m \varphi\|_2^2 = \frac{1}{2\pi} \int x^{2m} \exp(-x^2) dx \leq \frac{4}{\sqrt{2\pi}} \frac{m^m}{e^m}. \quad (36)$$

Equation [\(36\)](#) and [\(34\)](#) imply

$$\begin{aligned} & \int_{-c \log(n)}^{c \log(n)} k^2 M^2 \sup_{d \leq k-2} \left[2^{-8(k-d)} \left(\frac{k}{d}\right)^2 (k-d)^{k-d} |\phi^{(d)}(t)|^2 \right] dt \\ & \leq k^2 M^2 \sup_{d \leq k-2} \left[2^{-8(k-d)} \left(\frac{k}{d}\right)^2 (k-d)^{k-d} \frac{2d^d}{e^d} \right] \leq 2k^2 M^2 \sup_{d \leq k-2} \left[2^{-8(k-d)} e^{k-d} k! \left(\frac{k}{d}\right) \right] \\ & \leq 2k^2 k! M^2 \sup_{d \leq k-2} \left[2^{-4(k-d)} \left(\frac{k}{d}\right) \right] = 2k^2 k! M^2 \sup_{d \leq k-2} \left[2^{-4d} \left(\frac{k}{d}\right) \right], \end{aligned}$$

we used that $2^{-4}e \leq 1$. Therefore,

$$\begin{aligned} & \int_{-c \log(n)}^{c \log(n)} k^2 M^2 \sup_{d \leq k-2} \left[2^{-8(k-d)} \left(\frac{k}{d}\right)^2 (k-d)^{k-d} |\phi^{(d)}(t)|^2 \right] dt \\ & \leq 2k^2 k! M^2 \left[\sup_{d \leq k/4} \left[2^{-4d} \left(\frac{k}{d}\right) \right] \vee \sup_{k/4+1 \leq d \leq k-2} \left[2^{-4d} \left(\frac{k}{d}\right) \right] \right] \\ & \leq 2k^2 k! M^2 \left[\sup_{d \leq k/4} \left[2^{-4d} \overline{C} 2^k e^{-\bar{c}(|d-k/2|/\sqrt{k})^2} \right] \vee 1 \right], \end{aligned}$$

using the sub-Gaussian concentration of the Binomial distribution; there exists $\overline{C}, \bar{c} > 0$ universal constant such that $\frac{C_k^d}{2^k} \leq \overline{C} e^{-\bar{c}(|d-k/2|/\sqrt{k})^2}$. Finally, we get

$$\begin{aligned} \int_{-c \log(n)}^{c \log(n)} \left[k^2 M^2 \sup_{d \leq k-2} \left[2^{-8(k-d)} \binom{k}{d}^2 (k-d)^{k-d} |\phi^{(d)}(t)|^2 \right] \right] dt \\ \leq 2k^2 k! M^2 \left[\sup_{d \leq k/4} \left[\overline{C} 2^{k-4d} e^{-\bar{c}k/16} \right] \vee 1 \right] \leq 2k^2 k! M^2 \left[\left[\overline{C} 2^{k(1-\bar{c}/16)} \right] \vee 1 \right], \end{aligned}$$

where \overline{C} is a universal strictly positive constants.

A.1.5 Proof of Lemma 5

First, it holds that

$$\begin{aligned} \int_{|t| \leq c \log n} \exp(-t^2) e^{2|t|\tilde{c}_n} |H_k(t)|^2 dt &= 2 \int_{0 \leq t \leq c \log n} \exp(-t^2) e^{2t\tilde{c}_n} |H_k(t)|^2 dt \\ &= 2e^{\tilde{c}_n^2} \int_{0 \leq t \leq c \log n} \exp(-(t - \tilde{c}_n)^2) |H_k(t)|^2 dt \\ &= 2e^{\tilde{c}_n^2} \int_{-\tilde{c}_n \leq t \leq c \log n - \tilde{c}_n} \exp(-t^2) |H_k(t + \tilde{c}_n)|^2 dt \\ &\leq 2e^{\tilde{c}_n^2} \int_{\mathbb{R}} \exp(-t^2) |H_k(t + \tilde{c}_n)|^2 dt. \end{aligned}$$

Using the following property of Hermite polynomial we get

$$H_k(t + \tilde{c}_n) = \sum_{u=0}^k \binom{k}{u} H_u(t) \tilde{c}_n^{k-u}.$$

and the Cauchy Schwarz inequality leads to

$$H_k(t + \tilde{c}_n)^2 \leq k \sum_{u=0}^k \binom{k}{u}^2 \tilde{c}_n^{2k-2u} |H_u(t)|^2.$$

Equation (36) and the definition of Hermite polynomials imply that

$$\begin{aligned} \int_{\mathbb{R}} \exp(-t^2) |H_k(t + \tilde{c}_n)|^2 dt &\leq 2e^{\tilde{c}_n^2} k \sum_{u=0}^k \binom{k}{u}^2 \tilde{c}_n^{2k-2u} \int_{\mathbb{R}} \exp(-t^2) |H_u(t)|^2 dt \\ &\leq \frac{4e^{\tilde{c}_n^2}}{\sqrt{2\pi}} k \sum_{u=0}^k \binom{k}{u}^2 \tilde{c}_n^{2k-2u} u! = \frac{4e^{\tilde{c}_n^2}}{\sqrt{2\pi}} k(k!) \sum_{u=0}^k \frac{k!}{u!((k-u)!)^2} \tilde{c}_n^{2k-2u} \\ &\leq \frac{4e^{\tilde{c}_n^2}}{\sqrt{2\pi}} k(k!)(1 + \tilde{c}_n^2)^k, \end{aligned}$$

which completes the proof.

A.2 Proofs of the Propositions involved in the proof of Theorem 5

Notations. In the sequel, we exploit the approximation of $M_{\Delta}(\varepsilon)$ by a sequence of compound Poisson processes. For any $0 < \eta < \varepsilon$ define $M(\eta, \varepsilon)$, the centered compound Poisson process that approximates

$M(\varepsilon)$ as $\eta \downarrow 0$, by

$$M_t(\eta, \varepsilon) = \sum_{s \leq t} \Delta X_s \mathbf{1}_{\eta < |\Delta X_s| \leq \varepsilon} - t \int_{\eta < |x| \leq \varepsilon} x \nu(dx) = \sum_{i=1}^{N_t} Y_i - t \int_{\eta < |x| \leq \varepsilon} x \nu(dx)$$

where N is a Poisson process with intensity $\lambda_{\eta, \varepsilon} = \int_{\eta < |x| \leq \varepsilon} \nu(dx)$ and the $(Y_i)_{i \geq 1}$ are i.i.d. with jump measure

$$\mathbb{P}(Y_1 \in B) = \frac{1}{\lambda_{\eta, \varepsilon}} \int_{B \cap \{\eta < |x| \leq \varepsilon\}} \nu(dx) \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

Since it will be used several times in the rest of the paper, we write BCI for the Bienaymé-Chebyshev inequality which states that, if Z is a random variable with finite variance, then with probability larger than $1 - \alpha$, it holds

$$\mathbb{E}[Z] - \sqrt{\mathbb{V}(Z)/\alpha} \leq Z \leq \mathbb{E}[Z] + \sqrt{\mathbb{V}(Z)/\alpha}.$$

Finally, in several occasions we will use that $\sigma(u^+) \leq \sigma(u^*)$.

A.2.1 Proof of Proposition 3

Under H_0 : By means of Equations (21) and (22) we have that

$$\max_i Z_i \leq 4\sqrt{\Delta \Sigma^2 \log(n)}, \quad \text{on } \xi_n \quad \text{and} \quad S_n \geq \frac{\sqrt{\Delta \Sigma^2}}{\pi}, \quad \text{on } \xi'_n.$$

Therefore, for n strictly larger than $e^{4\sqrt{\pi}}$, on the event $\xi_n \cap \xi'_n$ we have that $Z_{(\tilde{n})} < \log(n)^{3/2} S_n$, and thus $\Phi_n^{(\max)} = 0$ as desired.

If a jump larger than u^* occurs : If $u^* = \varepsilon$, then Proposition 3 is satisfied as $\lambda_{\varepsilon, \varepsilon} = 0$, *i.e.* no jumps larger than ε happen. Assume from now on that $u^* < \varepsilon$. By definition of u^* , and since $\sigma(u)$ increases with u , we have that $u^* \geq \sqrt{(\Sigma^2 + \sigma^2(u^*)) \Delta \log(n)^2}$. Furthermore, let us assume that $N_{n\Delta}(u^*) \geq 1$, *i.e.* from now on we always condition by this event. This assumption, combined with (24), implies that on ξ_n there exists i such that $N_{i\Delta}(u^*) - N_{(i-2)\Delta}(u^*) = 1$, and therefore

$$|M_{i\Delta}(u^*, \varepsilon) - M_{(i-1)\Delta}(u^*, \varepsilon) - (M_{(i-1)\Delta}(u^*, \varepsilon) - M_{(i-2)\Delta}(u^*, \varepsilon))| \geq u^*.$$

In addition, by means of Equation (24), we also know that on ξ_n

$$|\overline{X}_{i\Delta}(u^*) - \overline{X}_{(i-1)\Delta}(u^*) - (\overline{X}_{(i-1)\Delta}(u^*) - \overline{X}_{(i-2)\Delta}(u^*))| \leq 2\sqrt{(\Sigma^2 + \sigma^2(u^*)) \Delta \log(n)}.$$

Recalling the definition of u^* and taking $n > e^2$ we can conclude that, on ξ_n , it holds that $Z_i \geq u^*/2$.

Furthermore, by Equation (25) we know that on ξ'_n

$$S_n \leq 2\sqrt{2\Delta(\Sigma^2 + \sigma^2(u^+))} \leq 2\sqrt{2\Delta(\Sigma^2 + \sigma^2(u^*))},$$

which allows to conclude that, for $n > e^2$, on $\xi_n \cap \xi'_n$, it holds

$$Z_i \geq \frac{S_n \log(n)^2}{2\sqrt{2}} > S_n \log(n)^{3/2},$$

that is $\Phi_n^{(\max)} = 1$, as desired.

A.2.2 Proof of Proposition 4

Under H_0 : By means of Equations (22) and (23), for any $\omega' \in \xi'_n$ and $\omega'' \in \xi''_n$, we have

$$S_n(\omega') \geq \frac{\sqrt{\Delta \Sigma^2}}{\sqrt{\pi}} \quad \text{and} \quad \bar{Y}_{n,6}(\omega'') \leq C(\sqrt{\Delta \Sigma^2})^6.$$

Therefore, on $\xi''_n \cap \xi'_n$, we have $\bar{Y}_{n,6} < C\pi^3 S_n^6$, and thus $\Phi_{n,c}^{(6)} = 0$, as desired.

If $\Delta\mu_6(u^*)$ is large and no large jump occurs: On the one hand, by Equation (23) we know that, on $\xi''_n \cap \{N_{n\Delta}(u^*) = 0\}$, it holds

$$\bar{Y}_{n,6} \geq \frac{1}{2} [2\Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3].$$

On the other hand, on ξ'_n , by means of Equation (25), we have that $S_n \leq 2\sqrt{2\Delta(\Sigma^2 + \sigma^2(u^*))}$. Thus, denoting by C_c an absolute constant depending only on c , whenever

$$\Delta\mu_6(u^*) \geq C_c \Delta^3 (\Sigma^2 + \sigma^2(u^*))^3,$$

it holds that $\bar{Y}_{n,6} > cS_n^6$, on $\xi''_n \cap \xi'_n \cap \{N_{n\Delta}(u^*) = 0\}$, for n larger than an absolute constant. We therefore conclude that $\Phi_{n,c}^{(6)} = 1$, as desired.

