Dynamics under location uncertainty: model derivation, modified transport and uncertainty quantification

Valentin Resseguier, Baylor Fox-Kemper, Etienne Mémin, Bertrand Chapron

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Motivations

- Rigorously identified subgrid dynamics effects
- Injecting likely small-scale dynamics
- Predict extreme events
- Quantification of modeling errors → Ensemble forecasts and data assimilation
- Studying different likely scenarios and attractors → Climate projections

Contents

Part I
Location uncertainty

Adding random velocity

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

Adding random velocity

Resolved
large scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma} \dot{\boldsymbol{B}}$$

Adding random velocity

Resolved
large scales

White-in-time
small scales

$$v = w + \sigma \dot{B}$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \dfrac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Adding random velocity

Resolved large scales

White-in-time small scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

Large scales:

$\boldsymbol{w}$

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Adding random velocity

Resolved large scales

White-in-time small scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

References :

Mikulevicius and Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz and Holm 2017

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Advection of tracer $\boldsymbol{\Theta}$

$$\frac{D\boldsymbol{\Theta}}{Dt} = 0$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Advection of tracer $\boldsymbol{\Theta}$

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Small scales:

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Advection of tracer $\boldsymbol{\Theta}$

$$\partial_t \Theta + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla} \Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \Theta \right)$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \dfrac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

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Advection of tracer $\boldsymbol{\Theta}$

Advection

Diffusion

$$\partial_t \Theta + \underbrace{\boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}_{\text{Diffusion}}$$

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Drift correction

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma} \dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \dfrac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

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Drift correction

Multiplicative random forcing

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Advection of tracer $\boldsymbol{\Theta}$

Advection

Diffusion

$$\partial_t \Theta + \underbrace{\boldsymbol{w}^{\star} \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}_{\text{Diffusion}}$$

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Drift correction

Multiplicative random forcing

Balanced energy exchanges

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma} \dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Derived random models

Conservations (mass, linear momentum, …)

$$\frac{D}{Dt}$$

Navier-Stokes

Derived random models

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Simplifications

Conservations (mass, linear momentum, …)

$$\frac{D}{Dt}$$

Navier-Stokes

Derived random models

Large scales:
$\boldsymbol{w}$

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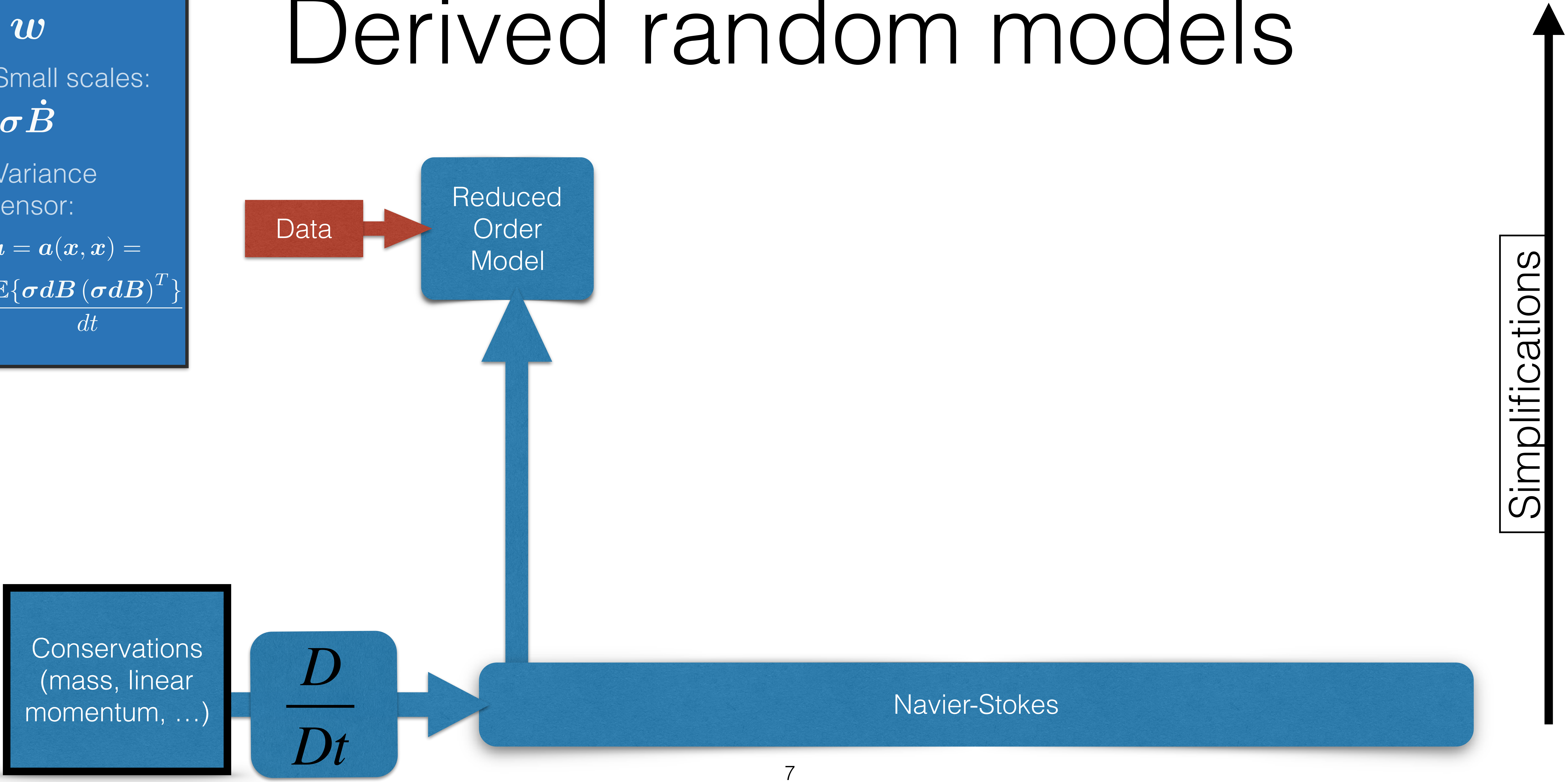

Derived random models

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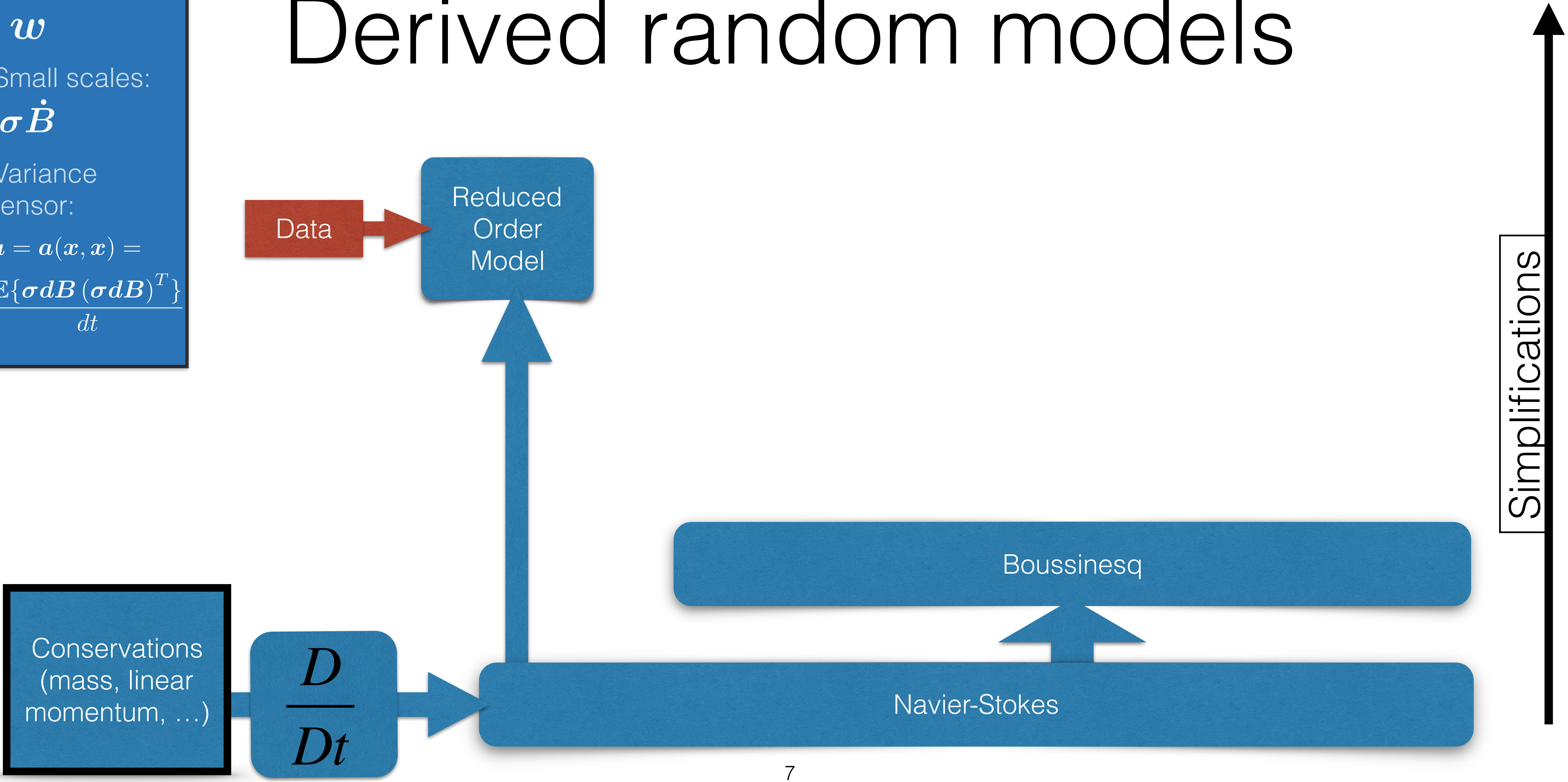

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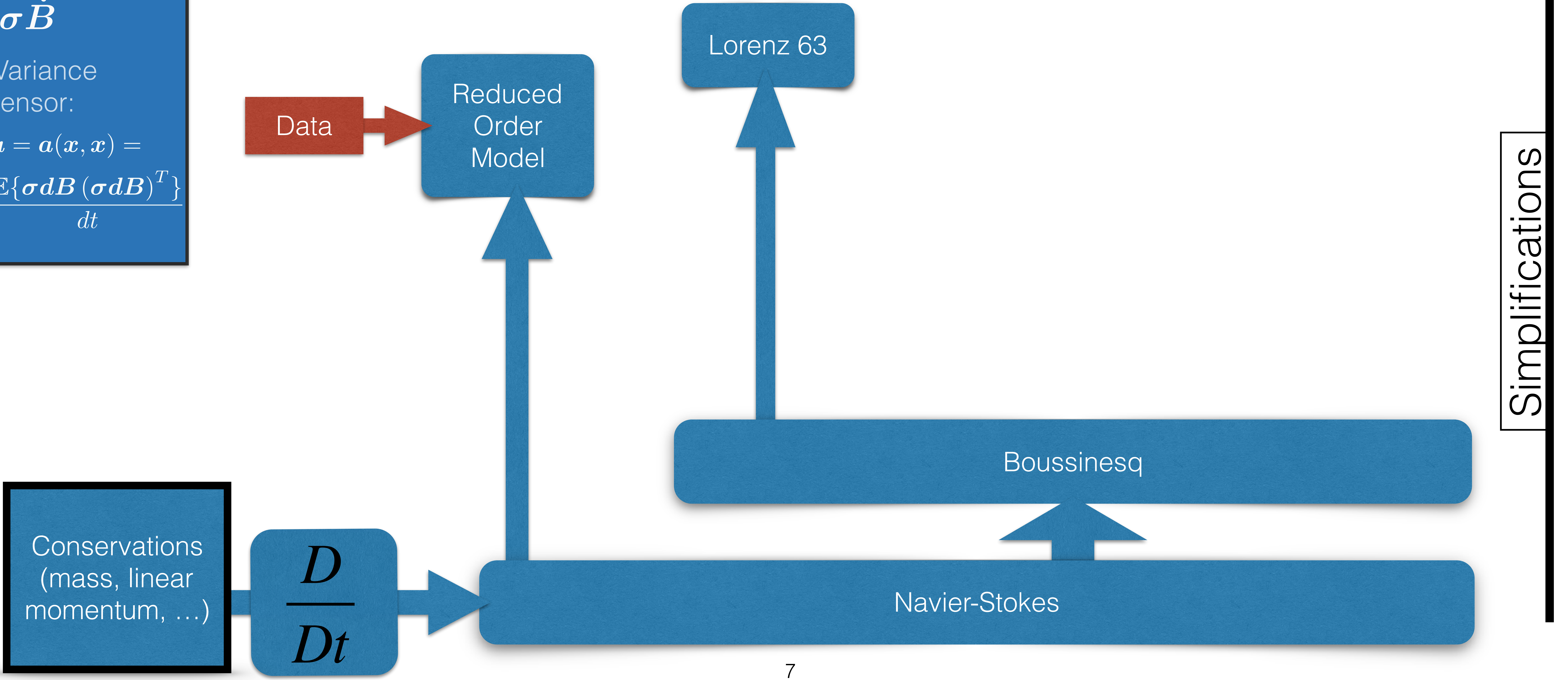

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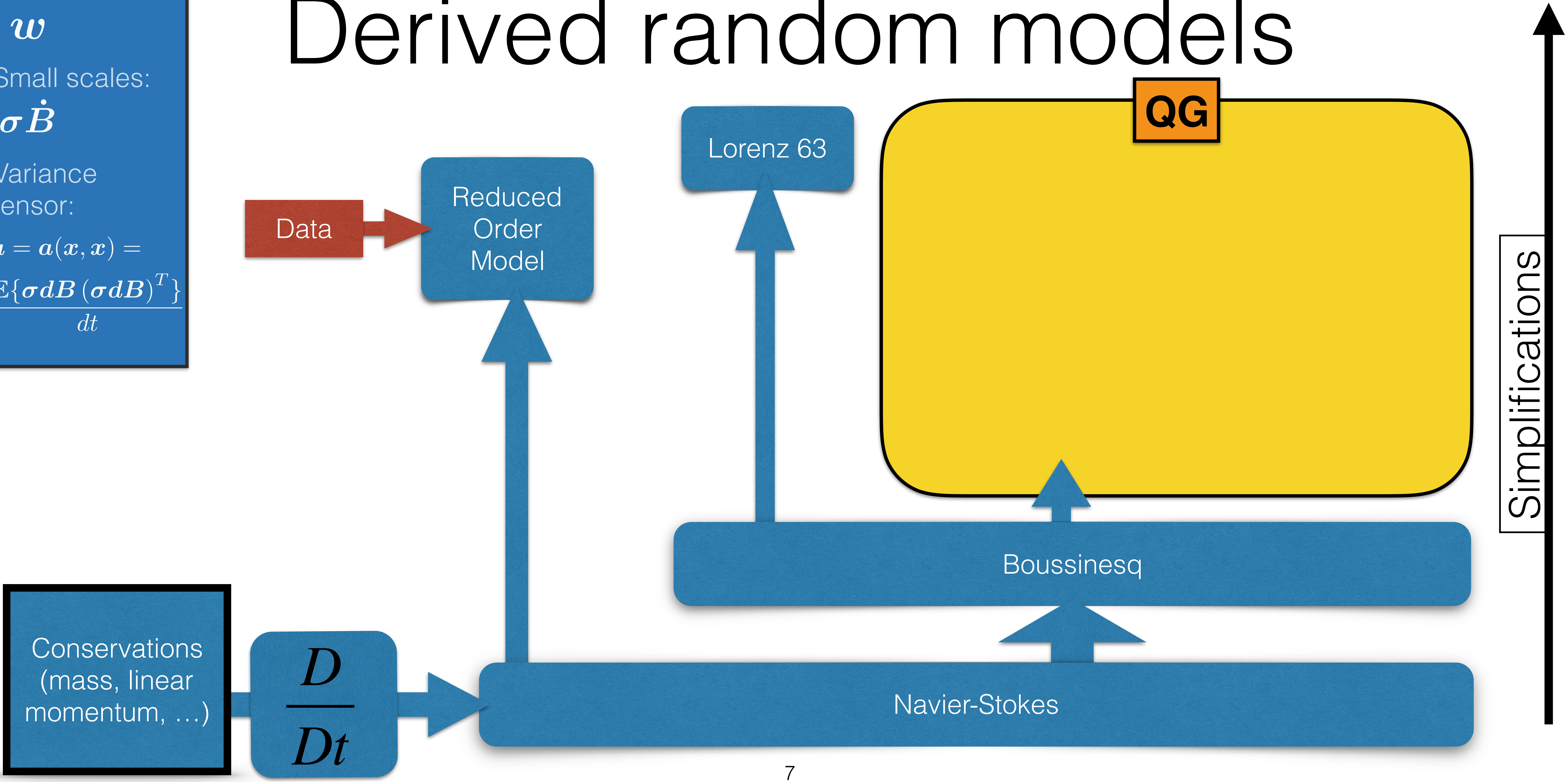

