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Numerical method for impulse control of Piecewise Deterministic Markov Processes

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Outline

Introduction

- Motivation

- Piecewise deterministic Markov processes

- Example

Impulse control

- Definition

- Quantization

- Discretization scheme

Numerical implementation

Conclusion and perspectives

Maintenance optimization

Equipments

- ▶ with several components
- ▶ subject to **random** failures

Choose the best **interventions**

- ▶ **when** ?
- ▶ what **type** : change or repair ?

In order to **optimize some criterion**

- ▶ minimize a **cost**: functioning, maintenance, ...
- ▶ maximize a **reward**: availability, ...

Our approach

- ▶ propose a general **model** for the evolution of the equipment state based on **PDMPs**
- ▶ **formalize** the maintenance problem as an **impulse control problem** for PDMPs
- ▶ **derive** a **numerical scheme** to approximate the value function (with error bounds)
- ▶ **compute** the approximate optimal maintenance cost

Piecewise deterministic Markov processes

[Davis 93] General class of **non-diffusion** dynamic stochastic **hybrid** models: **deterministic** motion punctuated by **random** jumps.

Hybrid process $X_t = (m_t, x_t)$

- ▶ **discrete** mode $m_t \in \{1, 2, \dots, p\}$
- ▶ **Euclidean** state variable $x_t \in \mathbb{R}^n$

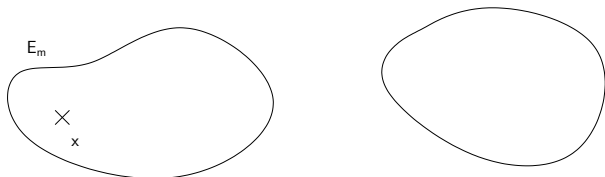
Local characteristics for each mode m

- ▶ E_m open subset of \mathbb{R}^d
- ▶ **Flow** $\phi_m: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ deterministic motion between jumps, one-parameter group of homeomorphisms
- ▶ **Intensity** $\lambda_m: \bar{E}_m \rightarrow \mathbb{R}_+$ intensity of random jumps
- ▶ **Markov kernel** Q_m on $(\bar{E}_m, \mathcal{B}(\bar{E}_m))$ selects the post-jump location

Dynamics

Starting point

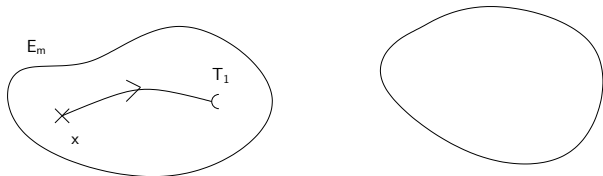
$$X_0 = (m, x)$$



Dynamics

X_t follows the deterministic flow until the first jump time $T_1 = S_1$

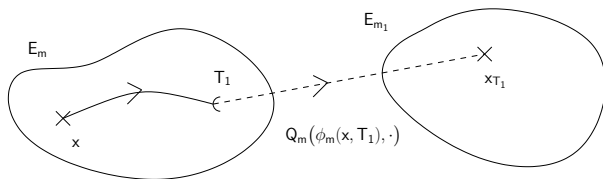
$$X_t = (m, \phi_m(x, t)), \quad \mathbb{P}_{(m,x)}(S_1 > t) = e^{-\int_0^t \lambda_m(\phi_m(x,s)) ds}$$



Dynamics

Post-jump location (m_1, x_{T_1}) selected by the **Markov kernel**

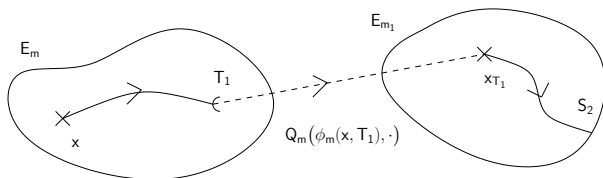
$$Q_m(\phi_m(x, T_1), \cdot)$$



Dynamics

X_t follows the **flow** until the next jump time $T_2 = T_1 + S_2$

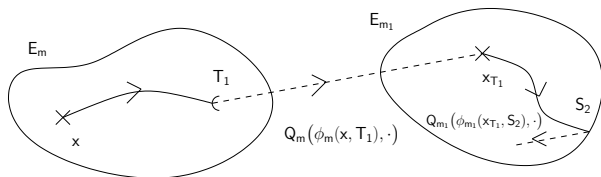
$$X_{T_1+t} = (m_1, \phi_{m_1}(x_{T_1}, t)), \quad t < S_2$$