A.2.3 Proof of Proposition 5

We begin with some preliminary results.

Lemma 11. *For n larger than an universal constant, $\varepsilon > 0$ and any $\log(n)^{-1} < \alpha \leq 1$ there exists an event ξ_n''' of probability larger than $1 - \alpha$ and two universal constants $c, C > 0$ such that the following holds:*

$$|\mathbb{E}[T_n^{(3)} | \xi_n'''] - \Delta\mu_3(\varepsilon)| \leq c \frac{\Delta^{3/2}(\Sigma^2 + \sigma^2(\varepsilon))^{3/2}}{\sqrt{n\alpha}},$$

and $\mathbb{V}(T_n^{(3)} | \xi_n''') \leq \frac{C}{n} [\Delta\mu_6(\varepsilon) + \Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^3].$

Corollary 4. *For any $\varepsilon > 0$ and for any $\log(n)^{-1} < \alpha \leq 1$, there exists an event ξ_n''' of probability larger than $1 - \alpha$ and two universal constants $c, C > 0$ such that the following holds:*

$$|\mathbb{E}[T_n^{(3)} | \xi_n''', N_{n\Delta}(u^*) = 0] - \Delta\mu_3(u^*)| \leq c \frac{\Delta^{3/2}(\Sigma^2 + \sigma^2(u^*))^{3/2}}{\sqrt{n\alpha}},$$

and $\mathbb{V}(T_n^{(3)} | \xi_n''', N_{n\Delta}(u^*) = 0) \leq \frac{C}{n} [\Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3].$

Proof of Proposition 5. For some given α , let ξ_n''' be an event as in Corollary 4. If $3 \leq k \leq 6$, thanks to the hypothesis (27) on $\Delta\mu_6(u^*)$, there exists an universal constant $C > 0$ such that

$$\mathbb{V}(T_n^{(3)} | N_{n\Delta}(u^*) = 0, \xi_n''') \leq \frac{C}{n} [\Delta(\Sigma^2 + \sigma^2(u^*))]^3.$$

Therefore, using BCI, we have

$$\mathbb{P}\left(\left|\frac{T_n^{(3)} - \Delta\mu_3(u^*)}{\sqrt{\frac{C}{n} [\Delta(\Sigma^2 + \sigma^2(u^*))]^3}}\right| > \frac{c/C + 1}{\sqrt{\alpha}} | N_{n\Delta}(u^*) = 0\right) \leq 2\alpha. \quad (37)$$

Under H_0 : $\mu_3(u^*)$ and $\sigma^2(u^*)$ are zero and thus

$$\mathbb{P}\left(\left|\frac{T_n^{(3)}}{\sqrt{\frac{c}{n}\Delta^3\Sigma^6}}\right| > \frac{1}{\sqrt{\alpha}}\right) \leq \alpha.$$

Therefore, recalling the definition of ξ'_n (see Equation (22)), we have that for $c > 0$ a large enough absolute constant, with probability larger than $1 - \alpha - \mathbb{P}(\xi'_n{}^c)$ it holds

$$|T_n^{(3)}| \leq \frac{c}{\sqrt{\alpha n}} S_n^3,$$

which means that the test is accepted with probability larger than $1 - 2\alpha$.

Under $H_{1,\rho}^{(3)}$ and conditional to $N_{\Delta n}(u^*) = 0$: Assume that for some $C_c > 0$ large enough absolute constant depending only on $c > 0$ we have by definition of u^* (since $u^* > u^+$)

$$\Delta|\mu_3(u^*)| \geq \rho \geq \frac{C_c}{\sqrt{n\alpha}} \sqrt{[\Delta(\Sigma^2 + \sigma^2(u^*))]^3}.$$

This implies by Equation (37) and for C_c large enough depending only on c that with probability larger than $1 - \alpha$

$$|T_n^{(3)}| \geq \frac{C_c}{2\sqrt{\alpha}} \left\{ \frac{\sqrt{\Delta^3(\Sigma^2 + \sigma^2(u^*))^3}}{\sqrt{n}} \right\}.$$

Therefore, for C_c large enough depending only on c , we have by definition of ξ'_n in Equation (22) that with probability larger than $1 - \alpha - \mathbb{P}(\xi'_n{}^c)$

$$|T_n^{(3)}| \geq \frac{c}{\sqrt{\alpha n}} S_n^3.$$

The test is thus rejected with probability larger than $1 - \alpha - \mathbb{P}(\xi'_n{}^c)$. □

A.2.4 Proof of Proposition 6

Proposition 6 can be proved with arguments very similar to those used in the proof of Proposition 5.

Lemma 12. *For any $\varepsilon > 0$ it holds $\mathbb{E}[T_n^{(4)}] = \Delta\mu_4(\varepsilon)$. For n larger than an absolute constant and for some universal constant $C > 0$, it holds*

$$\mathbb{V}(T_n^{(4)}) \leq \frac{C}{n} \left[\Delta\mu_8(\varepsilon) + [\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^4 \right].$$

Corollary 5. *For any $\varepsilon \in (0, 1]$, it holds $\mathbb{E}[T_n^{(4)} | N_{n\Delta}(u^*) = 0] = \Delta\mu_4(u^*)$. Moreover, there exists a universal constant $C > 0$ such that*

$$\mathbb{V}(T_n^{(4)} | N_{n\Delta}(u^*) = 0) \leq \frac{C}{n} \left[\Delta\mu_8(u^*) + [\Delta(\Sigma^2 + \sigma^2(u^*))]^4 \right]. \quad (38)$$

Proof of Proposition 6. The proof follows the same scheme as the one in Lemma 5. Here we only remark that Equation (38) implies

$$\mathbb{V}(T_n^{(4)} | N_{n\Delta}(u^*) = 0) \leq \frac{C}{n} \left[\Delta\mu_8(u^*) + [\Delta(\Sigma^2 + \sigma^2(u^*))]^4 \right] \leq \frac{C}{n} \left[(u^*)^4 \Delta\mu_4(u^*) + [\Delta(\Sigma^2 + \sigma^2(u^*))]^4 \right],$$

since $\Delta\mu_8(u^*) \leq (u^*)^4 \Delta\mu_4(u^*)$. By means of BCI we thus deduce that

$$\mathbb{P}\left(\left|\frac{T_n^{(4)} - \Delta\mu_4(u^*)}{\sqrt{\frac{C}{n}[(u^*)^4 \Delta\mu_4(u^*) + [\Delta(\Sigma^2 + \sigma^2(u^*))]^4]}}\right| > \frac{1}{\sqrt{\alpha}} |N_{n\Delta}(u^*) = 0\right) \leq \alpha,$$

which, using that $\alpha \geq \log(n)^{-1}$ and the definition of u^* , implies

$$\mathbb{P}\left(\frac{T_n^{(4)} - \frac{1}{2}\Delta\mu_4(u^*)}{\sqrt{\frac{C'}{n}[\Delta(\Sigma^2 + \sigma^2(u^*))]^4}} < \frac{1}{\sqrt{\alpha}} |N_{n\Delta}(u^*) = 0\right) \leq \alpha,$$

for some universal constant C' and for n larger than a universal constant. \square

A.3 Proofs of the Lemmas involved in the proof of Theorem 5

A.3.1 Proof of Lemma 1

Let Φ be such a test for $H_0 : \mathbb{P}$ against $H_1 : \mathbb{Q}$. The conditions on Φ lead to

$$\|\mathbb{P}^{\otimes n} - \mathbb{Q}^{\otimes n}\|_{TV} \geq |\mathbb{P}_{H_0}(\Phi = 0) - \mathbb{P}_{H_1}(\Phi = 0)| \geq 1 - \alpha_1 - \alpha_0.$$

A.3.2 Proof of Lemma 6

If $\nu_\varepsilon = 0$: Under H_0 we know that all Z_i are i.i.d. realizations of the absolute values of centered Gaussian random variables with variance $2\Delta\Sigma^2$. By Gaussian concentration we have that with probability larger than $1 - 1/n$, $\max_{i \leq \tilde{n}} Z_i \leq 4\sqrt{\Delta\Sigma^2 \log(n)}$, since $\tilde{n} = \lfloor n/2 \rfloor \leq n$.

If $\nu_\varepsilon \neq 0$: By BCI, with probability larger than $1 - \alpha$, it holds

$$|N_{n\Delta}(u^+) - \Delta n \lambda_{u^+, \varepsilon}| \leq \sqrt{\Delta n \lambda_{u^+, \varepsilon} / \alpha},$$

i.e. for $\alpha = 4 \log(n)^{-1}$ we have that with probability larger than $1 - 4 \log(n)^{-1}$

$$\log(n)/2 \leq N_{n\Delta}(u^+) \leq 2 \log(n).$$

Furthermore, we observe that

$$\begin{aligned} \mathbb{P}(N_{i\Delta}(u^+) - N_{(i-2)\Delta}(u^+) \leq 1) &= \exp(-2\Delta\lambda_{u^+, \varepsilon}) + 2\Delta\lambda_{u^+, \varepsilon} \exp(-2\Delta\lambda_{u^+, \varepsilon}) \\ &= \exp(-2 \log(n)/n) + 2 \frac{\log(n)}{n} \exp(-2 \log(n)/n). \end{aligned}$$

It thus follows

$$\begin{aligned} \mathbb{P}\left(\{\forall i \leq n, N_{i\Delta}(u^+) - N_{(i-2)\Delta}(u^+) \leq 1\}\right) &= \left(\exp(-2 \log(n)/n) + 2 \frac{\log(n)}{n} \exp(-2 \log(n)/n)\right)^n \\ &= \exp(-2 \log(n))(1 + 2 \log(n)/n)^n \rightarrow 1, \end{aligned}$$

at a rate which does not depend on $\nu, \varepsilon, b, \Sigma$.

Finally, by BCI, with probability larger than $1 - \alpha$, we have

$$|\bar{X}_{i\Delta}(u^*) - \bar{X}_{(i-1)\Delta}(u^*) - b\Delta| \leq \sqrt{(\Sigma^2 + \sigma^2(u^*))\Delta\alpha^{-1}}.$$

So, conditional on $\{1 \leq N_{n\Delta}(u^+) \leq 2\log(n)\}$, with probability larger than $1 - \log(n)^{-1}$, we have that $\forall i$ s.t. $N_{i\Delta}(u^+) - N_{(i-1)\Delta}(u^+) \neq 0$

$$|\bar{X}_{i\Delta}(u^*) - \bar{X}_{(i-1)\Delta}(u^*) - b\Delta| \leq \sqrt{(\Sigma^2 + \sigma^2(u^*))\Delta \log(n)}.$$

We conclude observing that, conditional on $\{1 \leq N_{n\Delta}(u^+) \leq 2\log(n)\}$, with probability larger than $1 - \log(n)^{-1}$, we have $\forall i$ s.t. $N_{i\Delta}(u^+) - N_{(i-2)\Delta}(u^+) \neq 0$

$$|\bar{X}_{i\Delta}(u^*) - \bar{X}_{(i-1)\Delta}(u^*) - (\bar{X}_{(i-1)\Delta}(u^*) - \bar{X}_{(i-2)\Delta}(u^*))| \leq 2\sqrt{(\Sigma^2 + \sigma^2(u^*))\Delta \log(n)}.$$

A.3.3 Proof of Lemma 7

Preliminary. Denote by $\tilde{Z}_i = |(X_{i\Delta}(u^+) - X_{(i-1)\Delta}(u^+)) - (X_{(i-1)\Delta}(u^+) - X_{(i-2)\Delta}(u^+))|$ and assume that ξ_n holds. We begin by observing that, for any $\omega \in \xi_n$, we have:

$$\frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}-4\log(n)} \tilde{Z}_{(i)}(\omega) \leq S_n(\omega) \leq \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} \tilde{Z}_i(\omega). \quad (39)$$

To show (39), let $I := \{i : \tilde{Z}_i = Z_i\}$, that is the set where no jumps of size larger than u^+ occur between $(i-2)\Delta$ and $i\Delta$. By means of the positivity of the variables \tilde{Z}_i and Z_i , we get

$$\frac{1}{\tilde{n}} \sum_{1 \leq i \leq \tilde{n}-2\log(n), i \in I} Z_{(i)} \leq S_n = \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}-2\log(n)} Z_{(i)}.$$

Moreover, since $\#I^c \leq 2\log(n)$ on ξ_n , we have

$$\frac{1}{\tilde{n}} \sum_{1 \leq i \leq \tilde{n}-2\log(n), i \in I} Z_{(i)} \leq S_n \leq \frac{1}{\tilde{n}} \sum_{i \in I} Z_i.$$

Using again that $\mathbb{P}\{\#I^c \leq 2\log(n) \cap \xi_n\} = 1$, the definition of I and the fact that \tilde{Z}_i, Z_i are positive, we obtain (39).