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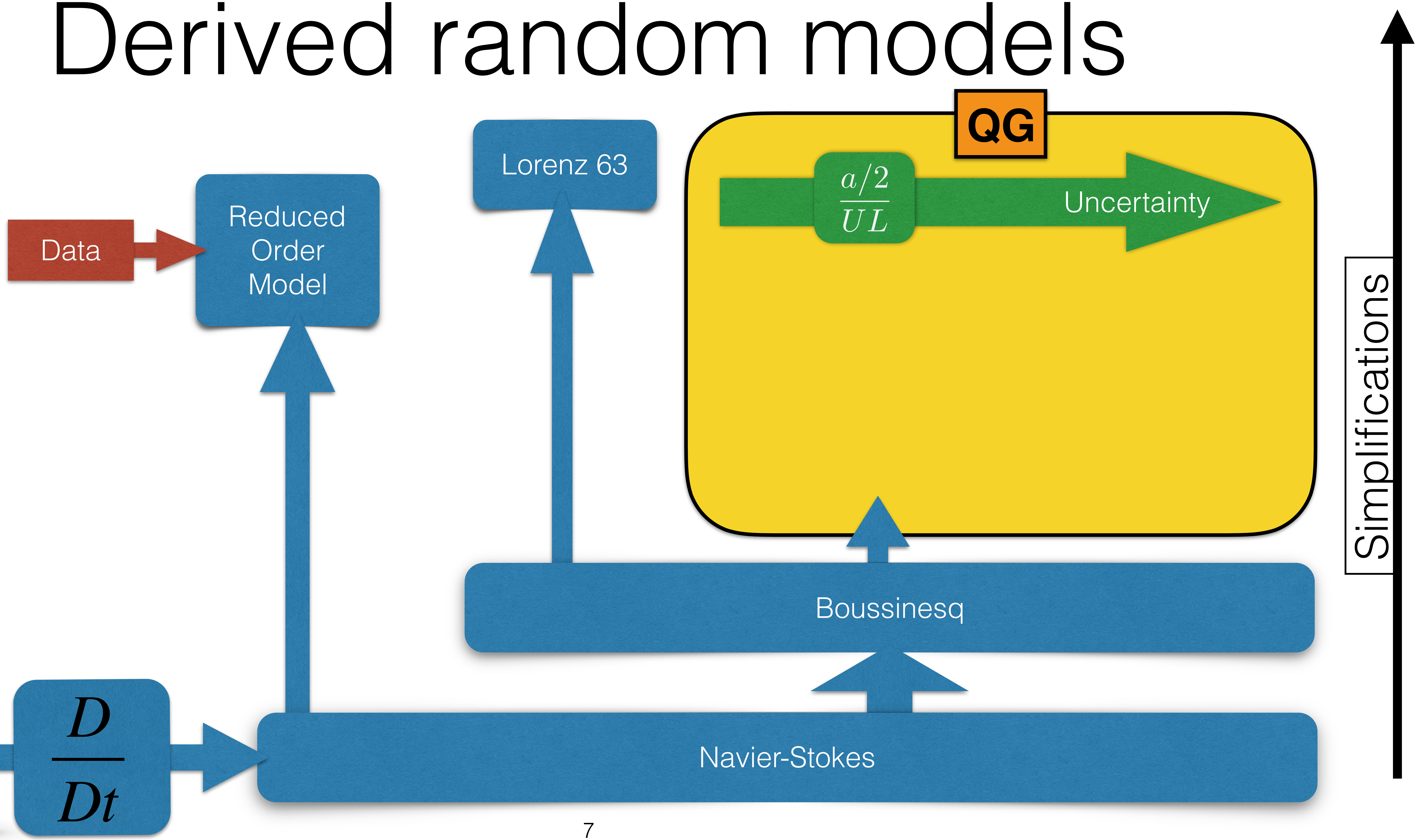

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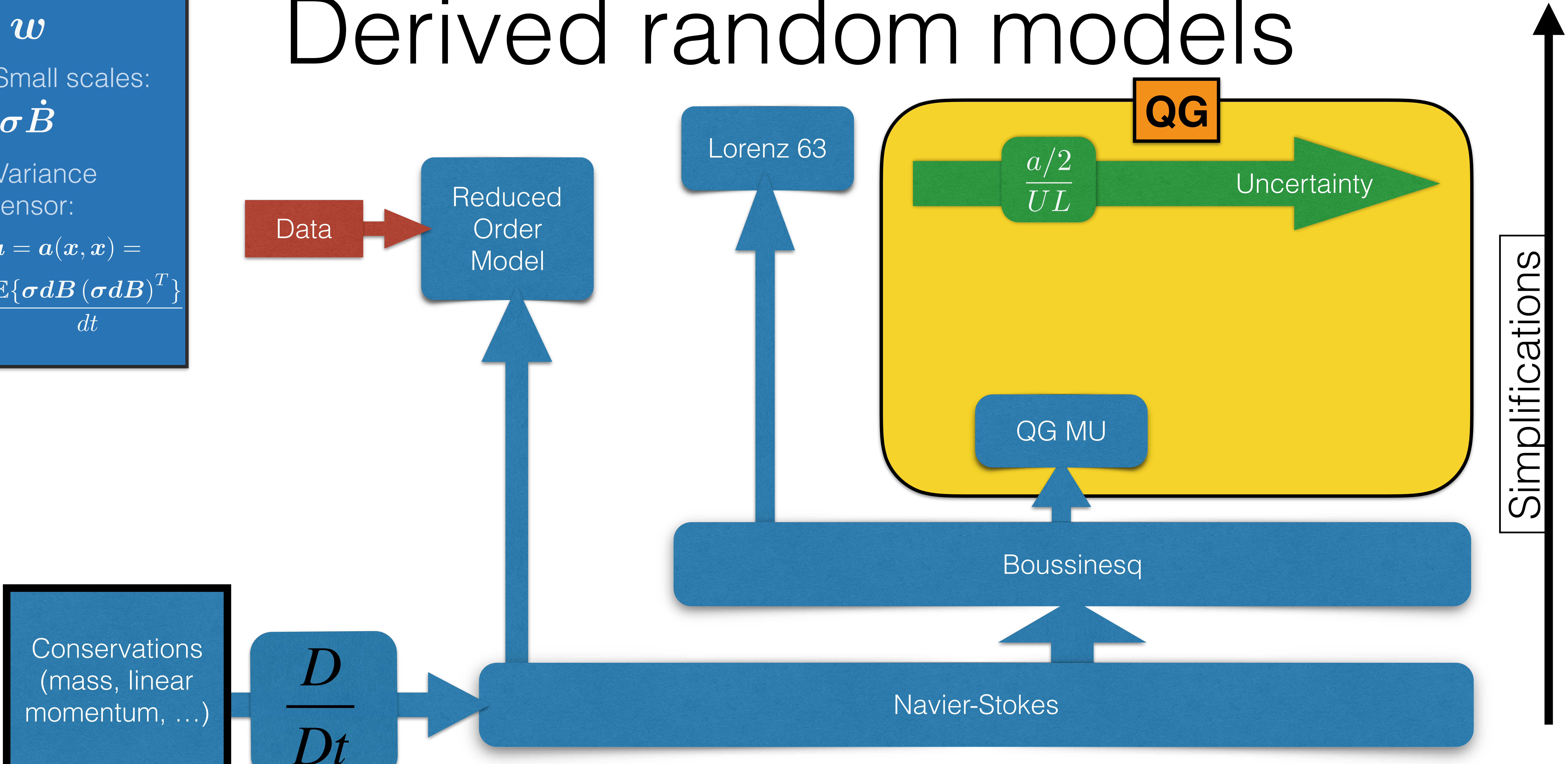

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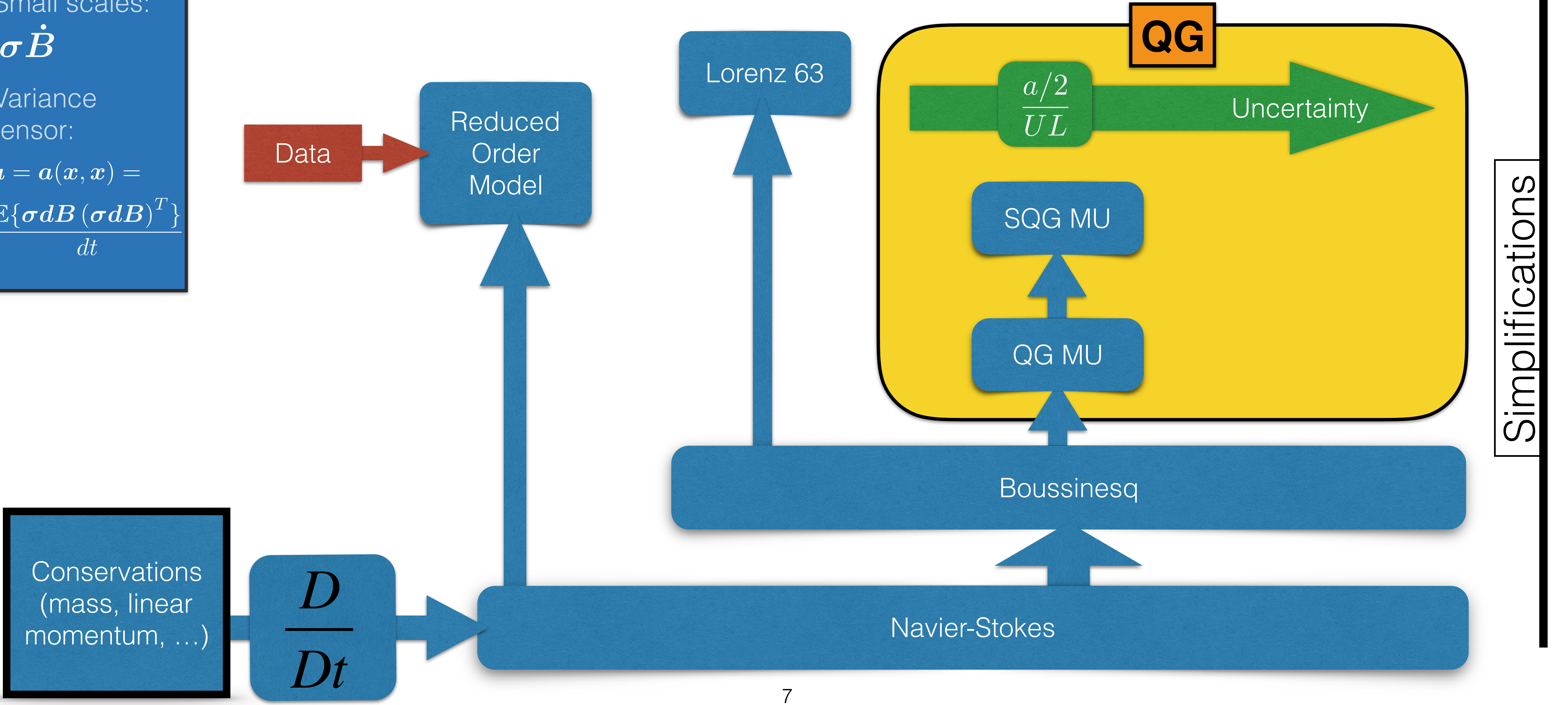
Derived random models



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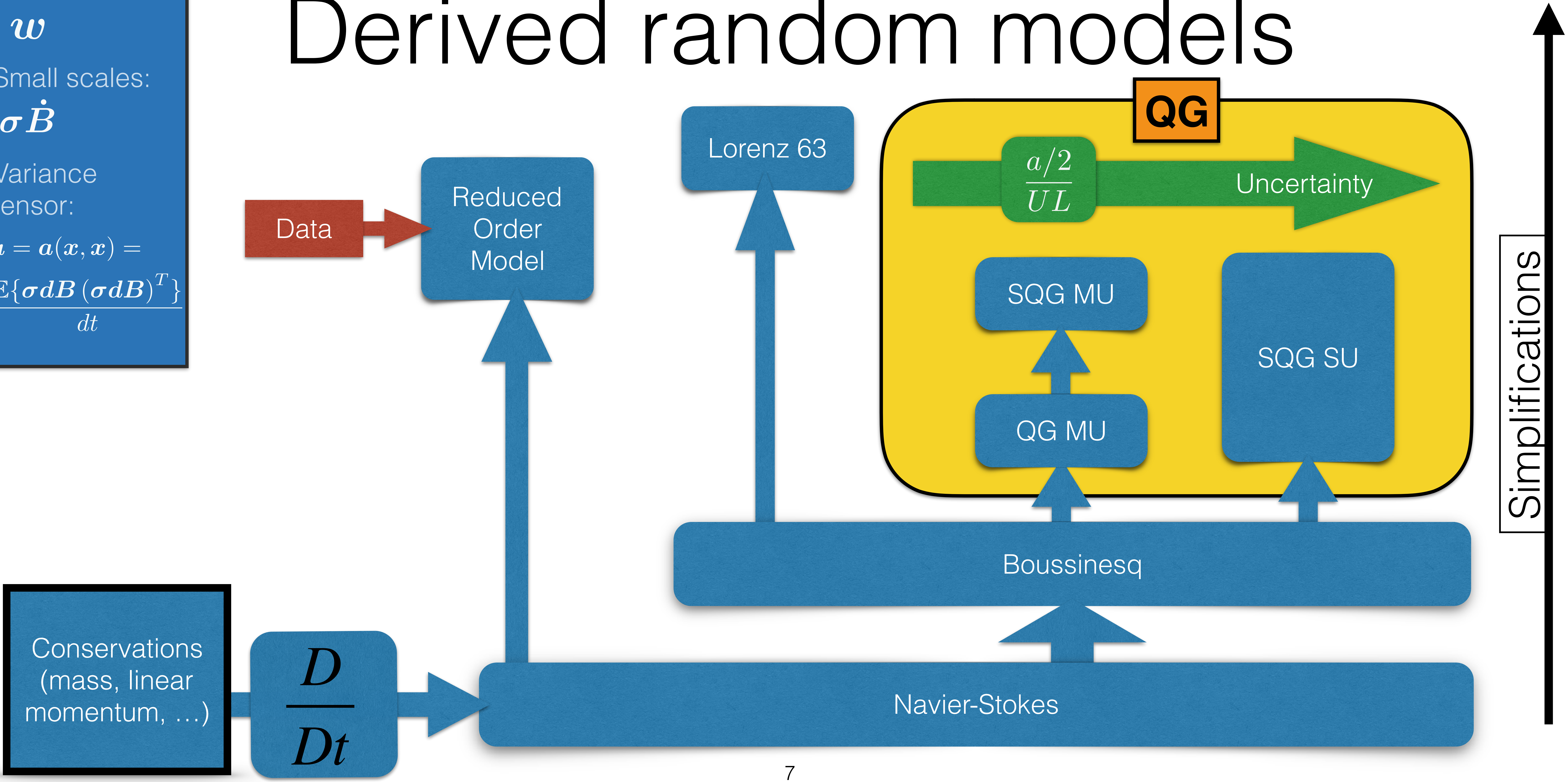

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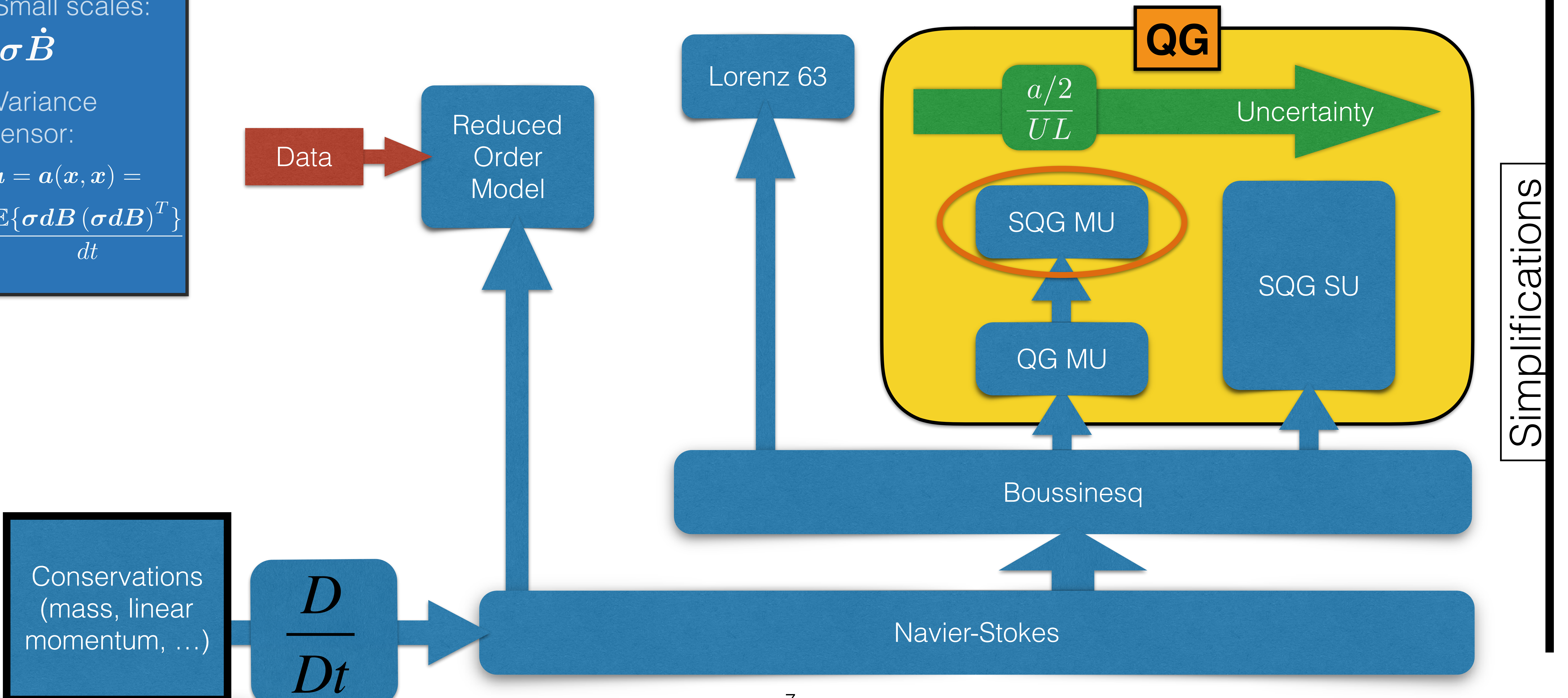
Derived random models