Dynamics

Post-jump location (m_2, x_{T_2}) selected by **Markov kernel**

$$Q_{m_1}(\phi_{m_1}(x_{T_1}, S_2), \cdot) \dots$$



Embedded Markov chain

$\{X_t\}$ strong Markov process [Davis 93]

Natural embedded Markov chain

- ▶ Z_0 starting point, $S_0 = 0$, $S_1 = T_1$
- ▶ Z_n new mode and location after n -th jump, $S_n = T_n - T_{n-1}$,
time between two jumps

Proposition

(Z_n, S_n) is a discrete-time Markov chain

Only source of randomness of the PDMP

Equipment model

Typical model with 4 components

- ▶ Component 1: 2 states – stable $\xrightarrow{\text{Exponential}}$ failed
- ▶ Component 2: 2 states – stable $\xrightarrow{\text{Weibull}}$ failed
- ▶ Components 3 and 4: 3 states
stable $\xrightarrow{\text{Weibull}}$ degraded $\xrightarrow{\text{Exponential}}$ failed

Equipment model

Typical model with 4 components

- ▶ Component 1: 2 states – **stable** $\xrightarrow{\text{Exponential}}$ **failed**
- ▶ Component 2: 2 states – **stable** $\xrightarrow{\text{Weibull}}$ **failed**
- ▶ Components 3 and 4: 3 states
stable $\xrightarrow{\text{Weibull}}$ **degraded** $\xrightarrow{\text{Exponential}}$ **failed**

Possible **maintenance** operations

- ▶ Components 1 and 2: **change**
- ▶ Components 3 and 4: **change** , **repair** (only in **stable** or **degraded** states)

Global equipment state

The equipment is globally

- ▶ **stable** if the 4 components are **stable**
- ▶ **degraded** if at least one component is **degraded** and the others are **stable** or **degraded**
- ▶ **failed** if at least one component is failed **failed**
- ▶ in the **workshop** if there is an ongoing maintenance operation of **change** or **repair**

PDMP model of the equipment

- ▶ Euclidean variables
 - ▶ functioning time of components 2, 3 and 4
 - ▶ calendar time
 - ▶ time spent in the workshop

- ▶ Discrete variables
 - ▶ state of the components / maintenance operations

Other applications of PDMPs

Applications of PDMPs

Engineering systems, operations research, management science, economics, internet traffic, dependability and safety, neurosciences, biology, ...

- ▶ mode: nominal, failures, breakdown, environment, number of individuals, response to a treatment, ...
- ▶ Euclidean variable: pressure, temperature, time, size, potential, protein level, ...

Impulse control problem

Impulse control

Select

- ▶ **intervention dates**
- ▶ new **starting point** for the process at interventions

to **minimize** a cost function

- ▶ **repair** a component before failure
- ▶ **change** treatment before relapse
- ▶ ...

[CD 89], [Davis 93], [dSDZ 14], ...

Mathematical definition

Strategy $\mathcal{S} = (\tau_n, R_n)_{n \geq 1}$

- ▶ τ_n intervention times
- ▶ R_n new positions after intervention

Value function

$$\mathcal{J}^{\mathcal{S}}(x) = E_x^{\mathcal{S}} \left[\int_0^{\infty} e^{-\alpha s} f(Y_s) ds + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i}^+) \right]$$

$$\mathcal{V}(x) = \inf_{\mathcal{S} \in \mathbb{S}} \mathcal{J}^{\mathcal{S}}(x)$$

- ▶ f, c cost functions, α discount factor
- ▶ Y_t controlled process, \mathbb{S} set of admissible strategies

Example of maintenance optimization

- ▶ τ_n : maintenance dates
- ▶ R_n : which components are to be changed/repaired

Value function

$$\mathcal{J}^S(x) = E_x^S \left[\int_0^\infty e^{-\alpha s} f(Y_s) ds + \sum_{i=1}^\infty e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i}^+) \right]$$

$$\mathcal{V}(x) = \inf_{S \in \mathcal{S}} \mathcal{J}^S(x)$$

- ▶ f **unavailability** cost proportional to time spend in **failed** state
- ▶ c fixed cost for going to the workshop + **repair** < **change** costs
- ▶ $\alpha = 0$ (finite horizon)

