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Stochastic Advection by Lie Transport and Location Uncertainty: a common ground for uncertainty quantification in fluid dynamics

Baylor Fox-Kemper
Darryl D. Holm
Wei Pan
Valentin Resseguier



Motivations

- Rigorously identified subgrid dynamics effects
- Injecting likely small-scale dynamics
- Studying bifurcations and attractors



Climate projections

- Quantification of modeling errors



Ensemble forecasts and data assimilation

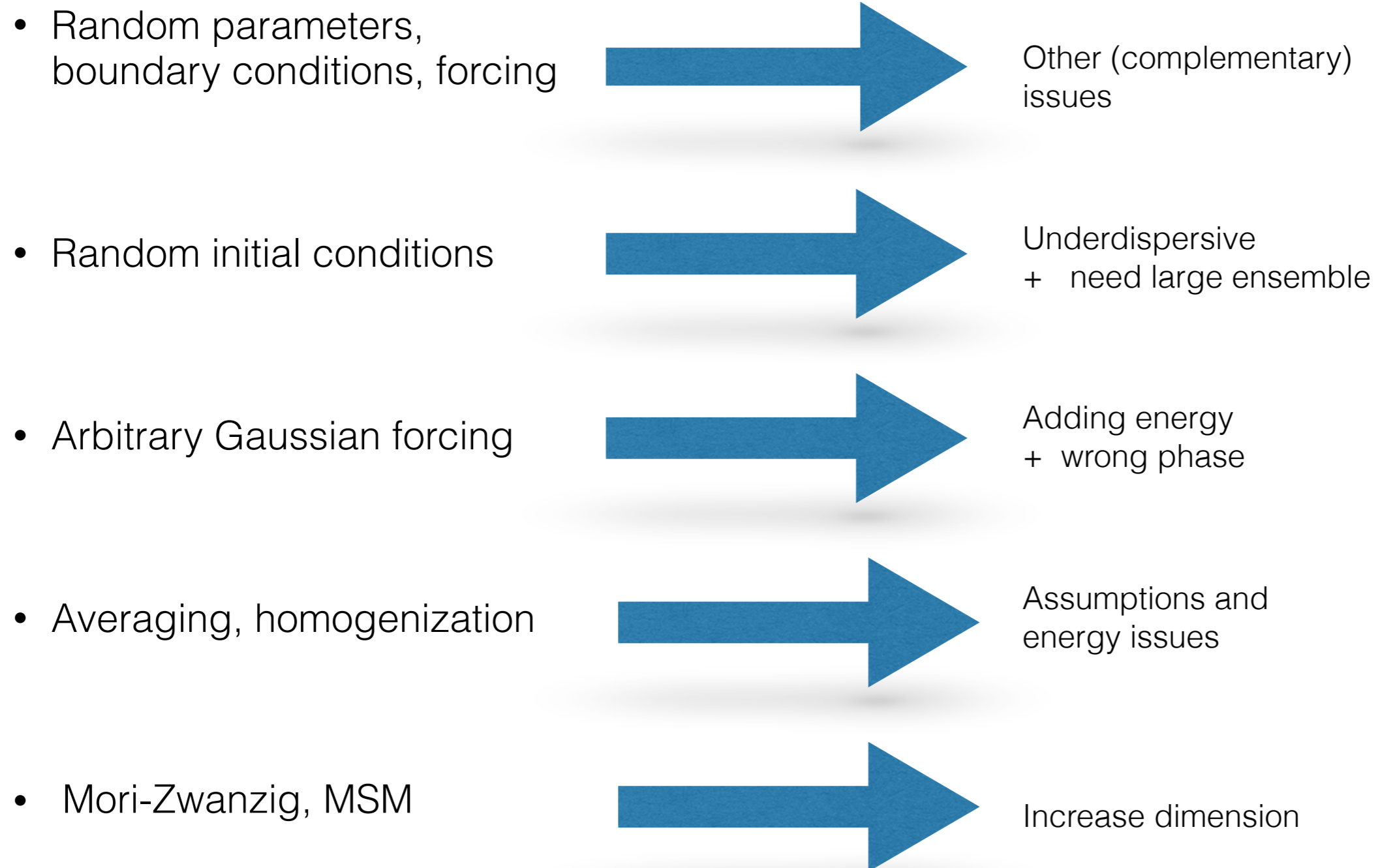
Contents

- Stochastic transport
- Stochastic Navier-Stokes : SALT vs LU
- Unresolved velocity parametrisation
- Unresolved velocity non-stationary heterogeneity

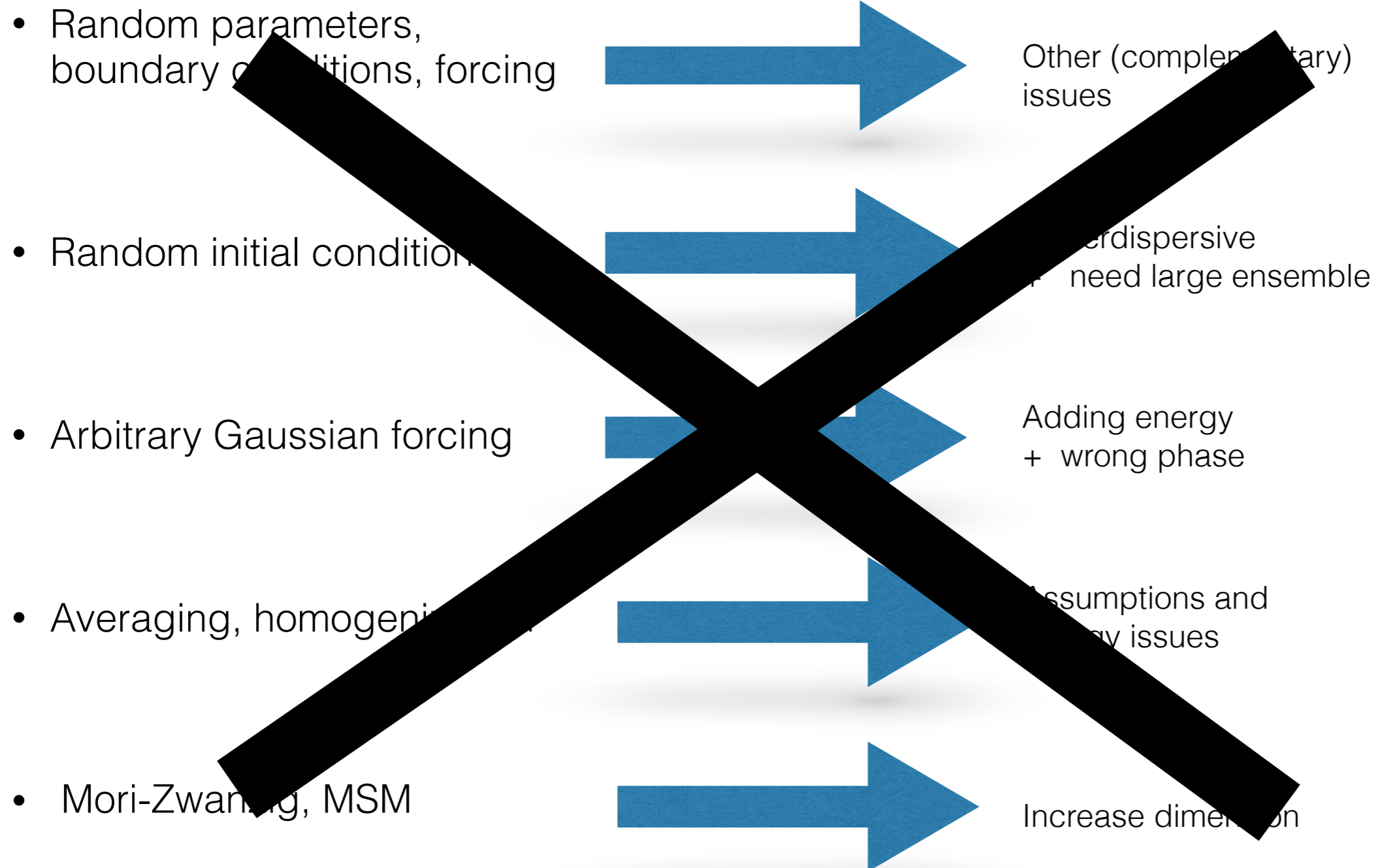
Part I

Stochastic transport

Usual random CFD



Usual random CFD



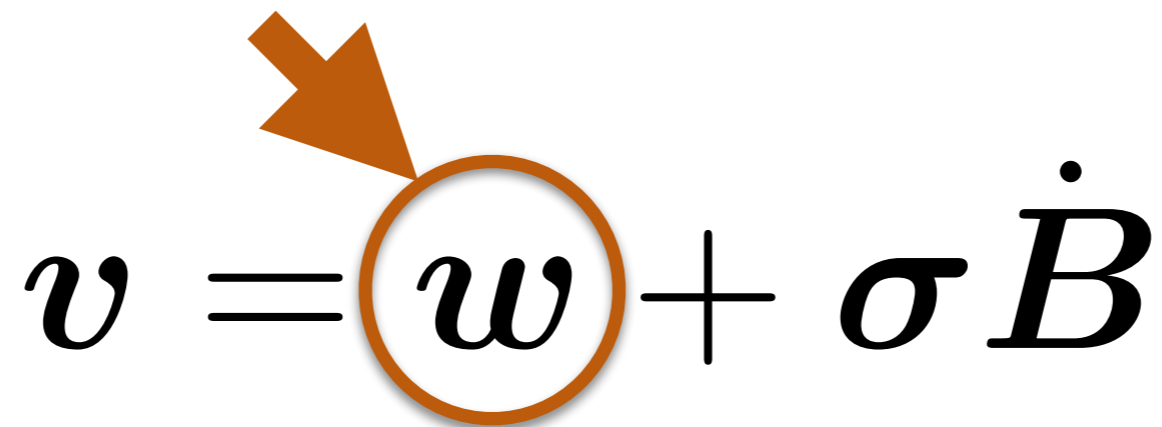
SALT & LU :
Adding random velocity

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$

SALT & LU :

Adding random velocity

Resolved
large scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$


SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

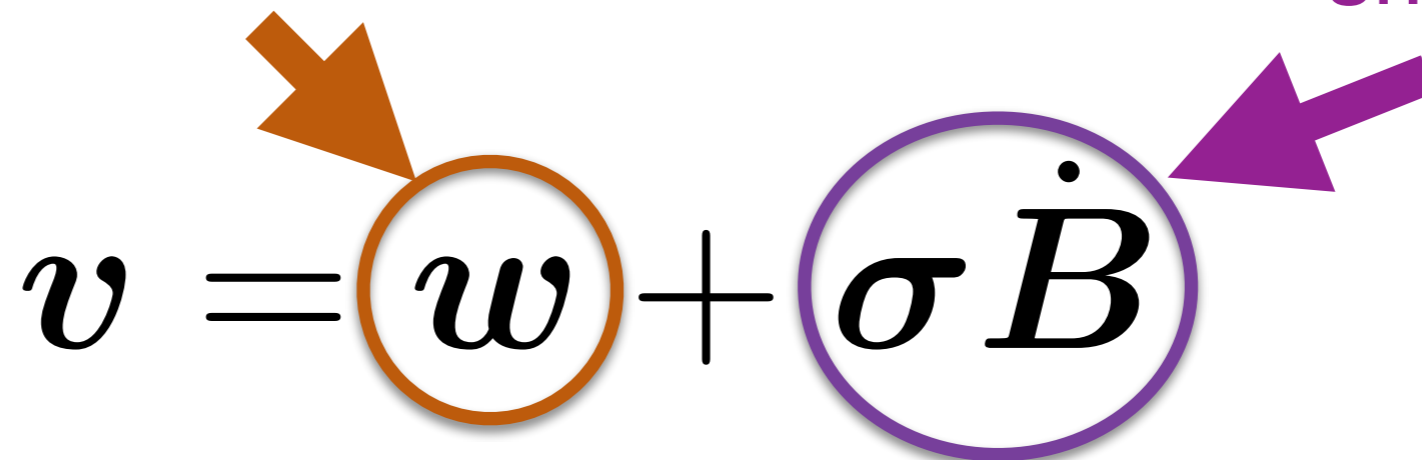
$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales

$$v = w + \sigma \dot{B}$$
The diagram shows the equation $v = w + \sigma \dot{B}$. The term w is enclosed in a brown circle, and a brown arrow points from the text "Resolved large scales" to it. The term $\sigma \dot{B}$ is enclosed in a purple circle, and a purple arrow points from the text "White-in-time small scales" to it.

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

SALT & LU :

Adding random velocity

$$d\mathbf{X}_t = \mathbf{w}^*(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

$$d\mathbf{X}_t = \begin{array}{c} \text{Stratonovich} \\ \mathbf{w}^*(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t) \circ d\mathbf{B}_t \end{array}$$

$$= \begin{array}{c} \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t \\ \text{Ito} \end{array}$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

References : Mikulevicius and Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al 2018 a, b

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

Ito

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

References : Mikulevicius and Rozovskii, 2004
Flandoli, 2011

LU

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

SALT

Holm , 2015	Cotter and al 2017
Holm and Tyranowski, 2016	Crisan et al., 2017
Arnaudon et al., 2017	Gay-Balmaz & Holm 2017
	Cotter and al 2018 a, b

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Advection of tracer Θ

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

$$\frac{D\Theta}{Dt} = 0$$

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
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Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Stratonovich notations:

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Stratonovich notations:

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Ito notations:

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

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Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}}$$

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w

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Ito notations:

$$\partial_t \Theta + \underbrace{w^*}_{\text{Advection}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)_{\text{Diffusion}}$$

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w
Small scales:
 $\sigma \dot{B}$
Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^*}_{\text{Advection}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)_{\text{Diffusion}}$$

Stratonovich drift : « Drift correction » in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w

Small scales:
 $\sigma \dot{B}$

Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction » in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w
Small scales:
 $\sigma \dot{B}$
Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Multiplicative random forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell formula
(Kunita 1990)

Large scales:
 w
Small scales:
 $\sigma \dot{B}$
Variance tensor:
 $a = a(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$

Multiplicative random forcing

Balanced energy exchanges

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction » in Ito notations

Large scales:

$$\mathbf{w}$$

Small scales:

$$\sigma \dot{B}$$

Variance tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp \mathbf{w}(\bullet, t), \forall t' \leq t$$

$$\begin{aligned}
 \partial_t \mathbb{E}(\Theta) + \mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta) &= \nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right) \\
 \partial_t Var(\Theta) + \mathbf{w}^* \cdot \nabla Var(\Theta) &= \nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla Var(\Theta) \right) + (\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta)
 \end{aligned}$$

The diagram illustrates the evolution of the mean and variance of a passive tracer Θ . The first equation shows the time derivative of the mean $\mathbb{E}(\Theta)$ is equal to the advection term $\mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta)$ plus the diffusion term $\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right)$. The second equation shows the time derivative of the variance $Var(\Theta)$ is equal to the advection term $\mathbf{w}^* \cdot \nabla Var(\Theta)$ plus the diffusion term $\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla Var(\Theta) \right)$ and a source term $(\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta)$.