Part II
SQG under Moderate Uncertainty

SQG MU

Code available online

SQG

$$\frac{Db}{Dt} = \boxed{-\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}}$$

$$\boldsymbol{u} = \left(\text{cst.} \boldsymbol{\nabla}^{\perp} \Delta^{-\frac{1}{2}} \right) b$$

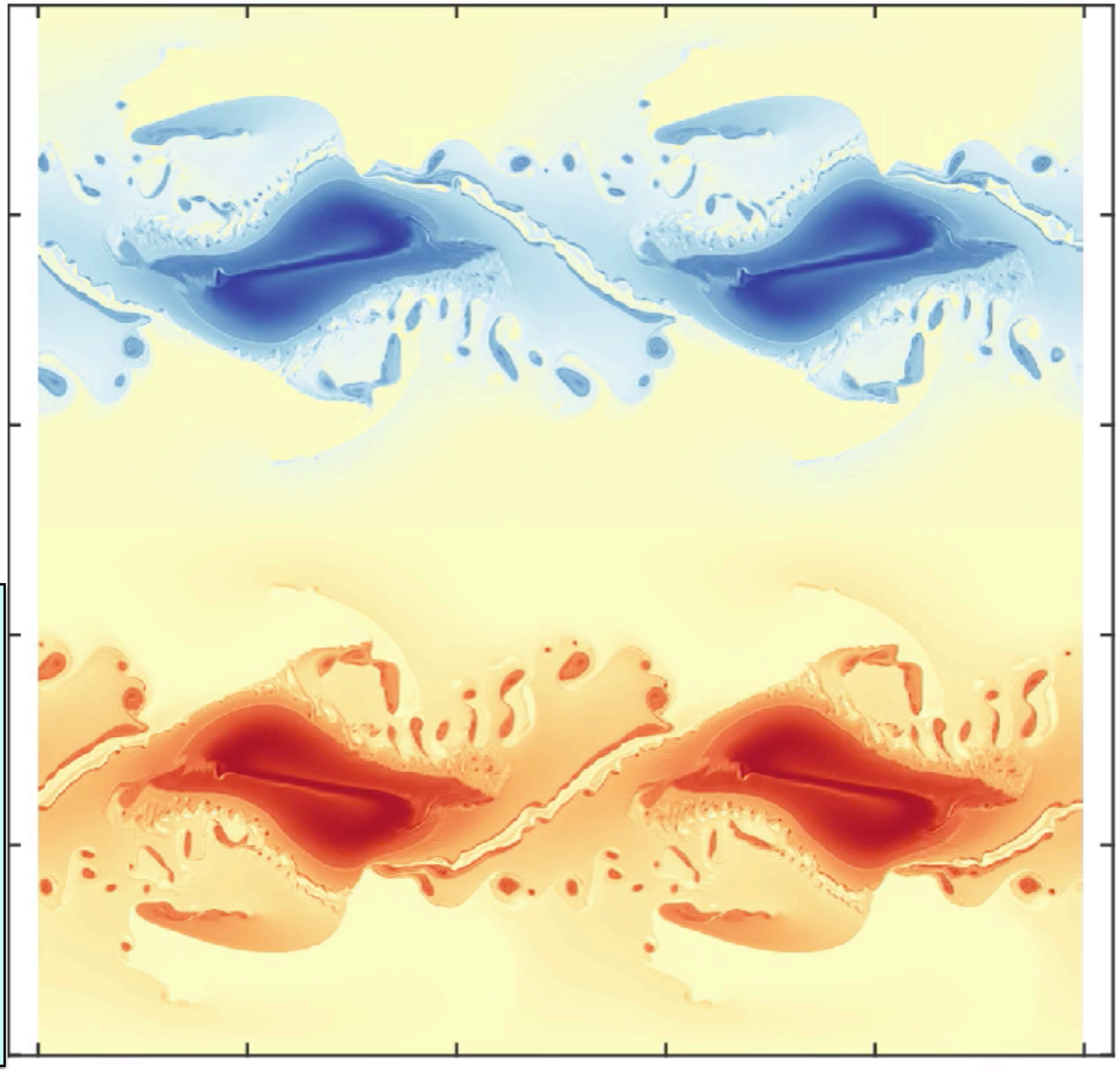
Reference flow:

deterministic

SQG

1024 x 1024

$t = 17$ day



SQG

$$\frac{Db}{Dt} = \boxed{-\alpha_{HV}\Delta^4 b \quad \text{Hyper-viscosity}}$$

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Reference flow:

deterministic

SQG

1024 x 1024

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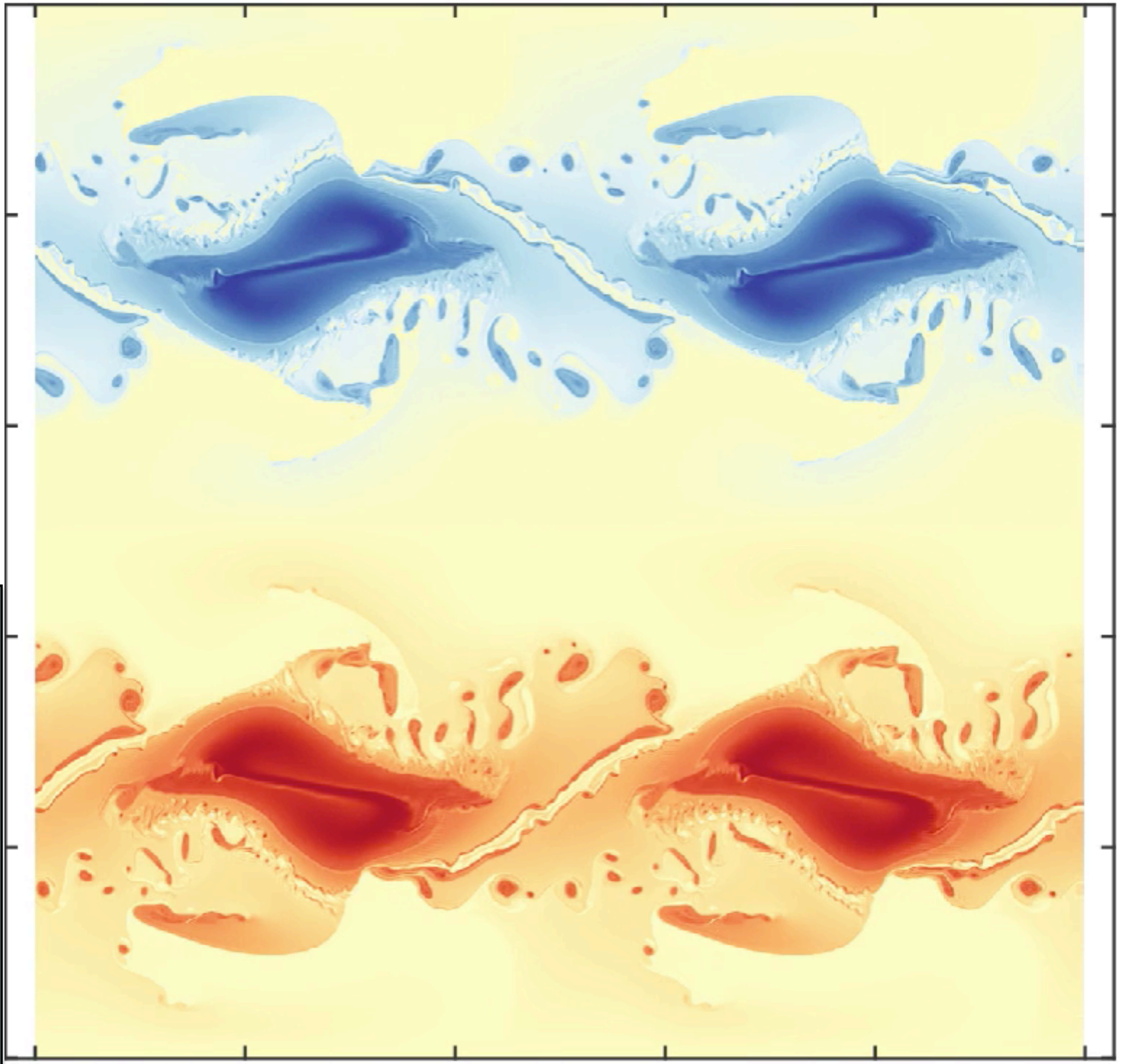
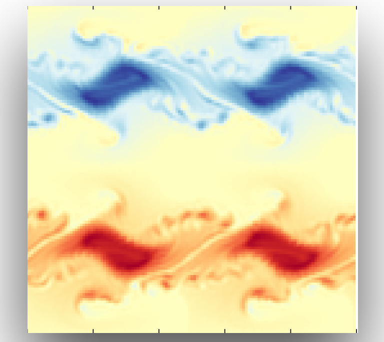
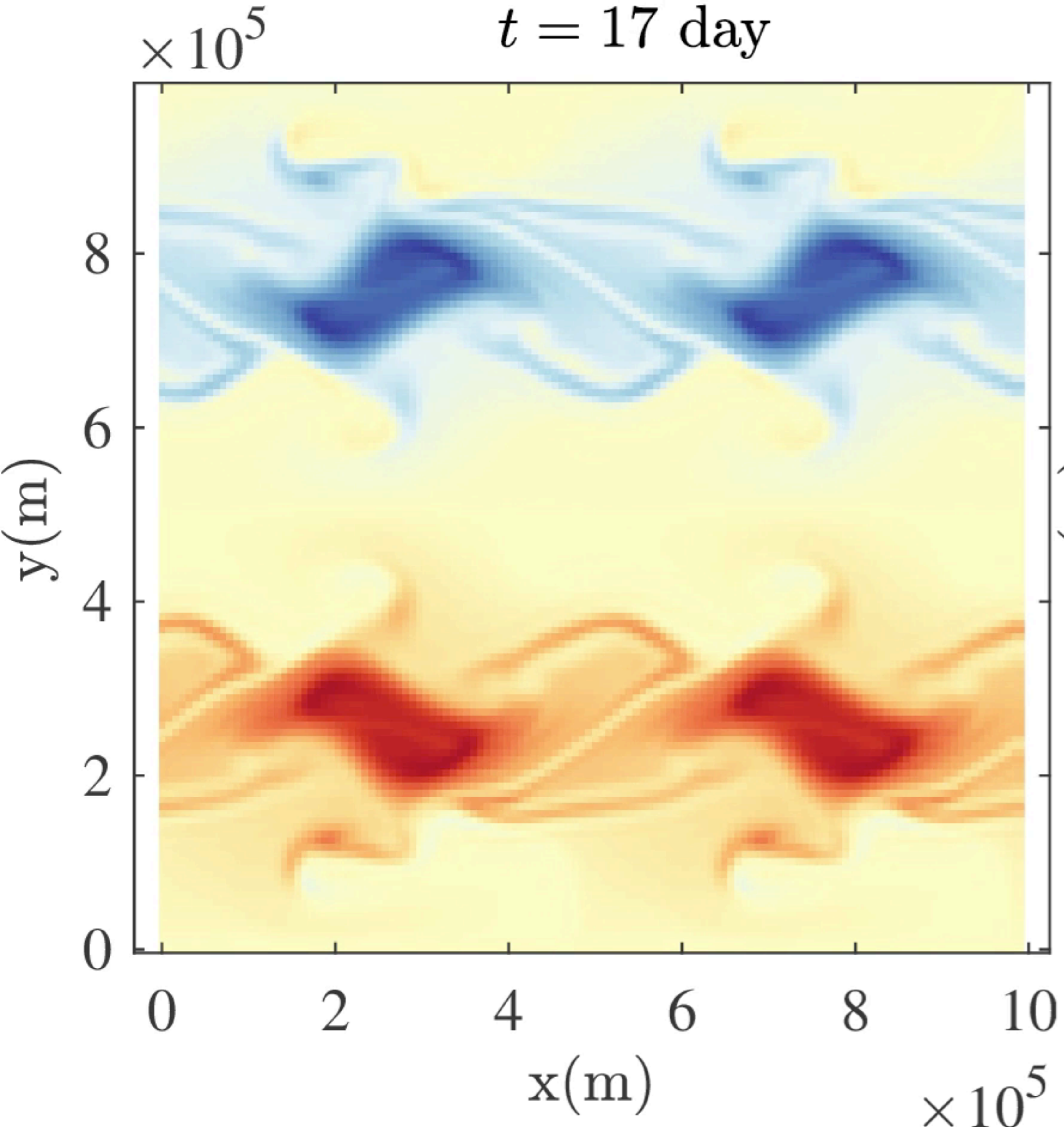
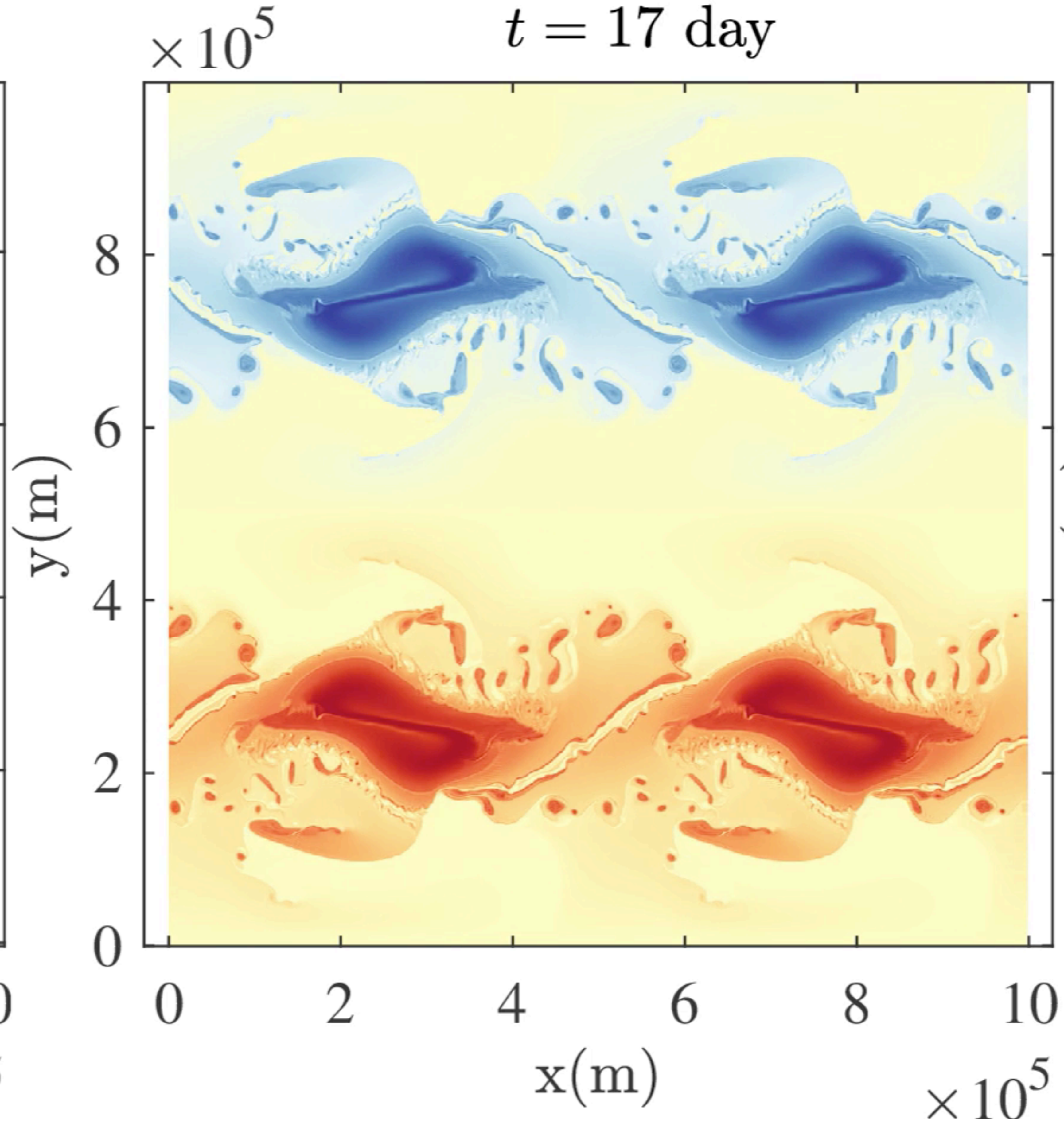