Control when ξ'_n is given by (25). Note that $\mathbb{E}\tilde{Z}_i^2 = 2\Delta(\Sigma^2 + \sigma^2(u^+))$, so by the Cauchy-Schwartz inequality $\mathbb{E}\tilde{Z}_i \leq \sqrt{2\Delta(\Sigma^2 + \sigma^2(u^+))}$. It follows by BCI, that with probability larger than $1 - 1/n$

$$\frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} \tilde{Z}_i \leq 2\sqrt{2\Delta(\Sigma^2 + \sigma^2(u^+))}.$$

Then, on ξ_n , by Equation (39), with probability larger than $1 - 1/n$ it holds $S_n \leq 2\sqrt{2\Delta(\Sigma^2 + \sigma^2(u^+))}$.

Control when ξ'_n is given by (22). In this case $\nu_\varepsilon = 0$, then $Z_i = \tilde{Z}_i$ which are i.i.d. and distributed like the absolute value of a centered Gaussian random variable with variance $2\Delta\Sigma^2$. By Gaussian concentration it then follows that with probability larger than $1 - \alpha$

$$\max_i |Z_i| \leq 2\sqrt{2\Delta\Sigma^2 \log(2/\alpha)}.$$

Using that $\mathbb{E}_{V \sim \mathcal{N}(0,1)} |V| = \sqrt{\frac{2}{\pi}}$ and BCI, we conclude that with probability larger than $1 - \alpha$

$$\frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}-4\log(n)} \tilde{Z}_{(i)} \geq \sqrt{\frac{2}{\pi}} \sqrt{2\Delta\Sigma^2} - \sqrt{\frac{1}{n\alpha}} 2\Delta\Sigma^2 - \frac{4\log(n)}{n} \sqrt{2\Delta\Sigma^2 \log(2/\alpha)}.$$

By Equation (39), with probability larger than $1 - 2^9 \pi n^{-1}$, it thus holds

$$S_n \geq \frac{\sqrt{\Delta \Sigma^2}}{\sqrt{\pi}}.$$

A.3.4 Preliminaries for the proofs of Lemmas 8, 11 and 12

If $Y \sim \mathcal{N}(m, \Sigma^2)$ its moments can be computed through the recursive formula:

$$\mathbb{E}[Y^k] = (k-1)\mathbb{V}(Y)\mathbb{E}[Y^{k-2}] + \mathbb{E}[Y]\mathbb{E}[Y^{k-1}], \quad k \in \mathbb{N}.$$

Lemma 13. *Let $Y \sim \mathcal{N}(m, \Sigma^2)$, then it holds*

$$\begin{aligned} \mathbb{E}[Y^3] &= 3\Sigma^2 m + m^3, & \mathbb{E}[Y^4] &= 3\Sigma^4 + 6m^2\Sigma^2 + m^4, & \mathbb{E}[Y^5] &= 15m\Sigma^4 + 10m^3\Sigma^2 + m^5, \\ \mathbb{E}[Y^6] &= 15\Sigma^6 + 45m^2\Sigma^4 + 15m^4\Sigma^2 + m^6, & \mathbb{E}[Y^7] &= 105m^4\Sigma^6 + 105m^3\Sigma^4 + 21m^5\Sigma^2 + m^7, \\ \mathbb{E}[Y^8] &= 105\Sigma^8 + 420m^2\Sigma^6 + 210m^4\Sigma^4 + 28m^6\Sigma^2 + m^8. \end{aligned}$$

Similarly, using the series expansion of the characteristic function, together with the Lévy Kintchine formula and $\varepsilon > 0$, we get

$$\mathbb{E}[M_\Delta(\varepsilon)^k] = \frac{d^k}{du^k} \exp\left(\Delta \int_{-\varepsilon}^{\varepsilon} (e^{iux} - 1 - iux)\nu_\varepsilon(dx)\right) \Big|_{u=0}. \quad (40)$$

Lemma 14. *For $\varepsilon > 0$, set $\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} y^2 \nu(dy)$ and $\mu_k(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} y^k \nu(dy)$, $k \geq 3$. We have*

$$\begin{aligned} \mathbb{E}[M_\Delta(\varepsilon)] &= 0, & \mathbb{E}[(M_\Delta(\varepsilon))^2] &= \Delta\sigma^2(\varepsilon), & \mathbb{E}[(M_\Delta(\varepsilon))^3] &= \Delta\mu_3(\varepsilon) \\ \mathbb{E}[(M_\Delta(\varepsilon))^4] &= \Delta\mu_4(\varepsilon) + 3\Delta^2\sigma^4(\varepsilon), & \mathbb{E}[(M_\Delta(\varepsilon))^5] &= \Delta\mu_5(\varepsilon) + 10\Delta^2\sigma^2(\varepsilon)\mu_3(\varepsilon), \\ \mathbb{E}[(M_\Delta(\varepsilon))^6] &= \Delta\mu_6(\varepsilon) + \Delta^2(10\mu_3(\varepsilon)^2 + 15\sigma^2(\varepsilon)\mu_4(\varepsilon)) + 15\Delta^3\sigma^6(\varepsilon), \\ \mathbb{E}[(M_\Delta(\varepsilon))^7] &= \Delta\mu_7(\varepsilon) + \Delta^2(21\sigma^2(\varepsilon)\mu_5(\varepsilon) + 35\mu_3(\varepsilon)\mu_4(\varepsilon)) + 105\Delta^3\sigma^4(\varepsilon)\mu_3(\varepsilon), \\ \mathbb{E}[(M_\Delta(\varepsilon))^8] &= \Delta\mu_8(\varepsilon) + \Delta^2(35\mu_4(\varepsilon)^2 + 56\mu_3(\varepsilon)\mu_5(\varepsilon) + 28\sigma^2(\varepsilon)\mu_6(\varepsilon)) \\ &\quad + \Delta^3(280\sigma^2(\varepsilon)\mu_3(\varepsilon)^2 + 210\sigma^4(\varepsilon)\mu_4(\varepsilon)) + 105\Delta^4\sigma^8(\varepsilon). \end{aligned}$$

More generally if ν_ε is a Lévy measure such that $\int_{-\varepsilon}^{\varepsilon} \nu(dx) > \Delta^{-1}$, then for any $k \geq 2$ even integer it holds that

$$\mathbb{E}[M_\Delta(\varepsilon)^k] \leq \tilde{C}_k \left(\Delta\mu_k(\varepsilon) + (\Delta\sigma^2(\varepsilon))^{k/2} \right),$$

where $\tilde{C}_k > 0$ is a constant that depends only on k .

Proof of Lemma 14. The explicit first 8 moments are computed using Equation (40). We prove now the last part of the Lemma. Denote $M(\eta, \varepsilon)$ the Lévy process with jump measure $\nu_\varepsilon \mathbf{1}_{[-\eta, \eta]^c}$, $N(\eta, \varepsilon)$ the corresponding Poisson process and let $\mathcal{L}(\eta, \varepsilon)$ be the law of the jumps given by $\frac{\nu_\varepsilon \mathbf{1}_{[-\eta, \eta]^c}}{\lambda_{\eta, \varepsilon}}$.

By Rosenthal's inequality there exists a constant \tilde{C}_k , depending on k only, such that for any $k \geq 2$, k even

$$\mathbb{E}[|M_\Delta(\eta, \varepsilon)|^k | N_\Delta(\eta, \varepsilon) = N] \leq \tilde{C}_k \max \left[N \mathbb{E}_{X \sim \mathcal{L}(\eta, \varepsilon)}[X^k], [N \mathbb{E}_{X \sim \mathcal{L}(\eta, \varepsilon)}[X^2]]^{k/2} \right].$$

Averaging over $N_\Delta(\eta, \varepsilon) \sim \mathcal{P}(\Delta\lambda_{\eta, \varepsilon})$, we have that

$$\begin{aligned} \mathbb{E}[|M_\Delta(\eta, \varepsilon)|^k] &\leq \tilde{C}_k \max \left[\Delta\lambda_{\eta, \varepsilon} \mathbb{E}_{X \sim \mathcal{L}(\eta, \varepsilon)}[X^k], \mathbb{E}[N_\Delta(\eta, \varepsilon)^{k/2}] [\mathbb{E}_{X \sim \mathcal{L}(\eta, \varepsilon)}[X^2]]^{k/2} \right] \\ &\leq \tilde{C}'_k \max \left[\Delta\lambda_{\eta, \varepsilon} \mathbb{E}_{X \sim \mathcal{L}(\eta, \varepsilon)}[X^k], [1 + \Delta\lambda_{\eta, \varepsilon}]^{k/2} [\mathbb{E}_{X \sim \mathcal{L}(\eta, \varepsilon)}[X^2]]^{k/2} \right], \end{aligned}$$

where we used that for a Poisson random variable it holds for any $k' \geq 1$ that $\mathbb{E}_{N \sim \mathcal{P}(\lambda)} X^{k'} \leq c_k (1 + \lambda)^{k'}$ where c_k is a constant that depends on k only. Setting $\mu_k(\eta, \varepsilon) = \int_{\eta \leq |x| \leq \varepsilon} y^k \nu(dy)$ we have for η small enough such that $\Delta \lambda_{\eta, \varepsilon} \geq 1$ (which always exists by assumption)

$$\begin{aligned} \mathbb{E}[|M_\Delta(\eta, \varepsilon)|^k] &\leq \tilde{C}_k'' \max \left[\Delta \mu_k(\eta, \varepsilon), [\Delta \lambda_{\eta, \varepsilon}]^{k/2} [\mathbb{E}_{X \sim \mathcal{P}(\eta, \varepsilon)} X^2]^{k/2} \right] \\ &\leq \tilde{C}_k'' \max \left[\Delta \mu_k(\eta, \varepsilon), [\Delta \mu_2(\eta, \varepsilon)]^{k/2} \right]. \end{aligned}$$

where \tilde{C}_k'' is some constant that only depends on k . Making η converge to 0 gives the result. \square

Corollary 6. For $\varepsilon > 0$, set $\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} y^2 \nu(dy)$ and $\mu_k(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} y^k \nu(dy)$, $k \geq 2$. It holds that

$$\begin{aligned} \mathbb{E}(X_\Delta - b\Delta) &= 0, & \mathbb{E}(X_\Delta - b\Delta)^3 &= \Delta \mu_3(\varepsilon), & \mathbb{E}(X_\Delta - \bar{X}_{\Delta, n}) &= 0, \\ \mathbb{E}(X_\Delta - \bar{X}_{\Delta, n})^3 &= \left(1 - \frac{1}{(n - \lfloor n/2 \rfloor)^2}\right) \Delta \mu_3(\varepsilon), & \mathbb{E}Z_1^2 &= 2\Delta(\Sigma^2 + \sigma^2(\varepsilon)) \\ \mathbb{E}Z_1^4 &= 4\Delta \mu_4(\varepsilon) + 12(\Delta(\Sigma^2 + \sigma^2(\varepsilon)))^2, & \mathbb{E}Z_1^6 &\geq 2\Delta \mu_6(\varepsilon) + (\Delta(\Sigma^2 + \sigma^2(\varepsilon)))^3. \end{aligned}$$

Proof. The result is obtained combining Lemmas 13 and 14 with the independence between $(X_{2\Delta} - X_\Delta)$ and X_Δ . \square

Corollary 7. For $\varepsilon > 0$, set $\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} y^2 \nu(dy)$ and $\mu_k(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} y^k \nu(dy)$, $k \geq 2$. It holds for any $k \geq 2$, k even integer that

$$\begin{aligned} \mathbb{E}(X_\Delta(\varepsilon) - b\Delta)^k &\leq \bar{C}_k(\Delta \mu_k(\varepsilon) + (\Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{k/2}), \\ \mathbb{E}(Z_1^k) &\leq \bar{C}_k(\Delta \mu_k(\varepsilon) + (\Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{k/2}), \end{aligned}$$

where \bar{C}_k is a constant that depends only on k .

