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# Quantification d'incertitude en mécanique des fluides : l'apport de la dynamique sous incertitude de position et de l'advection stochastique par transport de Lie

Baylor Fox-Kemper, Darryl D Holm, Wei Pan, Valentin Resseguier

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# Quantification d'incertitude en mécanique des fluides : l'apport de la dynamique sous incertitude de position et de l'advection stochastique par transport de Lie

Baylor Fox-Kemper

Darryl D. Holm

Wei Pan

**Valentin Resseguier**



# Motivations

- Rigorously identified subgrid dynamics effects
- Injecting likely small-scale dynamics
- Studying bifurcations and attractors



Climate projections

- Quantification of modeling errors



Ensemble forecasts and data assimilation







# Contents

- Stochastic transport
- Stochastic Navier-Stokes : SALT vs LU
- Unresolved velocity parametrisation
- Unresolved velocity non-stationary heterogeneity

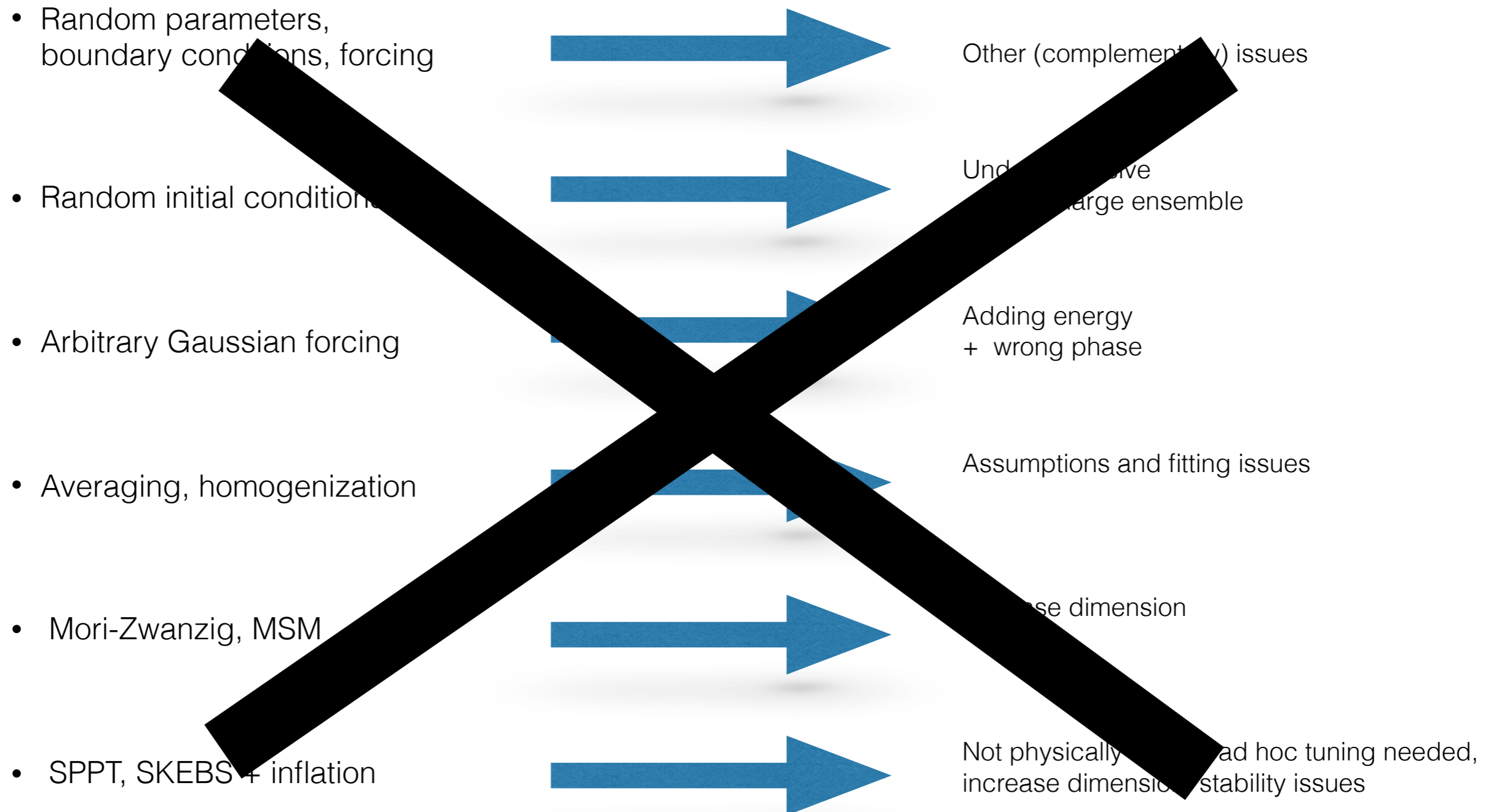
# Part I

## Stochastic transport

# Usual random CFD

• Random parameters, boundary conditions, forcing		Other (complementary) issues
• Random initial conditions		Underdispersive + need large ensemble
• Arbitrary Gaussian forcing		Adding energy + wrong phase
• Averaging, homogenization		Assumptions and fitting issues
• Mori-Zwanzig, MSM		Increase dimension
• SPPT, SKEBS + inflation		Not physically-based, ad hoc tuning needed, increase dimension, stability issues

# Usual random CFD



SALT & LU :  
Adding random velocity

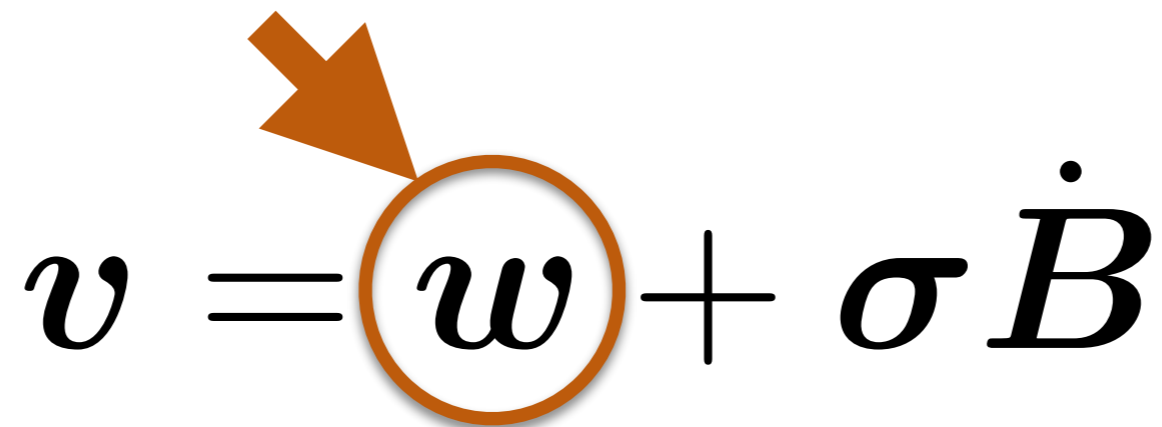
$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$



# SALT & LU :

## Adding random velocity

Resolved  
large scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$
The diagram shows the equation  $\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$ . The variable  $\boldsymbol{w}$  is enclosed in a brown circle, and a brown arrow points from the text "Resolved large scales" above to this circle.

# SALT & LU :

## Adding random velocity

Resolved  
large scales

White-in-time  
small scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

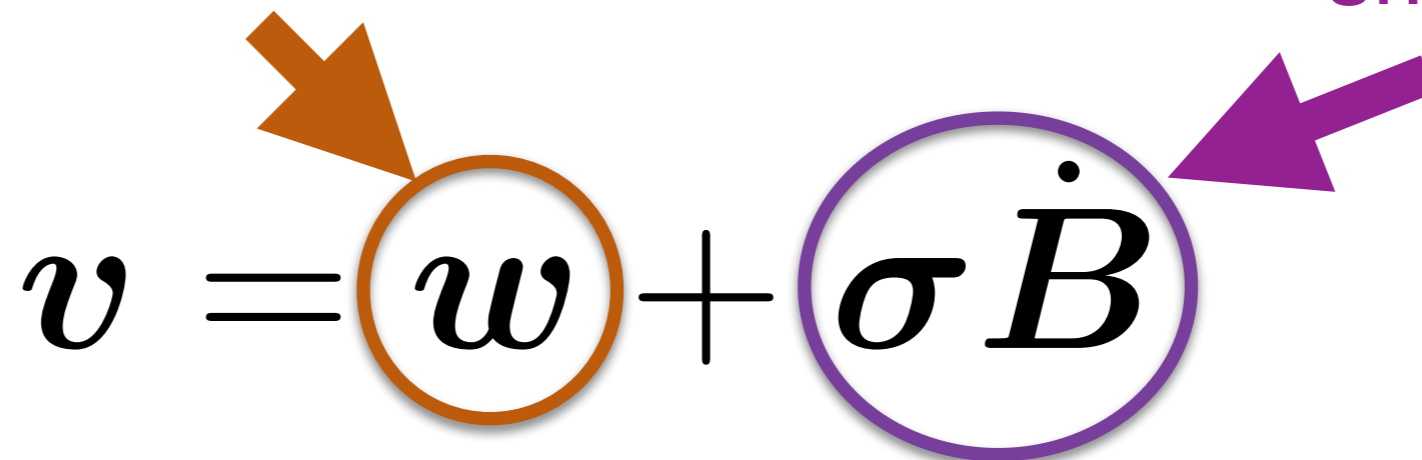
$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# SALT & LU :

## Adding random velocity

Resolved  
large scales

White-in-time  
small scales

$$v = w + \sigma \dot{B}$$
The diagram shows the equation  $v = w + \sigma \dot{B}$ . The term  $w$  is enclosed in a brown circle, and a brown arrow points from the text "Resolved large scales" to it. The term  $\sigma \dot{B}$  is enclosed in a purple circle, and a purple arrow points from the text "White-in-time small scales" to it.

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

# SALT & LU :

## Adding random velocity

$$d\mathbf{X}_t = \mathbf{w}^*(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

# SALT & LU :

## Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# SALT & LU :

## Adding random velocity

$$d\mathbf{X}_t = \overset{\text{Stratonovich}}{w^* (\mathbf{X}_t, t) dt + \sigma (\mathbf{X}_t, t) \circ d\mathbf{B}_t}$$

$$= \underset{\text{Ito}}{w (\mathbf{X}_t, t) dt + \sigma (\mathbf{X}_t, t) d\mathbf{B}_t}$$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# SALT & LU :

## Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

References : Mikulevicius and Rozovskii, 2004  
Flandoli, 2011

**Memin**, 2014  
Resseguier et al. 2017 a, b, c  
Chapron et al. 2017  
Cai et al. 2017

**Holm**, 2015  
Holm and Tyranowski, 2016  
Arnaudon et al., 2017

Cotter and al 2017  
Crisan et al., 2017  
Gay-Balmaz & Holm 2017  
Cotter and al 2018 a, b

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# SALT & LU :

## Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

References : Mikulevicius and Rozovskii, 2004  
Flandoli, 2011

### LU

**Memin**, 2014  
Resseguier et al. 2017 a, b, c  
Chapron et al. 2017  
Cai et al. 2017

### SALT

<b>Holm</b> , 2015	Cotter and al 2017
Holm and Tyranowski, 2016	Crisan et al., 2017
Arnaudon et al., 2017	Gay-Balmaz & Holm 2017
	Cotter and al 2018 a, b



Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Advection of tracer $\Theta$

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Advection of tracer $\Theta$

Ito-Wentzell  
formula  
(Kunita 1990)

$$\frac{D\Theta}{Dt} = 0$$

# Advection of tracer $\Theta$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell  
formula  
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# Advection of tracer $\Theta$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Ito-Wentzell  
formula  
(Kunita 1990)

Stratonovich notations:

# Advection of tracer $\Theta$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Ito-Wentzell  
formula  
(Kunita 1990)

Stratonovich notations:

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

# Advection of tracer $\Theta$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell  
formula  
(Kunita 1990)

Ito notations:

# Advection of tracer $\Theta$

Ito-Wentzell  
formula  
(Kunita 1990)

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$

# Advection of tracer $\Theta$

Ito-Wentzell  
formula  
(Kunita 1990)

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta}_{\text{Advection}} = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$



# Advection of tracer $\Theta$

Ito-Wentzell  
formula  
(Kunita 1990)

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}}$$

# Advection of tracer $\Theta$

Ito-Wentzell  
formula  
(Kunita 1990)

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + \underbrace{w^*}_{\text{Advection}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)_{\text{Diffusion}}$$

Stratonovich drift : « Drift correction »  
in Ito notations

# Advection of tracer $\Theta$

Ito-Wentzell formula  
(Kunita 1990)

Large scales:  
 $w$

Small scales:  
 $\sigma \dot{B}$

Variance tensor:  
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^*}_{\text{Stratonovich drift}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$

Advection

Diffusion

Stratonovich drift : « Drift correction »  
in Ito notations

# Advection of tracer $\Theta$

Ito-Wentzell formula  
(Kunita 1990)

Large scales:  
 $w$

Small scales:  
 $\sigma \dot{B}$

Variance tensor:  
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction » in Ito notations

# Advection of tracer $\Theta$

Ito-Wentzell formula  
(Kunita 1990)

Large scales:  
 $w$   
Small scales:  
 $\sigma \dot{B}$   
Variance tensor:  
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction » in Ito notations

# Advection of tracer $\Theta$

Ito-Wentzell formula  
(Kunita 1990)

Large scales:  
 $w$   
Small scales:  
 $\sigma \dot{B}$   
Variance tensor:  
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Multiplicative random forcing

Balanced energy exchanges

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction »  
in Ito notations

Large scales:

$$\mathbf{w}$$

Small scales:

$$\sigma \dot{B}$$

Variance tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

# Moments of a passive tracer $\Theta$

$$\Theta(\bullet, t') \perp \mathbf{w}(\bullet, t), \forall t' \leq t$$

$$\begin{aligned} \partial_t \mathbb{E}(\Theta) + \mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta) &= \nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right) \\ \partial_t \text{Var}(\Theta) + \mathbf{w}^* \cdot \nabla \text{Var}(\Theta) &= \nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \text{Var}(\Theta) \right) + (\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta) \end{aligned}$$

The diagram illustrates the evolution of the mean and variance of a passive tracer  $\Theta$ . The first equation shows the time derivative of the mean  $\mathbb{E}(\Theta)$  is equal to the advection term  $\mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta)$  plus the diffusion term  $\nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right)$ . The second equation shows the time derivative of the variance  $\text{Var}(\Theta)$  is equal to the advection term  $\mathbf{w}^* \cdot \nabla \text{Var}(\Theta)$  plus the diffusion term  $\nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \text{Var}(\Theta) \right)$  and a source term  $(\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta)$ .