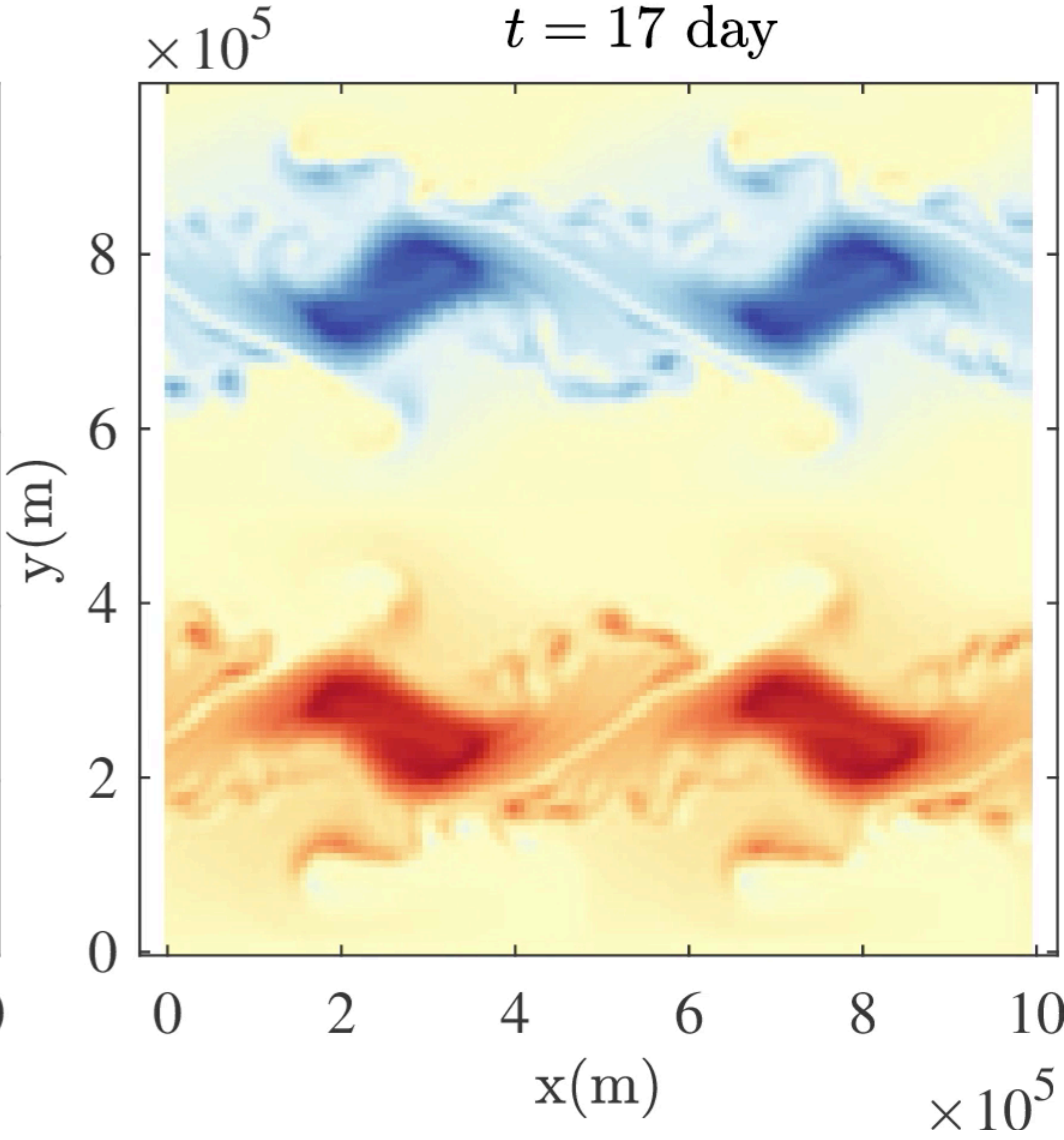


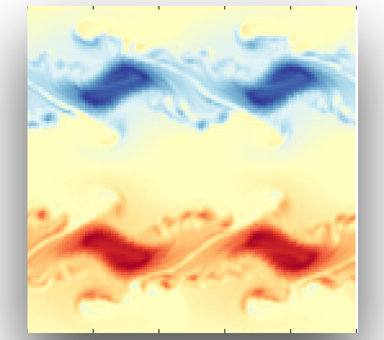
One realization : Stochastic destabilization

Deterministic 128 x 128	Deterministic 1024 x 1024	Location Uncertainty 128 x128







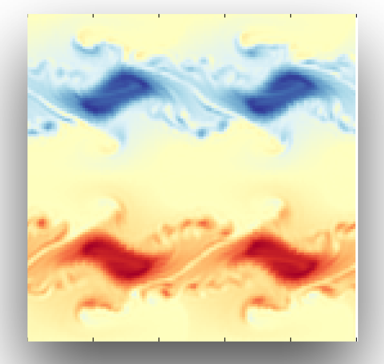


One realization : Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x128



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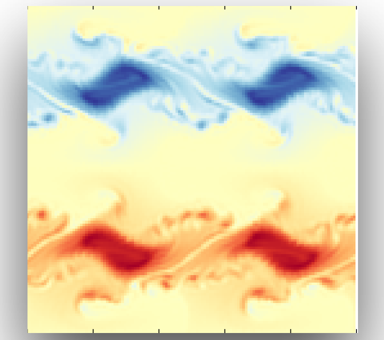

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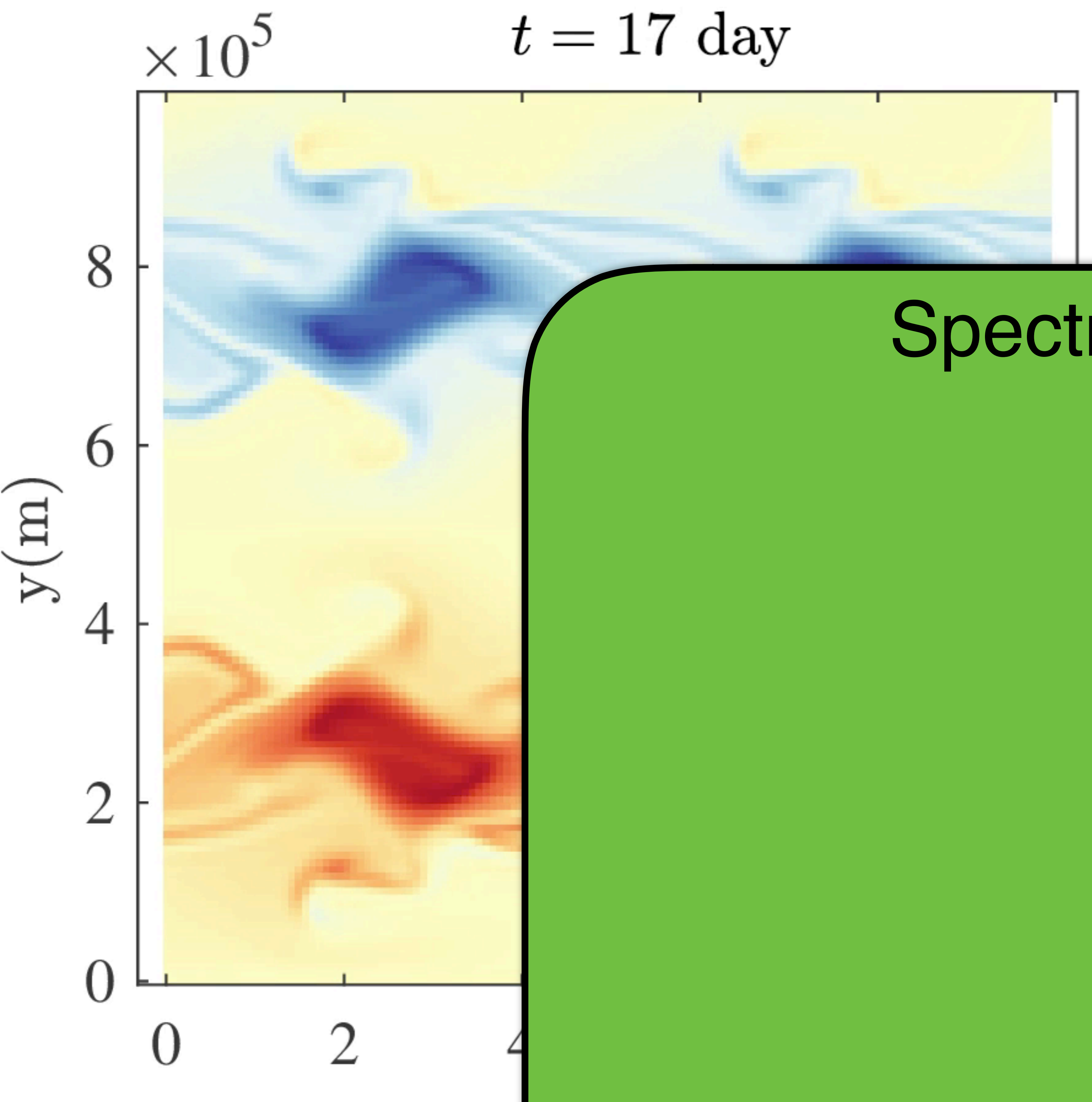

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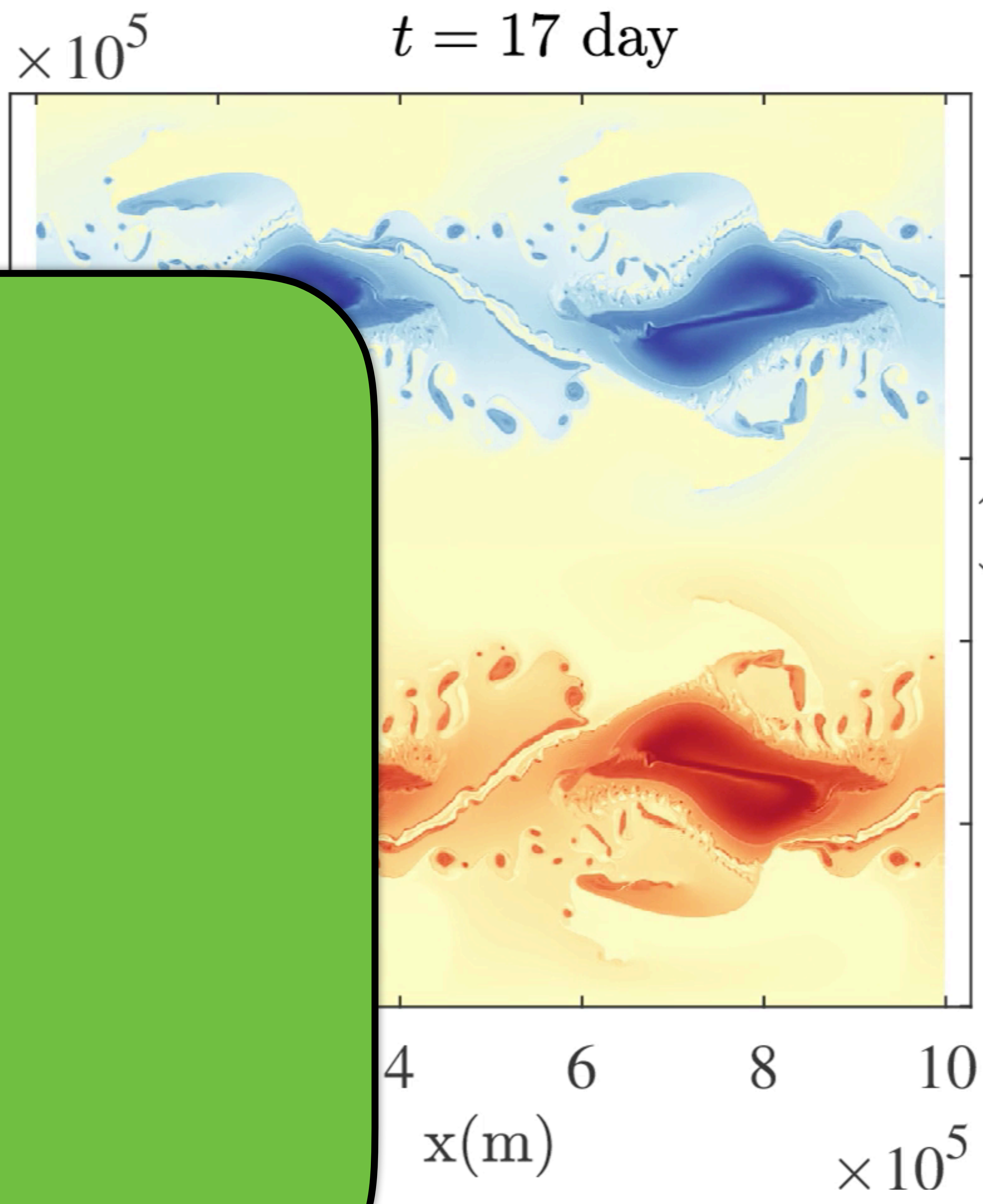

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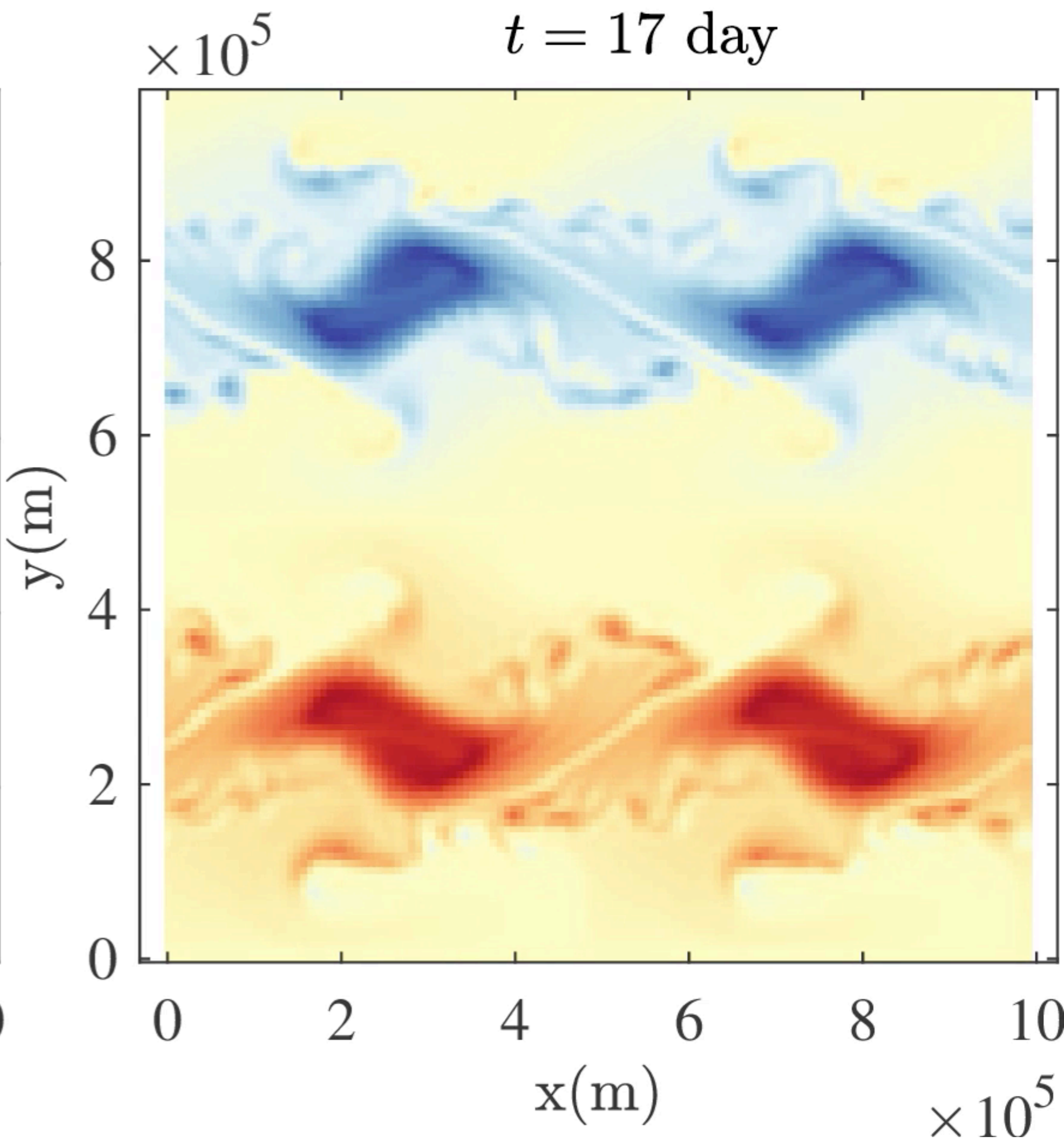
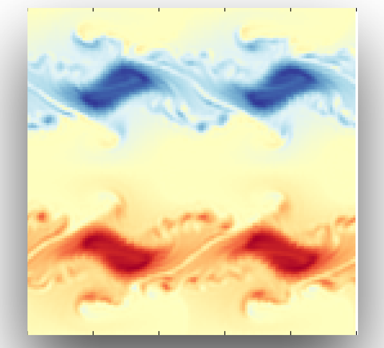


