



Stochastic Advection by Lie Transport and Location Uncertainty: a common ground for uncertainty quantification in fluid dynamics

Baylor Fox-Kemper, Darryl D Holm, Wei Pan, Valentin Resseguier

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Stochastic Advection by Lie Transport and Location Uncertainty: a common ground for uncertainty quantification in fluid dynamics

Baylor Fox-Kemper

Darryl D. Holm

Wei Pan

Valentin Resseguier



Motivations

- Rigorously identified subgrid dynamics effects
- Injecting likely small-scale dynamics
- Studying bifurcations and attractors



Climate projections

- Quantification of modeling errors



Ensemble forecasts and data assimilation

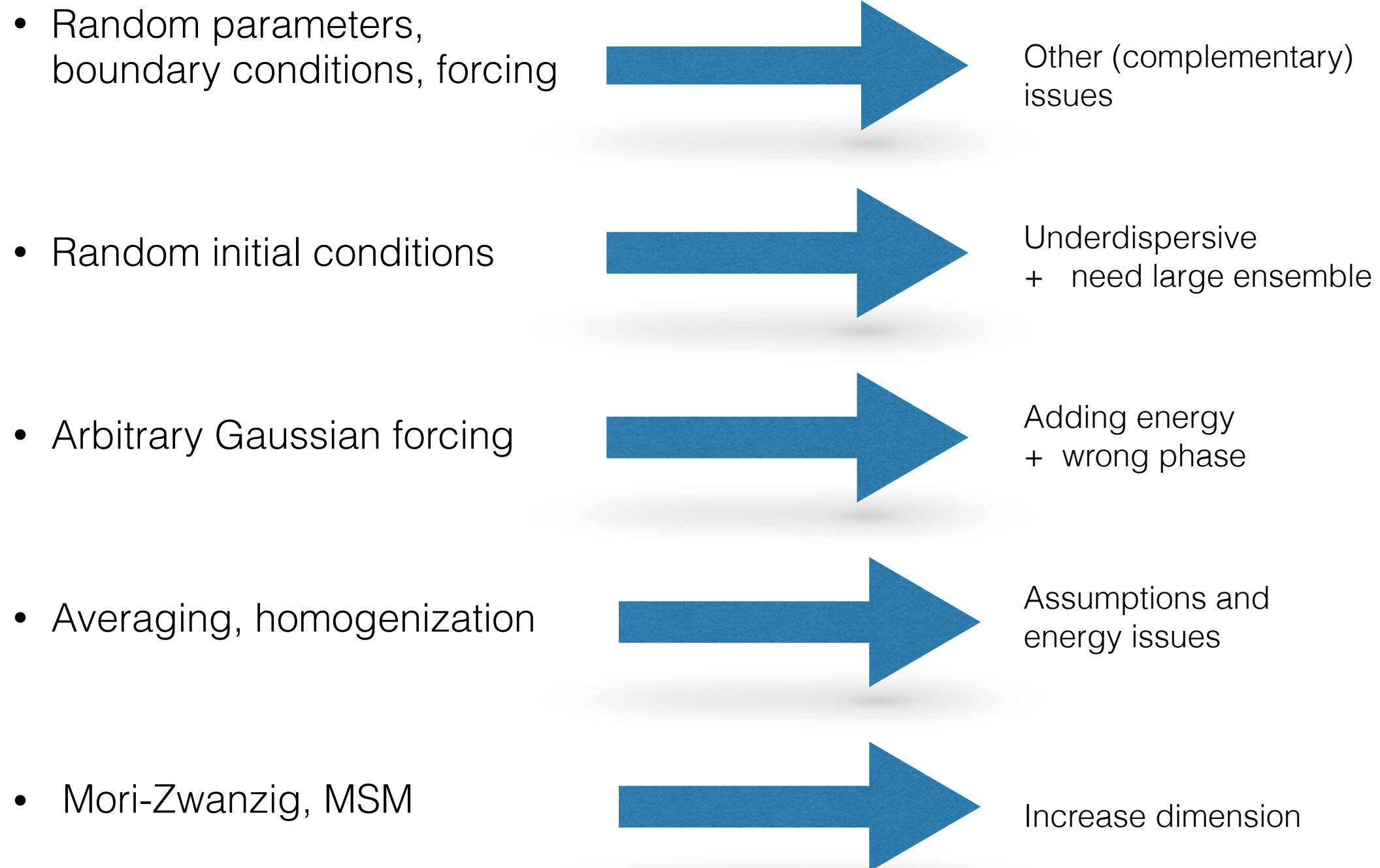
Contents

- Stochastic transport
- Stochastic Navier-Stokes : SALT vs LU
- Unresolved velocity parametrisation
- Unresolved velocity non-stationary heterogeneity

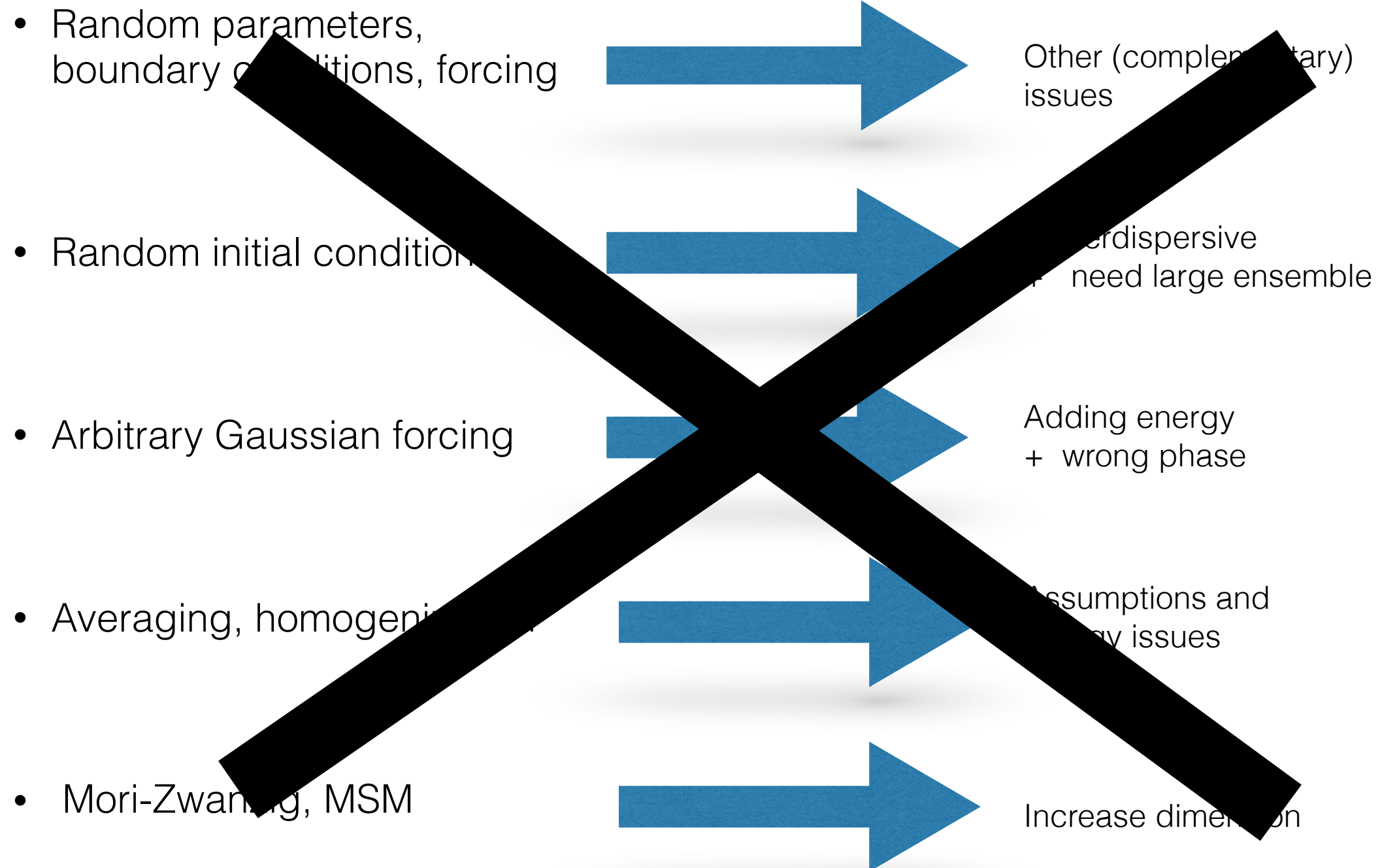
Part I

Stochastic transport

Usual random CFD



Usual random CFD



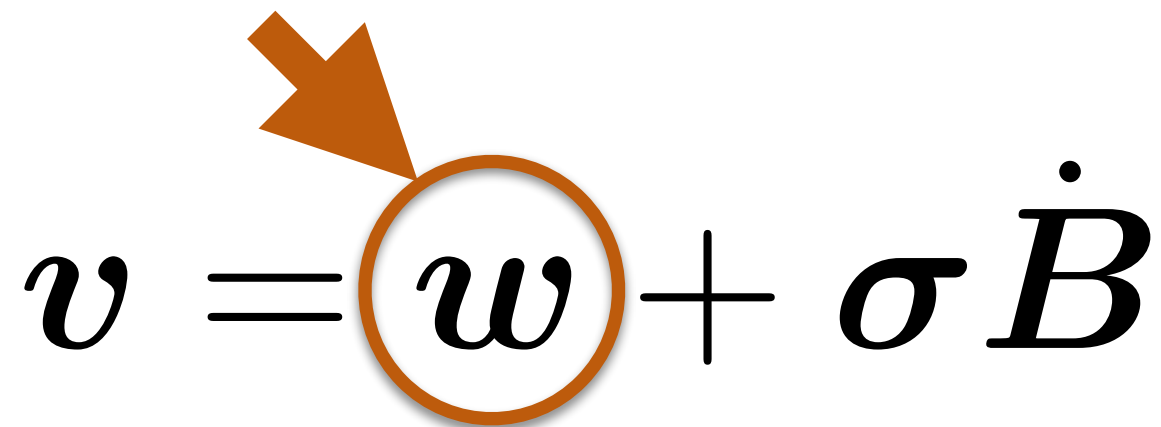
SALT & LU :
Adding random velocity

$$\boldsymbol{v} = \boldsymbol{w} + \sigma \dot{\boldsymbol{B}}$$

SALT & LU :

Adding random velocity

Resolved
large scales



The diagram shows the equation $v = w + \sigma \dot{B}$. The variable w is enclosed in a brown circle, and a brown arrow points from the text "Resolved large scales" to this circle.

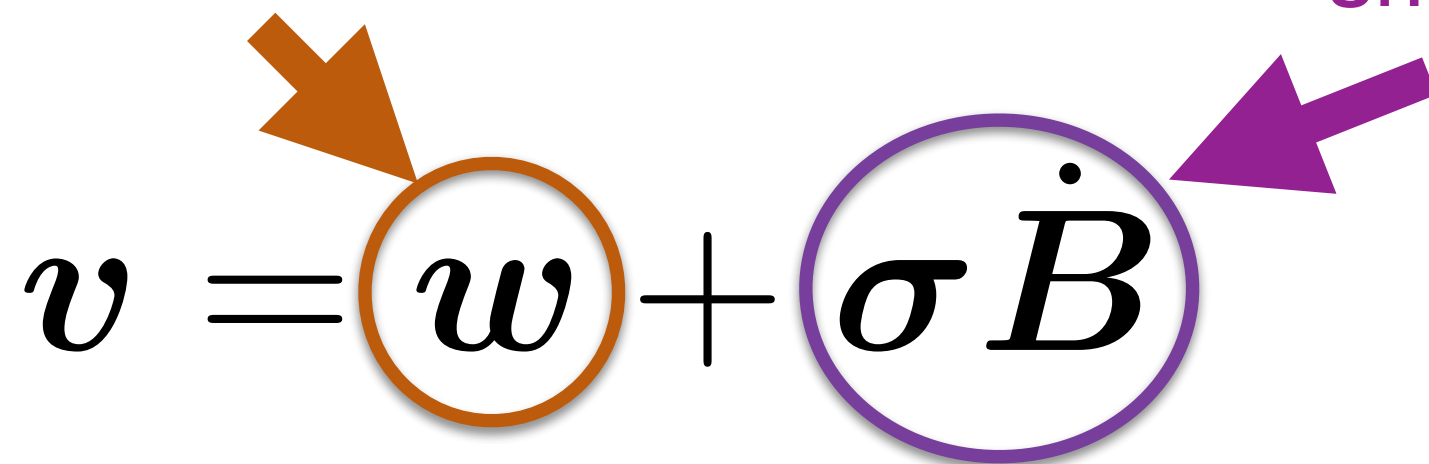
$$v = w + \sigma \dot{B}$$

SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales



The diagram illustrates the SALT & LU velocity equation. It features the equation $v = w + \sigma \dot{B}$ in a large black font. The term w is enclosed in an orange circle, and the term $\sigma \dot{B}$ is enclosed in a purple circle. An orange arrow points from the text "Resolved large scales" to the orange circle around w . A purple arrow points from the text "White-in-time small scales" to the purple circle around $\sigma \dot{B}$.

$$v = w + \sigma \dot{B}$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Resolved
large scales

White-in-time
small scales

$$v = \textcircled{w} + \textcircled{\sigma \dot{B}}$$

Large scales:

\boldsymbol{w}

Small scales:

$\boldsymbol{\sigma} \dot{\boldsymbol{B}}$

Variance
tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

SALT & LU :

