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Quantifying model errors for geophysical fluid dynamics data assimilation through Location Uncertainty or Stochastic Advection by Lie Transport

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Quantifying model errors for geophysical fluid dynamics data assimilation through Location Uncertainty or Stochastic Advection by Lie Transport

Baylor Fox-Kemper
Darryl D. Holm
Wei Pan
Valentin Resseguier



Motivations

- Rigorously identified subgrid dynamics effects
- Injecting likely small-scale dynamics
- Studying bifurcations and attractors



Climate projections

- Quantification of modeling errors



Ensemble forecasts and data assimilation







Contents

- Stochastic transport
- Stochastic Navier-Stokes : SALT vs LU
- Unresolved velocity parametrisation

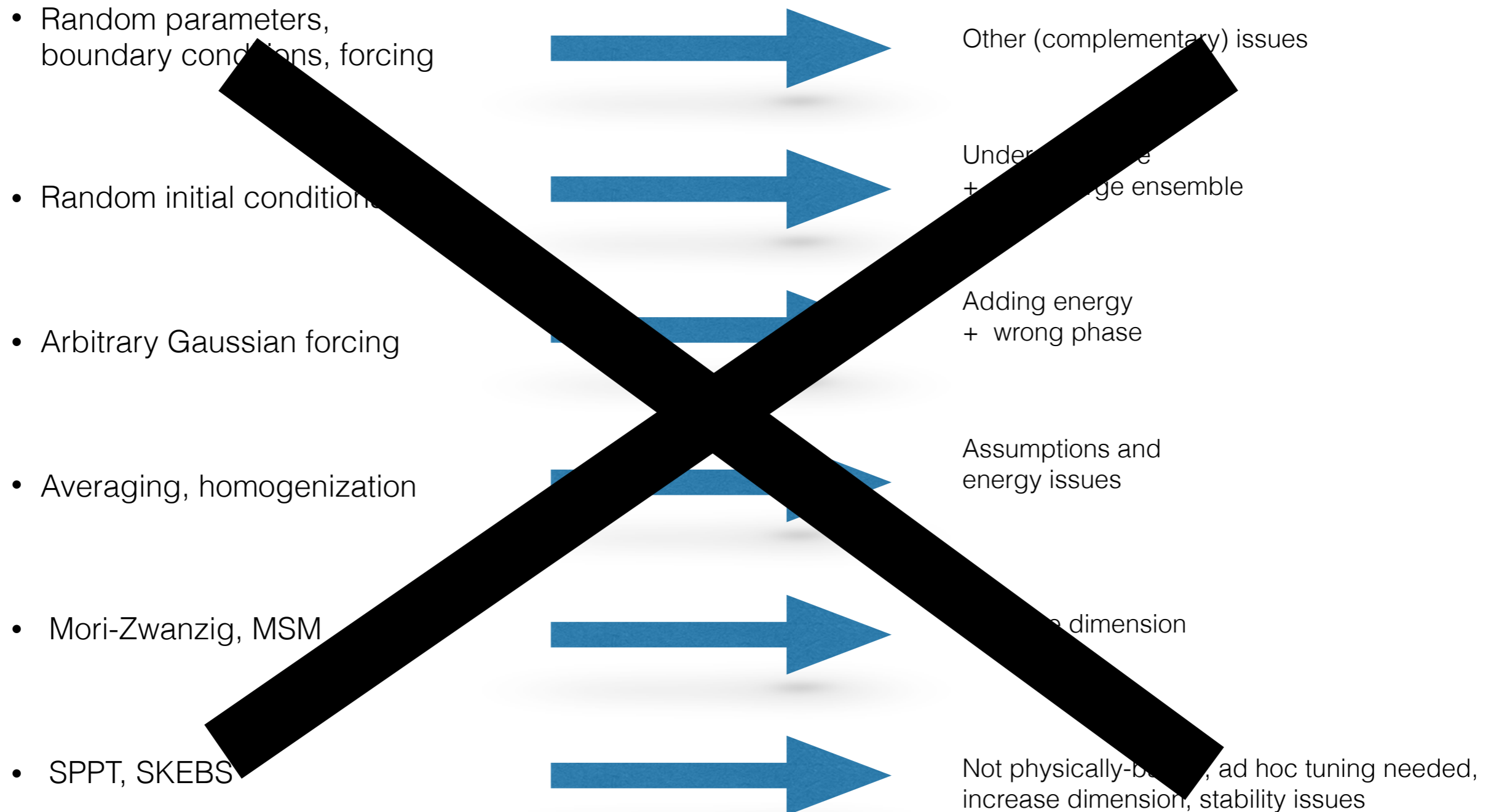
Part I

Stochastic transport

Usual random CFD

• Random parameters, boundary conditions, forcing		Other (complementary) issues
• Random initial conditions		Underdispersive + need large ensemble
• Arbitrary Gaussian forcing		Adding energy + wrong phase
• Averaging, homogenization		Assumptions and energy issues
• Mori-Zwanzig, MSM		Increase dimension
• SPPT, SKEBS		Not physically-based, ad hoc tuning needed, increase dimension, stability issues