Proof of Corollary 7.

$$\begin{aligned} \mathbb{E}(X_\Delta(\varepsilon) - b\Delta)^k &= \mathbb{E}(M_\Delta(\varepsilon) + \Sigma W_\Delta)^k \leq 2^{k+1}(\mathbb{E}M_\Delta(\varepsilon))^k + \Sigma^k \mathbb{E}W_\Delta^k \\ &\leq C_k(\Delta^{k/2} \Sigma^k + \Delta \mu_k(\varepsilon) + (\Delta \sigma^2(\varepsilon))^{k/2}), \end{aligned}$$

where C_k is a constant that depends on k only. \square

A.3.5 Proof of Lemma 8

If $\nu_\varepsilon = 0$. By Corollary 7, there exist universal constants \bar{C}_6 and \bar{C}_{12} such that $\mathbb{E}[Z_i^6] \leq \bar{C}_6 \Delta^3 \Sigma^6$ and $\mathbb{E}[Z_i^{12}] \leq \bar{C}_{12} \Delta^6 \Sigma^{12}$. Using that Z_i are i.i.d. we get

$$\mathbb{E}[\bar{Y}_{n,6}] \leq \bar{C}_6 \Delta^3 \Sigma^6, \quad \mathbb{V}(\bar{Y}_{n,6}) \leq \frac{\bar{C}_{12}}{n - \lfloor n/2 \rfloor} \Delta^6 \Sigma^{12}.$$

Therefore, by means of BCI, with probability larger than $1 - \log(n)^{-1}$ conditional to $N_{n\Delta}(u^*) = 0$ we have

$$\bar{Y}_{n,6} \leq \bar{C}_6 \Delta^3 \Sigma^6 + \sqrt{\frac{\bar{C}_{12} \log(n)}{(n - \lfloor n/2 \rfloor)}} (\Delta \Sigma^2)^6,$$

which allows to deduce that for n larger than a universal constant, with probability larger than $1 - \log(n)^{-1}$, we have $\bar{Y}_{n,6} \leq 2\bar{C}_6 \Delta^3 \Sigma^6$.

If $\nu_\varepsilon \neq 0$. By Corollaries 6 and 7, conditional to $N_{n\Delta}(u^*) = 0$, for any i it holds

$$\begin{aligned}\mathbb{E}[Z_i^6 | N_{n\Delta}(u^*) = 0] &\geq \Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3 \\ \mathbb{E}[Z_i^{12} | N_{n\Delta}(u^*) = 0] &\leq \bar{C}_{12}(\Delta\mu_{12}(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^6),\end{aligned}$$

where \bar{C}_{12} is the universal constant from Corollary 7. Using that the Z_i are i.i.d. we obtain

$$\begin{aligned}\mathbb{E}[\bar{Y}_{n,6} | N_{n\Delta}(u^*) = 0] &\geq \Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^6 \\ \mathbb{V}(\bar{Y}_{n,6} | N_{n\Delta}(u^*) = 0) &\leq \frac{\bar{C}_{12}}{n - \lfloor n/2 \rfloor} (\Delta\mu_{12}(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^6).\end{aligned}$$

It follows by BCI that with probability larger than $1 - \log(n)^{-1}$, conditional to $N_{n\Delta}(u^*) = 0$, we have

$$\bar{Y}_{n,6} \geq \Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3 - \sqrt{\frac{\bar{C}_{12} \log(n)}{(n - \lfloor n/2 \rfloor)} (\Delta\mu_{12}(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^6)}.$$

Since $\mu_{12}(u^*) \leq (u^*)^6 \mu_6(u^*)$, with probability larger than $1 - \log(n)^{-1}$ and conditional to $N_{n\Delta}(u^*) = 0$, we have

$$\bar{Y}_{n,6} \geq \Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3 - \sqrt{\frac{\bar{C}_{12} \log(n)}{(n - \lfloor n/2 \rfloor)} ((u^*)^6 \Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^6)}.$$

Finally, by means of the definition of u^* and for n larger than a universal constant, with probability larger than $1 - \log(n)^{-1}$ conditional to $N_{n\Delta}(u^*) = 0$, it holds

$$\bar{Y}_{n,6} \geq \frac{\Delta\mu_6(u^*) + (\Delta(\Sigma^2 + \sigma^2(u^*)))^3}{2}.$$

A.3.6 Proof of Lemma 11

Note that $\mathbb{E}[\bar{X}_{\Delta,n}] = b\Delta$ and that $\mathbb{V}(\bar{X}_{\Delta,n}) = (n - \lfloor n/2 \rfloor)^{-1} \Delta(\Sigma^2 + \sigma^2(\varepsilon))$. Write

$$\xi_n''' = \{|\bar{X}_{\Delta,n} - b\Delta| \leq r_n\} \quad \text{where} \quad r_n := r_n(\alpha) = \sqrt{\frac{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}{\alpha(n - \lfloor n/2 \rfloor)^{-1}}}.$$

By BCI we have for any $0 < \alpha \leq 1$

$$\mathbb{P}(\xi_n''') \geq 1 - \alpha. \tag{41}$$

Conditional on ξ_n''' we have by Corollary 6 and the definition of r_n

$$|\mathbb{E}[\bar{Y}_{n,3} | \xi_n'''] - \Delta\mu_3(\varepsilon)| \leq r_n^3 + 3\Delta(\Sigma^2 + \sigma^2(\varepsilon))r_n \leq 100 \frac{(\Delta(\Sigma^2 + \sigma^2(\varepsilon)))^{3/2}}{\sqrt{n\alpha}},$$

for $\alpha \geq \log(n)^{-1}$.

Now we compute $\mathbb{V}(T_n^{(3)})$. Since $\bar{X}_{\Delta,n}$ and $\bar{X}_{\Delta,n}^j$ are independent, as they are computed on two independent samples, the elements of the sum are independent of each other conditional on the second half of the sample. Then, conditional on the second half of the sample

$$\begin{aligned}\mathbb{V}(T_n^{(3)} | \xi_n''') &\leq \left(1 - \frac{1}{(n - \lfloor n/2 \rfloor)^2}\right)^2 \frac{1}{\lfloor n/2 \rfloor^2} \sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{E}[(X_{i\Delta} - X_{(i-1)\Delta} - \bar{X}_{\Delta,n})^3 | \xi_n'''] \\ &\leq \frac{16}{n} \mathbb{E}[X_{i\Delta} - X_{(i-1)\Delta} - \Delta b]^6 + [\bar{X}_{\Delta,n} - \Delta b]^6 | \xi_n'''] \\ &\leq \frac{16}{n} \left[\bar{C}_6 \left(\Delta\mu_6(\varepsilon) + [\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^3 \right) + r_n^6 \right],\end{aligned}$$

where \overline{C}_6 is the constant from Corollary 7. Hence, by Equation (41), there exists a universal constant C such that

$$\mathbb{V}(T_n^{(3)}|\xi_n''') \leq \frac{C}{n} \left[\Delta\mu_6(\varepsilon) + [\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^3 \right].$$

A.3.7 Proof of Lemma 12

The main ingredient of the proof consists in establishing expansions of $\mathbb{V}(T_n^{(4)})$. Computations are cumbersome but not difficult, we only give the main tools here but we do not provide all computations. By Corollary 6 and since $\overline{Y}_{n,2}$ and $\overline{Y}'_{n,2}$ are independent, we have

$$\begin{aligned} \mathbb{E}[\overline{Y}_{n,2}] &= 2\Delta(\Sigma^2 + \sigma^2(\varepsilon)), & \mathbb{E}[\overline{Y}_{n,2}\overline{Y}'_{n,2}] &= 4\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^2, \\ \mathbb{E}[\overline{Y}_{n,4}] &= 3 \times 4\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^2 + 4\Delta\mu_4(\varepsilon). \end{aligned}$$

In particular, $\mathbb{E}[T_n^{(4)}] = \Delta\mu_4(\varepsilon)$. Next, we have $\mathbb{V}[T_n^{(4)}] \leq 9\mathbb{V}[\overline{Y}_{n,2}\overline{Y}'_{n,2}] + \mathbb{V}[\overline{Y}_{n,4}]$. We analyse these two terms separately.

Since the Z_i in the sum composing $\overline{Y}_{n,4}$ are i.i.d. we have

$$\begin{aligned} \mathbb{V}(\overline{Y}_{n,4}) &\leq \frac{1}{[n/2]^2} \sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{E}(Z_i^8) \leq \frac{1}{[n/2]} 2^9 \overline{C}_8 \left(\Delta\mu_8(\varepsilon) + [\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^4 \right) \\ &\leq \frac{C}{n} \left(\Delta\mu_8(\varepsilon) + [\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^4 \right), \end{aligned}$$

where \overline{C}_8 is the constant from Corollary 7, and where C is a universal constant.

Similarly, as the Z_i in the sums composing $\overline{Y}_{n,2}$ and $\overline{Y}'_{n,2}$ are i.i.d. we have

$$\begin{aligned} \mathbb{V}(\overline{Y}_{n,2}\overline{Y}'_{n,2}) &\leq 4\mathbb{E}[\overline{Y}_{n,2}^2]\mathbb{V}(\overline{Y}'_{n,2}) \leq 4\frac{\overline{C}_4}{[n/4]} \left(\Delta\mu_4(\varepsilon) + [\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^2 \right) \\ &\quad \times \left[\frac{\overline{C}_4}{[n/4]} \left(\Delta\mu_4(\varepsilon) + [\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^2 \right) + 4\Delta^2(\Sigma^2 + \sigma^2(\varepsilon))^2 \right] \\ &\leq \frac{C'}{n} \left[[\Delta(\Sigma^2 + \sigma^2(\varepsilon))]^4 + \frac{\Delta^2\mu_4(\varepsilon)}{n} \right], \end{aligned}$$

where \overline{C}_4 is the constant from Corollary 7, and where C' is a universal constant. Combining this with the last displayed equation, completes the proof.