Large scales:

$$\mathbf{w}$$

Small scales:

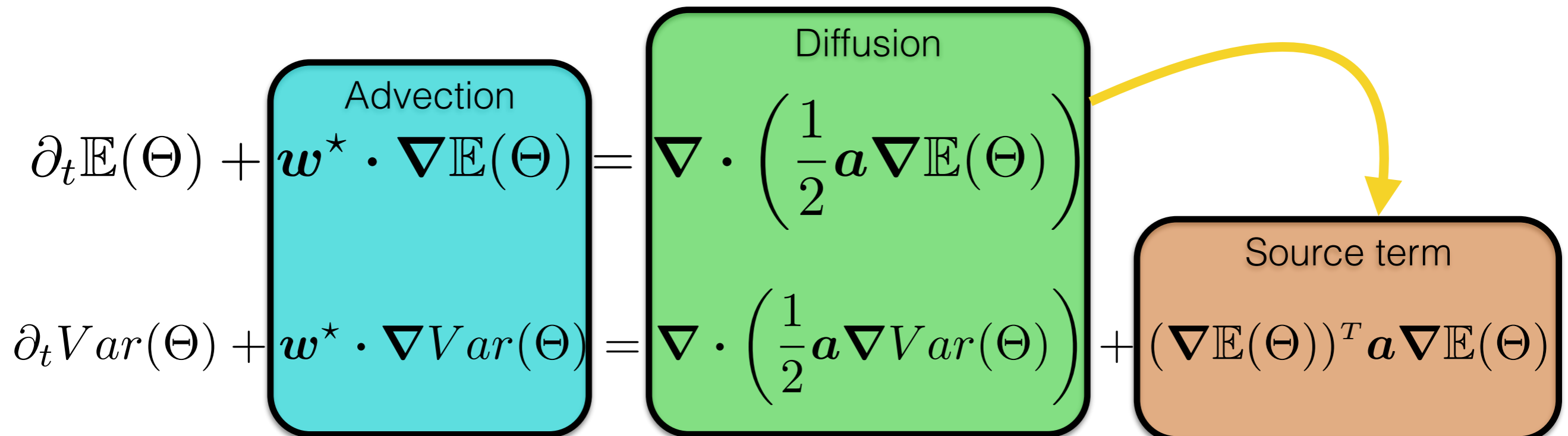
$$\sigma \dot{B}$$

Variance tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

# Moments of a passive tracer $\Theta$

$$\Theta(\bullet, t') \perp \mathbf{w}(\bullet, t), \forall t' \leq t$$





Large scales:

$w$

Small scales:

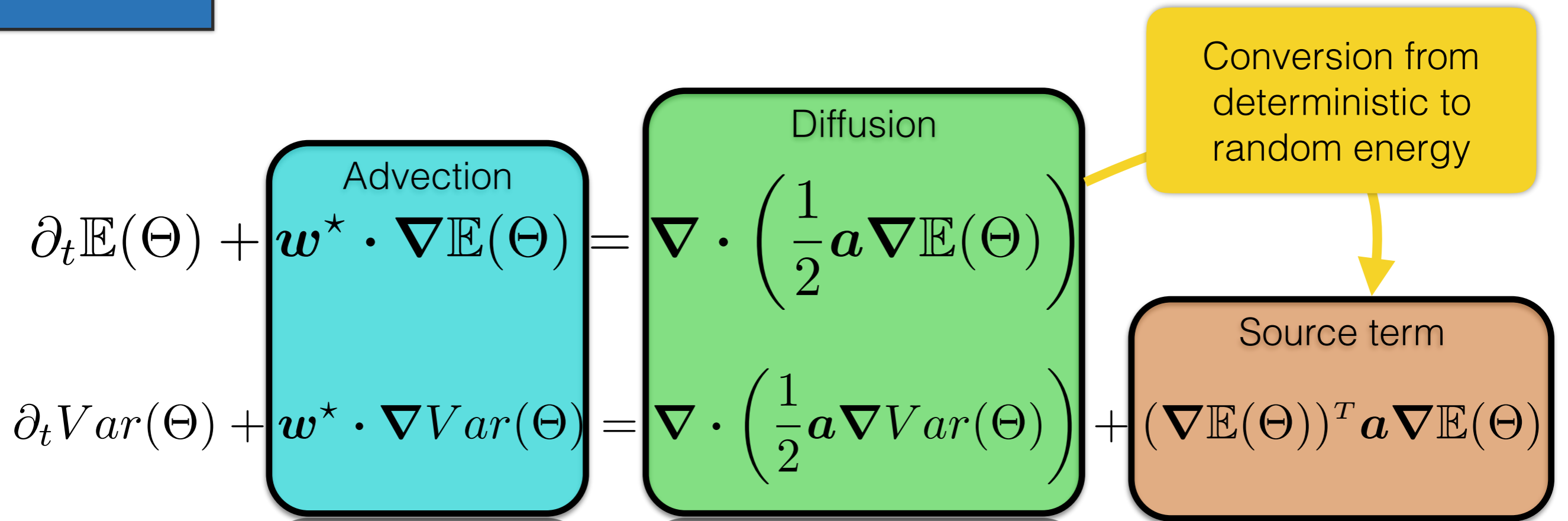
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Moments of a passive tracer $\Theta$

$$\Theta(\bullet, t') \perp\!\!\!\perp w(\bullet, t), \forall t' \leq t$$



Large scales:

$$\mathbf{w}$$

Small scales:

$$\sigma \dot{\mathbf{B}}$$

Variance tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

# Moments of a passive tracer $\Theta$

$$\Theta(\bullet, t') \perp \mathbf{w}(\bullet, t), \forall t' \leq t$$

Conservation of the energy  $\frac{d}{dt} \int_{\Omega} \Theta^2 = 0$

Advection  $\partial_t \mathbb{E}(\Theta) + \mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta) =$

Diffusion  $\nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right)$

Conversion from deterministic to random energy

$\partial_t Var(\Theta) + \mathbf{w}^* \cdot \nabla Var(\Theta) =$

$\nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla Var(\Theta) \right) +$

Source term  $(\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta)$

# Part II

## Stochastic Navier-Stokes

### SALT vs LU

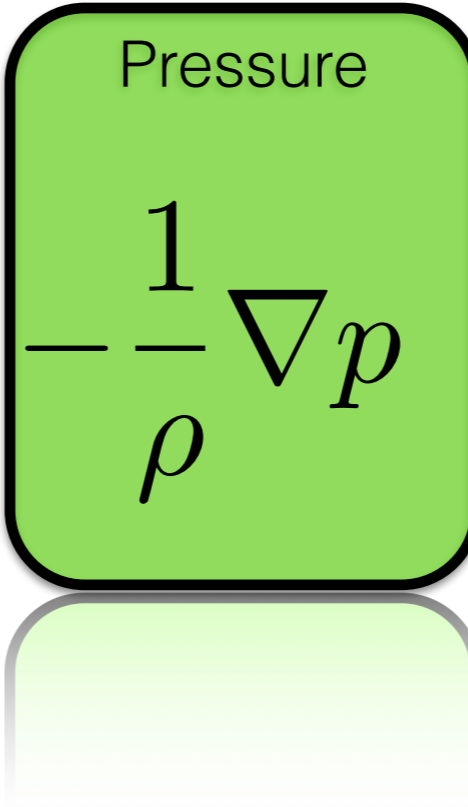
# Comparison

	LU	SALT
Scalar (e.g. SQG)	Identical	
Navier-Stokes	2 differences	
Vorticity	2 differences	
Kinetic energy conservation	✓	✗
Helicity (& 2D enstrophy) conservation	✗	✓

# Navier-Stokes LU

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \nabla p$$

# Navier-Stokes LU

$$\frac{Dw}{Dt} = \text{Pressure} \left[ -\frac{1}{\rho} \nabla p \right]$$


# Navier-Stokes LU

Stochastic  
transport  
of Itô drift

$$\frac{Dw}{Dt} =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

# Navier-Stokes LU

The diagram consists of two rounded rectangular boxes side-by-side. The left box is cyan and contains the text 'Stochastic transport of Ito drift' and the equation  $\frac{Dw}{Dt} =$ . The right box is green and contains the text 'Pressure' and the equation  $-\frac{1}{\rho} \nabla p$ . Below these boxes are two faint, semi-transparent versions of the same boxes. A purple arrow points upwards from the left faint box towards the cyan box.

Stochastic transport of Ito drift

$$\frac{Dw}{Dt} =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

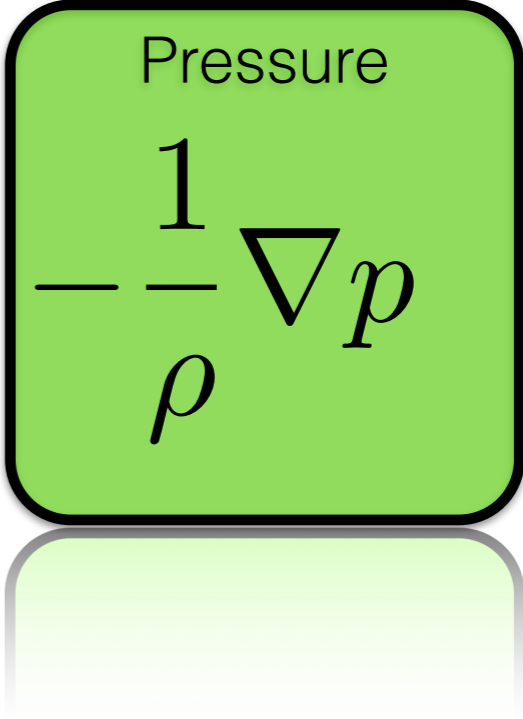
Conserve (Ito) kinetic energy



# Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

# Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = \text{Pressure} \frac{1}{\rho} \nabla p$$


# Navier-Stokes SALT

Stochastic  
transport of  
**Stratonovich** drift

$$\frac{Dw^*}{Dt}$$

$$+ \nabla (\sigma \circ \dot{B})^T w^* =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

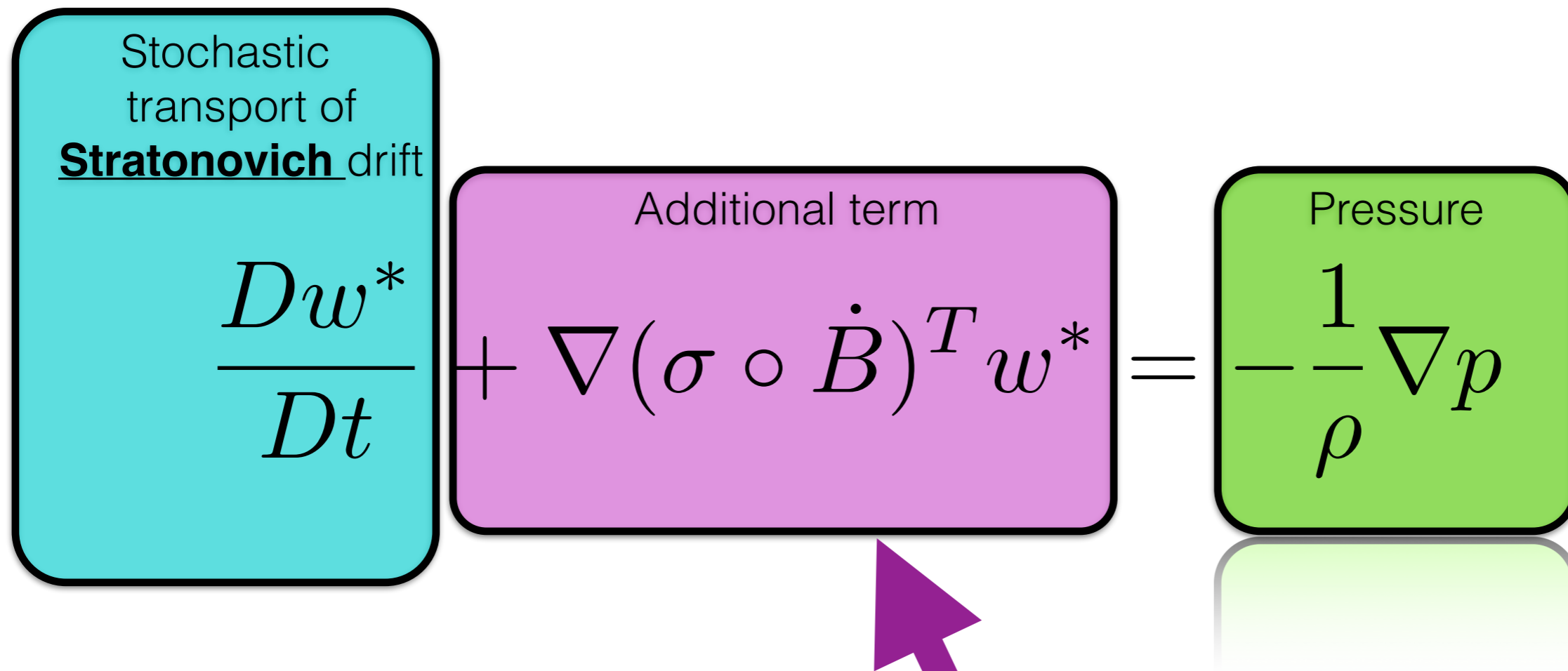
# Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Additional term

Pressure



Conserve (Stratonovich) helicity

# A word about reduced order models LU

- Very fast simulation of very complex system (e.g. for industrial application)
- Physical model (PDE) simplified using observations

## Forecast of reduced models LU

- Rigorous and low-cost estimators
- Stabilization of the unstable modes
- Maintain variability of stable modes
- Uncertainty quantification

## State of art

- Possible parametrization with eddy viscosity
- Impossible to parametrize with additional dissipation  
Need ad hoc closure like MQG  
(Sapsis and Majda, 2013a,b,c)

# Part III

## Unresolved velocity parametrisation

*Code available online*

Large scales:

$$w$$

Small scales:

$$\sigma \dot{B}$$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Parameter-free models for $\sigma$

	Cotter et al. 2018b	Resseguier et al. 2017b
<b>Applicable to :</b>	LU & SALT	LU & SALT
<b>Homogeneous</b>	No	Yes
<b>Stationary</b>	Yes	Not anymore
<b>Self-similar assumption</b>	No	Yes
<b>Data-driven</b>	Yes	No

# Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$



Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid  
Brownian  
motion

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Small scales:

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Variance  
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$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

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weighted  
EOFs learned  
on data

iid  
Brownian  
motion

# Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

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$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid  
Brownian  
motion

weighted  
EOFs learned  
on data

**PCA on :**  $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

# Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid  
Brownian  
motion

weighted  
EOFs learned  
on data

**PCA on :**  $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

Flow from **high-**  
resolution velocity

Flow from **low-**  
resolution velocity

(improvement of)  
 Resseguier et al. 2017b:  
 Self-similar model

Large scales:

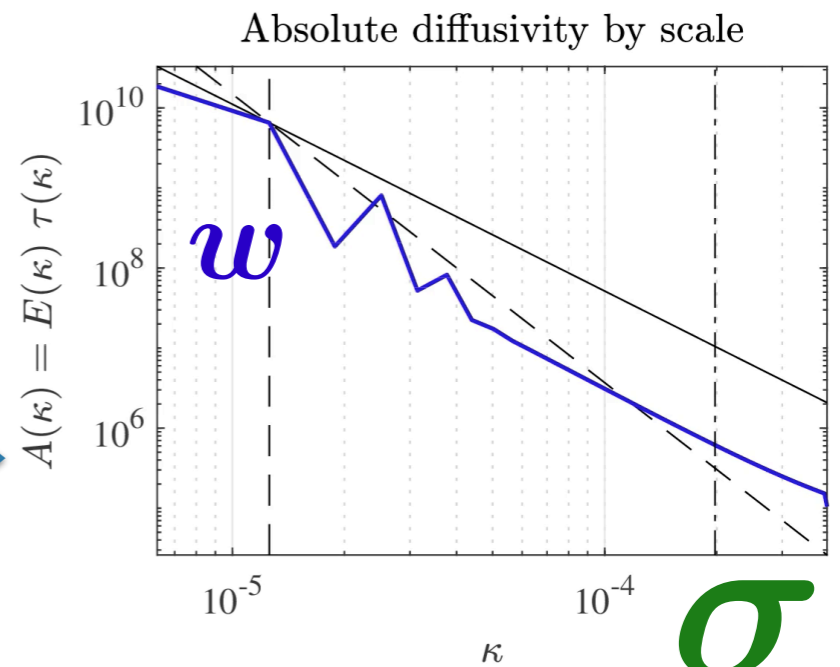
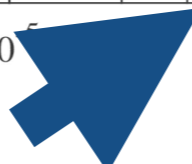
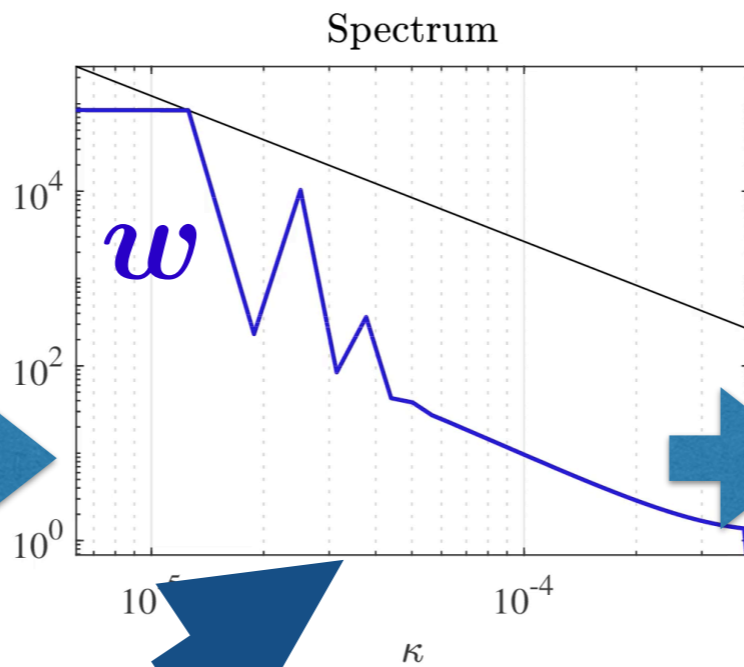
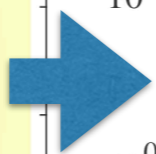
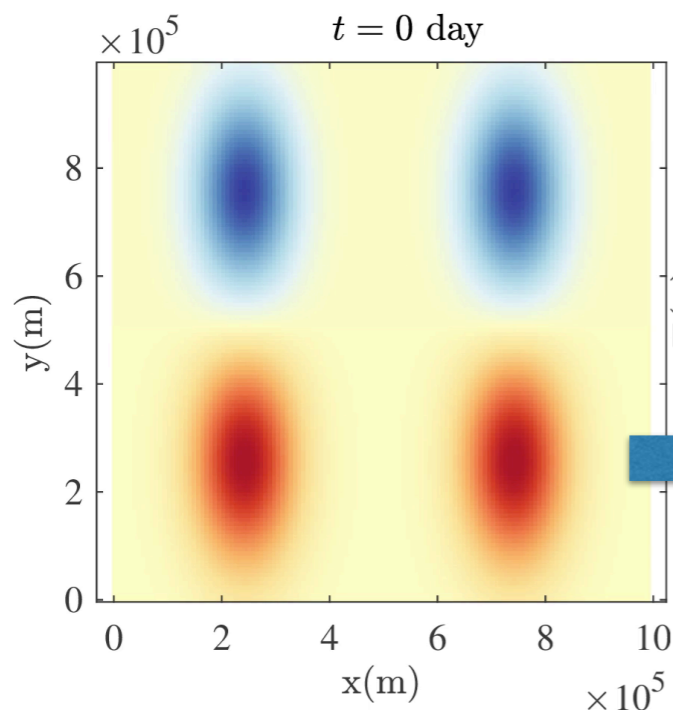
$w$