Usual random CFD



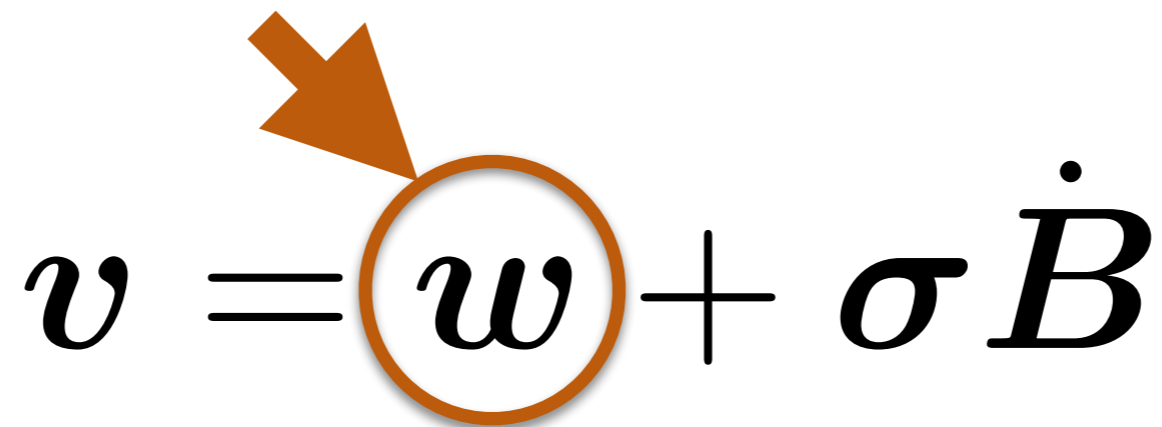
SALT & LU :
Adding random velocity

$$\mathbf{v} = \mathbf{w} + \sigma \dot{\mathbf{B}}$$

SALT & LU :

Adding random velocity

Resolved
large scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$


SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales

The diagram shows the equation $v = w + \sigma \dot{B}$. The term w is enclosed in a brown circle, and a brown arrow points to it from the text "Resolved large scales". The term $\sigma \dot{B}$ is enclosed in a purple circle, and a purple arrow points to it from the text "White-in-time small scales".

$$v = w + \sigma \dot{B}$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$a = a(x, x) =$

$\frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

SALT & LU :

Adding random velocity

$$d\mathbf{X}_t = \mathbf{w}^*(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

$$d\mathbf{X}_t = \overset{\text{Stratonovich}}{w^* (\mathbf{X}_t, t) dt + \sigma (\mathbf{X}_t, t) \circ d\mathbf{B}_t}$$

$$= \underset{\text{Ito}}{w (\mathbf{X}_t, t) dt + \sigma (\mathbf{X}_t, t) d\mathbf{B}_t}$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

References : Mikulevicius and Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al 2018 a, b

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

References : Mikulevicius and Rozovskii, 2004
Flandoli, 2011

LU

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

SALT

Holm , 2015	Cotter and al 2017
Holm and Tyranowski, 2016	Crisan et al., 2017
Arnaudon et al., 2017	Gay-Balmaz & Holm 2017
	Cotter and al 2018 a, b

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Advection of tracer Θ

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

$$\frac{D\Theta}{Dt} = 0$$

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Stratonovich notations:

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Stratonovich notations:

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
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(Kunita 1990)

Ito notations:

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
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Ito-Wentzell
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Large scales:

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Ito notations:

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Ito-Wentzell
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Large scales:

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Small scales:

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Ito notations:

$$\partial_t \Theta + \underbrace{w^*}_{\text{Advection}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)_{\text{Diffusion}}$$

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w

Small scales:
 $\sigma \dot{B}$

Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^*}_{\text{Stratonovich drift}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Advection

Diffusion

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w

Small scales:
 $\sigma \dot{B}$

Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction » in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w
Small scales:
 $\sigma \dot{B}$
Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w
Small scales:
 $\sigma \dot{B}$
Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Multiplicative random forcing