Spectrum



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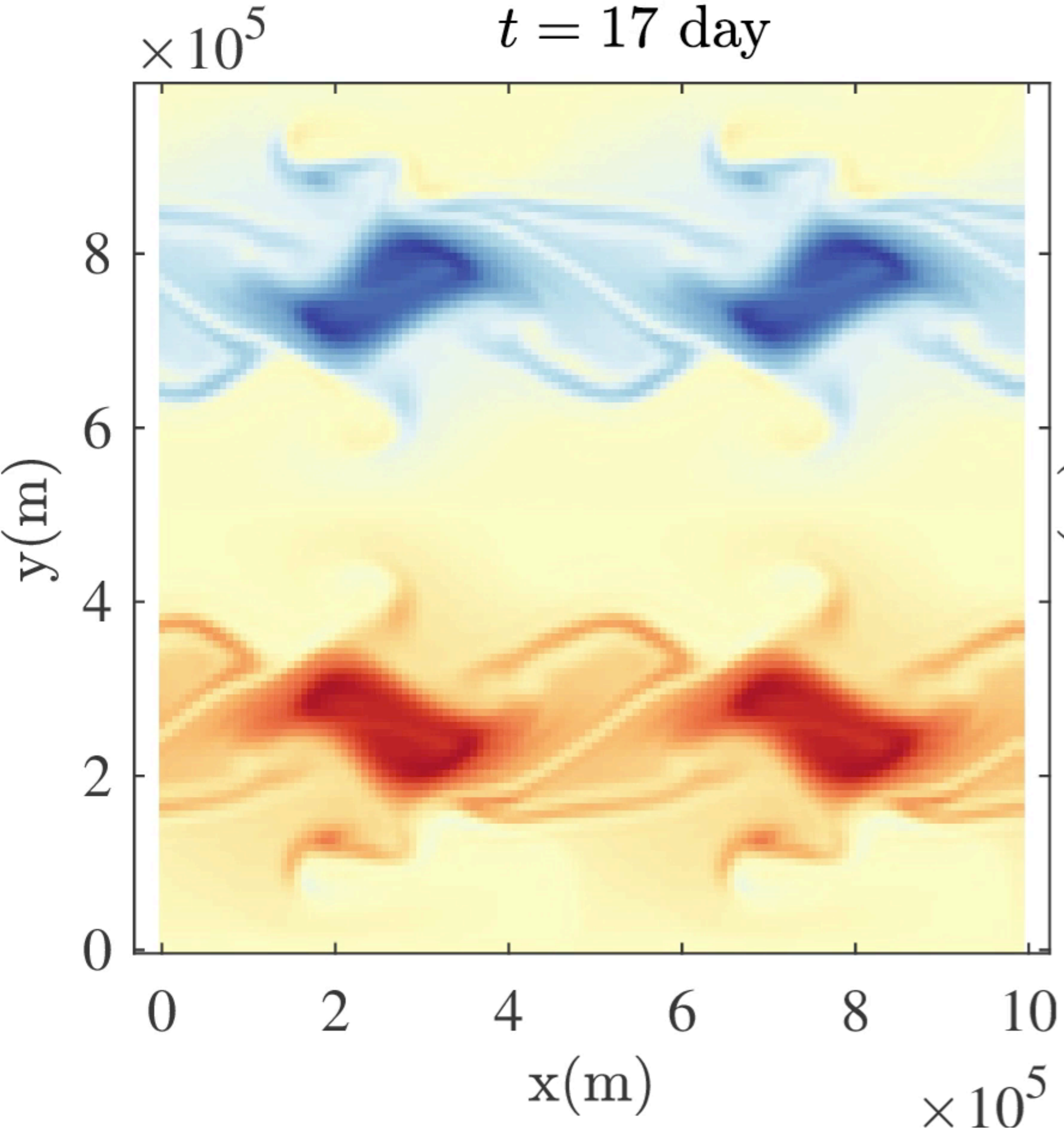

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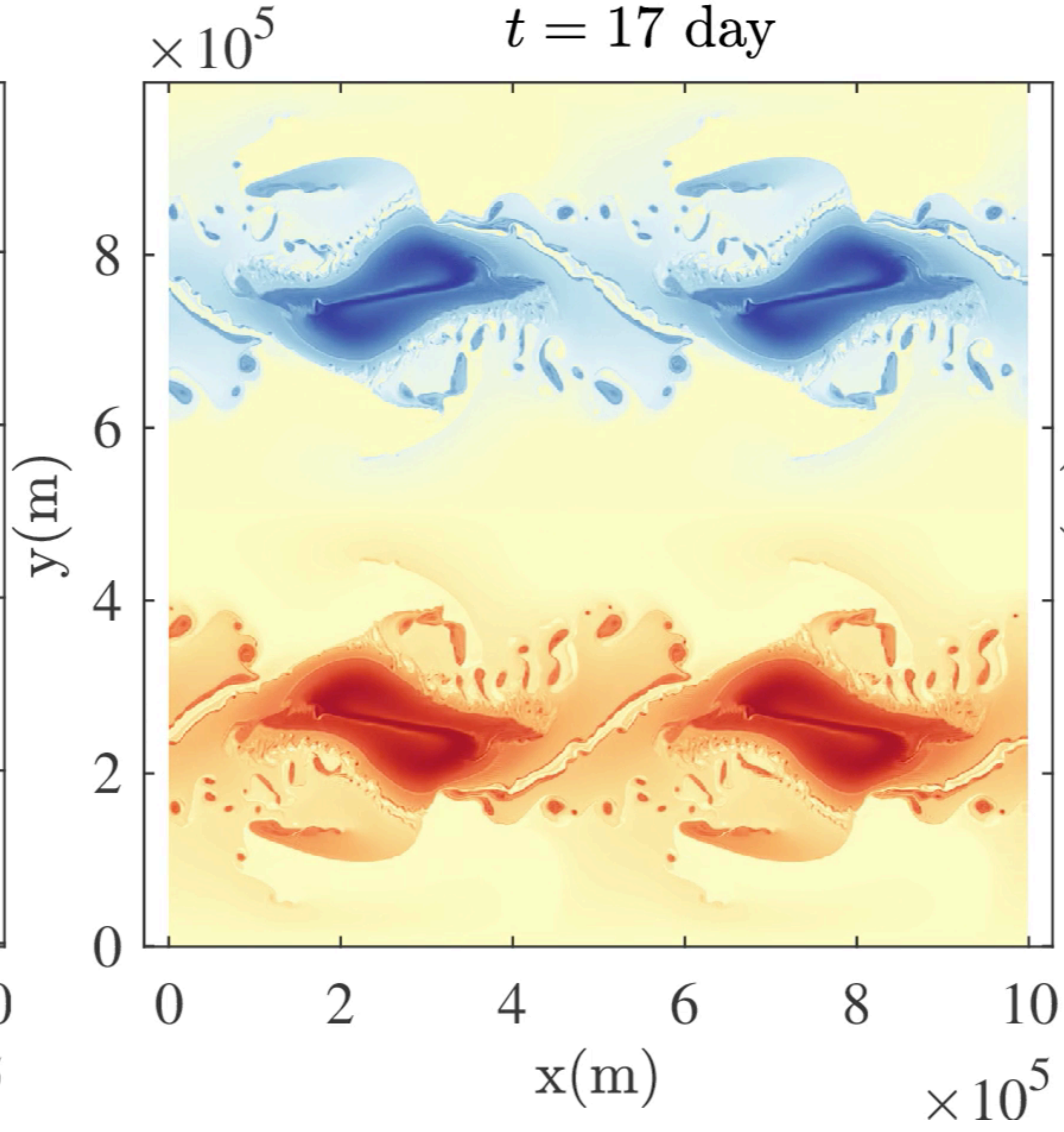
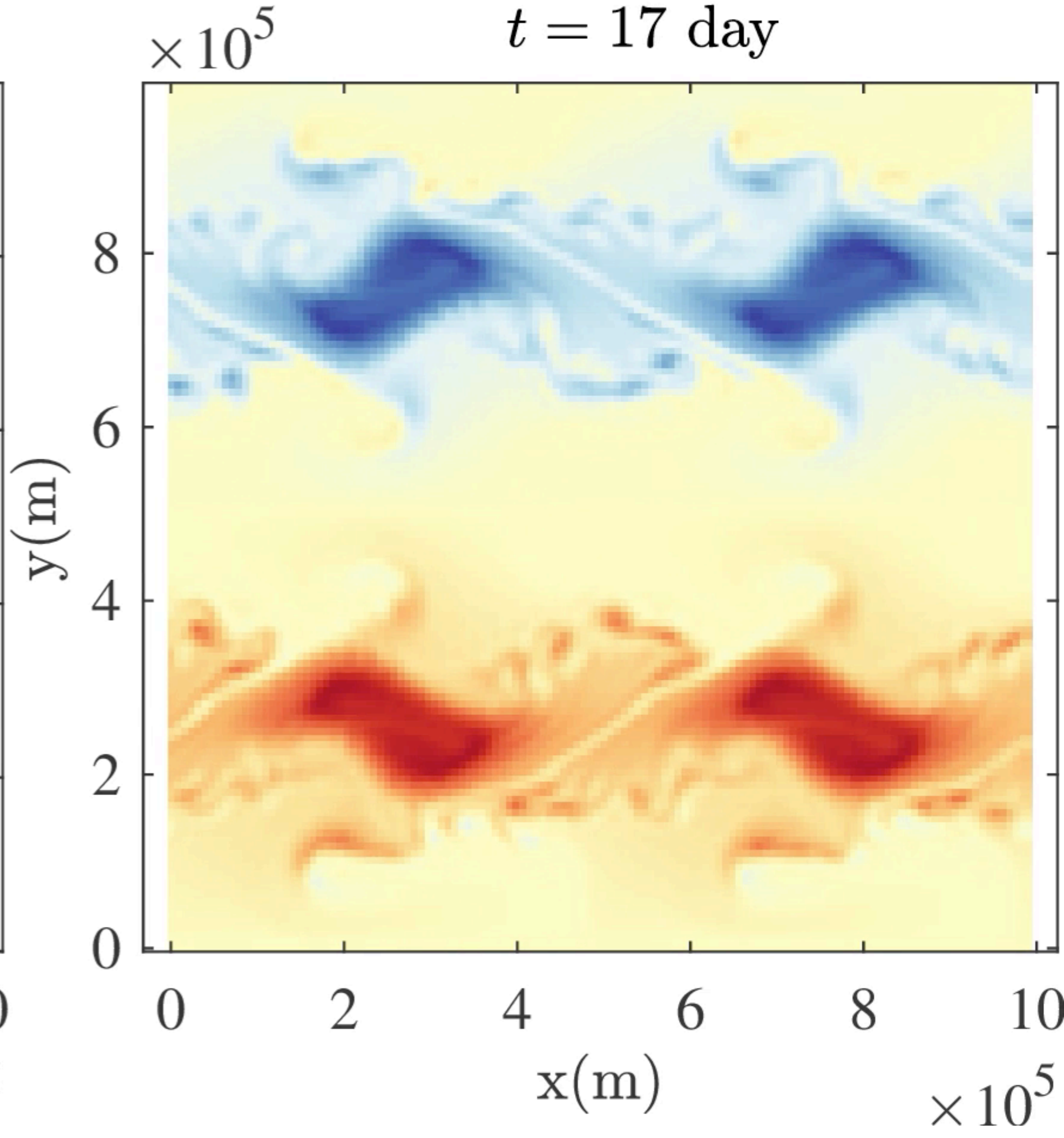
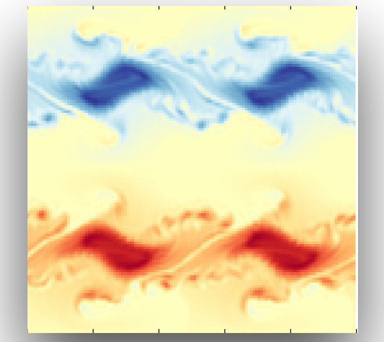


Location Uncertainty 128 x128





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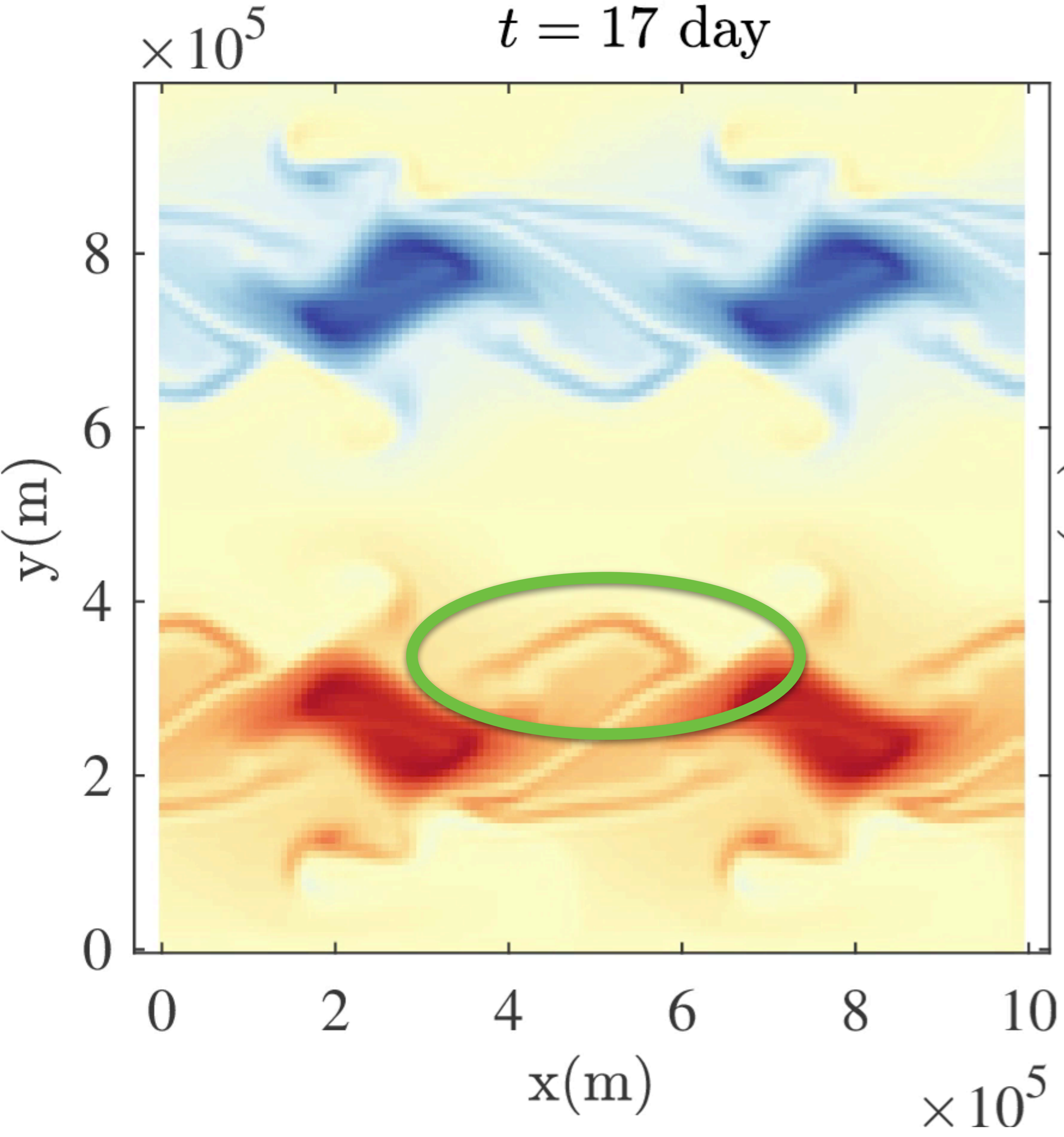