Small scales:

$\sigma \dot{B}$

Variance  
 tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$



Kinetic energy  
 Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \int_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

(improvement of)  
 Resseguier et al. 2017b:  
 Self-similar model

Large scales:

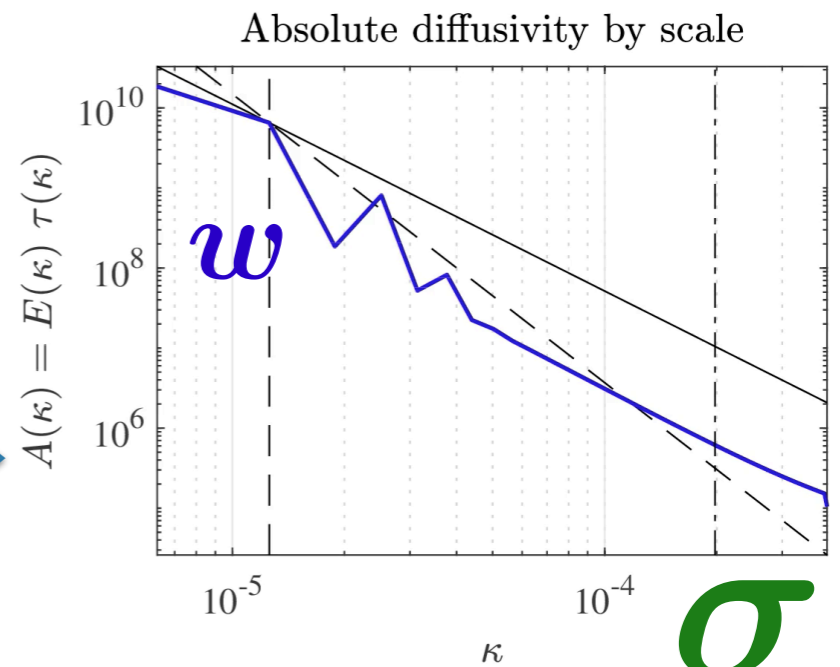
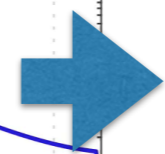
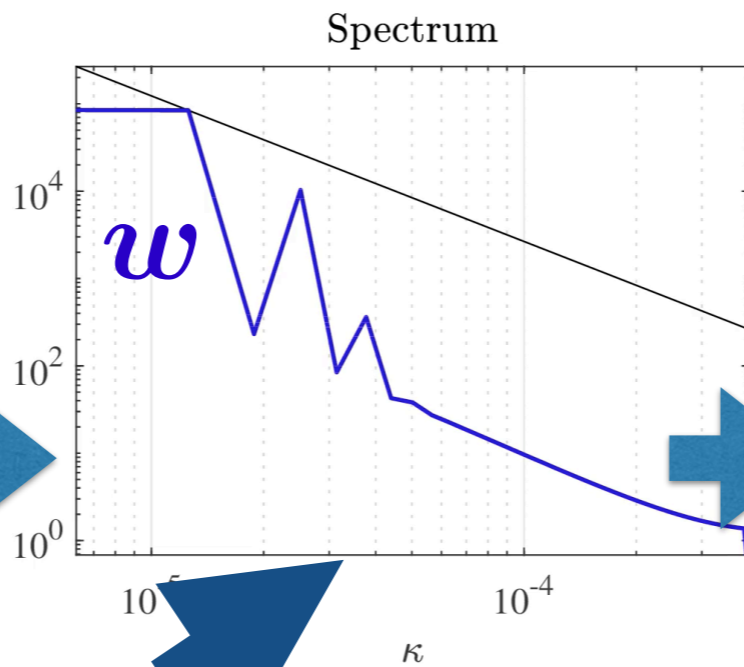
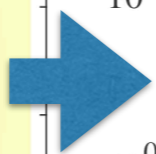
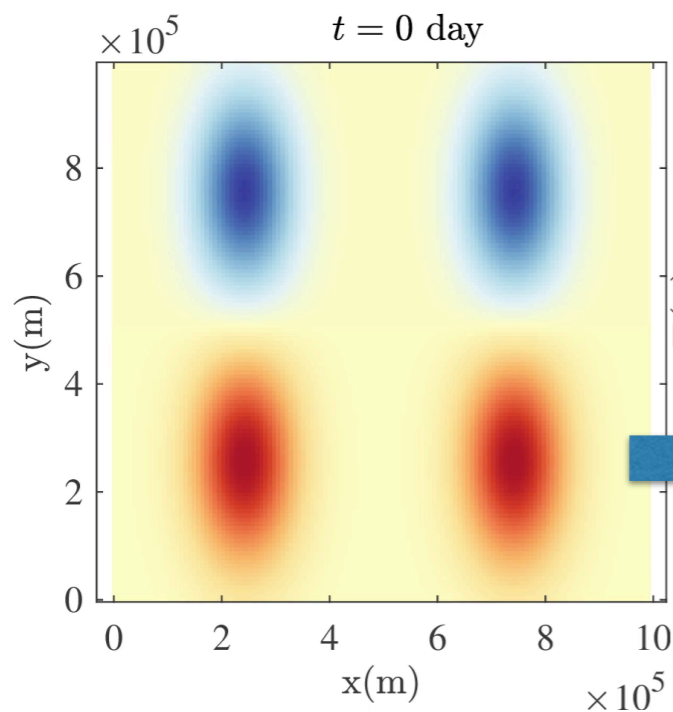
$w$

Small scales:

$\sigma \dot{B}$

Variance  
 tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$



Kinetic energy  
 Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \int_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

# (improvement of) Resseguier et al. 2017b: Self-similar model

Large scales:

$w$

Small scales:

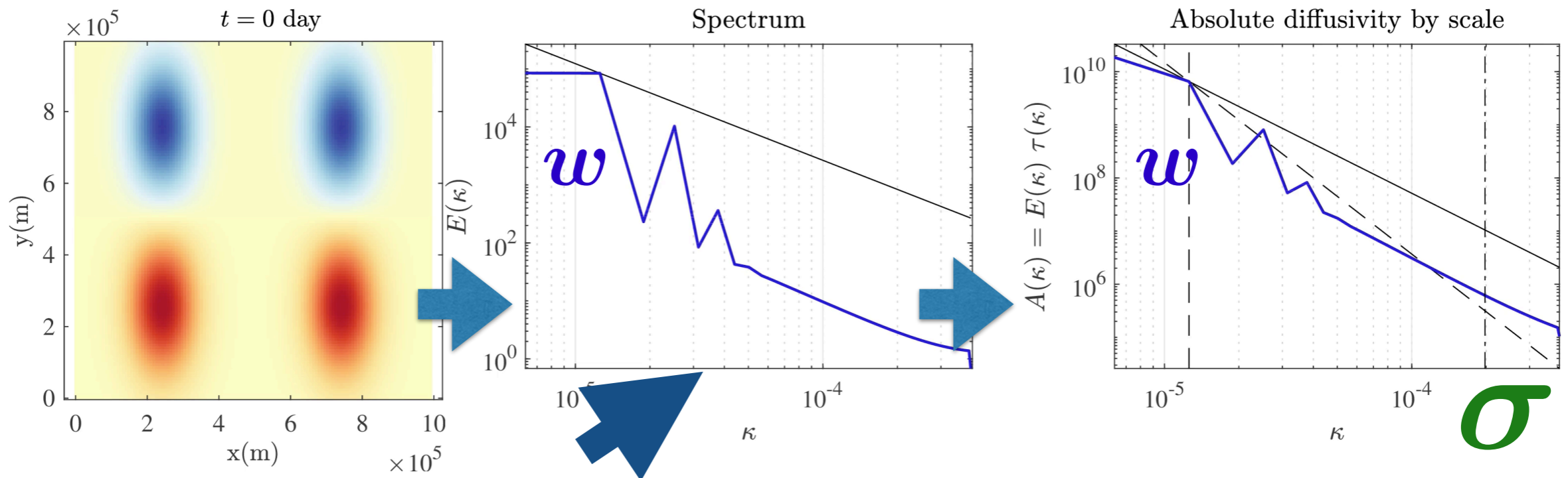
$\sigma \dot{B}$

Variance  
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Absolute Diffusivity  
Spectral Density

$$A(\kappa) = E(\kappa)\tau(\kappa) = \kappa^{-3/2} E^{1/2}(\kappa)$$



Kinetic energy  
Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \int_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

$t = 100$  day

## SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$-\frac{1}{\tau} (b - F) \quad \text{Forcing}$$

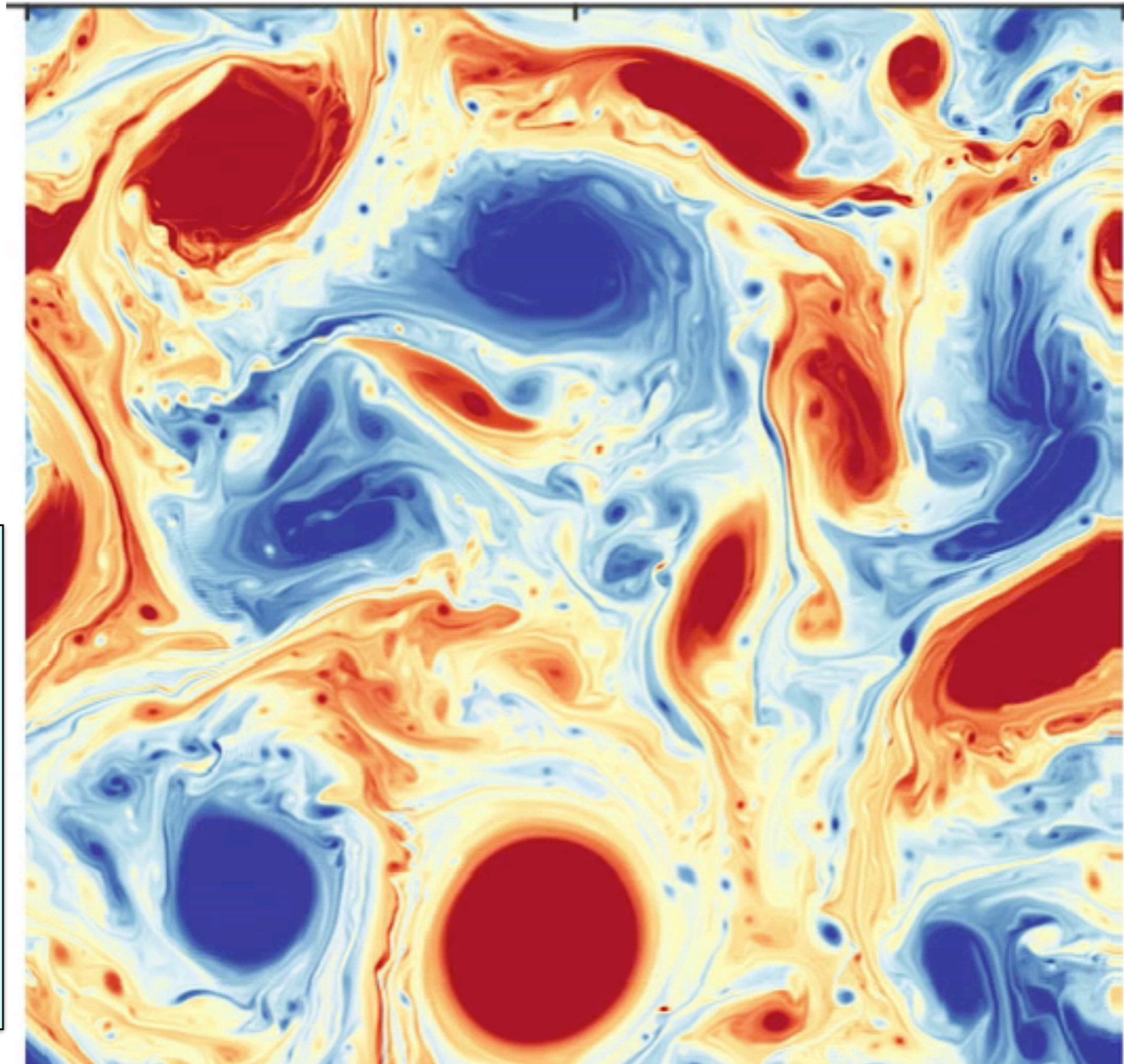
$$u = \left( \text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

deterministic

SQG

512 x 512





$t = 100$  day

## SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$- \frac{1}{\tau} (b - F) \quad \text{Forcing}$$

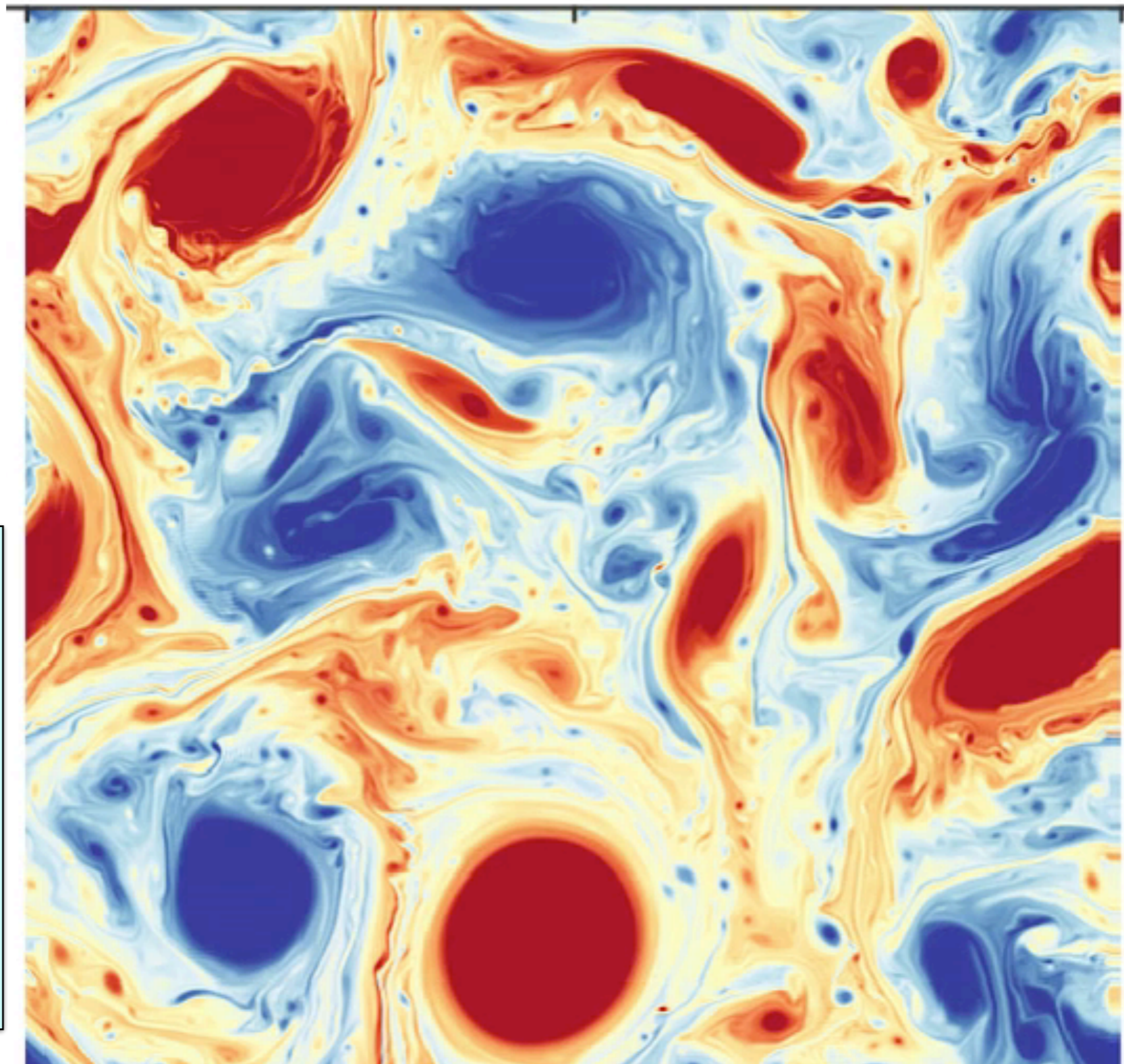
$$u = \left( \text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