Balanced energy exchanges

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction » in Ito notations

Part II

Stochastic Navier-Stokes

SALT vs LU

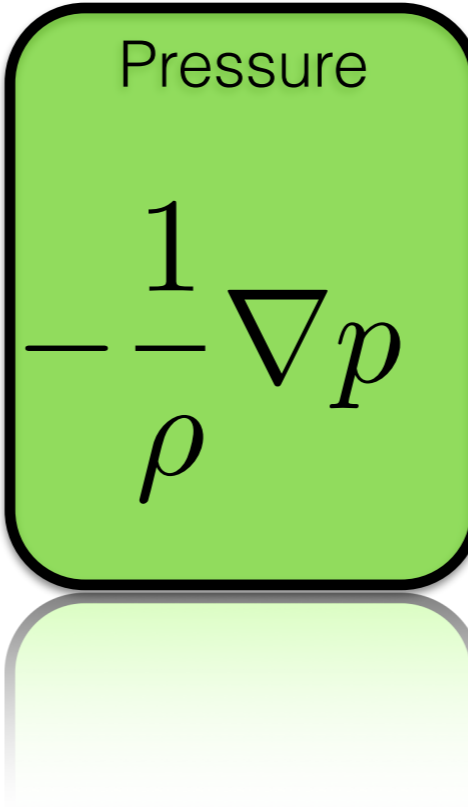
Comparison

	LU	SALT
Scalar (e.g. SQG)	Identical	
Navier-Stokes	2 differences	
Vorticity	2 differences	
Kinetic energy conservation	✓	✗
Helicity (& 2D enstrophy) conservation	✗	✓

Navier-Stokes LU

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

$$\frac{Dw}{Dt} = \text{Pressure} \left[-\frac{1}{\rho} \nabla p \right]$$


Navier-Stokes LU

Stochastic
transport
of Ito drift

$$\frac{Dw}{Dt} =$$

Pressure

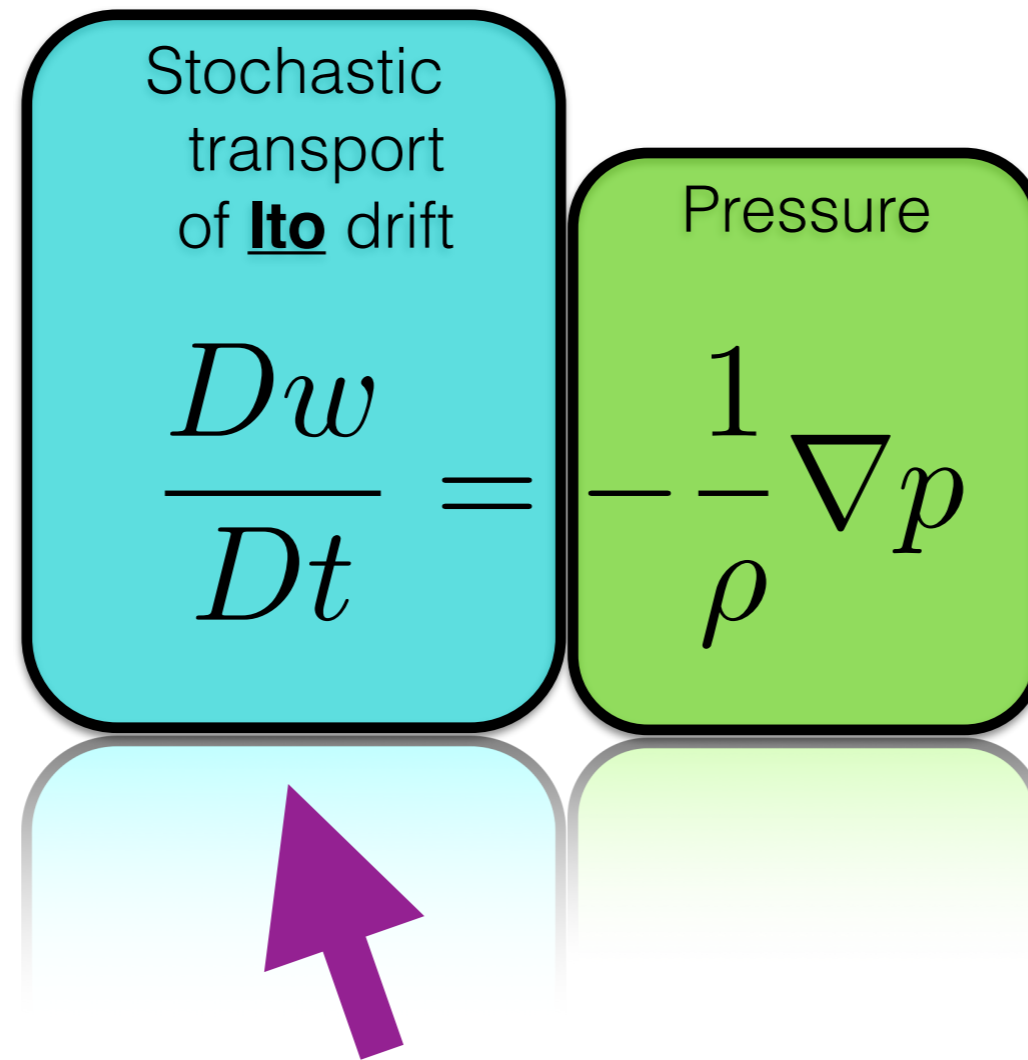
$$-\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