Dynamic programming

Costa, Davis, 1988

For any function $g \geq$ cost of the no-impulse strategy

- ▶ $v_0 = g$
- ▶ $v_n = \mathcal{L}(v_{n-1})$

$$v_n(x) \xrightarrow[n \rightarrow \infty]{} \mathcal{V}(x)$$

Dynamic programming operator

$$\begin{aligned}
 \mathcal{L}(w)(x) &= L(Mw, w)(x) \\
 &= \left(\inf_{t \leq t^*(x)} \mathbb{E}_x \left[F(x, t) + e^{-\alpha S_1} w(Z_1) \mathbb{1}_{\{S_1 < t \wedge t^*(x)\}} \right. \right. \\
 &\quad \left. \left. + e^{-\alpha t \wedge t^*(x)} Mw(\phi(x, t \wedge t^*(x))) \mathbb{1}_{\{S_1 \geq t \wedge t^*(x)\}} \right] \right) \\
 &\quad \wedge \mathbb{E}_x \left[F(x, t^*(x)) + e^{-\alpha S_1} w(Z_1) \right]
 \end{aligned}$$

with

$$\begin{aligned}
 F(x, t) &= \int_0^{t \wedge t^*(x)} e^{-\alpha s - \Lambda(x, s)} f(\phi(x, s)) ds \\
 Mw(x) &= \inf_{y \in \mathbb{U}} \{c(x, y) + w(y)\}
 \end{aligned}$$

Our aim

Propose a numerical method

- ▶ to compute an **approximation** of the value function
- ▶ with **error bounds**

Main difficulty

Discretization of the dynamic programming **operator**

Our approach

Discretize the underlying **Markov chain** (Z_n, S_n) by **optimal quantization**

Quantization

Quantization of a random variable X

Approximate X by \hat{X} taking **finitely** many values such that $\|X - \hat{X}\|_p$ is **minimum**

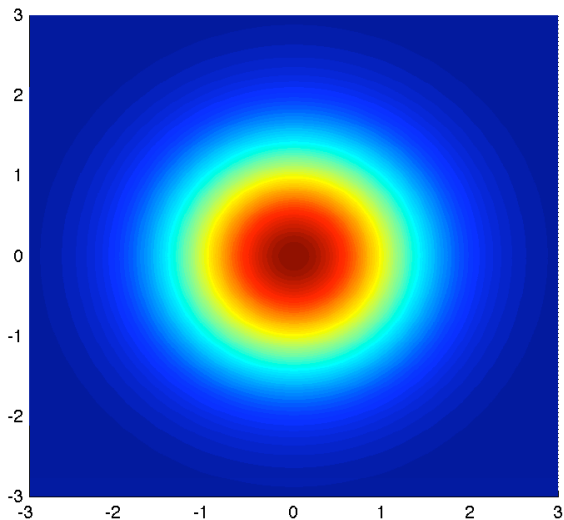
- ▶ Find a finite weighted grid Γ with $|\Gamma| = K$
- ▶ Set $\hat{X} = p_\Gamma(X)$ closest neighbor projection

Algorithms

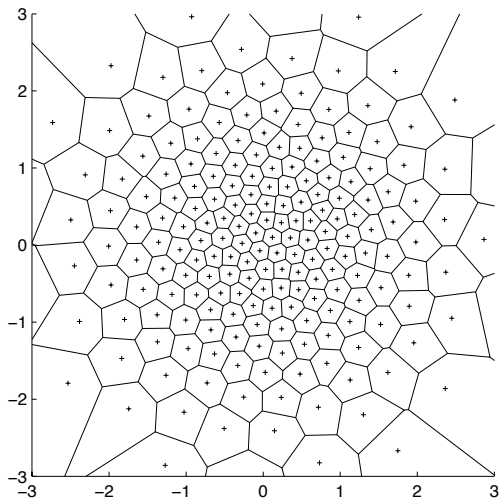
There exist algorithms providing

- ▶ Γ
- ▶ **law** of \hat{X}
- ▶ **transition probabilities** for quantization of Markov chains