Adding random velocity

$$d\boldsymbol{X}_t = \boldsymbol{w}^*(\boldsymbol{X}_t, t)dt + \boldsymbol{\sigma}(\boldsymbol{X}_t, t) \circ d\boldsymbol{B}_t$$

$$= \boldsymbol{w}(\boldsymbol{X}_t, t)dt + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)d\boldsymbol{B}_t$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

References : Mikulevicius and
Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and
Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al 2018 a, b

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

SALT & LU :

Adding random velocity

Stratonovich

$$d\mathbf{X}_t = w^*(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t) \circ d\mathbf{B}_t$$

$$= w(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t$$

Ito

LU

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

SALT

Holm, 2015
Holm and
Tyranowski, 2016
Arnaudon et al., 2017
Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al 2018 a, b

References : Mikulevicius and
Rozovskii, 2004
Flandoli, 2011

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Advection of tracer Θ

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Stratonovich notations:

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Stratonovich notations:

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Ito notations:

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Ito notations:

$$\partial_t \Theta + w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Advection of tracer Θ

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito-Wentzell
formula
(Kunita 1990)

Ito notations:

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Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}}$$

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

$$\partial_t \Theta + \underbrace{w^\star}_{\text{Stratonovich drift}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Advection

Diffusion

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Multiplicative
random
forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^\star}_{\text{Stratonovich drift}} \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Advection

Diffusion

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Multiplicative
random
forcing

Ito notations:

$$\partial_t \Theta + \underbrace{w^\star}_{\text{Stratonovich drift}} \cdot \nabla \Theta + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}}$$

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

Multiplicative
random
forcing

$$\partial_t \Theta + \underbrace{w^\star \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction »
in Ito notations

Advection of tracer Θ

Ito-Wentzell
formula
(Kunita 1990)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Ito notations:

Multiplicative
random
forcing

Balanced
energy
exchanges

$$\partial_t \Theta + \underbrace{w^\star \cdot \nabla \Theta}_{\text{Advection}} + \underbrace{\sigma \dot{B} \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)$$

Stratonovich drift : « Drift correction »
in Ito notations

Large scales:

\mathbf{w}

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp\!\!\!\perp \mathbf{w}(\bullet, t), \forall t' \leq t$$

$$\begin{aligned} \partial_t \mathbb{E}(\Theta) + \underbrace{\mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta)}_{\text{Advection}} &= \underbrace{\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right)}_{\text{Diffusion}} \\ \partial_t \text{Var}(\Theta) + \underbrace{\mathbf{w}^* \cdot \nabla \text{Var}(\Theta)}_{\text{Advection}} &= \underbrace{\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \text{Var}(\Theta) \right)}_{\text{Diffusion}} + \underbrace{(\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta)}_{\text{Source term}} \end{aligned}$$

Large scales:

\mathbf{w}

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Moments of a passive tracer Θ

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Large scales:

\mathbf{w}

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp \mathbf{w}(\bullet, t), \forall t' \leq t$$

$$\begin{aligned} \partial_t \mathbb{E}(\Theta) + \underbrace{\mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta)}_{\text{Advection}} &= \underbrace{\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right)}_{\text{Diffusion}} \\ \partial_t \text{Var}(\Theta) + \underbrace{\mathbf{w}^* \cdot \nabla \text{Var}(\Theta)}_{\text{Advection}} &= \underbrace{\nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \text{Var}(\Theta) \right)}_{\text{Diffusion}} + \underbrace{(\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta)}_{\text{Source term}} \end{aligned}$$

Conversion from deterministic to random energy

Large scales:

\mathbf{w}

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp \mathbf{w}(\bullet, t), \forall t' \leq t$$

Conservation
of the energy

$$\frac{d}{dt} \int_{\Omega} \Theta^2 = 0$$

$$\begin{aligned} \partial_t \mathbb{E}(\Theta) + \mathbf{w}^* \cdot \nabla \mathbb{E}(\Theta) &= \nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla \mathbb{E}(\Theta) \right) \\ \partial_t Var(\Theta) + \mathbf{w}^* \cdot \nabla Var(\Theta) &= \nabla \cdot \left(\frac{1}{2} \mathbf{a} \nabla Var(\Theta) \right) + (\nabla \mathbb{E}(\Theta))^T \mathbf{a} \nabla \mathbb{E}(\Theta) \end{aligned}$$

Advection

Diffusion

Conversion from
deterministic to
random energy

Source term

Part II

Stochastic Navier-Stokes

SALT vs LU

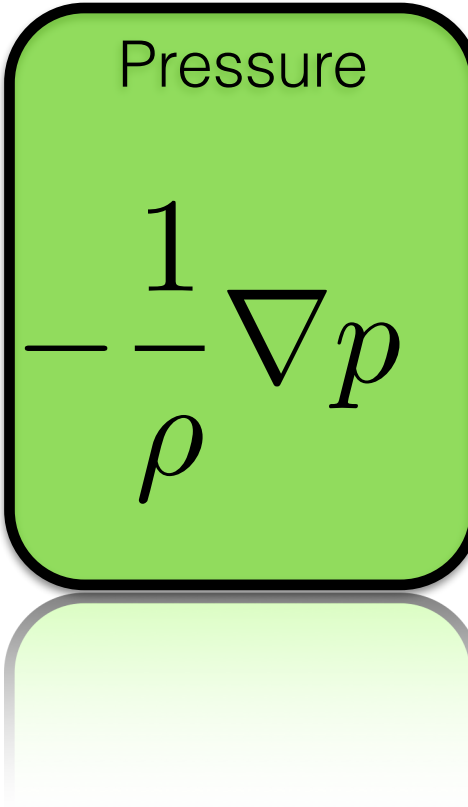
Comparison

	LU	SALT
Scalar (e.g. SQG)	Identical	
Navier-Stokes	2 differences	
Vorticity	2 differences	
Kinetic energy conservation	✓	✗
Helicity (& 2D enstrophy) conservation	✗	✓

Navier-Stokes LU

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

$$\frac{Dw}{Dt} = \text{Pressure} \left(-\frac{1}{\rho} \nabla p \right)$$


Navier-Stokes LU

Stochastic
transport
of lto drift

$$\frac{Dw}{Dt} =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

Navier-Stokes LU

The diagram illustrates the decomposition of the Navier-Stokes equation into two parts, each in a colored box. The left box is cyan and contains the text 'Stochastic transport of Ito drift' and the equation $\frac{Dw}{Dt} =$. The right box is green and contains the text 'Pressure' and the equation $-\frac{1}{\rho} \nabla p$. Below these boxes are two lighter-colored boxes, one cyan and one green, which are reflections of the ones above. A purple arrow points upwards from the bottom-left cyan box towards the top-left cyan box.

Stochastic transport of Ito drift

$$\frac{Dw}{Dt} =$$

Pressure

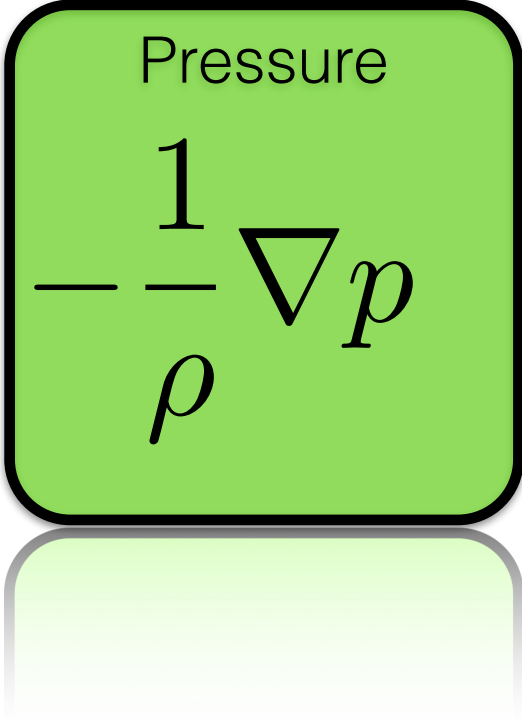
$$-\frac{1}{\rho} \nabla p$$

Conserve (Ito) kinetic energy

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho} \nabla p$$

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = \text{Pressure} \quad -\frac{1}{\rho} \nabla p$$


Navier-Stokes SALT

Stochastic
transport of
Stratonovich drift

$$\frac{Dw^*}{Dt}$$


$$+ \nabla (\sigma \circ \dot{B})^T w^* =$$

Pressure

$$-\frac{1}{\rho} \nabla p$$

Navier-Stokes SALT

$$\begin{array}{|c|} \hline \text{Stochastic} \\ \text{transport of} \\ \textbf{Stratonovich} \text{ drift} \\ \hline \frac{Dw^*}{Dt} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Additional term} \\ \hline \nabla (\sigma \circ \dot{B})^T w^* \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Pressure} \\ \hline -\frac{1}{\rho} \nabla p \\ \hline \end{array}$$


 Conserve (Stratonovich) helicity

Part III

Unresolved velocity parametrisation

Code available online

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$\mathbf{a} = \mathbf{a}(x, x) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Parameter-free models for σ

	Cotter et al. 2018b	Resseguier et al. 2017b
Applicable to :	LU & SALT	LU & SALT
Homogeneous	No	Yes
Stationary	Yes	Not anymore
Self-similar assumption	No	Yes
Data-driven	Yes	No

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

weighted
EOFs learned
on data

iid
Brownian
motion

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

weighted
EOFs learned
on data

PCA on : $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

Cotter et al. 2018b: Karhunen–Loève decomposition (EOF)

Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

$$\sigma dB_t = \sum_{i=1}^n \xi_i(x) dW_i(t)$$

iid
Brownian
motion

weighted
EOFs learned
on data

PCA on : $\Delta X_k(x) = X_{t_k, t_k + \Delta t}^{HR}(x) - X_{t_k, t_k + \Delta t}^{LR}(x)$

Flow from **high-**
resolution velocity

Flow from **low-**
resolution velocity

Large scales:

w

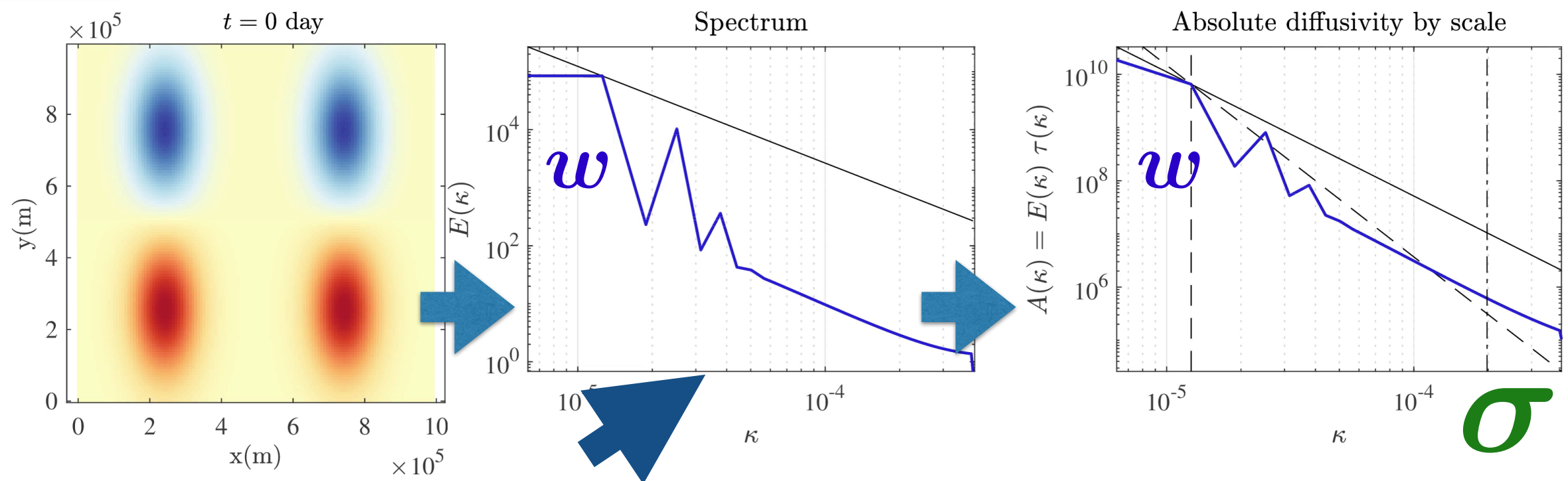
Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

(improvement of) Resseguier et al. 2017b: Self-similar model



Kinetic energy
Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

Large scales:

w

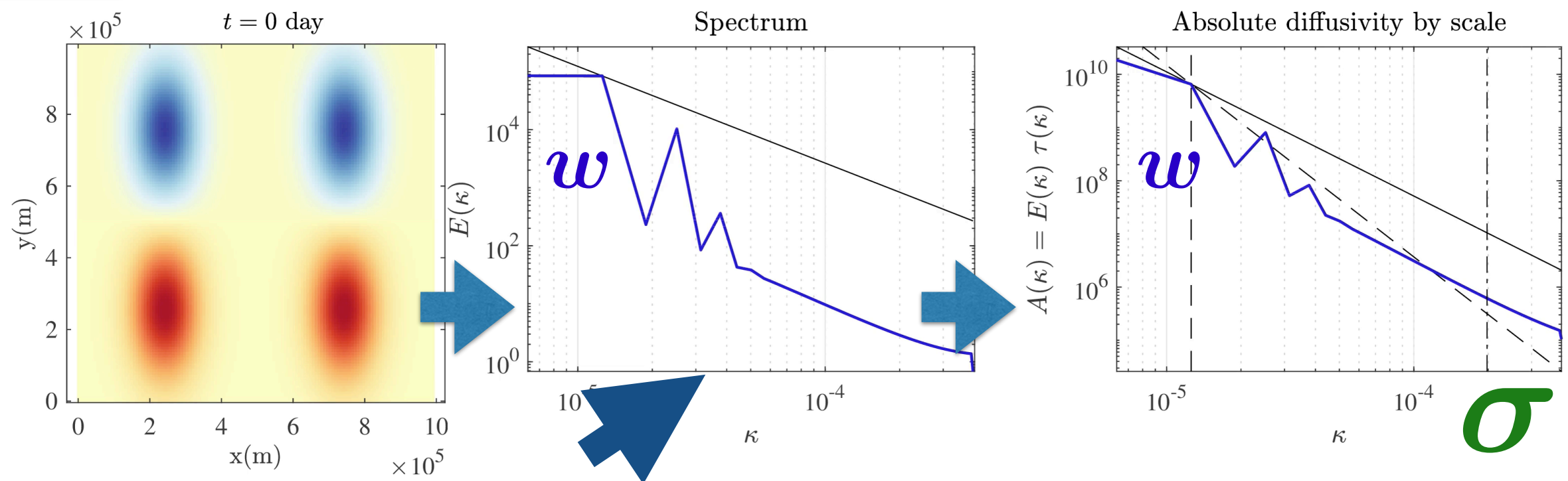
Small scales:

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Variance
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(improvement of) Resseguier et al. 2017b: Self-similar model



Kinetic energy
Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

Large scales:

w

Small scales:

$\sigma \dot{B}$

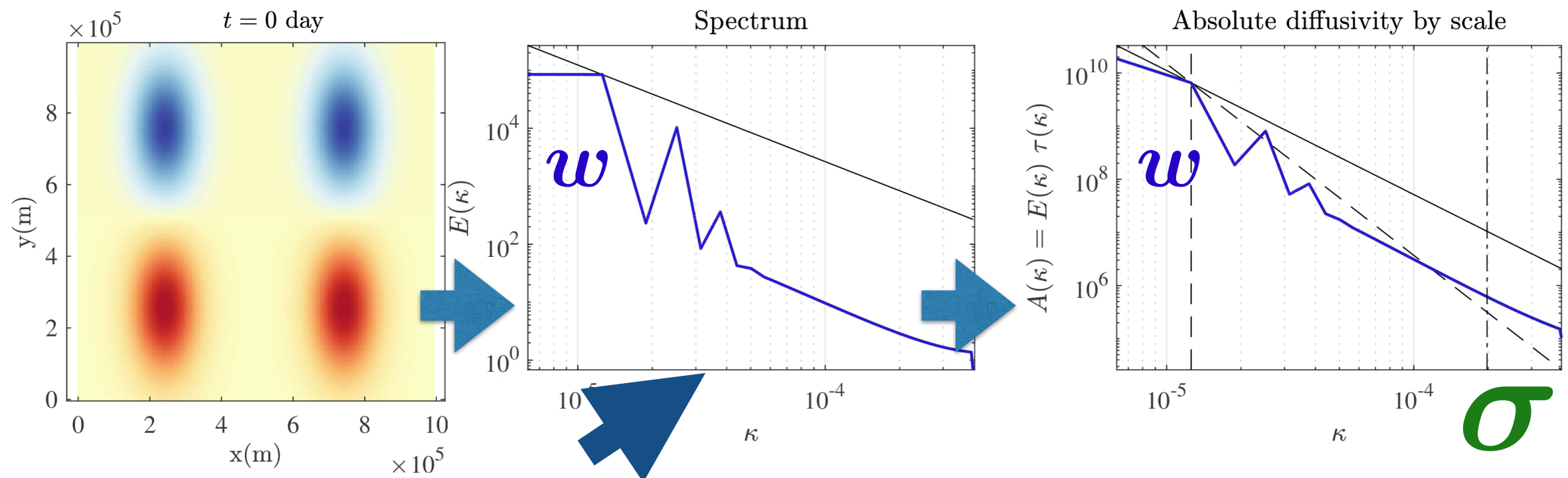
Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

(improvement of) Resseguier et al. 2017b: Self-similar model

Absolute Diffusivity
Spectral Density

$$A(\kappa) = E(\kappa)\tau(\kappa) = \kappa^{-3/2} E^{1/2}(\kappa)$$



Kinetic energy
Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \|\hat{v}(k)\|^2$$

$t = 100 \text{ day}$

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$-\frac{1}{\tau} (b - F) \quad \text{Forcing}$$

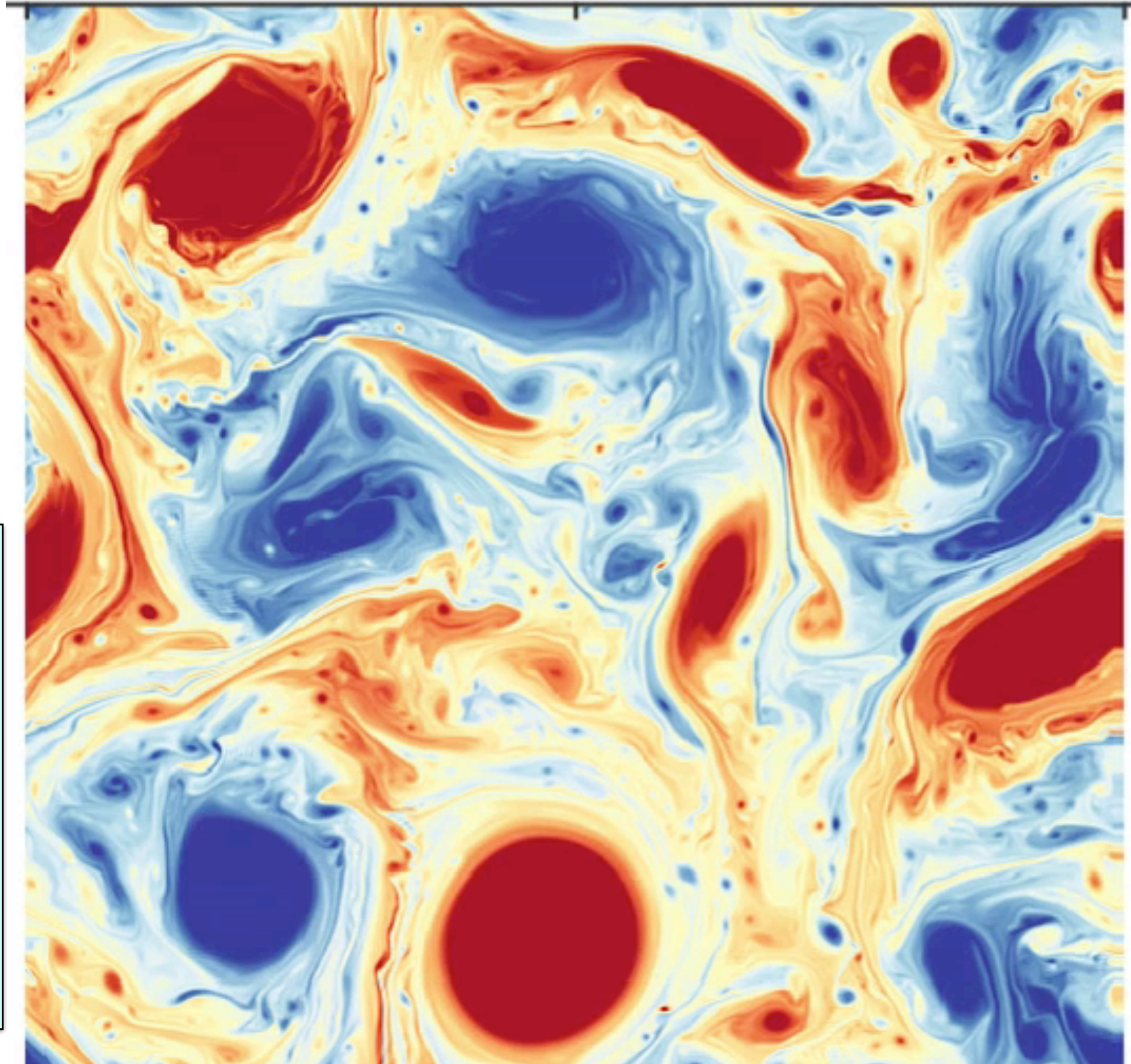
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

deterministic

SQG

512 x 512



$t = 100$ day

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$-\frac{1}{\tau} (b - F) \quad \text{Forcing}$$

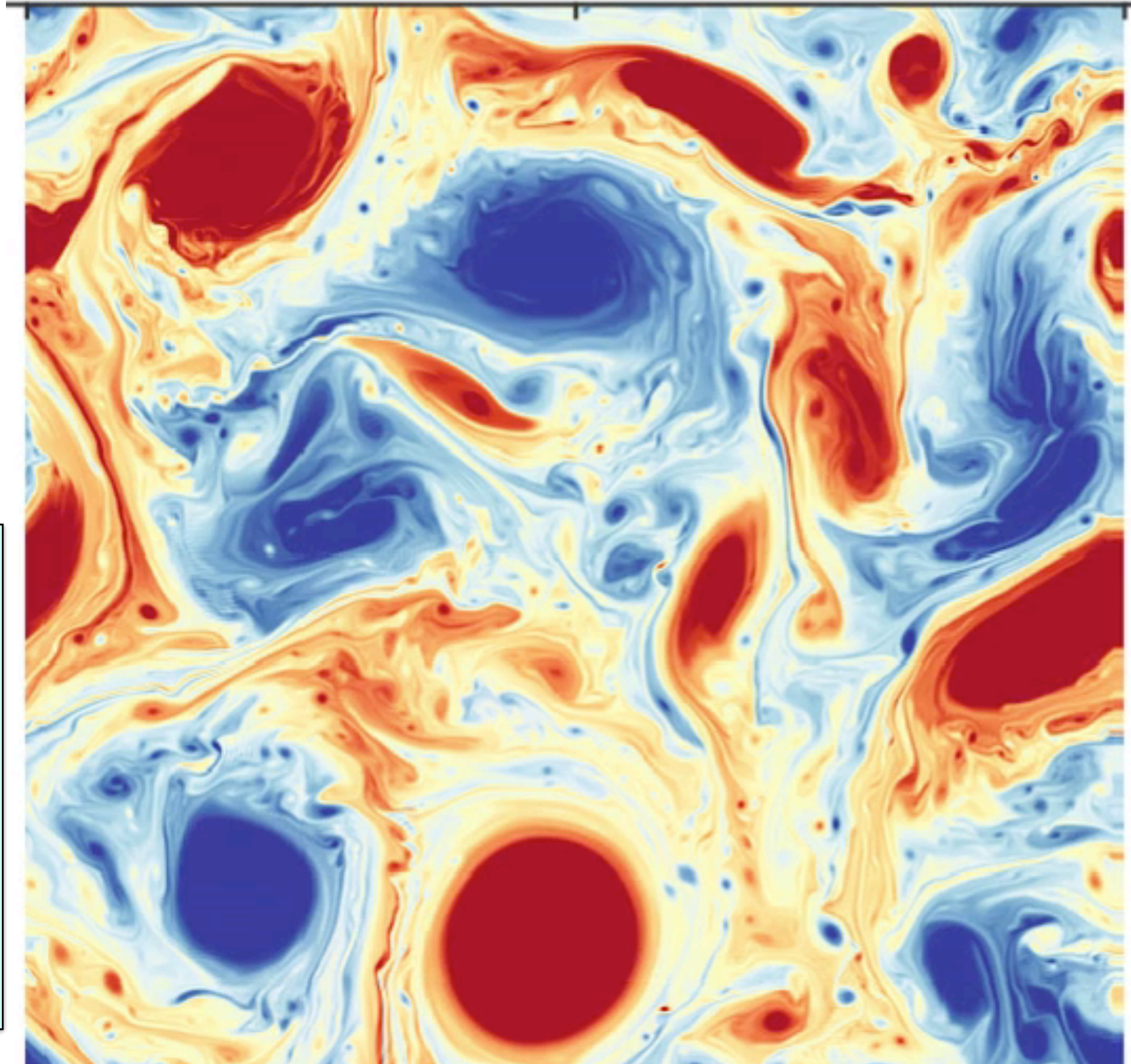
$$u = \left(\text{cst.} \nabla^\perp \Delta^{-1/2} \right) b$$

Reference flow:

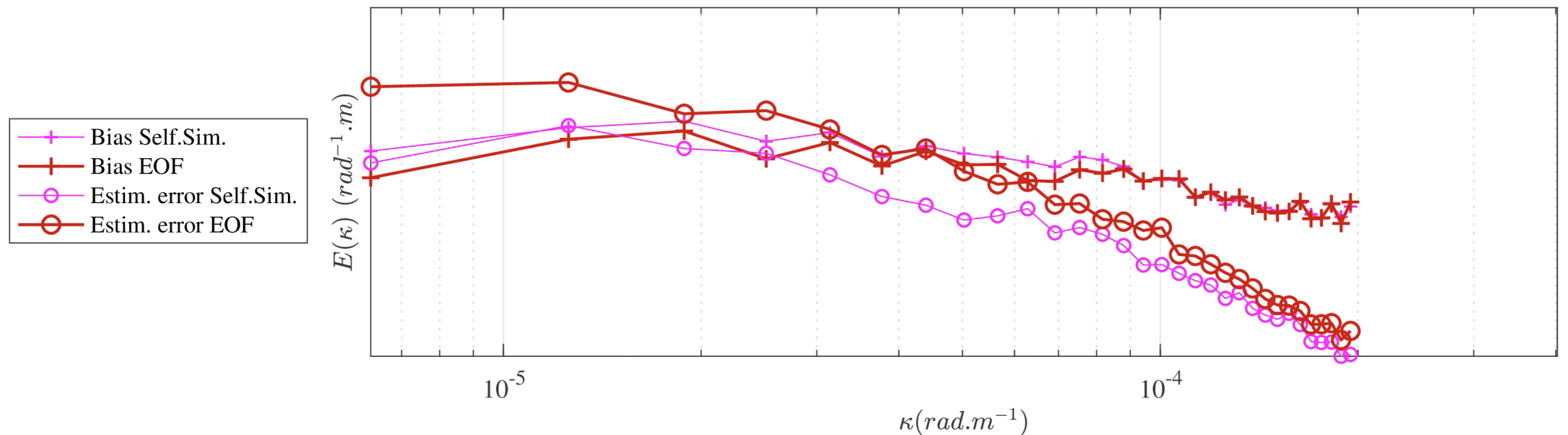
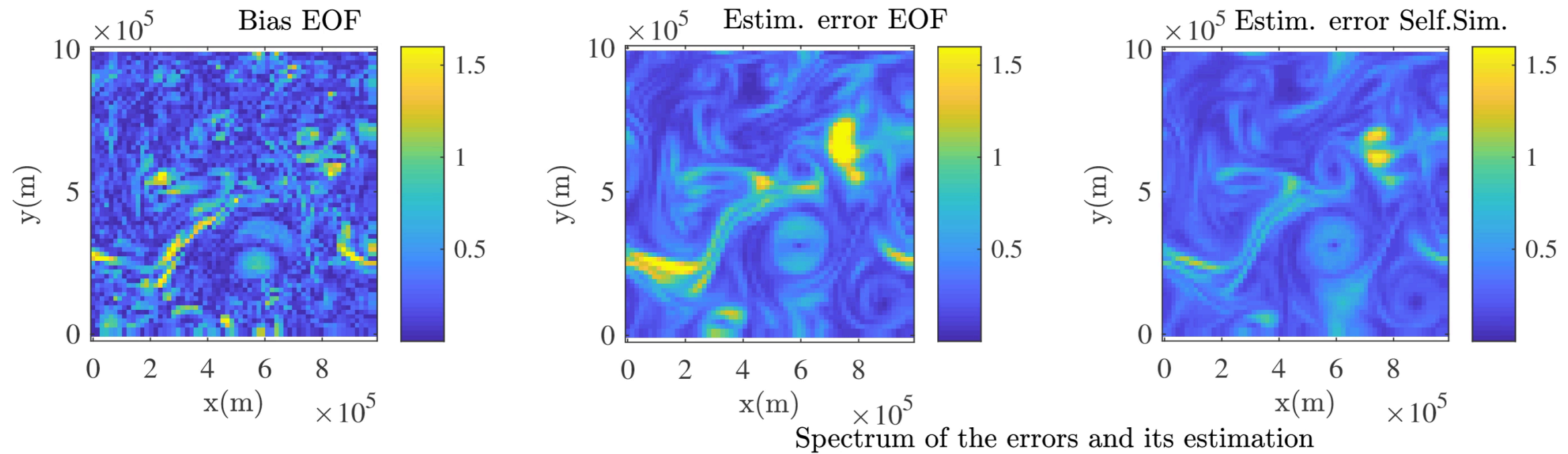
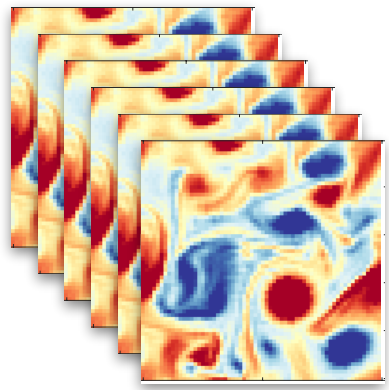
deterministic

SQG

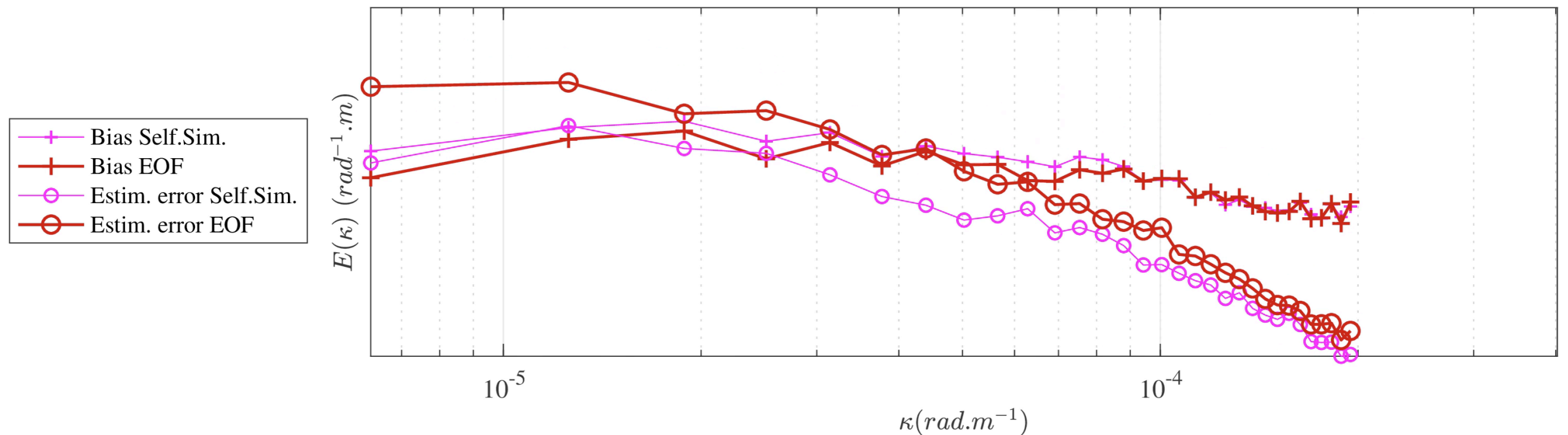
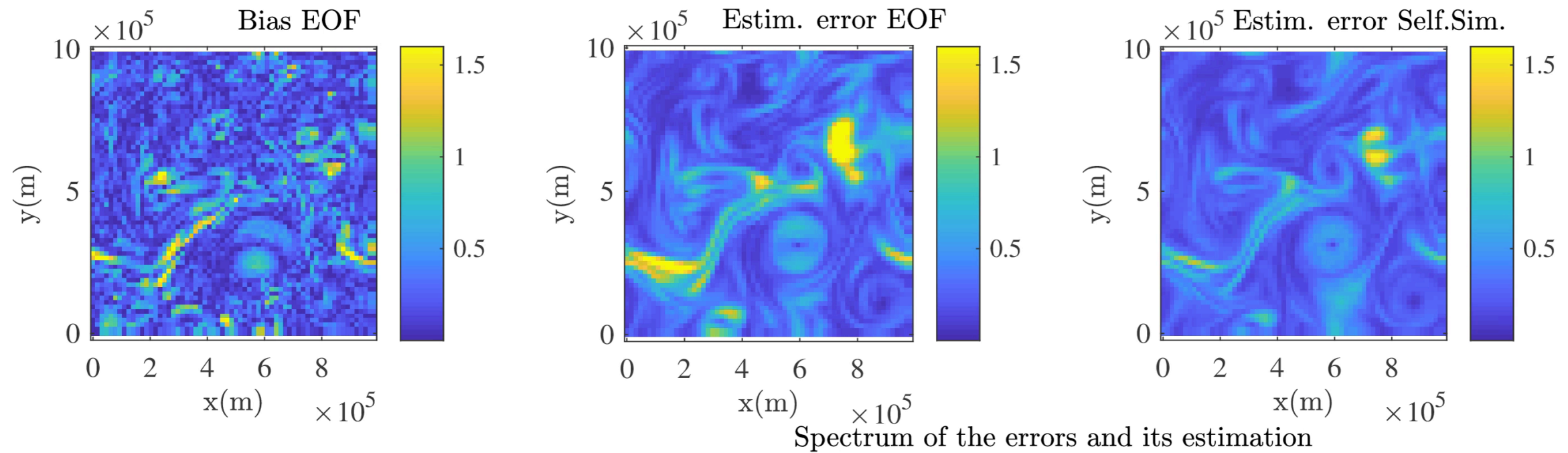
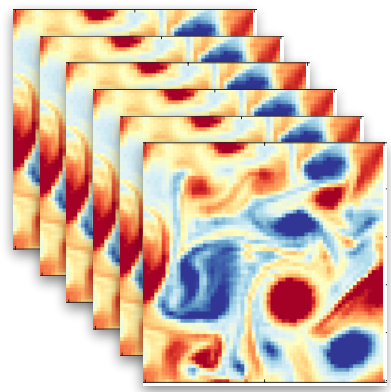
512 x 512



Ensemble : uncertainty quantification



Ensemble : uncertainty quantification



Part IV

Unresolved velocity
non-stationary heterogeneity

$t = 0$ day

SQG

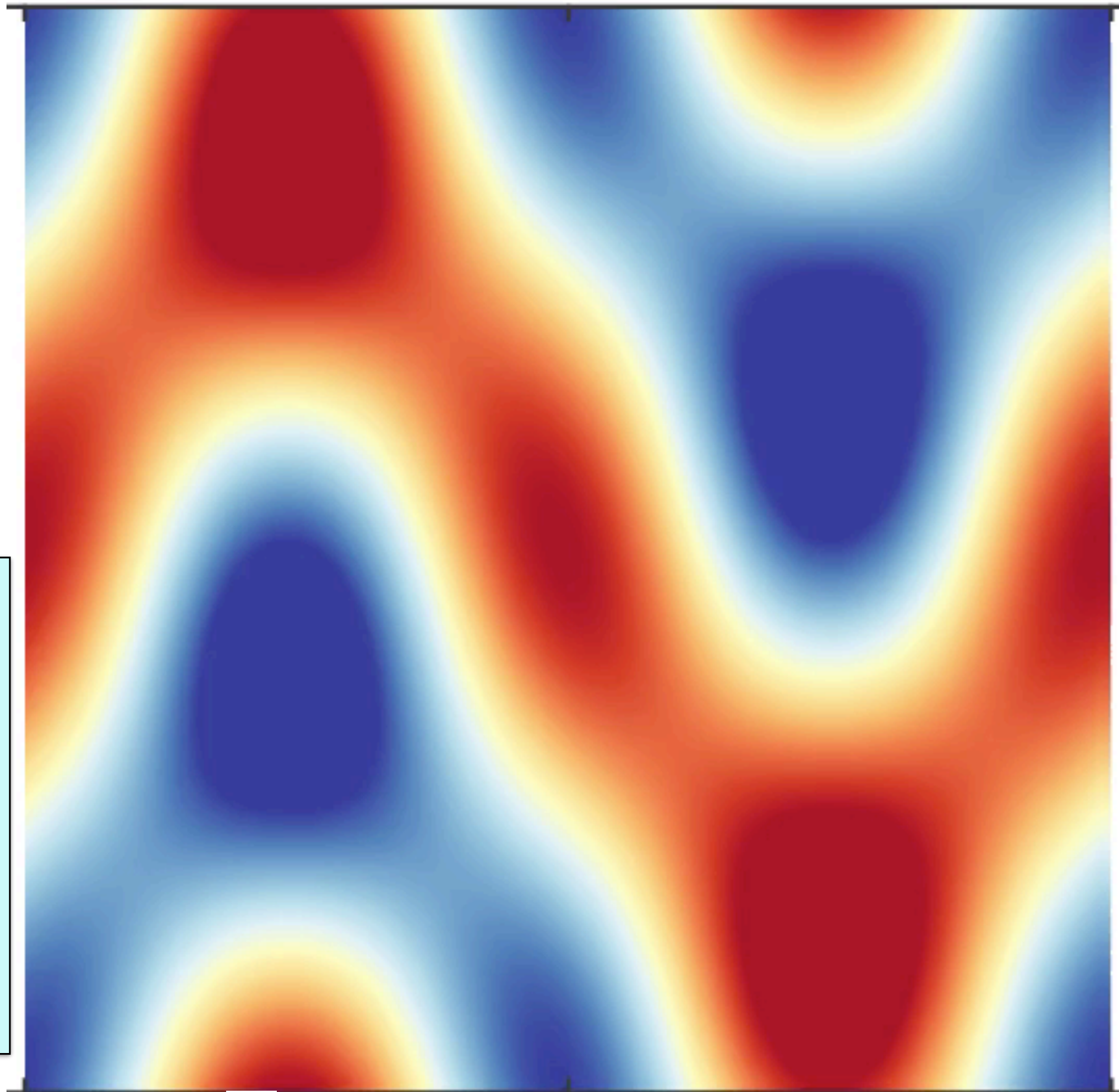
$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)