Stochastic transport of Ito drift

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \nabla p$$

Pressure

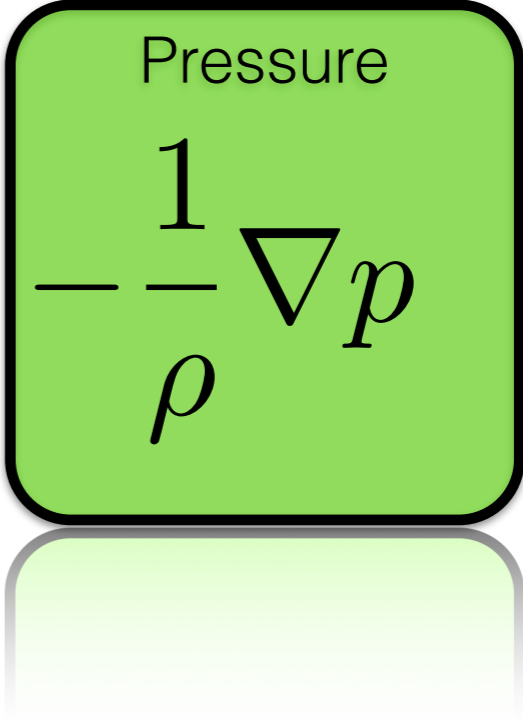


Conserve (Ito) kinetic energy

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = \text{Pressure} \frac{1}{\rho} \nabla p$$


Navier-Stokes SALT

Stochastic
transport of
Stratonovich drift

$$\frac{Dw^*}{Dt}$$

$$+ \nabla (\sigma \circ \dot{B})^T w^* =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

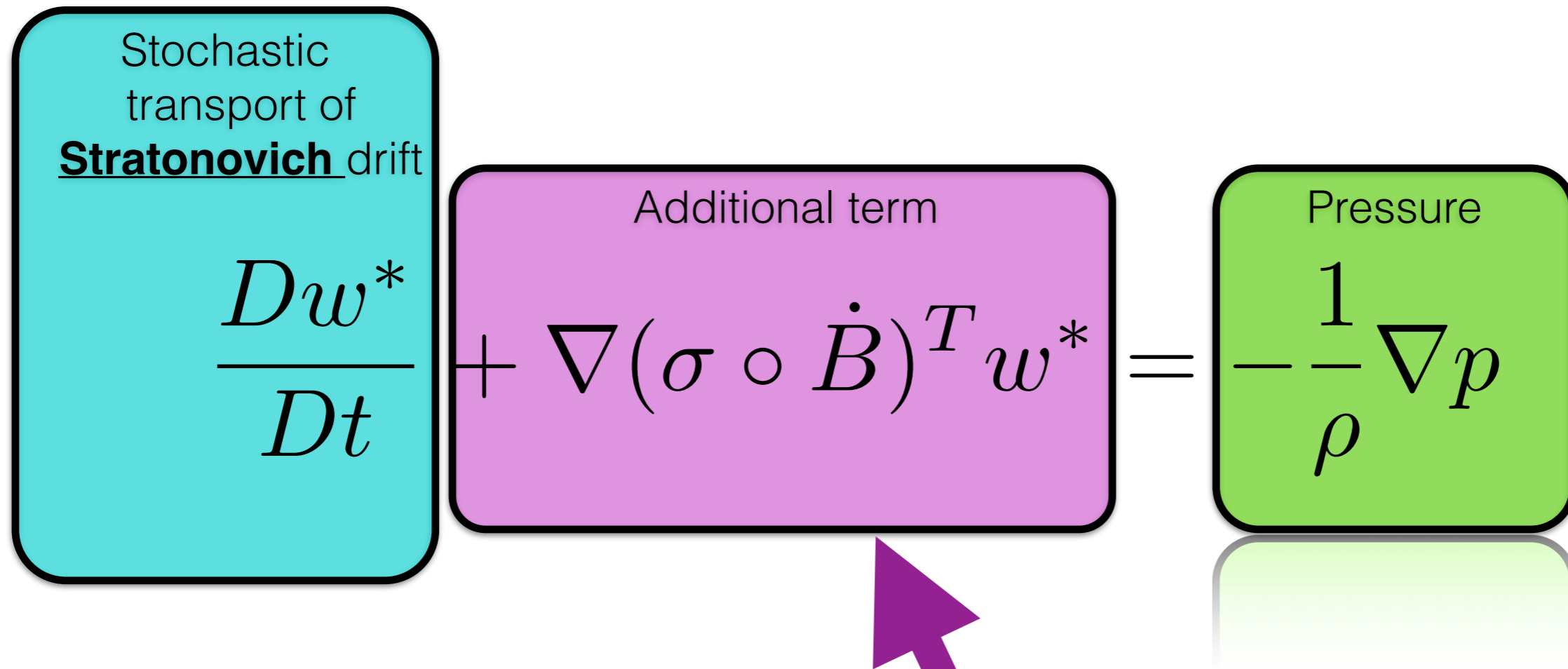
Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Additional term

Pressure



Conserve (Stratonovich) helicity

Part III

Unresolved velocity parametrisation

Code available online

Large scales:

$$w$$

Small scales:

$$\sigma \dot{B}$$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Parameter-free models for σ