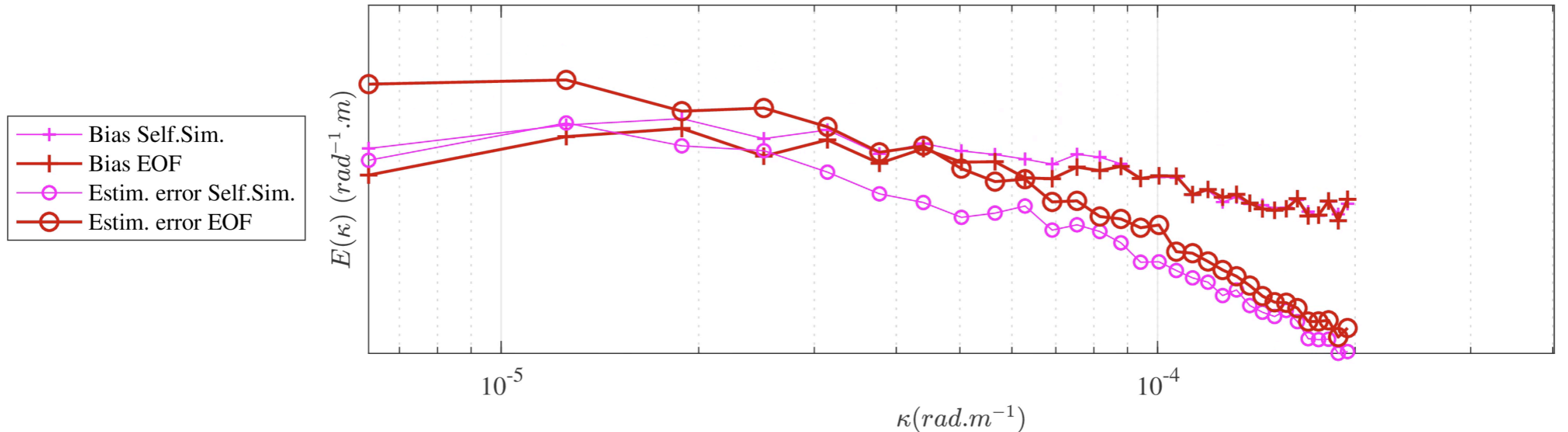
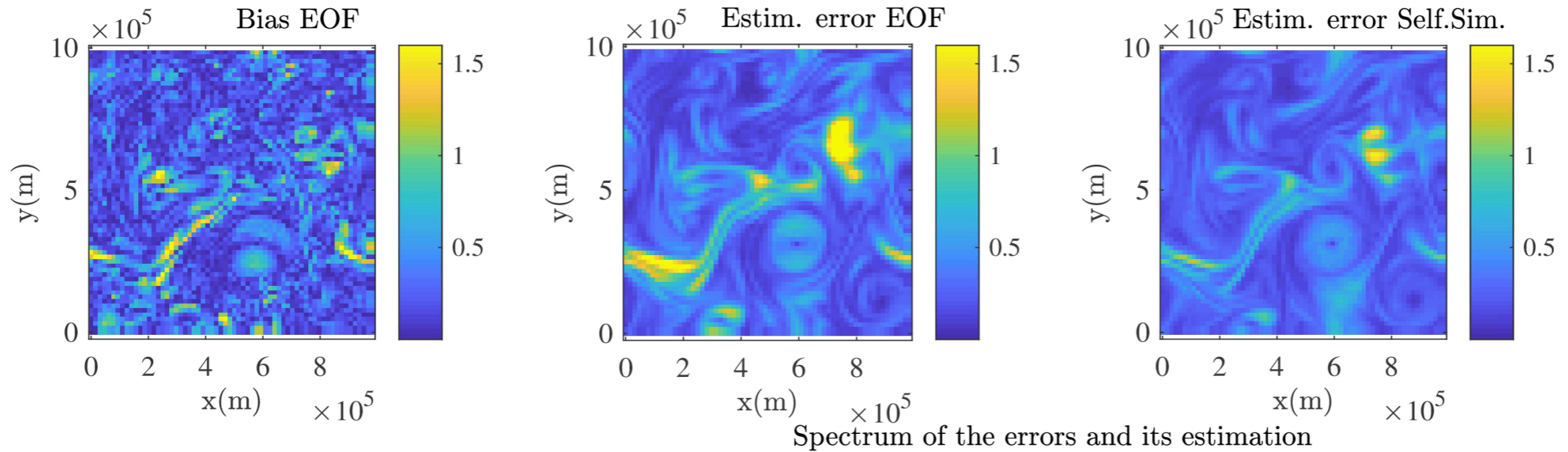
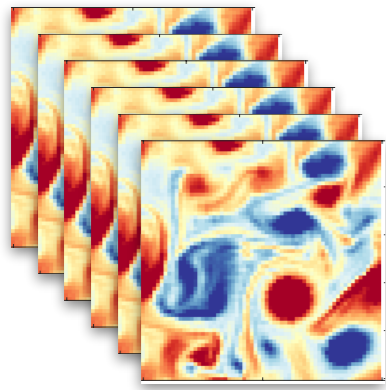
deterministic

SQG

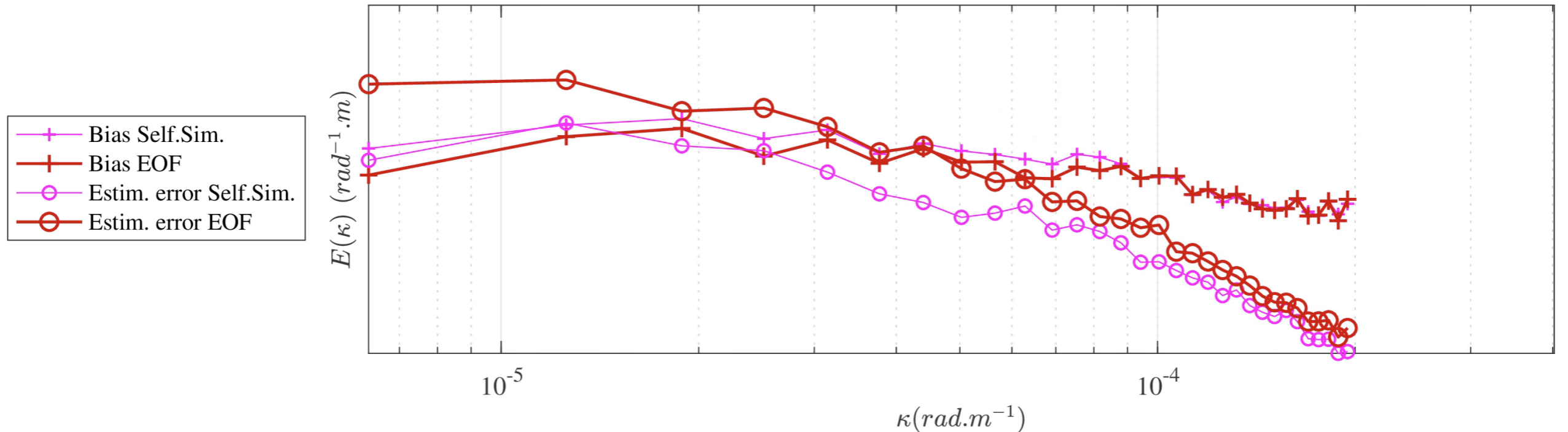
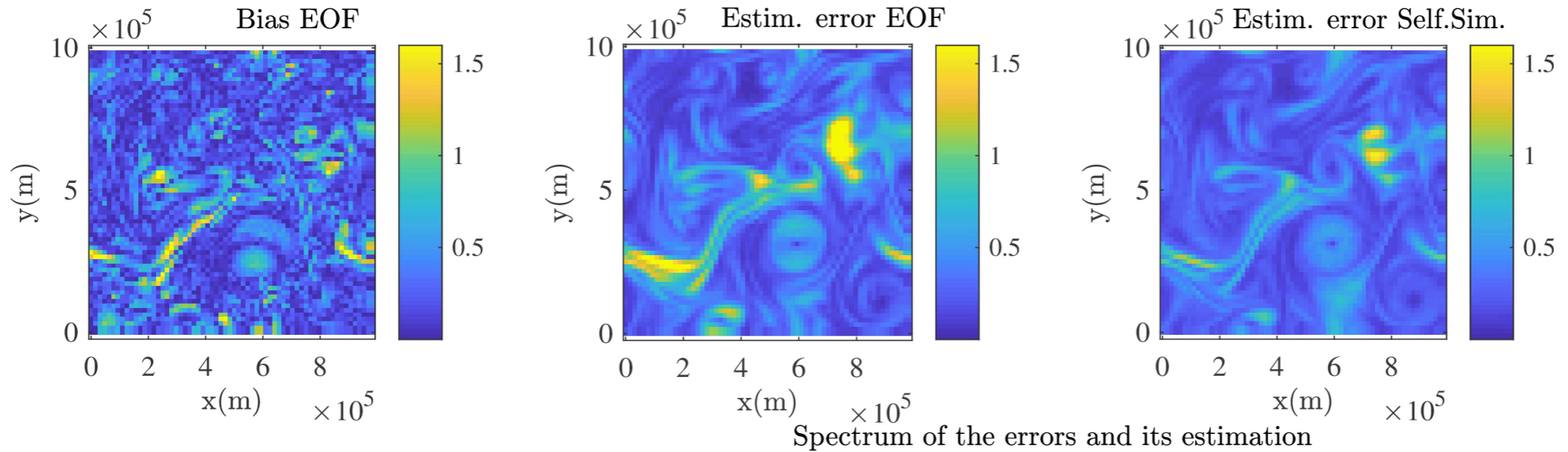
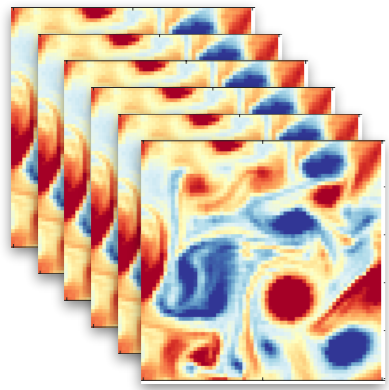
512 x 512

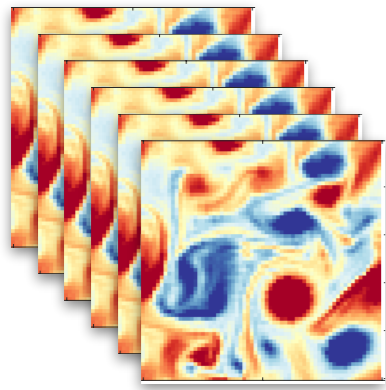


# Ensemble : uncertainty quantification



# Ensemble : uncertainty quantification



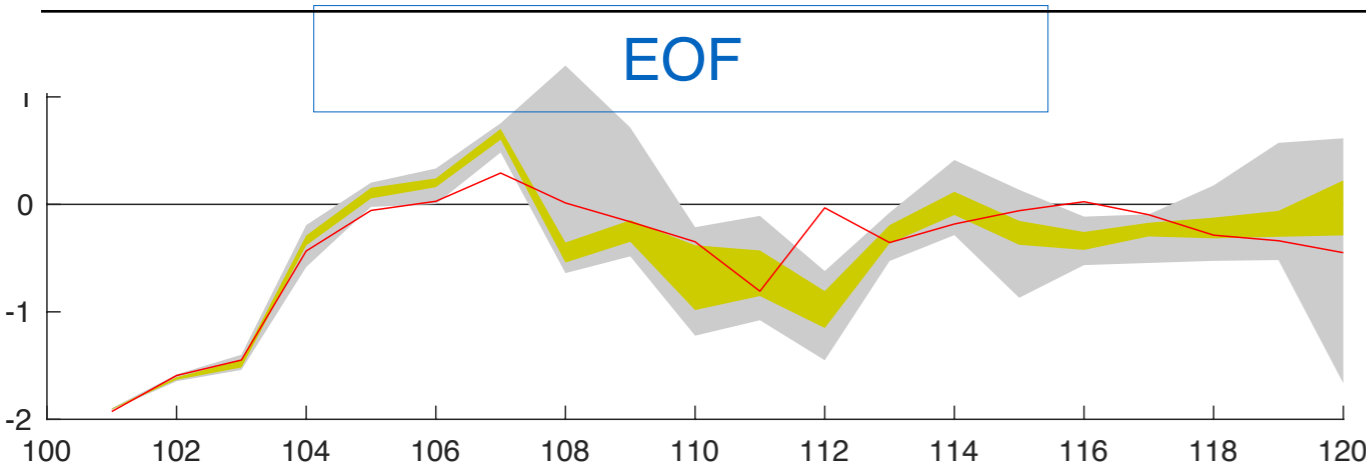


**20**  
**realisations**

# Ensemble : uncertainty quantification

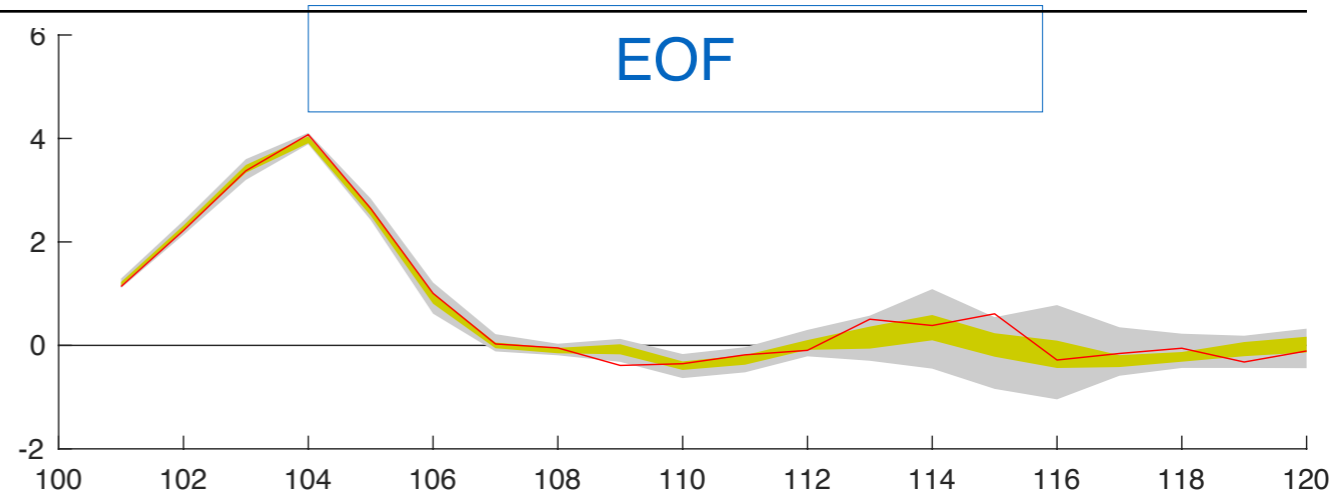
At point (250km,250km)

EOF

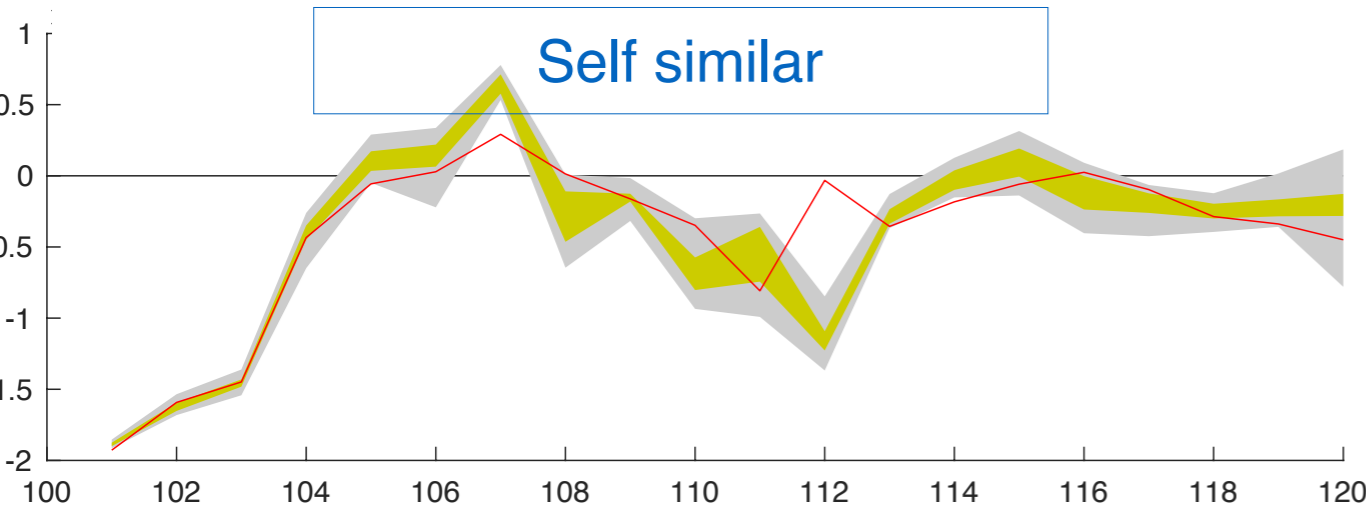


At point (500km,250km)