Large scales:

w

Small scales:

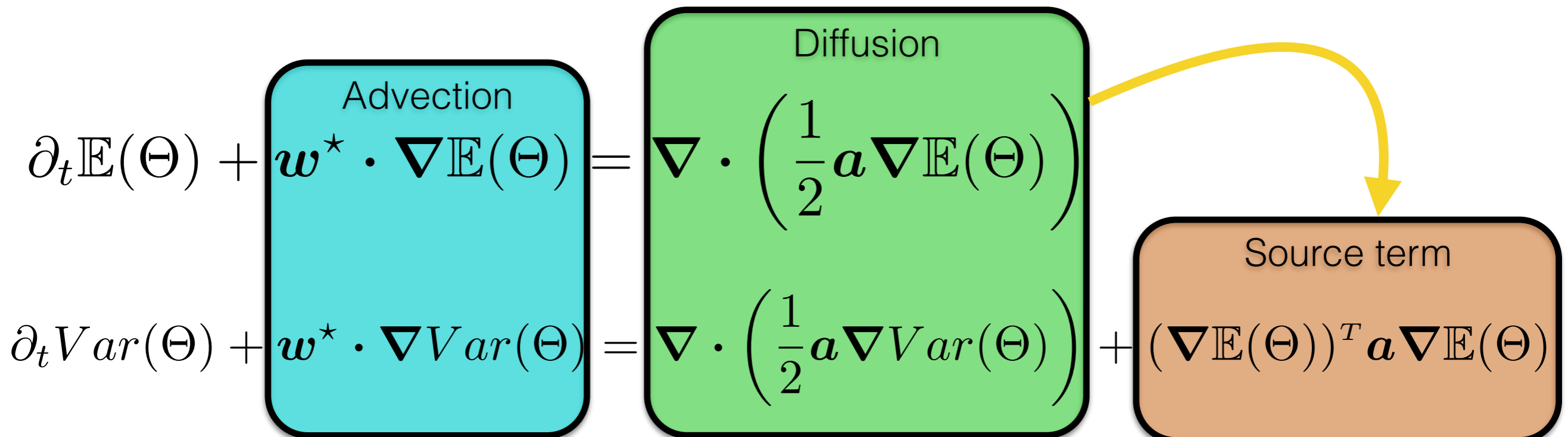
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp\!\!\!\perp w(\bullet, t), \forall t' \leq t$$



Large scales:

w

Small scales:

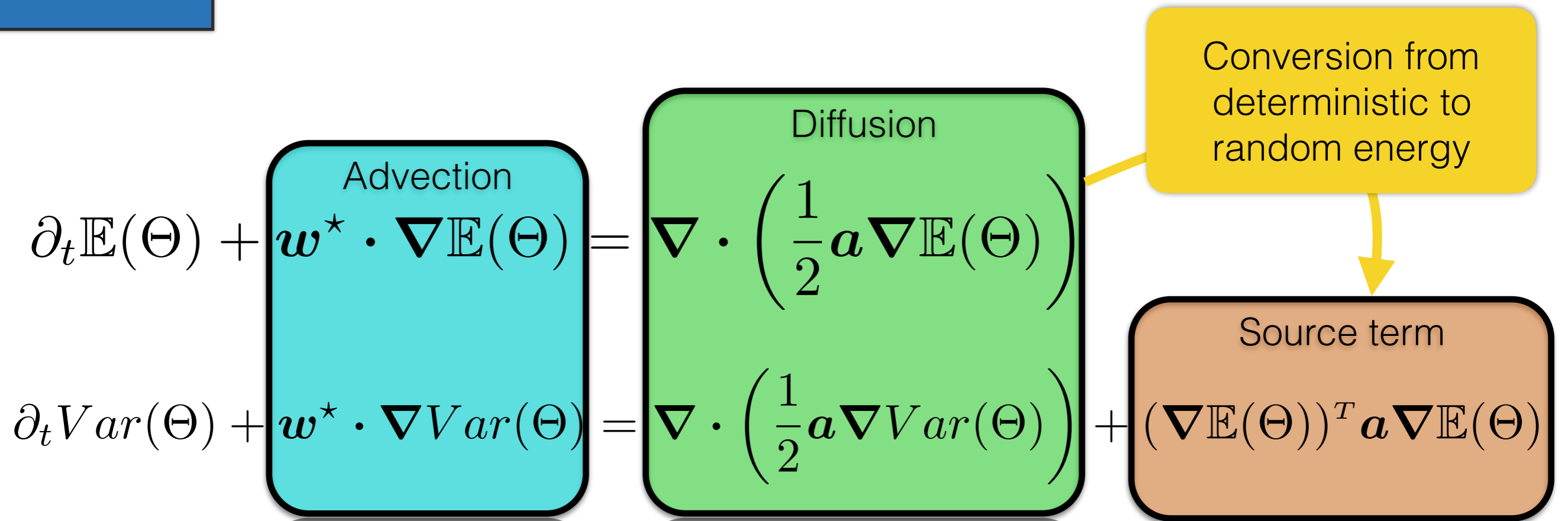
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp\!\!\!\perp w(\bullet, t), \forall t' \leq t$$



Large scales:

$$\mathbf{w}$$

Small scales:

$$\sigma \dot{B}$$

Variance tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp \mathbf{w}(\bullet, t), \forall t' \leq t$$

Conservation of the energy $\frac{d}{dt} \int_{\Omega} \Theta^2 = 0$

Advection $\partial_t \mathbb{E}(\Theta) + \mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta) =$

Diffusion $\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right)$

Conversion from deterministic to random energy

$\partial_t Var(\Theta) + \mathbf{w}^* \cdot \nabla Var(\Theta) =$

$\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla Var(\Theta) \right) +$

Source term

$(\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta)$

Part II

Stochastic Navier-Stokes

SALT vs LU

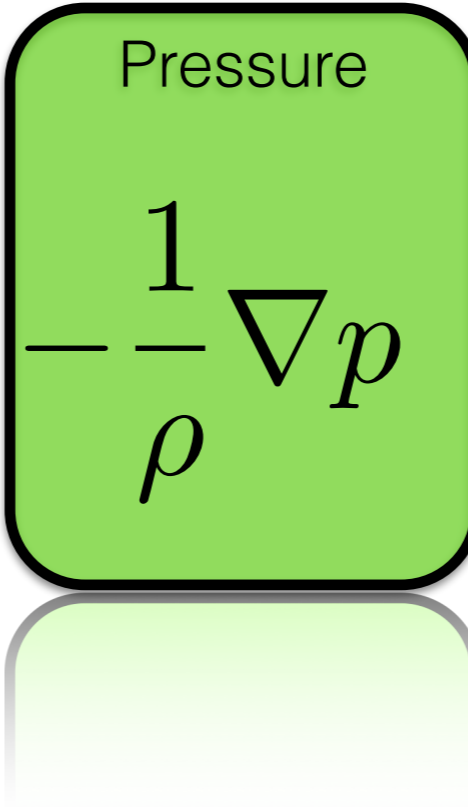
Comparison

	LU	SALT
Scalar (e.g. SQG)	Identical	
Navier-Stokes	2 differences	
Vorticity	2 differences	
Kinetic energy conservation	✓	✗
Helicity (& 2D enstrophy) conservation	✗	✓

Navier-Stokes LU

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

$$\frac{Dw}{Dt} = \text{Pressure} \left[-\frac{1}{\rho} \nabla p \right]$$


Navier-Stokes LU

Stochastic
transport
of Itô drift

$$\frac{Dw}{Dt} =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

Stochastic transport of Ito drift

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \nabla p$$

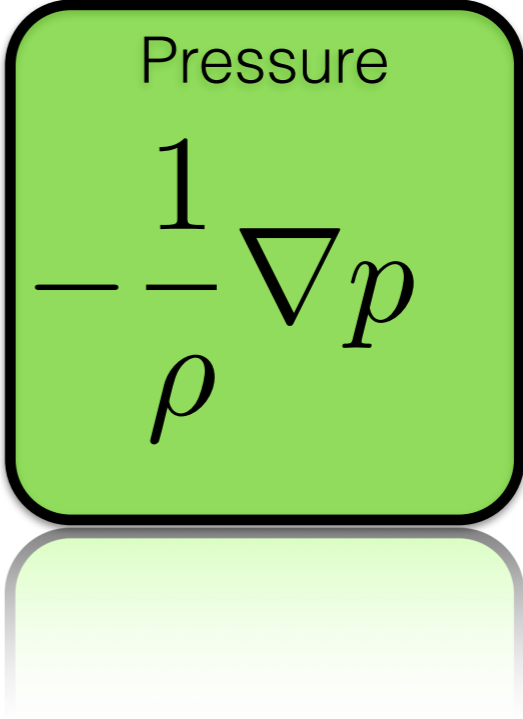
Pressure

Conserve (Ito) kinetic energy

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = \text{Pressure} \frac{1}{\rho} \nabla p$$


Navier-Stokes SALT

Stochastic
transport of
Stratonovich drift

$$\frac{Dw^*}{Dt}$$

$$+ \nabla (\sigma \circ \dot{B})^T w^* =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

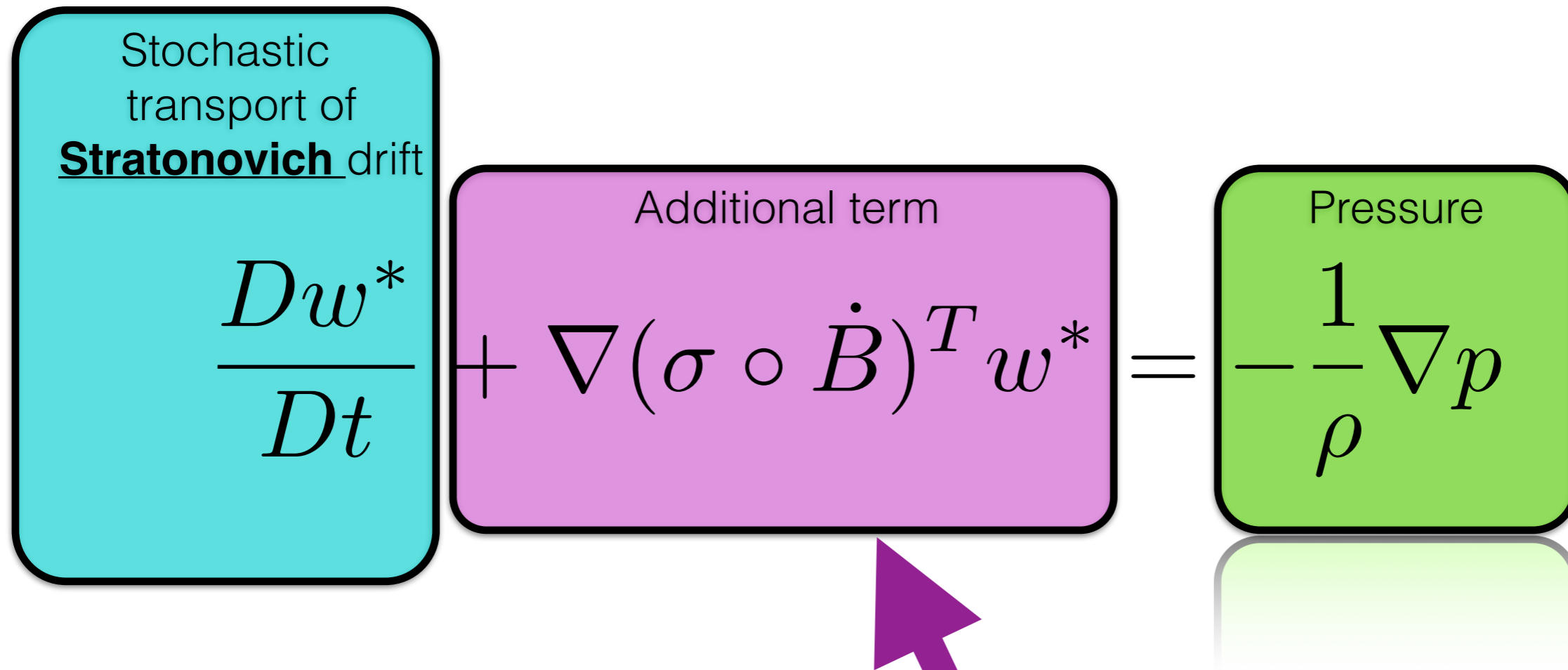
Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Additional term

Pressure



Conserve (Stratonovich) helicity

Part III

Unresolved velocity parametrisation

Code available online

Large scales:

$$w$$

Small scales:

$$\sigma \dot{B}$$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Parameter-free models for σ

	Cotter et al. 2018b	Resseguier et al. 2017b
Applicable to :	LU & SALT	LU & SALT
Homogeneous	No	Yes
Stationary	Yes	Not anymore
Self-similar assumption	No	Yes
Data-driven	Yes	No

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

weighted
EOFs learned
on data

iid
Brownian
motion

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

weighted
EOFs learned
on data

PCA on : $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

weighted
EOFs learned
on data

PCA on : $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

Flow from **high-**
resolution velocity

Flow from **low-**
resolution velocity

(improvement of)
 Resseguier et al. 2017b:
 Self-similar model

Large scales:

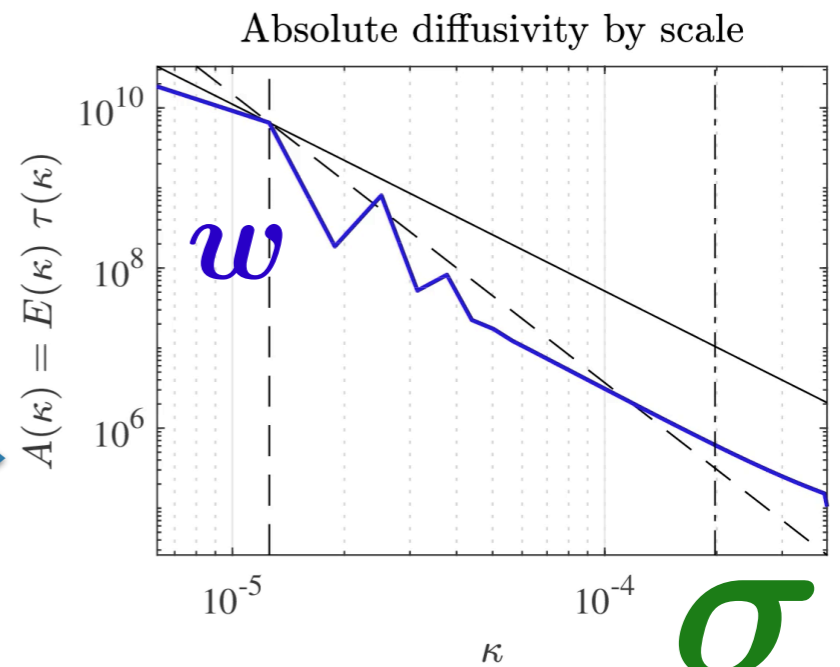
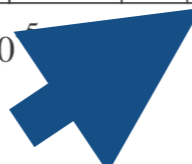
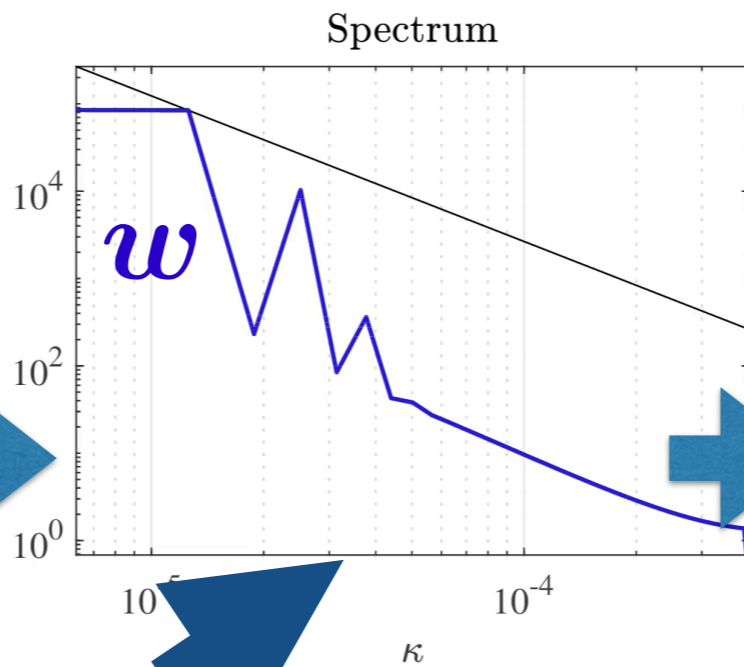
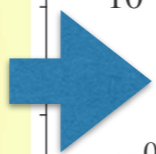
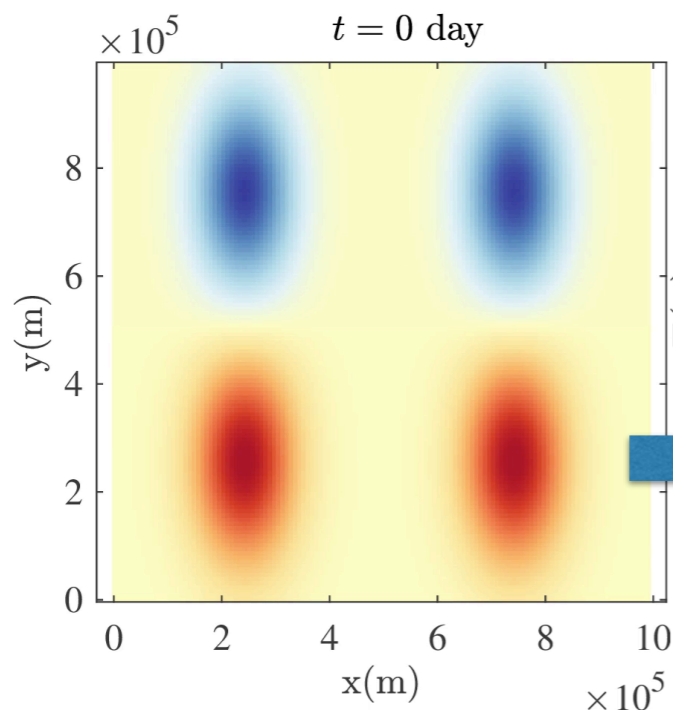
w

Small scales:

$\sigma \dot{B}$

Variance
 tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$



Kinetic energy
 Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \int_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

(improvement of)
 Resseguier et al. 2017b:
 Self-similar model

Large scales:

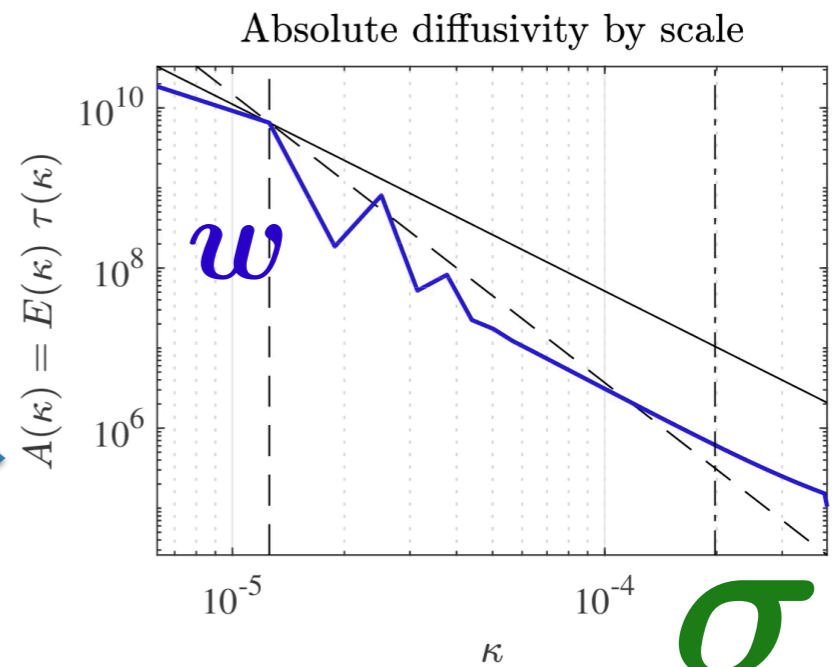
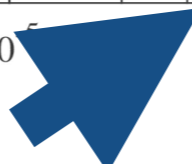
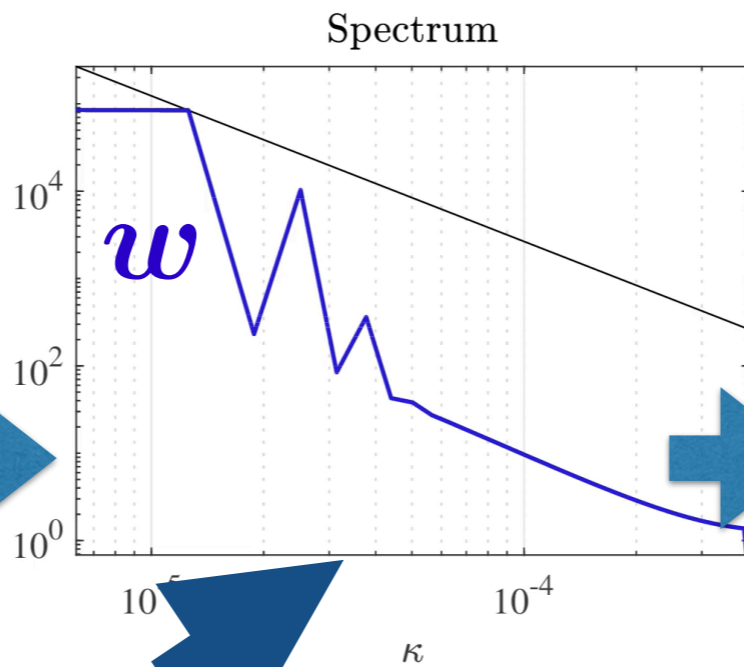
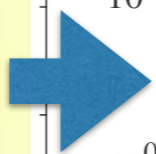
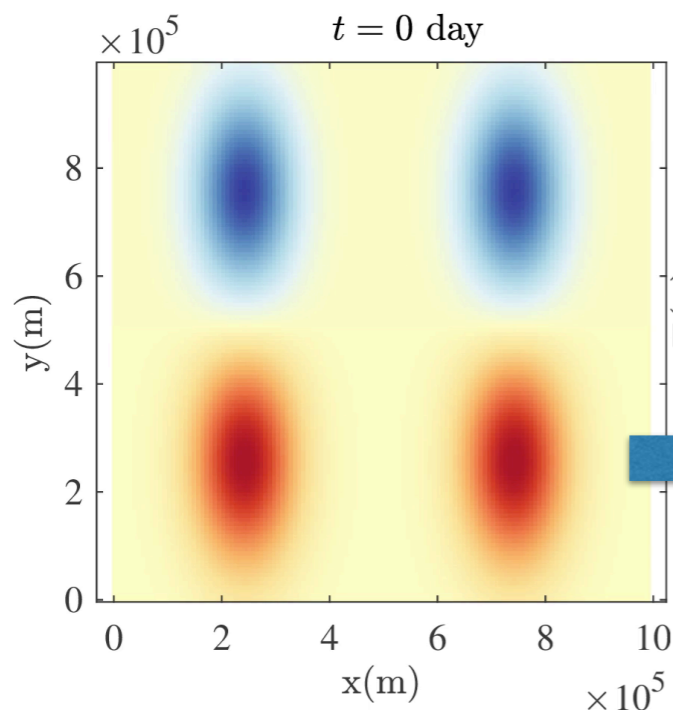
w

Small scales:

$\sigma \dot{B}$

Variance
 tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$



Kinetic energy
 Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \int_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

(improvement of) Resseguier et al. 2017b: Self-similar model

Large scales:

w

Small scales:

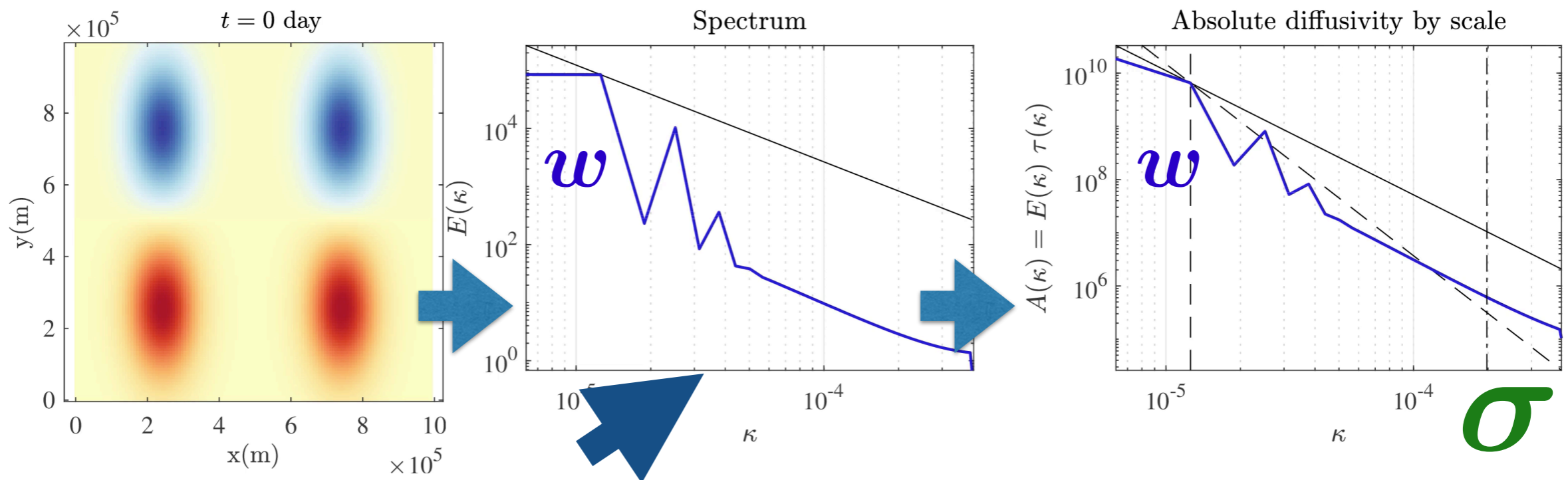
$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Absolute Diffusivity
Spectral Density

$$A(\kappa) = E(\kappa)\tau(\kappa) = \kappa^{-3/2} E^{1/2}(\kappa)$$



Kinetic energy
Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \int_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

$t = 100$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$- \frac{1}{\tau} (b - F) \quad \text{Forcing}$$

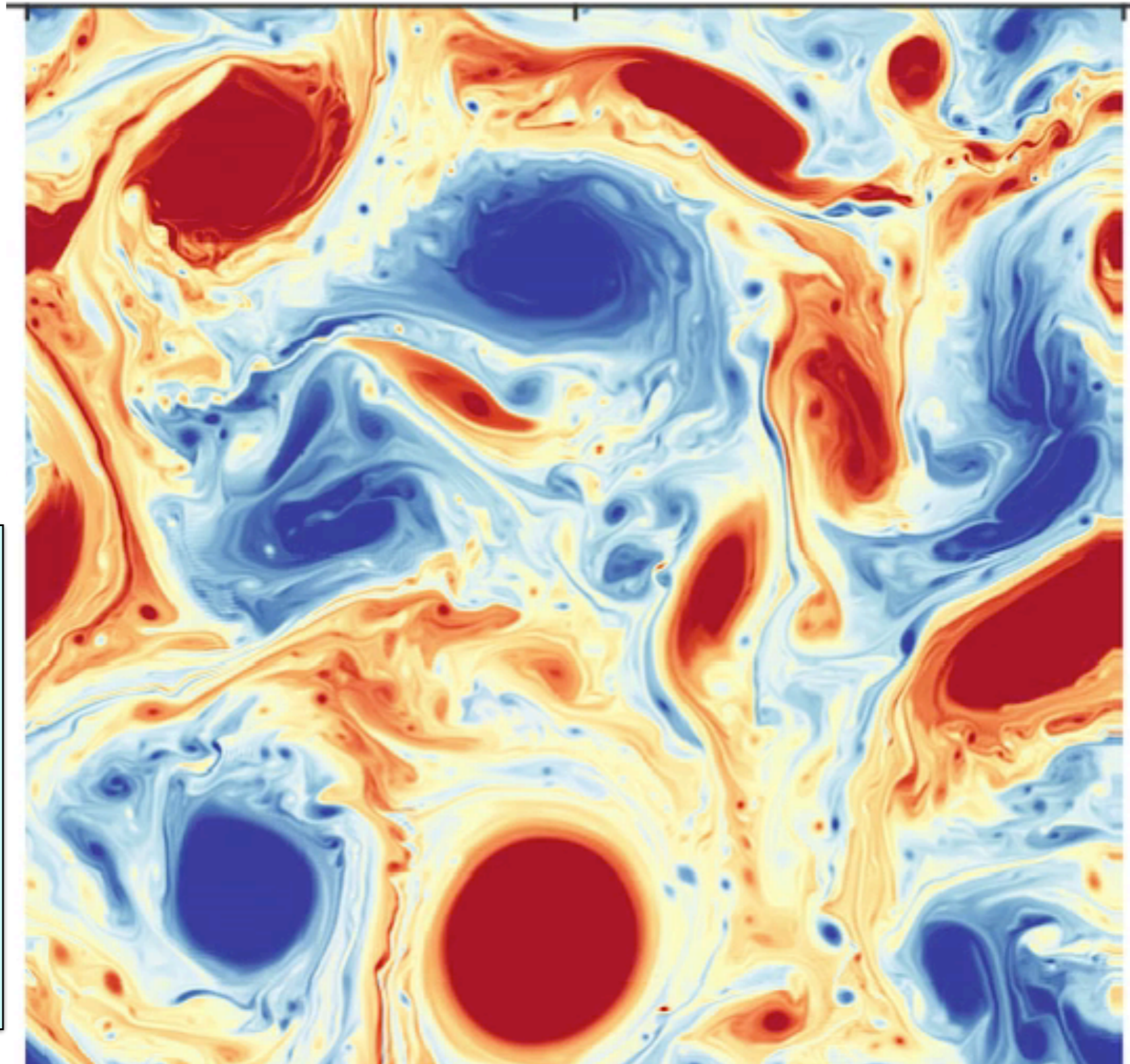
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

deterministic

SQG

512 x 512



$t = 100$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$-\frac{1}{\tau} (b - F) \quad \text{Forcing}$$