	Cotter et al. 2018b	Resseguier et al. 2017b
Applicable to :	LU & SALT	LU & SALT
Homogeneous	No	Yes
Stationary	Yes	Not anymore
Self-similar assumption	No	Yes
Data-driven	Yes	No

$t = 100$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$-\frac{1}{\tau} (b - F) \quad \text{Forcing}$$

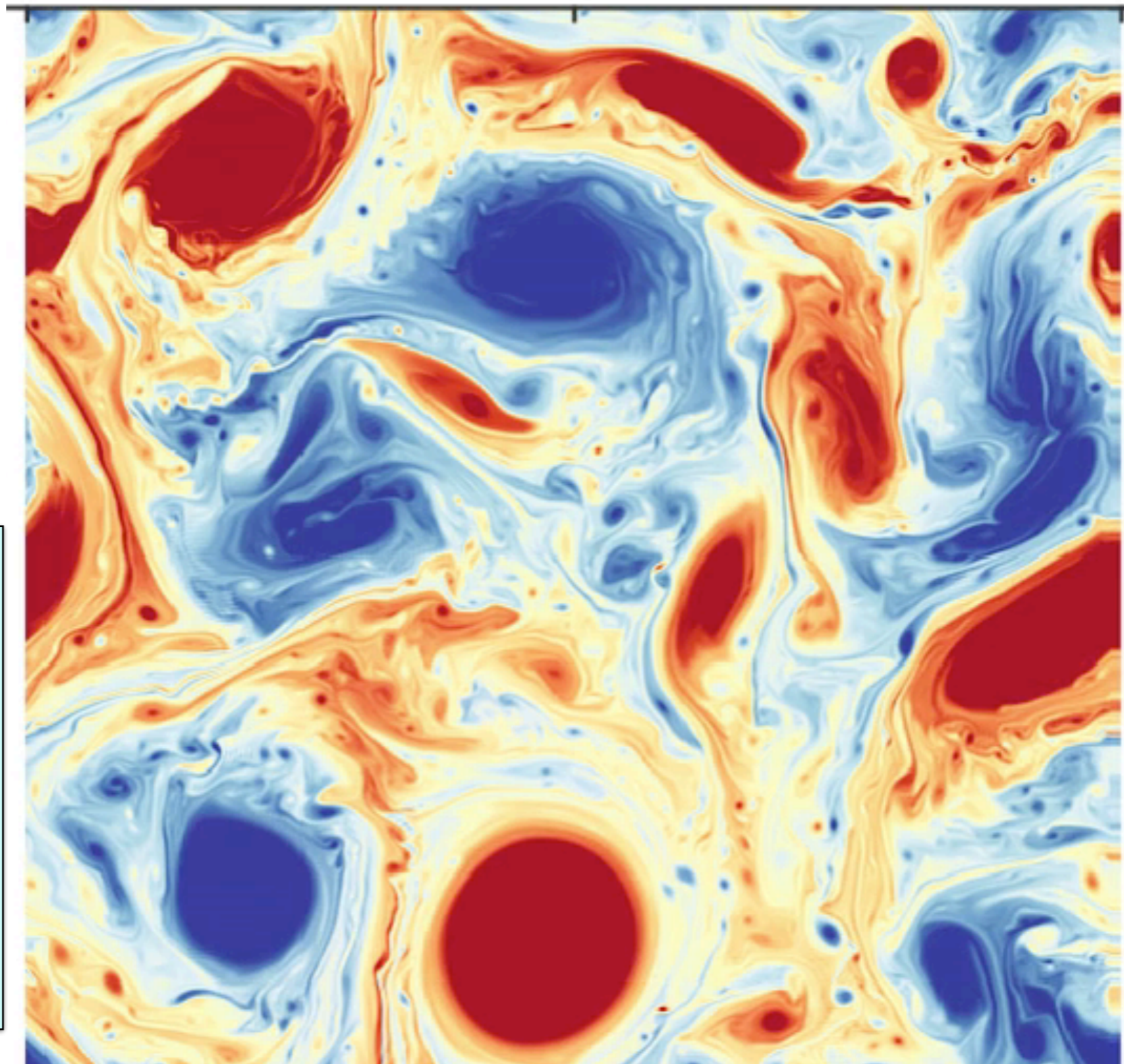
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

deterministic

SQG

512 x 512



$t = 100$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$- \frac{1}{\tau} (b - F) \quad \text{Forcing}$$