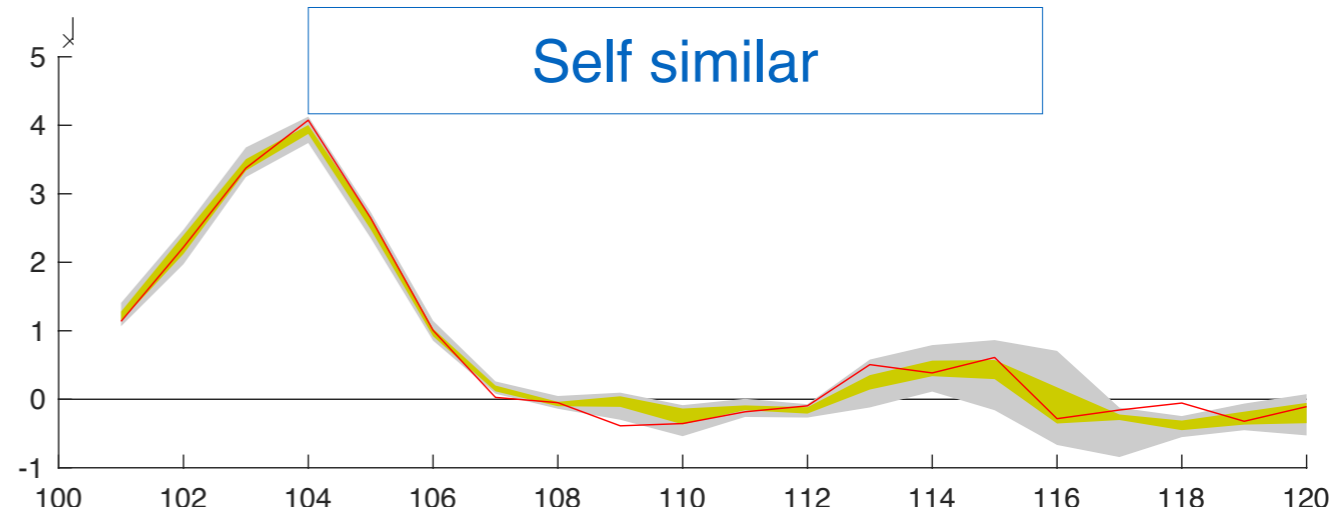
EOF

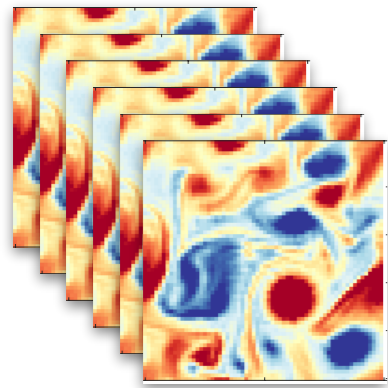


Self similar



Self similar





**20**  
**realisations**

# Ensemble : uncertainty quantification

At point (250km,500km)

EOF

Self similar

At point (500km,500km)

EOF

Self similar

# Part IV

Unresolved velocity  
non-stationary heterogeneity

$t = 0$  day

## SQG

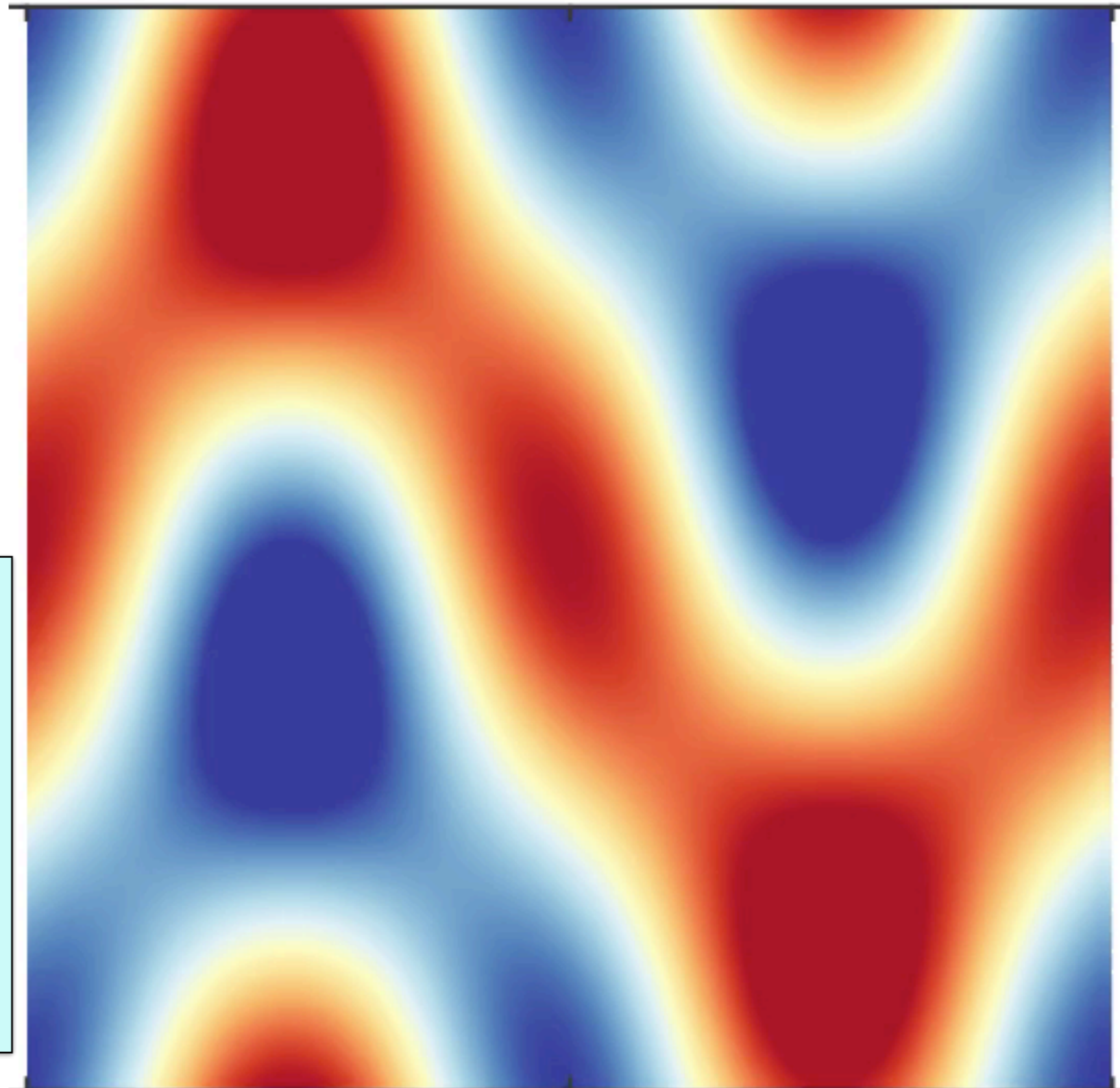
$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG  
1024 x 1024

Constantin et al.  
(1994, 1999, 2012)



$t = 0$  day

## SQG

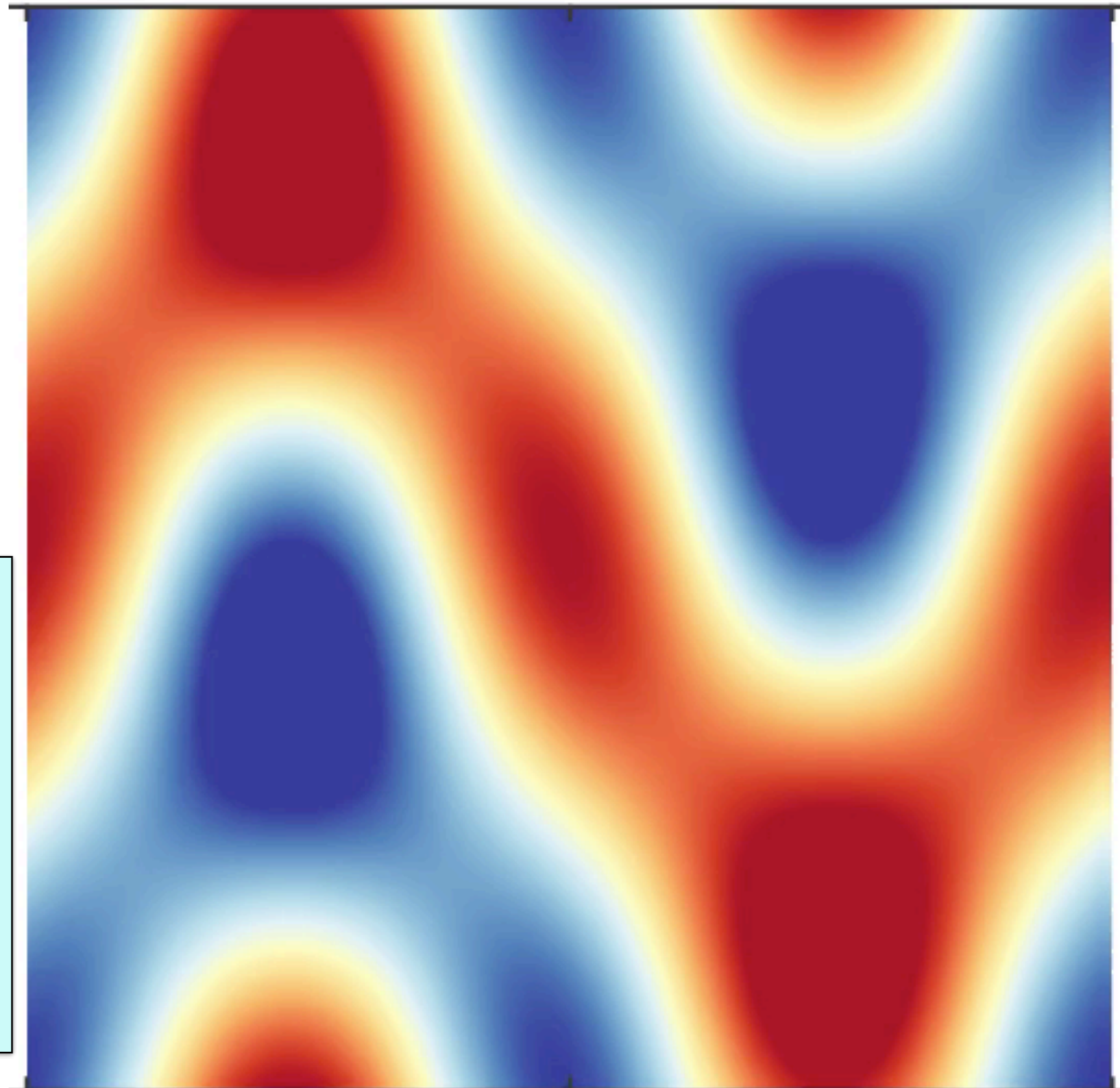
$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG  
1024 x 1024

Constantin et al.  
(1994, 1999, 2012)





$t = 15 \text{ day}$

## SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

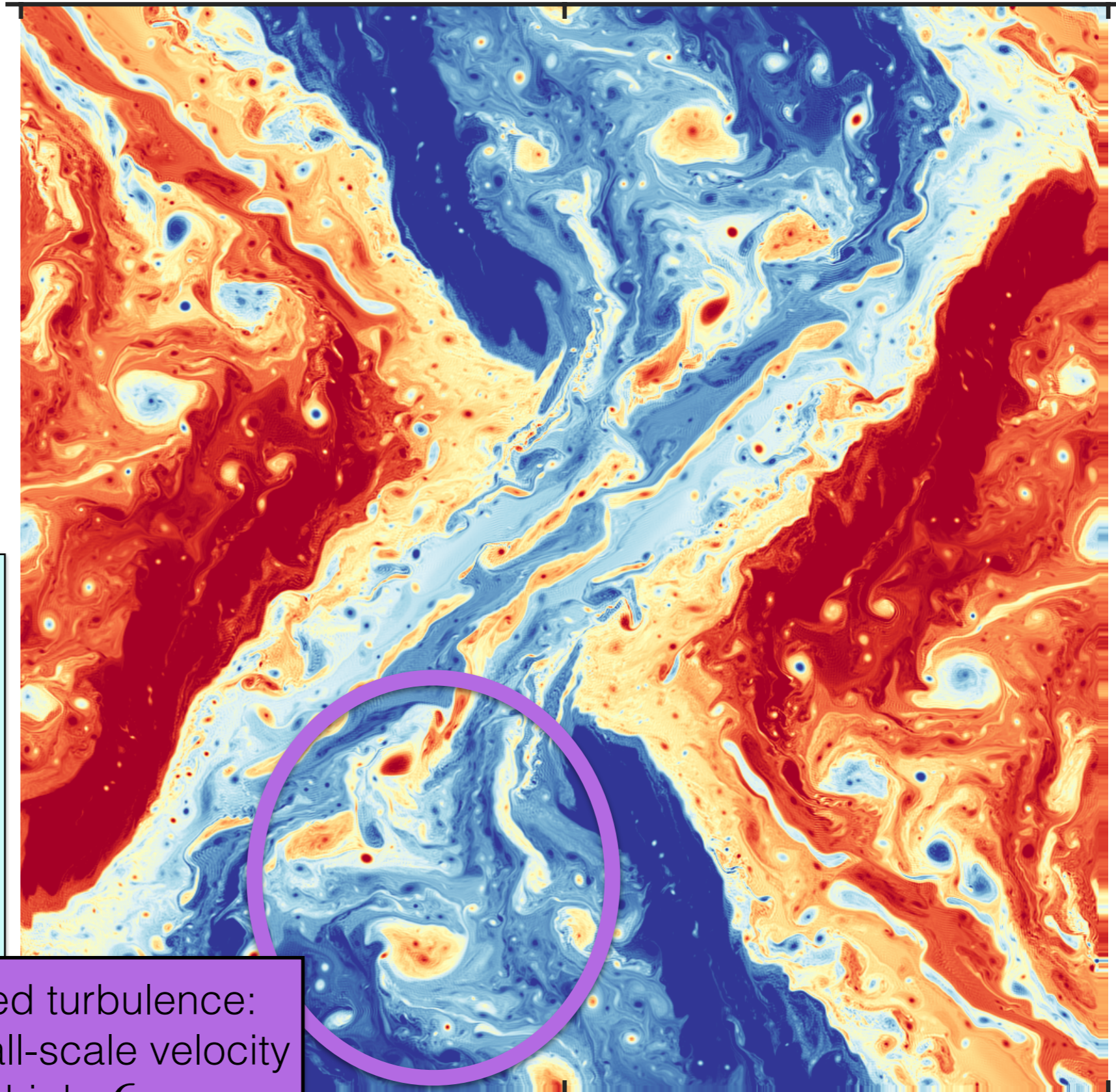
$$\mathbf{u} = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG  
1024 x 1024

Constantin et al.  
(1994, 1999)