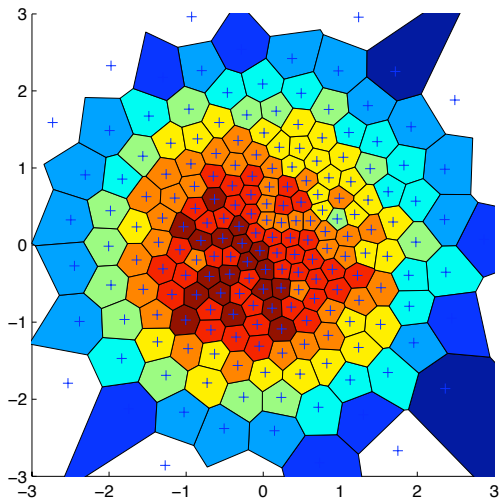
Example: $\mathcal{N}(0, I_2)$:



Example: $\mathcal{N}(0, I_2)$:



Example: $\mathcal{N}(0, I_2)$:



Horizon and control set

- ▶ finite set \mathbb{U} of new starting points
- ▶ select horizon N such that $v_N(x) - \mathcal{V}(x)$ small enough

→ numerical approximation of $v_N(x)$

Main idea

Replace the dynamic programming iteration of **functions** by an iteration of **random variables**

Backward dynamic programming

Change of notation

For a well chosen function g and large enough N

- ▶ $v_N = g$
- ▶ $v_n = \mathcal{L}(v_{n+1})$

$$v_0(x) \simeq \mathcal{V}(x)$$

Dynamic programming

Markov property

$$\begin{aligned}
 v_n(Z_n) &= \mathcal{L}(Mv_{n+1}, v_{n+1})(Z_n) \\
 &= \left(\inf_{t \leq t^*(Z_n)} \mathbb{E} \left[F(Z_n, t) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) \mathbb{1}_{\{S_{n+1} < t \wedge t^*(Z_n)\}} \right. \right. \\
 &\quad \left. \left. + e^{-\alpha t \wedge t^*(Z_n)} Mv_{n+1}(\phi(Z_n, t \wedge t^*(Z_n))) \mathbb{1}_{\{S_{n+1} \geq t \wedge t^*(Z_n)\}} \mid Z_n \right] \right. \\
 &\quad \left. \wedge \mathbb{E} \left[F(Z_n, t^*(Z_n)) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) \mid Z_n \right] \right)
 \end{aligned}$$

with

$$\begin{aligned}
 F(x, t) &= \int_0^{t \wedge t^*(x)} e^{-\alpha s - \Lambda(x, s)} f(\phi(x, s)) ds \\
 Mv_{n+1}(x) &= \inf_{y \in \mathbb{U}} \{c(x, y) + v_{n+1}(y)\}
 \end{aligned}$$

Recursion on random variables

$v_n(Z_n)$ expression of $v_{n+1}(Z_{n+1})$, Z_n , S_{n+1}
+ $v_{n+1}(y)$ for all y in \mathbb{U}

Numerical scheme

- ▶ first compute recursively $\tilde{v}_n(y)$ approximation of $v_n(y)$ for all y in \mathbb{U}
- ▶ then compute recursively $\hat{v}_n(\hat{Z}_n)$ approximation of $v_n(Z_n)$

Discretization

In the expression of operator \mathcal{L} replace

- ▶ inf by min over a discretized grid
- ▶ Z_n , Z_{n+1} , S_{n+1} by their quantized approximation **starting from $Z_0 \in \mathbb{U}$**

Recursion on random variables

$v_n(Z_n)$ expression of $v_{n+1}(Z_{n+1})$, Z_n , S_{n+1}
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Discretization

In the expression of operator \mathcal{L} replace

- ▶ inf by min over a discretized grid
- ▶ Z_n , Z_{n+1} , S_{n+1} by their quantized approximation starting from $Z_0 = x$

Properties of the numerical scheme

- ▶ the quantized process has **no Markov property** \rightarrow a different approximation of \mathcal{L} for each time step and each starting point
- ▶ Under Lipschitz regularity assumption, convergence of the scheme with **errors bounds** depending on
 - ▶ the time discretization step $\inf \rightarrow \min$
 - ▶ the quantization error $(Z_n, S_n) \rightarrow (\widehat{Z}_n, \widehat{S}_n)$