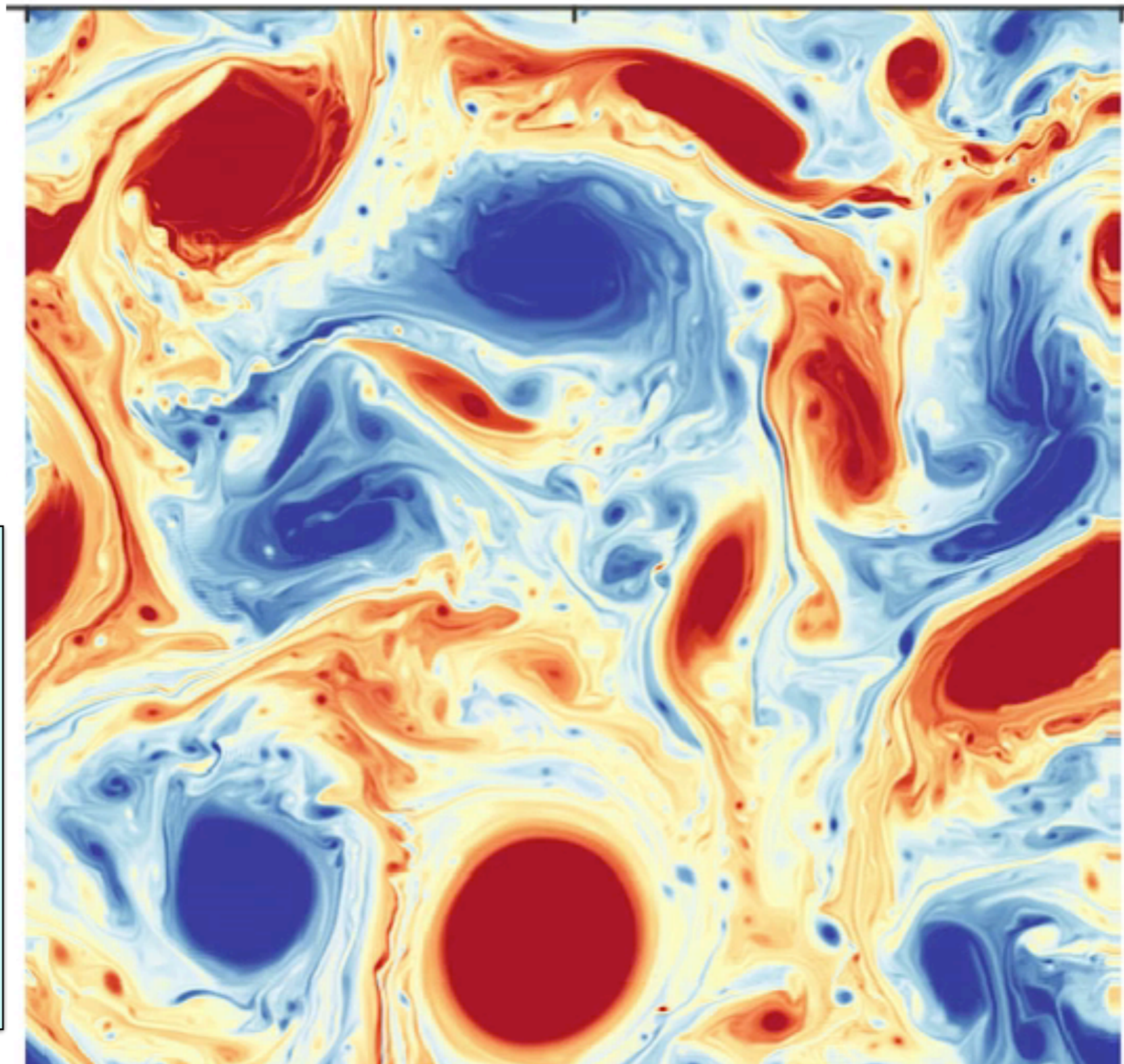
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