Developed turbulence:  
strong small-scale velocity  
high  $\epsilon$



$t = 15$  day

## SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

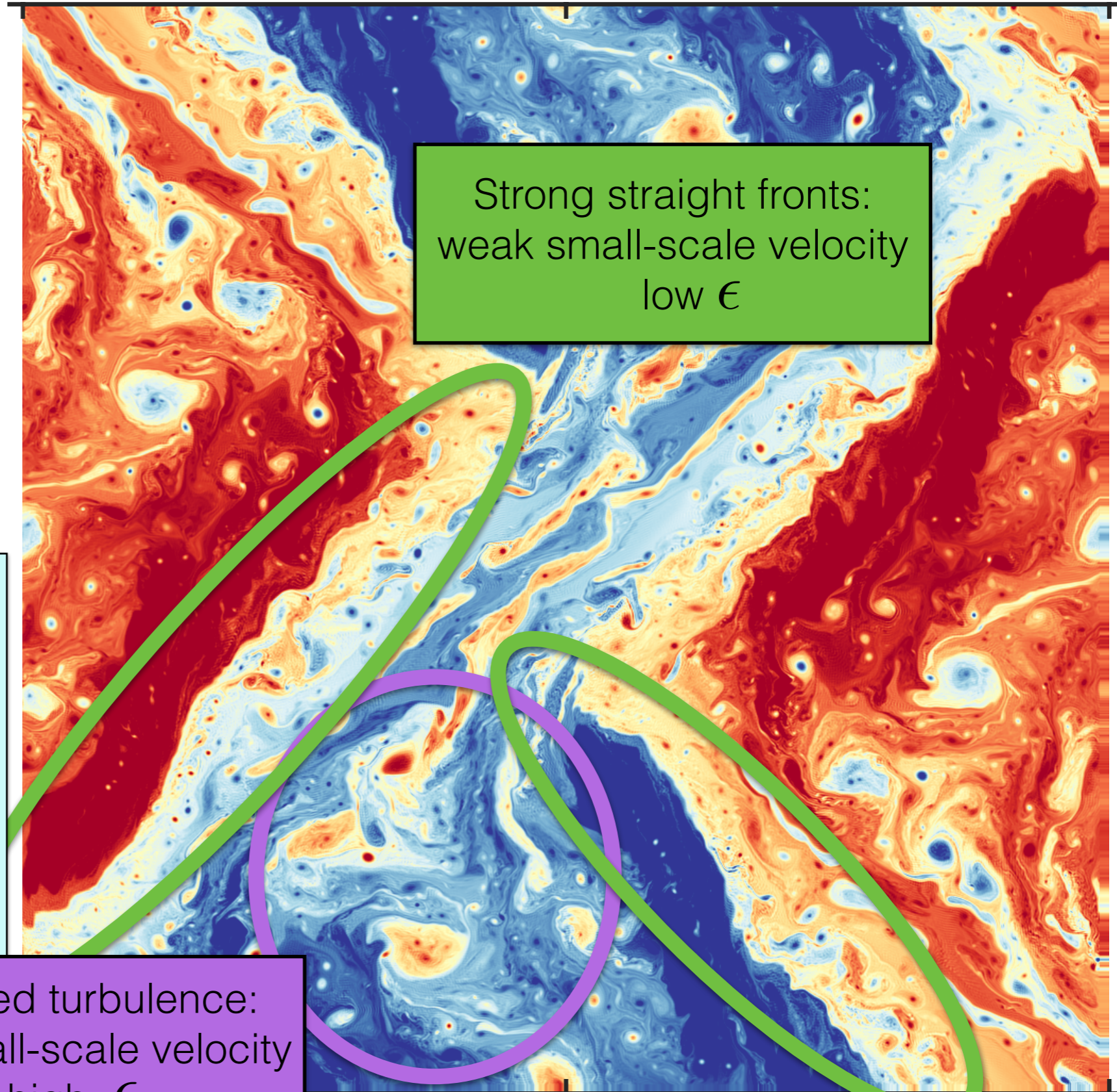
Reference flow:

deterministic SQG  
1024 x 1024

Constantin et al.  
(1994, 1999)

Developed turbulence:  
strong small-scale velocity  
high  $\epsilon$

Strong straight fronts:  
weak small-scale velocity  
low  $\epsilon$



Large scales:

$w$

Small scales:

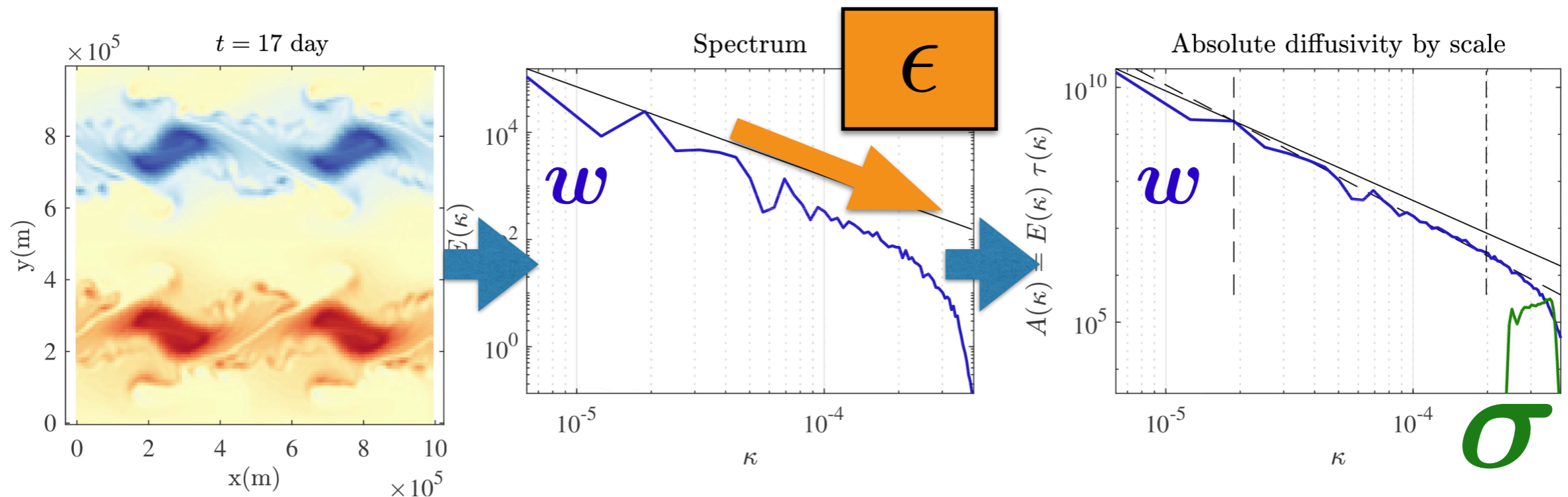
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Heterogeneous modulation of $\sigma$

(for Resseguier et al. 2017b)



Large scales:

$w$

Small scales:

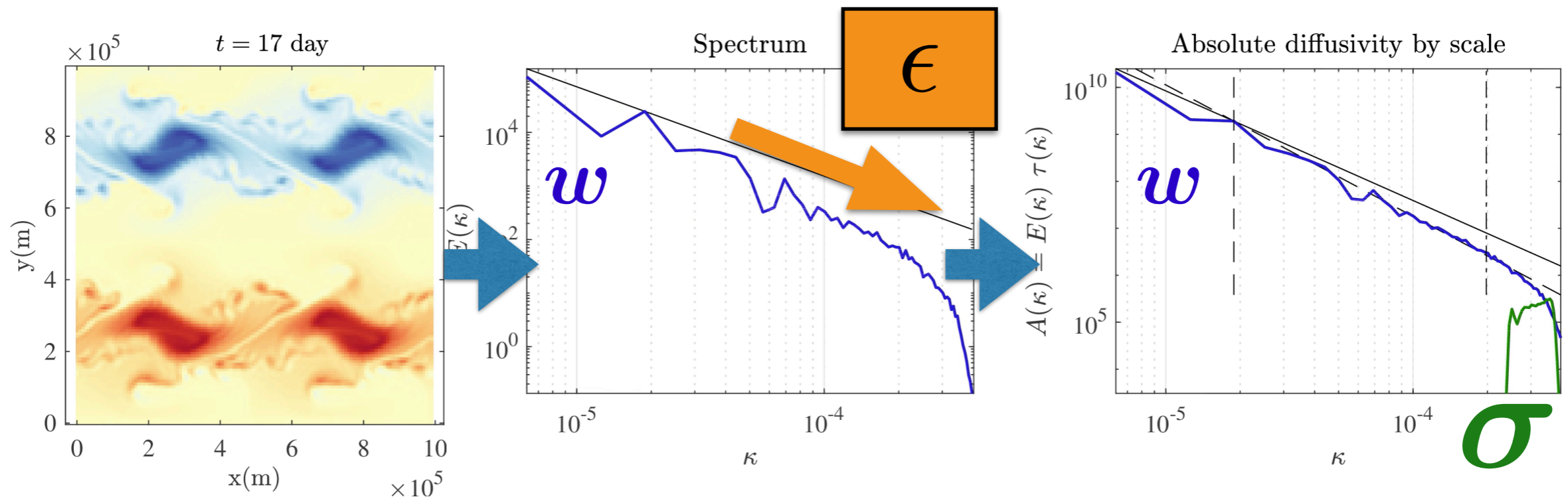
$\sigma \dot{B}$

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# Heterogeneous modulation of $\sigma$

(for Resseguier et al. 2017b)



Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance tensor:

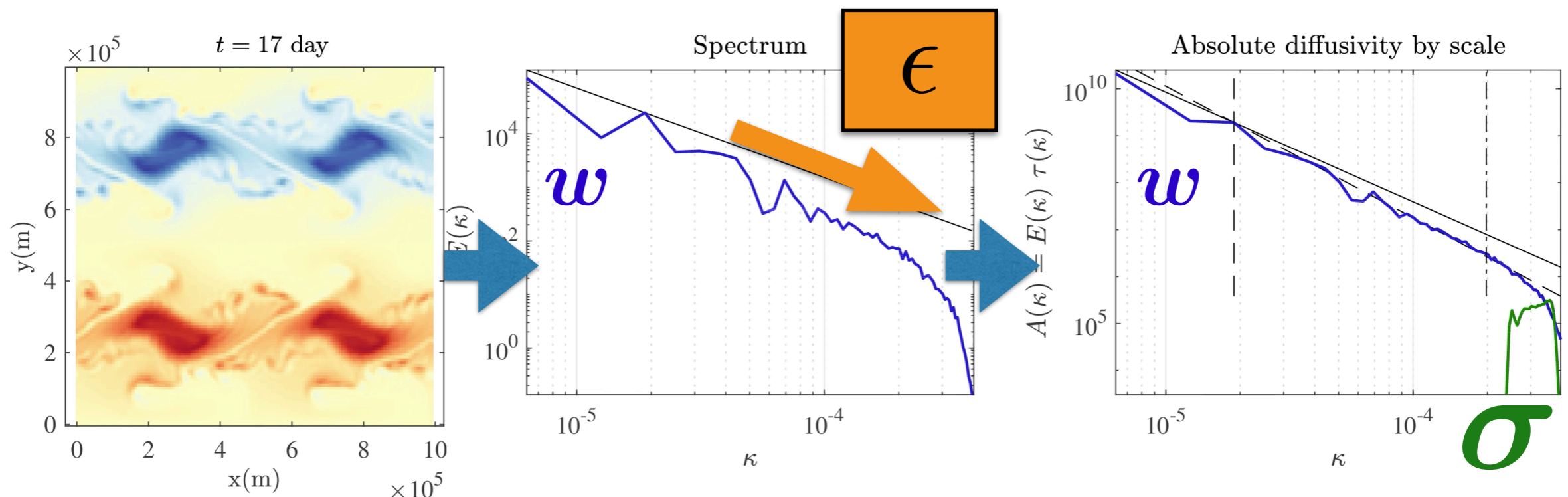
$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Heterogeneous

# modulation of $\sigma$

(for Resseguier et al. 2017b)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst.} \epsilon^p(x) \kappa^{-q}$$



Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

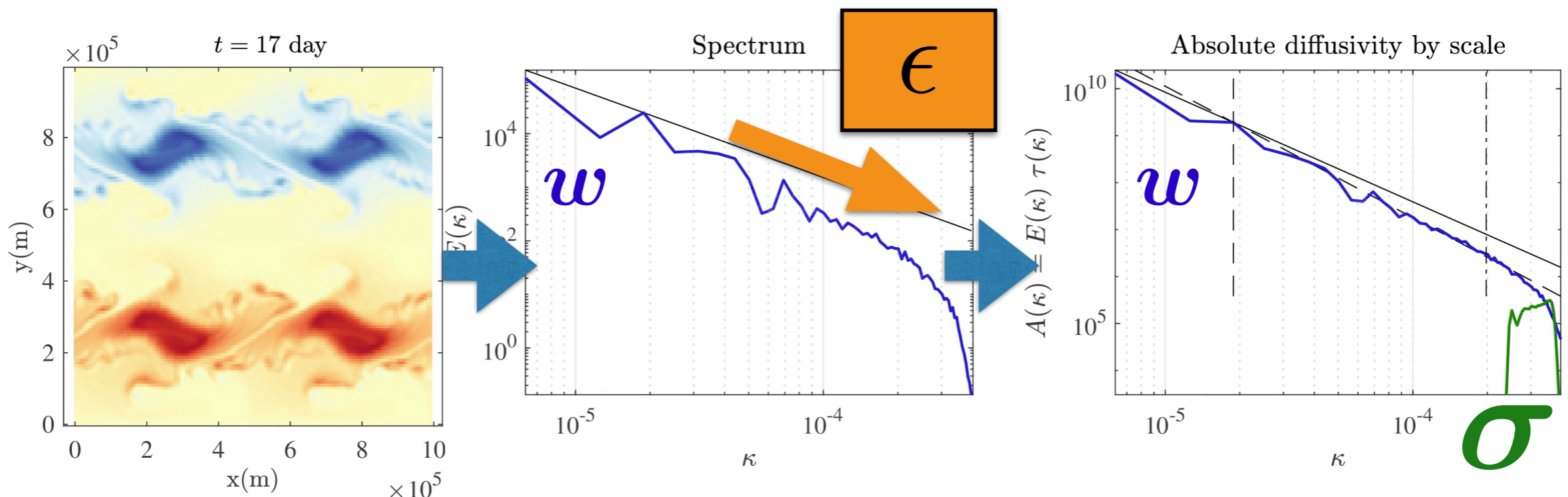
# Heterogeneous

# modulation of $\sigma$

(for Resseguier et al. 2017b)

Dynamics  
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst.} \cdot \epsilon^p(x) \kappa^{-q}$$



Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

# Heterogeneous

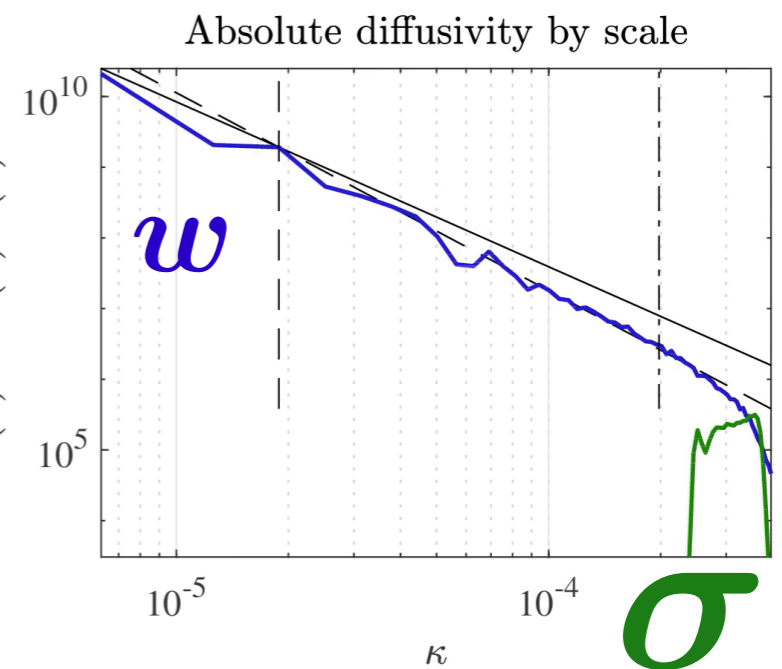
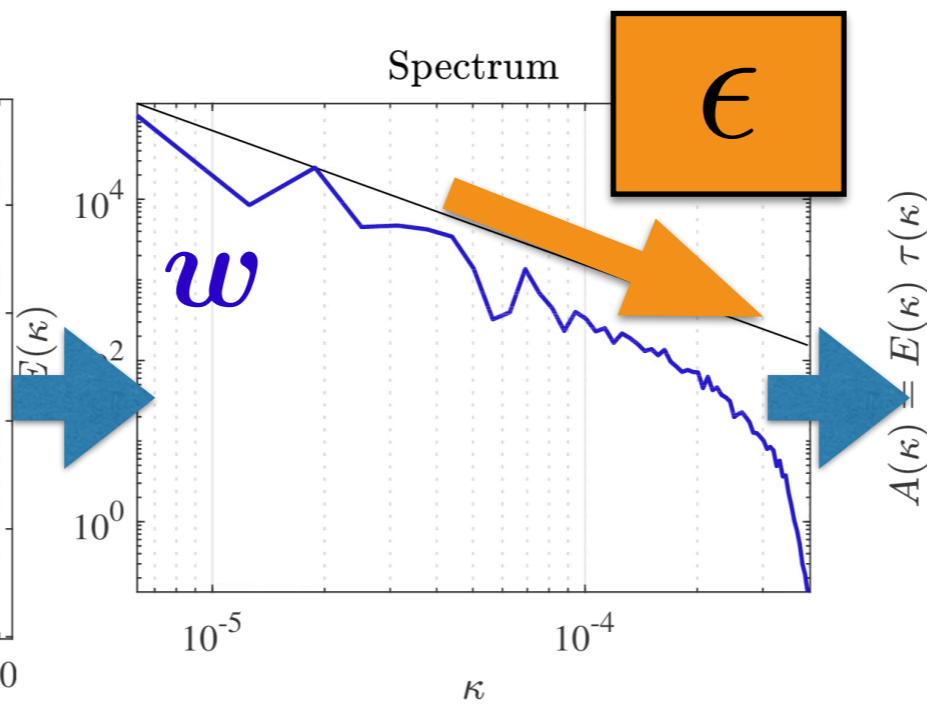
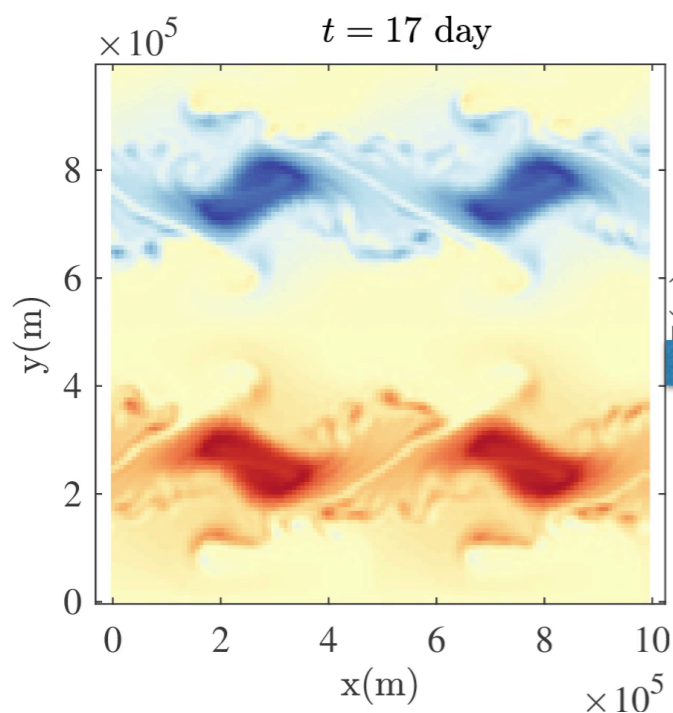
# modulation of $\sigma$