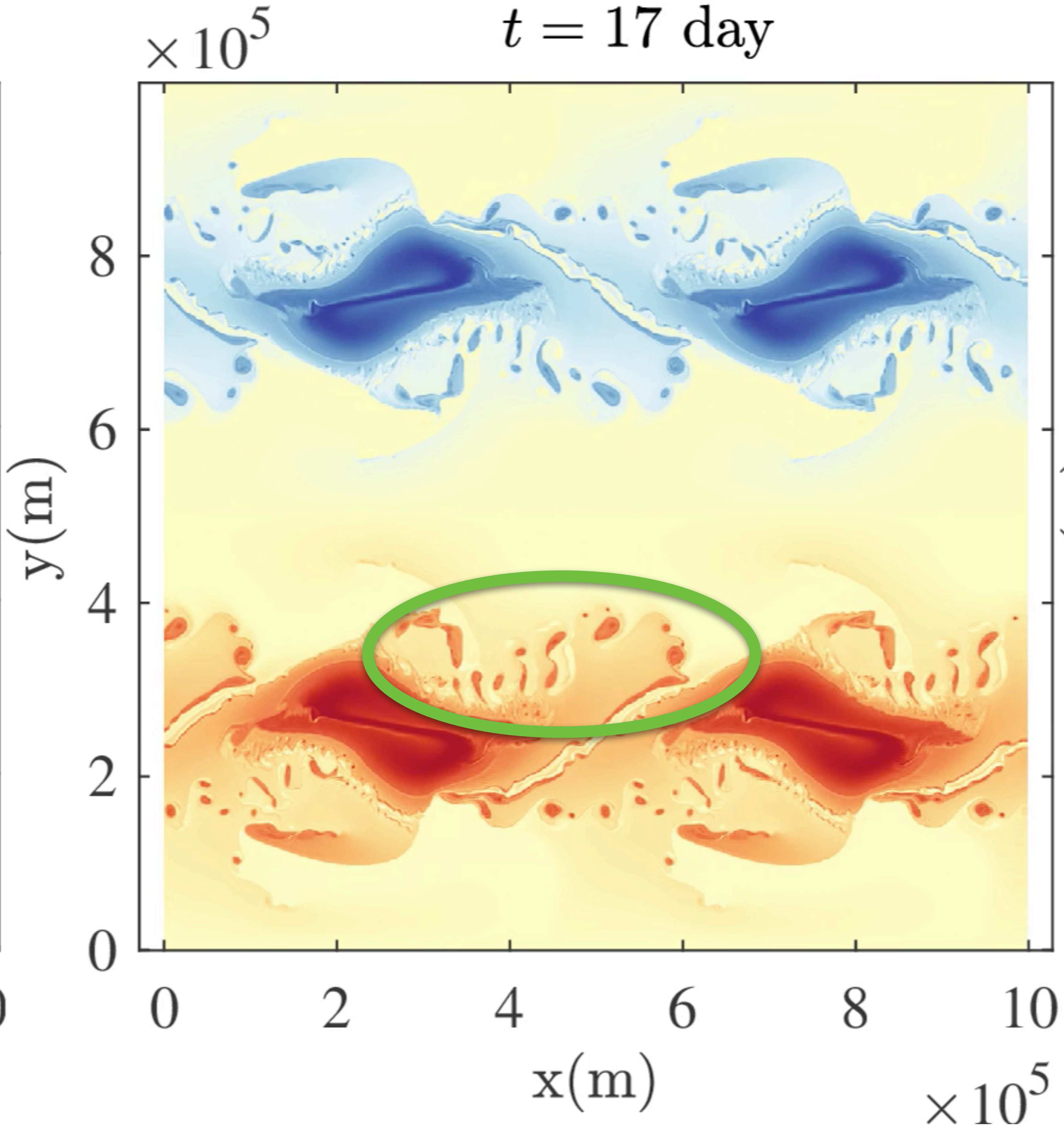

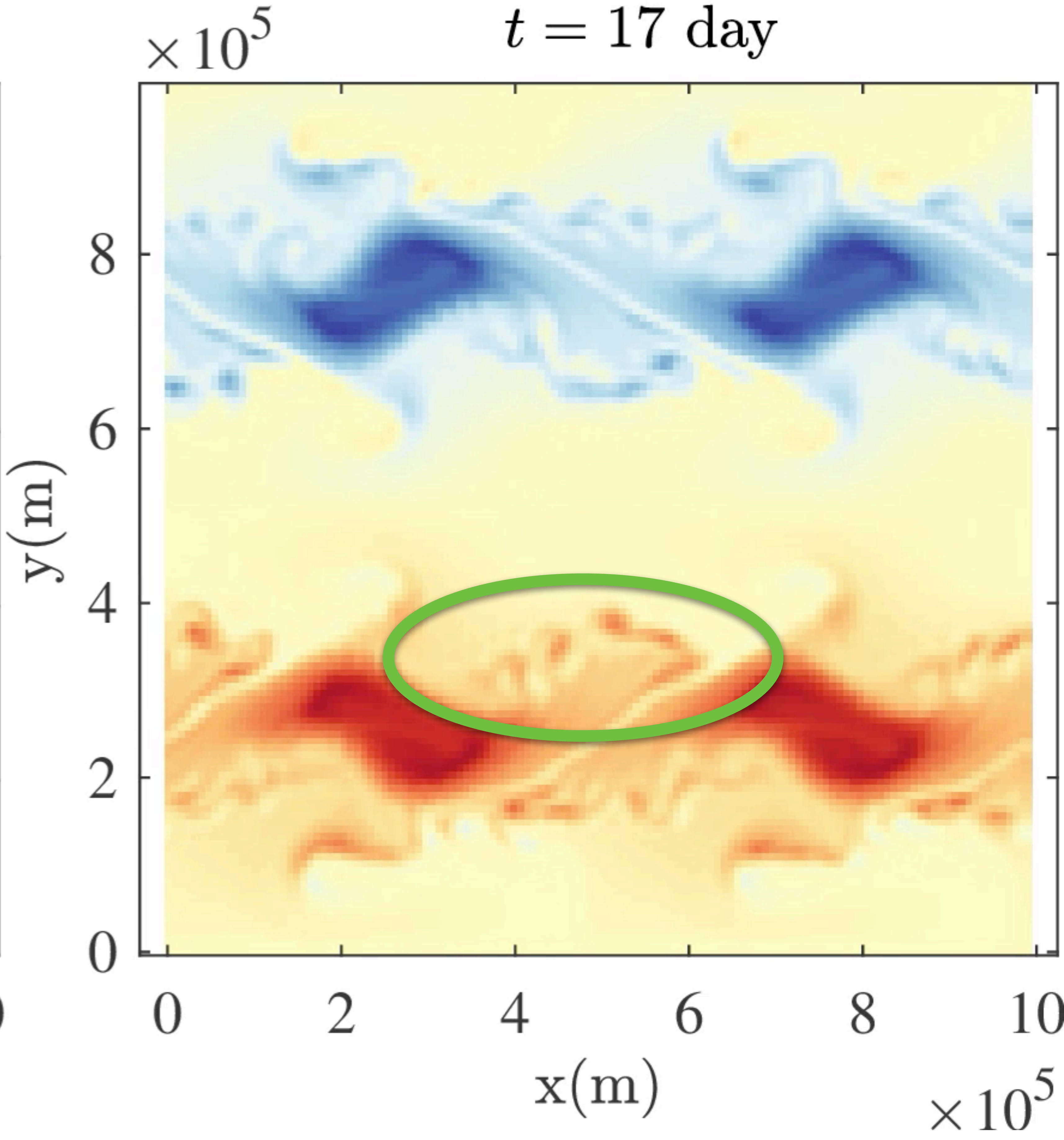

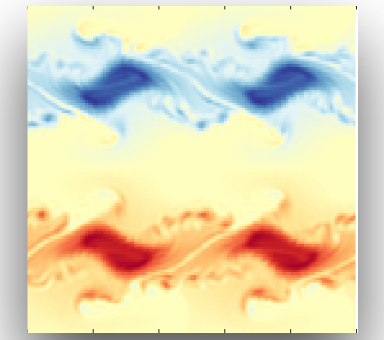

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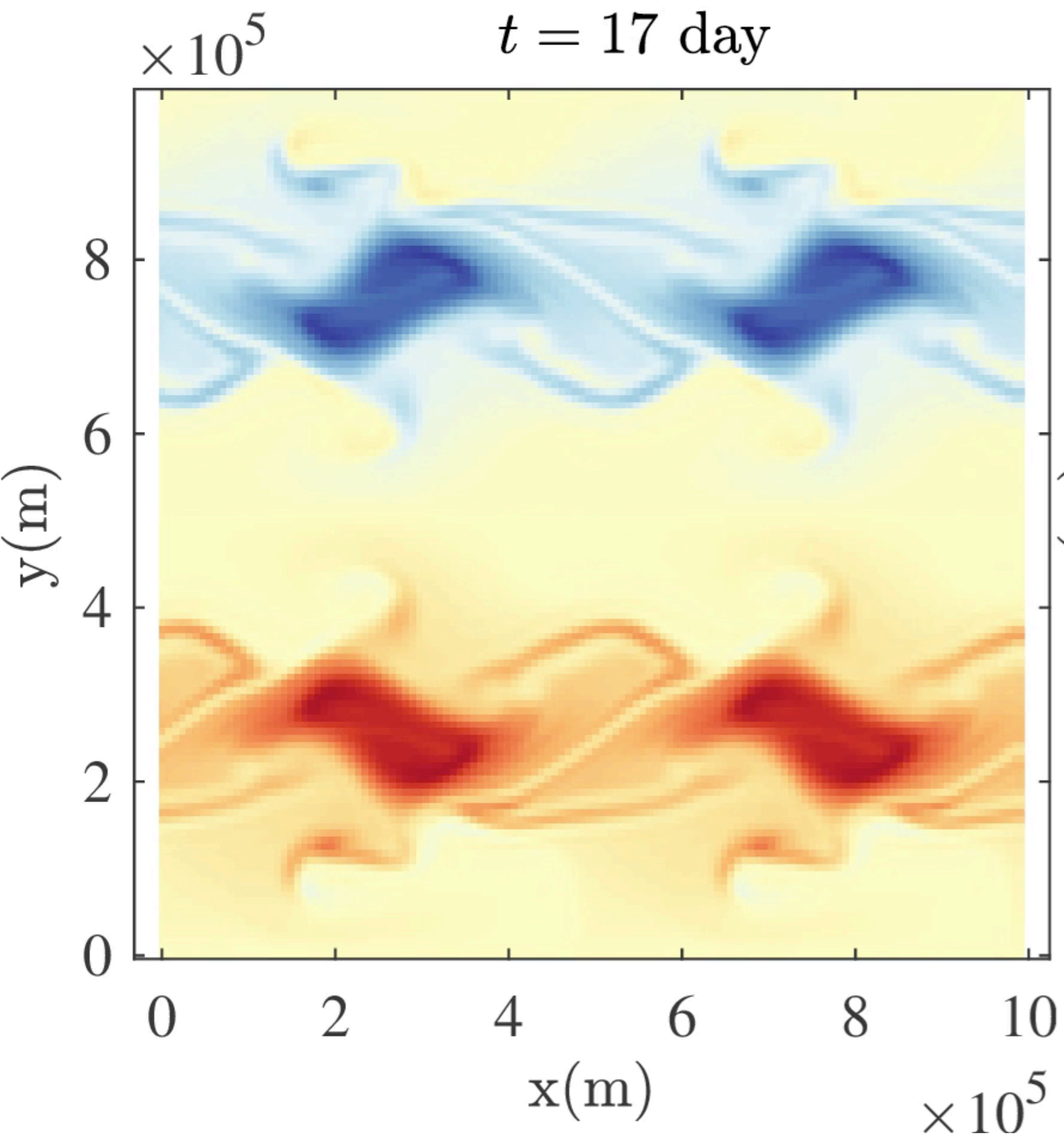
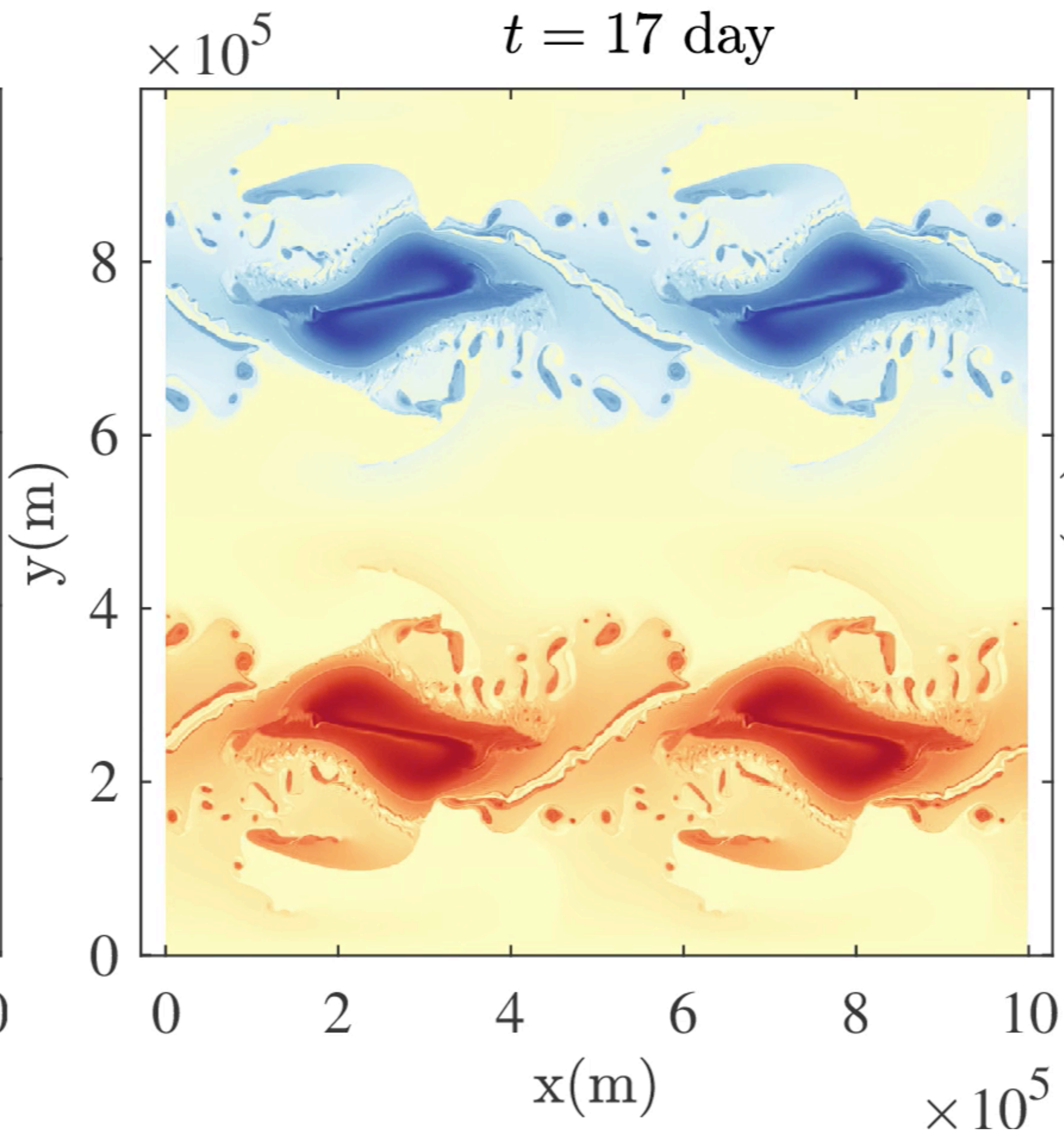
Deterministic 128 x 128



Deterministic 1024 x 1024



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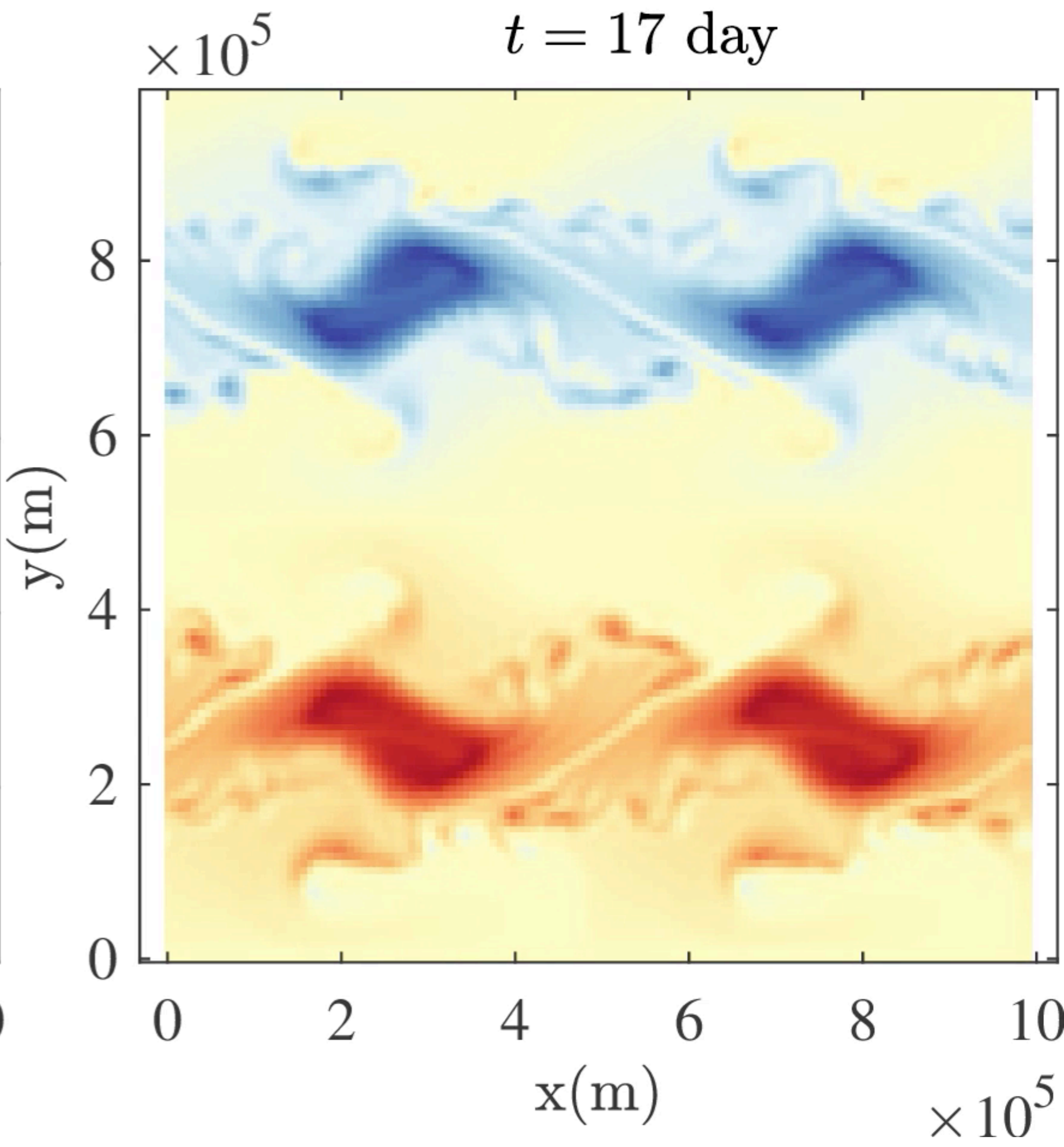