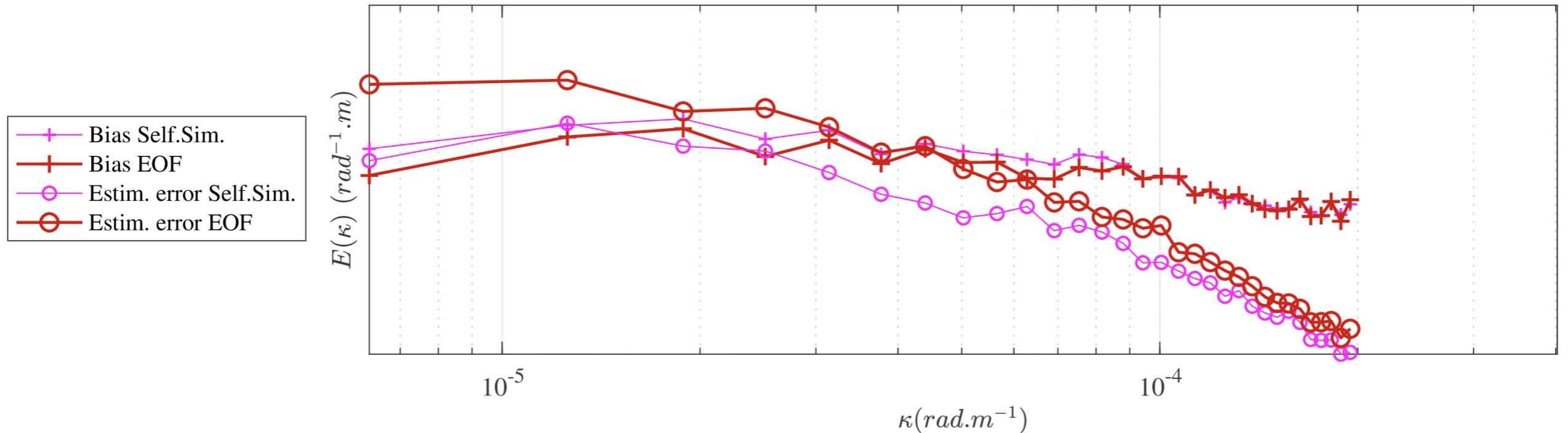
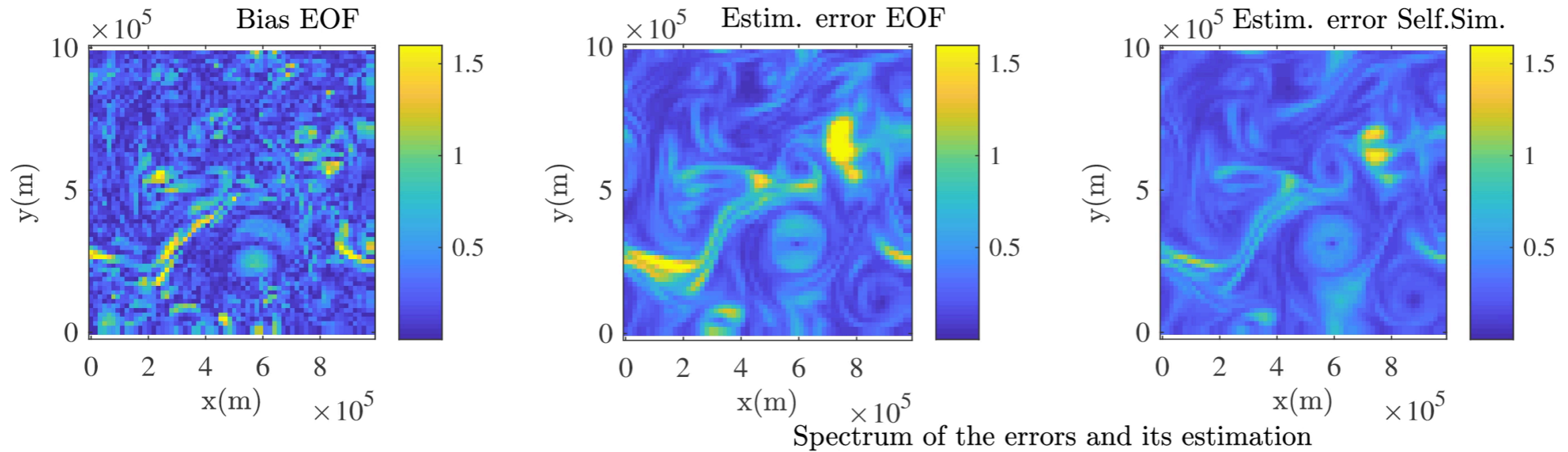
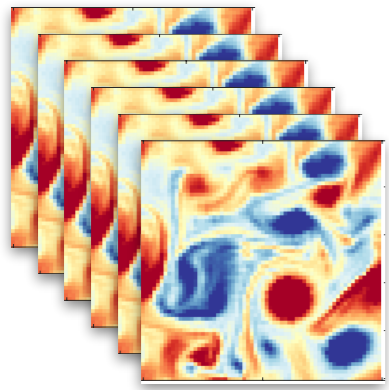
deterministic