A.4 Proof of Proposition 2

Let $\beta \in (0, 2)$ and consider a Lévy measure ν that has a density with respect to the Lebesgue measure such that there exist two positive constants $c_+ > c_- > 0$ with

$$\frac{c_-}{|x|^{\beta+1}} dx \leq d\nu(x) \leq \frac{c_+}{|x|^{\beta+1}} dx.$$

The characteristic function of the increment of this Lévy process with Brownian part of variance Σ^2 is $\tilde{\Psi} = \exp(\tilde{\psi})$, with

$$\tilde{\psi}(t) = -\Delta t^2 \Sigma^2 / 2 + \Delta \int_{-\varepsilon}^{\varepsilon} (e^{itu} - 1 - itu) d\nu(u).$$

The rescaled increment has characteristic function $\Psi = \exp(\psi)$, with

$$\psi(t) = -t^2 \frac{\Sigma^2}{2(\Sigma^2 + \sigma^2(\varepsilon))} + \Delta \int_{-\varepsilon}^{\varepsilon} (e^{itu/\sqrt{\Delta s^2}} - 1 - itu/\sqrt{\Delta s^2}) d\nu(u),$$

where $\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} u^2 d\nu(u) \in [\frac{2c_-}{2-\beta}\varepsilon^{2-\beta}, \frac{2c_+}{2-\beta}\varepsilon^{2-\beta}]$. From now on, write $s^2 = \Sigma^2 + \sigma^2(\varepsilon)$. We have

$$\psi(t) = -t^2 \frac{\Sigma^2}{2s^2} + \Delta \int_{-\varepsilon}^{\varepsilon} (e^{itu/\sqrt{\Delta s^2}} - 1 - itu/\sqrt{\Delta s^2}) d\nu(u).$$

Note that the cumulants of \tilde{X}_Δ are such that $\mu_1(\varepsilon) = 0$, $\mu_2(\varepsilon) = 1$ and for all $k \geq 3$

$$|\mu_k(\varepsilon)| \in \left[\frac{2c_-}{k-\beta} \frac{\varepsilon^{k-\beta}}{(\sqrt{\Delta s})^k}, \frac{2c_+}{k-\beta} \frac{\varepsilon^{k-\beta}}{(\sqrt{\Delta s})^k} \right].$$

In the sequel we show that Assumption $(\mathcal{H}_\Psi(\varepsilon))$ holds, for that we use the assumption

$$s^2 \Delta / \varepsilon^2 \geq \bar{C} \log(n), \quad (42)$$

where $\bar{C} = \tilde{c}^{-1} > 0$ is a large enough constant, i.e. $a := \frac{\varepsilon}{s\sqrt{\Delta}} \leq \frac{\tilde{c}}{\sqrt{\log(n)}}$.

A.4.1 Preliminary technical Lemmas

Lemma 15. *There exists $c_\beta > 0$ a constant that depends only on β, c_+, c_- such that the following holds*

$$\operatorname{Re}(\psi(t)) \leq -c_\beta t^2 \mathbf{1}_{\{t\varepsilon \leq \sqrt{\Delta s^2}\}} - c_\beta \frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} \mathbf{1}_{\{t\varepsilon > \sqrt{\Delta s^2}\}} - \frac{\Sigma^2}{2s^2} t^2 \mathbf{1}_{\{t\varepsilon > \sqrt{\Delta s^2}\}}, \quad t > 0.$$

where $\operatorname{Re}(y)$ is the real part of y .

Proof of Lemma 15. It holds

$$\psi(t) = -t^2 \frac{\Sigma^2}{2s^2} + \Delta \int_{-\varepsilon}^{\varepsilon} (e^{itu/\sqrt{\Delta s^2}} - 1 - itu/\sqrt{\Delta s^2}) d\nu(u) = -t^2 \frac{\Sigma^2}{2s^2} + I.$$

We now focus on the study of I . Doing the change of variable $v = tu/\sqrt{\Delta s^2}$ we get

$$I = \Delta \sqrt{\Delta s^2} \int_{-t\varepsilon/\sqrt{\Delta s^2}}^{t\varepsilon/\sqrt{\Delta s^2}} (e^{iv} - 1 - iv) \frac{d\nu(v\sqrt{\Delta s^2}/t)}{t}.$$

If $t\varepsilon \leq \sqrt{\Delta s^2}$, for $0 \leq v \leq 1$, there exists an absolute constant $c > 0$ with $\operatorname{Re}(e^{iv} - 1 - iv) = \cos(v) - 1 \leq -cv^2$, then,

$$\begin{aligned} \operatorname{Re}(I) &\leq 2\Delta \sqrt{\Delta s^2} c_- t^\beta \int_0^{t\varepsilon/\sqrt{\Delta s^2}} \frac{-cv^2}{(v\sqrt{\Delta s^2})^{\beta+1}} dv \\ &= -2\Delta c c_- \frac{t^\beta}{\sqrt{\Delta s^2}^\beta} \int_0^{t\varepsilon/\sqrt{\Delta s^2}} v^{1-\beta} dv = -\frac{2c c_-}{2-\beta} \frac{t^2 \varepsilon^{2-\beta}}{s^2}. \end{aligned}$$

Since $\sigma^2(\varepsilon) \in [2c_- \frac{\varepsilon^{2-\beta}}{2-\beta}, 2c_+ \frac{\varepsilon^{2-\beta}}{2-\beta}]$, whenever $t\varepsilon \leq \sqrt{\Delta s^2}$ we have

$$\operatorname{Re}(\psi(t)) \leq -\frac{c c_-}{c_+} \frac{\sigma^2(\varepsilon)}{s^2} t^2 - \frac{\Sigma^2}{s^2} t^2 \leq -\left(\frac{c c_-}{c_+} \wedge 1\right) t^2,$$

using $s^2 = \sigma^2(\varepsilon) + \Sigma^2$.

If $t\varepsilon > \sqrt{\Delta s^2}$, then $\operatorname{Re}(e^{iv} - 1 - iv) = \cos(v) - 1 \leq 0$ for any $v \in \mathbb{R}$ then,

$$\begin{aligned} \operatorname{Re}(I) &\leq 2\Delta c_- \frac{t^\beta}{\sqrt{\Delta s^2}^\beta} \int_0^{t\varepsilon/\sqrt{\Delta s^2}} \frac{\cos(v) - 1}{v^{\beta+1}} dv \leq 2\Delta c_- \frac{t^\beta}{\sqrt{\Delta s^2}^\beta} \int_0^1 \frac{\cos(v) - 1}{|v|^{\beta+1}} dv \\ &\leq -2\Delta c c_- \frac{t^\beta}{\sqrt{\Delta s^2}^\beta} \int_0^1 v^{1-\beta} dv = -\frac{2\Delta c c_-}{(2-\beta)\sqrt{\Delta s^2}^\beta} t^\beta, \end{aligned}$$

where we used the previous bound on $\cos(v)$. It follows that whenever $t\varepsilon \geq \sqrt{\Delta s^2}$

$$\operatorname{Re}(\psi(t)) \leq -t^2 \frac{\Sigma^2}{2s^2} - 2\Delta \frac{c c_-}{2-\beta} \frac{t^\beta}{\sqrt{\Delta s^2}^\beta}.$$

Putting together the cases $\{t\varepsilon > \sqrt{\Delta s^2}\}$ and $\{t\varepsilon \leq \sqrt{\Delta s^2}\}$ completes the proof. \square

Lemma 16. *There exists $c_\beta > 0$ a constant that depends only on β, c_+, c_- such that the following holds for $t > 0$*

$$|\psi^{(1)}(t)| \leq c_\beta t \mathbf{1}_{\{t\varepsilon \leq \sqrt{\Delta s^2}\}} + c_\beta \Delta \frac{t^{\beta-1}}{\sqrt{\Delta s^2}^\beta} \mathbf{1}_{\{t\varepsilon > \sqrt{\Delta s^2}\}} + \frac{\Sigma^2}{s^2} t \mathbf{1}_{\{t\varepsilon > \sqrt{\Delta s^2}\}}.$$

Proof of Lemma 16. First, it holds

$$\Psi^{(1)}(t) = -\frac{\Sigma^2}{s^2} t + \Delta \int_{-\varepsilon}^{\varepsilon} \frac{i u}{\sqrt{\Delta s^2}} (e^{i u t / \sqrt{\Delta s^2}} - 1) d\nu(u)$$

Proof is similar as the one of Lemma 15 replacing $e^{itv} - 1 - itv$ with $ive^{itv} - iv$ that is of order tv^2 close to 0, leading to

$$|\psi^{(1)}(t)| \leq t \frac{\Sigma^2}{s^2} + \frac{c_\beta t \sigma^2(\varepsilon)}{s^2} \mathbf{1}_{\{t\varepsilon \leq \sqrt{\Delta s^2}\}} + c_\beta \Delta \frac{t^{\beta-1}}{\sqrt{\Delta s^2}^\beta} \mathbf{1}_{\{t\varepsilon > 1\}}.$$

\square

Lemma 17. *For any integer $d \geq 2$ we have*

$$|\psi^{(d)}(t)| \leq c_\beta \left(\frac{\varepsilon}{\sqrt{\Delta s^2}} \right)^{d-2} + \frac{\Sigma^2}{s^2} \mathbf{1}_{\{d=2\}} \leq c_\beta \left(\frac{\varepsilon}{\sqrt{\Delta s^2}} \right)^{d-2} := c_\beta a^{d-2},$$

where $c_\beta > 0$ is a constant that depends only on β, c_+, c_- .

Proof of Lemma 17. The proof follows since for any $m \geq 2$

$$\psi^{(m)}(t) = \frac{\Delta}{\sqrt{\Delta s^2}^m} \int_{-\varepsilon}^{\varepsilon} e^{i t u / \sqrt{\Delta s^2}} (u i)^m d\nu(u).$$

We use the change of variable setting $v = tu/\sqrt{\Delta s^2}$, we upper bound $|e^{itv}|$ by 1 and use that $\beta < 2$. \square

We remind that $a := \frac{\varepsilon}{\sqrt{\Delta s^2}}$. Then, $K a^2 = c_{int}^2 \log n \frac{\varepsilon}{\sqrt{\Delta s^2}} \leq c_{int}^2 \tilde{c}^2 = \frac{c_{int}^2}{\tilde{C}^2} \leq 1$ by Equation (42) and for $\tilde{C} \geq c_{int}$.

Lemma 18. Assume that $\bar{C} \geq c_{int}$ (which implies $Ka^2 \leq 1$), there exists $C_\beta > 0$ that depends only on β, c_+, c_- such that the following holds. For all $m \leq K$

$$\left| \frac{(\exp(\psi))^{(m)}}{\exp(\psi)} \right| \leq C_\beta^m f^m,$$

where

$$f(t) = (c_\beta \vee 1) \sqrt{\Delta s^2} \varepsilon^{-1} \mathbf{1}_{\{t\varepsilon/\sqrt{\Delta s^2} \leq 1\}} + (c_\beta \vee 1) \Delta \frac{t^{\beta-1}}{\sqrt{\Delta s^2}^\beta} \mathbf{1}_{\{t\varepsilon > \sqrt{\Delta s^2}\}} + \frac{\Sigma^2}{s^2} t \mathbf{1}_{\{t\varepsilon > \sqrt{\Delta s^2}\}},$$

where c_β is defined in Lemma 16.

Proof of Lemma 18. The proof is similar to the proof of Lemma 10 considering instead the induction hypothesis $H(m) : \forall d \in \mathbb{N}, |R_m^{(d)}| \leq (4(c_\beta \vee 1))^m f^m (1 + ma)^d$. Assumption $H(1)$ holds since $R_1 = \psi^{(1)}$ and Lemma 17 gives $|\psi^{(d)}| \leq c_\beta a^{d-2}$ for d larger than 2, to prove the induction we use Lemma 17, $0 \leq f^{-1} \leq a$ and $ma^2 \leq 1$. \square

Set $t_{\min} = c \log(n)$.