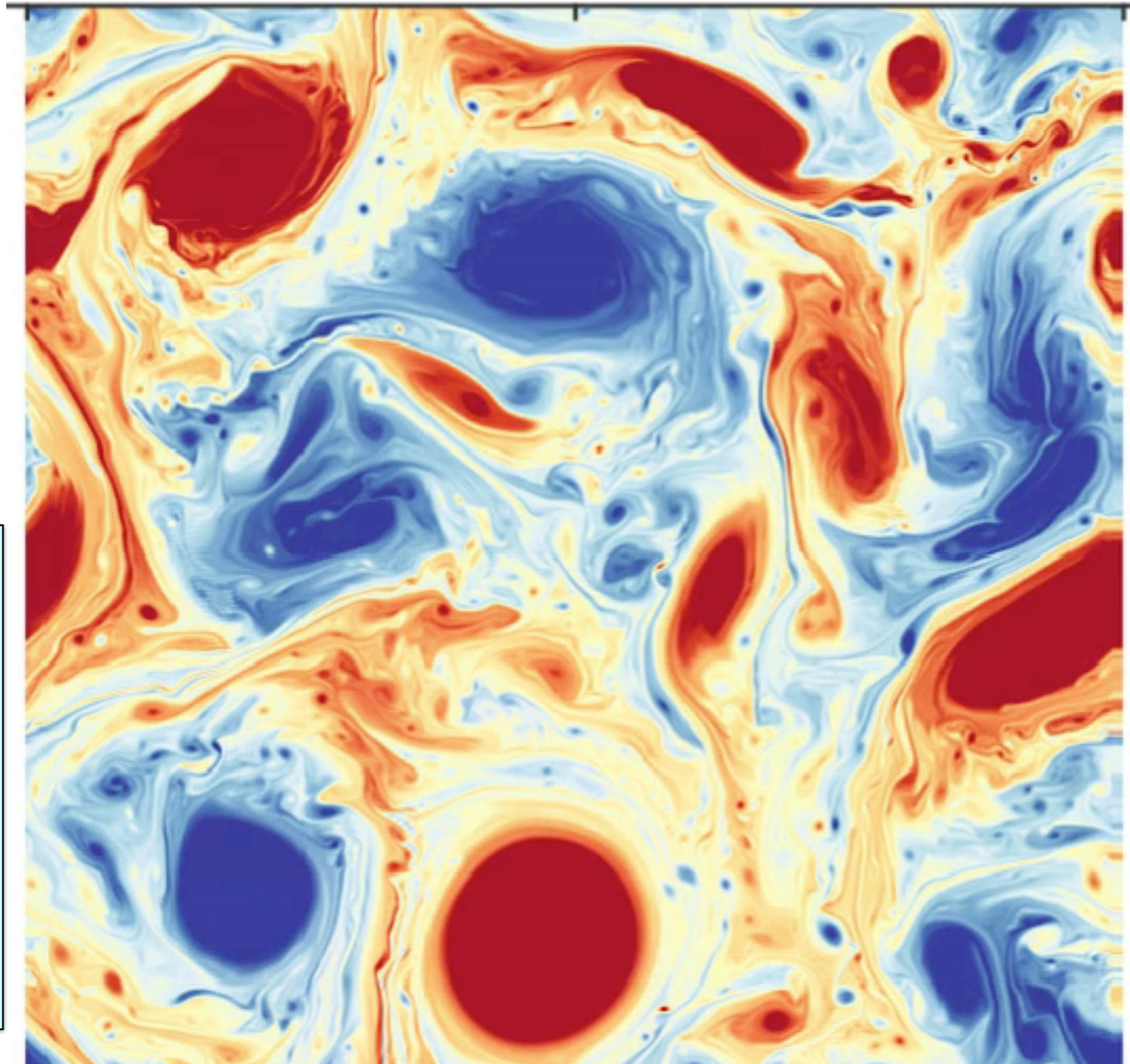
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

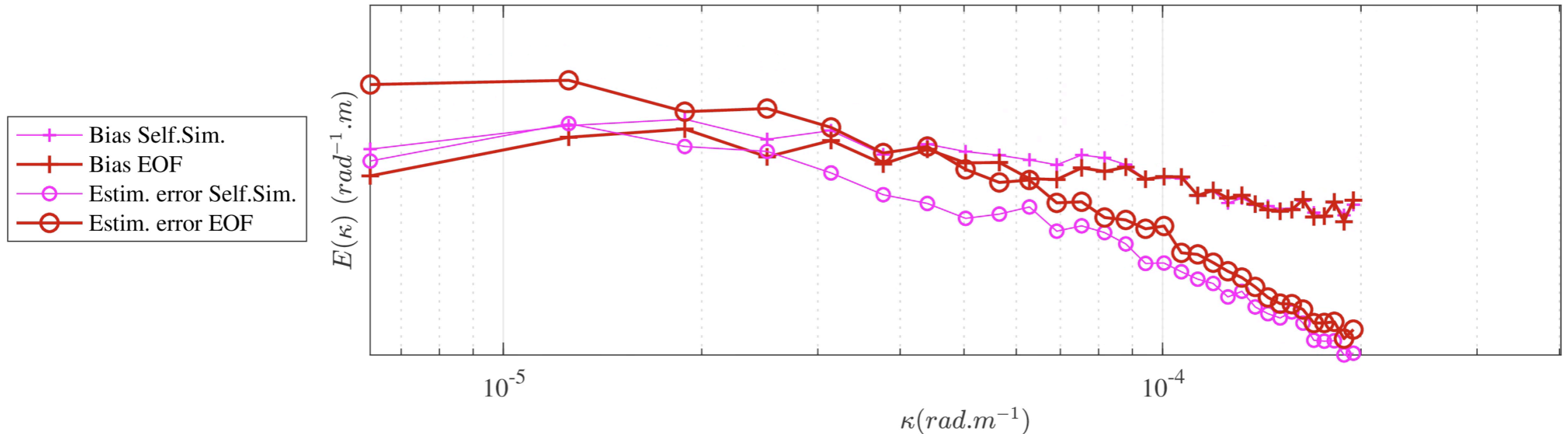
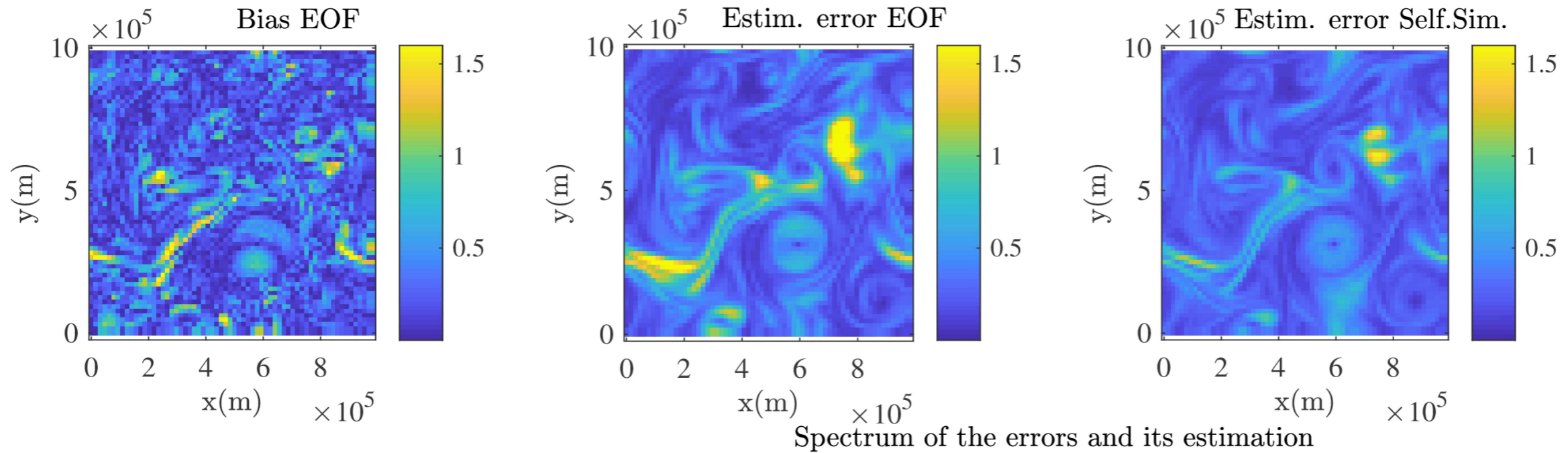
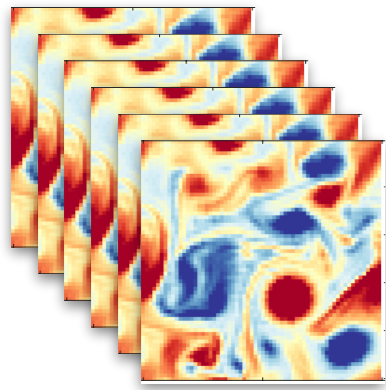
deterministic

SQG

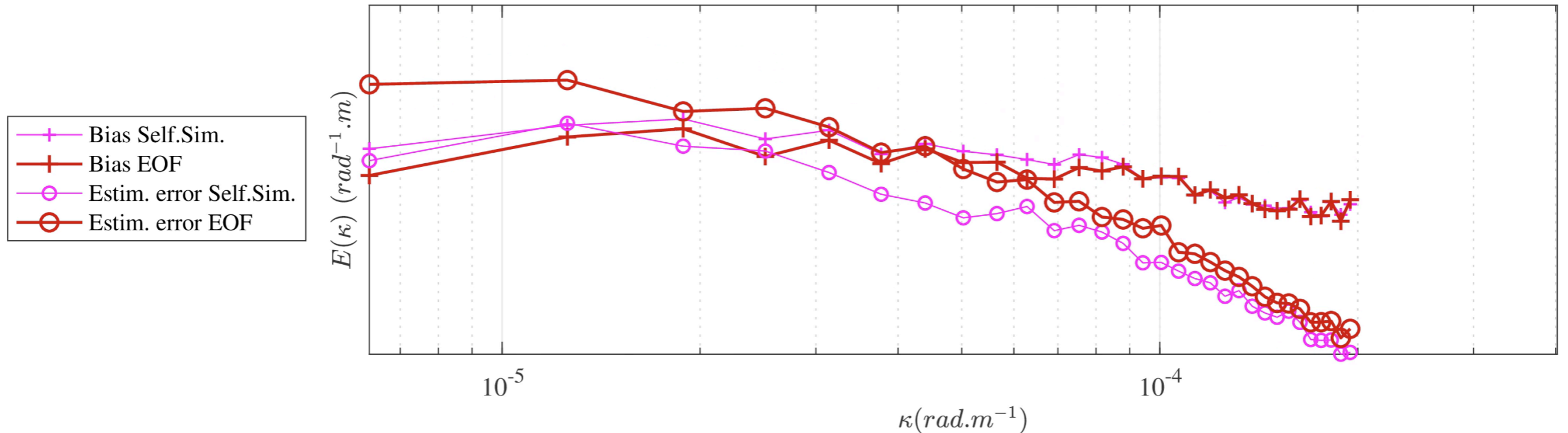
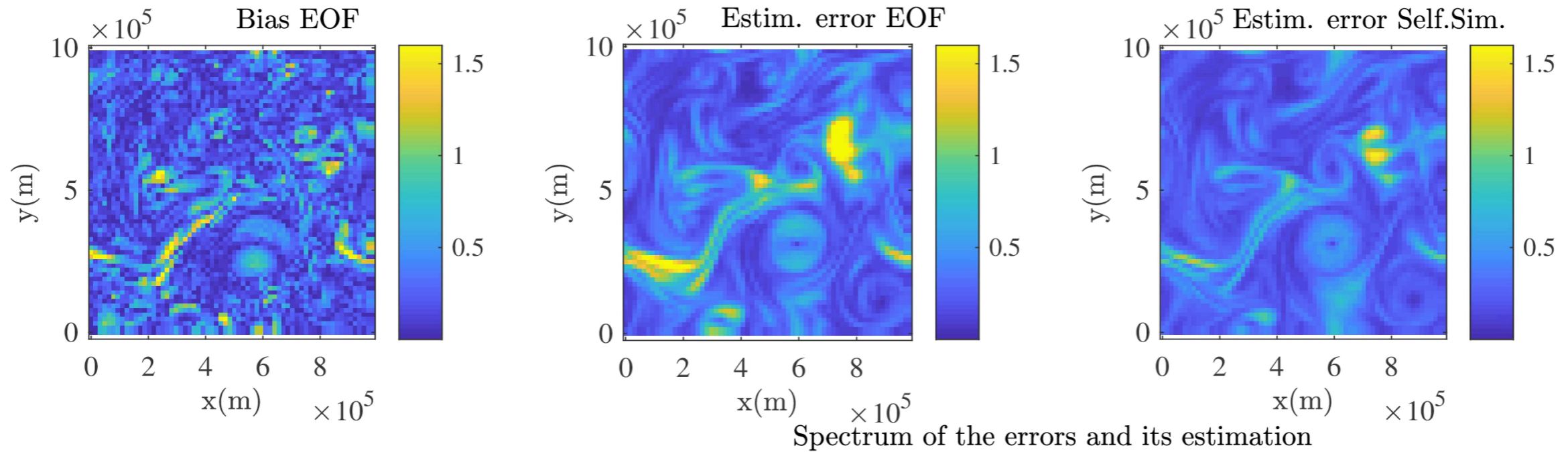
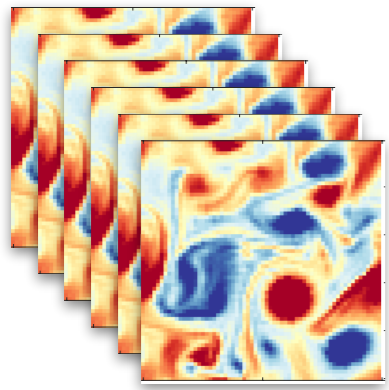
512 x 512