SQG

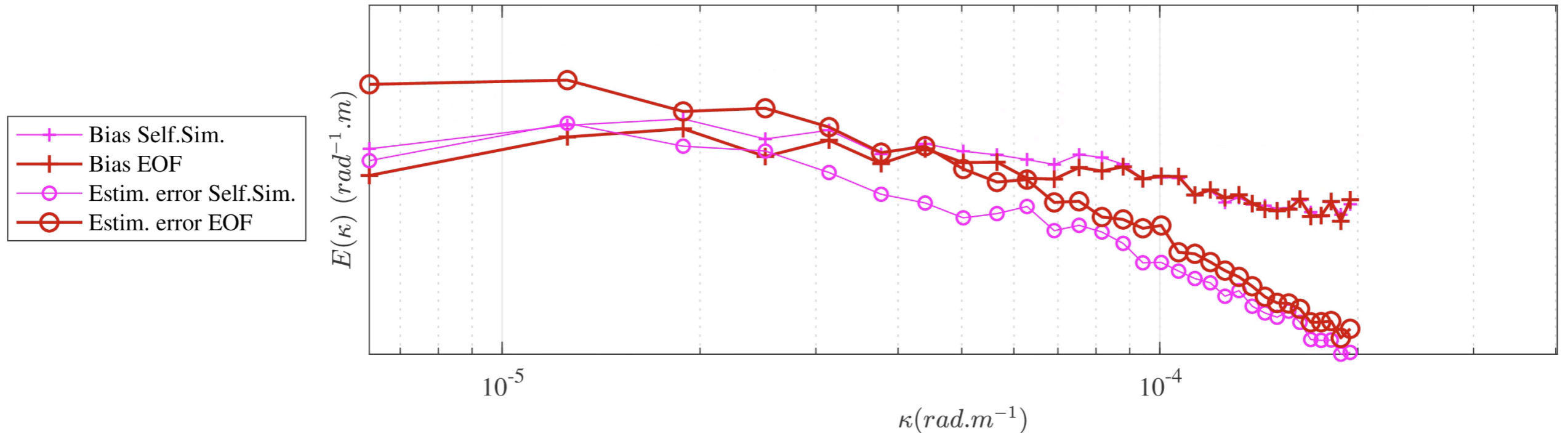
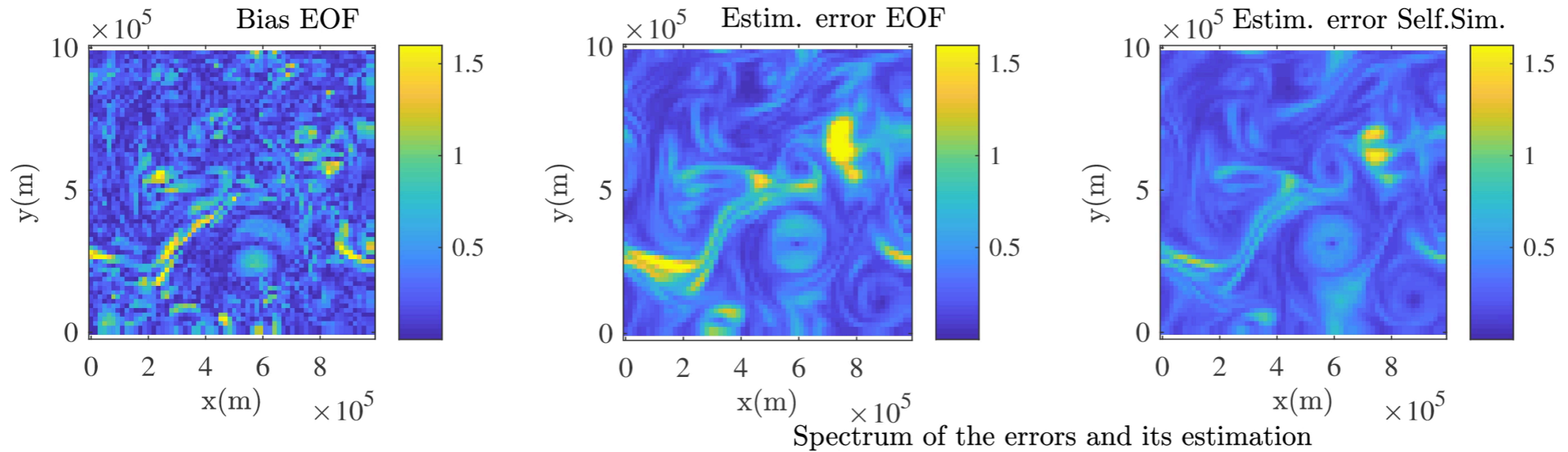
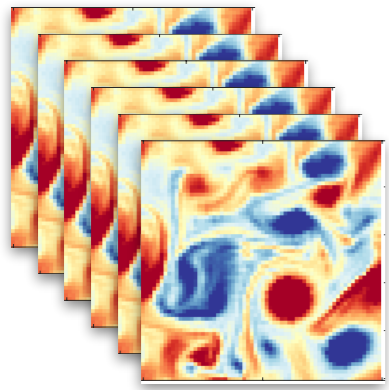
512 x 512



Ensemble : uncertainty quantification



Ensemble : uncertainty quantification



Conclusion

Conclusion

- Stochastic transport (in LU & SALT) blindly describes unresolved triades
- LU conserves kinetic energy / SALT conserves helicity
- For the small-scale velocity parametrisation, both the data-driven (Cotter et al. 2018b) and the self-similar method (Resseguier et al. 2017b) lead to accurate uncertainty quantification (to address filter divergence)
- Additional results:
Energy-flux modulation improves the simulations

Bibliography

- Cotter, C. J., Crisan, D., Holm, D. D., Pan, W., & Shevchenko, I. (2018). Numerically modelling stochastic Lie transport in fluid dynamics. *arXiv preprint arXiv: 1801.09729*.
- Kunita, H., Stochastic Flows and Stochastic Differential Equations, 1990. (Cambridge: Cambridge University Press).
- Resseguier, V., Mémin, E., & Chapron, B. (2017). Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophysical & Astrophysical Fluid Dynamics*, 111(3), 177-208.