$t = 0$ day

SQG

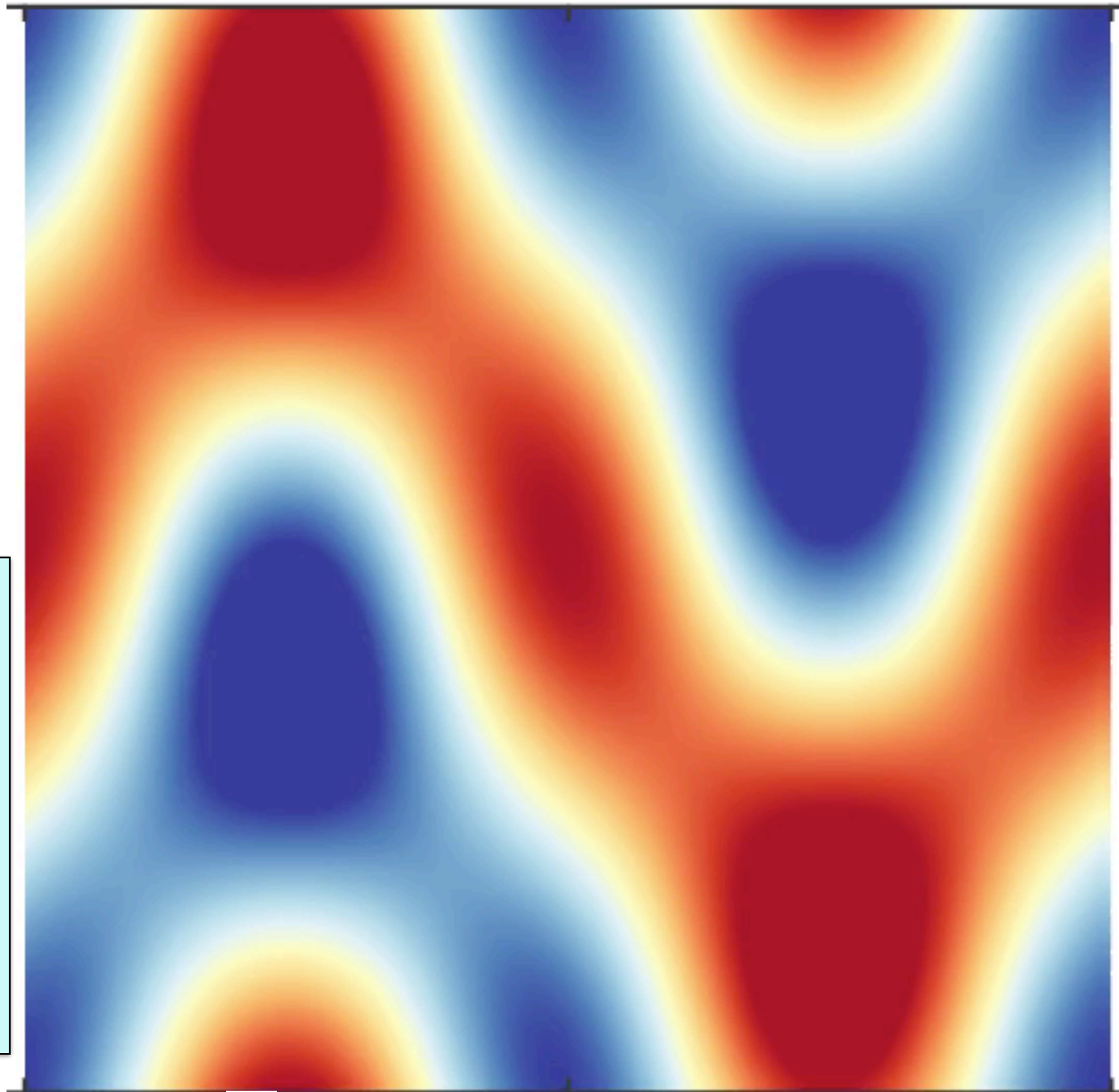
$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)



$t = 15 \text{ day}$

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

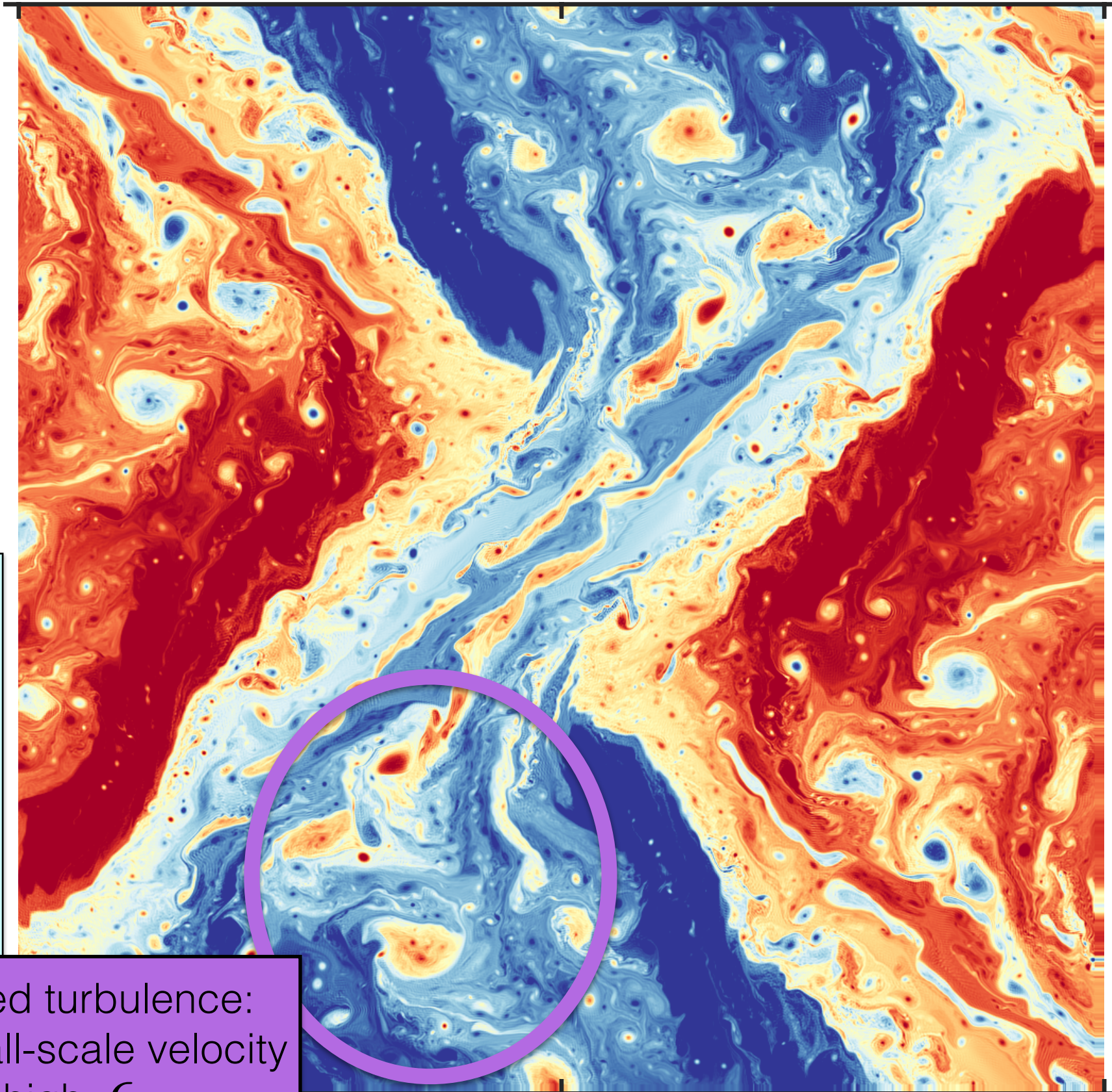
$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999)

Developed turbulence:
strong small-scale velocity
high ϵ



$t = 15 \text{ day}$

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

$$\mathbf{u} = \left(\text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b$$

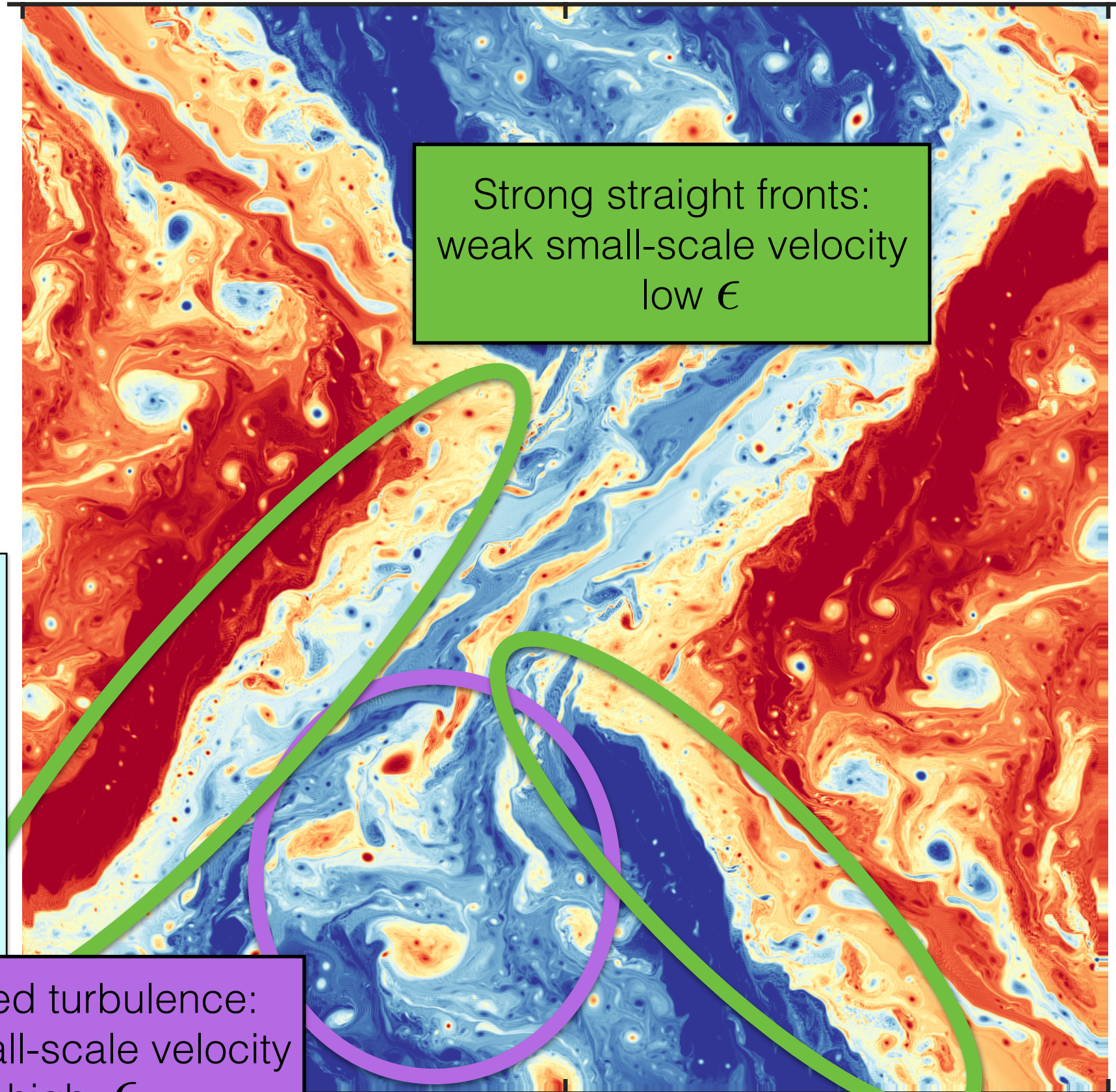
Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1995)

Developed turbulence:
strong small-scale velocity
high ϵ

Strong straight fronts:
weak small-scale velocity
low ϵ



Large scales:

w

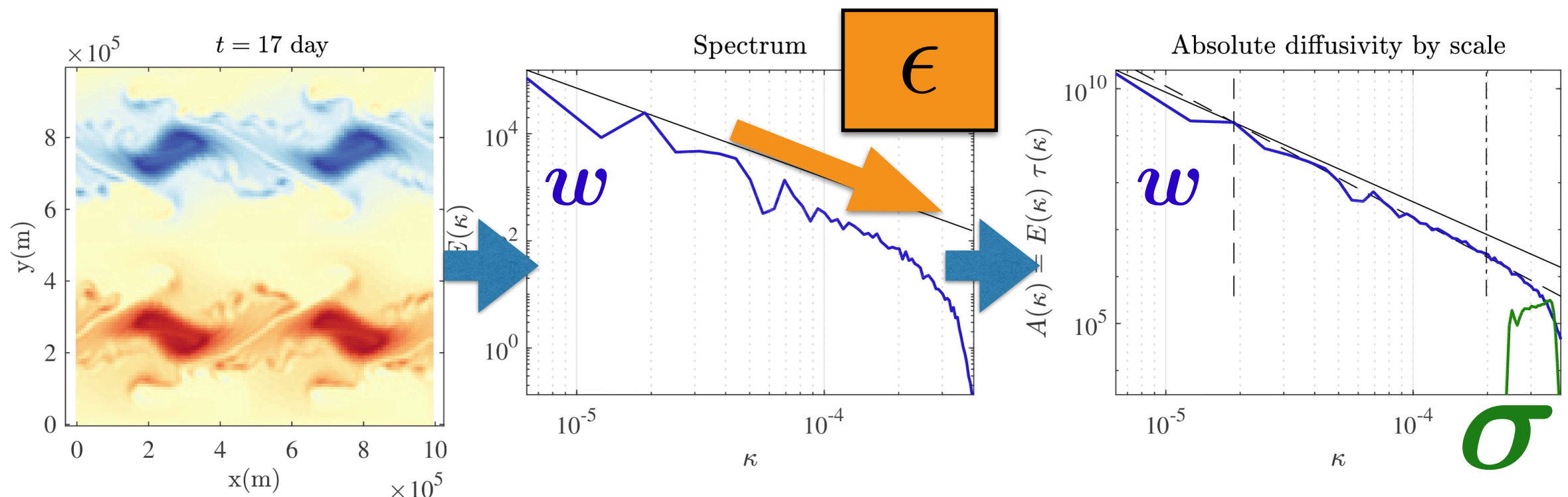
Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{\mathbb{E}\{\sigma d\mathbf{B} (\sigma d\mathbf{B})^T\}}{dt}$$

Heterogeneous modulation of σ (for Resseguier et al. 2017b)