Ensemble : uncertainty quantification



Ensemble : uncertainty quantification



Part IV

Unresolved velocity
non-stationary heterogeneity

$t = 0$ day

SQG

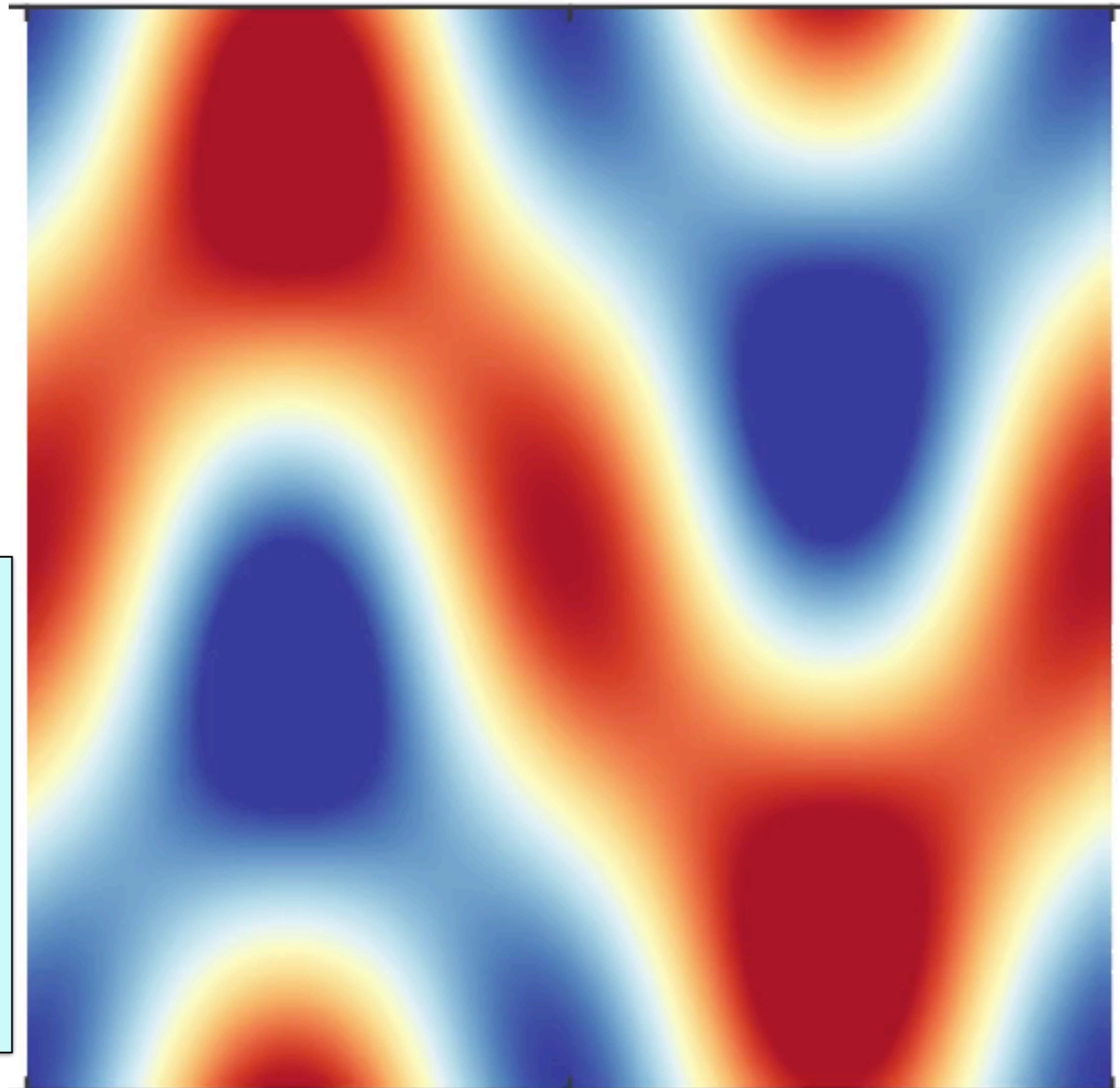
$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)



$t = 0$ day

SQG

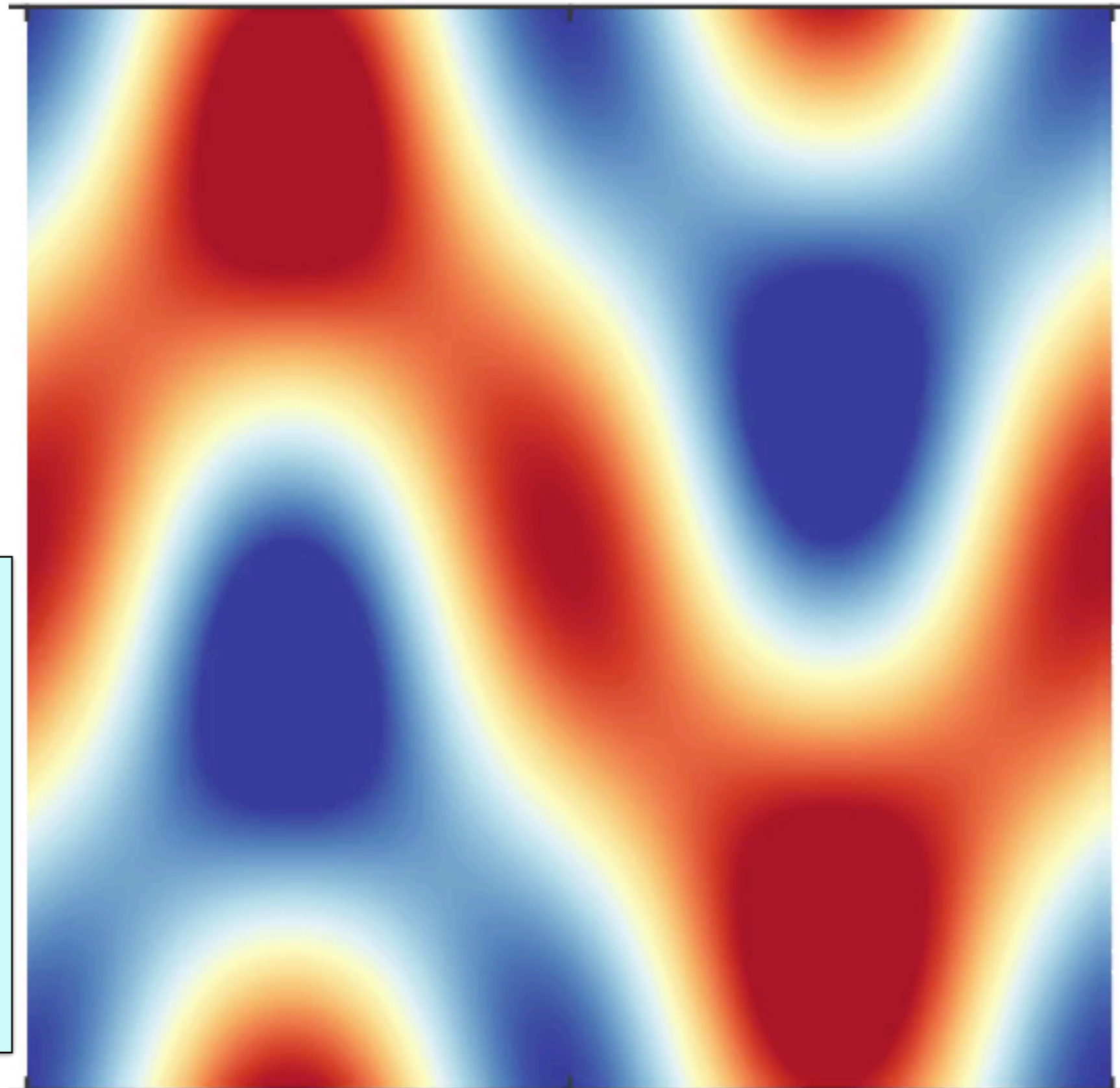
$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-} \\ \text{viscosity}$$

$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)



$t = 15 \text{ day}$

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

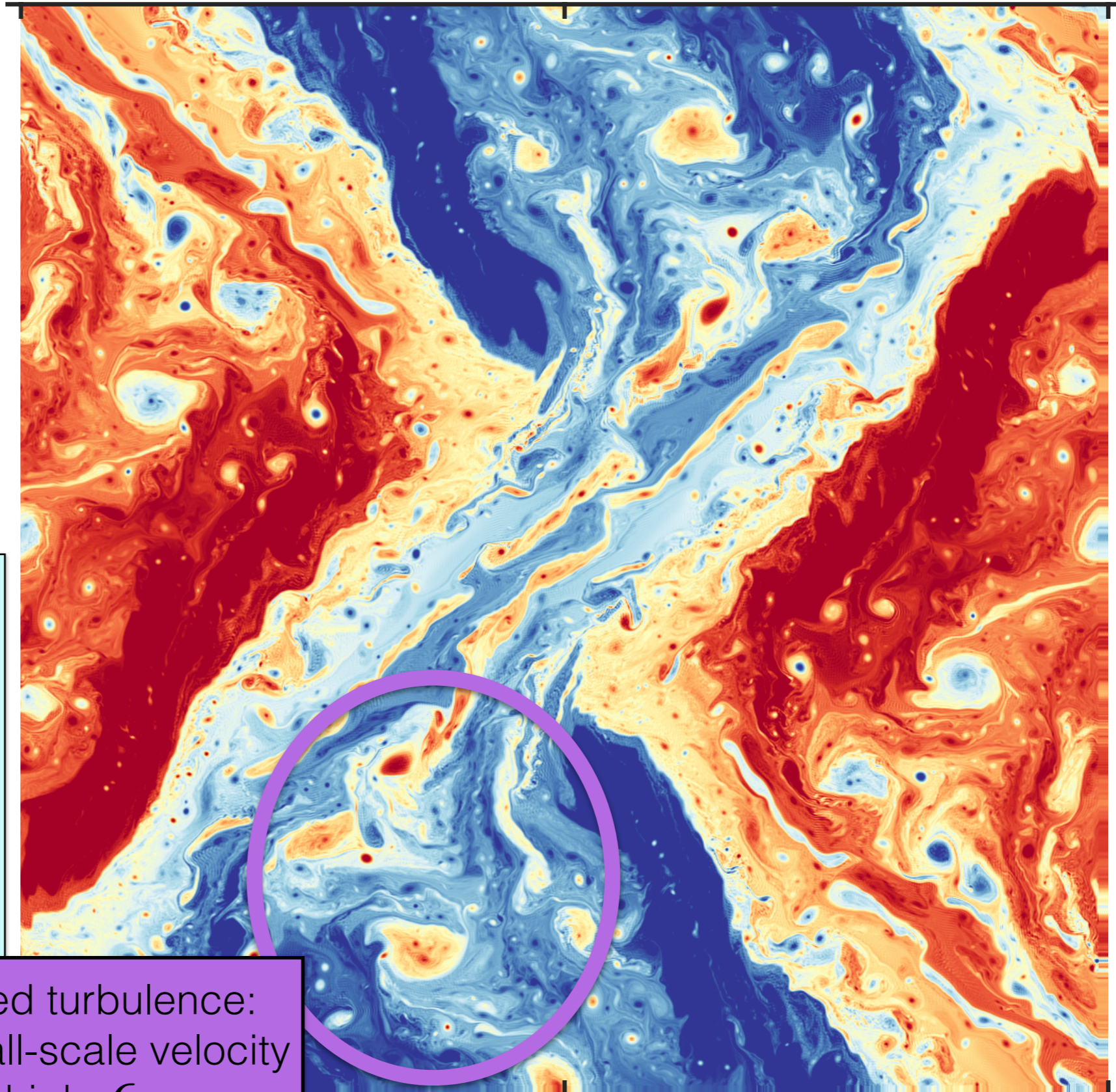
$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999)

Developed turbulence:
strong small-scale velocity
high ϵ



$t = 15$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

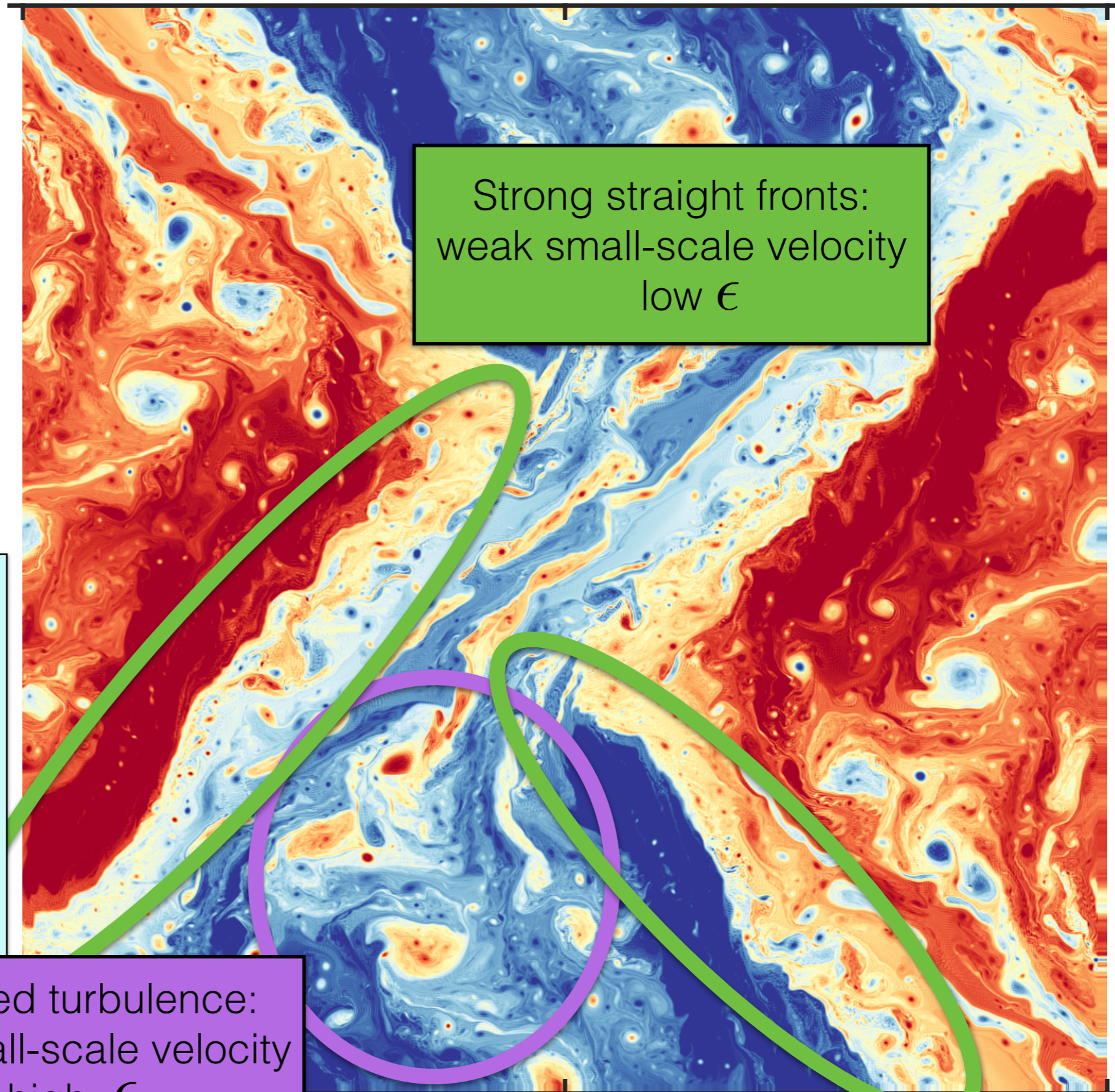
Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999)

Developed turbulence:
strong small-scale velocity
high ϵ

Strong straight fronts:
weak small-scale velocity
low ϵ



Large scales:

w

Small scales:

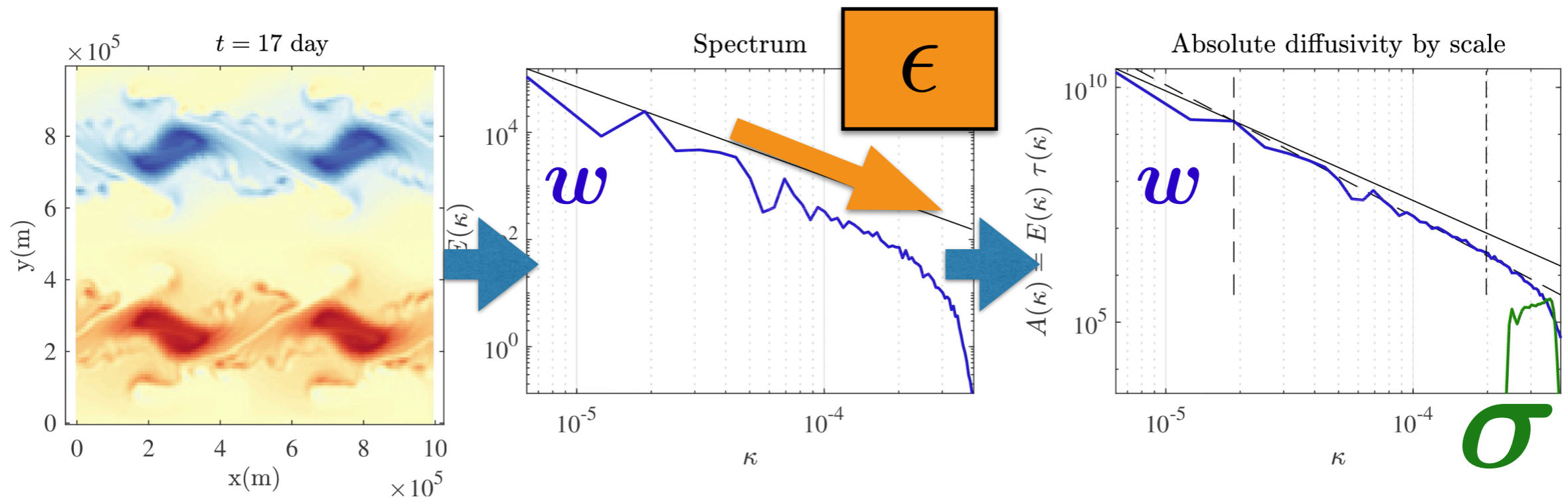
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Heterogeneous modulation of σ

(for Resseguier et al. 2017b)



Large scales:

w

Small scales:

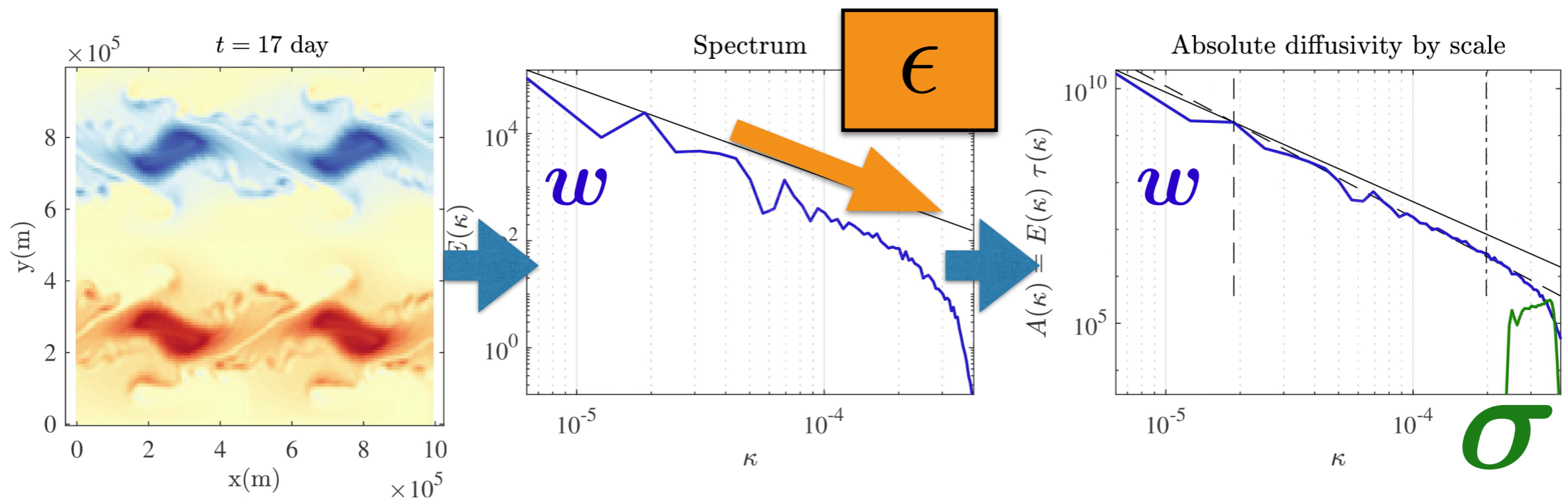
$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Heterogeneous modulation of σ

(for Resseguier et al. 2017b)



Heterogeneous modulation of σ

(for Resseguier et al. 2017b)

Large scales:

w

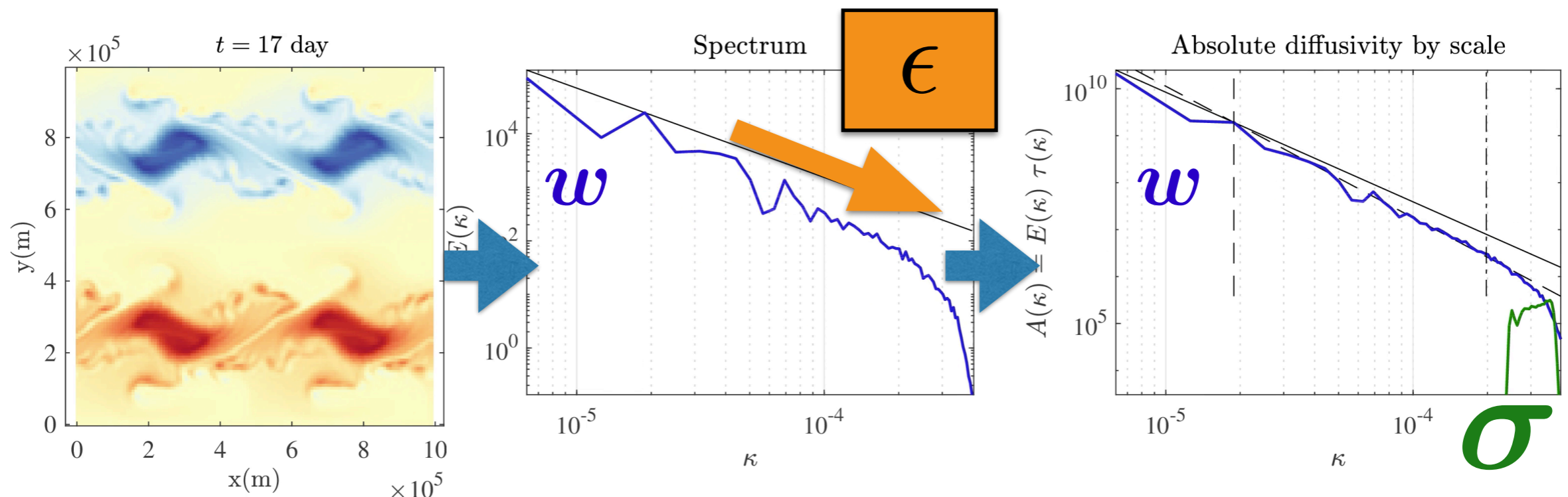
Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst.} \epsilon^p(x) \kappa^{-q}$$



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

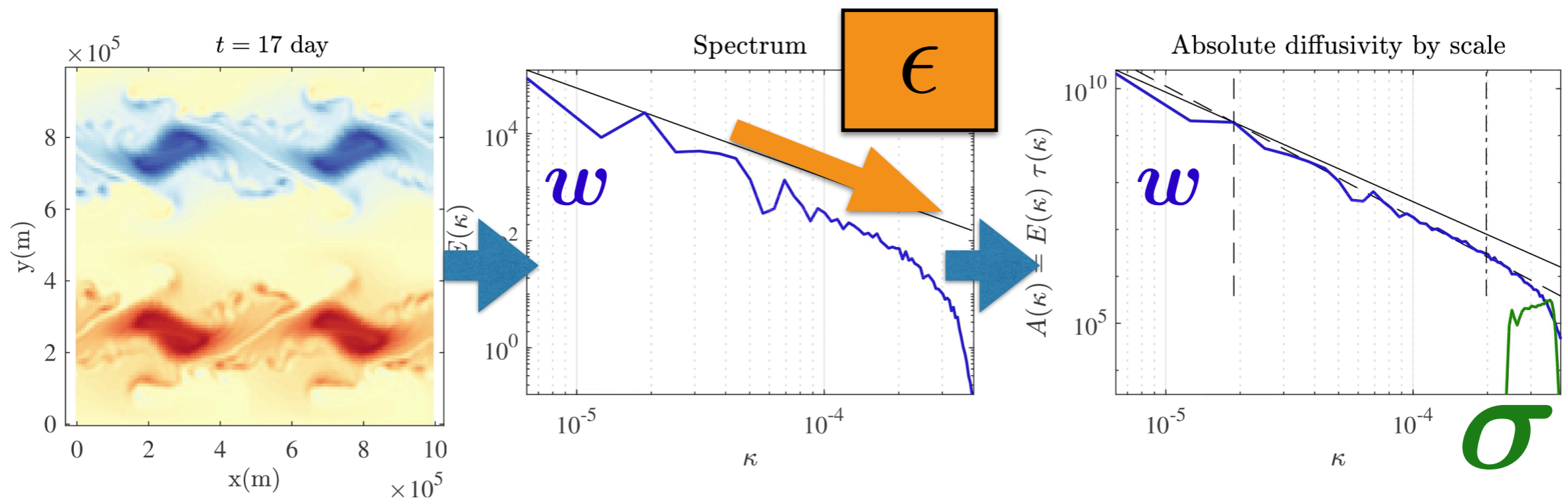
Heterogeneous

modulation of σ

(for Resseguier et al. 2017b)

Dynamics
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst.} \cdot \epsilon^p(x) \kappa^{-q}$$



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Heterogeneous

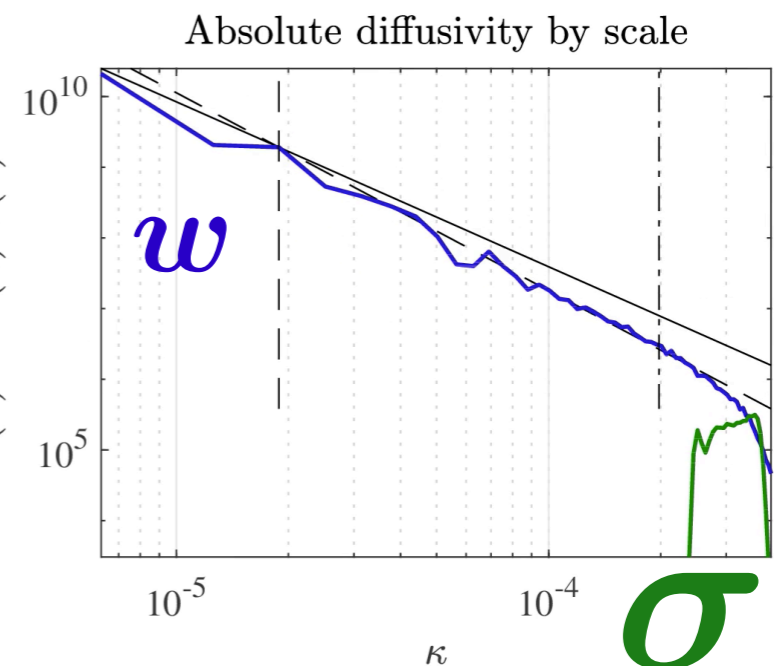
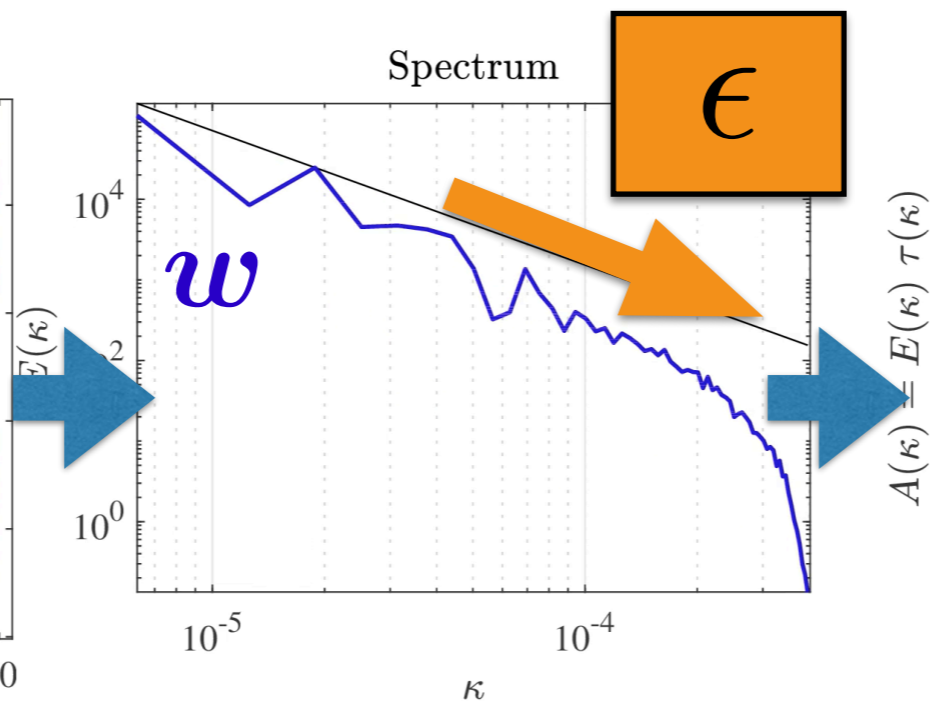
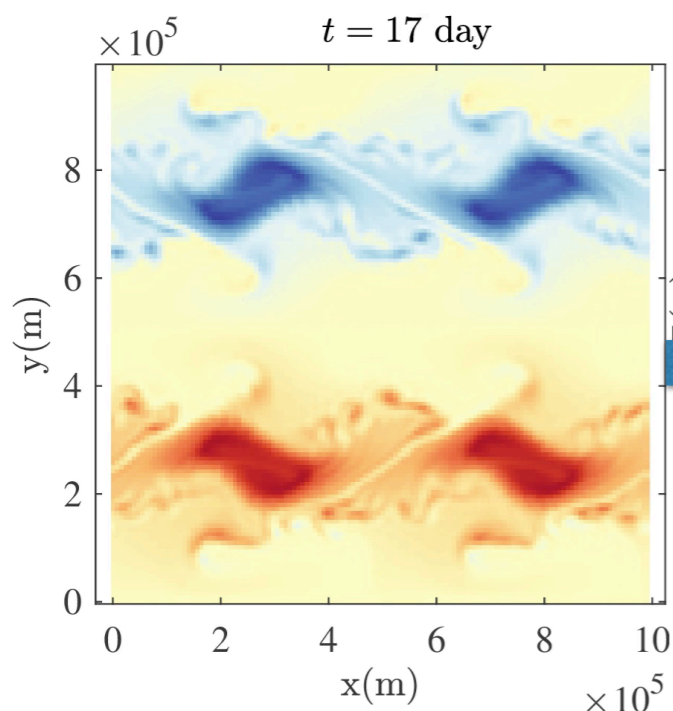
modulation of σ