Lemma 19. Assume that $1 \leq c_{int}^2 \leq c$ (which implies $K < t_{\min}$). For any $m \leq K$, and any $t \in [t_{\min} \wedge a^{-1}, a^{-1}]$ there exists a constant $C_\beta > 0$, depending only on β, c_+, c_- , such that

$$\left| \frac{(\exp(\psi))^{(m)}}{\exp(\psi)} \right| \leq C_\beta^m \tilde{f}^m,$$

where $\tilde{f}(t) = (c_\beta \vee 1)t$, where c_β is defined in Lemma 16.

Proof of Lemma 19. The proof is similar to the proof of Lemma 10 considering instead the induction hypothesis, for $C_\beta \geq 4$ and $t \in [t_{\min} \wedge a^{-1}, a^{-1}]$, $H(m) : \forall d \in \mathbb{N}, |R_m^{(d)}| \leq C_\beta^m \tilde{f}^m (1 + m)^d$. The results holds since for all $m \leq K$, we have by assumption that $(1 + m) \leq 1 + K \leq 2t_{\min}$. \square

A.4.2 Proof Proposition 2

First part of the integral $ta > 1$. First, on the interval $t \in [\frac{1}{a}, \infty)$, by Lemma 18 there exists a constant \bar{c}_β that depends only on β, c_+, c_- such that

$$f(t) \leq \bar{c}_\beta \left[\Delta \frac{t^{\beta-1}}{\sqrt{\Delta s^2}^\beta} + \frac{t \Sigma^2}{s^2} \right].$$

There exists by Lemma 15 a constant $\bar{c}'_\beta > 0$ that also depends only on β, c_+, c_- such that

$$|\exp(\psi(t))| \leq \exp \left(-2\bar{c}'_\beta \left(\frac{t^2 \Sigma^2}{s^2} \right) \vee \left(\frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} \right) \right), \quad ta > 1,$$

and also such that

$$|\exp(\psi(t))| \leq \exp(-2\bar{c}'_\beta/a^2), \quad ta > 1.$$

Indeed, the bound on $\text{Re}(\psi)$ given in this lemma is (up to a multiplicative constant) decreasing with t , and at the junction point $t = a^{-1}$ between the two parts of the upper bound it holds

$$\Delta \frac{a^{-\beta}}{\sqrt{\Delta s^2}^\beta} = \Delta \varepsilon^{-\beta} = \frac{\varepsilon^{2-\beta}}{s^2} a^{-2} \geq \frac{2c_-}{2-\beta} \frac{\sigma^2}{s^2} a^{-2}.$$

Therefore, by Equation (42)

$$|\exp(\psi(t))| \leq n^{-\bar{C}^2 \bar{c}'_\beta} := n^{-\kappa(\bar{C})}, \quad ta > 1. \quad (43)$$

Using the previous inequalities, it follows from Lemma 18 that there exists $C_\beta > 0$ depending only on β, c_+, c_- such that

$$\begin{aligned} \int_{(1/a) \vee (c \log(n))}^{+\infty} |\Psi^{(m)}(t)|^2 dt &\leq \exp(-\bar{c}'_\beta/a^2) \int_{1/a}^{+\infty} C_\beta^{2m} \bar{c}_\beta^{2m} \left[\frac{\Delta t^{\beta-1}}{\sqrt{\Delta s^2}^\beta} + \frac{t \Sigma^2}{s^2} \right]^{2m} \exp\left(-\bar{c}'_\beta \left(\frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} + \frac{t^2 \Sigma^2}{s^2} \right)\right) dt \\ &\leq \exp(-\bar{c}'_\beta/a^2) (C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left[\frac{\Delta t^{\beta-1}}{\sqrt{\Delta s^2}^\beta} \right]^{2m} \exp\left(-\bar{c}'_\beta \left(\frac{t^2 \Sigma^2}{s^2} \right) \vee \left(\frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} \right)\right) dt \\ &\quad + \exp(-\bar{c}'_\beta/a^2) (C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left[\frac{t \Sigma^2}{s^2} \right]^{2m} \exp\left(-\bar{c}'_\beta \left(\frac{t^2 \Sigma^2}{s^2} \right) \vee \left(\frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} \right)\right) dt \\ &:= A_1 + A_2. \end{aligned}$$

We first consider the term A_1 . By definition of A_1 , we immediately get

$$A_1 \leq \exp(-\bar{c}'_\beta/a^2) (C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left[\frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} \times t^{\beta-2} \right]^m \exp\left(-\bar{c}'_\beta \frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta}\right) dt$$

using $\beta < 2$ and (42) which implies $1/a \geq \sqrt{\bar{C} \log(n)} > 1$ we obtain

$$\begin{aligned} A_1 &\leq \exp(-\bar{c}'_\beta/a^2) (C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left[\frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} \times a^{2-\beta} \right]^m \exp\left(-\bar{c}'_\beta \frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta}\right) dt \\ &\leq \exp(-\bar{c}'_\beta/a^2) (C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left[\frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta} \right]^m \exp\left(-\bar{c}'_\beta \frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta}\right) dt. \end{aligned}$$

Consider the change of variable, $v = \Delta t^\beta / \sqrt{\Delta s^2}^\beta$, and use the (43) to get

$$A_1 \leq n^{-\kappa(\bar{C})} (C_\beta \bar{c}_\beta)^{2m} \frac{\sqrt{\Delta s^2}}{\beta \Delta^{1/\beta}} \int_{\Delta/\varepsilon^\beta}^{+\infty} v^{m+1/\beta-1} \exp(-\bar{c}'_\beta v) dv.$$

Finally,

$$A_1 \leq n^{-\kappa(\bar{C})} \frac{\sqrt{\Delta s^2}}{\beta \Delta^{1/\beta}} (C_\beta \bar{c}_\beta)^{2m} (\bar{c}'_\beta)^{-m-1/\beta} \Gamma(m+1/\beta).$$

It follows that for any $m \leq K = c_{int}^2 \log(n)$, $\Delta n^{c_{\max}} \geq 1$ and $\log(s)/\log(n) \leq c_{\max}$ we have if \bar{C} is large enough depending only on $c_{int}, \beta, c_+, c_-, c_{\max}$ (see (43)) that

$$A_1 \leq n^{-4} m!.$$

Now, we turn to the term A_2 . Note that if $\Sigma = 0$ then $A_2 = 0$, in the sequel let $\Sigma \neq 0$. It holds

$$\begin{aligned} A_2 &\leq \left[\exp(-\bar{c}'_\beta/a^2) (C_\beta \bar{c}_\beta)^{2m} \int_{s^2/\Sigma^2 \vee 1/a}^{+\infty} \left[\frac{t \Sigma^2}{s^2} \right]^{2m} \exp\left(-(\bar{c}'_\beta \frac{t^2 \Sigma^2}{s^2})\right) dt \right] \\ &\quad + \left[\exp(-\bar{c}'_\beta/a^2) (C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{s^2/\Sigma^2 \vee 1/a} \left[\frac{t \Sigma^2}{s^2} \right]^{2m} \exp\left(-(\bar{c}'_\beta \frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta})\right) dt \right] := A_{2,1} + A_{2,2}. \end{aligned}$$

We first consider the term $A_{2,1}$, using (35), the fact that $s^2 \geq \Sigma^2$ and $a \geq 1$ (from (42))

$$\begin{aligned}
A_{2,1} &= \exp(-\bar{c}'_\beta/a^2)(C_\beta \bar{c}_\beta)^{2m} \left[\frac{\Sigma^2}{s^2} \right]^{2m} \int_{s^2/\Sigma^2}^{+\infty} t^{2m} \exp(-\bar{c}'_\beta \frac{t^2 \Sigma^2}{s^2}) dt \\
&\leq \exp(-\bar{c}'_\beta/a^2)(C_\beta \bar{c}_\beta)^{2m} \left[\frac{\Sigma^2}{s^2} \right]^{2m} \exp(-\bar{c}'_\beta \frac{s^2}{\Sigma^2}) \int_0^{+\infty} t^{2m} \exp(-\bar{c}'_\beta \frac{t^2 \Sigma^2}{2s^2}) dt \\
&\leq \exp(-\bar{c}'_\beta/a^2)(C_\beta \bar{c}_\beta)^{2m} \left[\frac{\Sigma^2}{s^2} \right]^{2m} \exp(-\bar{c}'_\beta \frac{s^2}{\Sigma^2}) \times 2\sqrt{2\pi} 2^m m! \left[\frac{s^2}{\bar{c}'_\beta \Sigma^2} \right]^{m+\frac{1}{2}} \\
&\leq \exp(-\bar{c}'_\beta/a^2) \left[\frac{s^2}{\Sigma^2} \right]^{1/2} \exp(-\bar{c}'_\beta \frac{s^2}{\Sigma^2}) \times \frac{10}{\sqrt{\bar{c}'_\beta}} \left[\frac{2C_\beta^2 \bar{c}_\beta^2}{\bar{c}'_\beta} \right]^m m!.
\end{aligned}$$

Then, from (43) and for a constant \bar{C} large enough depending only on β, c_+, c_-, c_{int} , it holds for any $m \leq K = c_{int}^2 \log(n)$ that $A_{2,1} \leq n^{-4} m!$. Now, we consider the second term,

$$\begin{aligned}
A_{2,2} &= \exp(-\bar{c}'_\beta/a^2)(C_\beta \bar{c}_\beta)^{2m} \left[\frac{\Sigma^2}{s^2} \right]^{2m} \int_{1/a}^{s^2/\Sigma^2 \vee 1/a} t^{2m} \exp(-\bar{c}'_\beta \frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta}) dt \\
&\leq \exp(-\bar{c}'_\beta/a^2)(C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{s^2/\Sigma^2 \vee 1/a} \exp(-\bar{c}'_\beta \frac{\Delta t^\beta}{\sqrt{\Delta s^2}^\beta}) dt.
\end{aligned}$$

We apply the change of variable $v = \Delta t^\beta / \sqrt{\Delta s^2}^\beta$

$$\begin{aligned}
A_{2,2} &\leq \exp(-\bar{c}'_\beta/a^2) \frac{\sqrt{\Delta s^2}^\beta}{\beta \Delta^{1/\beta}} (C_\beta \bar{c}_\beta)^{2m} \int_{\Delta/\varepsilon^\beta}^{+\infty} \left(\frac{\sqrt{\Delta s^2}^\beta}{\Delta} v \right)^{1/\beta-1} \exp(-\bar{c}'_\beta v) dv \\
&\leq \exp(-\bar{c}'_\beta/a^2) \frac{1}{\beta} (C_\beta \bar{c}_\beta)^{2m} \frac{\sqrt{\Delta s^2}}{\Delta^{1/\beta}} (\bar{c}'_\beta)^{-1/\beta+1} \Gamma(1/\beta).
\end{aligned}$$

So for any $m \leq K = c_{int}^2 \log(n)$, $n^{c_{\max}} \Delta \geq 1$ and $\log s / \log n \leq c_{\max}$ we have if \bar{C} is large enough depending only on $c_{int}, \beta, c_+, c_-, c_{\max}$ (see (43)) that

$$A_{2,2} \leq n^{-4} m!.$$

Gathering both bounds on $A_{2,1}, A_{2,2}$, for any $m \leq K = c_{int}^2 \log(n)$ we have if \bar{C} large enough depending only on $c_{int}, \beta, c_+, c_-, c_{\max}$ that

$$A_2 = A_{2,1} + A_{2,2} \leq 2n^{-4} m!.$$

Finally, gathering all terms, we derive that $(\mathcal{H}_\Psi(\varepsilon))$ holds on the set $ta \geq 1$

$$\int_{(1/a) \vee (c \log(n))}^{+\infty} |\Psi^{(m)}(t)|^2 dt \leq A_1 + A_2 \leq 3n^{-4} m!, \quad m \leq K = c_{int}^2 \log(n),$$

which is the desired result.