Step 1: Exact simulation of the PDMP

Implementation of an exact simulator for reference strategies to serve as benchmark

	1	2	3	4	5
intervention	never	1 day failed	1 day failed	1 day degraded or failed	1 day degraded or failed
C1 failed	nothing	change	change	change	change
C3 degraded	nothing	change	repair	change	repair
C3 failed	nothing	change	change	change	change
C2 failed and C4 stable	nothing	change 2+4	change 2+4	change 2+4	2+4 2+4
C2 failed and C4 degraded	nothing	change 2+4	change 2+4	change 2+4	change 2+4
C2 stable and C4 degraded	nothing	change 2+4	repair 4	change 2+4	repair 4
C2 stable and C4 failed	nothing	change 2+4	change 2+4	change 2+4	change 2+4
Mean cost	19680	11184	11114	11521	8359

Step 2 : Discretisation of the control set \mathbb{U}

Finite control set \mathbb{U}

\implies discretize the functioning times at interventions

\implies project the real times on the grid feasibly

Compromise between precision and computation time

Tests on strategy 5

Grid	Number of points	relative error
$3 \times 3 \times 3 \times 5$	419	0.1458
$4 \times 4 \times 4 \times 5$	627	0.1331
$5 \times 5 \times 5 \times 5$	1055	0.1235
$3 \times 3 \times 3 \times 11$	788	0.0962
$4 \times 4 \times 4 \times 11$	1219	0.0819
$5 \times 5 \times 5 \times 11$	1855	0.0730
$6 \times 6 \times 6 \times 11$	2790	0.0672
$7 \times 7 \times 7 \times 11$	3570	0.0634
$8 \times 8 \times 8 \times 11$	4647	0.0604
$3 \times 3 \times 3 \times 21$	1403	0.0775
$4 \times 4 \times 4 \times 21$	2195	0.0626
$5 \times 5 \times 5 \times 21$	3423	0.0534
$6 \times 6 \times 6 \times 21$	4900	0.0436
$7 \times 7 \times 7 \times 21$	6489	0.0384
$8 \times 8 \times 8 \times 21$	8399	0.0350

Step 3: Discretizing the embedded Markov chain

- calibration on reference strategies

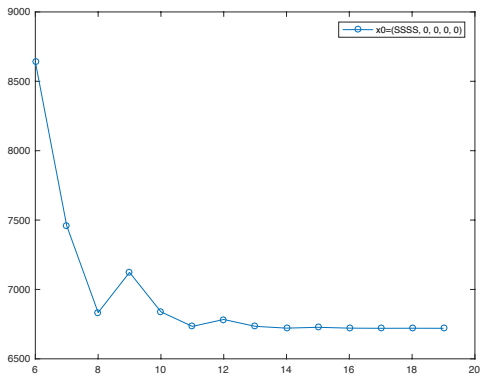
Compromise between precision and computation time

Number of points	Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5
50	19680	11145	11075	11485	8326
100	19680	11207	11134	11509	8367
200	19680	11173	11104	11531	8361
400	19680	11193	11124	11531	8366
1000	19680	11180	11109	11517	8355
Exact cost	19680	11184	11114	11521	8359

Step 4: Calibrating N the number of allowed jumps + interventions

Horizon N (number of iterations)

- ▶ 5 for Strategy 1
- ▶ up to 30 for Strategies 2 and 3 (mean 6)
- ▶ up to 25 for Strategies 4 and 5 (mean 6)



Step 5: Approximation of the value function

Strategy 1	Strategy 2	Strategy 3	Strategy 4	Strategy 5	Approx. Value function
19680	11184	11114	11521	8359	6720

- ▶ relative gain of 19.6% vs Strategy 5
- ▶ numerical **validation** of the algorithm with various starting points: consistent results

Conclusion and perspectives

Numerical method to approximate the value function

- ▶ rigorously validated
- ▶ with general error bounds
- ▶ numerical demanding but viable in low dimensional examples

Conclusion and perspectives

Numerical method to approximate the value function

- ▶ rigorously validated
- ▶ with general error bounds
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Work in progress

- ▶ Approximation of an ϵ -optimal strategy: numerical and theoretical study - PGMO grant 2018-2019

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