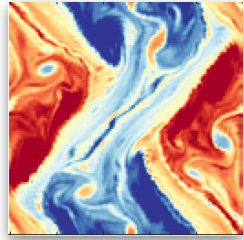
(for Resseguier et al. 2017b)

Dynamics
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst.} \epsilon^p(x) \kappa^{-q}$$

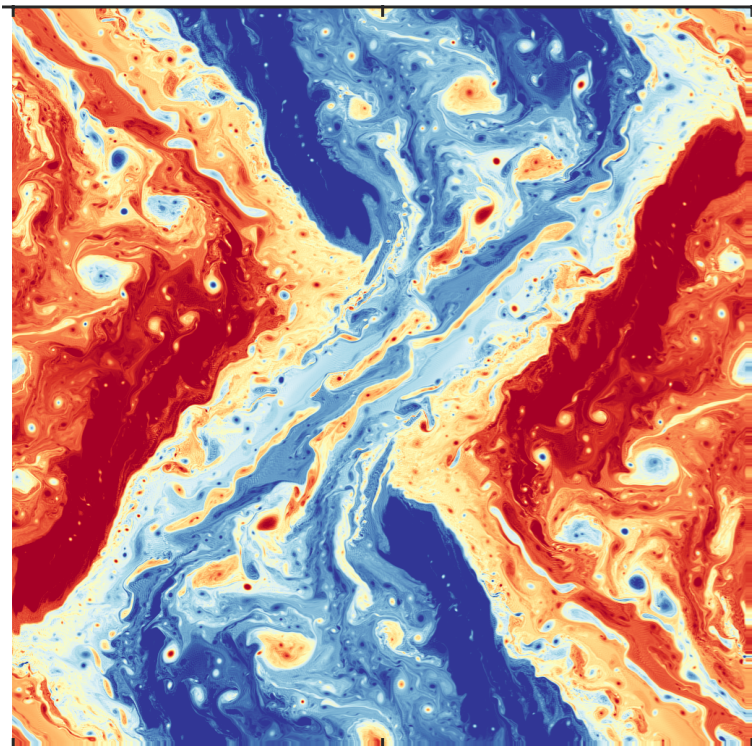
Learned on large scales at each time t



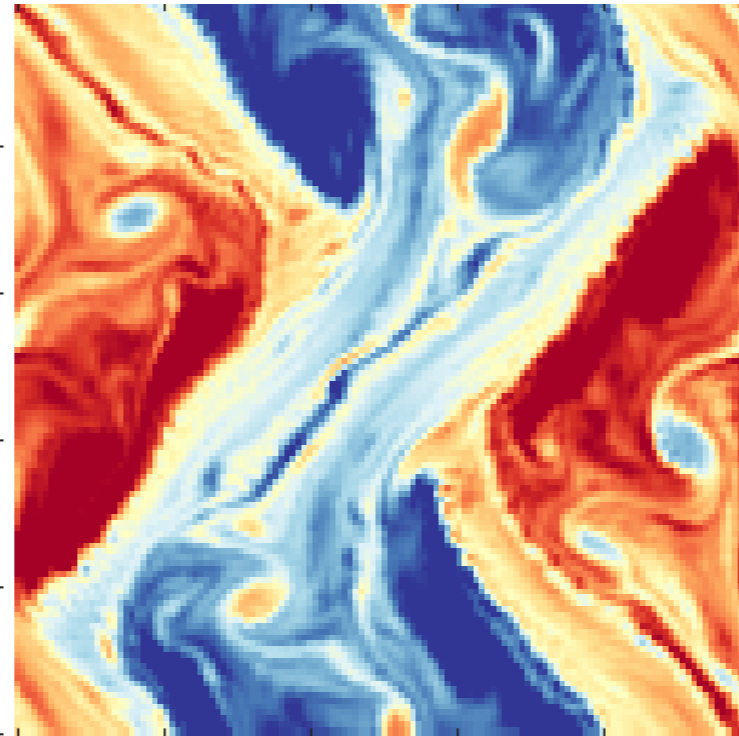


Heterogenous modulation

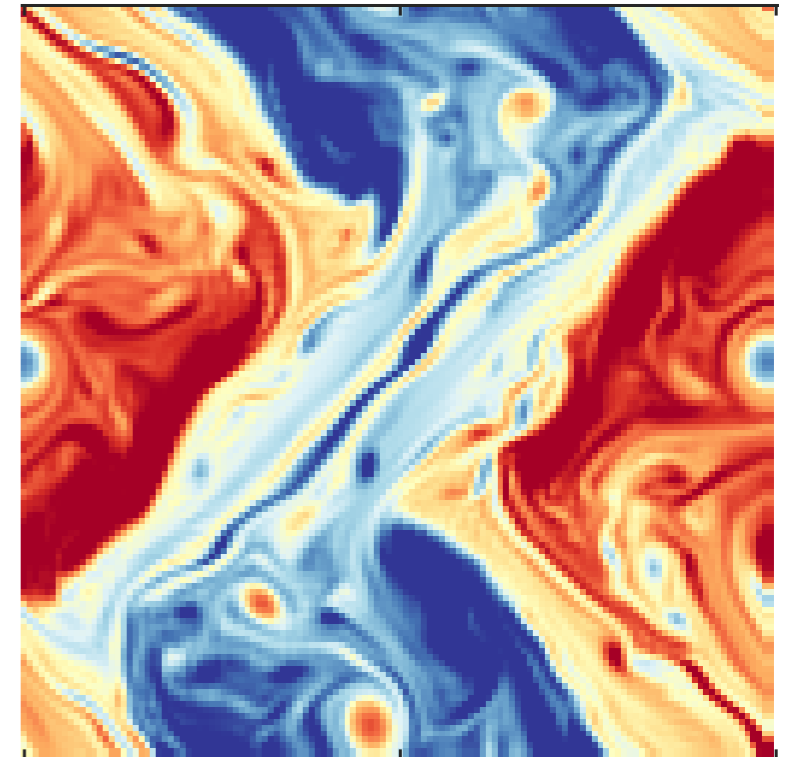
Deterministic 1024 x 1024

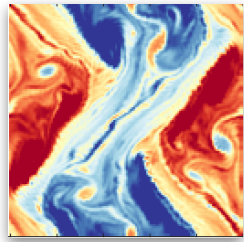


Stochastic 128 x 128
with Homogeneous
small-scale velocity



Stochastic 128 x 128
with Energy-flux-based
modulation





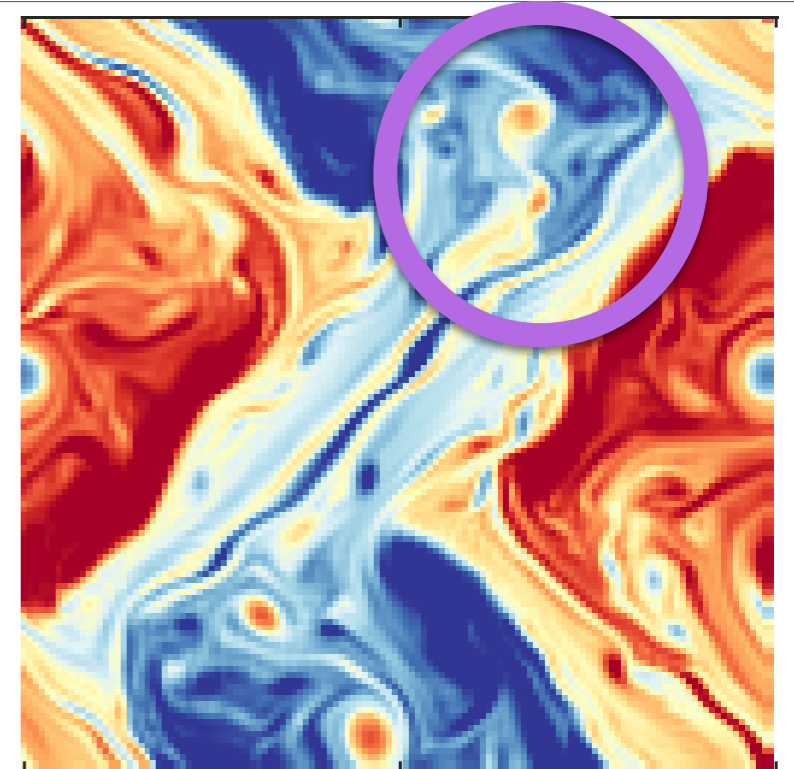
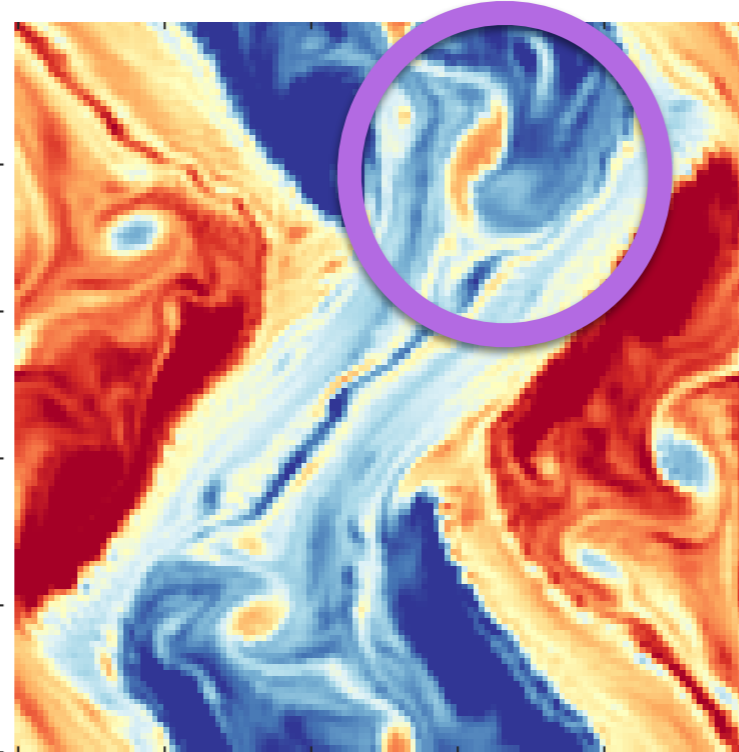
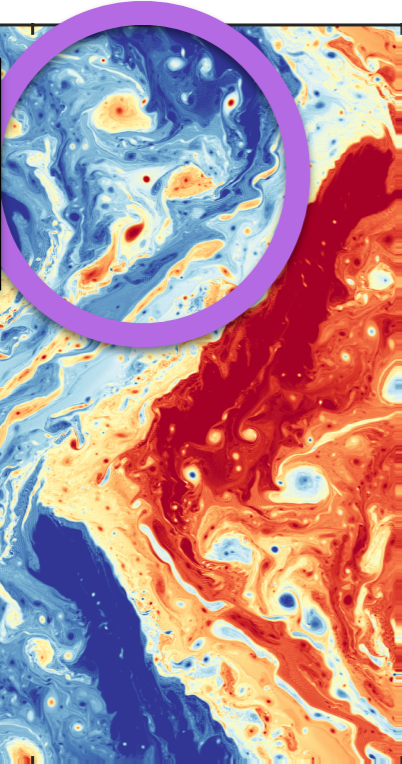
Heterogenous modulation