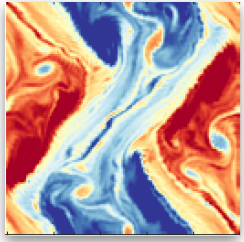
(for Resseguier et al. 2017b)

Dynamics  
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst.} \epsilon^p(x) \kappa^{-q}$$

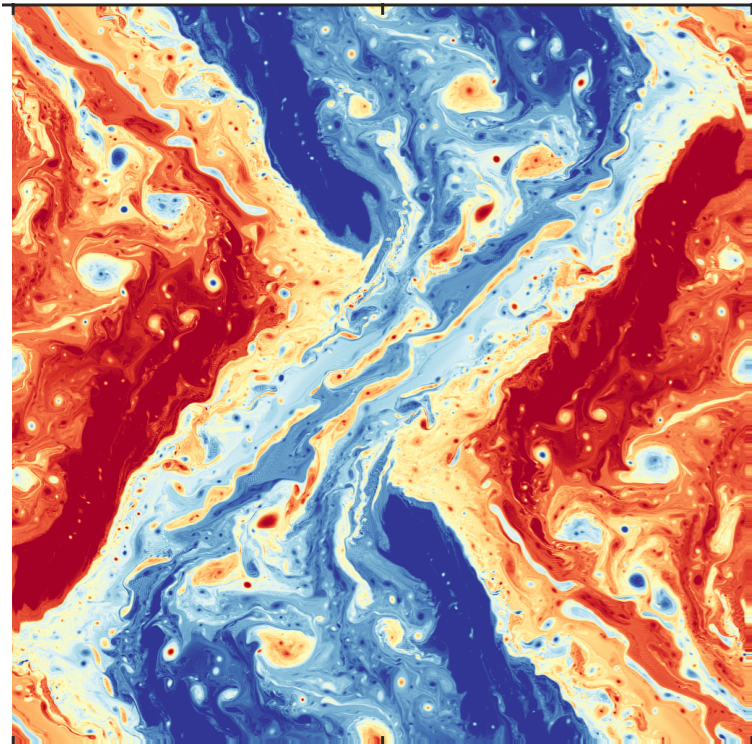
Learned on large scales at each time  $t$



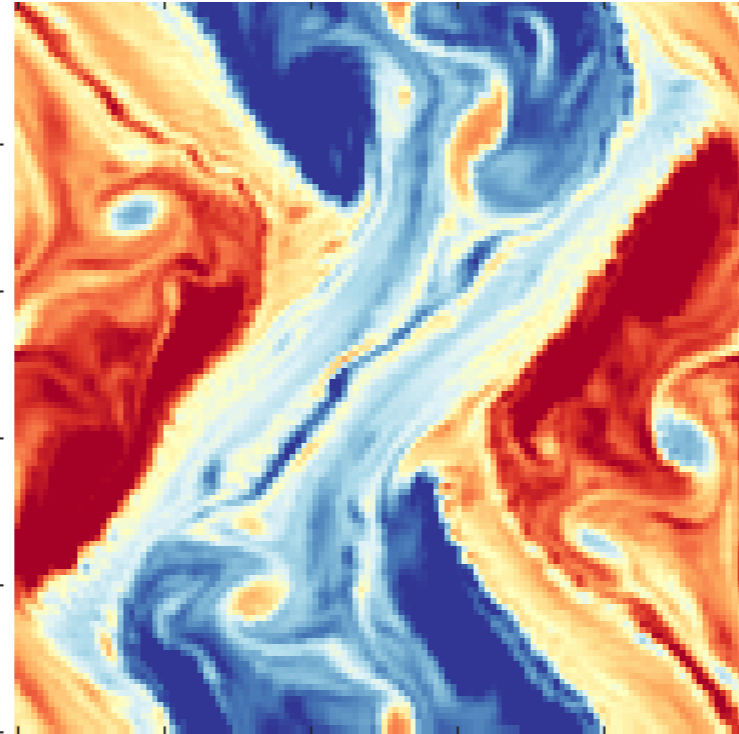


# Heterogenous modulation

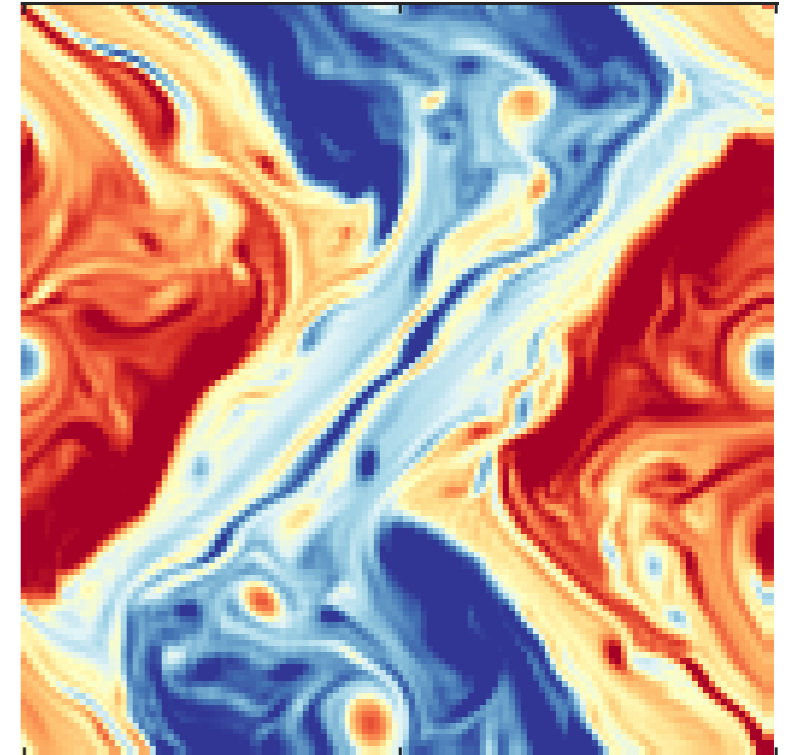
Deterministic 1024 x 1024



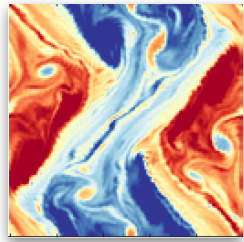
Stochastic 128 x 128  
with Homogeneous  
small-scale velocity



Stochastic 128 x 128  
with Energy-flux-based  
modulation

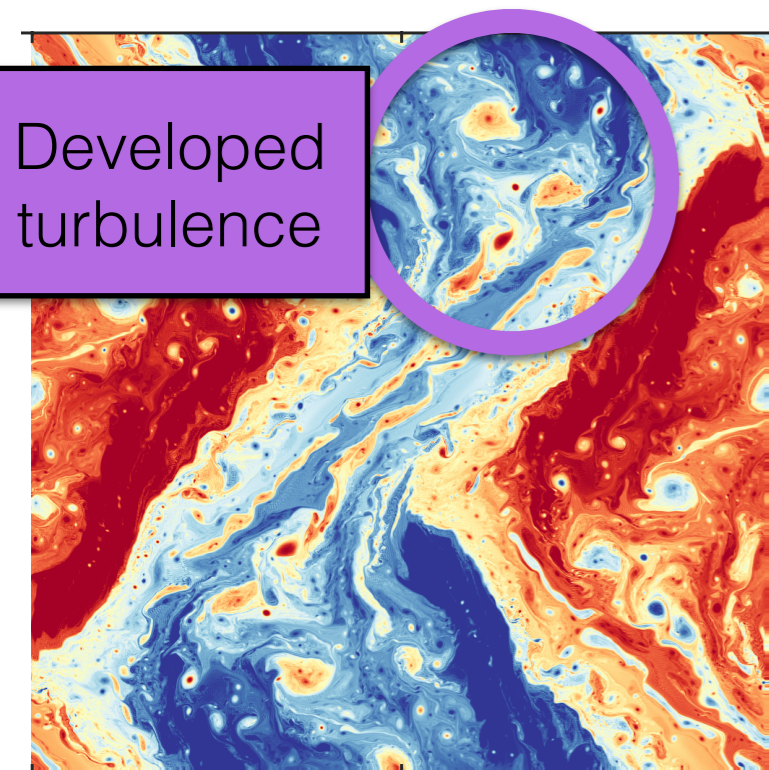




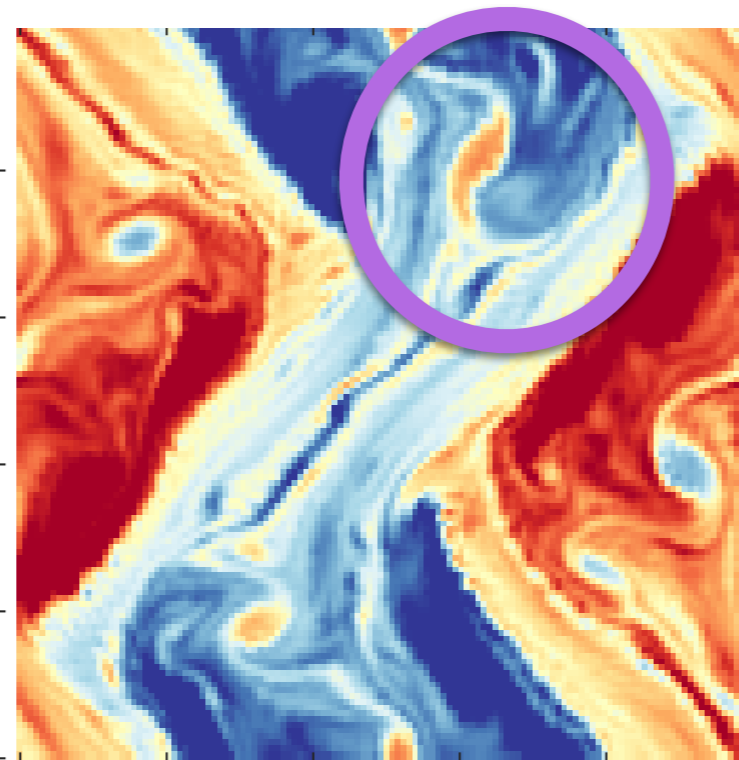


# Heterogenous modulation

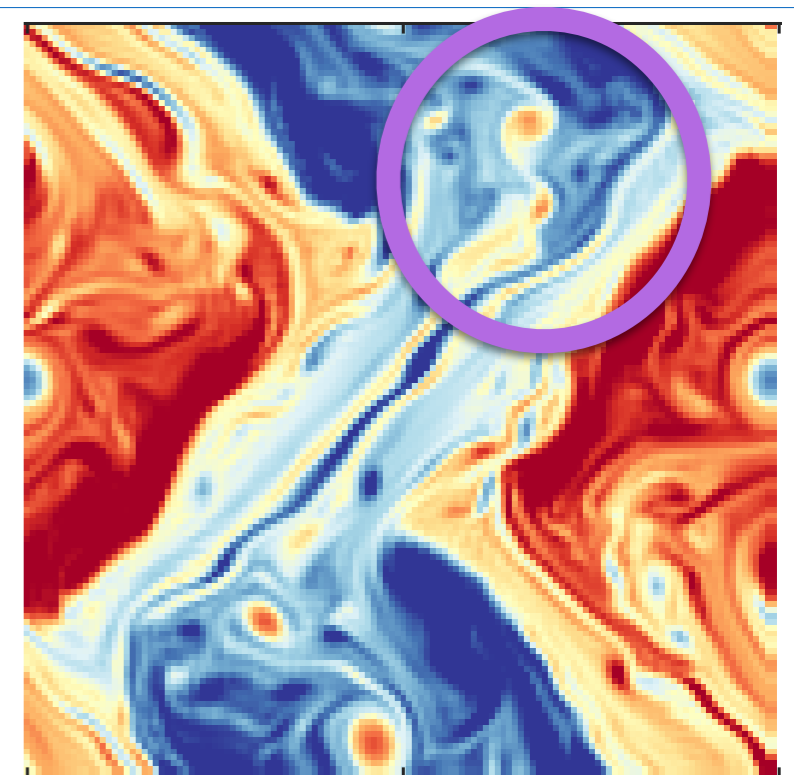
Deterministic 1024 x 1024

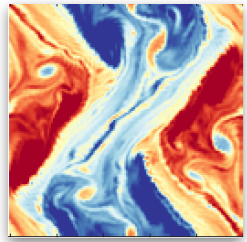


Stochastic 128 x 128  
with Homogeneous  
small-scale velocity



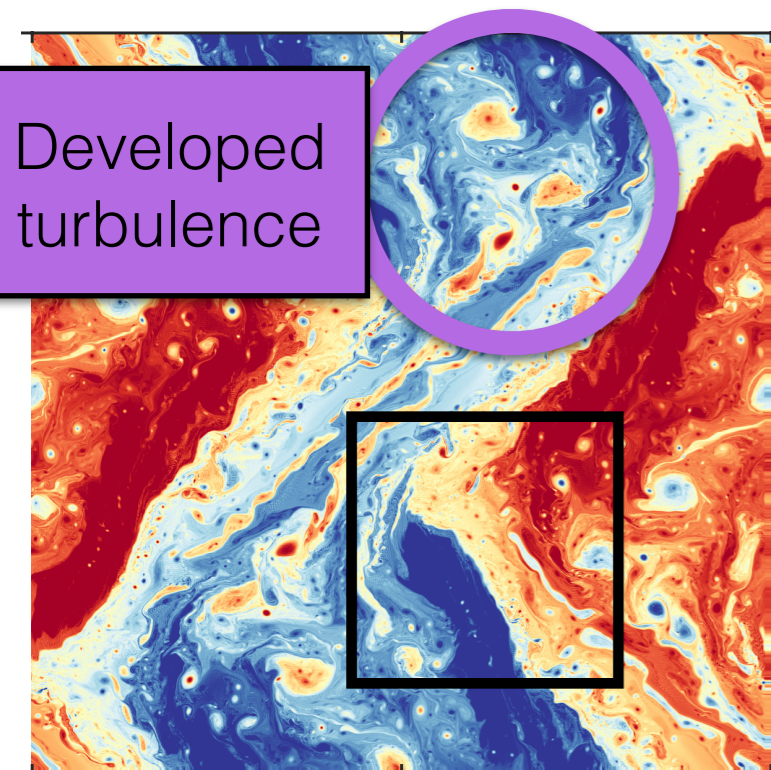
Stochastic 128 x 128  
with Energy-flux-based  
modulation