Second part of the integral $ta \leq 1$. Whenever $1/a \leq t_{\min} = c \log(n)$, this part of the integral is 0. In what follows, assume that $1/a > t_{\min} = c \log(n)$. We have by definition of \tilde{f} in Lemma 19 that there exists $c_\beta > 0$ depending only on β, c_+, c_- such that $\tilde{f}(t) \leq \bar{c}_\beta t$. Moreover, Lemma 15 implies that there exists $\bar{c}'_\beta > 0$ depending only on β, c_+, c_- such that $|\exp(\psi(t))| \leq \exp(-2\bar{c}'_\beta t_{\min}^2)$, $t \in [t_{\min}, 1/a]$. Then, by (42)

$$|\exp(\psi(t))| \leq n^{-c^2 \bar{c}'_\beta} := n^{-\kappa(c)}, \quad t \in [t_{\min}, 1/a]. \quad (44)$$

This together with Lemma 19 imply that there exists a constant $C_\beta > 0$ that depends only on β, c_+, c_- such that

$$\int_{c \log(n)}^{1/a} |\Psi^{(m)}(t)|^2 dt \leq C_\beta^{2m} \exp(-\bar{c}'_\beta t_{\min}^2) \int_{c \log(n)}^{1/a} (\bar{c}_\beta t)^{2m} \exp(-\bar{c}'_\beta t^2) dt.$$

Using (35) and (44) we get

$$\begin{aligned} \int_{c \log(n)}^{1/a} |\Psi^{(m)}(t)|^2 dt &\leq \exp(-\bar{c}_\beta t_{\min}^2) C_\beta^{2m} (2\bar{c}_\beta)^{2m} \times 2\sqrt{2\pi} 2^m m! (\bar{c}'_\beta)^{-m+1/2} \\ &\leq n^{-\kappa(c)} m! \times 10\sqrt{\bar{c}'_\beta} (4C_\beta \bar{c}_\beta / \sqrt{\bar{c}'_\beta})^{2m}. \end{aligned}$$

We conclude taking $c \geq 0$ a large enough constant depending only on β, c_+, c_-, c_{int} (see (44))

$$\int_{c \log(n)}^{1/a} |\Psi^{(m)}(t)|^2 dt \leq n^{-4} m!, \quad m \leq K = c_{int}^2 \log(n).$$

Conclusion. Putting both parts of the integral together we have

$$\int_{c \log(n)}^{+\infty} |\Psi^{(m)}(t)|^2 dt \leq 3n^{-4} m! (s+1).$$

This finishes the proof of Proposition 2.

A.5 Proof of Proposition 1

Let $\varepsilon > 0$ and consider a Lévy measure ν . Assume that Σ is such that $\Sigma \geq c_\Sigma \sigma(\varepsilon) \sqrt{\log(n)}$ where $c_\Sigma > 0$ is an absolute constant. Consider the events for $i \leq n$

$$A_i = \left\{ \left| \frac{M_{i\Delta}(\varepsilon) - M_{(i-1)\Delta}(\varepsilon)}{\sqrt{\Delta(\Sigma^2 + \sigma^2(\varepsilon))}} \right| \leq \frac{c_{sup}}{c_\Sigma} \right\}.$$

By the equation leading to Equation (29) we immediately get for $\lambda_{0,\varepsilon} \geq 24 \log(n)/\Delta$ and since $\tilde{c} \leq 1$

$$\mathbb{P} \left(\bigcap_{i=1}^n \left| \frac{M_{i\Delta}(\varepsilon) - M_{(i-1)\Delta}(\varepsilon)}{\sqrt{\Delta\sigma^2(\varepsilon)}} \right| \geq c_{sup} \sqrt{\log n} \right) \leq 2n^{-2}.$$

Then, it follows from $\Sigma \geq c_\Sigma \sigma(\varepsilon) \sqrt{\log(n)}$ that $\mathbb{P}(\bigcap_i A_i) \geq 1 - 2/n^2$. Note that the proof of Theorem 2 would hold exactly in the same way if one consider $(\mathcal{I}^c)^n \cap (\bigcap_i A_i)$ instead of $(\mathcal{I}^c)^n$. For this reason, we focus only on the event $\bigcap_i A_i$ in what follows.

The rescaled increment $\tilde{X}_\Delta(\varepsilon)$, on the event $\bigcap_i A_i$, has characteristic function given by $\Psi = \Phi_1 \Psi_2$, where $\Phi_1(t) = \exp(-\frac{t^2 \Sigma^2}{2(\Sigma^2 + \sigma^2(\varepsilon))})$ and Ψ_2 is the characteristic function of the rescaled jump component $\tilde{M}_\Delta(\varepsilon)$ on A_1 . Namely, for $d\mu$ being the measure corresponding to the distribution of the rescaled jump component $\tilde{M}_\Delta(\varepsilon)$ restricted to A_1 , it holds

$$\Psi_2(t) = \int_{-c_{sup}/c_\Sigma}^{c_{sup}/c_\Sigma} \exp(iut) d\mu(s\sqrt{\Delta}u),$$

where $s^2 = \Sigma^2 + \sigma^2(\varepsilon)$.

Lemma 20. For any $d \in \mathbb{N}$, it holds $|\Psi_2^{(d)}(t)| \leq (c_{sup}/c_\Sigma)^d$.

Proof. It immediately follows from $\int d\mu(s\sqrt{\Delta}u) = 1$ and $(e^{iut})^{(d)} = (ui)^d e^{uit}$. \square

Using that

$$\Psi^{(k)}(t) = \sum_j \binom{k}{j} \Phi_1^{(j)} \Psi_2^{(k-j)},$$

and Lemma 20 imply

$$|\Psi^{(k)}(t)|^2 \leq \left(1 + \frac{c_{sup}}{c_\Sigma}\right)^{2k} \max_{j \leq k} |\Phi_1^{(j)}|^2,$$

From Lemmas 19 and 18, with $c_+ = c_- = 0, \beta = 1$, we get

$$|\Phi_1^{(j)}(t)|^2 \leq \exp(-c_1 t^2/2) t^{2j} c_2^j,$$

where $c_1, c_2 > 0$ depend only on c_{sup}, c_Σ . Therefore, for any $j \in \mathbb{N}$ we have proceeding as in the bound of $A_{2,1}$ in the proof of Proposition 2 that

$$\int_{c \log(n)}^{+\infty} |\Phi_1^{(j)}(t)|^2 dt \leq j! C_1^j \exp(-c_3 c \log(n)^2),$$

where $C_1, c_3 > 0$ depend only on c_{sup}, c_Σ . This concludes the proof, taking c large enough depending only on $c_{sup}, c_\Sigma, c_{int}$.

B Some bounds in total variation distance

Lemma 21. *If $\mu_1, \mu_2 \in \mathbb{R}$, $0 \leq \Sigma_1^2 \leq \Sigma_2^2$ and $\Sigma_2 \in \mathbb{R}_{>0}$, then*

$$\|\mathcal{N}(\mu_1, \Sigma_1^2)^{\otimes n} - \mathcal{N}(\mu_2, \Sigma_2^2)^{\otimes n}\|_{TV} \leq 1 - \left(\frac{\Sigma_1}{\Sigma_2}\right)^n + \sqrt{n} \frac{|\mu_1 - \mu_2|}{\sqrt{2\pi\Sigma_2^2}}, \quad \forall n \geq 1.$$

Proof. Since the Lemma is trivially true if $\Sigma_1 = 0$ we can assume that $\Sigma_1 > 0$ without loss of generality. By triangular inequality, for any $n \geq 1$ we have:

$$\|\mathcal{N}(\mu_1, \Sigma_1^2)^{\otimes n} - \mathcal{N}(\mu_2, \Sigma_2^2)^{\otimes n}\|_{TV} \leq \|\mathcal{N}(\mu_1, \Sigma_1^2)^{\otimes n} - \mathcal{N}(\mu_1, \Sigma_2^2)^{\otimes n}\|_{TV} + \|\mathcal{N}(\mu_1, \Sigma_2^2)^{\otimes n} - \mathcal{N}(\mu_2, \Sigma_2^2)^{\otimes n}\|_{TV}.$$

Denote by $\varphi_n(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{x_1^2 + \dots + x_n^2}{2}}$ the density of a standard normal distribution on \mathbb{R}^n and observe that

$$\begin{aligned} \|\mathcal{N}(\mu_1, \Sigma_1^2)^{\otimes n} - \mathcal{N}(\mu_1, \Sigma_2^2)^{\otimes n}\|_{TV} &= \|\mathcal{N}(0, \Sigma_1^2)^{\otimes n} - \mathcal{N}(0, \Sigma_2^2)^{\otimes n}\|_{TV} \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \left| \varphi_n(z_1, \dots, z_n) - \left(\frac{\Sigma_1}{\Sigma_2}\right)^n \varphi_n\left(\frac{\Sigma_1}{\Sigma_2} z_1, \dots, \frac{\Sigma_1}{\Sigma_2} z_n\right) \right| dz_1 \dots dz_n \\ &\leq \frac{1}{2} \int_{\mathbb{R}^n} \left| 1 - \left(\frac{\Sigma_1}{\Sigma_2}\right)^n \right| \varphi_n(z_1, \dots, z_n) dz_1 \dots dz_n \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \left(\frac{\Sigma_1}{\Sigma_2}\right)^n \left| \varphi_n(z_1, \dots, z_n) - \varphi_n\left(\frac{\Sigma_1}{\Sigma_2} z_1, \dots, \frac{\Sigma_1}{\Sigma_2} z_n\right) \right| dz_1 \dots dz_n \\ &= \frac{1 - \left(\frac{\Sigma_1}{\Sigma_2}\right)^n}{2} + \frac{1}{2} \int_{\mathbb{R}^n} \left(\frac{\Sigma_1}{\Sigma_2}\right)^n \left(\varphi_n\left(\frac{\Sigma_1}{\Sigma_2} z_1, \dots, \frac{\Sigma_1}{\Sigma_2} z_n\right) - \varphi_n(z_1, \dots, z_n) \right) dz_1 \dots dz_n \\ &= \frac{1 - \left(\frac{\Sigma_1}{\Sigma_2}\right)^n}{2} + \frac{1}{2} \left(\frac{\Sigma_1}{\Sigma_2}\right)^n \left(\left(\frac{\Sigma_2}{\Sigma_1}\right)^n - 1 \right) = 1 - \left(\frac{\Sigma_1}{\Sigma_2}\right)^n. \end{aligned}$$