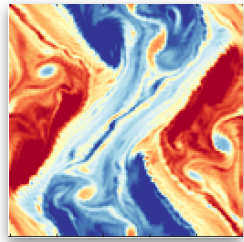
Deterministic 1024 x 1024

Stochastic 128 x 128
with Homogeneous
small-scale velocity

Stochastic 128 x 128
with Energy-flux-based
modulation

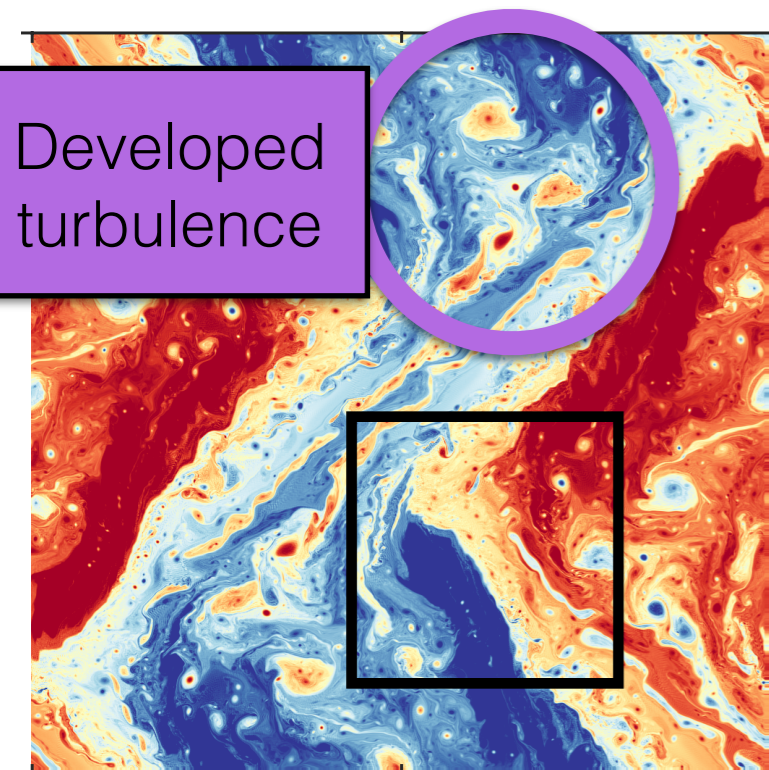
Developed
turbulence



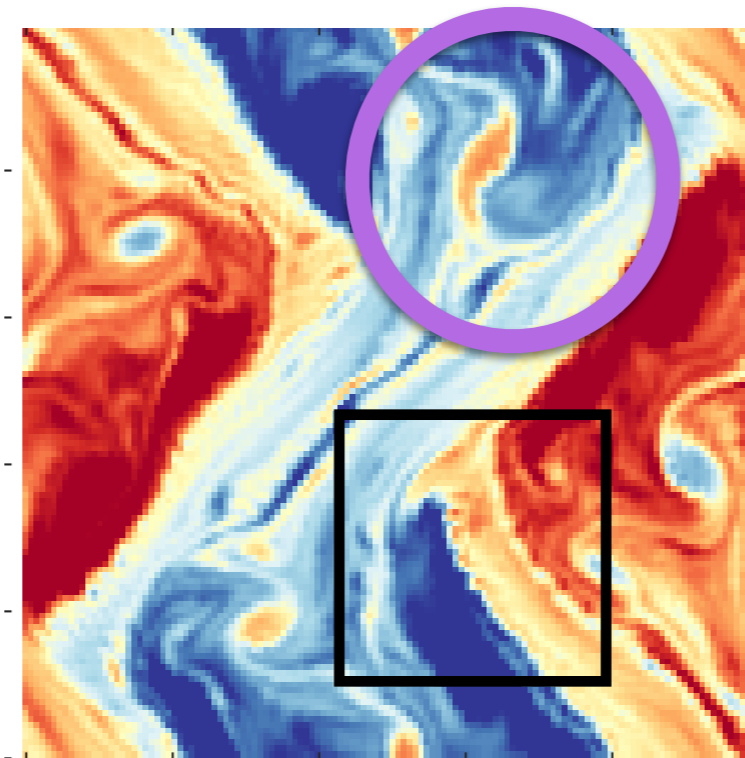


Heterogenous modulation

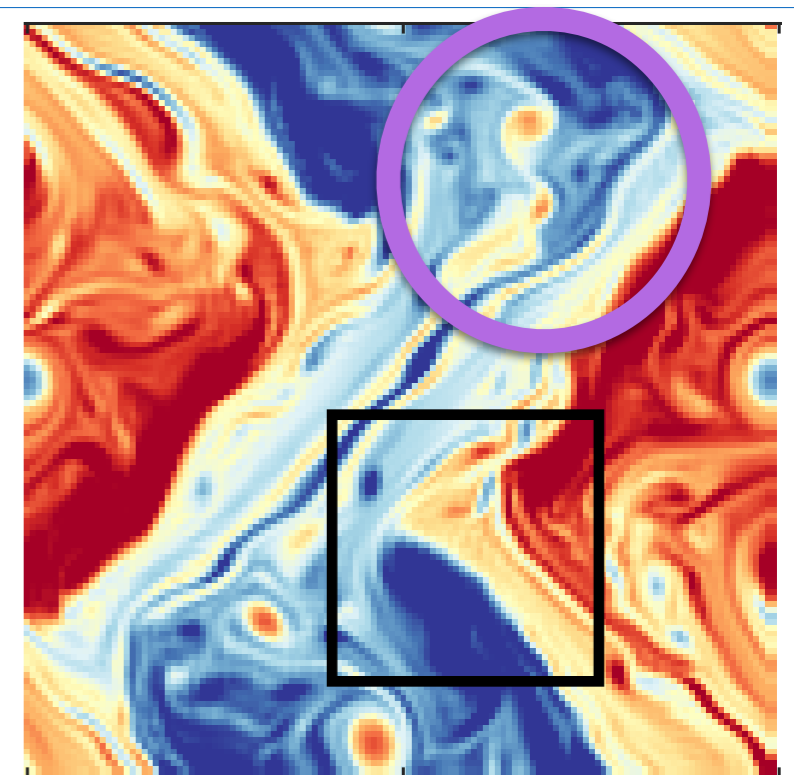
Deterministic 1024 x 1024

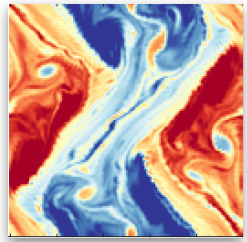


Stochastic 128 x 128
with Homogeneous
small-scale velocity



Stochastic 128 x 128
with Energy-flux-based
modulation





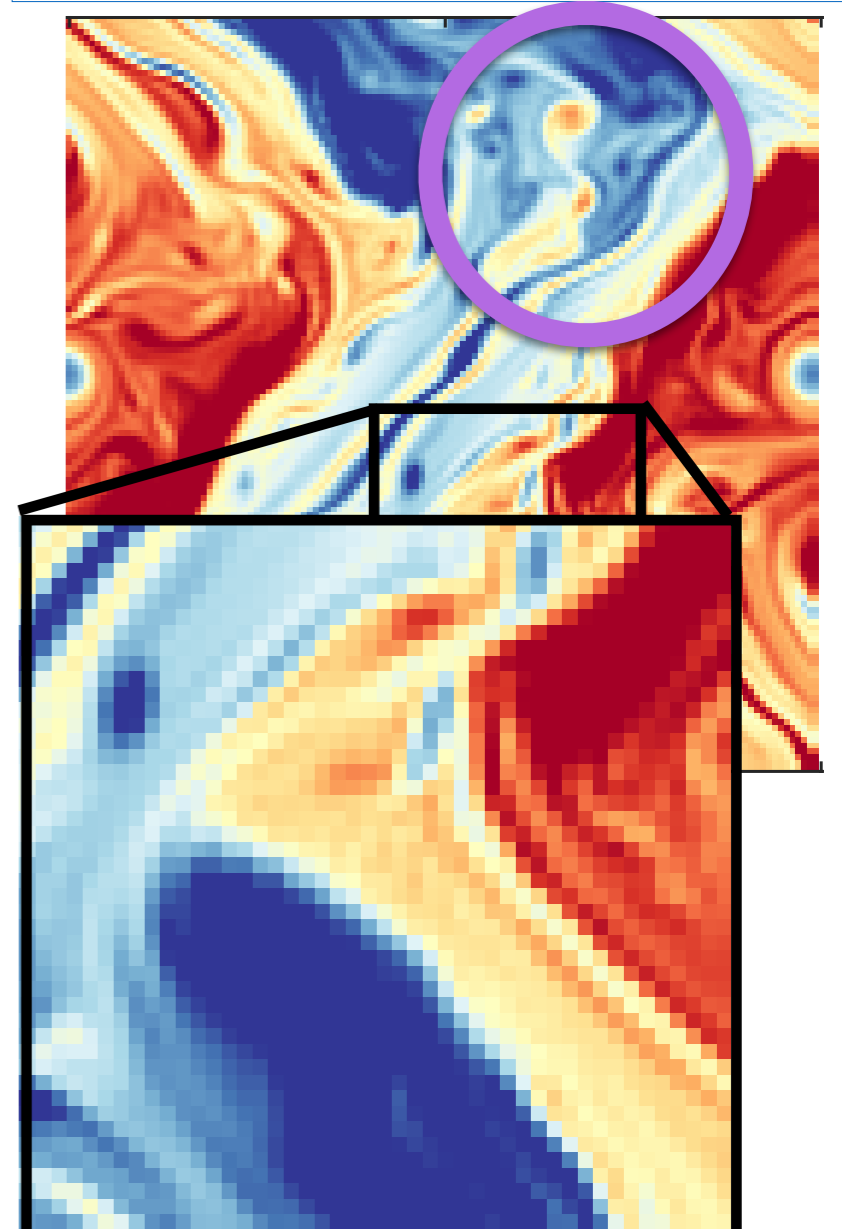
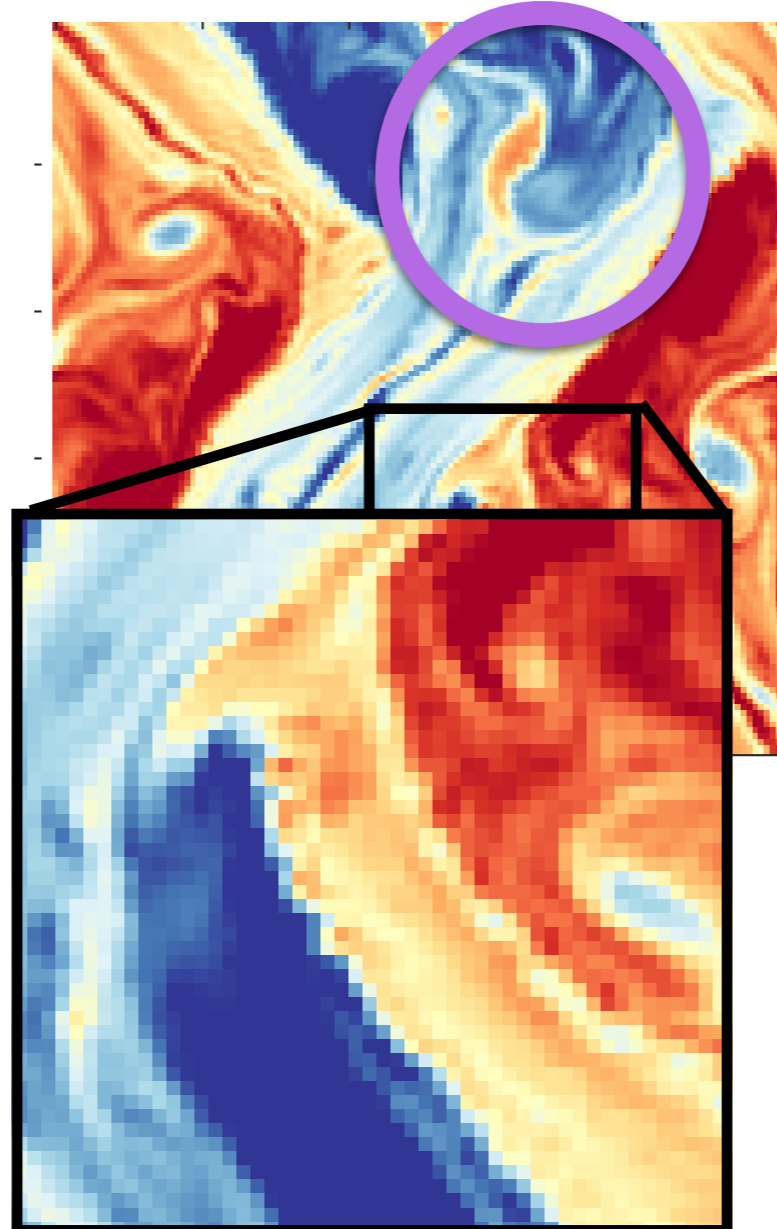
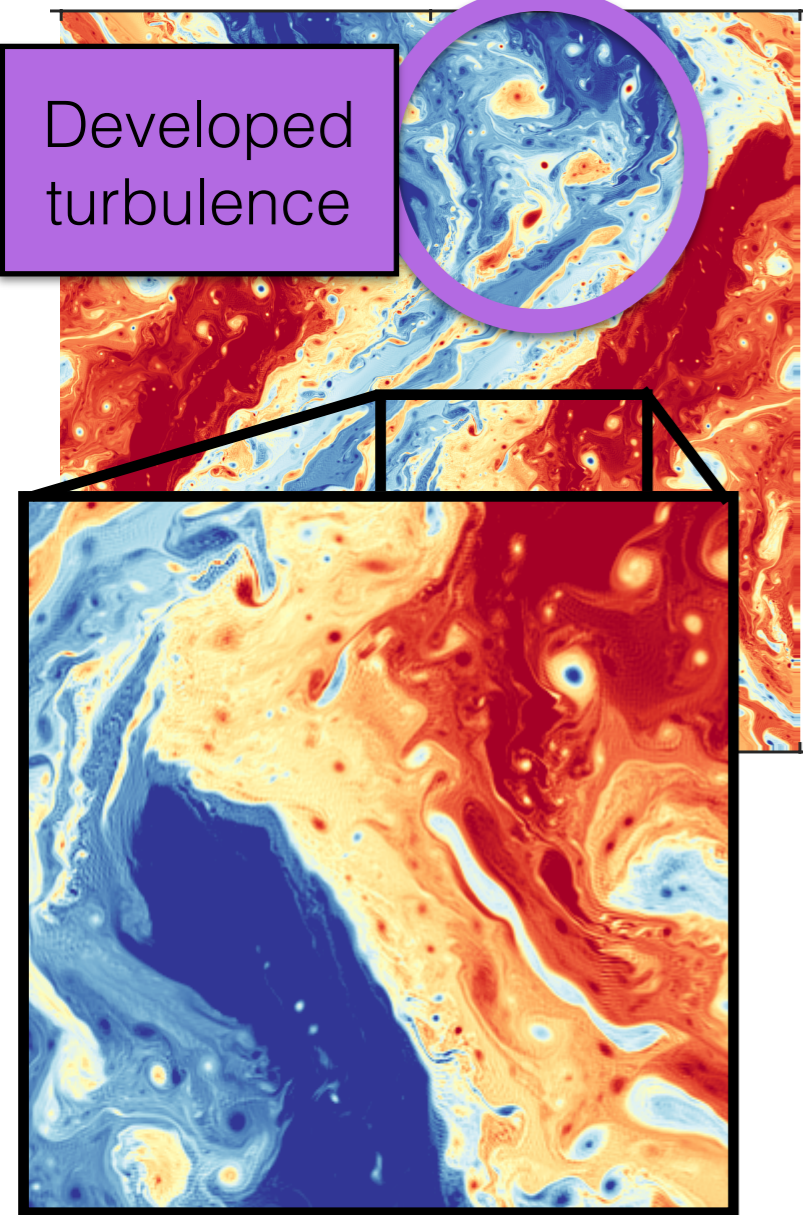
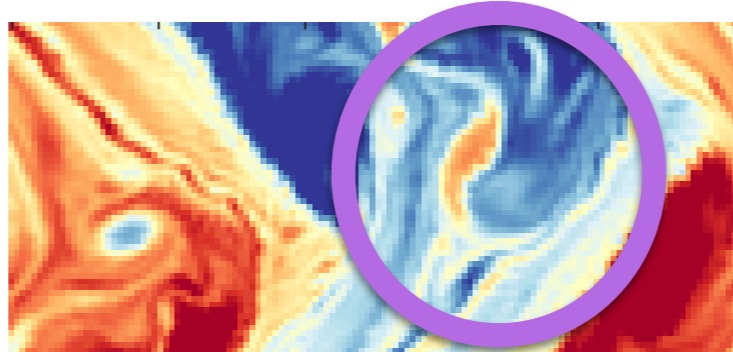
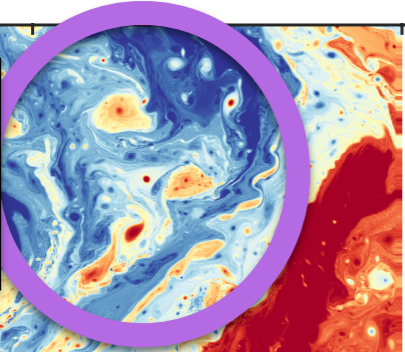
Heterogenous modulation

Deterministic 1024 x 1024

Stochastic 128 x 128
with Homogeneous
small-scale velocity

Stochastic 128 x 128
with Energy-flux-based
modulation

Developed
turbulence



Conclusion

Conclusion

- Stochastic transport (in LU & SALT) blindly describes unresolved triades
- LU conserves kinetic energy / SALT conserves helicity
- For the small-scale velocity parametrisation, both the data-driven (Cotter et al. 2018b) and the self-similar method (Resseguier et al. 2017b) lead to accurate uncertainty quantification (to address filter divergence)
- Energy-flux modulation improves the simulations

Bibliography

- Cotter, C. J., Crisan, D., Holm, D. D., Pan, W., & Shevchenko, I. (2018). Numerically modelling stochastic Lie transport in fluid dynamics. *arXiv preprint arXiv: 1801.09729*.
- Kunita, H., Stochastic Flows and Stochastic Differential Equations, 1990. (Cambridge: Cambridge University Press).
- Resseguier, V., Mémin, E., & Chapron, B. (2017). Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophysical & Astrophysical Fluid Dynamics*, 111(3), 177-208.