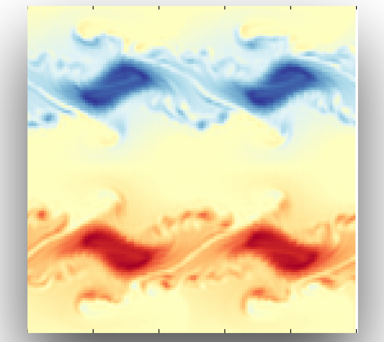

One realization : Stochastic destabilization

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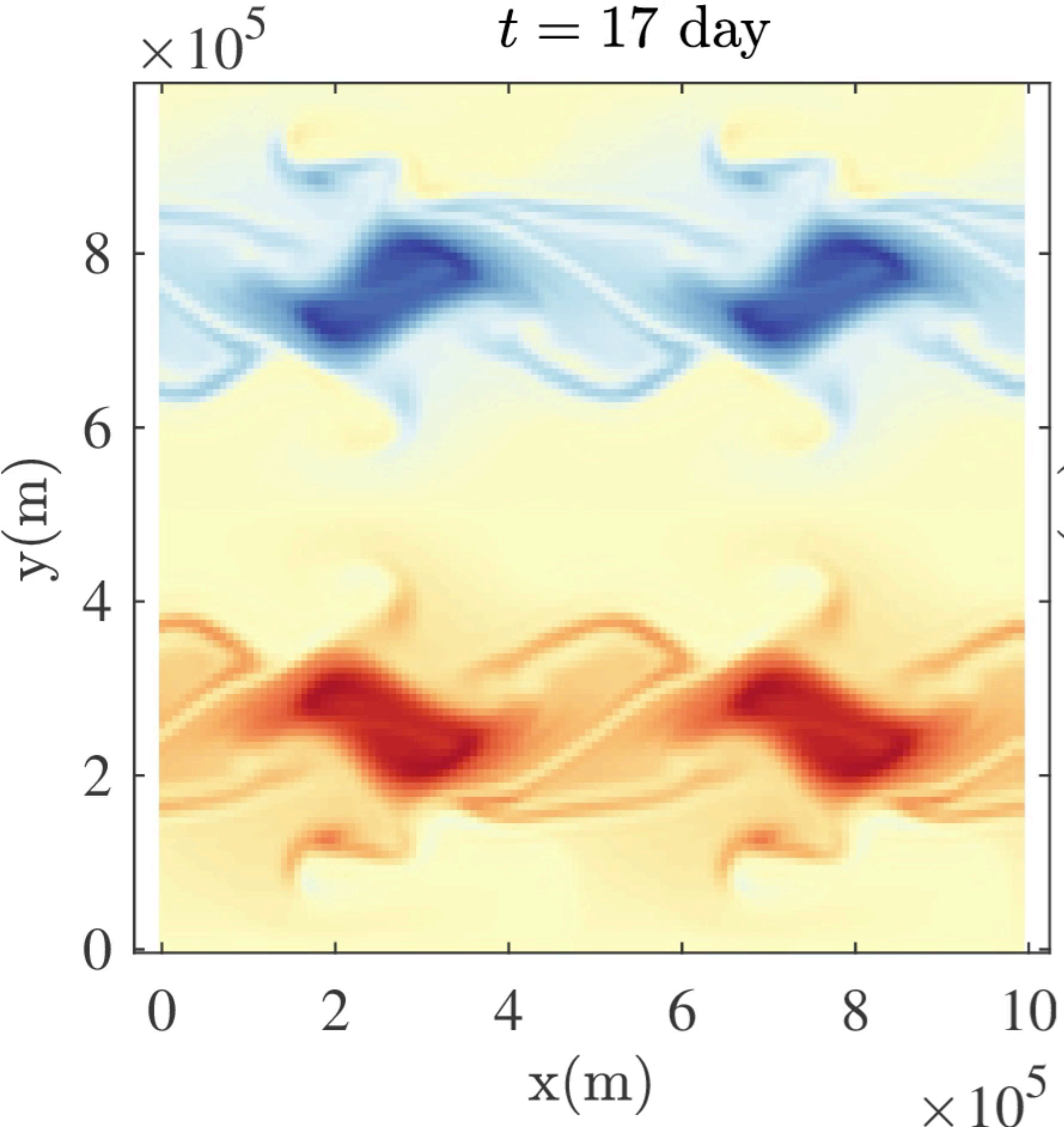

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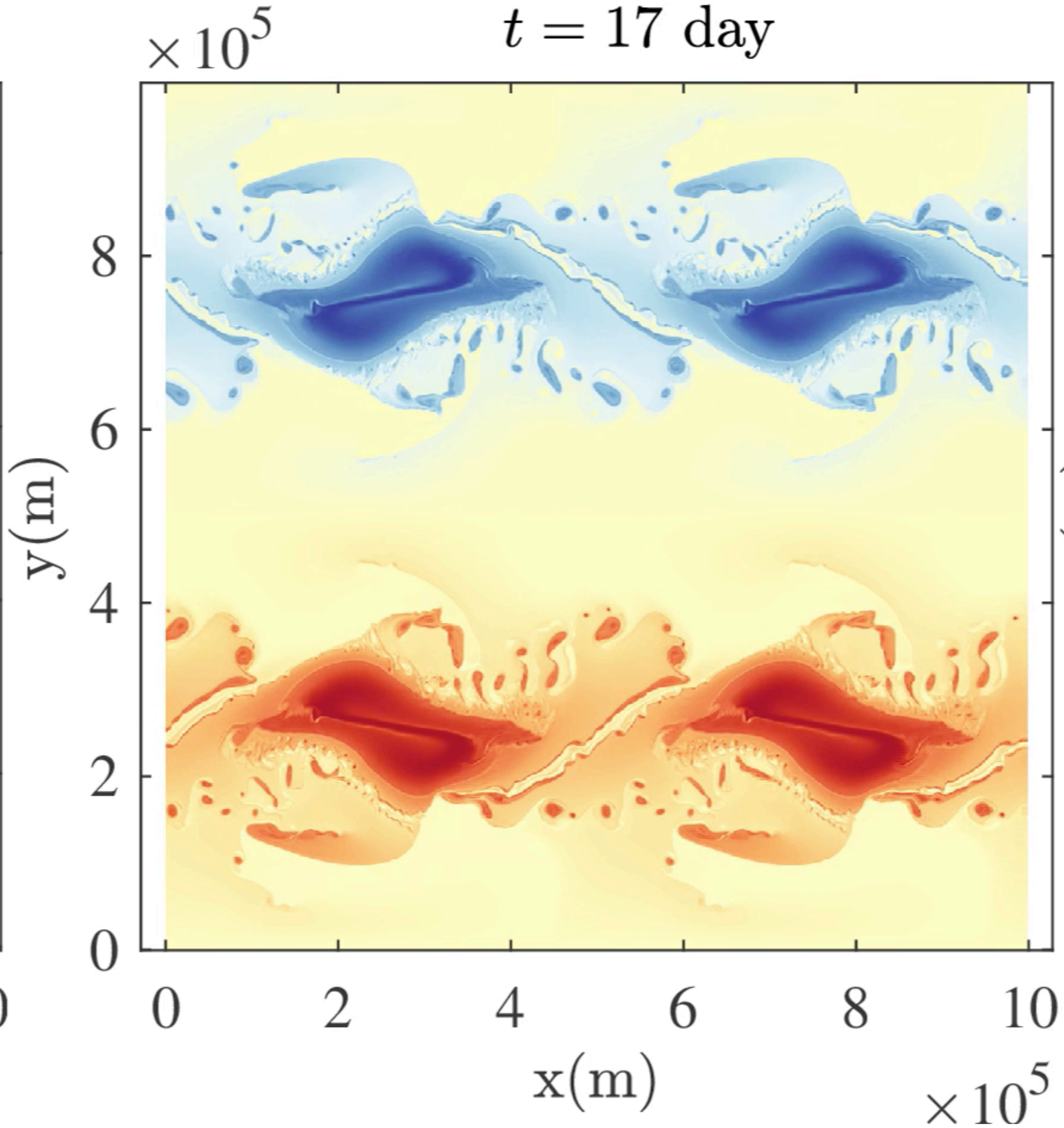

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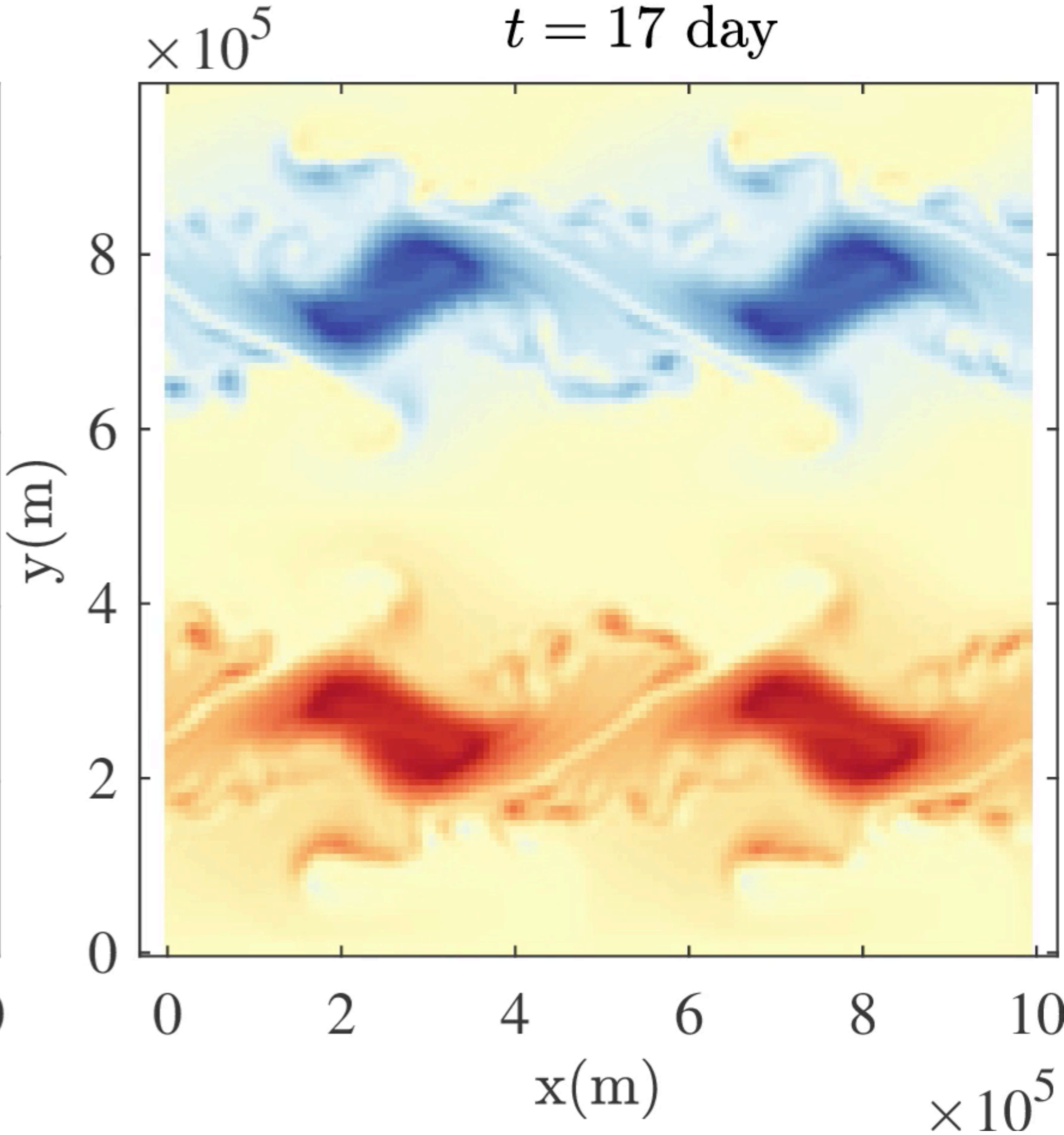
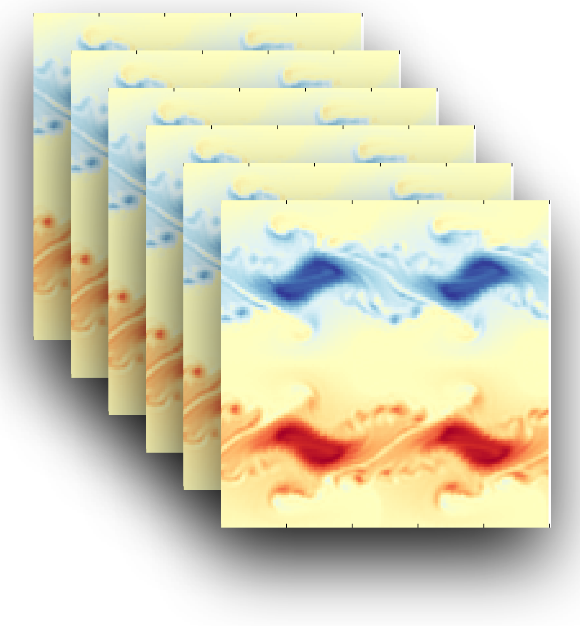
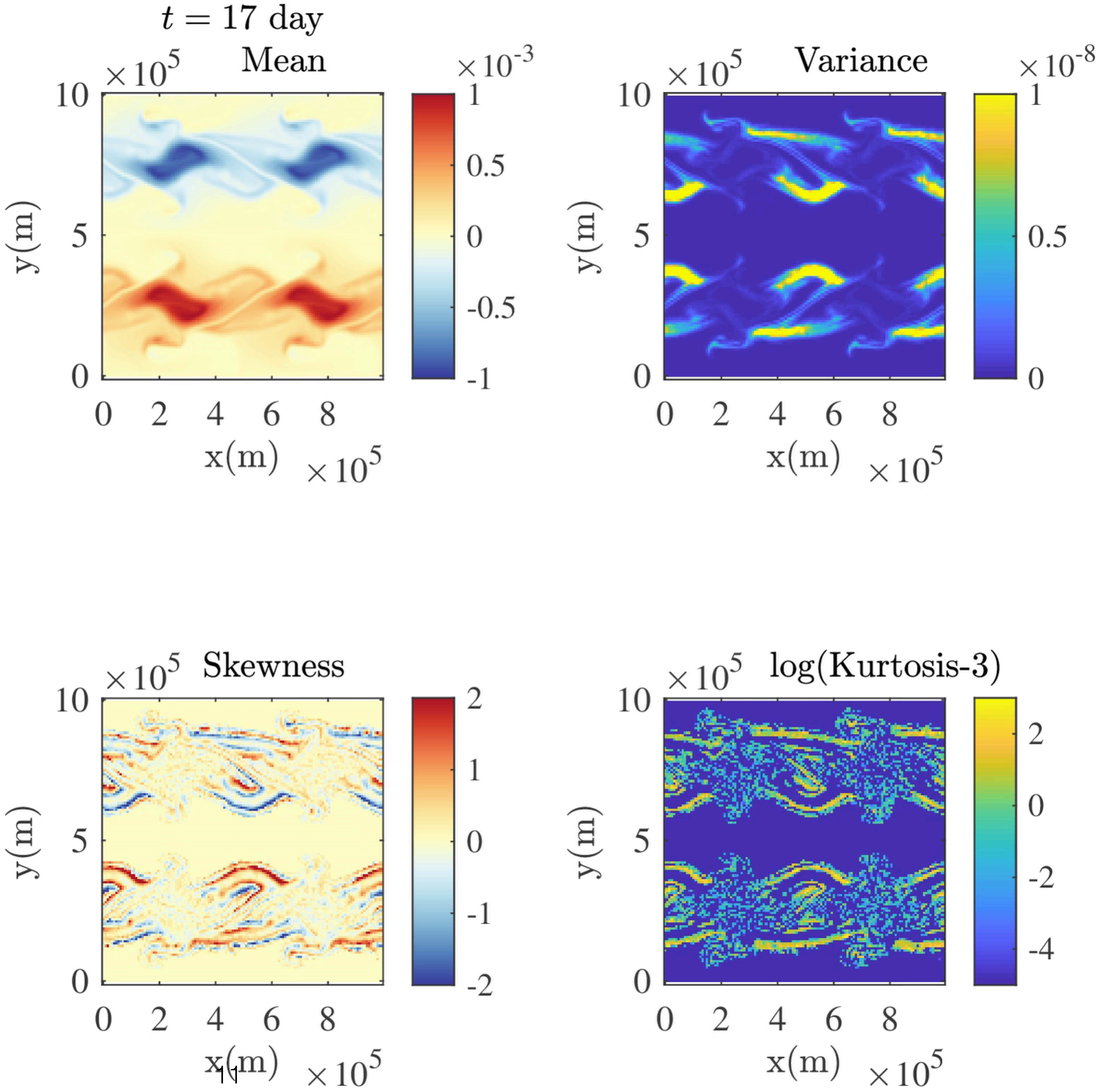
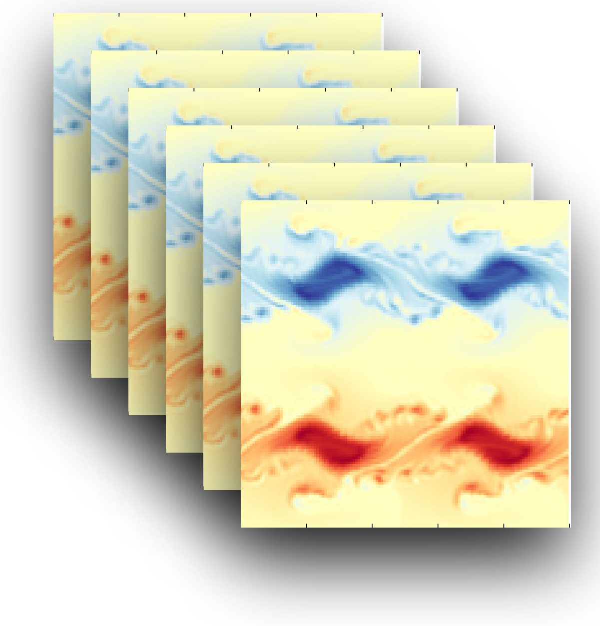