Let U be any $n \times n$ orthonormal matrix with first column given by $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^t$, set $\alpha = \frac{\mu_1 - \mu_2}{\Sigma_2}$ and $\beta = U^t(\alpha, \dots, \alpha)^t = (\sqrt{n}\alpha, 0, \dots, 0)^t$. Observe that $\varphi_n(Ux) = \varphi_n(x)$ for any $x \in \mathbb{R}^n$. We have:

$$\begin{aligned} \|\mathcal{N}(\mu_1, \Sigma_2^2)^{\otimes n} - \mathcal{N}(\mu_2, \Sigma_2^2)^{\otimes n}\|_{TV} &= \frac{1}{2\Sigma_2^n} \int_{\mathbb{R}^n} \left| \varphi_n\left(\frac{x_1 - \mu_1}{\Sigma_2}, \dots, \frac{x_n - \mu_1}{\Sigma_2}\right) - \varphi_n\left(\frac{x_1 - \mu_2}{\Sigma_2}, \dots, \frac{x_n - \mu_2}{\Sigma_2}\right) \right| dx_1 \dots dx_n \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \left| \varphi_n(z_1 - \alpha, \dots, z_n - \alpha) - \varphi_n(z_1, \dots, z_n) \right| dz_1 \dots dz_n. \end{aligned}$$

Let $z = (z_1, \dots, z_n)^t$. By the change of variable $y = U^t z$ we get

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \varphi_n(z_1 - \alpha, \dots, z_n - \alpha) - \varphi_n(z_1, \dots, z_n) \right| dz_1 \dots dz_n &= \int_{\mathbb{R}^n} |\varphi_n(Uy - (\alpha, \dots, \alpha)^t) - \varphi_n(Uy)| dy_1 \dots dy_n \\ &= \int_{\mathbb{R}^n} |\varphi_n(U(y - \beta)) - \varphi_n(Uy)| dy_1 \dots dy_n \\ &= \int_{\mathbb{R}^n} |\varphi_n(y_1 - \sqrt{n}\alpha, y_2, \dots, y_n) - \varphi_n(y_1, \dots, y_n)| dy_1 \dots dy_n \\ &= \int_{\mathbb{R}^n} \left| \varphi_1(y_1 - \sqrt{n}\alpha) \prod_{j=2}^n \varphi_1(y_j) - \prod_{j=1}^n \varphi_1(y_j) \right| dy_1 \dots dy_n \\ &= \int_{\mathbb{R}} |\varphi_1(y_1 - \sqrt{n}\alpha) - \varphi_1(y_1)| dy_1 = 2 \int_{-\frac{\sqrt{n}|\alpha|}{2}}^{\frac{\sqrt{n}|\alpha|}{2}} \varphi_1(y) dy \leq \frac{2\sqrt{n}|\alpha|}{\sqrt{2\pi}}. \end{aligned}$$

We deduce that

$$\|\mathcal{N}(\mu_1, \Sigma_2^2)^{\otimes n} - \mathcal{N}(\mu_2, \Sigma_2^2)^{\otimes n}\|_{TV} \leq \frac{\sqrt{n}|\mu_1 - \mu_2|}{\sqrt{2\pi}\Sigma_2}.$$

□

Lemma 22. Let $(X_i)_{i \geq 1}$ and $(Y_i)_{i \geq 1}$ be sequences of i.i.d. random variables a.s. different from zero and N, N' be two Poisson random variables with N (resp. N') independent of $(X_i)_{i \geq 1}$ (resp. $(Y_i)_{i \geq 1}$). Denote by λ (resp. λ') the mean of N (resp. N'). Then, for any $n \geq 1$

$$\left\| \mathcal{L}\left(\sum_{i=1}^N X_i\right)^{\otimes n} - \mathcal{L}\left(\sum_{i=1}^{N'} Y_i\right)^{\otimes n} \right\|_{TV} \leq n(\lambda \wedge \lambda') \|X_1 - Y_1\|_{TV} + 1 - e^{-n|\lambda - \lambda'|}.$$

Proof. The proof is a minor extension of the proof of Theorem 13 in [14]. For the ease of the reader we repeat here the essential steps. Without loss of generality, let us suppose that $\lambda \geq \lambda'$ and write $\lambda = \alpha + \lambda', \alpha \geq 0$. By triangle inequality,

$$\begin{aligned} \left\| \mathcal{L}\left(\sum_{i=1}^N X_i\right)^{\otimes n} - \mathcal{L}\left(\sum_{i=1}^{N'} Y_i\right)^{\otimes n} \right\|_{TV} &\leq \left\| \mathcal{L}\left(\sum_{i=1}^N X_i\right)^{\otimes n} - \mathcal{L}\left(\sum_{i=1}^{N''} X_i\right)^{\otimes n} \right\|_{TV} \\ &\quad + \left\| \mathcal{L}\left(\sum_{i=1}^{N''} X_i\right)^{\otimes n} - \mathcal{L}\left(\sum_{i=1}^{N'} Y_i\right)^{\otimes n} \right\|_{TV}, \end{aligned} \quad (45)$$

where N'' is a random variable independent of $(X_i)_{i \geq 1}$ and with the same law as N' .

Let P be a Poisson random variable independent of N'' and $(X_i)_{i \geq 1}$ with mean α . Then,

$$\begin{aligned} \left\| \mathcal{L} \left(\sum_{i=1}^N X_i \right)^{\otimes n} - \mathcal{L} \left(\sum_{i=1}^{N''} X_i \right)^{\otimes n} \right\|_{TV} &= \left\| \mathcal{L} \left(\sum_{i=1}^{N''+P} X_i \right)^{\otimes n} - \mathcal{L} \left(\sum_{i=1}^{N''} X_i \right)^{\otimes n} \right\|_{TV} \\ &\leq \left\| \delta_0 - \mathcal{L} \left(\sum_{i=1}^P X_i \right)^{\otimes n} \right\|_{TV} \\ &= \mathbb{P} \left(\left(\sum_{i=1}^P X_i \right)^{\otimes n} \neq (0, \dots, 0) \right) \leq 1 - e^{-n\alpha}, \end{aligned} \quad (46)$$

where (46) follows by Lemma 23.

In order to bound the second addendum in (45) we condition on N' and use again Lemma 23 joined with the fact that $\mathcal{L}(N') = \mathcal{L}(N'')$. Denoting by N'_1, \dots, N'_n n independent copies of N' and by $X_{i,j}$ (resp. $Y_{i,j}$), $j = 1, \dots, n$ and $i \geq 1$, n independent copies of X_i (resp. Y_i), we have:

$$\begin{aligned} &\left\| \mathcal{L} \left(\sum_{i=1}^{N''} X_i \right)^{\otimes n} - \mathcal{L} \left(\sum_{i=1}^{N'} Y_i \right)^{\otimes n} \right\|_{TV} \\ &= \sum_{m_1 \geq 0, \dots, m_n \geq 0} \left\| \mathcal{L} \left(\sum_{i=1}^{m_1} X_{i,1}, \dots, \sum_{i=1}^{m_n} X_{i,n} \right) - \mathcal{L} \left(\sum_{i=1}^{m_1} Y_{i,1}, \dots, \sum_{i=1}^{m_n} Y_{i,n} \right) \right\|_{TV} \mathbb{P}(N'_1 = m_1) \dots \mathbb{P}(N'_n = m_n) \\ &\leq \sum_{m_1 \geq 0, \dots, m_n \geq 0} (m_1 \|X_{1,1} - Y_{1,1}\|_{TV} + \dots + m_n \|X_{1,n} - Y_{1,n}\|_{TV}) \mathbb{P}(N'_1 = m_1) \dots \mathbb{P}(N'_n = m_n) \\ &= \|X_1 - Y_1\|_{TV} (\mathbb{E}[N'_1] + \dots + \mathbb{E}[N'_n]) = n\lambda' \|X_1 - Y_1\|_{TV}. \end{aligned}$$

□

Lemma 23. Let X, Y and Z be three random variables on \mathbb{R}^N , $N \geq 1$, with Z independent of X and Y . Then,

$$\|\mathcal{L}(X + Z) - \mathcal{L}(Y + Z)\|_{TV} \leq \|X - Y\|_{TV}. \quad (47)$$

Let $(X_i)_{i=1}^m$ and $(Y_i)_{i=1}^m$ be independent random variables on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$, $N \geq 1$. Then, for any $m \geq 1$,

$$\left\| \sum_{i=1}^m X_i - \sum_{i=1}^m Y_i \right\|_{TV} \leq \sum_{i=1}^m \|X_i - Y_i\|_{TV}. \quad (48)$$

Proof. To prove (47) one can use the variational definition of the total variation. Denoting by P_A the law of a random variable A , it holds

$$\begin{aligned} 2\|\mathcal{L}(X + Z) - \mathcal{L}(Y + Z)\|_{TV} &= \sup_{\|f\|_\infty \leq 1} \int f(x) (P_{X+Z}(dx) - P_{Y+Z}(dx)) \\ &= \sup_{\|f\|_\infty \leq 1} \int f(x) \left(\int P_Z(dx - z) (P_X(dz) - P_Y(dz)) \right) \\ &= \sup_{\|f\|_\infty \leq 1} \int \int f(x) P_Z(dx - z) (P_X(dz) - P_Y(dz)). \end{aligned}$$

Denote by $g_f(z) = \int f(x) P_Z(dx - z)$ and observe that g_f is measurable with $\|g_f\|_\infty \leq \|f\|_\infty$. It then follows that

$$\sup_{\|f\|_\infty \leq 1} \int g_f(z) (P_X(dz) - P_Y(dz)) \leq \sup_{\|g\|_\infty \leq 1} \int g(z) (P_X(dz) - P_Y(dz)) = 2\|X - Y\|_{TV}.$$

Equation (48) is straightforward using (47). Indeed, by induction, it suffices to prove the case $n = 2$. Let \tilde{X}_2 be a random variable equal in law to X_2 and independent of Y_1 and of X_1 . By triangle inequality we deduce that

$$\|\mathcal{L}(X_1 + X_2) - \mathcal{L}(Y_1 + Y_2)\|_{TV} \leq \|\mathcal{L}(X_1 + X_2) - \mathcal{L}(Y_1 + \tilde{X}_2)\|_{TV} + \|\mathcal{L}(Y_1 + \tilde{X}_2) - \mathcal{L}(Y_1 + Y_2)\|_{TV}$$

and by means of (47) we conclude that

$$\begin{aligned} \|\mathcal{L}(Y_1 + \tilde{X}_2) - \mathcal{L}(Y_1 + Y_2)\|_{TV} &\leq \|\tilde{X}_2 - Y_2\|_{TV} = \|X_2 - Y_2\|_{TV} \\ \|\mathcal{L}(X_1 + X_2) - \mathcal{L}(Y_1 + \tilde{X}_2)\|_{TV} &\leq \|X_1 - Y_1\|_{TV}. \end{aligned}$$

□

Lemma 24. *Let P and Q be probability density. For any $n \geq 1$ it holds:*

$$\|P^{\otimes n} - Q^{\otimes n}\|_{TV} \leq \sqrt{2n\|P - Q\|_{TV}}.$$

Proof. Let $H(P, Q)$ denote the Hellinger distance between P and Q , i.e.

$$H(P, Q) = \sqrt{\int \left(\sqrt{\frac{dP}{d\mu}} - \sqrt{\frac{dQ}{d\mu}} \right)^2 d\mu}$$

where μ it is a common dominating measure for P and Q . It is well known (see e.g. Lemma 2.3 and Property (iv) page 83 in [19]) that

$$\frac{H^2(P, Q)}{2} \leq \|P - Q\|_{TV} \leq H(P, Q), \quad H^2(P^{\otimes n}, Q^{\otimes n}) = 2 \left(1 - \left(1 - \frac{H^2(P, Q)}{2} \right)^n \right).$$

In particular $H^2(P^{\otimes n}, Q^{\otimes n}) \leq nH^2(P, Q)$ and $\|P^{\otimes n} - Q^{\otimes n}\|_{TV} \leq \sqrt{nH^2(P, Q)} \leq \sqrt{2n\|P - Q\|_{TV}}$, as desired. □