Large scales:

w

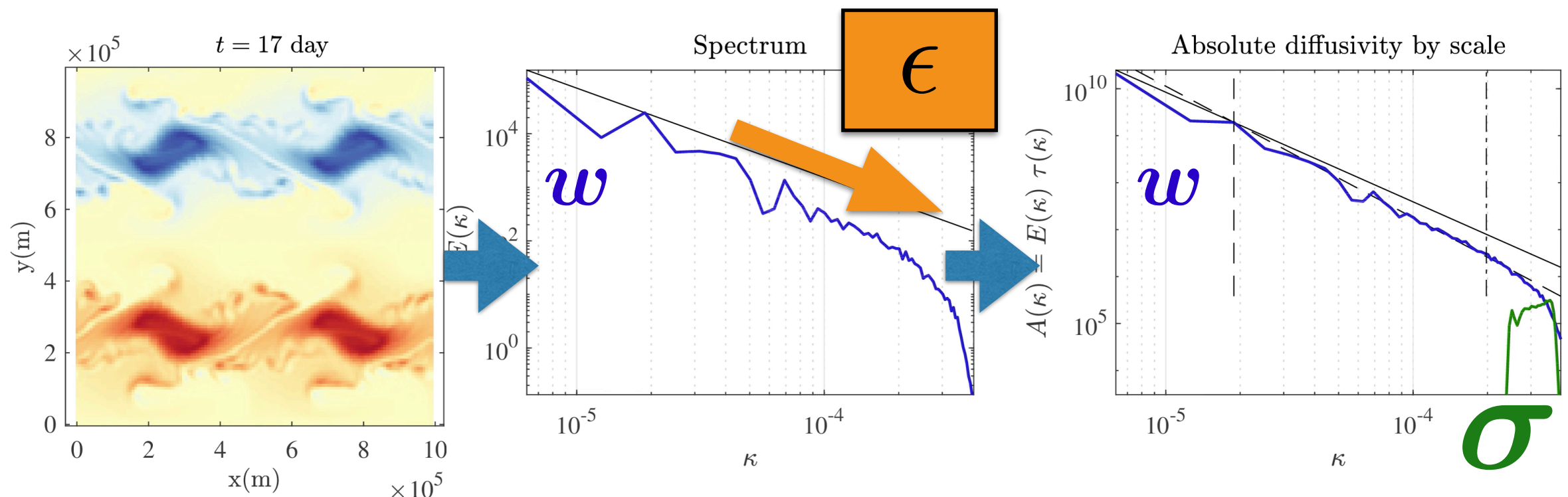
Small scales:

$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Heterogeneous modulation of σ (for Resseguier et al. 2017b)



Large scales:

w

Small scales:

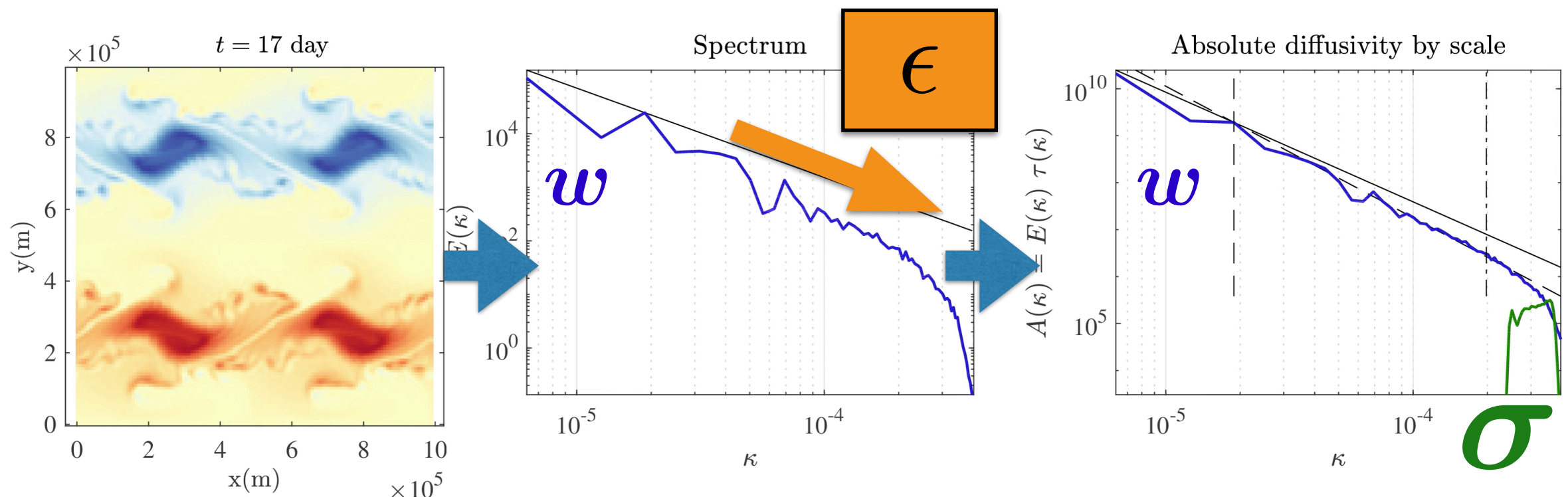
$\sigma \dot{B}$

Variance
tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Heterogeneous modulation of σ (for Resseguier et al. 2017b)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst. } \epsilon^p(x) \kappa^{-q}$$



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

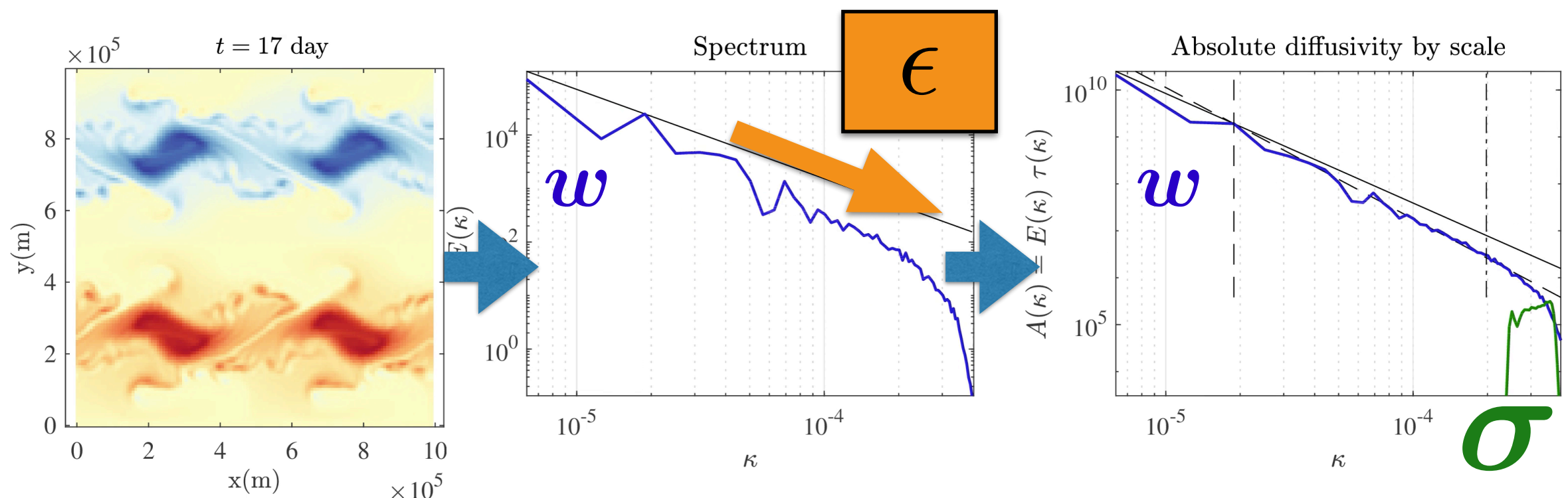
$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

Heterogeneous modulation of σ

(for Resseguier et al. 2017b)

Dynamics
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst. } \epsilon^p(x) \kappa^{-q}$$



Large scales:

w

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$

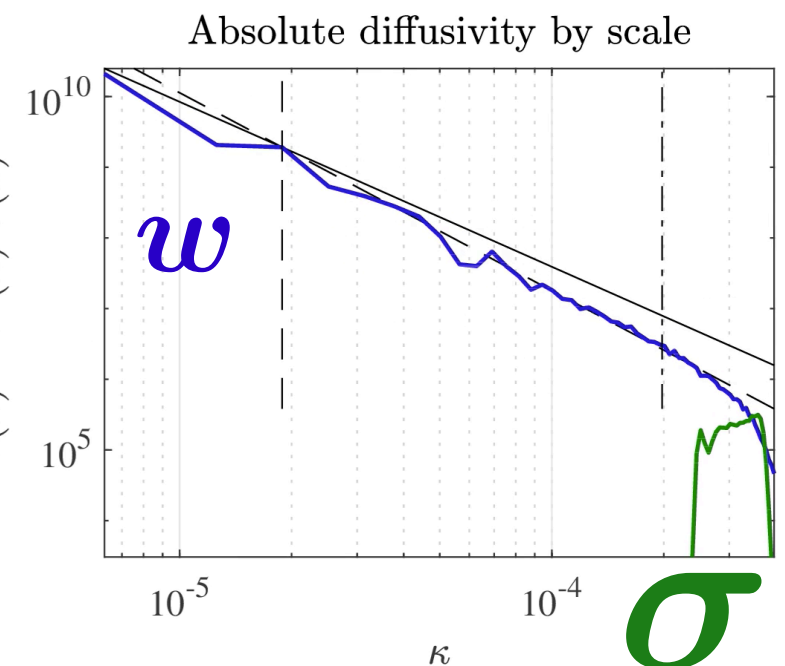
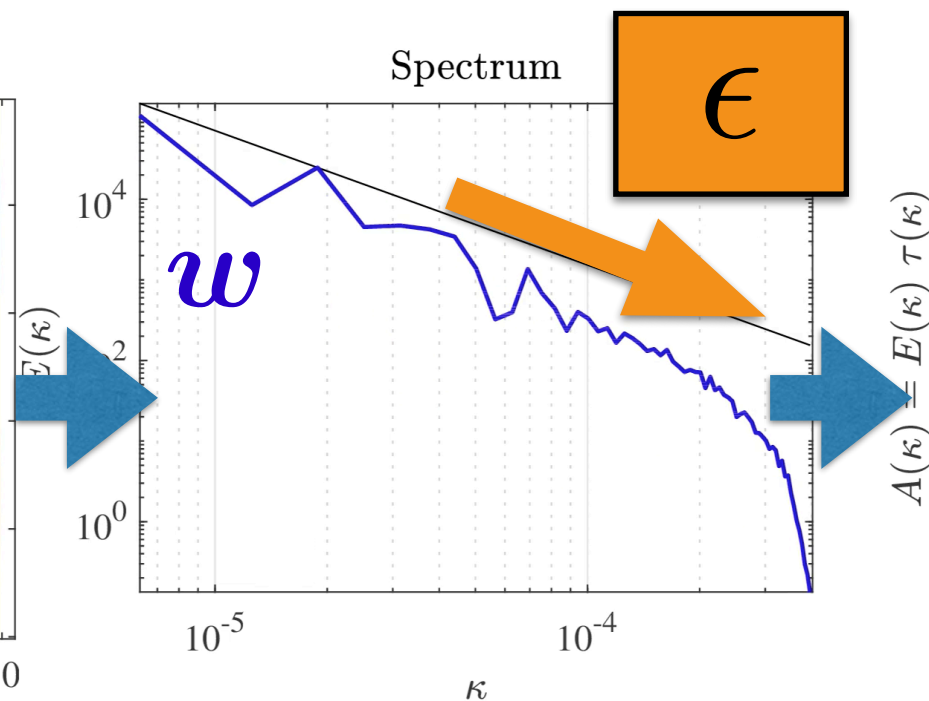
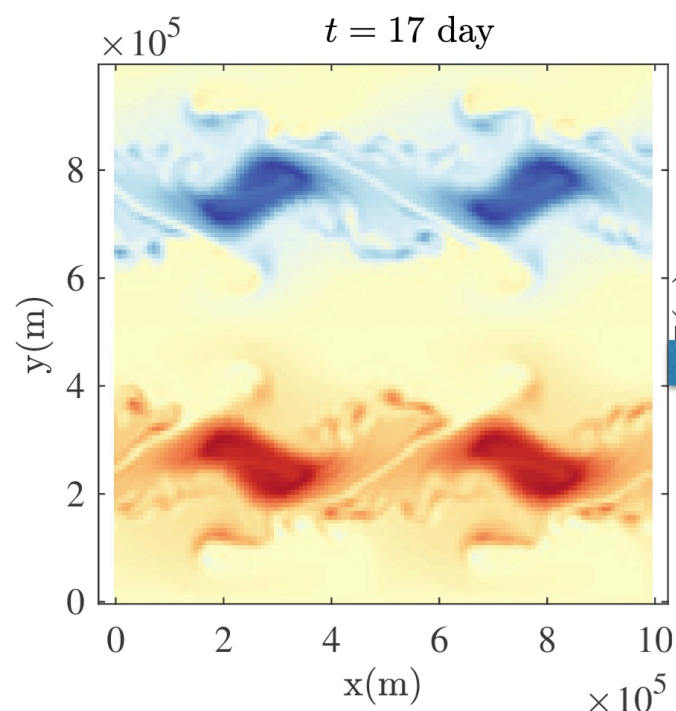
Heterogeneous modulation of σ

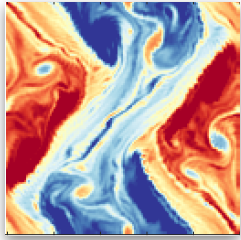
(for Resseguier et al. 2017b)