Ensemble: random coherent structures





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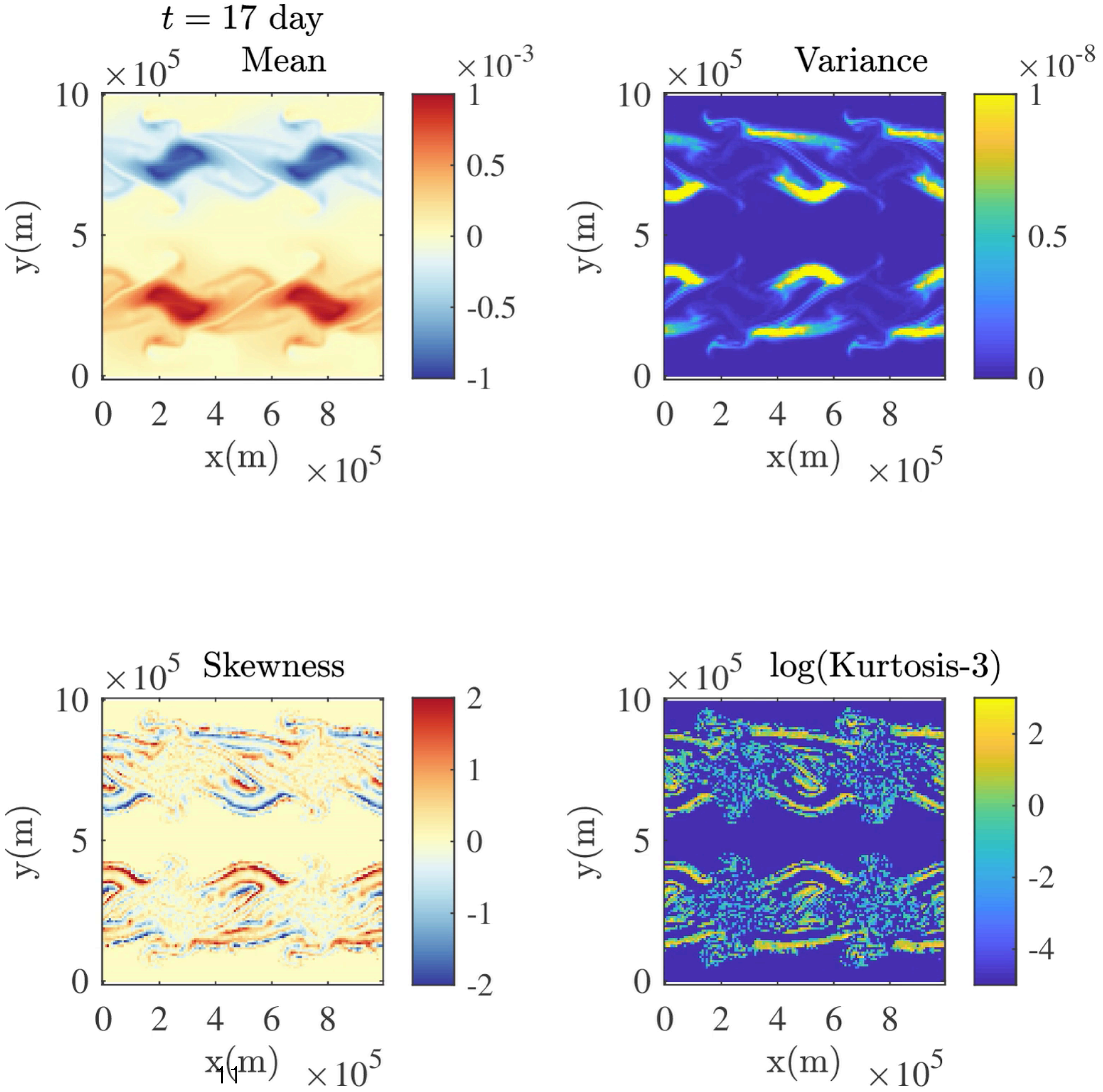
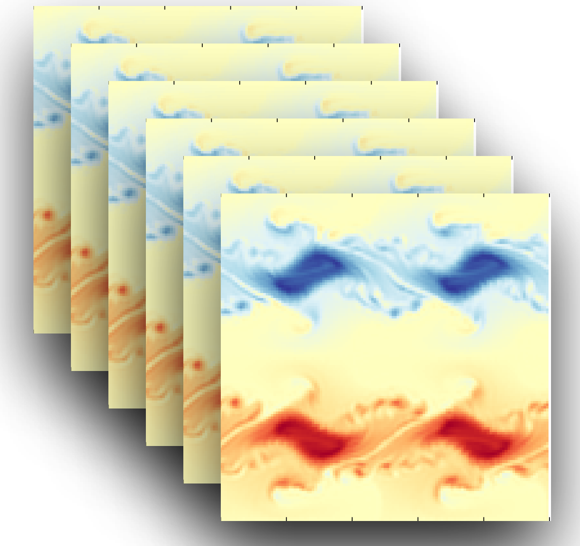
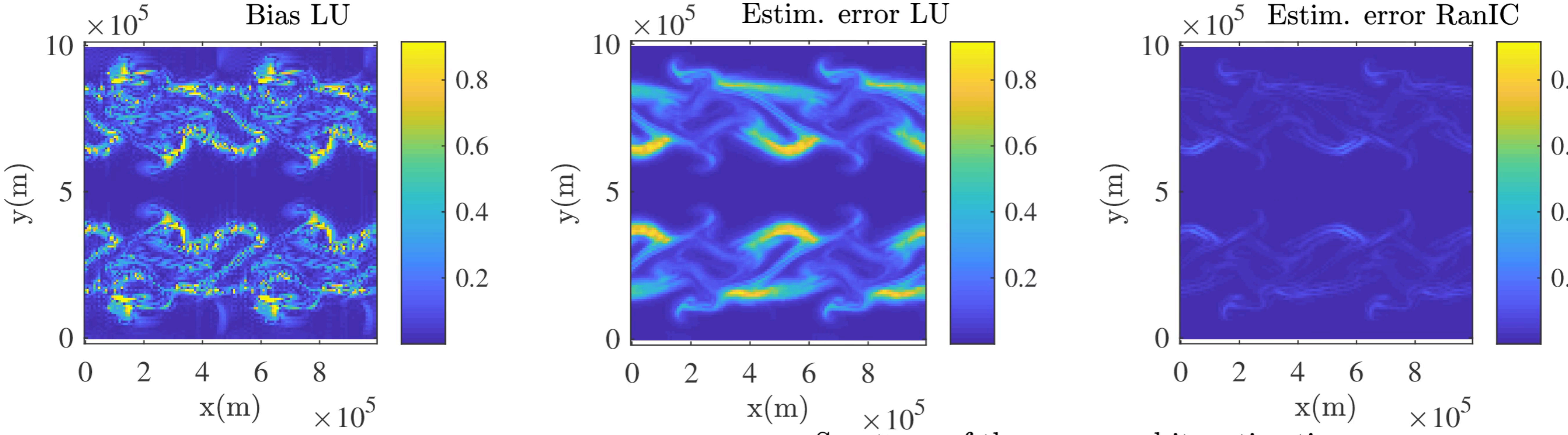
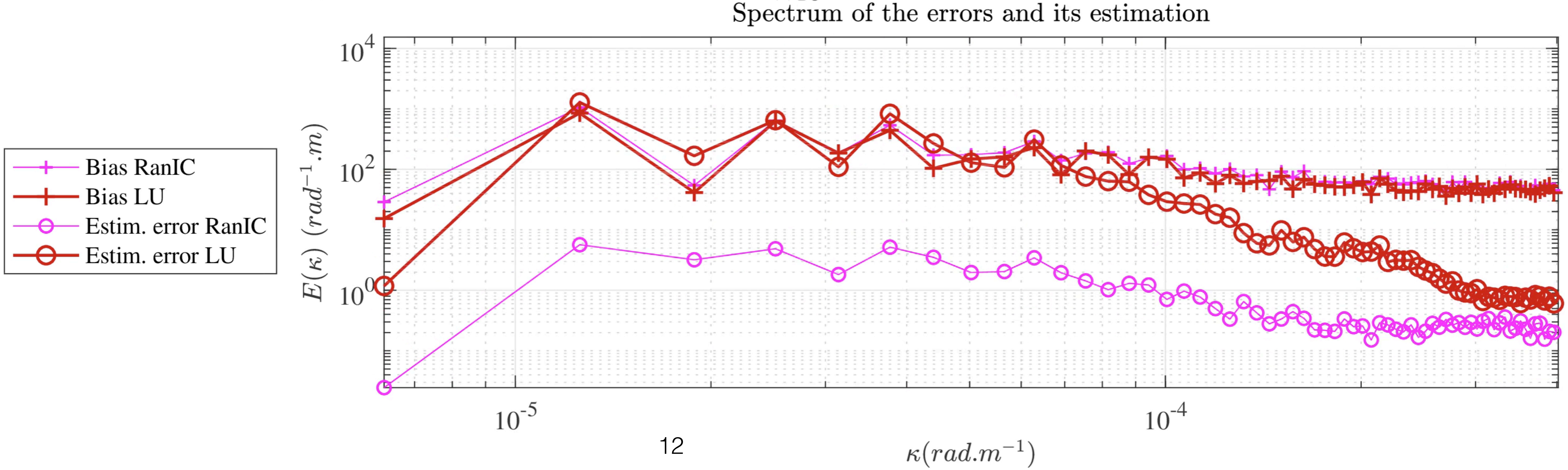
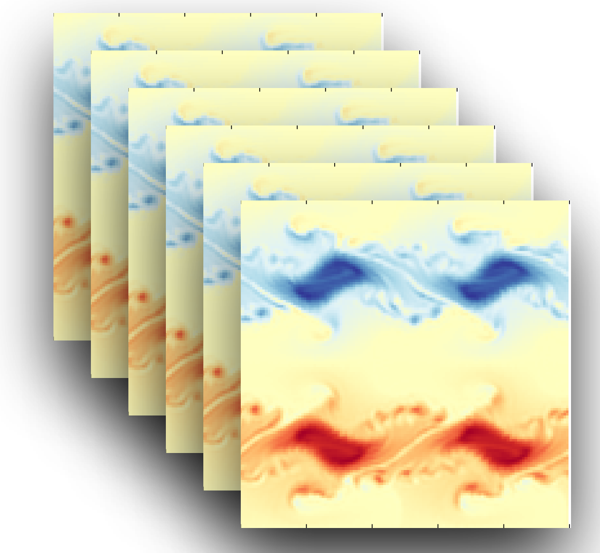




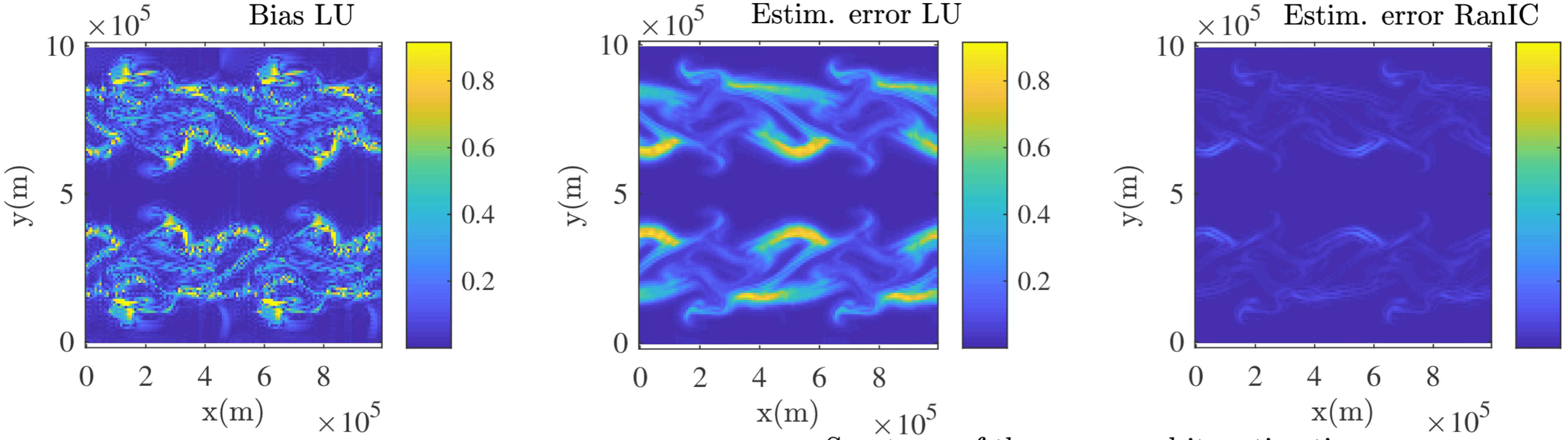
Ensemble : uncertainty quantification

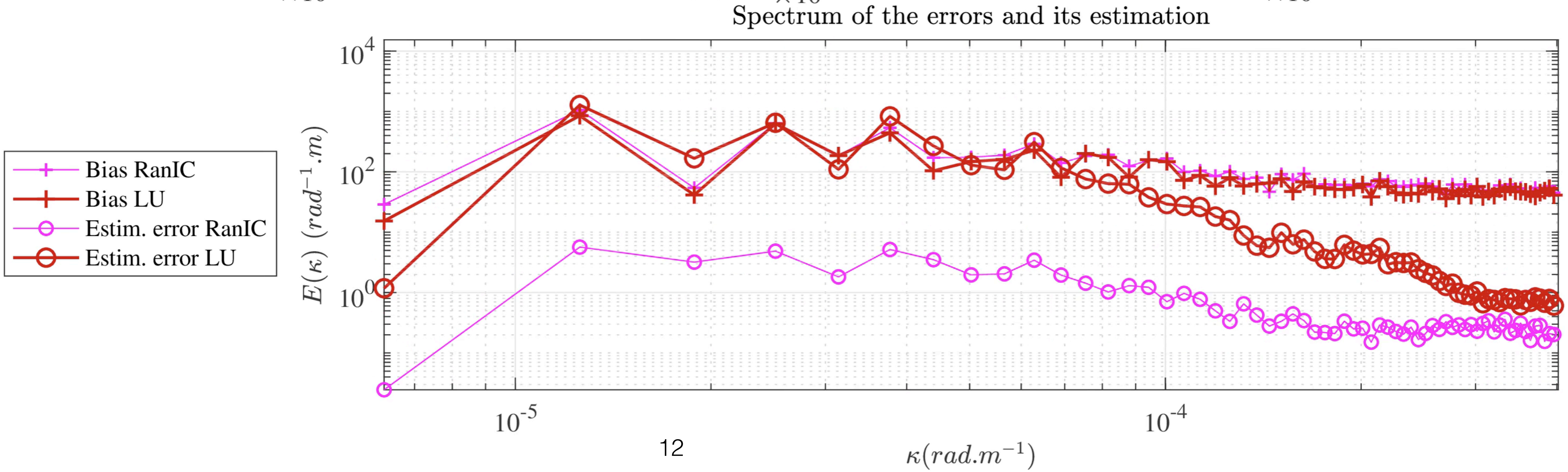






Ensemble : uncertainty quantification





Conclusion

Models under location uncertainty blindly describe unresolved triades

- Conserve energy
- Model derivation
- Instabilities triggered,
  possibly followed by extreme events
- Uncertainty quantification to address filter divergence

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Related works and outlooks

- Bifurcations (SQG) and attractor (Lorenz 63) exploration
- Stabilization / destabilization of Reduced Order Model
- Comparisons with data-driven $\boldsymbol{\sigma}$ and Stochastic Lie Derivative approaches (Holm and coauthors)
- Parametrization and tests based on higher-order statistics
  (curvature, energy flux through scales, bispectrum, …)
- Mimic barotropization
- Girsanov theorem for MLE and Bayesian estimations with satellite images
- Learning $\boldsymbol{\sigma}$ on SWOT data
- Filtering / smoothing with (reduced) models under location uncertainty

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