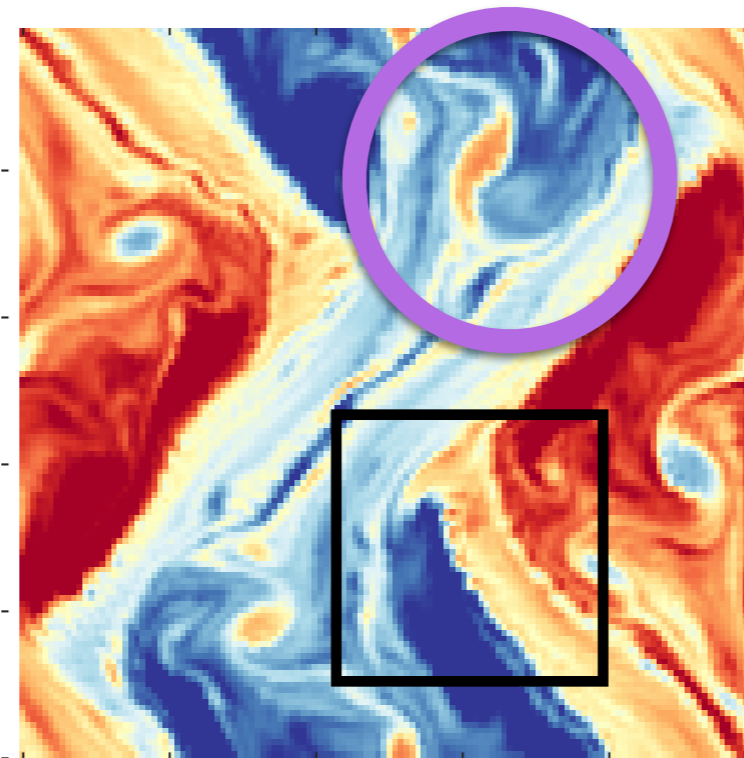


# Heterogenous modulation

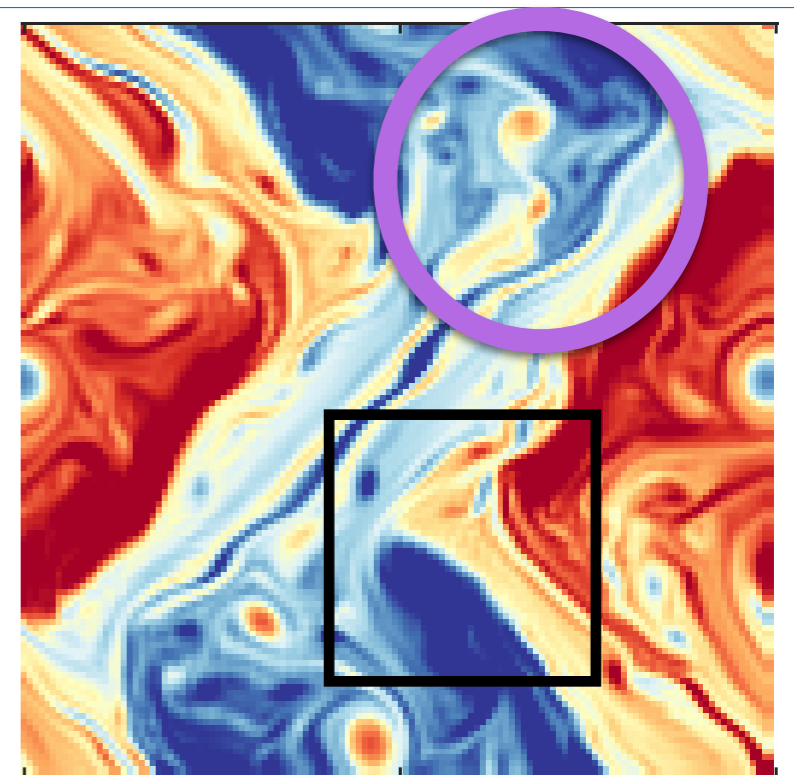
Deterministic 1024 x 1024

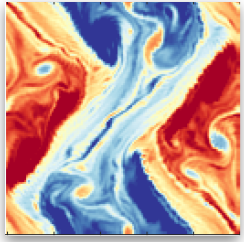


Stochastic 128 x 128  
with Homogeneous  
small-scale velocity



Stochastic 128 x 128  
with Energy-flux-based  
modulation





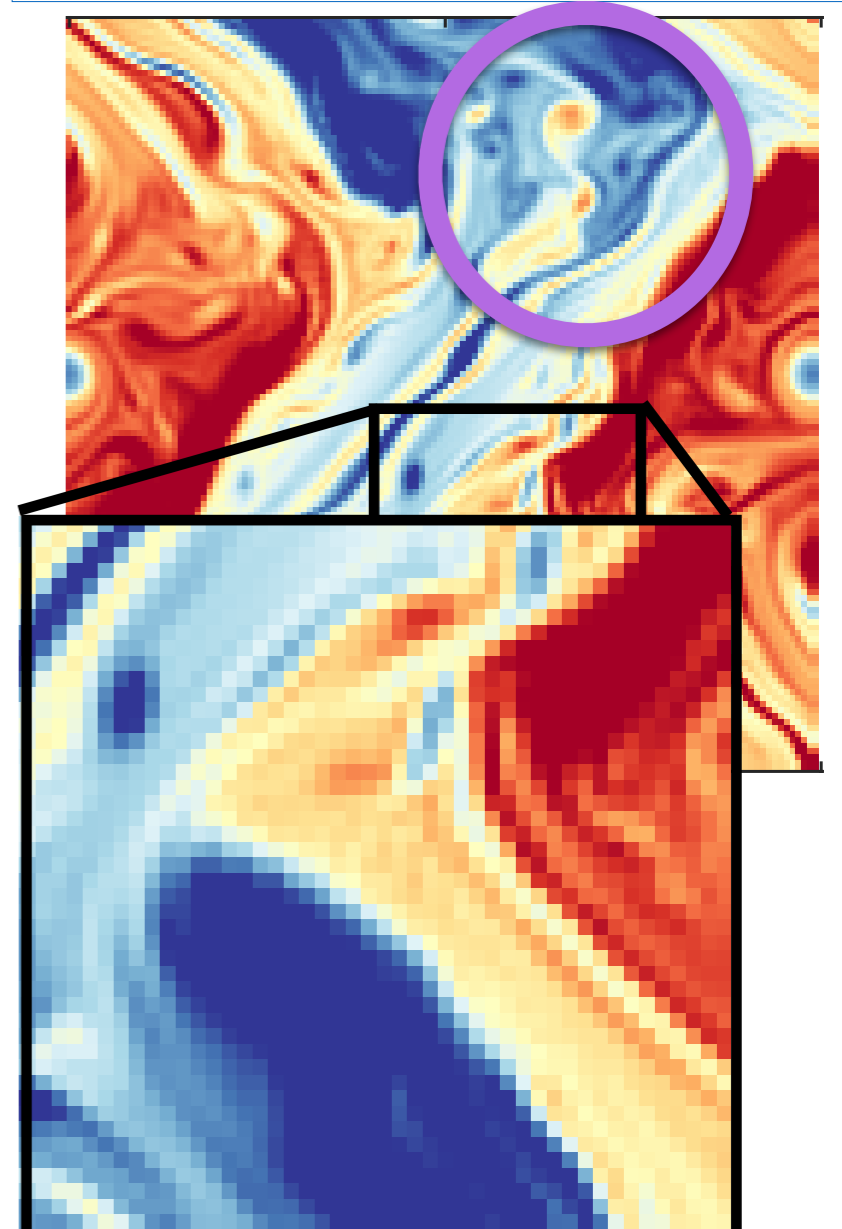
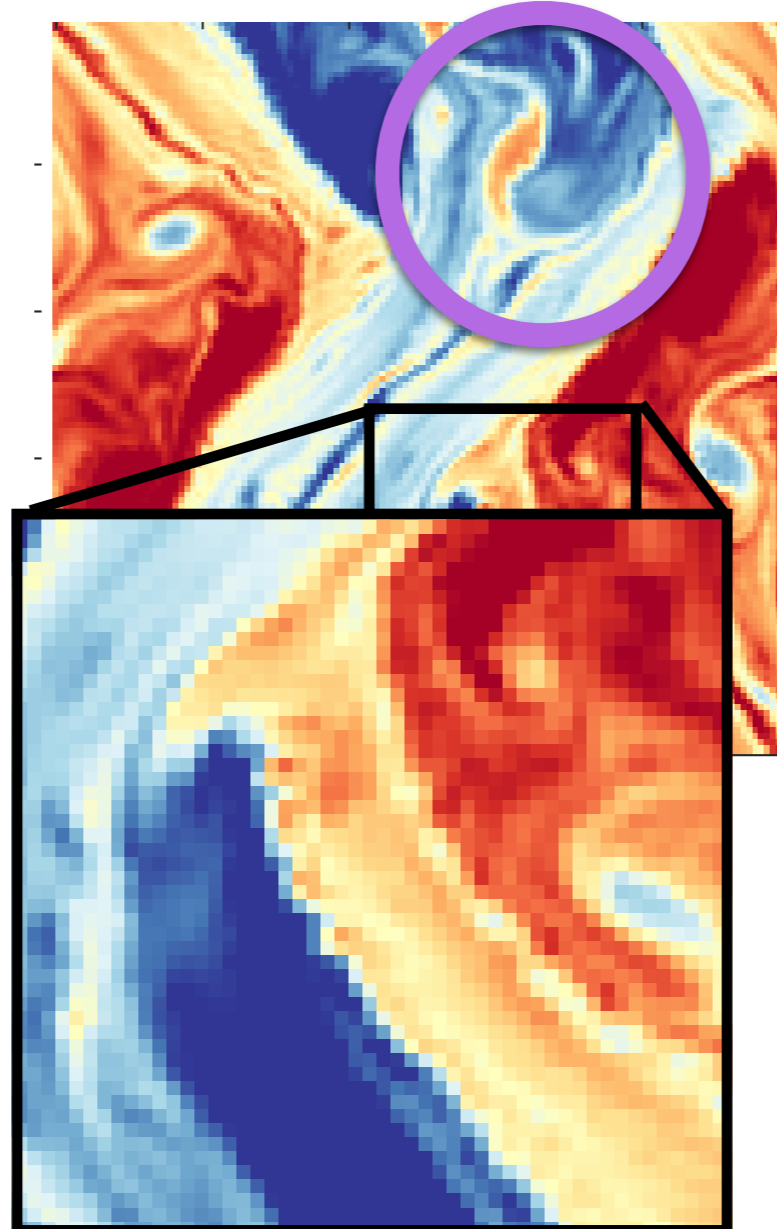
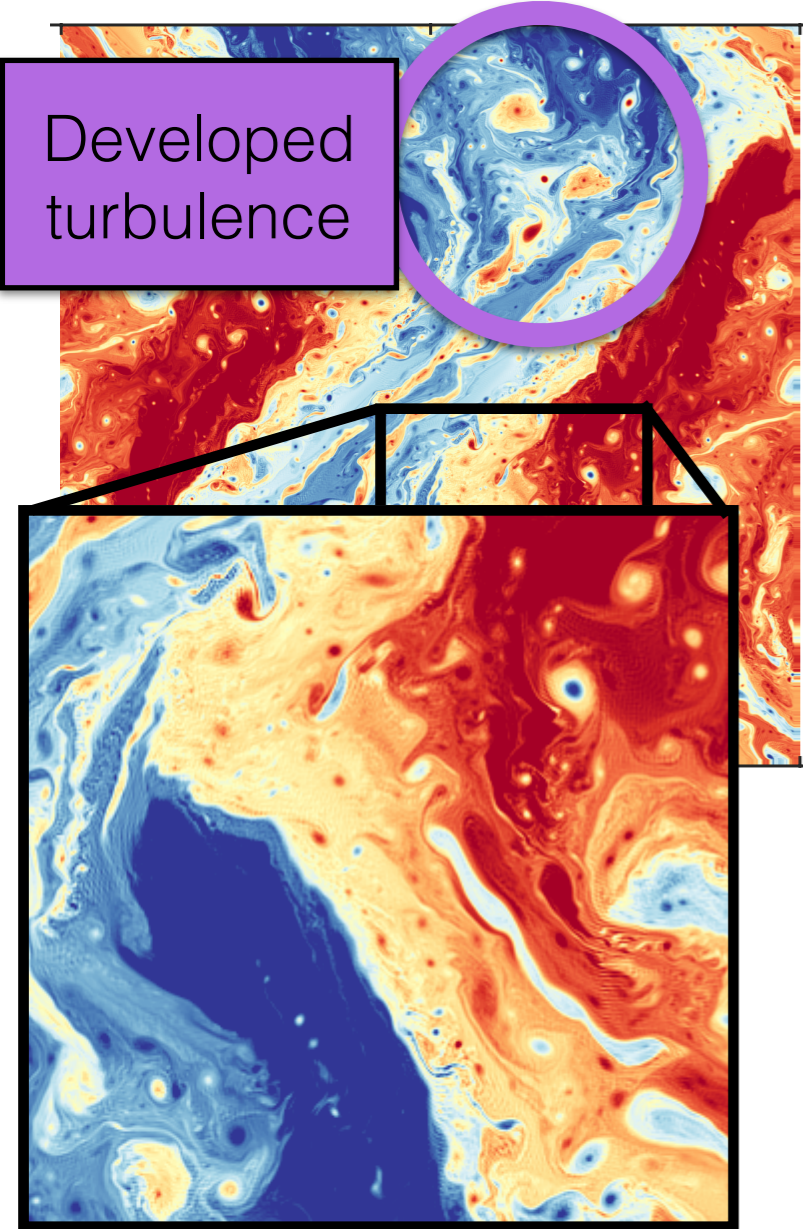
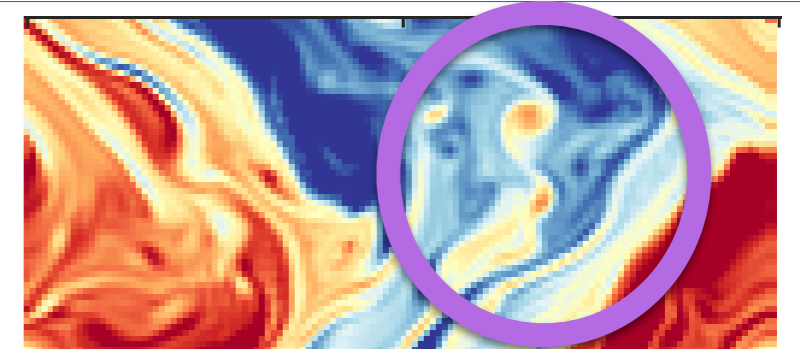
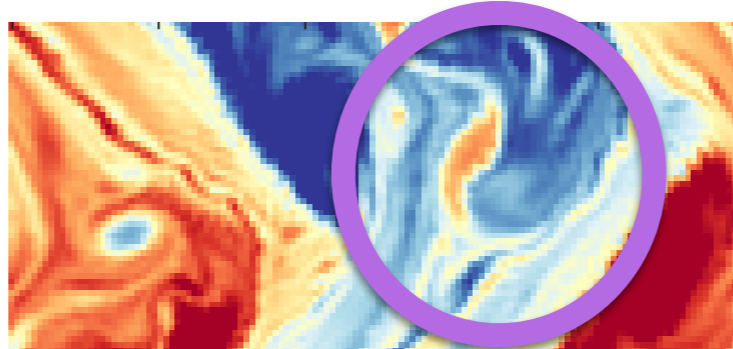
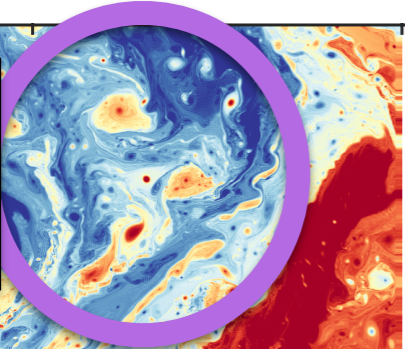
# Heterogenous modulation

Deterministic 1024 x 1024

Stochastic 128 x 128  
with Homogeneous  
small-scale velocity

Stochastic 128 x 128  
with Energy-flux-based  
modulation

Developed  
turbulence



# Conclusion

# Conclusion

- Stochastic transport (in LU & SALT) blindly describes unresolved triades
- LU conserves kinetic energy / SALT conserves helicity
- For the small-scale velocity parametrisation, both the data-driven (Cotter et al. 2018b) and the self-similar method (Resseguier et al. 2017b) lead to accurate uncertainty quantification (to address filter divergence)
- Energy-flux modulation improves the simulations

# Bibliography

- Cotter, C. J., Crisan, D., Holm, D. D., Pan, W., & Shevchenko, I. (2018). Numerically modelling stochastic Lie transport in fluid dynamics. *arXiv preprint arXiv: 1801.09729*.
- Kunita, H., *Stochastic Flows and Stochastic Differential Equations*, 1990. (Cambridge: Cambridge University Press).
- Resseguier, V., Mémin, E., & Chapron, B. (2017). Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophysical & Astrophysical Fluid Dynamics*, 111(3), 177-208.