Dynamics
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x) \tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst.} \epsilon^p(x) \kappa^{-q}$$

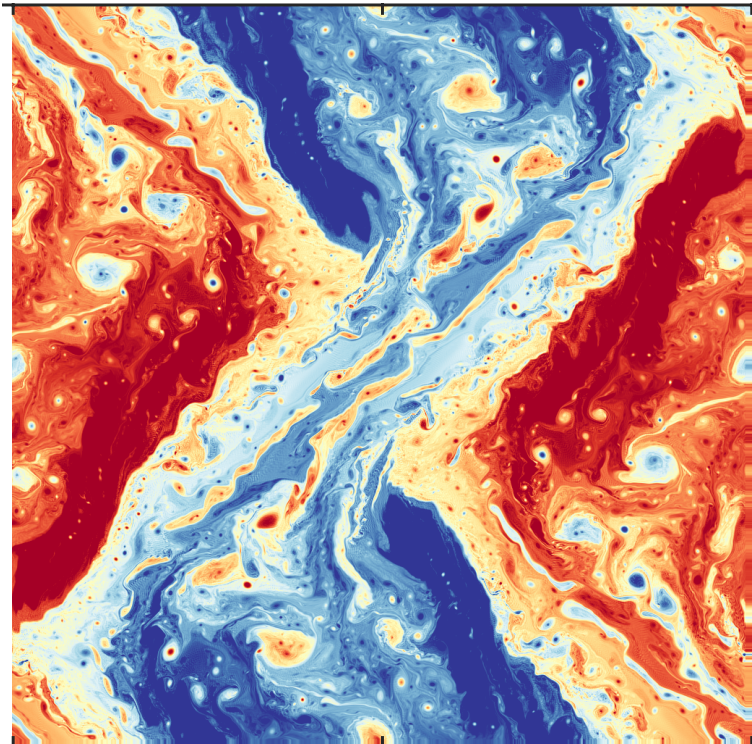
Learned on large scales at each time t



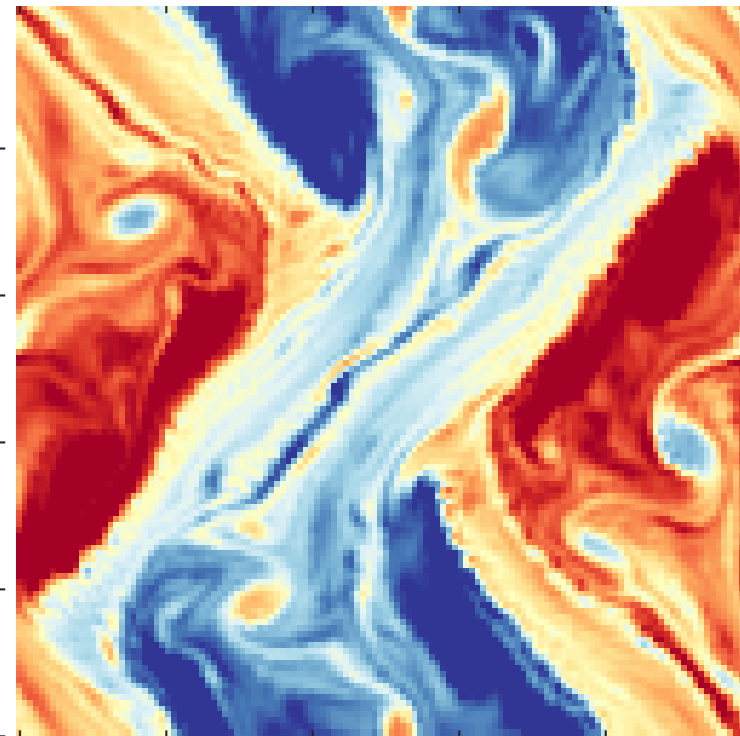


Heterogenous modulation

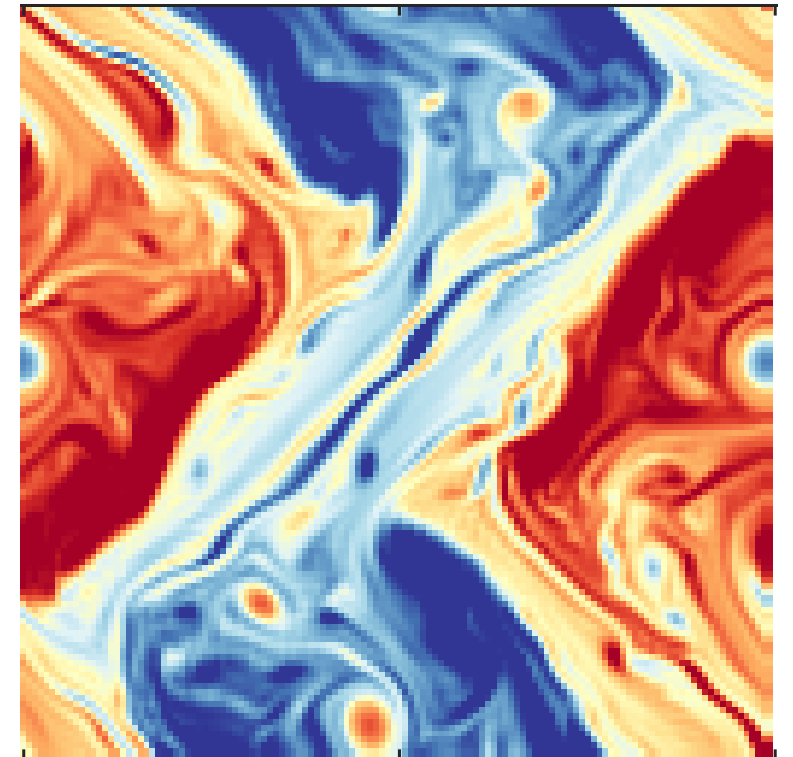
Deterministic 1024 x 1024

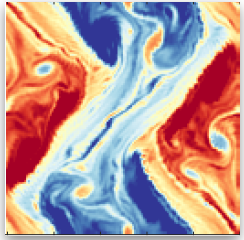


Stochastic 128 x128
with Homogeneous
small-scale velocity



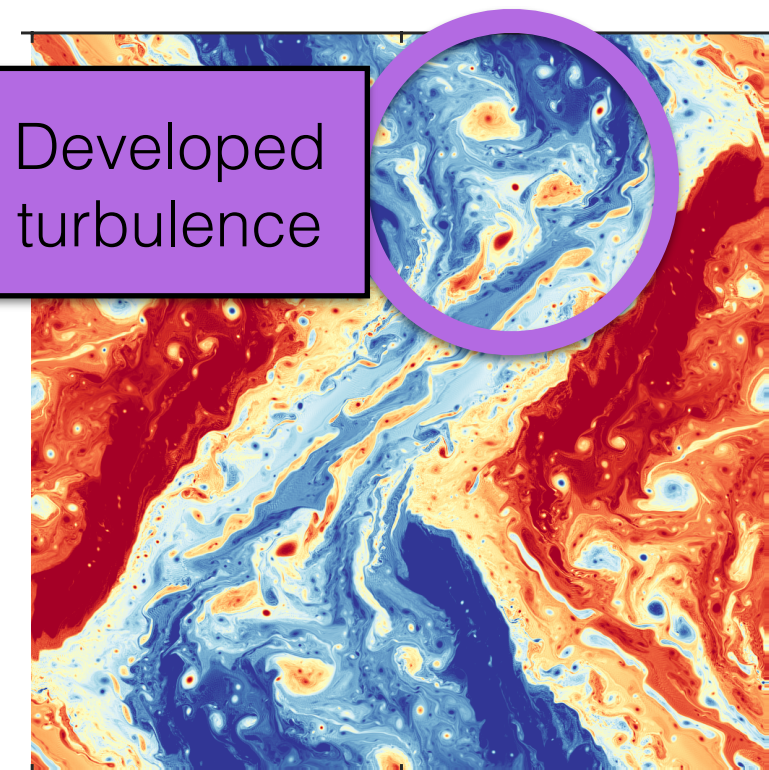
Stochastic 128 x128
with Energy-flux-based
modulation



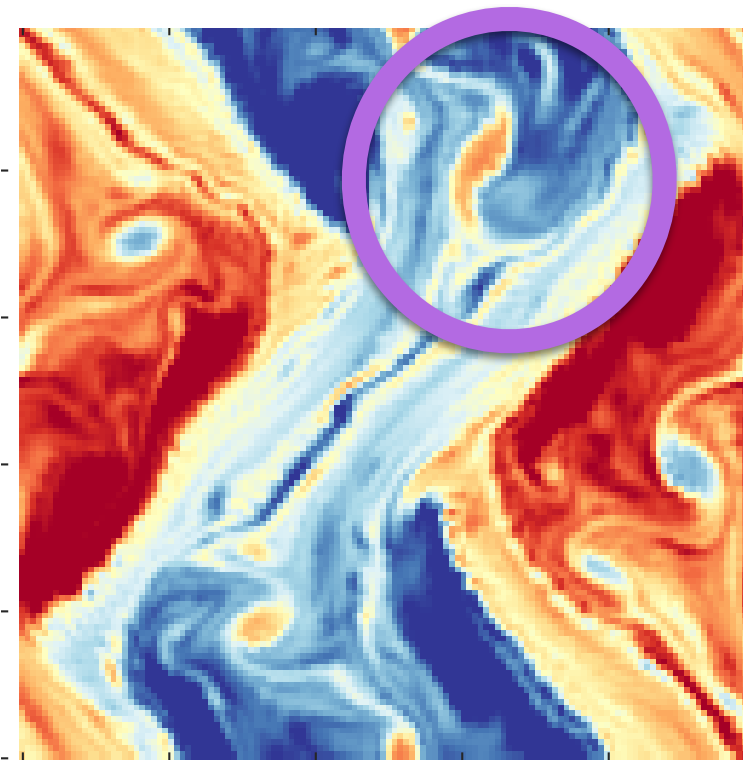


Heterogenous modulation

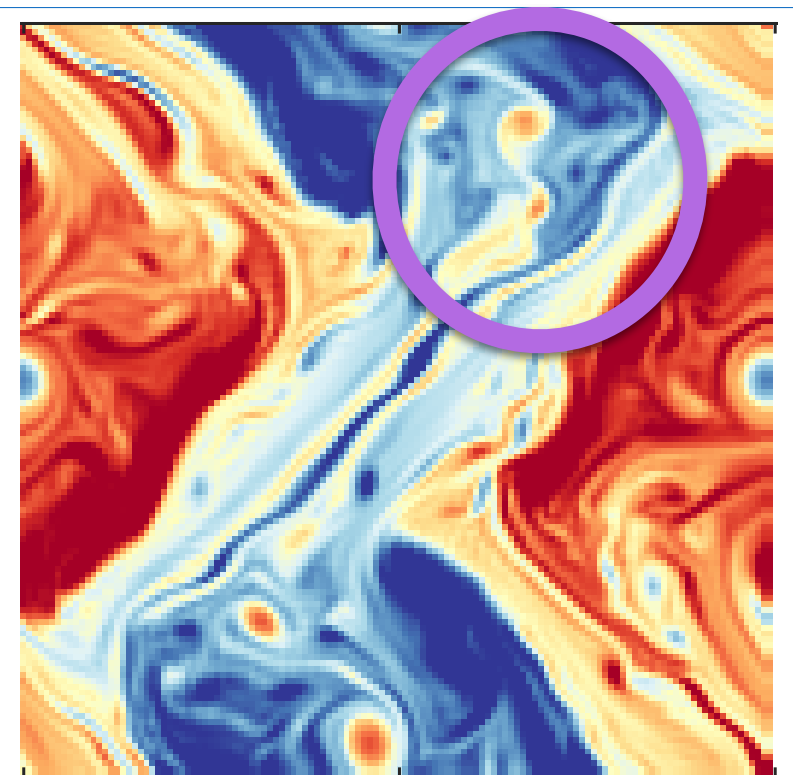
Deterministic 1024 x 1024

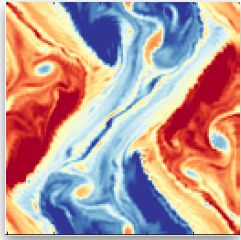


Stochastic 128 x128
with Homogeneous
small-scale velocity



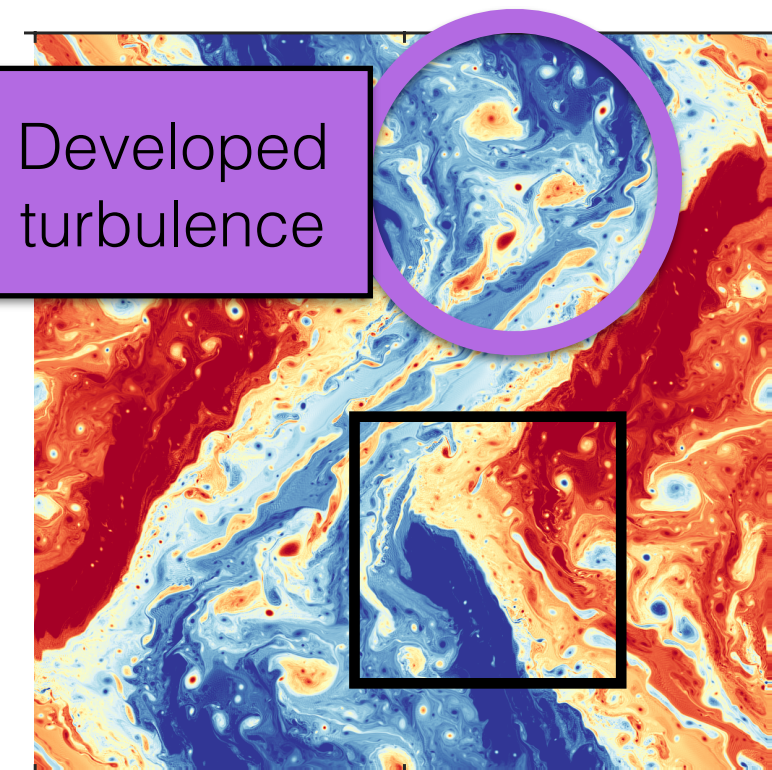
Stochastic 128 x128
with Energy-flux-based
modulation



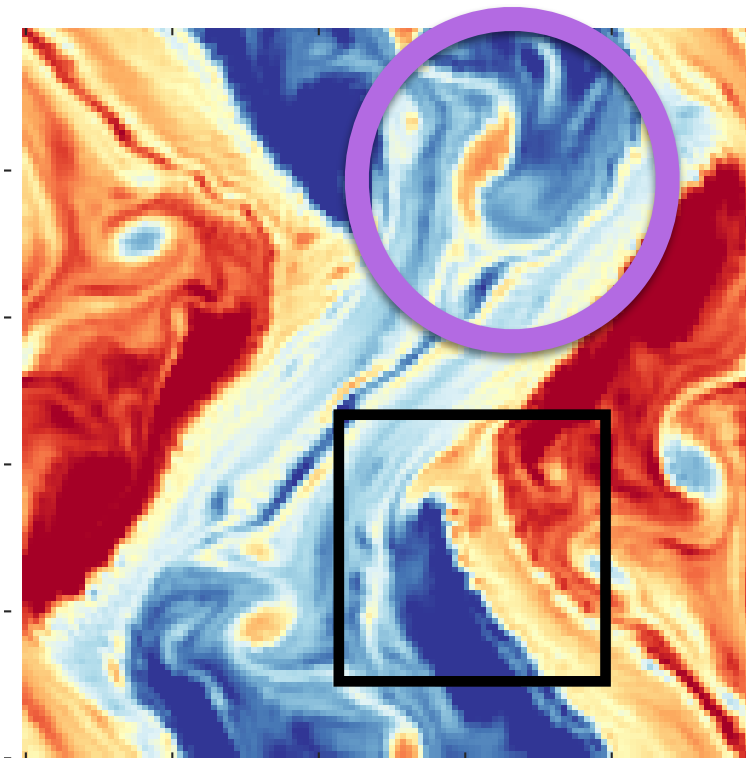


Heterogenous modulation

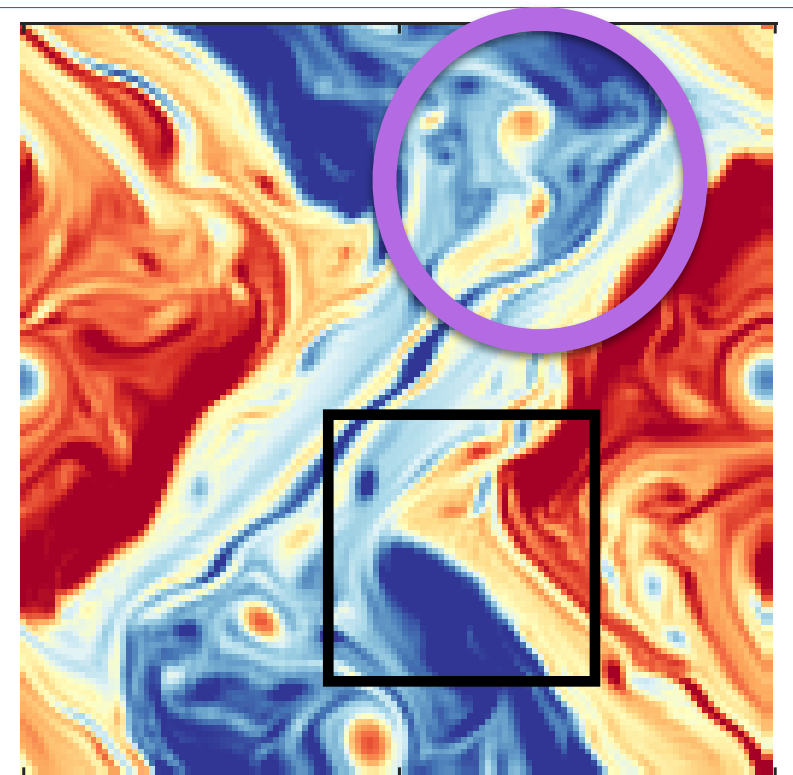
Deterministic 1024 x 1024

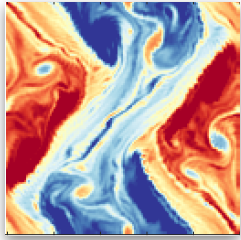


Stochastic 128 x128
with Homogeneous
small-scale velocity



Stochastic 128 x128
with Energy-flux-based
modulation





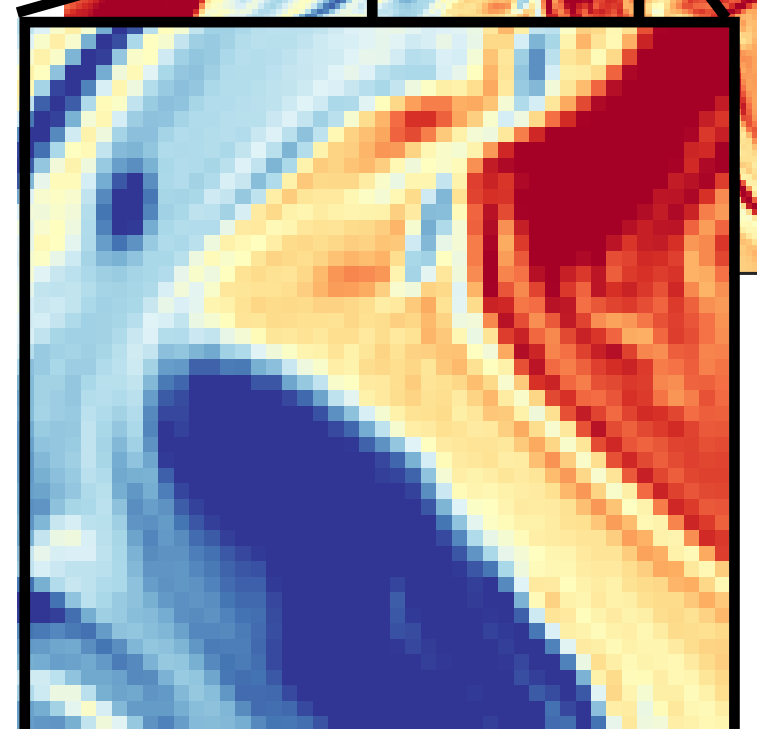
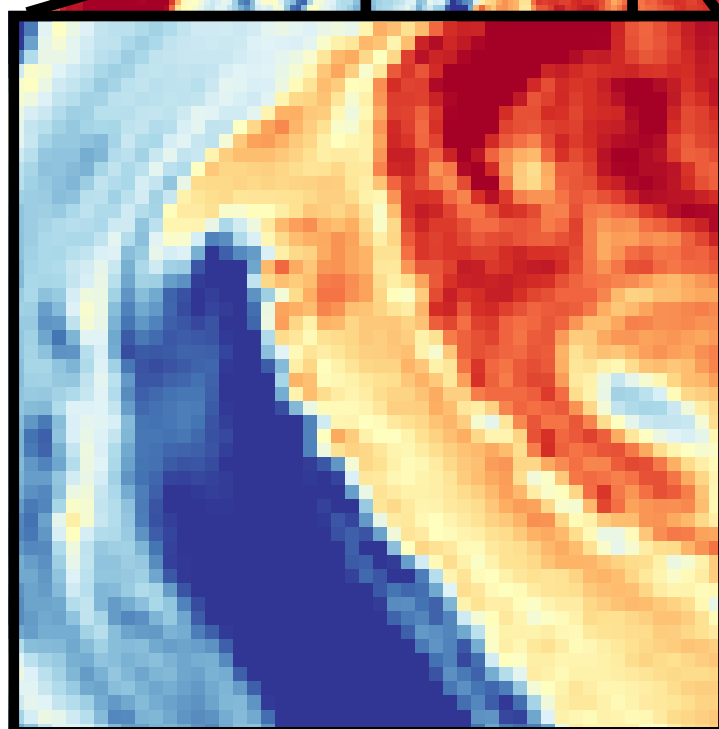
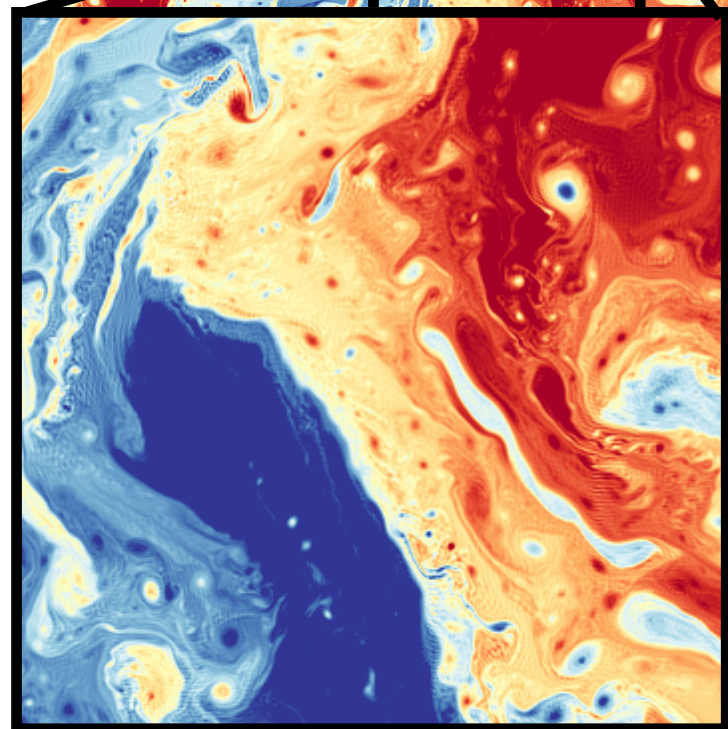
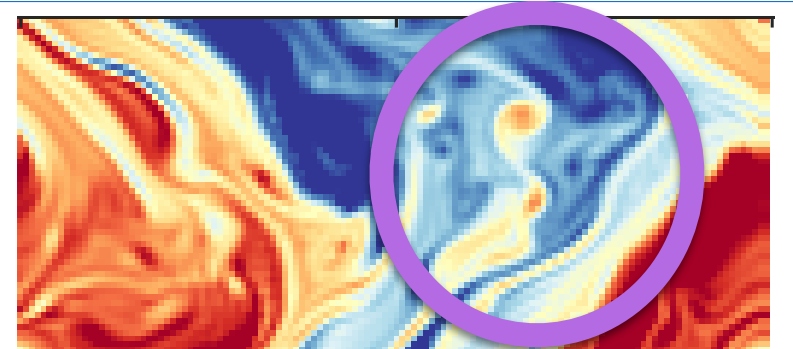
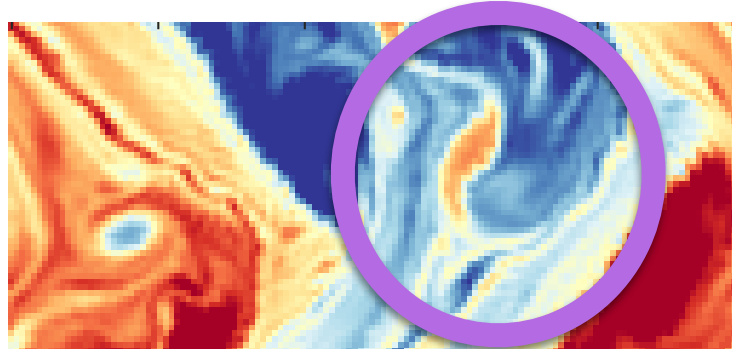
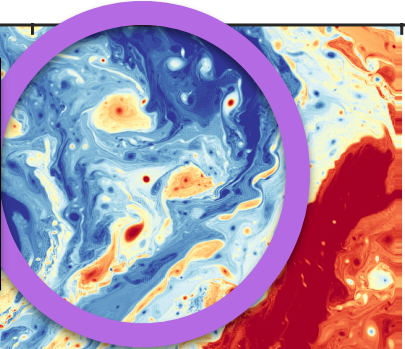
Heterogenous modulation

Deterministic 1024 x 1024

Stochastic 128 x 128
with Homogeneous
small-scale velocity

Stochastic 128 x 128
with Energy-flux-based
modulation

Developed
turbulence



Conclusion

Conclusion

- Stochastic transport (in LU & SALT) blindly describes unresolved triades
- LU conserves kinetic energy / SALT conserves helicity
- For the small-scale velocity parametrisation, both the data-driven (Cotter et al. 2018b) and the self-similar method (Resseguier et al. 2017b) lead to accurate uncertainty quantification (to address filter divergence)
- Energy-flux modulation improves the simulations

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- Cotter, C. J., Crisan, D., Holm, D. D., Pan, W., & Shevchenko, I. (2018). Numerically modelling stochastic lie transport in fluid dynamics. *arXiv preprint arXiv: 1801.09729*.
- Kunita, H., Stochastic Flows and Stochastic Differential Equations, 1990. (Cambridge: Cambridge University Press).
- Resseguier, V., Mémin, E., & Chapron, B. (2017). Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophysical & Astrophysical Fluid Dynamics*, 111(3), 177-208.